SCHOOL OF GRADUATE STUDIES

Faculty of computer and mathematical sciences

Department of mathematics

A project report on:

The Tennis ball problem and its variations

A project submitted to department of mathematics in partial fulfillment of the requirements for the degree of Master of Science in mathematics.

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Declaration

I declare that this project is compiled by me and that no part of the project has formed the basis for the award of any Degree, Diploma, Associateship, Fellowship, or any other title to me.

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Name signature

Permission:

This is to certify that this project is compiled by Demelash Ashagrie in the Department of Mathematics, Addis Ababa University, under my supervision. I hereby also confirm that the project can be submitted for evaluation by examiner and eventual defense.

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Advisor's name	Signature

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Finally, I need to say thank you to my family especially my father, my sister (Tigist Ashagrie) and my friends.

Summary of the project

This project is concerned with the tennis ball problem and its variations and we use k trees, lattice paths, Motzkin paths and ballot problem. In the tennis ball problem one is given successive pairs of balls numbered {1,2},{3,4},{5,6},... and at each stage throw one out the window on to the lawn.

Mallow and Shapiro obtained a formula for the sum over all possible arrangements of the numbers on the balls numbered {1, 2}, {3, 4},... at each stage we throw one balls out of the window .After n stages some set of n balls is on the lawn. There is also a generating function and a closed formula for the sequence 3, 23, 131, 664, 3166, 14545, 65187, 287060, ... The nth term of which gives the sum over all possible arrangements of the total of the numbers on the balls on the lawn .The problem has connections with "bi colored Motzkin paths", Lattice path and the" ballot ball problem".

Merlini, Sprugnoli ,and Verri generalized the tennis ball problem to balls numbered {1, 2, 3,... ,k},{k+1, k+2, ...,2k},..., in this project work there is an alternative bijective proof of the generalized tennis ball problem using k-trees . And also obtain summation formula for all possible arrangements of balls out on the lawn using k trees and lattice paths.

Preliminaries

Tennis ball problem The tennis ball problem goes as follows. At the first turn you are given balls numbered 1 and 2. You throw one of them out the window on to the lawn. At the second turn balls numbered 3 and 4 and now you throw out on the lawn any of the three balls in the room with you. Then balls 5 and 6 are brought in and you throw out one of the four available balls. The game is continuous for n turns. The first question is how many different arrangements on the lawn are possible. It is easy to see that there are 2, 5, and 14 possibilities after 1, 2, and 3 turns. This suggests the Catalan numbers which turns out to be the case.

Motzkin path A path in the xy plane is called Motzkin path if it satisfies the following

- The possible steps are u=up=(1,1) ,D=down(1,-1) and level=(1,0) we allow the level steps to be one of two colors ,L or l .
- The path starts at (0,0) and consists of n steps
- The paths never go below the x-axis.

Ballot ball problem Suppose that in an election, candidate A receives a vote and candidate B receives b votes where a≥kb for some positive integer k. Compute the number of ways the ballots can be ordered. So, then A maintains more than k times as many votes as B throughout the counting of the ballots.

Lattice path Given two such points (p, q) and (r, s), with p ≥ r and q ≥ s, a rectangular lattice path from (r, s) to (p, q) that is made up of horizontal steps H= (1, 0) and vertical steps V=(0,1). Thus, rectangular lattice paths from (r, s) to (p, q) using unit horizontal and vertical segments.

K tree A k-tree is constructed from a single distinguished k-cycle by repeatedly gluing other k-cycles to existing one along an edge. If K is any non empty subset of {2, 3, 4...} , then a K-tree is obtained as above using k-cycles with k ∈ K.

Ordered tree Ordered trees are trees with distinguished vertex called the root where the children of each internal vertex are linearly ordered. Ordered trees are drawn so that the children of each internal vertex are shown in order from left to right.

Lagrange inversion formula

Let f(u) and □(u) be formal power series in u, with □(0)=0. Then there is a unique formal power series u=u(t) that satisfies u=z□(u). Further, the value f(u(t)) of f at the root u=u(t),when expanded in a power series in t about t=0 satisfies

$$[t^n]\{f(u(t))\} = \frac{1}{n}[u^{n-1}]\{f'(u)(u)^n\}$$

SECTION ONE

1.1 Introduction

The tennis ball problem goes as follows. At the first turn you are given balls numbered 1 and 2. You throw one of them out the window on to the lawn. At the second turn balls numbered 3 and 4 and now you throw out on the lawn any of the three balls in the room with you. Then balls 5 and 6 are brought in and you throw out one of the four available balls. The game continuous for n turns. The first question is how many different arrangements on the lawn are possible. It is easy to see that there are 2, 5, and 14 possibilities after 1, 2, and 3 turns. This suggests the Catalan numbers which turns out to be the case. A more basic question is "what is the total sum of the balls on the lawn over all these possibilities"? Here the first few terms are 3, 23, 131, and 664. As an example consider the two possibilities after the first turn they are {1} or {2} the total sum is 1+2 = 3 and consider the five possibilities after two turns. They are {1, 2}, {1, 3}, {1, 4}, {2, 3} or {2, 4}. The total sum is (1+2) + (1+3) + (1+4) + (2+3) + (2+4) = 23.

Mallows and Shapiro considered a more basic question of finding a formula for the sum over all possible arrangements after each throw. They obtained that the sum over all possible arrangements after nth throw is given by

$$M_n = \frac{2n^2+5n+4}{n+2}\binom{2n+1}{n} - 2^{2n+1}$$

Merli, Sprugnoli, and Verri generalized the tennis ball problem to balls numbered {1, 2, 3, ..., k}, {k+1, k+2,..., 2k},... using the notion of generating trees. In this project, we given an alternative bijective proof and show that the number of arrangements of balls on the lawn after the nth throw is the (n+1)st generalized Catalan number

$$C_{k,\,n+1}=\frac{1}{(n+1)k+1}\binom{(n+1)k+1}{n+1}$$

We also obtain a summation formula for all possible arrangements of balls out on the lawn in the generalized case using k-trees and lattice paths.

1.2 Ordered trees

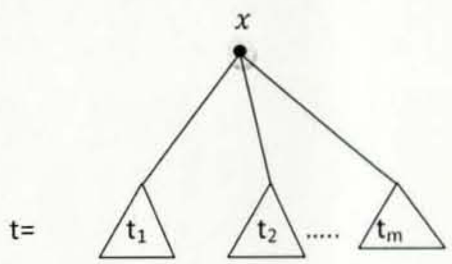
Ordered trees are trees with distinguished vertex called the root where the children of each internal vertex are linearly ordered. Ordered trees are drawn so that the children of each internal vertex are shown in order from left to right.

1.2.1 Enumeration of ordered trees by number of edges

The combinatorial structures that we shall be dealing with are (unlabeled) ordered trees. Let T_n be the class of all ordered trees with n edges. Our terminology is borrowed from Knuth [1]

Ordered trees may be defined recursively as follows:

If $t_1, t_2, \dots, t_m$ are ordered trees, m≥0, then



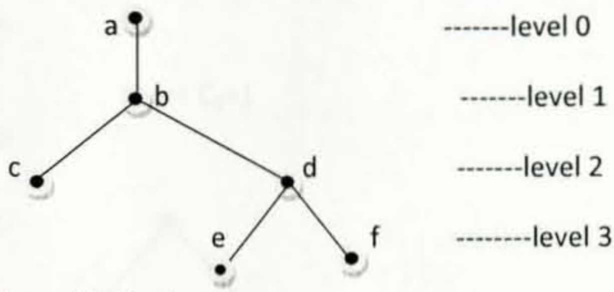
is also an ordered trees

The trees are ordered in the sense that the order among sub trees (or children) is significant. With each node x in a tree t, we associate two values.

1) Its degree
2) Its level

The degree of x (sometimes known as degree) is the number of children it has, and the level of x is the distance (the number of edges separating it from the roots of t). A node of degree 0 is referred to as leaf. Otherwise it is called an internal node. The root is the only node at level 0.

Example



An ordered tree with 5 edges

Leaves (nodes of degree 0): c, e, f

Internal nodes of degree 1: a

Internal nodes of degree 2: b, d

Level 0(root): a

Level 1: b

Level 2: c, d

Level 3: e, f

Let T_n ($n\geq0$) denote the set of ordered trees with n edges .the enumeration of ordered trees on n edges can be performed as follows.

Let C_n=number of ordered trees on n edges.

Let $C(z) = \sum_{n=0}^{\infty} C_n z^n$ be the generating function of $\{C_n\}$

Illustration:

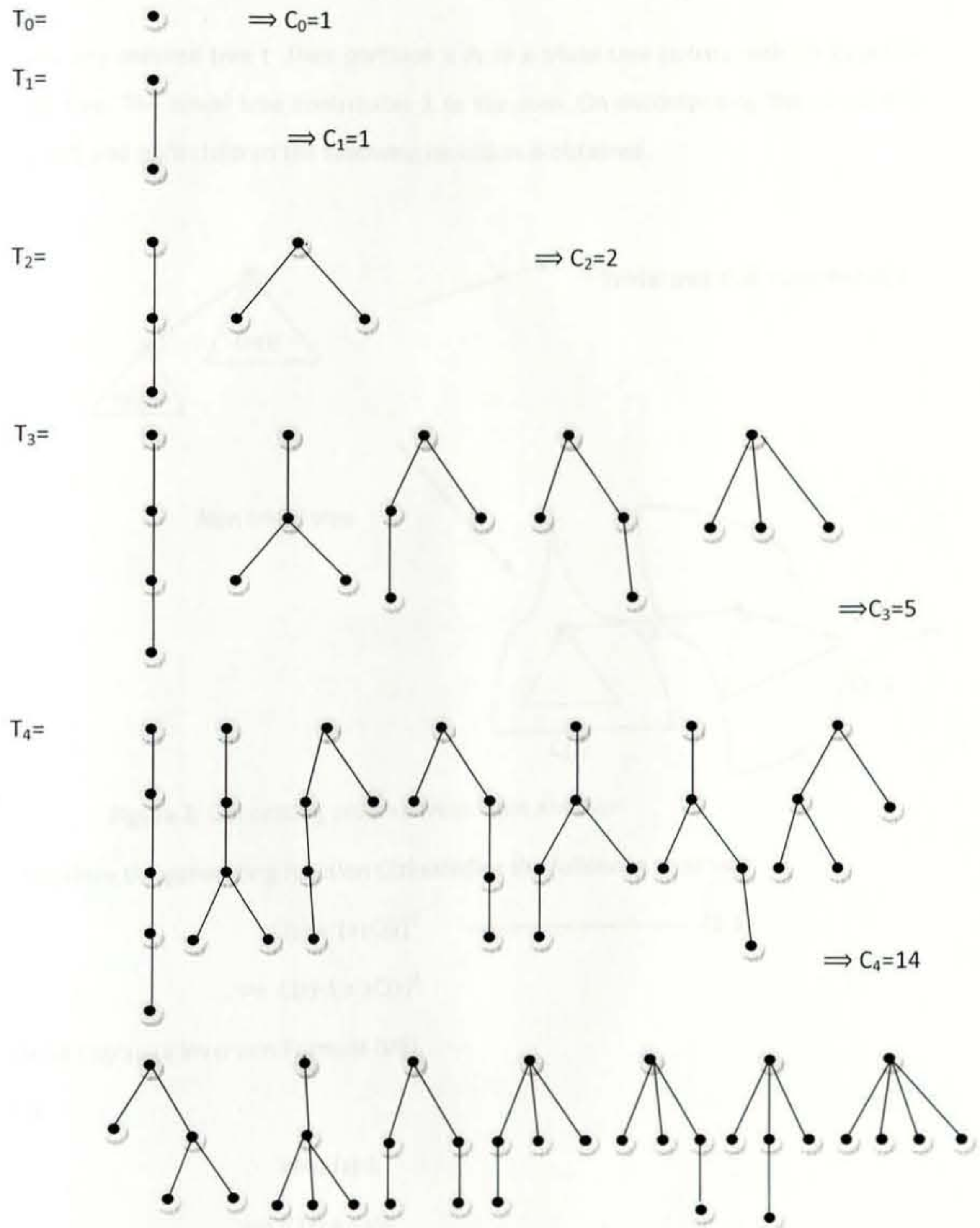


Figure 1.T_n-{ordered trees with n edges}

Theorem 1: number of ordered trees on n edges, is given by $C_n = \frac{1}{n+1}\binom{2n}{n}$

Proof: take any ordered tree t .Then partition it in to a trivial tree (a tree with no edge) or non-trivial tree. The trivial tree contributes 1 to the sum .On decomposing the non-trivial part in to left and right children the following recursion is obtained.

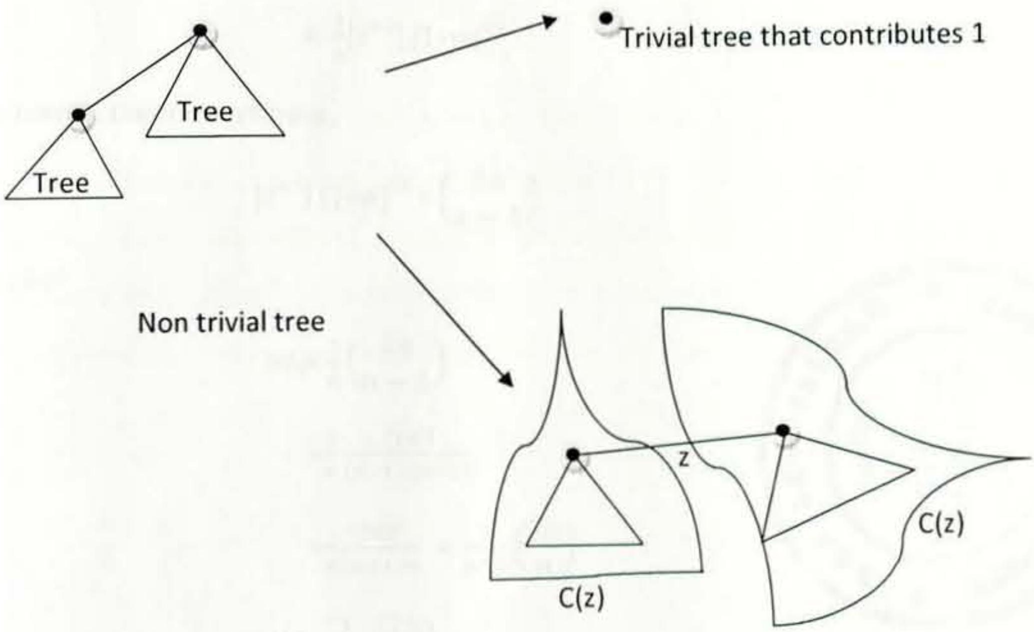


Figure 2: Generating ordered trees from another

Therefore the generating function C(z) satisfies the following recursion.

$$C(z) = 1+zC(z)^2 \quad \text{-----------------------------(1.1)}$$

$$\Rightarrow C(z)-1 = zC(z)^2$$

Using Lagrange Inversion Formula (LIF).

Let

$$W=C(z)-1$$

$$\Rightarrow C(z) = 1+W$$

Then from (1.1)

$$W = z(1+W)^2$$

Let

$$= (1+w)^2$$

This implies that

$$W_n = \frac{1}{n}[z^{n-1}]\ ((z))^n$$

$$= \frac{1}{n}[z^{n-1}]\ (1+w)^{2n}$$

From binomial theorem we have,

$$[z^{n-1}]\ (1+w)^{2n} = \binom{2n}{n-1}$$

This implies

$$W_n = \frac{1}{n}\binom{2n}{n-1}$$

$$= \frac{1}{n}\frac{(2n)!}{(n-1)!(n+1)!}$$

$$= \frac{(2n)!}{n!(n+1)n!} = \frac{1}{n+1}\binom{2n}{n}$$

$$\Rightarrow W_n = \frac{1}{n+1}\binom{2n}{n}$$

But $\mathbf{C_n} \leftrightarrow \mathbf{W_n}$, and hence

$$C_n = \frac{1}{n+1}\binom{2n}{n} \text{ ------------------------------ (1.2)}$$

Or applying quadratic formula on

$$zC(z)^2 - C(z) + 1 = 0$$

We obtain

$$C(z) = \frac{1 \pm \sqrt{1-4z}}{2z}$$

C(0)=1 implies that

$$C(z) = \frac{1-\sqrt{1-4z}}{2z}$$

But this is the well known generating function of the sequence of Catalan numbers and therefore the explicit formula for the sequence is

$$C_n = \frac{1}{n+1}\binom{2n}{n}$$

This implies that the number of ordered trees on n edges is given by the explicit formula (1.2) and this proves the theorem.

1.3 K-Trees

A k-tree is constructed from a single distinguished k-cycle by repeatedly gluing other k-cycles to existing one along an edge. If K is any non empty subset of {2, 3, 4, ... }, then a K-tree is obtained as above using k-cycles with k Є K. An edge of the distinguished k-cycle designated as a root and all children of the internal edges are shown in order in the plane containing the distinguished k cycle. If an internal edge has more than one child, we glue the cycles to the internal edge with diminishing size 2-cycles are glued to the midpoint of an edge of k cycles as shown in figure 4.

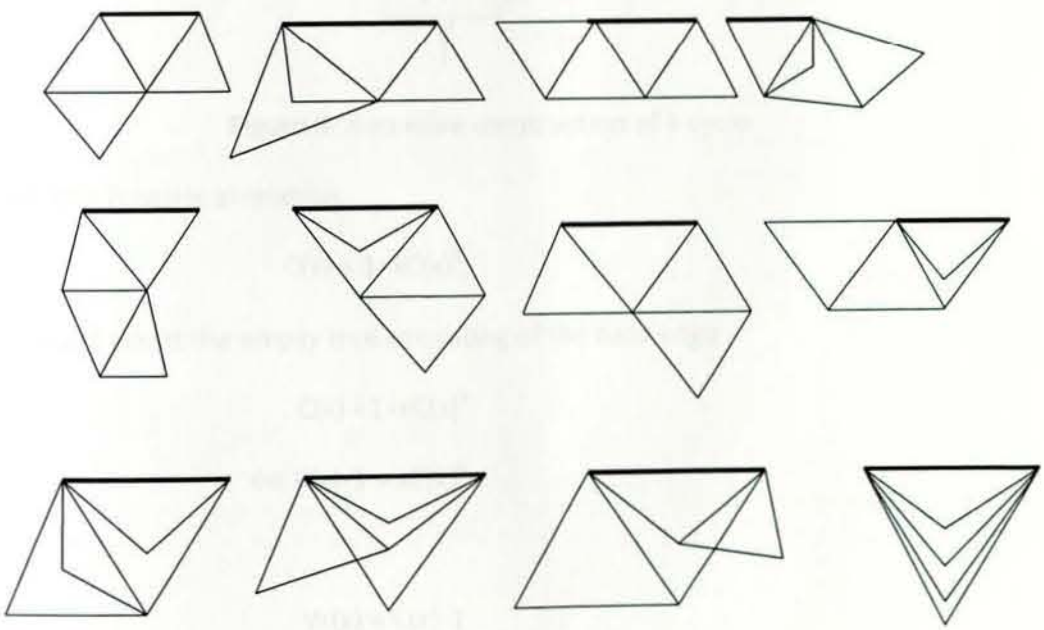


Figure 3: An example of the twelve 3 trees containing 4 3-cycles.

1.3.1 Enumeration of ordered k trees

Let the generating function of k trees be

$$C(x) = \sum_{n=0}^{\infty} C_{k,n} x^n$$

Where

$C_{k,n}$ = the number of k trees with exactly n k-trees with exactly n k-cycles.

If we begin with a distinguished k-cycle and construct an ordered k tree recursively by attaching another k cycles to one of the k edge of the distinguished as shown in the figure below.

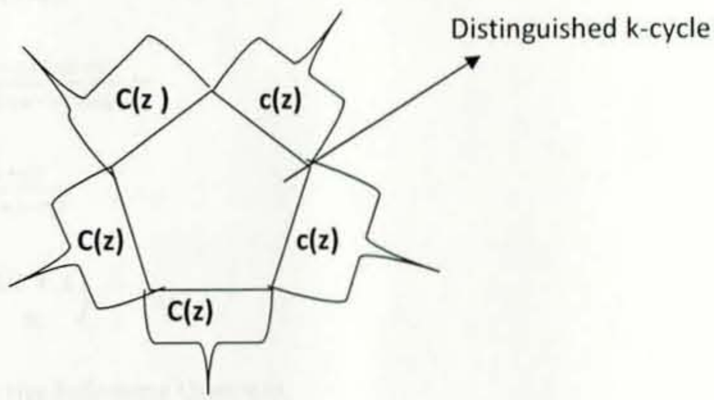


Figure 5: Recursive construction of k cycle

We obtain a functional relation

$$C(x) = 1 + xC(x)^k,$$

Where the 1 count the empty tree consisting of the root edge

$$C(x) = 1 + xC(x)^k$$

$$\Leftrightarrow C(x) - 1 = xC(x)^k$$

Let

$$W(x) = C(x) - 1$$

And

$$□(x) = (1 + w(x))^k$$

$F(x) = (w+1)^s = c^s$ Using Lagrange inversion formula (LIF)

$$[x^n]c^s(x) = \frac{1}{n}[w^{n-1}]s(w+1)^{s-1}((w+1)^k)^n$$

$$= \frac{1}{n}[w^{n-1}]s(w+1)^{nk+s-1}$$

$$= \frac{s}{n}[w^{n-1}]\sum_{i=0}^{nk+s-1}\binom{nk+s-1}{i}w^i$$

$$= \frac{s}{n}\binom{nk+s-1}{n-1}$$

$$= \frac{s}{n}\frac{(nk+s-1)!}{(n-1)!(nk+s-n)!}$$

$$= \frac{s}{n}\frac{(nk+s-1)!(nk+s)}{(n-1)!(nk+s-n)!(nk+s)}$$

$$= \frac{s}{nk+s}\frac{(nk+s)!}{n!(nk+s-n)!}$$

$$= \frac{s}{nk+s}\binom{nk+s}{n}$$

Hence we have proven the following theorem.

Theorem1.1 The generating function $c(x)=\sum_{n=0}^{\infty} c_{k,n}x^n$ satisfies the recurrence relation

$$c(x) = 1 + xc^k(x),$$

And

$$[z^n]c^s(x) = \frac{s}{nk+s}\binom{nk+s}{n}.$$

The idea used to prove theorem1.1 can be modified to obtain an enumeration formula for mixed trees or k-trees.

Remark. If we let s=1 and k ≥ 3 in theorem, we obtain sequences of numbers given by

$$c_{k,n} = \frac{1}{kn+1}\binom{kn+1}{n},$$

And these sequences of numbers count the number of homogeneous k-trees. We refer to these sequences of numbers as higher order Catalan numbers because of the similarities of the functional equation they satisfy.

SECTION TWO

2.1 The Catalan numbers and a tennis ball problem (The simplest case)

For n ∈ N, let $C_n = \frac{1}{(n+1)}\binom{2n}{n}$ denote the n^{th} Catalan number we find that the first seven Catalan numbers are $C_0=1$, $C_1=1$, $C_2=2$, $C_3=5$, $C_4=14$, $C_5=42$ and $C_6=132$.

In Ralph P.Grimaldi and Joseph G.Moser paper they introduce another application where the Catalan numbers arise. This application is motivated by the

"Tennis ball problem"

The application in question

Imagine yourself standing inside of a box someone outside the box tosses in two tennis balls labeled 1,2; you then toss out one of these balls. The person outside the box now tosses in two more tennis balls –these labeled 3, 4; you now toss out one of the three balls in the box. At this point the following five sets of two labels (on tosses –out tennis balls) are possible:

{1, 2}	{2, 3}
{1, 3}	{2, 4}
{1, 4}	
3 +	2 = 5

[Note: we consider 1 (on the first ball tosses out) followed by 2 (on the second ball tossed out) the same as 2 (on the first ball tossed out) followed by 1(on the second ball tossed out) so {1, 2} accounts for both of these situation, but is only counted once.]If the process is repeated a third time (with the introduction of tennis ball labeled 5, 6), we find 14 possible sets of three labels (on tossed –out tennis balls) these sets are

{1, 2, 3,} {1, 3, 4} {1, 4, 5} {2, 3, 4} {2, 4, 5}

{1, 2, 4} {1, 3, 5} {1, 4, 6} {2, 3, 5} {2, 4, 6}

{1, 2, 5} {1, 3, 6} {2, 3, 6}

{1, 2, 6}

(4 + 3 + 2 + 3 + 2) =14

We claim that if this process is repeated n times (where the i^{th} application of the process introduces the tennis balls labeled 2i-1,2i and the person inside the box then tosses out one of the i+1 tennis balls in the box) then the number of sets on ,n labels (on tossed –out tennis balls) is C_{n+1} ,the $(n+1)^{st}$ Catalan number .[in addition ,these C_{n+1}

Sets can be arranged in C_n columns where the sets (with elements arranged in C_n columns where the sets (with elements arranged in ascending order) in a column all have the same first n-1 elements.]

Throughout the discussion it is easier to follow the argument if we agree to list the entries in any set in ascending order. In order to establish the general result we refer to the article by R.Grimaldi [11]. This article develops the Catalan numbers, via a certain type of partition, as follows:

(i) Start with partition $p_2 = 2 = c_2$;
(ii) Replace the 2 in p_2 3 + 2. This result in the partition $p_3 = 3 + 2$ which sums to 5 = c_3;
(iii) The 3 in p_3 is now replaced with 4 + 3 +2 and the 2 (in p_3) with 3 + 2. The result is $p_4 = (4 + 3 + 2) + (3 + 2)$ which sums to 14 = c_4;
(iv) The 4 in p_4 is now replaced with 5+4+3+2, the 3 (in p_4) with 4+3+2 and the 2 (in p_4) with 3+2. The result in $p_5 = (5 + 4 + 3 + 2 + 4 + 3 + 2 + 3 + 2) + (4 + 3 + 2 + 3 + 2)$ which sums to 42=c_5; and,
(v) In general, once the partition $p_n, n \geq 2$, as described above, is obtained, and generate the partitions p_{n+1} by replacing each summand s in p_n by the sum (s + 1) + s + (s - 1) + ... + 3 + 2.

This partition (namely, p_{n+1}) then sums to c_{n+1}.

We now establish our claim about the number of sets generated by n applications of this two-in one-out process for the labeled tennis balls. For this we consider the open statement

S(n): Following n applications of the two-in one-out process (for tennis balls labeled 1, 2; 3, 4; 5, 6; ... ; 2n-1, 2n) we obtain c_{n+1} sets arranged in c_n columns. Further, if a column contains k sets, where $2\leq k \leq n+1$, then the largest entries in these k sets are $2n-k+1, 2n-k+2, \dots, 2n-1, 2n$. for n=1 we see that S(1) is valid and to check the validity of s(2) and s(3) assuming the result true for any n $\in z^{+}$, now go to n+1. Consider any of the c_n columns-say one with k sets. When the tennis balls labeled 2n + 1, 2n + 2 are introduced; these k sets give rise to k columns of sets: the first column containing k+1 sets (where the last entry in the i-th set is $2n-k+i-1, 1\leq i \leq k+1$), the second column containing k sets (where the last entry in the i-th set is $2n-k+i, 1 \leq i \leq k$), ..., the k-th (last) column containing two sets (with last entries $2n-1$ *and* $2n$). Hence the original k sets of size $n+1$. Consequently, it now follows [from (v) of our previous discussion on Catalan numbers via a certain type of partition] that S(n) $\Rightarrow$ S($n+1$), so the result is true for all n≥1 by the principle of mathematical induction.

2.2 Enumeration of balls on the lawn

2.2.1 A bijection between a ball arrangement on the lawn and k-trees

We begin by establishing a one-to-one correspondence between the Tennis Ball arrangements out on the lawn and k-trees.

Suppose $B_1=\{x_1, x_2, x_3, \dots, x_n\}$ is an arrangements of tennis balls out on the lawn after the n^{th} throw then $x_j \in \{1, 2, 3, \dots, jk\} \setminus \{x_1, x_2, x_3, \dots, x_{j-1}\}$ where $1 \leq j \leq n$ and $x_j < x_{j+1}$

Now, for each $x_j \in B_1=\{x_1, x_2, x_3, \dots, x_n\}$ subtract j and prepend 0 to obtain

$B2=\{0, y_1, y_2, y_3, \dots, y_n\}$ write elements of B_2 in reverse order and use $y_n\, y_{n-1}\, y_{n-2}, \dots, y_2\, y_1\, 0$

As an inversion table to construct a sequence of the first n+1 positive integers in which each integers repeat k-1 times .

For example:

If we are given 3 tennis balls at a time for four consecutive times and we throw the balls numbered 1, 3, 5, and 10 on the lawn then we obtain B_1= {1, 3, 5, 10}

Verification on the above examples

We have given 3 balls for four consecutive times

That implies

{1, 2, 3} , {4, 5, 6}, {7, 8, 9}, and {10, 11, 12}

Out of this we take balls numbered 1 and 3 from {1, 2, 3}

Take ball numbered 5 from {4, 5, 6}

No balls are drawn from {7, 8, 9}

Take ball numbered 10 from {10, 11, 12}

So, we obtain B_1= {1, 3, 5, 10} where k=3 and n=4 now from the above procedures we can get B_2= {0-0, 1-1, 3-2, 5-3, 10-4}

B_2= {0, 0, 1, 2, 6}

Write B_2 in reverse order and we get 62100.

Using 62100 as an inversion table we get the following.

Go to the sequence of numbers 1, 2, 3, 4, 5 in which every numbers appears (k-1) times

In our case k=3 so the numbers appears (3-1)=2(twice) let us take this ten blank spaces for our brief description those are the following x x x x x x x x x x or _ _ _ _ _ _ _ _ _ _

Now, 6 2 1 0 0
↓ ↓ ↓ ↓ ↓
1 2 3 4 5

From the above we have the following

- There are six numbers which are greater than 1

  $\Rightarrow$ _ _ _ _ _ _ 1 1 _ _

- There are two numbers which are greater than 2

  $\Rightarrow$ _ _ 2 2 _ _ 1 1 _ _

- There is one number which is greater than 3

  $\Rightarrow$ _ 3 2 2 3 _ 1 1 _ _

- There is no any number which is greater than 4

  $\Rightarrow$ 4 3 2 2 3 4 1 1 _ _

- There is also no number which is greater than 5

  $\Rightarrow$ 4 3 2 2 3 4 1 1 5 5

At the last we get the sequence of the form **4322341155**

As further examples

B_1= {3, 6, 7} yields 44331122 ------------------ (1.1)

And

B_2= {2, 3, 6} yields 42211334 ----------------- (1.2)

To show the sequence in (1.1) is right we have the following

First let us find B_2, we just applying the above steps that we are applying to find the sequence for B_1= {1, 3, 5, 10} those are

B_2= {0, 3-1, 6-2, 7-3}

B_2= {0, 2, 4, 4}

Write B_2 in the reverse order and we get 4420.using 4420 as an inversion table go to the sequence of numbers 1,2,3 and 4 in which every number appears (k-1)=2 times where k=3.

Let us have 8 blank spaces _ _ _ _ _ _ _ _

Now, 4 4 2 0

$\downarrow\ \downarrow\ \downarrow\ \downarrow$

1 2 3 4

From the above we have,

- There are four numbers which are greater than 1

⇒ _ _ _ _ 11_ _

- ❖ There are four numbers which are greater than 2

  ⇒ _ _ _ _ 1122

- ❖ There are two numbers which are greater than 3

  ⇒ _ _331122

- ❖ There is no any number which is greater than 4

  ⇒ 44331122

So we have the following sequence for B_1= {3, 6, 7} which is **44331122**

To show the sequence in (1.2) is right again we have the following

We have to find B_2 to do this

B_2 = {0, 2-1, 3-2, 6-3}

B_2 = {0, 1, 1, 3}

Write B2 in reverse order and we get 3110.using 3110 as an inversion table we get the following, go to the sequence of numbers 1,2,3, and 4 in which every numbers appears k-1 times which is equal to twice times.

Let us consider 8 free spaces _ _ _ _ _ _ _ _ for those numbers 1, 2, 3, and 4

Now, 3 1 1 0

↓↓↓↓

1 2 3 4

2 From the above we consider these points

- ❖ There are three numbers which are greater than 1

  ⇒ ___ ___ ___ 11___ ___ ___

- ❖ There is one number which is greater than 2

  ⇒ ___ 2211___ ___ ___

- ❖ There is one number which is greater than 3

  ⇒ ___ 221133 ___

- ❖ There is no number which greater than 4

  ⇒ 42211334

So, for B1= {2, 3, 6} we have a sequence **42211334**

Now, we construct a k tree consisting of n+1 k cycles from the sequence obtained above using the following recursive algorithm. Start with a k cycle, label it n+1, and distinguish an edge since the largest number n+1 appears k-1 times in the sequence there are k different positions to place the other numbers in the sequence relative to n+1.scan the sequence left-to-right and attach each sub tree obtained from a subsequence not containing n+1 to a side of the distinguished k- cycle in a counterclockwise direction. For example, the three trees that corresponds to the sequences **4322341155, 44331122**, and **42211334** are shown below.

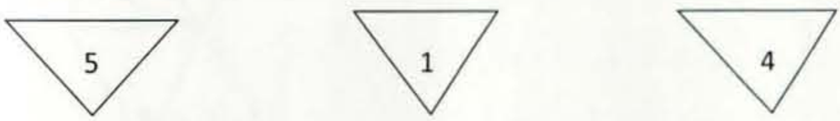
We are going to show the relationship between those trees and sequences by having the following procedures

Left ↔ right

Center ↔ right

Right ↔ center

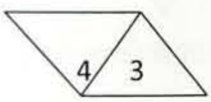
- We have seen that B_1 = {1, 3, 5, 10} yields that is 4322341155 for this we have 3 sub-sequences those are



Number three is located in the middle of 4 so, by our procedures we have

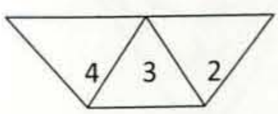
Center ↔ right

This implies



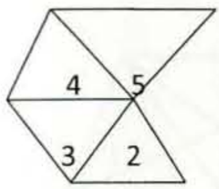
Number two is located in the center of 3

Center ↔ right



Now the sub sequence 432234 is in the left of 5 so, Left ↔ left

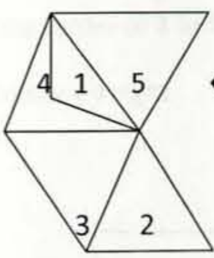
This implies,



Sub-sequences 11 is in the left of 55 and in the right of 432234 so,

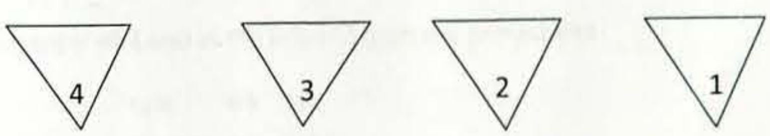
Left ↔left and right ↔ center

From this we get



↔ **4322341155**

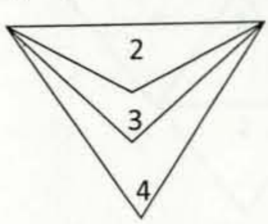
- For the second sequence with B_1={3, 6, 7} which is 44331122.we have four subsequences that is 44 33 11 22 and their representation is,



The subsequence 33 is to the right of 44 since we have

Right ↔ center

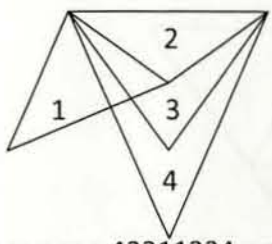
We get the following representation



The subsequence 11 is in the left of subsequence 22, by our procedures

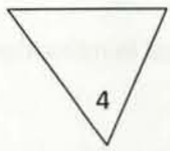
Left ↔ left

For this we have a representation below



↔ **44331122**

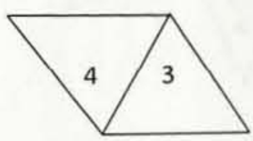
- For the sequence 42211334 we have the following construction



3 is located in the center of 4 so from the procedure we get

Center ↔ right

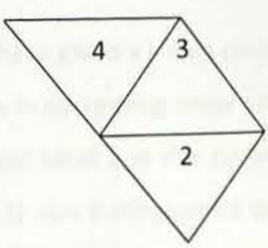
This implies that,



2 is located in the center of 4 and in the left of 3 from the procedures

Left ↔ left
Center ↔ right

This implies that,

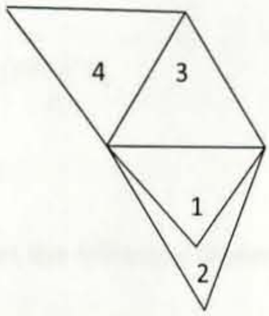


1 is located in the right of 2 and in the left of 3 and by applying the procedures we get,

Left ↔ left and

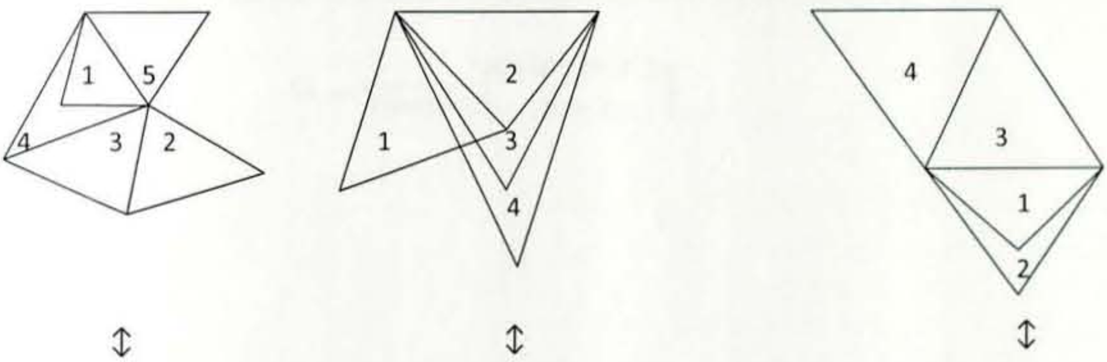
Right ↔ center

This implies that,



↔ **42211334**

This is all about the construction of sequences from k trees by applying the given procedures.



4322341155 44331122 4221133

Figure 6: 3 trees corresponding to the sequences

The labels are used to facilitate the construction of k trees from the sequence and dropping the labels we obtain unlabeled ordered k trees that corresponds to the sequence obtained from arrangements of tennis balls out on the lawn.

Conversely, suppose we have given a k tree consisting of n+1 k cycles use a preorder traversal of the k tree and label its cycles in decreasing order with the integers n+1,n,n-1,...,1.thus the distinguished cycles receives the highest label and the cycles visited last time receive the label 1 .we use this numbers to label the (k-1) non distinguished edges of each cycle ,and use post order traversal to obtain a sequence of the first n+1 numbers in which each number appears k-1 times .we then use the inversion table of this sequence and reverse all the steps in the previous algorithms to obtain an arrangements of balls on the lawn that corresponds to the given tree.

Using the method of generating function and Lagrange inversion formula the number of k trees consisting of n k- cycles is

$$C_{k,n} = \frac{s}{nk+s}\binom{nk+s}{n}$$

This is already done on page 9.

Hence we have proven the following theorem.

Theorem 2:suppose we are given balls numbered {1,2,...,k},{k+1,k+2,...,2k},... at a time and we throw one of the k balls out window onto the lawn .then the number of all possible arrangements of balls out on the lawn after the nth throw is

$$C_{k,\,n+1}=\frac{1}{(n+1)k+1}\binom{(n+1)k+1}{n+1}.$$

SECTION THREE

3.1 A connection between tennis ball problem with "bicolored Motzkin paths", "lattice path" and "ballot problem"

3.1.1 The sum of ball arrangements out on the lawn for n=2, where n is the number of balls on the lawn.

Definition: (bi colored Motzkin paths)

A path in the xy plane is called Motzkin path if it satisfies the following

- The possible steps are u=up=(1,1) ,D=down(1,-1) and level=(1,0) we allow the level steps to be one of two colors ,L or l .
- The path starts at (0,0) and consists of n steps
- The paths never go below the x-axis.

  If there had been only one kind of level step and the paths ended on the x-axis we would have regular Motzkin path. The generating function for Motzkin path

$$\sum_{n=0}^{\infty} M_n x^n = \frac{1 - x - \sqrt{1 - 2x - 3x^2}}{2x^2}$$

For n= 0, 1, 2, 3, 4, 5, 6, 7, ... we have the following Motzkin numbers
The Motzkin number M_n=1, 1, 2, 4, 9, 21, 51, 127, ... counts Motzkin n paths:

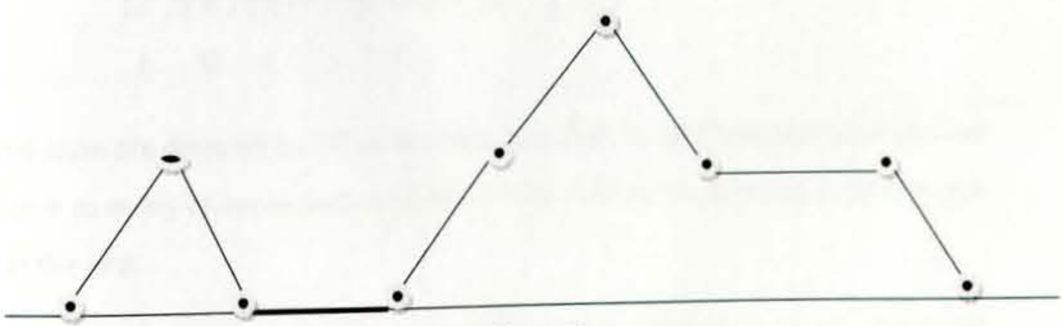


Figure 7: Motzkin path

Let b(n, k) be the number of possible paths that end at (n, k).

Here is a table of small values

n\k	0	1	2	3	4
0	1	0	0	0	0
1	2	1	0	0	0
2	5	4	1	0	0
3	14	14	6	1	0
4	42	48	27	8	1

We can make three observations:

One: we get the recursion

$$b(n+1,k)= b(n,k-1)+ 2b(n, k)+ b(n,k+1)$$

This holds for k, n≥0 if we specify that b (0, 0) = 1 and b(n,-1) = 0 both cases (conditions) make sense in terms of these paths.

Secondly we have

$$b(n, k) =\frac{k+1}{n+1}\binom{2n+2}{n-k}$$

Thirdly

$b(n, 0)=\frac{1}{(n+2)}\binom{2(n+1)}{n+1}$), the n+1st Catalan number which was Proved by J.mose and Grimaldi. Now we return to the balls on the lawn after n turns. Let us look at a typical example after 6 turns.

12' /3'4' /5'6 /7'8 /9, 10 /11, 12'

I U L L D I

The balls on the lawn are denoted by *"I" as* we read from left to right one pair at a time we must always have as many or more pairs with both balls selected as with no balls selected with equality at the end.

If both balls are selected mark the pair with U, if neither is selected mark with a D, if the odd member is the one selected use an L, if the even number was selected then use an I. This sets up an obvious correspondence with the bi colored Motzkin paths ending at height zeros and this shows that the number of possibilities after n turn is the Catalan number C_{n+1}.

We now want to shift back to sub diagonal path from (0, 0) to (n+1, n+1) using unit east and north steps .If we number the steps along each such path starting at the origin using the numbers to 2n+1, then the numbers of the horizontal steps correspond to the number of the balls on the lawn except that we ignore the initial horizontal step numbered zero .all sub diagonal paths must go from (0,0) to (1,0) at the first step so let us look on (1,0) as our starting point and (n+1,n) as the terminal point.

We then look at steps in pairs to set up a correspondence with bi colored Motzkin paths.

EE $\leftrightarrow$ U

EN $\leftrightarrow$ L

NE $\leftrightarrow$ I

NN $\leftrightarrow$ D

Where E= (0, 1) and N= (1, 0)

Lattice path

Given two such points (p, q) and (r, s), with p ≥ r and q ≥ s, a rectangular lattice path from (r, s) to (p, q) that is made up of horizontal steps H= (1,0) and vertical steps V=(0,1). Thus, rectangular lattice paths from (r, s) to (p, q) using unit horizontal and vertical segments.

Example. In figure 8 we show a rectangular lattice path from (0, 0) to (7, 5), consisting of 7 horizontal steps and 5 vertical steps. Given that the paths starts at (0, 0), it is uniquely determined by the sequence

H, V, V, H, H, H, V, V, H, V, H, H

Of 7 H'S and 5 V'S.

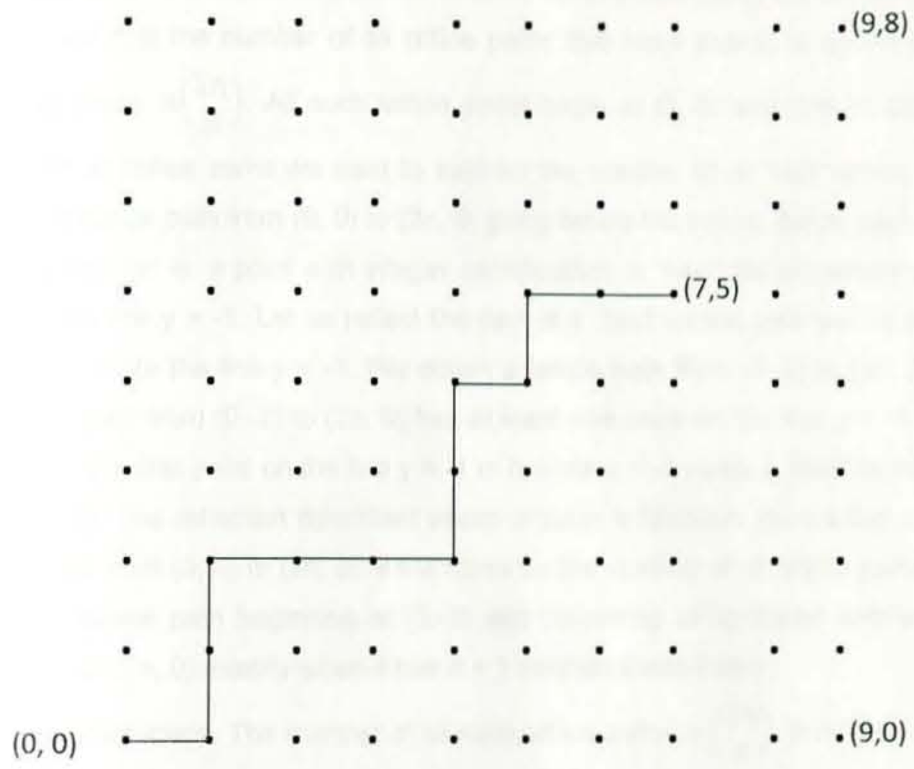


Figure 8: a rectangular lattice path from (0, 0) to (7, 5)

Moreover,

Imagine that there are n persons holding a 5 dollar bill and n persons holding s 10 dollar bill in front of a box office. The ticket costs 5 dollars and at the beginning the box office has no cash. How many ways are there to line up the 2n people if we want to make sure the box office never runs out of change? We define the Catalan number C_n as the answer to this question.

The problem is equivalent to the following lattice path enumeration problem: how many lattice paths are there, starting from (0, 0) consisting of n northeast steps (from (x, y) to (x + 1, y + 1)) and of n southeast steps (from (x, y) to (x + 1, y - 1)) that never go below the x-axis. Indeed, we can associate a northeast step to each person holding a 5 dollar bill, and southeast step to each person holding a 10 bill. As we parse the people standing in line, we obtain a lattice path. The box office never runs out of change exactly when the lattice path remains above the x-axis.

The lattice path visualization allows us to solve the problem using the reflection principle. Let us note first that the number of all lattice paths that have exactly n northeast steps and n southeast steps is$\binom{2n}{n}$. All such lattice paths begin at (0, 0) and end at (2n, 0). From the number of all lattice paths we want to subtract the number of all "bad" lattice paths, i.e., the number of lattice path from (0, 0) to (2n, 0) going below the x axis. Since each step ends at a lattice point (that is, a point with integer coordinates), a "bad" lattice path necessarily has a point on the line y = -1. Let us reflect the part of a "bad" lattice path before its first point on the line y = -1 to the line y = -1. We obtain a lattice path from (0,-2) to (2n, 0). Conversely, any lattice path from (0,-2) to (2n, 0) has at least one point on the line y = -1. Reflecting the part before the first point on the line y = -1 to the line y = -1 yields a "bad" lattice path from (0, 0) to (2n, 0). The reflection described above provide a bijection. Hence the number of \bad" lattice paths from (0, 0) to (2n, 0) is the same as the number of all lattice paths from (0,-2) to (2n, 0). A lattice path beginning at (0,-2) and consisting of northeast and southeast steps only, ends at (2n, 0) exactly when it has n + 1 northeast steps and

n - 1 southeast steps. The number of all such lattice paths is $\binom{2n}{n}$. thus the Catalan number C_n is given by

$$C_n = \binom{2n}{n} - \binom{2n}{n+1} = \binom{2n}{n}\left(1 - \frac{n}{n+1}\right) = \frac{1}{n+1}\binom{2n}{n}$$

Theorem 3:

The number of rectangular lattice paths from (r, s) to (p, q) equals to the binomial coefficient

$$\binom{p-r+q-s}{p-r} = \binom{p-r+q-s}{q-s}$$

Proof.

The two binomial coefficients in the statement of the theorem are equal. A rectangular lattice paths from (r, s) to (p, q) is uniquely determined by its sequence of p-r horizontal steps H and q-s vertical steps V, and every such sequences determines a rectangular lattice paths from (r, s) to (p, q). Hence, the number of paths equals the number of permutations of p-r + q-s objects of which p-r are H's q-s are V's. And we know that this number to be the binomial coefficient

$$\binom{p-r+q-s}{p-r}.$$

Consider a rectangular lattice path from (r, s) to (p, q), where $p \geq r$ and $q \geq s$. Such a path uses exactly (p-r) + (q-s) steps, there is no loss in generality in assuming that (r, q) = (0, 0). This is because we may simply translate (r, s) back to (p, q) and those from (0, 0) to (p-r, q-s). by theorem 3 , if $p \geq 0$ and $q \geq 0$,the number of rectangular lattice paths from (0, 0) to (p, q) equals

$$\binom{p+q}{p} = \binom{p+q}{q}$$

We now consider rectangular lattice paths from (0, 0) to (p, q) that are restricted to lie on or Below the line y=x in the coordinate plane. We call such paths sub diagonal rectangular lattice paths from (0, 0) to (9, 9) in figure 9.

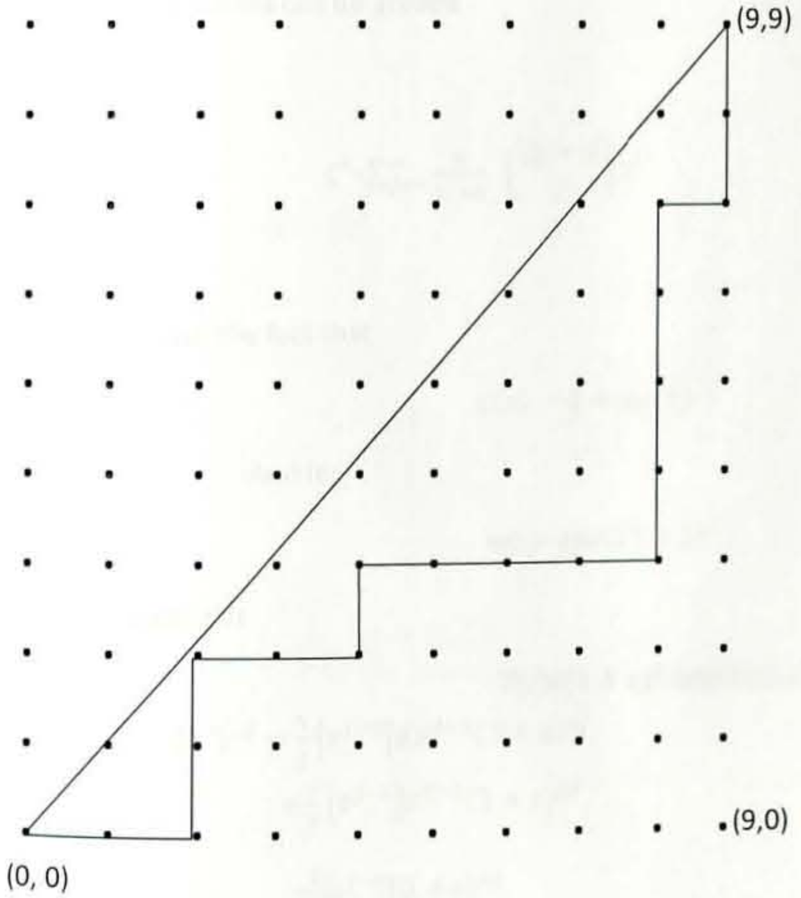


Figure 9

To evaluate the sum over all possible sets of balls on the lawn takes a bit more doing and it's worthwhile to separate out some definition and lemmas first.

Definition and notation

The nth Catalan number is $C_n = \frac{1}{n+1}\binom{2n}{n}$ the generating function for the sequence of Catalan number is C=C(z)

$$C = C(z) = \sum_{n=0}^{\infty} C_n z^n = \frac{1-Q}{2z} \text{ where } Q = \sqrt{1-4z}$$

Similarly $B = B(z) = \sum_{n=0}^{\infty}\binom{2n}{n} z^n = \frac{1}{Q}$ is the generating function for the sequence of centernal binomial coefficients.

The following lemma can be proved

Lemma 1

$$C^d = \sum_{j=0}^{\infty} \frac{d}{2j+d}\binom{2j+d}{j} z^j$$

Proof:

We use the fact that

$$c(z) - 1 = zc(z)^2$$

And let

$$w(z) = z(w(z)+1)^2$$

This implies that

$$⍰(z) = (1+z)^2 \text{ and } F(z) = z^d$$

$$[z^j]c^d = \frac{1}{j}[z^{j-1}]dz^{d-1}(1+z)^{2j}$$

$$= \frac{d}{j}[z^{j-1}]z^{d-1}(1+z)^{2j}$$

$$= \frac{d}{j}[z^{j-d}](1+z)^{2j}$$

$$= \frac{d}{j}[z^{j-d}]\sum_{i=0}^{2j}\binom{2j}{i} z^i$$

$=\frac{d}{j}\binom{2j}{j-d}$

$=\frac{d}{2j+d}\binom{2j+d}{j}$

Therefore,

$$C^d=\sum_{j=0}^{\infty}\frac{d}{2j+d}\binom{2j+d}{j}z^j$$

Lemma 2

$$C^d=\sum_{j=0}^{\infty}\frac{d}{j+d}\binom{2j+d-1}{j+d-1}z^j$$

Proof:

We again use the fact that

$$c(z)-1=zc(z)^2$$

And let

$$w(z)=z(w(z)+1)^2$$

So, we have,

⍰(z)=$(1+z)^2$ and F(z)=z^d

$[z^j]c^d=\frac{1}{j}[z^{j-1}]dz^{d-1}(1+z)^{2j}$

$=\frac{d}{j}[z^{j-1}]z^{d-1}(1+z)^{2j}$

$=\frac{d}{j}[z^{j-d}](1+z)^{2j}$

$=\frac{d}{j}[z^{j-d}]\sum_{i=0}^{2j}\binom{2j}{i}z^i$

$=\frac{d}{j}\binom{2j}{j-d}$

$=\frac{d}{2j+d}\binom{2j+d}{j}$

$$=\frac{d}{j+d}\binom{2j+d-1}{j+d-1}$$

Therefore,

$$C^d=\sum_{j=0}^{\infty}\frac{d}{j+d}\binom{2j+d-1}{j+d-1}z^j$$

Lemma 3

$$BC^d=\sum_{j=0}^{\infty}\binom{2j+d}{j}z^j$$

Lemma 4

$$B=\frac{C}{1-Zc^2} \text{ or alternatively } \frac{1}{1-zc^2}=\frac{B}{C}$$

Proof:

$$\frac{1}{1-zc^2}=\frac{1}{1-Z(\frac{1-\sqrt{1-4Z}}{2Z})^2}$$

$$=\frac{1}{1-Z(\frac{1+(1-4Z)-2\sqrt{1-4Z}}{4z^2})}$$

$$=\frac{1}{\frac{4Z-1+2\sqrt{1-4Z}-1+4Z}{4Z}}$$

$$=\frac{4Z}{4Z-2+2\sqrt{1-4Z}}$$

$$=\frac{4Z}{8Z-2+2\sqrt{1-4Z}}$$

$$=\frac{2Z}{4Z-2+2\sqrt{1-4Z}}$$

$$=\frac{2Z}{\sqrt{1-4Z}(1-\sqrt{1-4Z})}$$

$$=\frac{\frac{1}{\sqrt{1-4Z}}}{\frac{1-\sqrt{1-4Z}}{2Z}}$$

$$=\frac{B}{C}$$

Therefore,

$$\frac{B}{C}=\frac{1}{1-ZC^2}$$

Lemma 5

2B=C(1+B)

Proof:

$$C(1+B)=\frac{1-\sqrt{1-4Z}}{2Z}(1+\frac{1}{\sqrt{1-4Z}})$$

$$=\frac{1-\sqrt{1-4Z}}{2Z}+\frac{1-\sqrt{1-4Z}}{2Z\sqrt{1-4Z}}$$

$$=\frac{\sqrt{1-4Z}(1-\sqrt{1-4Z})+1-\sqrt{1-4Z}}{2Z\sqrt{1-4Z}}$$

$$=\frac{\sqrt{1-4Z}-1+4Z+1-\sqrt{1-4Z}}{2Z\sqrt{1-4Z}}$$

$$=\frac{4Z}{2Z\sqrt{1-4Z}}=\frac{2}{\sqrt{1-4Z}}$$

$$=2.\frac{1}{\sqrt{1-4Z}}$$

=2.B=2B

Therefore,

2B=C (1+B).

Lemma 6:

$$B^3=\sum_{n=0}^{\infty}(2n+1)\binom{2n}{n}z^n$$

Proof:

We are going to find the coefficient of z^n in $(1-4z)^{\frac{-3}{2}}=\sum_{i=0}^{\infty}\binom{\frac{-3}{2}}{i}(-4z)^i$ which is

=

=

Since we have, $\binom{2n}{n} = \frac{2^n(2n-1)(2n-3)\cdots 3\cdot 1}{n!}.$

= (2n+1)

Therefore we have,

=

Lemma 7

The number of sub diagonal path from (0, 0) to (i, j) will be denoted N(i, j) will be denoted N(i, j) and —— ( )

Proof:

We know that the Catalan number counts the number of sub diagonal rectangular lattice paths from (0, 0) to (n, n), now we determine the number l(i, j) of rectangular lattice paths from (0, 0) to (i, j) that cross the diagonal, and then subtract l(i, j) from the total number of rectangular lattice paths from (0, 0) to (i, j). The number l(i, j) is the same as the number of rectangular lattice paths from (0, -1) to (i, j-1) that touch (possibly cross) the diagonal line y=x. This follows by shifting paths down one unit, thereby shifting a path γ in to a path γ', and this establishes a one to one correspondence between the two kinds of path.

Consider a path γ' from (0, -1) to (i, j-1) that touches the diagonal line y=x. Let γ_1 be the sub path of γ' from (0, -1) to the first diagonal point (d, d) touched by γ'. Let γ_2' be the sub path of γ' from (d, d) to (i, j-1). You reflect γ_1' about the line y=x and obtain a path γ_1^* from (-1, 0) to (d, d). Following γ_1^* with γ_2, we get a path γ^* from (-1, 0) to (i,j-1). This construction is illustrated in figure 10.

Now every rectangular lattice path θ from (-1, 0) to (i, j-1) must cross the diagonal line y=x, since (-1, 0) is above the line and (i, j-1) is below. If you reflect the part of θ that goes from (-1, 0) to the first crossing point, we get a path from (0, -1) to (i, j-1) that touches the line y=x. This shows that the correspondence γ' to γ^* is a one to one correspondence, and hence that I(i, j) equals the number of rectangular lattice path from (-1, 0) to (i, j-1) . By theorem 3 we have

$$N(i,j) = \binom{i+1+j-1}{j-1} = \binom{i+j}{j-1}$$

Therefore, the number of sub diagonal rectangular lattice paths from (0, 0) to (i, j) equals

$\binom{i+j}{j} - I(i,j) = \binom{i+j}{j} - \binom{i+j}{j-1} = \frac{(i+j)!}{i!j!} - \frac{(i+j)!}{(j-1)!(i+1)!}$, which simplifies to

$\frac{i-j+1}{i+1}\binom{i+j}{j}$.

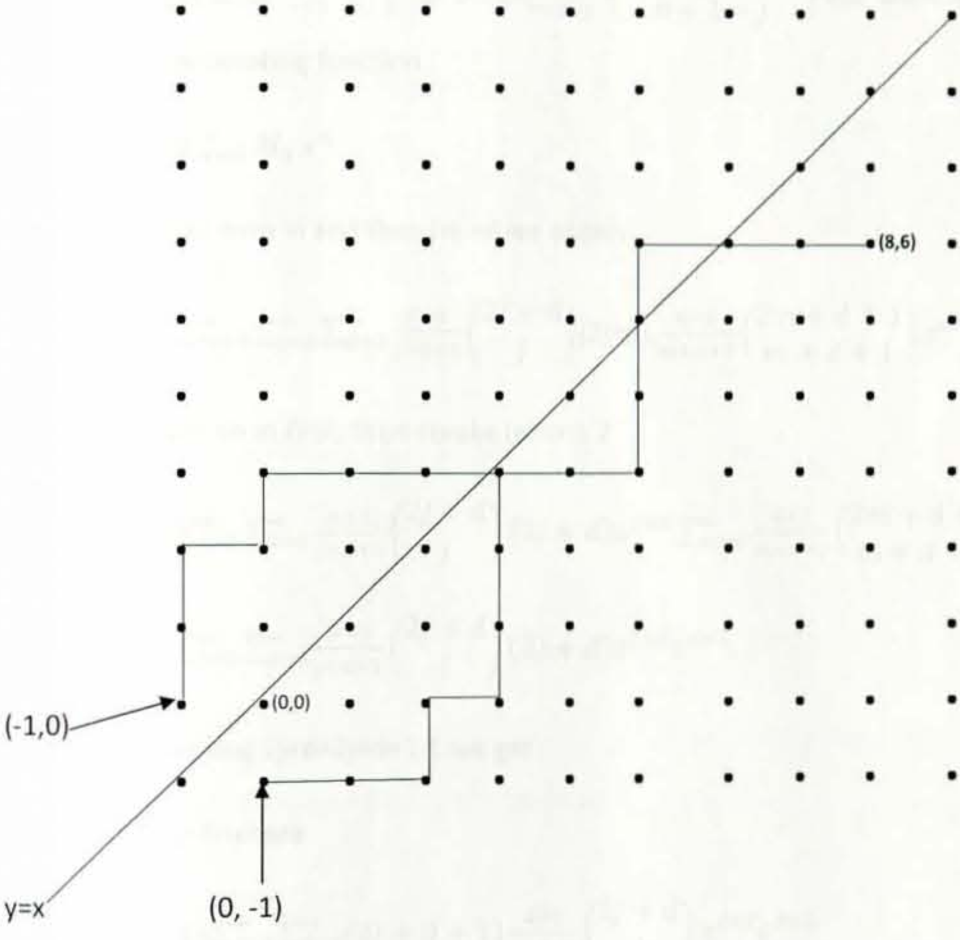


Figure 10

We now want to get on to the main computation. Note first that

$M_n = \sum_{i=0}^{n} \sum_{j=0}^{i} N(i,j).(i+j)N(n+1-j, n-i)$

Before launching in to the evaluation let's look at each term there are N(i, j) paths from (0, 0) to (i, j) and N(n+1-j,n-i) paths from (i+1, j) to (n+1,n+1). What does it mean for a path to have arrived at (i, j)? Of the balls {1, 2, ..., i+j-1},i-1 of them are on the lawn and j of them are to stay in the room. The horizontal step (i, j)→ (i+1,j) indicates the ball numbered i+1 is to be on the lawn ,and hence the term (i+j)

By lemma 7 we then get

$M_n=\sum_{i=0}^{n}\sum_{j=0}^{i}\frac{i+1-j}{i+1}\binom{i+j}{i}$. (i +j).$\frac{i+2-j}{n+2-j}\binom{2n+1-i-j}{n+1-j}$ We want to find a closed form for the generating function

M(z)=$\sum_{n=0}^{\infty} M_n\, z^n$

If we set n=m +i and then i=j +d we obtain

M(z)=$\sum_{m=0}^{\infty}\sum_{j=0}^{\infty}\sum_{d=0}^{\infty}\frac{d+1}{j+d+1}\binom{2j+d}{j}$(2j+d).$\frac{d+2}{m+d+2}\binom{2m+d+1}{m+d+1}z^{m+j+d}$.

We sum on m first; then invoke lemma 2

M(z)=$\sum_{j=0}^{\infty}\sum_{d=0}^{\infty}\frac{d+1}{j+d+1}\binom{2j+d}{j}(2j+d)z^{j+d}\sum_{m=0}^{\infty}\frac{d+2}{m+d+2}\binom{2m+d+1}{m+d+1}z^{m}$

M(z)=$\sum_{j=0}^{\infty}\sum_{d=0}^{\infty}\frac{d+1}{j+d+1}\binom{2j+d}{j}(2j+d)z^{j+d}c^{d+2}$.

By rewriting 2j+d=2j+d+1-1 we get

M(z)=P-R where

P=$\sum_{j=0}^{\infty}\sum_{d=0}^{\infty}(2j+d+1)\frac{d+1}{j+d+1}\binom{2j+d}{j}z^{j+d}c^{d+2}$

And

R=$\sum_{j=0}^{\infty}\sum_{d=0}^{\infty}\frac{d+1}{j+d+1}\binom{2j+d}{j}z^{j+d}c^{d+2}$

However, with the aid of lemma 3 and 4, we obtain

P=$\sum_{d=0}^{\infty}(d+1)C^{d+2}\sum_{j=0}^{\infty}\frac{2j+d+1}{j+d+1}\binom{2j+d}{j}z^{j+d}$

=$\sum_{d=0}^{\infty}(d+1)C^{d+2}z^{d}\sum_{j=0}^{\infty}\frac{2j+d+1}{j+d+1}\binom{2j+d}{j+d}z^{j}$ since, $\binom{n}{k}=\binom{n}{n-k}$

=$\sum_{d=0}^{\infty}(d+1)C^{d+2}z^{d}\sum_{j=0}^{\infty}\binom{2j+d}{j+d}z^{j}$ since $\binom{n}{k}=(\frac{n}{k})\binom{n-1}{k-1}$

$=\sum_{d=0}^{\infty} d+1\, C^{d+2}z^{d}\sum_{j=0}^{\infty}\binom{2j+d+1}{j+d+1}z^{j}$

$=\sum_{d=0}^{\infty}(d+1)C^{d+2}z^{d}\ BC^{d+1}$---------------- By lemma 3

$=BC^{3}\sum_{d=0}^{\infty}(d+1)C^{2d}z^{d}$

$=BC^{3}\sum_{d=0}^{\infty}(d+1)(C^{2}z)^{d}$

$=BC^{3}.\frac{1}{(1-ZC^{2})^{2}}$ Since $\sum_{d=0}^{\infty}(d+1)(C^{2}z)^{d}=\frac{1}{(1-ZC^{2})^{2}}$ by applying derivative rule

$=BC^{3}(\frac{B}{C})^{2}$ ----------------- By lemma 4

$=B^{3}C$

Therefore,

P=B^{3}C

From the second term we have via lemma 2 and 4

$R=\sum_{j=0}^{\infty}\sum_{d=0}^{\infty}\frac{d+1}{j+d+1}\binom{2j+d}{j}z^{j+d}c^{d+2}$

$=\sum_{d=0}^{\infty}c^{d+2}z^{d}\sum_{j=0}^{\infty}\frac{d+1}{j+d+1}\binom{2j+d}{j}z^{j}$

$=\sum_{d=0}^{\infty}c^{d+2}z^{d}\sum_{j=0}^{\infty}\frac{d+1}{j+d+1}\binom{2j+d}{j+d}z^{j}$ since$\binom{n}{k}=\binom{n}{k-1}$

$=\sum_{d=0}^{\infty}c^{d+2}z^{d}\ c^{d+1}$ ---------------by lemma 2

$=\sum_{d=0}^{\infty}c^{3}c^{2d}z^{d}$

$=c^{3}\sum_{d=0}^{\infty}c^{2d}z^{d}$

$=c^{3}\sum_{d=0}^{\infty}(c^{2}z)^{d}$

$=c^{3}\frac{1}{1-zc^{2}}=c^{3}\frac{B}{c}$--------------------by lemma 4

$=C^2B$

Therefore,

$$\mathbf{R=}\boldsymbol{C^2B}$$

Thus,

$$\mathbf{M\ (z) =}\ \boldsymbol{B^3C - C^2B}$$

Remarks:

- $M(Z)=\frac{C^3B^3}{Z}\left(1+\frac{2}{B}\right)=\frac{C^3}{z(1-4z)}(B+2)$
- $M_n=\frac{2n^2+5n+4}{n+2}\binom{2n+1}{n}-2^{2n+1}$

The second follows from the first as follows:

$$zc^3B^3(1+\frac{2}{B})=\frac{zC^3}{Q^3}(1+2Q)$$

$$=\frac{z}{Q^3}(1+2Q)(\frac{1-Q}{2z})^3 \text{ where } B=\frac{1}{Q} \text{ and } C=\frac{1-Q}{2z}.$$

$$=\frac{z}{Q^3}(1+2Q)\frac{(1-2Q+Q^2)(1-Q)}{8z^3}$$

$$=\frac{z}{8z^3Q^3}(1+2Q)(1-Q-2Q+2Q^2+Q^2-Q^3)$$

$$=\frac{1}{8Z^2Q^3}(1+2Q)(1-3Q+3Q^2-Q^3)$$

$$=\frac{1}{8Z^2Q^3}(1-Q-3Q^2+5Q^3-2Q^4)$$

$$=\frac{1}{2Z^2Q^3}(-1+7z-8z^2+Q-5Qz)$$

by taking $z=\frac{1-Q^2}{4}$ *we can simply show that*

$$\frac{1}{4}(1 - Q - 3Q^2 + 5Q^3 - 2Q^4) = (-1 + 7z - 8z^2 + Q - 5Qz)$$

So we get,

$$=\frac{1}{2}[-\frac{B^3}{z^2} + 7\frac{B^2}{z} - 8B^3 + \frac{B^2}{z^2} - 5\frac{B^2}{z}]$$

Since $Q^2 = 1 - 4z, but\ B^2 = \sum_{n=0}^{\infty} 4^n z^n$ while $B^3 = \sum_{n=0}^{\infty}(2n+1)\binom{2n}{n}$ then extracting n[th]

Terms yields

$$M(z)=\frac{1}{2}[\sum_{n=0}^{\infty}(2n+1)\binom{2n}{n}z^{n-2} + \sum_{n=0}^{\infty} 7(2n+1)\binom{2n}{n}z^{n-1} -$$

$$\sum_{n=0}^{\infty} 8(2n+1)\binom{2n}{n}z^n + \sum_{n=0}^{\infty} 4^n z^{n-2} - \sum_{n=0}^{\infty} 5(4^n)z^{n-1}]$$

Now, extract the n[th] terms

$$M_n = \frac{1}{2}\left[-(2n+5)\binom{2n+4}{n+2} + 7(2n+3)\binom{2n+2}{n+1} - 8(2n+1)\binom{2n}{n} + 4^{n+2} - 5.4^{n+1}\right]$$

$$=\frac{1}{2}[\frac{-(2n+5)(2n+4)(2n+3)(2n+2)}{(n+2)(n+2)(n+1)} + 7(2n+3)\frac{(2n+2)}{(n+1)} - 8(2n+1)\left(\frac{n+1}{2n+1}\right)]\binom{2n+1}{n} +$$

$$\frac{1}{2}(4^n.4^2 - 5.4^n.4)$$

$$= \frac{1}{2}\left[\frac{-16n^2-64n-60}{n+2} + 28n + 42 - 8n - 8\right] + \frac{1}{2}(4^n(16-20))$$

$$= \binom{2n+1}{n}\left(\frac{-8n^2-32n-30}{n+2} + 10n + 17\right) + 2^{2n}2^{-1}.-2^2$$

$$=\binom{2n+1}{n}\frac{-8n^2-32n-30+10n^2+37n+34}{n+2} - 2^{2n-1+2}$$

$$=\frac{2n^2+5n+4}{n+2}\binom{2n+1}{n} - 2^{2n+1}$$

Therefore,

$$M_n = \frac{2n^2+5n+4}{n+2}\binom{2n+1}{n} - 2^{2n+1}$$

Two other remarks can be made here first, an asymptotic result.

$$M_n \sim 4^n\left(4\sqrt{\frac{n}{\pi}} - 2\right)$$

Secondly the expected value of the balls on the lawn is $\frac{n(4n+5)}{6}$ if we assume each available ball is equally likely to be picked at each turn.

The ballot problem

Suppose that in an election, candidate A receives a vote and candidate B receives b votes where a≥kb for some positive integer k .compute the number of ways the ballots can be ordered .so, then A maintains more than k times as many votes as B throughout the counting of the ballots.

Theorem 2 (the ballot theorem):

The solution to the ballot problem is $\frac{a-kb}{a+b}\binom{a+b}{a}$.

Proof:

Let $N_k(a,b)$denote the number of ways the a + b ballots (a≥ kb) can be ordered so that candidate A maintain more than k times as many votes as B throughout the counting of the ballots. The conditions $N_k(a,b)=1$ for all a>0 and $N_k(kb,b)=0$ for all b>0, are easily verified by considering the statements of ballot problem, and they both satisfy

$N_k(a,b) = \frac{a-kb}{a+b}\binom{a+b}{a}$ for b>0 and a>kb, we see that

$N_k(a,b) = N_k(a,b-1) + N_k(a-1,b)$. By considering the last vote in a ballot permutation

$= \frac{a-k(b-1)}{a+b-1}\binom{a+b}{a} + \frac{a-1-kb}{a+b-1}\binom{a+b}{a}.$

This implies,

The solution to the ballot problem k is $\frac{a-kb}{a+b}\binom{a+b}{a}.$

The weak ballot problem, Catalan numbers

The ballot problem is often stated in a" weak" version :suppose that candidates B receive n votes , where m ≥ kn for some positive integers k , and compute the number of ways the ballots can be ordered so that A always has at least k times as many votes as B throughout the counting of the ballots.

Any ballot permutation in which a maintains at least k-times the number of votes for B can be converted in to one in which A maintain move that k times the number of votes for B by simply appending a vote for A to the beginning of the permutation. This process is reversible, and hence the solution to the weak version is the same as the "strict" version when A receives m+1 votes and B receives n votes

$\frac{(m+1)-kn}{(m+1)+n}\binom{(m+1)+n}{m+1} = \frac{m+1-kn}{m+1}\binom{m+n}{m}$

Put k=1 and m=n produces the well known Catalan numbers, $c_n = \frac{1}{n+1}\binom{2n}{n}$

If we put m=kn produces the generalized Catalan numbers also called the k-Catalan numbers

$C_{n,k} = \frac{1}{n+1}\binom{(k+1)n}{n}$

3.2 The sum of ball arrangements out on the lawn for generalized case

We now return our attention to a sequence obtained from the arrangements of balls out on the lawn by taking sums after each throw in the generalized case. For example , if we are given 3 balls at a time the possible numbers out on the lawn after the first throw are {1},{2},or{3} , and the sum is 1+2+3=6 similarly ,after the second throw the sum over all arrangements is 1+2+1+3+1+4+1+5+1+6+2+3+2+4+5+2+6+3+4+3+5+3+6=75.We obtain a summation formula for such sequences using a method similar to the one used by Mallows and Shapiro for the balls (n=2)

The following two lemmas about lattice paths that use unit East and North steps are crucial to our result.

Lemma 1

The number of lattice paths p(i, j) from (0, 0) to (i, j) not crossing the line y=mx is given by

$$P(i, j)=\begin{cases}\binom{i+j-1}{i-1}, & 0\le j\le \\ \binom{i+j-1}{i-1}-\sum_{t=1}^{\frac{j-m}{m}}\frac{1}{mt+1}\binom{(m+1)t}{t}\binom{(i+j-1)-(m+1)t}{i-t} & \\ & , j\geq m+1\end{cases}$$

Proof:

If 0≤j≤m, all paths from (0, 0) to (i, j), do not cross the line y= mx, and hence P(i, j)= $\binom{i+j-1}{i-1}$.This follows from Bizleys theorem. That for every t ∈ {1, 2,... ⌈$\frac{j-m}{m}$⌉} the number of paths crossing the line y=mx exactly t times is given by

$$\frac{1}{mt+1}\binom{(m+1)t}{t}\binom{(i+j-1)-(m+1)t}{i-t}$$

For j≥m+1, P (i, j) is then obtained by subtracting the paths that cross the line y=mx from the total paths from (0, 0) to (i, j)

That means,

$$\binom{i+j-1}{i-1}-\sum_{t=1}^{\frac{j-m}{m}}\frac{1}{mt+1}\binom{(m+1)t}{t}\binom{(i+j-1)-(m+1)t}{i-t}.$$

Lemma 2

The number of lattice paths $P_{i,j}^{n,nm}$ from (i, j) to (n, mn) not crossing the line y=mx is given by

$$P_{i,j}^{n,nm} = \frac{mi-j+1}{mn-j+1}\binom{(m+1)n-(i+j)}{n-i}.$$

This lemma can be verified easily by induction using the obvious recurrence relation that $P_{i,j}^{n,nm}$ satisfies.

Now each arrangements of balls on the lawn after the n^{th} throw in the generalized case can be represented by lattice paths from (0, 0) to (n+1, (k-1)(n+1)) that uses unit East and North steps not crossing the line y=(k-1)x, such that the horizontal steps represent the balls on the lawn and the vertical ones represent balls in the room .this process is reversible and establishes a one-to-one correspondence.

Notice that all paths not crossing the line y=(k-1)x must go from (0, 0) to (1,0) in the first step ,and the number of paths that end at (n+1,(k-1)(n+1))is the same as the ones that end at (n+1,(k-1)n).hence the number of paths from (0, 0) to (n+1,(k-1)(n+1)) not crossing the line y=(k-1)x is the same as the number of paths from (1,0) to (n+1,(k-1)n).

If we label the unit steps along such paths from (1, 0) to (n+1, (k-1)n) using the numbers 1 through kn (the total number of balls).then the label of the horizontal steps correspond to the balls on the lawn . Hence finding the sum over all arrangements of balls on the lawn is equivalent to finding the sum of all possible labeling of horizontal steps.

Suppose there is a horizontal step from (i, j) to (i+1, j), then its label is i+j, and such labeling can be found in P(i, j). $P_{i+1,j}^{n+1,(k-1)(n+1)}$ different ways . Hence we have the following result

Theorem 4

Suppose we are given balls numbered {1,2,...,k},{k+1,k+2,...,2k},... at a time and we throw one of the k balls out window on to the lawn .then sum over all possible arrangements of balls out on the lawn after the n[th] throw is given by

$$S_{n,k} = \sum_{i=1}^{n} \sum_{j=0}^{(k-1)i} P(i,j)\ x\ (i+j)\ x\ P_{i+1,j}^{n+1,(k-1)(n+1)}$$

Where

$$P(i,j)) = \begin{cases} \binom{i+j-1}{i-1}, & 0 \le j \le k-1 \\ \binom{i+j-1}{i-1} - \sum_{t=1}^{\frac{j-(k-1)}{k-1}} \frac{1}{(k-1)t+1} \binom{kt}{t} \binom{(i+j-1)-kt}{i-t}, & j \ge k \end{cases}$$

And

$$p_{i+1,j}^{n+1,(k-1)(n+1)} = \frac{(k-1)(i+1)-j+1}{(k-1)(n+1)-j+1} \binom{k(n+1)-(i+j+1)}{n-i}$$

The following table gives the sums obtained from theorem 3 for $2 \leq k \leq 6$ and $1 \leq n \leq 6$ $using$ Maple.

K/n	1	2	3	4	5	6
2	3	23	131	664	3166	14545
3	6	75	708	5991	47868	369315
4	10	174	2298	27258	305574	3309444
5	15	335	5690	86860	1253515	17478840
6	21	573	11901	222210	3922680	66909378

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