GRADUATE SEMINAR REPORT

ON

ASYMPTOTIC EXPANSIONS

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Preface

A complete understanding of mathematical problem always requires interpretation, and most often this requires approximation or numerical evaluation of complicated expressions. In this report, we tired to show how to get meaningful approximations of functions defined by a complicate definite integrals or contour integral expressions by means of asymptotic expansions.

The whole work of this report is divided in to two parts:

The first part contains a brief introduction to the general theory of asymptotic expansions. Here definitions, examples and basic properties (linear combination, differentiation, integration) of asymptotic and order relations, asymptotic sequence and asymptotic expansions are discussed in detail.

In part II, the most important methods of obtaining asymptotic expansions for the functions defined by integrals are developed. In this part various methods like method Integration by parts, Laplace's method, Method of steepest descents and Method of stationary phase are discussed.

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I. Introduction

There are series w hose t erm a t f irst d ecreases r apidly b ut l ater s tarts increasing again. Such series are used to be called semi-divergent (Stieltjes) and numerical computers often talk of convergently beginning series; but Mathematical literatures call them asymptotic series or expansions (Poincare').

Although, the theory of asymptotic series were introduced by many Mathematicians like Stirling,Euler,Legendre ;the modern theory of asymptotic series was initiated by Stieltjes(1886) and more developed by Poincare'(1854-1912) great French Mathematician in 1886.

Asymptotic expansions are generally divergent series, which has a great value in many computations.

- They play an important role in the solution of linear differential equations (about irregular singular points).
- They help to construct and investigate the series, which represents function given by complicated integral expression, or power series or else appears as solutions of differential equations, asymptotically.
- They help to compute values of function f(z) for large value of arguments(z)
- They were used by Astronomers.

1. Asymptotic and Order Relations

1.1. Definition and Examples

In order to describe the behavior of a function f(z) (as $z \to z_0$), in terms of a known function $\phi(z)$ we shall often use the following notations:

Suppose f (z) and $\phi(z)$ defined on a region R in the complex plane and let z_0 be a limit point of R, which may or may not belong to R, possibly the point at infinity, and the function $\phi(z)$ may vanish, tends to infinity or has another behavior at z_0, then we have the following definitions.

Definition 1: We say that f (z) is of order $\phi(z)$ (or f is "big Oh" of ϕ or f is of order not exceeding ϕ) as $z \to z_0$, denoted by f(z)=O($\phi(z)$) or simply f=O(ϕ) if there is a positive constant A > 0 and a neighbourhood U of z_0 such that $\frac{|f(z)|}{|g(z|} \le A \ , \forall z \in U \cap R.$

In other words if $\phi \neq 0$ in R, f (z)=O($\phi(z)$) means $\left|\frac{f(z)}{g(z)}\right|$ is bounded in the neighborhood of z_0.

Definition 2: We say that f(z) is of order less than $\phi(z)$ (or f is "little oh " of ϕ) as $(z \to z_0)$ denoted by f(z)=o($\phi(z)$) or f=o(ϕ), if for any $\varepsilon > 0$ there is a neighbourhood U_ε of z_0 such that $\left|\frac{f(z)}{g(z)}\right| \le \varepsilon \ , \forall z \in U_\varepsilon \cap R.$

In other words for $\phi \neq 0$ in R, f = o(ϕ) means $\left|\frac{f(z)}{g(z)}\right| \to 0 \ as\ z \to z_o$.

These two relations (O and o) which are given in definition 1 & definition 2 respectively are called order relations.

Remarks:

If we write as f = O(1) $(z \to z_0)$ it simply to mean that $|f|$ is bounded as $z \to z_0$ and f = o(1) $(z \to z_0)$ is to mean f vanishes as $z \to z_0$.

If the functions involved in order relation depends on parameters, then, in general also the constant A, and the neighborhood U, U_ε involved in the definition will depends on the parameters. If A, U, U_ε may be chosen to be independent of the parameters, then the order relation is said to be hold uniformly in parameters.

<u>Definition 3:</u> we say a function f (z) is asymptotic (or asymptotically equal or equivalent) to a function $\phi(z)$ or ϕ is an asymptotic approximation of f as $z \to z_0$, denotes as $f(z) \sim \phi(z)$ if $\left|\frac{f(z)}{g(z)}\right| \to 1$ (unity) as $z \to z_0$ or

$f(z) \sim \phi(z)$ iff $\lim_{z \to z_0}\left|\frac{f(z) - \phi(z)}{\phi(z)}\right| = 0$, where z_0 can be a constant or $+\infty$, and the relation is called an asymptotic relation.

Moreover from the definition it follows that if $f(z) \sim \phi_1(z)$ for $\theta_1 < \arg z < \theta_2$ and $f(z) \sim \phi_2(z)$ for $\theta_3 < \arg z < \theta_4$, where $\theta_3 < \theta_2$ and $\phi_1 \neq \phi_2$, then $\phi_1(z) \sim \phi_2(z)$ for $\theta_3 < \arg z < \theta_2$.

And also we note that an asymptotic relation is not unique relation; for instance

$2\cosh z \sim e^z$ for $|\arg z| < \dfrac{\pi}{2}$ as $z \to \infty$, and also

$2\cosh z \sim e^z + p_n(z)$ for $|\arg z| < \pi/2$ where $p_n(z)$ is any polynomial in z.

In the above all definition the function $\phi(z)$ is called the GAUGE function, being the function against which the behavior of f (z) is gauged.

Examples:

1. Since $\left|\dfrac{\sinh z}{e^z}\right| = \left|\dfrac{e^z - e^{-z}}{2e^z}\right| = \left|\dfrac{1}{2} - \dfrac{1}{2e^{2z}}\right| \le \frac{1}{2}$

   as $z \to \infty$.

   $\Rightarrow \left|\dfrac{\sinh z}{e^z}\right|$ is bounded as $z \to \infty$.

   Thus $\sinh z = O(e^z)$ as $z \to \infty$.

2. Since $\left|\dfrac{1/z^2}{1/z}\right| = \left|\dfrac{1}{z}\right| \to 0$ as $z \to \infty$.

   Thus $1/z^2 = o(1/z)$ as $z \to \infty$.

3. Since $\left|\dfrac{(z+1)^2}{z^2}\right| = \left|\dfrac{z^2 + 2z + 1}{z^2}\right| = \left|\dfrac{1}{z^2} + \dfrac{2}{z} + 1\right|$.

   $\Rightarrow \left|\dfrac{(z+1)^2}{z^2}\right| \to 1 \quad z \to \infty$.

   Hence $(z+1)^2 \sim z^2 \quad z \to \infty$.

Note that:

1. If f(z) =O($\phi(z)$) where z ∈ [a, ∞) = R we mean simply $\left|\frac{f(z)}{g(z)}\right|$ is

   bounded though the region R.

   i.e. It asserts that there is a number A > 0 such that

   $\left|\frac{f(z)}{g(z)}\right| \leq A \;\; (z \geq a)$ with out giving information concerning the actual size of A. Of course if the above formula holds a certain value of A, then it holds for every large value; thus there is an infinite set of positive A's.

   The least member of this set is the supremum (Least upper bound) of $\left|\frac{f(z)}{g(z)}\right|$ in the interval [a, ∞), we call it the **implied** constant of the O term for this interval.

2. The notation $O(\phi)$and $o(\phi)$can be also used to denote classes of function f with the properties of the definitions 1 and 2 respectively or unspecified functions with the property.

1.2. Operations in Asymptotic and Order Relations.

Operations with order relations are governed by a number of simple rules. We shall set out the more frequently used rules for the O-symbol:

The corresponding rules holds for the o-symbol. In the following rules the region R and z_0 are fixed, and the qualifying phrase "as "is omitted though out.

1. If f=O$(\phi)$and a > 0,then$|f|^a = O(|\phi|^a)$ (I)

2. If f_i =O(ϕ_i) i = 1,2...k and a_i's are constants, then

$$\sum_{i=1}^{k} a_i f_i = O(\sum_{i=1}^{k} a_i \phi_i) \tag{II}$$

This relation holds also for infinite series provided that f_i=O(ϕ_i) uniformly in i. In the case of infinite series, equation (II) and similar statements will be interpreted in the following manner.
If $\sum |a_i \phi_i|$ converges then so does $\sum a_i f_i$ and (II) $\sum_{i=1}^{\infty} |a_i f_i| = O(\sum_{i=1}^{\infty} |a_i \phi_i|)$ is true and if $\sum |a_i \phi_i|$ diverges then there nothing to state.

3. If f_i =O(ϕ_i), for finite i, a_i are constant, and $|\phi_i| \le \phi$, then ∀z common to R and to some neighbourhood U_0 and z_0 we have

$$\sum_{i=1}^{k} a_i f_i = O(\phi) \tag{III}$$

This relation holds for infinite series provided that f_i =O (ϕ_i) uniformly in i and $\sum |a_i| < \infty$.

4. If f_i =O (ϕ_i) i = 1,2...k, then

$$\prod_i f_i = O(\prod_i \phi_i) \tag{IV}$$

Proof: The proof of (1) is immediate.

To proof (2), by assumption, $f_i = O(\phi_i)$

$\Rightarrow$ there are numbers A_i and a neighbourhood U_i of z_0 associated with f_i such that $\left|\frac{f_i}{\phi_i}\right| \leq A_i, \forall i$

i) If the number of f_i is finite, there is an $A \geq A_i$, $\forall$i and a neighbourhood

$U \subseteq U_i$, $\forall$i such that $\left|\sum a_i f_i\right| \leq \sum |a_i| A_i |\phi_i| \leq A \sum |a_i| |\phi_i|$

for $z \in R \cap U$.

$$\Rightarrow \quad \left|\frac{\sum a_i f_i}{\sum a_i \phi_i}\right| \leq A \quad \Rightarrow \quad \sum_i a_i f_i = O\{\sum a_i \phi_i\}$$

ii) If the number of f_i is infinite, then the existence of A and U follows from the uniformity in i, of the order relation. The proof (3) can be deduced from the proof of (2) since under the circumstances enucsaged we may take U about to be contained in U_0 and then $A\sum |a_i||\phi_i| \leq A\sum |a_i| \phi = A_1\phi$, where $A_1 = A\,\Sigma|a_i|$ is a finite number (constant).The proof of (4) is similar to that of (2).

5.We have also the following combinations of order relations, which can be easily verified.

(i) $O(O(f)) = O(\phi)$.

(ii) $O(o(f)) = o(O(f)) = o(o(\phi)) = o(\phi)$.

(iii) $O(f)\, O(\phi) = O(f\phi)$.

(iv) $O(f)\, o(\phi) = o(f)\, o(\phi) = o(f\phi)$.

(v) $O(f) + O(f) = O(f) + o(f) = O(f)$.

(vi) $o(f) + o(f) = o(f)$.

1:3 Differentiation and Integration of Asymptotic and Order Relations.

1.3.1 Differentiation

Differentiation of asymptotic or order relations is not always permissible.

For example, if $f(z) = z + \cos z$, then f (z) ~ z as z→ ∞,

but it is not true that $f'(z) \sim 1$.

i.e. $f'(z)$= 1- sin z is not asymptotic to $\dot{z} = 1$ as z→ ∞.

To assume the legitimacy of differentiation further conditions, are needed. For real variables, these conditions can be expressed in terms of the monotonicity of derivatives.

Now we have the following theorem:

Theorem 1: Let f (z) be continuously differentiable and f (z) ~ z^p as z→ ∞, where p ≥ 1 is constant. Then $f'(z) \sim pz^{p-1}$, provided that $f'(z)$ is non - decreasing for all sufficiently large z.

Proof: Let $f(z) = z^p\{1 + \eta(z)\}$ where $|\eta(z)| \le \varepsilon$ when z > Z, assignable and positive, and ε being an arbitrary number in the interval (0,1). If h > 0, then

$$hf'(z) \le \int_z^{z+h} f'(t)dt = f(z+h) - f(z)$$

$$= \int_z^{z+h} pt^{p-1}dt + (z+h)^p\eta(z+h) - z^p\eta(z) \le hp(z+h)^{p-1} + 2\varepsilon(z+h)^p$$

Let $h = \sqrt{\varepsilon}\, z$ then we have

$$f'(z) \le pz^{p-1}\left[\left(1+\sqrt{\varepsilon}\right)^{p-1} + 2p^{-1}\sqrt{\varepsilon\left(1+\sqrt{\varepsilon}\right)^p}\right], \text{(z > Z)} \qquad (1)$$

Similarly $f'(z) \geq pz^{p-1}\left[\left(1-\sqrt{\varepsilon}\right)^{p-1} - 2p^{-1}\sqrt{\varepsilon}\right]$, for $\left(z > \dfrac{Z}{1-\sqrt{\varepsilon}}\right)$ (2)

Thus from (1) and (2), we have

$$pz^{p-1}\left[\left(1-\sqrt{\varepsilon}\right)^{p-1} - 2p^{-1}\sqrt{\varepsilon}\right] \leq f'(z) \leq pz^{p-1}\left[\left(1+\sqrt{\varepsilon}\right)^{p-1} + 2p^{-1}\sqrt{\varepsilon\left(1+\sqrt{\varepsilon}\right)^{p}}\right]$$

$$\Rightarrow \; f'(z) \sim pz^{p-1}.$$

In complex plane, differentiation of asymptotic and order relations is generally permissible in sub regions of the original region of validity.

Theorem 2: - Let f(z) be holomorphic (i.e. f is analytic and free from singularity) function in a region containing a closed annular sector S and f (z) =O(z^p) (or f (z) =o(z^p) (1)
as z→ ∞ in S, where p is any fixed real number, then

$$f^{(m)}(z) = O(z^{p-m}) \quad \text{or} \quad f^{(m)}(z) = o(z^{p-m}) \qquad (2)$$

as z→ ∞ in any closed annular sector C properly interior to S and having the same vertex.

Proof: Consider the cauchy's integral formula for the m$^{\text{th}}$ derivative of an analytic function given by $f^{(m)} = \dfrac{m!}{2\pi i}\displaystyle\int_{\varphi}\frac{f(t)dt}{(t-z)^{m+1}}$ (3)

in which the path φ is chosen to be circle enclosing t=z .

The essential reason z is restricted to an interior region in the final result is to permit inclusion of φ in S. Since $|z-c|^{p} \sim |z|^{p}$, for a constant number C, the vertex of S may be taken to be the region with out loss of generality.

Let S be defined by $\alpha \leq ph\ z \leq \beta$, $|z| \geq R$, and consider the annular sector S' to be defined by $\alpha + \delta \leq ph\ z \leq \beta - \delta$, $|z| \geq R'$, where δ is a positive acute and $R' = \dfrac{R}{1 - \sin\ \delta}$.

By taking δ small enough we can ensure that S' contains C in (3) take φ to be $|t - z| = |z| \sin \delta$.

Than $|z|\ (1 - \sin\ \delta) \leq |t| \leq |z|\ (1 + \sin\ \delta)$.

Hence $t \in S$ whenever $z \in S'$. Moreover, if K is the implied constant of (1) for S, then

$\left|f^{(m)}(z)\right| \leq \dfrac{m!}{(|z| \sin\ \delta)^m} K|z|^p (1 \pm \sin\ \delta)^p$, the upper or lower sign being taken a ccording a s p $\geq$ o r p< 0 .In e ither c ase $f^{(m)}(z)$ *is* $O(z^{p-m})$, a s required. The proof in when the symbol O in (1) and (2) is replaced by o is similar. We have shown, incidentally that t he i mplied c onstant o f (2) i n S ' does not exceed $m!\,(\csc\ \delta)^m\ (1 \pm \sin\ \delta)^p\ K$, but because this bound tends to infinity as $\delta \to 0$ we cannot infer that (2) is valid in S.

1.3.2 Integration

Order relation may be integrated either with respect to the independent variable or with respect to parameters. For the sake of simplicity we shall restrict our selves to integrals with respect to real variables. Extension to complex and abstract variables is possible.

Theorem 3: - Let z be a real variable and R be the interval a < z < b and let $f = O(\phi)$ as z→ b .

If f and ϕ are measurable in R, then

$$\int_z^b f(t)dt = O\left(\int_z^b \phi(t)\, dt\right), \text{ as } z \to b.$$

Proof: -If $\int_z^b |\phi(t)|\, dt = \infty$ there is nothing to proof.

Suppose $\int_z^b |\phi(t)|\, dt = \infty$ for some z ∈ R .Now since f = O(f), there is a positive number A and Z, so that $\int_z^b |\phi(t)|\, dt = \infty$

and $|f(z)| \le A|\phi(z)|$ for Z < z < b.

$$\Rightarrow \left|\int_z^b f(t)\, dt\right| \le \int_z^b |f(t)|\, dt \le \int_z^b |\phi(t)|\, dt \le A\int_z^b |\phi(t)|\, dt, \text{ for } Z < z < b.$$

$$\Rightarrow \left|\frac{\int_z^b f(t)\, dt}{\int_z^b \phi(t)\, dt}\right| \le \frac{\int_z^b |f(t)|\, dt}{\int_z^b |\phi(t)|\, dt} \le A$$

$$\Rightarrow \int_z^b f(t)dt = O\left(\int_z^b \phi(t)\, dt\right) \text{ as } z \to b.$$

with respect to parameters we state the following theorem with out the proof.

Theorem 4: - Let z be a variable in the set R, let γ be a real parameter, $\alpha < \gamma < \beta$, and let $f(z, \gamma) = O(\phi(z, \gamma))$, uniformly in γ, as $z \to z_0$.

If for each fixed z ∈ R, f and ϕ are measurable functions of γ in $\alpha < \gamma < \beta$,

then $\int_\alpha^\beta f(z,\gamma)\, d\gamma = O\left(\int_\alpha^\beta f(z,\gamma)\, d\gamma\right)$ as $z \to z_0$.

2. Asymptotic sequences

2.1. Definition and examples.

A sequence $\{\phi_n(z)\}$ of gauge functions defined on a region R is called an asymptotic sequence as $z \to z_0$, if for each n

$$\phi_{n+1} = o(\phi_n) \text{ as } z \to z_0 \text{ in R.} \quad \text{i.e. if } \left|\frac{\phi_{n+1}}{\phi_n}\right| \to 0 \quad \text{as} \quad z \to z_0 .$$

Examples: a) the sequence $\{(z-z_0)^n\}$ is an asymptotic sequence as $z \to z_0$, since for ε > 0, there is a neighborhood U_ε of z_0, such that

$$\lim_{z \to z_0} \frac{(z-z_0)^{n+1}}{(z-z_0)^n} \le \varepsilon \qquad \text{i.e.} \left|\frac{(z-z_0)^{n+1}}{(z-z_0)^n}\right| \to 0 \text{ as } \quad z \to z_0$$

b) $\{z^{-n}\}$ is also an asymptotic sequence as $z \to \infty$ as can be easily verified.

2:2 properties of asymptotic sequences.

(i) Any sub sequence of asymptotic sequence is again asymptotic sequence.

(ii) If $\{\phi_n\}$ is an asymptotic sequence and a > 0, then $\{|\phi_n|^a\}$ is also an asymptotic sequence.

(iii) If $\{\phi_n\}$ and $\{\varphi_n\}$ are asymptotic sequence containing the same number of functions, then $\{\phi_n \varphi_n\}$ is an asymptotic sequence . i.e. the product of asymptotic sequence of the same function is an asymptotic sequence.

(iv) The definite integral of the asymptotic sequence (either with respect of variables or parameters) is again an asymptotic sequence if the corresponding integrals exist i.e.

- If z is a variable in a region R= $a < z < b$, $\{\phi_n\}$ be an asymptotic sequence as $z \to b$ and if all integrals $\varphi_{n}(z) = \int_z^b |\phi(t)|\, dt$ exist, then $\{\varphi_n\}$ is an asymptotic sequence as $z \to b$.
- If $\{\phi_n(z,\gamma)\}$ be an asymptotic sequence uniformly in a parameter $\gamma \in (\alpha,\beta)$ for $z \to z_0$ in R ,and if all integrals $\varphi_{n}(z) = \int_\alpha^\beta |\phi_n(z,\gamma)|\, d\gamma$ exist, then $\{\varphi_n\}$ is an asymptotic sequence.

(ii) Two sequences $\{\phi_n\}$ and $\{\varphi_n\}$ are said to be equivalent to each other, if for each n, $\phi_n = O(\varphi_n)$ and $\varphi_n = O(\phi_n)$. Now if $\{\phi_n\}$ and $\{\varphi_n\}$ are equivalent to each other and if $\{\phi_n\}$ is an asymptotic sequence, then $\{\varphi_n\}$ is also an asymptotic sequence.

Proof: -The proof of each properties (i-v) follows directly from the properties of order relation discussed under 2:2 above.

Now we prove property (v) as follows:

Since the two sequences are equivalent and $\{\phi_n\}$ is asymptotic sequence

$$\Rightarrow \quad \varphi_n = O(\phi_n) \text{ and } \phi_{n+1} = o(\phi_n)$$

Now using order property 5(ii), we have

$\varphi_{n+1} = O(\phi_{n+1}) = O(o(\phi_n)) = O(o(O(\varphi_n))) = O(o(\phi_n))$, since by 5(ii) $o(O(\varphi_n)) = o(\phi_n)$

$= o(\varphi_n)$------- again by 5(ii) $o(O(\varphi_n)) = o(\phi_n)$

Hence $\varphi_{n+1} = o(\varphi_n)$, which implies $\{\varphi_n\}$ is an asymptotic sequence.

The differentiation of asymptotic sequence does not necessarily yields an asymptotic sequence.

For instance the sequence $\{\phi_n(z)\} = \{z^{-n}(a + \cos(z^n))\}$ is an asymptotic sequence for n= 1,2,3... as $z \to \infty$.

But $\{\phi_n'\}$ is not asymptotic sequence.

3. Asymptotic Expansions

3.1 Definition and Examples

Definition 1: - Let f(z) be a function real or complex variables defined on a region R and let $\{\phi_n\}$ be an asymptotic sequence for $z \to z_0$ in R;

and let a be a constant (independent of the variable z). Then the series $\sum_{n=0}^{\infty} a_n \phi_n(z)$ (which my be convergent or divergent) is said to be an asymptotic expansions (to m terms) of the function f (z) (with respect to the asymptotic sequence $\{\phi_n(z)\}$) if for every value of m,

$$f(z) - \sum_{n=0}^{m} a_n \phi_n(z) = o(\phi_m) \text{ as } z \to z_0.$$

i.e. if
$$\lim_{z \to z_0} \left| \frac{f(z) - \sum_{n=0}^{m} a_n \phi_n(z)}{\phi_m(z)} \right| = 0$$

Notation:

a) if f (z) has an asymptotic expansion to m terms, we denote it as

$$f(z) \sim \sum_{n=0}^{m} a_n \phi_n(z) \text{ as } z \to z_0 .$$

b) If f(z) possesses an asymptotic an expansion to any number of terms (i.e. with m = ∞) we denote it as $f(z) \sim \sum_{n=0}^{\infty} a_n \phi_n(z)$ as $z \to z_0$ and called an asymptotic expansion of f (z).

c) The partial sum of this formal series will be called an asymptotic approximation (representation) to f (z) .The first term $(a_0\phi_0(z))$ is called dominant term and written as $f(z) \sim a_0\phi_0(z)$ meaning

$$\frac{f(z)}{\phi_0(z)} \to a_0 \text{ as } z \to z_0.$$

If an asymptotic expansion to m terms, with m finite, involves certain parameters, w e s hall s ay t hat i t h olds u niformity i n t hese p arameters i f t he reminder in $o(\phi_0)$ uniformly in the parameters.

An asymptotic expansion (m = ∞) involving certain parameters will be said to hold uniformly in these parameters if $f - \sum_{n=o}^{m} a_n\phi_n = o(\phi_m)$,uniformly in the parameters for each sufficiently large m (but not necessarily uniformly in m).

Remark: If our asymptotic sequence $\{\phi_n(z)\} = \{z^{-n}\}$ we say that the series $\sum_{n=0}^{\infty} a_n \phi_n(z) = \sum_{n=0}^{\infty} z^{-n}$ is asymptotic expansion to the function f(z) if

$$\lim_{z\to\infty} \left| z^m \left(f_m(z) - f(z)\right) \right| = 0 \quad \text{, for fixed n where } f_m(z) = \sum_{n=0}^{m} z^{-n}$$

in this case we call the series $\sum_{n=0}^{\infty} z^{-n}$ as an asymptotic power series.

Or simply an asymptotic expansion with respect to the asymptotic sequence $\{z^{-n}\}$ is called asymptotic power series.

Examples of asymptotic expansion:

1. $\frac{1}{z-1} \sim \sum_{n=1}^{\infty} z^{-n}$ as $z \to \infty$.

2. $\frac{z+1}{z^2-1} \sim \sum_{n=1}^{\infty} \frac{z+1}{z^{2n}}$ as $z \to \infty$.

3. $\frac{\sin z}{z} \sim \sum_{n=1}^{\infty} \frac{n!\, e^{\frac{-(n+1)z}{2n}}}{(\ln z)^n}$ $\quad for\ \phi_n(z) = \{(\ln z)^{-n}\}$ as $z \to \infty$.

In general asymptotic expansions are divergent.

3.2. Properties of Asymptotic Expansions.

Theorem 1:- If a function f (z) possesses an asymptotic expansion

(to m terms) then the coefficients of an asymptotic expansion are given

successfully by $$a_m = \lim_{z \to z_0} \left\{ \frac{f(z) - \sum_{n=0}^{m-1} a_n \phi_n(z)}{\phi_m(z)} \right\} \qquad (*)$$

and conversely suppose we have functions $f(z), \phi_0(z), \phi_1(z), \phi_2(z), \ldots, \phi_m(z)$ defined in some region R, and if

$$a_m = \lim_{z \to z_0} \left\{ \frac{f(z) - \sum_{n=0}^{m-1} a_n \phi_n(z)}{\phi_m(z)} \right\}$$ and $a_m \neq 0$,then $\{\phi_n\}$ is an

asymptotic sequence and $\sum_{n=1}^{\infty} a_n \phi_n$ is an asymptotic expansion (to m terms) of

f(z) as $z \to z_0$.

Proof: - (⇒) suppose an asymptotic expansion of f(z) exists.

$$\Rightarrow f(z) \sim \sum_{n=0}^{\infty} a_n \phi_n(z)$$

which intern implies for a fixed m, $f(z)-\sum_{n=0}^{m} a_n\phi_n(z) = o(\phi_m(z))$

$$\Leftrightarrow \lim_{z\to z_0}\left(\frac{f(z)-f_m(z)}{\phi_m(z)}\right)=0 \text{, where } f_m = \sum_{n=0}^{m} a_n\phi_m .$$

$$\Rightarrow f_m = f_{m-1} + a_m\,\phi_m = \sum_{n=0}^{m-1} a_n\phi_n + a_m\,\phi_m \text{, so that } a_m = \frac{f_m - f_{m-1}}{\phi_m}$$

$$\Rightarrow \lim_{z\to z_0} = \lim_{z\to z_0}\left(\frac{f_m(z) - f_{m-1}(z)}{\phi_m(z)}\right)$$

$$\therefore\ a_m = \lim_{z\to z_0}\left(\frac{f(z)-f_{m-1}(z)}{\phi_m(z)}\right) = \lim_{z\to z_0}\left\{\frac{f(z)-\sum_{n=0}^{m-1} a_n\phi_n(z)}{\phi_m(z)}\right\}$$

Note that:

since $f(z) \sim \sum_{n=0}^{m} a_n\,\phi_n(z) = f_m(z)$

$\Rightarrow f(z) \sim f_m(z)$ as $z \to z_0$. or $\frac{f(z)}{f_m(z)} \to 1$ as $z \to z_0$.

$\Rightarrow f(z) \to f_m(z)$ as $z \to z_0$.

i.e. f(z)and $f_m(z)$ are asymptotically equivalent or equal.

(⇐) Conversely suppose $f(z), \phi_0(z), \phi_1(z), \phi_2(z), \phi_3(z), \ldots, \phi_m(z)$ defined in some region R, and let $a_m = \lim_{z \to z_0} \left\{ \dfrac{f(z) - \sum_{n=0}^{m-1} a_n \phi_n(z)}{\phi_m(z)} \right\}, a_m \neq 0$, then

Claim i) $\{\phi_n\}$ is an asymptotic sequence. i.e. we went to show $\phi_{m+1} = o(\phi_m)$.

Now, since $a_m = \lim_{z \to z_0} \left\{ \dfrac{f(z) - \sum_{n=0}^{m-1} a_n \phi_n(z)}{\phi_m(z)} \right\}$ (*)

$$\Rightarrow \lim_{z \to z_0} \left\{ \frac{f(z) - \sum_{n=0}^{m-1} a_n \phi_n(z)}{\phi_m(z)} \right\} - a_m = 0$$

$$\Rightarrow \lim_{z \to z_0} \left(f - \sum_{n=0}^{m-1} a_n \phi_n - a_m \phi_m \right) = 0 \quad \Rightarrow \lim_{z \to z_0} \left\{ \frac{f(z) - \sum_{n=0}^{m} a_n \phi_n(z)}{\phi_m(z)} \right\} = 0$$

$$\Rightarrow f - \sum_{n=o}^{m} a_n \phi_n = o(\phi_m) \qquad (1)$$

Similarly if we replace m by m+1 in (*) we have

$$f - \sum_{n=o}^{m} a_m \phi_m = a_{m+1} \phi_{m+1} + o(\phi_{m+1}) \qquad (2)$$

Thus, from (1) and (2) we get

$$a_{m+1} + o(\phi_{m+1}) = f - \sum_{n=o}^{m} a_n \phi_n = o(\phi_m)$$

$$\Rightarrow [a_{m+1} + o(1)] \phi_{m+1} = o(\phi_m) \qquad (**)$$

Now if $a_{m+1} \neq 0$, then $(a_{m+1} + o(1)) \neq 0$ for some z in the neighborhood of z_0

and thus from (**) it follows that $\lim_{z \to z_0} \frac{[a_{m+1} + o(1)]\phi_{m+1}}{\phi_m} = 0$

$$= (a_{m+1} + o(1)) \lim_{z \to z_0} \frac{\phi_{m+1}}{\phi_m} = 0$$

$\Rightarrow \quad \lim_{z \to z_0} \frac{\phi_{m+1}}{\phi_m} = 0$ or $\left|\frac{\phi_{m+1}}{\phi_m}\right| \to 0$ as $z \to z_0$.

$\Rightarrow \quad \phi_{m+1} = o(\phi_m)$. Hence $\{\phi_m\}$ is an asymptotic sequence.

ii) More over from (*), i.e. from $a_m = \lim_{z \to z_0} \left\{ \frac{f(z) - \sum_{n=0}^{m-1} a_n \phi_n(z)}{\phi_m(z)} \right\}$ we get

(1) i.e. $f - \sum_{n=o}^{m} a_n \phi_n = o(\phi_m)$. Hence $\sum_{n=0}^{\infty} a_n \phi_n(z)$ is asymptotic expansion of f (to m terms) as $z \to z_0$.

2. If $\sum_{n=0}^{\infty} a_n \phi_n(z)$ is asymptotic expansion of f (to m terms), then the same formal series will also provide an asymptotic expansions any lesser number of terms of the same function. i.e.

$f - \sum_{n=o}^{m-1} a_n \phi_n = O(\phi_m)$, as $z \to z_0$.Since from theorem 1 above if f possess an asymptotic expansion ,then the coefficient a_m is given by

$$a_m = \lim_{z \to z_0} \left\{ \frac{f(z) - \sum_{n=0}^{m-1} a_n \phi_n(z)}{\phi_m(z)} \right\}, a_m \neq 0$$

$$\Rightarrow \left| \frac{f - \sum_{n=0}^{m+1} a_n \phi_n}{\phi_m} \right|$$ is bounded by a constant a_m as $z \to z_0$.

Hence $f - \sum_{n=o}^{m-1} a_n \phi_n = O(\phi_m)$.

3. The asymptotic expansion of a given function is unique, although it depends on the choice of the asymptotic sequence.

Example: 1. $\frac{1}{z-1} \sim \sum_{n=1}^{\infty} z^{-n}$ and $\frac{1}{z-1} \sim \sum (z+1)\ z^{-2n}$ as $z \to \infty$.

2. $\frac{1}{z+1} \sim \sum (-1)^{n-1}\ z^{-n}$ *and* $\frac{1}{z+1} \sim (-1)^{n+1} \frac{(z^2 - z + 1)}{z^{3n}}$ as $z \to \infty$.

From property (3) we see that the asymptotic expansion depends on the choice of asymptotic sequence, but if we fix the sequence the expansion for that function is unique (because each coefficients of an asymptotic expansion are uniquely determined by theorem 1 by limit that is unique if it exists). But the converse of the property (3) is not true, so we have

4. Two distinct functions f & g may have the same asymptotic expansion.

Example: If $f(z)=\dfrac{1}{z+1}$ and $g(z)=\dfrac{1}{z+1}+e^{-z}$, then both f & g have the same asymptotic expansion $\sum_{n=1}^{\infty}(-1)^{n+1} z^{-n}$ as $z\to\infty$.

In general if we let g (z)=e $^{-z}$, then from theorem 1 above, we have

$$\lim_{z\to\infty} e^{-z} = 0 = a_0 \quad \to g(z)-a_0 \to 0;$$

$$a_1 = \lim_{z\to\infty} z\, e^{-z} = \lim_{z\to\infty}(e^{-z}-a_0)z = 0 = a_1 \quad \to (g(z)-a_0-\frac{a_1}{z})z \to 0$$

$\vdots$

and inductively

$$a_m = \lim_{z\to\infty} z^m\, e^{-z} = \lim_{z\to\infty}(e^{-z}-[a_0+\frac{a_1}{z}+\cdots+\frac{a_{m-1}}{z^{m-1}})z^m = a_m = 0$$

$$\Rightarrow \left(g(z)-\sum_{n=0}^{m} a_n z^{-n}\right) z^m \to 0 \text{ as } z\to\infty.$$

Thus our $g(z)=e^{-z} \sim \sum_{n=0}^{\infty} a_n z^{-n} = 0+\frac{0}{z}+\cdots+\frac{o}{z^n}+\cdots$ or simply $e^{-z}\sim 0$.

The function e^{-z} is said to be transcendentally small for large z.

Therefore If f(z) has an asymptotic expansion ,then f(z) + g(z) = f(z) $+e^{-z}$ has the same asymptotic expansion as that of f(z) in terms of the gauge function $\phi_n(z) = z^{-n}$.

5. Asymptotic expansion can be combined linearly.

Theorem 2: - if $f(z) \sim \sum_{m=0}^{\infty} \frac{a_m}{z^m}$ and $g(z) \sim \sum_{m=0}^{\infty} \frac{b_m}{z^m}$ and A & B are any constants, then

$$(1)\ Af(z)+Bg(z) \sim \sum_{m=0}^{\infty} \frac{(Aa_m + Bb_m)}{z^m} \qquad \text{and}$$

$$(2)\ f(z).g(z) \sim \sum_{m=0}^{\infty} \frac{c_m}{z^m}, \text{ where } c_m = a_0 b_m + \cdots + a_m b_0 .$$

Proof: - The proof of (1) is simply follows from definition.

To prove (2), choose a_n arbitrary for fixed n and write

(3)

$$\begin{cases} f(z) = s_n(z) + \dfrac{h(z)}{z^n} \quad where \quad s_n(z) = \sum\limits_{m=0}^{n} \dfrac{a_m}{z^m} \quad and \\ g(z) = s'_n(z) + \dfrac{l(z)}{z^n} \quad where \quad s'_n(z) = \sum\limits_{m=0}^{n} \dfrac{b_m}{z^m} \end{cases}$$

then by definition an asymptotic expansion

$$(4) \begin{cases} (f(z) - s_n(z))z^n = h(z) \to 0 and \\ (g(z) - s'_n(z))z^n = l(z) \to 0 \end{cases} \quad \text{as } z \to \infty.$$

Now let $S_n + T_n$, where S_n is the sum of the terms involving the powers $1, \frac{1}{z}, \frac{1}{z^2}, \cdots, \frac{1}{z^n}$ and T_n is the sum of other terms, which involves $\frac{1}{z^{n+1}}, \frac{1}{z^{n+2}}, \cdots, \frac{1}{z^{2n}}$. Note that thus $T_n z^n \to 0$ as $z \to \infty$.

Multiplying $S_n S'_n$ term-by-term and collecting like powers, we can readily verify that $s_n(z) = \sum_{m=0}^{n} \frac{c_m}{z^m}$ with $c_0, c_1, \text{---}, c_n$ as given in (2).

By definition of asymptotic expansion have to show that $(fg - s_n)z^n \to 0$

as z→ ∞, so from (3) $fg = s_n s_n' + \frac{h+l}{z^n} + \frac{hl}{z^{2n}} = T_n + \frac{h+l}{z^n} + \frac{hl}{z^{2n}}$.

Now since $T_n z^n \to 0$ as z→ ∞ and using (3), we obtain then desired result, which gives (2). i.e. $(fg - s_n)z^n = T_n z^n + h + l + \frac{hl}{z^n} \to 0$

Note that property (1) can be generalized to asymptotic series, which are not asymptotic power series provided the sequence $\{\phi_n(z)\}$ is the same for both series.

If $f(z) \sim \sum_{n=0}^{\infty} a_n \phi_n(z)$ and $g(z) \sim \sum_{n=0}^{\infty} b_n \varphi_n(z)$ we can often produce a composite set $\{\phi_n\} \cup \{\varphi_n\}$ and regard each expansion as a composite sequence, with zero coefficients. But (2) is only true for asymptotic power series.

3.3. Integration and Differentiation of Asymptotic Expansions.

3.3.1. Integration

Theorem .1 suppose that for all sufficiently large values of the positive real z, f(z) is continuous function and

$f(z) \sim \sum_{n=0}^{\infty} a_n z^{-n}$ as z→ ∞, then $f(t) - a_0 - \frac{a_1}{t}$

integerable and $F(z) = \int_z^{\infty} \left(\mathrm{f(t)} - \mathrm{a}_0 - \frac{a_1}{t} \right) dt \sim \sum_{n=1}^{\infty} \frac{a_{n+1}}{nz^n}$ as z→ ∞.

Proof: (i) existence; since $f(t)-a_0-\frac{a_1}{t}$ is continuous when t > 0 and is $O(1/t^2)$ as $t\to\infty$, hence integrable:

$\Rightarrow$ the integral F(z) exists for z > 0.

Now from the definition if $f(z)\sim\sum_{n=0}^{\infty}a_n z^{-n}$,then $f(z)=\sum_{n=o}^{k}\frac{a_n}{z^n}+O\left(\frac{1}{z^{k+1}}\right)$

so that $F(z)=\int_z^{\infty}\left(\text{f(t) - a}_0-\frac{a_1}{t}\right)dt=\int_z^{\infty}\left[\sum_{n=o}^{k}\frac{a_n}{t^n}+O\left(\frac{1}{t^{k+1}}\right)-\text{a}_0-\frac{a_1}{t}\right]dt$

$$=\int_z^{\infty}\left[\sum_{n=o}^{k}\frac{a_n}{t^n}-\text{a}_0-\frac{a_1}{t}+O\left(\frac{1}{t^{k+1}}\right)\right]dt$$

$$=\int_z^{\infty}\left[\sum_{n=o}^{k}\frac{a_n}{t^n}+O\left(\frac{1}{t^{k+1}}\right)\right]dt \quad \text{for } k\geq 2$$

$$=\sum_{n=o}^{k}\int_z^{\infty}\frac{a_n}{t^n}dt+\int_z^{\infty}O\left(\frac{1}{t^{k+1}}\right)dt=\sum_{n=2}^{k}\frac{a_n}{(n-1)z^{n-1}}+\int_z^{\infty}O\left(\frac{1}{t^{k+1}}\right)dt$$

Now $O\left(\frac{1}{z^{k+1}}\right)$ is bounded by $\frac{A}{z^{k+1}}$ for large z.

Therefore

$$\left|\int_z^{\infty}O\left(\frac{1}{t^{k+1}}\right)dt\right|\leq+\int_z^{\infty}\left|O\left(\frac{1}{t^{k+1}}\right)\right|dt\leq A\int_z^{\infty}\left(\frac{1}{t^{k+1}}\right)dt=\frac{A}{K\,z^k}=O\left(\frac{1}{z^k}\right)$$

$$\therefore\quad F(z)=\sum_{n=2}^{k}\frac{a_n}{(n-1)z^{n-1}}+O\left(\frac{1}{z^k}\right)=\sum_{n=2}^{k-1}\frac{a_{n+1}}{n\,z^n}+O\left(\frac{1}{z^k}\right),$$

and thus the result follows:

$$\text{i.e. } F(z)=\int_z^{\infty}\left(\text{f(t) - a}_0-\frac{a_1}{t}\right)dt\sim\sum_{n=1}^{\infty}\frac{a_{n+1}}{nz^n}\,.$$

3.3.2 Differentiation

Differentiation of asymptotic expansion may be invalid. For instance ,

if $f(z)=e^{-z}\sin(e^{z})$ for positive real z, then f (z) ~ $\frac{0}{z}+\frac{0}{z^{2}}+\ldots$as z → ∞.

Since from property (4) we have $e^{-z} \sim \frac{0}{z}+\frac{0}{z^{2}}+\ldots.$

But $f'(z)=\cos(e^{z})-e^{-z}\sin\left(e^{z}\right)$ oscillates as z → ∞,

and therefore has no asymptotic expansion.

Theorem 2. If f (z) has a continuous derivative f '(z) and f '(z) possesses an asymptotic expansion as z → ∞, then

$$(1)\ f(z) \sim \sum_{n=0}^{\infty} \frac{a_n}{z^n} \quad \Rightarrow \quad (2)\ f'(z) \sim -\sum_{n=2}^{\infty} \frac{(n-1)a_{n-1}}{z^n}$$

$$= -\sum_{n=1}^{\infty} \frac{na_n}{z^{n+1}} \quad (\text{as } z \to \infty).$$

Proof: suppose $f'(z) \sim \sum_{m=0}^{\infty} \frac{b_n}{z^n}$ as z → ∞. (*)

Claim: (*) is identical to (2).

Now since $f'(z)$ is continuous, ⇒ $f'(z)$ is integerable and from (*)

$$f(y)-f(z)=\int_z^y f'(t)dt=\int_z^y\left(\sum_{n=o}^{k}\frac{b_n}{t^n}+O\left(\frac{1}{t^{k+1}}\right)\right)dt \qquad (**)$$

$$=\int_z^y\left[b_0+\frac{b_1}{t}+\sum_{n=2}^{k}\frac{b_n}{t^n}+O\left(\frac{1}{t^{k+1}}\right)\right]dt$$

$$= \int_z^y b_0 \, dt + \int_z^y b_1 \, dt + \int_z^y \left[\sum_{n=2}^{k} \frac{b_n}{t^n} + O\left(\frac{1}{t^{k+1}} \right) \right] dt$$

$$= b_0 (y-z) + b_1 \ln(y/z) + \int_z^y \left[\sum_{n=2}^{k} \frac{b_n}{t^n} + O\left(\frac{1}{t^{k+1}} \right) \right] dt$$

$$\therefore \; f(y) - f(z) = b_0 (y-z) + b_1 \ln(y/z) + \int_z^y \left(f'(t) - b_0 - \frac{b_1}{t} \right) dt .$$

But as $y \to \infty$, $f(y) \to a_0$ and $\int_z^\infty \left(f'(t) - b_0 - \frac{b_1}{t} \right) dt$ is convergent. Since the integrand is $O\left(\frac{1}{z^2} \right)$. It follows that $b_0 = b_1 = 0$ (otherwise it diverges).

Therefore $a_0 - f(z) = \int_z^\infty \left(f'(t) - b_0 - \frac{b_1}{t} \right) dt$.

But by theorem above $\int_z^\infty \left(f'(t) - b_0 - \frac{b_1}{t} \right) dt \sim \sum_{n=1}^{\infty} \frac{b_{n+1}}{n z^n}$

Hence $a_0 - f(z) \sim \sum_{n=1}^{\infty} \frac{b_{n+1}}{n z^n}$ as $z \to \infty$.

But we know that

$$a_0 - f(z) \sim - \sum_{n=1}^{\infty} \frac{a_n}{z^n}$$

(as if $f(z) \sim \sum_{n=0}^{\infty} a_n \phi_n(z)$, then $af(z) \sim a \sum_{n=0}^{\infty} a_n \phi_n(z)$)

Again we know that asymptotic expansion is unique

$$\Rightarrow \sum_{n=1}^{\infty} \frac{b_{n+1}}{n\, z^n} = \sum_{n=1}^{\infty} \frac{-a_n}{z^n}$$

$$\Rightarrow \frac{b_{n+1}}{n} = -a_n \qquad \text{or} \qquad b_{n+1} = -na_n$$

$$\Rightarrow \quad a_0 - f(z) \sim \sum_{n=1}^{\infty} \frac{-na_n}{nz^n} = \sum_{n=1}^{\infty} \frac{-a_n}{z^n}$$

Now derivating both sides, we get

$$0 - f'(z) \sim \sum_{n=1}^{\infty} -a_n \left(\frac{-n}{z^{n+1}} \right) = \sum_{n=1}^{\infty} \frac{na_n}{z^{n+1}} \; .$$

Hence $f'(z) \sim -\sum_{n=1}^{\infty} \frac{na_n}{z^{n+1}} = -\sum_{n=2}^{\infty} \frac{(n-1)a_{n-1}}{z^n}$.

Thus, in other words, the asymptotic expansion is obtained by term-by-term differentiation.

4. Methods of obtaining asymptotic expansion (approximation)

4.1 Integration by parts

There are several methods of obtaining asymptotic expansion of functions defined by definite integrals.

One of these methods is the method of integration by parts. In this method, the successive terms of the asymptotic series are produced by repeated integration by parts; the asymptotic character of the series is then proved by examining the reminders, which is in the form of a definite integral. The field of application of the method is rather restricted, and it is difficult to formulate precise theorems of any degree of generality. Instead of attempting this, we try to make the idea clear by discussing particular examples.

Example: 1 ***Exponential integral***

The exponential integral $Ei(x)$ is defined by the formula $Ei(x) = \int_x^\infty \frac{e^{-t}}{t}\, dt$.

To obtain the asymptotic expansion of $Ei(x)$, integrate $\int_x^\infty \frac{e^{-t}}{t}\, dt$ by parts

repeatedly and obtain that $Ei(x) = \frac{-e^{-x}}{x} - \int_x^\infty \frac{e^{-t}}{t^2}\, dt$

$$= \frac{-e^{-x}}{x} + \frac{-e^{-x}}{x^2} + 2\int_x^\infty \frac{e^{-t}}{t^2}\, dt \quad = - - -$$

$$= e^{-x} \sum_{n=1}^{k} \frac{(-1)^{n+1}(n-1)!}{x^n} + (-1)^k k! \int_x^\infty \frac{e^{-t}}{t^{k+1}}\, dt .$$

The reminder term $R_n(x)=(-1)^n n!\int_x^\infty \frac{e^{-t}}{t^{n+1}}\,dt$ is bounded for large x

(as x→∞) by $\left|R_n(x)\right| \le \frac{n!}{x^{n+1}}\int_x^\infty e^{-t}\,dt = \frac{n!\,e^{-x}}{x^{n+1}}$.

$$\Rightarrow R_n(x) = O\left(\frac{n!\,e^{-x}}{x^{n+1}}\right).$$

Thus $Ei(x) = e^{-x}\sum_{n=1}^{k}\frac{(-1)^{n+1}(n-1)!}{x^n} + O\left(\frac{n!\,e^{-x}}{x^{n+1}}\right)$.

$\Rightarrow \quad Ei(x) \sim e^{-x}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}(n-1)!}{x^n}$ and it is divergent.

<u>Example 2</u>. Similarly to find the asymptotic expansion of a function

defined by $f(x) = \int_0^\infty \frac{e^{-t}}{1+xt}\,dt$, integrate by parts repeatedly to obtain,

$$f(x) = 1 - x\int_0^\infty \frac{e^{-t}}{(1+xt)^2}\,dt \qquad = 1 - x + 2x^2\int_0^\infty \frac{e^{-t}}{(1+xt)^3}\,dt \ \ldots$$

$$= e^{-x}\sum_{n=1}^{k}(-1)^n n!\,x^n + (-1)^{k+1}(k+1)!\,x^{k+1}\int_0^\infty \frac{e^{-t}}{(1+xt)^{k+2}}\,dt$$

and the remainder term

$R_n(x) = (-1)^{k+1}(k+1)!\,x^{k+1}\int_0^\infty \frac{e^{-t}}{(1+xt)^{k+2}}\,dt$ is proved to be O(1) as x→∞.

$$\Rightarrow f(x) = e^{-x}\sum_{n=0}^{k}(-1)^n n!\,x^n + O(1).$$

Hence $f(x) \sim \sum_{n=0}^{\infty}(-1)^n n!\,x^n$.

Example3. ***The incomplete gamma function***

Consider the incomplete gamma function

$\gamma(a,x) = \int_0^x e^{-t} t^{a-1} dt$ where x and a are positive.

i) If x is small, we can deduce a series simply by using the expansion

$e^{-t} = \sum_{n=1}^{\infty} \frac{(-1)^n t^n}{n!}$ and integration by parts as

$$\gamma(a,x) = \int_0^x \sum_{n=0}^{\infty} \frac{(-1)^n t^{n+a-1}}{n!} dt$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n t^{n+a}}{n!(n+a)}$$ Which converges for all x.

In fact the power series converges for all bounded complex numbers. However, its usefulness as an approximation is limited (if x is small x), since it converges very slowly (it is alternating series) for the large x.

ii) To find a representation that is useful for large x, we define, the complementary gamma function

$$\Gamma(a,x) = \Gamma(a) - \gamma(a,x) = \int_x^{\infty} e^{-t} t^{a-1} dt$$

where, $\Gamma(a) = \int_0^{\infty} e^{-t} t^{a-1} dt$ is a gamma function. Now integrating it by

parts n times we obtain $\Gamma(a,x) = \sum_{k=1}^{n} \frac{\Gamma(a)}{\Gamma(a-k+1)} e^{-x} x^{a-k} + \frac{\Gamma(a)}{\Gamma(a,n)} \Gamma(a-n,x)$

But $\left|\Gamma(a-n,x)\right| = \left|\int_x^\infty e^{-t} t^{a-n-1} dt\right| \le x^{a-n-1} \int_x^\infty e^{-t} dt = x^{a-n-1} e^{-x}$ for large x, provided n > a-1, thus for the reminder term $R_n(x) = \dfrac{\Gamma(a)}{\Gamma(a,n)} \Gamma(a-n,x)$

We have $\left|R_n(x)\right| = \left|\dfrac{\Gamma(a)}{\Gamma(a,n)} \Gamma(a-n,x)\right| \le \left|\dfrac{\Gamma(a)}{\Gamma(a-n)}\right| x^{a-n-1} e^{-x}$

$$\Rightarrow \quad \Gamma(a,x) = \sum_{k=1}^{n} \frac{\Gamma(a)}{\Gamma(a-k+1)} e^{-x} x^{a-k} + O\left(\left|\frac{\Gamma(a)}{\Gamma(a-n)}\right| e^{-x} x^{a-1}\right)$$

$$\Rightarrow \Gamma(a,x) \sim \sum_{k=1}^{\infty} \frac{\Gamma(a)}{\Gamma(a-k+1)} e^{-x} x^{a-k} \quad \text{as } x \to \infty \qquad (*)$$

$$= e^{-x} x^{a-1} \sum_{k=0}^{\infty} \frac{(a-1)(a-2)---(a-k)}{x^k}$$

As a specific case, consider the complementary error function,

$$Erfc\ (x) = \int_x^\infty e^{-t^2} dt \ .$$

For this if we let $t^2 = s$, then $Erfc\ (x) = \frac{1}{2} \int_{x^2}^\infty e^{-s} s^{-1/2} ds = \frac{1}{2} \Gamma(1/2, x^2)$ so that from (*) we have

$$Erfc(x) = \frac{1}{2} \Gamma(\frac{1}{2}, x^2) \sim \frac{1}{2} e^{-x^2} \sum_{k=1}^{\infty} \frac{\Gamma(1/2)}{(1-k-1/2)} (x^2)^{-k+1/2}$$

$$= \frac{\sqrt{\pi}}{2} e^{-x^2} \sum_{k=1}^{\infty} \frac{1}{\Gamma(-k+3/2)} x^{1-2k}$$

$$= \frac{\sqrt{\pi}}{2} e^{-x^2} \sum_{k=1}^{\infty} \frac{1}{\Gamma(-k+3/2)} \frac{1}{x^{2k-1}}$$

$$= \frac{1}{x\sqrt{\pi}} e^{-x^2} \sum_{k=0}^{\infty} \frac{(-1)^k 1.3.---(2k-1)}{(2x^2)^k} \quad \text{as } x \to \infty. \mathbf{4.2}$$

4.2 Laplace's method

This method is applied to find an asymptotic expansion of the integral function of the form $f(z) = \int_{\alpha}^{\beta} \phi(x)\, e^{z\,h(x)} dx$

and the expansion obtained by this method is called laplace approximation.

Theorem: 1 (***Laplace's theorem***) Let $\phi(x)$ and h (x) two real continuous functions defined in the finite or semi-finite interval $\alpha \leq x \leq \beta$, such that

1. $\phi(x)\, e^{zh(x)}$ is absolutely integerable over the interval [α,β] for every value of z;
2. $h(x)$ has a single maximum value in the interval [α,β], namely at x = α and the supremum of h (x) in any closed subinterval not containing α is less than h(α) ;
3. $h''(x)$ is continuous ,and $h'(\alpha)=0$, $h''(\alpha) \leq 0$,then

$$\int_{\alpha}^{\beta} \phi(x)\, e^{z\,h(x)} dx \sim \phi(\alpha)\, e^{z\,h(\alpha)} \left[\frac{-\pi}{2zh''(\alpha)} \right]^{1/2} \quad \text{as } z \to \infty.$$

Proof: under condition stated, for an arbitrary number ε > 0, choose a positive number δ(< β-α) such that: $\begin{cases} h''(\alpha)-\varepsilon \leq h''(x) \leq h''(\alpha)+\varepsilon < 0 \\ h''(\alpha)-\varepsilon \leq h''(x) \leq h''(\alpha)+\varepsilon < 0 \end{cases}$,when $\alpha \leq x \leq \alpha+\delta$. Since in this sub interval

$h(x) = h(\alpha) + \frac{1}{2}(x-\alpha)^2 h''(\zeta)$, where $\alpha < \zeta < \alpha + \delta$,

h(x)-h(α) lies between $-\frac{1}{2}B(x-\alpha)^2$ and $-\frac{1}{2}A(x-\alpha)^2$ where A & B are positive constants; with $A = -h''(\alpha) - \varepsilon$ and $B = -h''(\alpha) + \varepsilon$.

Hence $\int_{\alpha}^{\alpha+\delta} \phi(x)\, e^{z\,h(x)} dx$ lies between

$$\{h(\alpha)-\varepsilon\}e^{z\,h(\alpha)} \int_{\alpha}^{\alpha+\delta} e^{\frac{-1}{2}zB(x-\alpha)^{2}} dx \text{ and}$$

$$\{h(\alpha)+\varepsilon\}e^{z\,h(\alpha)} \int_{\alpha}^{\alpha+\delta} e^{\frac{-1}{2}zA(x-\alpha)^{2}} dx$$

But since $\int_{\alpha}^{\alpha+\delta} e^{\frac{-1}{2}zA(x-\alpha)^{2}} dx = \int_{0}^{\infty} e^{\frac{-1}{2}zAu^{2}} du - \int_{\delta}^{\infty} e^{\frac{-1}{2}zAu^{2}} du$

$$= \left\{\frac{\pi}{2\,zA}\right\}^{1/2} \left\{1 + O\left(e^{\frac{-1}{2}zA\,\delta^{2}}\right)\right\}$$

When z is large, it follows that

$$\int_{\alpha}^{\alpha+\delta} \phi(x)\, e^{z\,h(x)} dx \le \{h(\alpha)+\varepsilon\}e^{z\,h(\alpha)} \left\{\frac{\pi}{2\,zA}\right\}^{1/2} \left\{1 + O\left(e^{\frac{-1}{2}zA\,\delta^{2}}\right)\right\},$$

and similarly

$$\int_{\alpha}^{\alpha+\delta} \phi(x)\, e^{z\,h(x)} dx \ge \{h(\alpha)-\varepsilon\}e^{z\,h(\alpha)} \left\{\frac{\pi}{2\,zB}\right\}^{1/2} \left\{1 + O\left(e^{\frac{-1}{2}zB\,\delta^{2}}\right)\right\}.$$

For the rest of the interval, we have

$$\left|\int_{\alpha+\delta}^{\beta} \phi(x)\, e^{z\,h(x)} dx\right| \le \int_{\alpha+\delta}^{\beta} |\phi(x)|\, e^{h(x)} e^{(z-1)M} dx,$$

Where $M = \sup_{\alpha+\delta \le x \le \beta} h(x) < h(\alpha)$ by (2).

Hence by (1) $\left|\int_{\alpha+\delta}^{\beta} \phi(x)\, e^{z\,h(x)} dx\right| \le e^{(z-1)M} \int_{\alpha+\delta}^{\beta} |\phi(x)|\, e^{\,h(x)} dx = Ke^{(z-1)M}$.

So we have $\{\phi(\alpha) - \varepsilon\}\left\{\dfrac{\pi}{2B}\right\}^{1/2}\left\{1 + O(e^{\frac{-1}{2}zB\delta^2})\right\} - \dfrac{K\sqrt{z}}{e^M} e^{z\{M-h(\alpha)\}}$

$$\le \sqrt{z}\, e^{-z\,h(\alpha)} \int_{\alpha}^{\beta} \phi(x)\, e^{z\,h(x)} dx$$

$$\le \{\phi(\alpha) + \varepsilon\}\left\{\frac{\pi}{2A}\right\}^{1/2}\left\{1 + O(e^{\frac{-1}{2}zA\delta^2})\right\} + \frac{K\sqrt{z}}{e^M} e^{z\{M-h(\alpha)\}}$$

Now if we let z→∞, then

$$(\phi(\alpha)-\varepsilon)\left(\frac{-\pi}{2h''(\alpha)-2\varepsilon}\right)^{1/2} \le \overline{\lim_{z}} \int_{\alpha}^{\beta} \phi(x)\, e^{z\,h(x)} \sqrt{z} e^{-z\,h(\alpha)} dx \le (\phi(\alpha)+\varepsilon)\left(\frac{-\pi}{2h''(\alpha)+2\varepsilon}\right)^{1/2}$$

Since ε was an arbitrary, we have

$$\Rightarrow \lim_{z\to\infty} \int_{\alpha}^{\beta} \phi(x)\, e^{zh(x)} \sqrt{z} e^{-zh(\alpha)} dx = \phi(\alpha)\left(\frac{-\pi}{2h''(\alpha)}\right)^{1/2}.$$

$$\Rightarrow \int_{\alpha}^{\beta} \phi(x)\, e^{z\,h(x)} dx \sim \phi(\alpha) e^{zh(\alpha)}\left(\frac{-\pi}{2h''(\alpha)}\right)^{1/2}.$$

An alternative form of this approximation is obtained by putting $e^{h(x)} = g(x)$, namely that if g(x) attains its maximum at x = α where $g'(\alpha) = 0, g''(\alpha) < 0$,then

$$\int_{\alpha}^{\beta} \phi(x)\, (g(x))^z\, dx \sim \phi(\alpha)(g(x))^{z+1/2}\left(\frac{-\pi}{2zg''(\alpha)}\right)^{1/2} \text{ as } z\to\infty.$$

Example: 1 Consider Bessel function $I_n(z)$ of integer order n, which has an integral representation $I_n(z) = \frac{1}{\pi}\int_0^{\pi} e^{z\cos x} \cos nx \, dx$

Here $\phi(x) = \cos nx$; $h(x) = \cos x$ which decreases steadily from a maximum at x=0.Since h(0)=1, $h'(0) = 0$ *and* $h''(0) = -1 < 0$,it follows that by above theorem $I_n(z) \sim \frac{e^z}{(2\pi z)^{1/2}}$, as z→∞.

Example: 2 approximate the gamma function defined by Euler's integral $\Gamma(z+1) = \int_0^{\infty} e^{-x} x^z dx = z!$ for large positive integer z.

Solution: In its current form, it is not possible to apply laplace's method, since x has no maximum. But if we put x= z t, we obtain

$$\Gamma(z+1) = z^{z+1}\int_0^{\infty} e^{z(-t+\ln t)} dt.$$

Now this integral is of the desired form with $h(t) = -t + \ln t$, which has a single maximum (-1) at t=1, with $h'(1) = 0$ *and* $h''(1) = -1 < 0$

Therefore applying the theorem to the intervals 0 ≤ t ≤1 & t ≥ 1, we obtain

$$\Gamma(z+1) \sim 2z^{z+1}\left[1.e^{-z}\left(\frac{\pi}{2z.1}\right)\right]^{1/2}$$

Which when simplified gives $\Gamma(z+1) \sim (2\pi z)^{1/2} z^z e^{-z}$ as z→∞.

(Note that this approximation of $\Gamma(z+1)$ is known as Starling's formula that is given as $n! \sim \sqrt{(2\pi n)} n^n e^{-n}$ or $z! = \Gamma(z+1) \sim \sqrt{(2\pi z)} z^z e^{-z}$

Example: 3 As an example for an alternative formulation of the theorem, consider laplace's first integral for the legender polynomial, namely

$$P_n(\mu) = \frac{1}{\pi}\int_0^{\pi}\left(\mu + \left(\mu^2 - 1\right)^{1/2}\cos x\right)^n dx$$, Where $\mu > 1$ and the square root is positive.

Here $\phi(x) = 1$ & $g(x) = \mu + \left(\mu^2 - 1\right)^{1/2}\cos x$. And since g(x) has its greatest value at x=0 on [0,π] with $g'(0) = 0$ *and* $g''(0) = -\left(\mu^2 - 1\right)^{1/2} < 0$, we get

$$P_n(\mu) \sim \frac{1}{\sqrt{2\pi n}}\frac{\left(\mu + \left(\mu^2 - 1\right)^{1/2}\right)^{n+1/2}}{\left(\mu^2 - 1\right)^{1/4}}$$ as z→∞.

Generalizations of laplace's Method.

In our discussion of the lap laces method, we considered integrals of the form $\int_{\alpha}^{\beta}\phi(x)\, e^{z\,h(x)}dx$ as z→∞,

where h (x) has a single maximum at a point ξ in the finite or infinite interval $\alpha \le x \le \beta$.for convenience we broke the interval up so that the maximum was attained at an end point ,but this was not essential. It would be possible to generalize the method to cover integrals of the form

$\int_{\alpha}^{\beta}\phi(x, z)\, e^{\,h(x,z)}dx$, where ϕ(x, z) is bounded as z→∞ and h (x, z) has a single maximum ξ; but this stationary point ξ would no longer be fixed, it would vary with z .

The expression of the integrand as a product ϕe^{h} is some what arbitrary, and different factorizations may lead to different asymptotic formula valid in different circumstances. It is customary to change the variable, if it is possible, to make the stationary point independent of z.

For example, in the case of the gamma function,

$\int_0^\infty e^{-x} x^z dx = \Gamma(z+1) = \int_0^\infty e^{-x} e^{z\ln x} dx = \int_0^\infty e^{z\ln x - x} dx$, we could not take

$\phi(x) = e^{-x}, h(x) = \ln x$, because $\ln x$ has no stationary point. But we could take $\phi(x) = 1 \; and \; h(x,z) = z\ln x - x$

The function h(x,z) has a single maximum point at x = z, and the asymptotic expansion could be worked out on that basis. We avoided this by making the change of variable x = z t (in example 2 above).

Example: 2

consider the Bessel function

$$K_z(a) = \frac{1}{2}\int_{-\infty}^{\infty} e^{zx - a\cosh x} dx = \frac{1}{2}\int_{-\infty}^{\infty} e^{zx} \, e^{-a\cosh x} dx,$$

where z and a are positive and z is large.

Here, laplace's method is not applicable with the factorization $h(x) = x, \phi(x) = e^{-a\cosh x}$ as h(x) has no maximum. But $zx - a\cosh x$ has a single maximum at $x = \sinh^{-1}(z/a)$, which varies with z.

If we put $x = \sinh^{-1}(z/a) + t$, we obtain

$$K_z(a) = \frac{1}{2}\left(z + \sqrt{z^2 + a^2}\right)^z \int_{-\infty}^{\infty} e^{z(t-e^t)} \phi(t,z)\, dt$$

Where $\phi(t,z) = e^{\left(\frac{-a^2 \cosh t}{z+\sqrt{z^2+a^2}}\right)}$. The function $t - e^t$ has a single maximum at t=0, but we have obtained this simple form at the expense of introducing z into ϕ. This is quite harmless, since in any finite interval $-\alpha \le t \le \alpha$, $\phi(t, z)$ is continuous and $\phi(\alpha, z) \le \phi(t, z) \le \phi(0, z)$;

Hence $\phi(t, z) \to 1$ as $z \to \infty$ uniformly in t, and as $\phi(t, z) \le 1$ for all t, z.

Now we have $\int_{-\alpha}^{\alpha} e^{z(t-e^t)} \phi(t,z)\, dt = \phi(t_0, z) \int_{-\alpha}^{\alpha} e^{z(t-e^t)}\, dt$, where $-\alpha \le t_0 \le \alpha$,

and so $\int_{-\alpha}^{\alpha} e^{z(t-e^t)} \phi(t,z)\, dt \sim \int_{-\infty}^{\infty} e^{z(t-e^t)}\, dt$

$$\sim e^{-z} \int_{-\infty}^{\infty} e^{\frac{-1}{2} z t^2}\, dt = e^{-z} \left(\frac{2\pi}{z}\right)^{1/2}.$$

For the intervals $t \ge \alpha$ and $t \le -\alpha$, write $t = \alpha + \gamma$ and recall that $\phi(t, z) \le 1$ for all t and for all z. Then using the inequality $e^{\gamma} \ge 1 + \gamma$, $(\gamma \ge 0)$, we have

$$0 \le \int_{\alpha}^{\infty} e^{z(t-e^t)} \phi(t,z)\, dt \le e^{z(\alpha - e^{\alpha})} \int_{0}^{\infty} e^{z\gamma - ze^{\alpha}\left(e^{\gamma}-1\right)}\, d\gamma$$

$$\le e^{z(\alpha - e^{\alpha})} \int_{0}^{\infty} e^{-z\gamma\left(e^{\gamma}-1\right)}\, d\gamma = \frac{e^{-z\left(e^{\alpha}-\alpha\right)}}{z\left(e^{\alpha}-1\right)} = o\left(\frac{e^{-z}}{z}\right),$$

since $e^{\alpha} - \alpha > 1$. Similarly fort $\le -\alpha$.

Thus, when a is positive and z→∞

$$K_z(a) \sim \left(z + \sqrt{z^2 + a^2}\right)^z e^{-z} \sqrt{\frac{2\pi}{z}}.$$

Or more conveniently $K_z(a) \sim \dfrac{2^z z^z e^{-z}}{a^z} \sqrt{\dfrac{\pi}{2z}}$, as z→∞.

Theorem (*Watson's lemma*)

Suppose $\phi(t)$ is analytic in the sector $0 < |t| < R, |\arg t| \le \delta < \pi$ and suppose

$$\phi(t) = \sum_{k=1}^{\infty} a_k t^{(k/n)-1} \; for \; |t| < R \text{ and that } |\phi(t)| \le k e^{bt}$$

where k and b are positive numbers independent of t for R≤ t ≤T,

then $\int_0^T e^{-zt} \phi(t) dt \sim \sum_{k=1}^{\infty} \frac{a_k \Gamma(k/n)}{z^{k/n}}$

as $|z| \to \infty$ in the sector $|\arg\ z| \le \delta < \pi/2$.

Proof: The proof is direct applications of the stated hypothesis. We divide the interval of integration in to two parts.

$$f(z) = \int_0^R e^{-zt} \phi(t) dt + \int_R^T e^{-zt} \phi(t) dt = \int_0^T e^{-zt} \phi(t) dt. \quad (1)$$

The second integral we bound by

$$\left| \int_R^T e^{-zt} \phi(t) dt \right| \le \int_R^T e^{-xt} |\phi(t)| dt \quad \text{,for } x = \text{Re } z > 0 \quad (2)$$

$$\le K \int_R^T e^{-(x-b)t} dt \le K_1 e^{-xR} \text{ for sufficiently large x.}$$

Thus the second integral is of order of e^{-xR} which is small (because as $x \to \infty, e^{-xR} \to 0$). For the first integral we note that since the power series of $\phi(t)$ is convergent on the real axis,

$\phi(t) = \sum_{k=1}^{m-1} a_k t^{(k/n)-1} + R_m(t)$ Where the reminder term has the bound

$|R_m(t)| \leq ct^{(m/n)} - 1$ for the interval 0 <t ≤R.

Therefore for the first interval we have

$$\int_0^R e^{-zt}\phi(t)dt$$

$$= \left[\int_0^\infty e^{-zt}\sum_{k=1}^{m-1} a_k t^{(k/n)-1}dt - \int_R^\infty e^{-zt}\sum_{k=1}^{m-1} a_k t^{(k/n)-1}dt.\right] + \int_0^R e^{-zt} R_m(t)\,dt$$

(3)

of these three integrals, we evaluate the first and estimate the second and the third:

$$\int_0^\infty e^{-zt}\sum_{k=1}^{m-1} a_k t^{(k/n)-1}dt = \sum_{k=1}^{m-1}\frac{a_k\Gamma(k/n)}{z^{k/n}} \quad (4)$$

$$\int_R^\infty e^{-zt}\sum_{k=1}^{m-1} a_k t^{(k/n)-1}dt = \sum_{k=1}^{m-1}\frac{a_k\Gamma(k/n, zR)}{z^{k/n}} \quad (5)$$

The function $\Gamma(k/n, zR)$ is the incomplete gamma function discussed earlier in section (5.1), and $\Gamma(k/n, zR) = O(e^{-xR})$ for $x = \operatorname{Re} z > 0$.

Finally

$$\left|\int_0^R e^{-zt} R_m dt\right| \leq c\int_0^R e^{-xt} t^{(m/n)-1}dt = \frac{c}{x^{m/n}}\int_0^{xR} e^{-t} t^{(m/n)-1}dt = o(x^{-m/n}) \quad (6)$$

Now taking all these estimates together we have

$$|f(z)| = \left|\int_0^T e^{-zt}\phi(t)dt\right| = \left|\int_0^R e^{-zt}\phi(t)dt + \int_R^T e^{-zt}\phi(t)dt\right|$$

$$\leq \left|\int_0^R e^{-zt}\phi(t)dt\right| + \left|\int_R^T e^{-zt}\phi(t)dt\right| \leq \left|\int_0^R e^{-zt}\phi(t)dt\right| + O(e^{-xR})$$

$$\leq \left|\sum_{k=1}^{m-1} \frac{a_k\Gamma(k/n)}{z^{k/n}} - O(e^{-xR}) + o(x^{-m/n})\right| + O(e^{-xR}) \leq \left|\sum_{k=1}^{m-1} \frac{a_k\Gamma(k/n)}{z^{k/n}})\right| + O(1).$$

$$\Rightarrow f(z) = \int_0^T e^{-zt}\phi(t)dt \sim \sum_{k=1}^{\infty} \frac{a_k\Gamma(k/n)}{z^{k/n}} \text{ as } |z| \to \infty.$$

Watson's lemma can be stated in a slightly different, but equivalent form as follows,

Theorem: 3

suppose $\phi(t)$ is analytic in a neighborhood of t = 0

$\phi(t) = \sum_{k=0}^{\infty} a_k t^k \ for \ |t| \leq R$ and suppose that $|\phi(t)| \leq ke^{bt^n}$ for R≤ t ≤T,

then,

$$\int_0^T e^{-zt}\phi(t)dt \sim \frac{1}{n}\sum_{m=1}^{\infty} \frac{a_{m-1}\Gamma(m/n)}{z^{m/n}} \text{ as } |z| \to \infty$$

in the sector $|\arg z| \leq \delta < \pi/2$.

Note

This lemma actually shows that lapace's method can be extended in to a sector in the complex plane, and it is not restricted to real valued functions.

Example 1: Obtain an asymptotic expansion of

$$f(z)=\int_0^\infty \frac{e^{-zt}}{1+t^2}dt \text{ for } |z|\to\infty \text{ in } |\arg z|\le \frac{\pi}{2}-\delta, \delta>0.$$

Solution: To apply the lemma let $\phi(t)=\frac{1}{1+t^2}$, then

$$\phi(t)=\sum_{n=0}^{\infty}(-1)^n t^{2n}=\sum_{n=1}^{\infty}(-1)^{n+1}t^{2n-2} \quad |t|<1 .$$

Now for t ≥ ½, $e^t>1$ and $\frac{1}{1+t^2}<1 \quad \Rightarrow \phi(t)=\frac{1}{1+t^2}<e^t$ which fulfils the lemma.

So by Watson's lemma, we have

$$\int_0^\infty \frac{e^{-zt}}{1+t^2}dt=\int_0^\infty e^{-zt}\phi(t)dt=\int_0^\infty e^{-zt}\sum_{n=1}^{\infty}(-1)^{n+1}a_{2n-1}t^{2n-2}dt$$

$$\sim \sum_{n=1}^{\infty}(-1)^{n+1}\frac{\Gamma(2n-1)}{z^{2n-1}}.$$

$$=\sum_{n=0}^{\infty}(-1)^n\frac{\Gamma(2n+1)}{z^{2n+1}}=\sum_{n=0}^{\infty}(-1)^n\frac{(2n)!}{z^{2n+1}}$$

Hence $\int_0^\infty \frac{e^{-zt}}{1+t^2}dt \sim \sum_{n=0}^{\infty}(-1)^n\frac{(2n)!}{z^{2n+1}}$ as $|z|\to\infty$ in $|\arg z|\le\frac{\pi}{2}-\delta,\ \delta>0.$

Example 2: consider the integral $F(z)=\int_0^\infty \frac{e^t}{t+z}dt$ for complex valued z.

If we make the change of variables $t=|z|s$, we find

$$F(z)=\frac{|z|}{z}\int_0^\infty \frac{e^{-|z|s}}{1+\frac{|z|}{z}s}ds$$

The function $\phi(s) = \left(1 + \frac{|z|}{z} s\right)^{-1}$ is analytical for $|s| < 1$, and bounded for large real s,

using its power series expansion we have

$$F(z) \sim \frac{|z|}{z} \int_0^\infty e^{-|z|s} \sum_{n=0}^{\infty} \left(-\frac{|z|}{z} s\right)^n ds$$

$$= \frac{1}{z} \int_0^\infty e^{-\gamma} \sum_{n=0}^{\infty} \left(-\frac{\gamma}{z}\right)^n d\gamma \quad = \quad \sum_{n=0}^{\infty} (-1)^n \frac{n!}{z^n}.$$

Which is an asymptotic expansion of F(z) valid for $|z| \to \infty$.

4.3 Method of steepest descents

This method is developed by Debye, and is applied to find the asymptotic expansion of integrals of the form,

$$f(z) = I = \int_\alpha^\beta e^{zh(t)} \phi(t) dt \quad \text{or} \quad I = \int_c e^{zh(t)} \phi(t) dt$$

when z is large and positive. Where the path of integration is an arc or a closed curve in the complex plane.

The functions $h(t)$ and $\phi(t)$ are independent of z and are analytic functions of a complex variable t, regular in a domain which contains the path of integration. Those points of the t plane at which $h'(t) = 0$ are called **saddle points** of $h(t)$. The surface representing $\left|e^{zh(t)}\right|$ as a function of Re t and Im t is called **the relief** of $e^{zh(t)}$.

In the t-plane, curves along which $\text{Re}\, h(t)$ is constant are called **level curves**: along such curves $h(t)$ has a constant modulus (they are contour lines of the relief), and the phase e^h changes as rapidly as possible. Those along which $\text{Im}\, h(t)$ is constant are called **steepest paths**: along such curves e^h has a constant phase, and the modulus of e^h changes as rapidly as possible (they are the gradient of the relief).

The main aim of the method of steepest descents is to deform the path of integration so as to make it coincides as far as possible with arcs of steepest paths in order to apply Watson's lemma.

i.e. to satisfy the conditions

- The path passes through a zero t_0 of $h'(t)$ ...(the saddle point).
- The imaginary part of $h(t)$ is constant on the path (the steepest path).

If $\alpha \,\&\, \beta$ lie on steepest arcs through the saddle points, for instance if $\alpha \,\&\, \beta$ are singularities of $h(t)$, then the path of integration may be deformed so as to consist entirely of steepest paths through saddle points; other wise two steepest arcs may occur which do not pass through saddle points. This letter case may be described by references to the relief by saying that we first descend along a gradient line to a singularity and then climb the saddle along anther gradient line.

In any event, $\text{Re}\, h(t)$ is monotonic along any steepest path (except at saddles), and Laplace's method may be used to evaluate the integral asymptotically.

Now if t_0 is a saddle point for $h(t)$ the paths of steepest descent through a saddle point t_0 are given by $h(t) = h(t_0) - \tau$,

where τ is real positive, and so, if s is the arc of the path, $\frac{d\tau}{ds} = \pm|h'(t)|$.

Therefore $\frac{d\tau}{ds}$ can only change sign if the path goes through anther seddle point or a singularity of $h'(t)$. This variable τ is usually monotonic on a steepest path from a saddle point and either increases to ∞ or decreases to $-\infty$.

But since the integrand is $e^{zh(t_0)-z\tau}\phi$,

a path on which $\tau \to -\infty$ would be lead to a divergent integral. Hence we try to choose a path of integration on which τ is positive – these are the paths of steepest decent from the saddle point.

Therefore if it is possible to deform the path of integration and express the integral as the sum of integrals along paths of steepest descent from a saddle point, all that remains is to consider the asymptotic behavior of integrals of the form $e^{zh(t_o)}\int_0^\infty e^{-z\tau}\phi(t)\frac{dt}{d\tau}d\tau$,

where z is large and positive, and, to each of those, we can usually apply Watson's lemma.

Example: 1

Consider the integral $f(z)=\dfrac{1}{2\pi i}\int_C \dfrac{e^{z(t-\sqrt{t})}}{t}dt$

Where the contour C is any vertical line of the form $t=x_0+iy, x_0>0$ fixed, $-\infty<y<\infty$.

Now we see that our $h(t)=t-\sqrt{t}$ and $\phi(t)=\dfrac{1}{\sqrt{t}}$ since $h'(t)=0$ at t= ¼.

Hence ¼ is the saddle point of h(t) ,and thus the paths of steepest descent through the saddle point t=1/4 is given by $t-\sqrt{t}=\dfrac{-1}{4}-\tau^2, t=(\dfrac{1}{2}+i\tau)^2$ for real positive number τ .

Therefore we fulfills all the criteria to deform the path of integration, that is

- The path passes through t =1/4 when τ = 0.
- Im $(t-\sqrt{t})$ is constant on the path.

Hence in terms of the new variables , τ ,our integral becomes

$$f(z)=\frac{1}{2\pi i}\int_0^\infty \frac{e^{z(-\tau^2-1/4)}}{(i\tau+1/2)^2}(1+2i\tau)\, id\tau$$

$$=\frac{2}{\pi}e^{-z/4}\int_0^\infty \frac{e^{-z\tau^2)}}{1+2i\tau}d\tau$$

$$f(z)=\frac{2}{\pi}e^{-z/4}\int_0^\infty e^{-z\tau^2}\sum_{n=0}^\infty (-1)^n(2i\tau)^n d\tau.$$

Now applying Watson's lemma, we have

$$f(z)\sim\frac{1}{2\pi}e^{-z/4}\sum_{n=0}^\infty(-4)^n\frac{\Gamma(n+1/2)}{x^{n+1/2}} \text{ as } z\to\infty.$$

Example: **2**

Hankel's integral $\frac{1}{\Gamma(z)} = \frac{1}{2\pi i}\int_c e^s s^{-z}\, ds$

where the contour C traverses from $s = \infty e^{-i\pi}$ to $s = \infty e^{i\pi}$ around the origin on the Re s < 0 axis.

To approximate this integral substitute s= z t so that

$$\frac{1}{\Gamma(z)} = \frac{1}{2\pi i}\int_c e^{zt}(zt)^{-z}\, zdt = \frac{1}{2\pi i}\int_c e^{zt} z^{-z+1} e^{-z\ln t}\, dt$$

$$= \frac{1}{2\pi i} z^{1-z}\int_c e^{z(t - t\ln t)}\, dt$$ with the same path as before.

Hence we have $h(t) = t - \ln t$ & $\phi(t) = 1$ and h (t) has only one saddle point namely at $t_0 = 1$ and $h(t_0) = h(1) = 1$, and can be represented as a power series near $t_0 = 1$ by

$$h(t) = h(t_0) + (t - t_0)h'(t_0) + \frac{(t - t_0)^2 h''(t_0)}{2!} + - - -$$

$$= 1 + \sum_{n=2}^{\infty}\frac{(1 - t)^n}{n}$$

When z is positive, the paths of steepest descent through the saddle point are given by $t - \ln t = 1 - \tau$ where $\tau > 0$,satisfies the equation of the steepest path $\text{Im}(t - \ln t) = 0$, which, with $t = x + iy$ can be represented as the curve $x = y\cot y$.

Now to deform the path of integration in to the steepest descent, we make the exact change of variables τ in to η^2, so that $h(t) = 1 - \eta^2$.

i.e. we want a curve, parameterized by η, passing through t =1 at $\eta = 0$,with $\text{Im}(h) = 0$ Re(h) < 1 and with locally quadratic behavior in η.

With this change of variables the integrals becomes

$$\frac{1}{\Gamma(z)} = \frac{1}{2\pi i} z^{1-z} \int_0^{\infty} e^{z(1-\eta^2)} \left(\frac{dt}{d\eta}\right) d\eta \text{ which is still exact.}$$

Now to find $\frac{dt}{d\eta}$, we must make some approximation. We first invert the change of variables to find t=t (η) and then differentiate to find $\frac{dt}{d\eta}$.

We determined that

$$t(\eta) = 1 + i\sqrt{2}\,\eta - \frac{2}{3}\eta^2 - \frac{i}{9\sqrt{2}}\eta^3 - \frac{2}{135}\eta^4 + \frac{i}{540\sqrt{2}}\eta^5 - \frac{4}{8505}\eta^6 + ---$$

$$\Rightarrow \frac{dt}{d\eta} = t'(\eta) = i\sqrt{2} - \frac{3i}{9\sqrt{2}}\eta^2 + \frac{5i}{540\sqrt{2}}\eta^4 + ---$$

$$= i\sqrt{2}\left[1 - \frac{\eta^2}{6} + \frac{\eta^4}{216} + ---\right].$$

Thus $$\frac{1}{\Gamma(z)} = \frac{i\sqrt{2}}{2\pi i} e^z z^{1-z} \int_0^{\infty} e^{z\eta^2} \left(1 - \frac{\eta^2}{6} + \frac{\eta^4}{216} + ---\right) d\eta$$

$$= \frac{1}{\pi\sqrt{2}} e^z z^{1-z} \int_0^{\infty} e^{z\eta^2} \left(1 - \frac{\eta^2}{6} + \frac{\eta^4}{216} + ---\right) d\eta$$

Therefore using Watson's lemma to integrate term by term, we get

$$\frac{1}{\Gamma(z)} \sim e^z z^{-z} \left(\frac{z}{2\pi}\right)^{1/2} \left(1 - \frac{1}{12z} + \frac{1}{288z^2} + ---\right) \text{ as } z \to \infty.$$

or $$\frac{1}{\Gamma(z)} \sim e^z z^{-z} \left(\frac{z}{2\pi}\right)^{1/2} \sum_{n=1}^{\infty} 1.3.5....(2n+1).(-1)^n.a_{2n+1}$$

Example: 3 Consider the integral

$$F(z) = \int_0^1 e^{iz[t+t^2]} \frac{dt}{\sqrt{t}} \qquad \text{for large z.}$$

The path of integration is the straight-line segment on the real t axis between 0 & 1 which we want to deform in to steepest descent paths.

The function $f(t) = i(t+t^2)$ has a saddle point at t = -1/2 which is of no use for us; the "saddle point " method does not help here. Instead we look for steepest descent paths through t = 0,where f(0)= 0,so the steepest descent path has $\operatorname{Im} f(t) = 0$.

If we set $t = x + iy$ we found that $\operatorname{Re} f(t) = -y(2x+1)$

and $\operatorname{Im} f(t) = x + x^2 - y^2)$. The steepest descent paths though t = 0 which satisfies $\operatorname{Im} f(t) = 0$ is the curve $y^2 = x + x^2$

Similarity the steepest descent paths though t =1 satisfies $\operatorname{Im} f(t) = 2\ (f(1) = 2i)$ is given by $x + x^2 - y^2 = 2$.

So we can deform the path of integration in to two integrals, one on the path

$$c_1 : f(t) = f(0) - \tau \ for \tau \geq 0 \quad \Rightarrow \quad c_1 : i(t+t^2) = -\tau$$

a steepest descent paths through t = 0 and the other on the

$$c_2 : f(t) = f(1) - \eta \ for \eta \geq 0$$

$\Rightarrow \quad c_2 : i(t+t^2) = 2i - \eta$ the steepest descent paths through t =1.

These exact changes of variables make our integral as

$$F(z) = \int_0^\infty \frac{ie^{z\tau}}{\left[\frac{-1}{2} + \frac{1}{2}\sqrt{1+4i\tau}\right]^{1/2} (1+4i\tau)^{1/2}} d\tau - \frac{1}{3}\int_0^\infty \frac{e^{2iz} e^{-z\eta} \quad id\eta}{\left[\frac{-1}{2} + \frac{3}{2}\sqrt{1+4i\eta/9}\right]^{1/2} (1+4i\eta/9)^{1/2}}$$

Now we can apply Watson's lemma.

We expand the integrands, excluding the exponential, in power of τ & η and then integrate term-by-term. After a tedious calculation, we found that:

$$F(z)=\sqrt{i}\int_0^{\infty}\frac{e^{-z\tau}}{\sqrt{\tau}}\left(1-\frac{3i\tau}{2}-\frac{35\tau^2}{8}+\frac{231\tau^3}{16}+---\right)d\tau-\frac{i}{3}e^{2iz}\int_0^{\infty}e^{-z\eta}\left(1-\frac{7i\eta}{18}-\frac{37\eta^2}{216}-\frac{911\eta^3}{11664}+---\right)d\eta$$

$$F(z)\sim\left(\frac{\pi}{z}\right)^{1/2}e^{i\pi/4}\left(1-\frac{3i}{4z}-\frac{108}{32z^2}+\frac{3465i}{128z^3}\right)-\frac{i}{3}e^{2iz}\left(\frac{1}{z}-\frac{7i}{18z^2}-\frac{37}{108z^3}\right)+o(x^{-4})$$

4:4 Method of stationary phase

The method of stationary phase is applied to find an asymptotic expansion of integrals with oscillating integrand. It takes note of the fact that if an integrand is rapidly oscillating, the integral is nearly zero except for contributions those points where the phase of the integrand is stationary.

Example 1: consider the integral $G(z)=\int_0^{\infty}\cos z(\frac{t^3}{3}-t)dt$.

For large z, the integrand is rapidly oscillating as long as the function $\frac{t^3}{3}-t$ is changing in t.

At t = 1,the function $\frac{t^3}{3}-t$ has a zero derivative, and thus, the phase of the integrand is stationary.

The most significant contribution to the integral comes from the neighborhood of this point, and all other regions are inconsequential comparison.

Now expanding $\frac{t^3}{3}-t$ about this point (t =1)

i.e. $\frac{t^3}{3}-t=\frac{-2}{3}+(t-1)^2+---,$

Truncate the approximation at the quadratic term and using this approximation in the integral, we get $G(z)\sim\int_0^\infty \cos(\frac{-2z}{3}+z(t-1)^2)dt$.

If our argument is correct, changing the lower limit from 0 to -∞ makes no substantial difference for large z, and we have

$$G(z)\sim\cos\frac{2z}{3}\int_{-\infty}^{\infty}\cos zt^2dt+\sin\frac{2z}{3}\int_{-\infty}^{\infty}\sin zt^2dt.$$

But we know that $\int_{-\infty}^{\infty}\cos at^2dt=\int_{-\infty}^{\infty}\sin at^2dt=\left(\frac{\pi}{2a}\right)^{1/2}$,

$$\therefore G(z)\sim\left(\frac{\pi}{2z}\right)^{1/2}\left[\cos\frac{2z}{3}+\sin\frac{2z}{3}\right]=\left(\frac{\pi}{z}\right)^{1/2}\frac{1}{\sqrt{2}}\left[\cos\frac{2z}{3}+\sin\frac{2z}{3}\right]$$

$$=\left(\frac{\pi}{z}\right)^{1/2}\left[\cos\frac{2z}{3}.\frac{\sqrt{2}}{2}+\sin\frac{2z}{3}.\frac{\sqrt{2}}{2}\right]$$

$$=\left(\frac{\pi}{z}\right)^{1/2}\left[\cos\frac{2z}{3}.\cos\frac{\pi}{4}+\sin\frac{2z}{3}.\sin\frac{\pi}{4}\right]=\left(\frac{\pi}{z}\right)^{1/2}\cos(\frac{2z}{3}-\frac{\pi}{4})$$

Hence $G(z)\sim\left(\frac{\pi}{z}\right)^{1/2}\cos(\frac{2z}{3}-\frac{\pi}{4}).$

Example 2: consider the integral

$J_0(z) = \frac{2}{\pi}\int_0^{\pi/2}\cos\,(z\cos\theta)d\theta$ (the zeroth order Bessel function $J_0(z)$.)

For large z, the integrand is rapidly oscillating provided $\frac{d}{d\theta}\cos\theta \neq 0$ and at θ = 0, the phase of the integrand is stationary and we expect significant contributions to the integral.

We approximate $\cos\theta$ about $\theta = 0$ as

$\cos\theta \sim 1-\theta^2/2$ and obtain $J_0(z) \sim \frac{2}{\pi}\int_0^{\pi/2}\cos z\,(1-\theta^2/2)d\theta$.

Finally by similar way as example 1 above,

If move the upper limit from $\frac{\pi}{2}$ to ∞, there should be a little charge in this approximation, and we obtain a result of

$$J_0(z) \sim \frac{2}{\pi}\int_0^{\infty}\cos z\,(1-\frac{\theta^2}{2})d\theta = \left(\frac{2}{\pi z}\right)^{1/2}\cos\,(z-\frac{\pi}{4})\text{ for large z.}$$

In general the method of stationary phase is applied to find asymptotic expansion of integrals of the from $f(z) = \int_\alpha^\beta g(t)\,e^{izh(t)}dt$ in which z is a large positive variable and h(t) is a real function of real variable t.

And according to Stokes and Kelvin, the dominant terms in the asymptotic expansion of integral arises from the immediate neighborhood of the end points of the interval and from the immediate neighborhood of those points at which h(t) is stationary i.e. $h'(t) = 0$; and in the first approximation the contribution of stationary point, if there are any, is more important then the contribution of the end points.

Suppose that g is continuous and h is twice continuously differentiable, let τ be the only stationary point of h, for $\alpha < \tau < \beta$, $h'(\tau) = 0$ *and* $h''(\tau) > 0$.

In the assumption that the neighborhood of τ will give the principal contribution to the integral, we introduce a new variable of integration u by the substitution $u^2 = h(t) - h(\tau)$ and obtain

$$f(z) \sim \int_{\tau-\varepsilon}^{\tau+\varepsilon} g(t)\, e^{izh(t)}\, dt = \int_{-u_1}^{u_2} 2u \frac{g(t)}{h'(t)}\, e^{[iz(h(\tau)+u^2)]} du$$

where $u_1 = [h(\tau - \varepsilon) - h(\tau)]^{1/2}$ and $u_2 = [h(\tau + \varepsilon) - h(\tau)]^{1/2}$

Since he neighbouhood of u = 0 matters, we may replace g(t) by g(τ) and

$\frac{2u}{h'(t)}$ by $\left(\frac{2}{h''(\tau)}\right)^{1/2}$,which is the limit of $\frac{2u}{h'(t)}$ as t→ τ ,so that

$$f(z) \sim \left(\frac{2}{h''(\tau)}\right)^{1/2} g(\tau) \int_{-u_1}^{u_2} e^{izh(\tau)+izu^2}\, du \, .$$

By the same argument we may expand the integration from -∞ to ∞ and finally obtain

$$f(z) \sim \left(\frac{2\pi}{zh''(\tau)}\right)^{1/2} g(\tau) e^{izh(\tau)+i\pi/4} \quad \text{as } z \to \infty. \qquad (*)$$

For instance consider our example 1 above

$G(z) = \int_0^\infty \cos\, z(\frac{t^3}{3} - t)dt$ Which can be rewritten as

$$G(z) = \mathrm{Re} \int_0^\infty e^{iz(\frac{t^3}{3} - t)} .$$

Here $h(t) = \frac{t^3}{3} - t$ *and* $g(t) = 1$ thus $h'(t) = 0$ at t=1 $\Rightarrow$ τ=1 and 0<1<∞

(i.e.1 is the only stationary point of h, in (0,∞)).

$h''(t) = 2t \Rightarrow h''(1) = 2 > 0$ and h (1)= -1+1/3 = -2/3.

Therefore applying (*) we get,

$$G(z) \sim \left(\frac{2\pi}{2z}\right)^{1/2} e^{i\left(z(\frac{-2}{3}) + \pi/4\right)} = \left(\frac{\pi}{z}\right)^{1/2} e^{i\left(z(\frac{-2}{3}) + \pi/4\right)}$$

$$= \left(\frac{\pi}{z}\right)^{1/2} \cos\left[-\left(\frac{2z}{3} - \pi/4\right)\right]$$

$$= \left(\frac{\pi}{z}\right)^{1/2} \cos\left[\frac{2z}{3} - \pi/4\right].$$

As we can see, the method of stationary phase gives a quick estimate of the integrals of rapidly varying integrals.

Unfortunately, the method provides no check on the validity of the answer, and no way to get a new more terms. This limitation is remedied by the method of steepest descent and Watson's lemma. Since for any analytic function, a point of stationary phase is exactly a saddle point in complex plane. Therefore it makes sense to deform the path of integration in to a steepest descent path through the saddle point, invoking Watson's lemma along the way.

For instance consider our first example again

$$\text{i.e. } G(z) = \int_0^\infty \cos z(\frac{t^3}{3} - t)dt = \text{Re}\int_0^\infty e^{iz(\frac{t^3}{3}-t)} dt,$$

we want to deform the path of integration (the real axis $0 \le t < \infty$)into a steepest descent path. For this let $h(t) = i\left(\frac{t^3}{3} - t\right)$.

So the steepest descent path for h(t) through the origin is $t = iy$, y > 0 and through the saddle point (t = 1) is $x(x^2 - 3 - 3y^2) = -2$ where $t = x + iy$.

So we deform the path of integration in to the two paths

$$c_1 : i(\frac{t^3}{3} - t) = -\tau,$$

$$c_2 : i\left(\frac{t^3}{3} - 3\right) = -2i - \eta^2,$$

To invert these changes of variables for t = t (τ) and t = t (η) we make use of the power series

$$c_1 : i\left(-\tau + \frac{\tau^3}{3} - \frac{\tau^5}{3} + \frac{4\tau^7}{9} - \frac{55\tau^9}{81} + ---\right)$$

$$c_2 : 1 + \sqrt{i}\,\eta - \frac{i\eta^2}{6} - \frac{5i\sqrt{i}}{72}\eta^3 - \frac{1}{27}\eta^4 + \frac{77\sqrt{i}}{3456}\eta^5 + ---.$$

Substituting this change of variables into the original and integrating term by term, we find

$$\operatorname{Re}\int_0^\infty e^{iz(\frac{t^3}{3}-t)}dt = \operatorname{Re}\left[\int_0^\infty e^{-z\tau}\left(\frac{dt}{d\tau}\right)d\tau + e^{-2iz/3}\int_{-\infty}^{\infty} e^{-z\eta^2}\left(\frac{dt}{d\eta}\right)d\eta\right]$$

$$= \operatorname{Re}\left[\left(\frac{\pi}{z}\right)^{1/2} e^{-i\left(\frac{2z}{3}-\frac{\pi}{4}\right)}\left(1-\frac{5i}{50\,z}+\frac{385}{4608\,z^2}+---\right)\right]$$

$$=\left(\frac{\pi}{z}\right)^{1/2}\left[\cos\left(\frac{2z}{3}-\frac{\pi}{4}\right)\left(1-\frac{385}{4608z^2}+---\right)-\frac{1}{10z}\sin\left(\frac{2z}{3}-\frac{\pi}{4}\right)+---\right]$$

There is no contribution from the integral along c_1 since there the integrand is purely imaginary.

As last example consider also example 2 above:

$$J_0(z) = \frac{2}{\pi}\operatorname{Re}\int_0^{\pi/2} e^{iz\cos\theta}\,d\theta$$

Here the function $f(\theta) = i\cos\theta$ has a saddle point at $\theta = 0$ in $(0, \pi/2)$.

Thus he steepest descant path through $\theta = 0$ is the carve

$\cos x \cosh\ y = 1, \theta = x + iy$

and the steepest descant path through the end point $\theta = \pi/2$ (not a saddle point) is $\theta = \frac{\pi}{2} - iy\ , y > 0$.

Therefore the original path of integration is deformed into two paths

$$c_1 : i\cos\theta = i - \tau^2$$

$$c_2 : i\cos\theta = -\eta$$

To invert these expirations for $\theta = \theta(\tau)$ and $\theta = \theta(\eta)$ we use

$$\cos\theta = \sum_{n=0}^{\infty} (-1)^n \frac{\theta^{2n}}{(2n)!} \text{ and } \cos\theta = -\sum_{n=0}^{\infty} (-1)^n \frac{(\theta - \pi/2)^{2n+1}}{(2n+1)!},$$

which enables us to find power series expansion for the inverse functions

$\theta = \theta(\tau)$ and $\theta = \theta(\eta)$

$$c_1 : \theta = \frac{2}{\sqrt{2i}}\tau - \frac{1}{6\sqrt{2i}}\tau^3 - \frac{3}{6\sqrt{2i}}\tau^5 + \frac{5i}{448\sqrt{2i}}\tau^7 + ---$$

$$c_2 : \theta = \frac{\pi}{2} + i\left(\eta + \frac{1}{6}\eta^3 - \frac{3}{40}\eta^5 + \frac{5}{112}\eta^7 + ---\right).$$

as a consequence

$$J_0(z) = \frac{2}{\pi}\mathrm{Re}\left[e^{iz}\int_0^{\infty} e^{-z\tau^2}\left(\frac{d\theta}{d\tau}\right)d\tau - \int_0^{\infty} e^{-z\eta}\left(\frac{d\theta}{d\eta}\right)d\eta\right]$$

$$\sim \left(\frac{2}{\pi z}\right)^{1/2}\left[\cos\left(z - \frac{\pi}{4}\right)\left(1 - \frac{45}{64z^2} + ---\right) + \sin\left(z - \frac{\pi}{4}\right)\left(\frac{1}{8z} - \frac{525}{7168z^3} + ---\right)\right]$$

References

1. Jemes P.Keener, **Principles of applied mathematics: Transfromation and Approximation**, Addison-Wesley publishing Company, 1988.
2. Erwin Kreyszig, **Advanced engineering mathematics**, New International (p) limited, publishers, 1997.
3. A.Erde`lyi, **Asymptotic Expansions**, Doverpublications, INC, 1956,USA.
4. E.T.Copson, **Asymptotic Expansions**, Cambridge university press, 1965.
5. F.W.J.Olver, **Introduction to Asymptotic and Special Functions**, Academic press, 1974.
6. Earl D. Rainsville, **Special Functions**, The Macmillan Company, 1960, New York.
7. Mary L. Boas, **Mathematical Methods in the Physical Sciences** second edition, John Wiley and sons, 1983.Canada.
8. Barry Spain and M.G.Smith, **Functions of Mathematical Physics**, Van Nostrand Reinhold Company, 1970,London.
9. Sir Harold Jeffreys and Bertha Swirces (Lead Jeffreys), **Methods of Mathematical Physics**, Camberedge University press, 1962,New York.