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Addis Ababa University

Addis Ababa University

Department of Mathematics

The undersigned hereby certify that they have read and recommend to the school of graduate studies for acceptance of a project Entitled "**Clifford semigroups and seminear-rings of endomorphism"** by Dejenehunegnaw in partial fulfillment of the requirements for the degree of Master of Science in Mathematics.

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Advisor: ______________________

Dr. TilahunAbebaw

Examining Committee: ________________________________

Dr. BrhanuBekele

____

Dr. YibeltalYitayew

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Abstract

In this project we consider the structure of semi groups of self mapping of a semi group S under pointwise composition, generated by the endomorphisms of S. It is shown that,ifS is Clifford semi groups with underlying semilattice , the endomorphisms of S generate a Clifford semigroup E^+ (S) whose underlying semilattice is the set of endomorphisms of $\wedge$.These results contribute to the wider theory of seminear-rings of endomorphisms, Since E^+ (S) has a natural structure as distributively generated seminear-ring.

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Introduction

Let G be a group and M(G) be the set of all functions from G into G. Then M(G) admits two natural binary operations. It is a semigroup under composition of functions (written multiplicatively) and a groupunder point-wise composition (written additively) using the group operation in G.that is (f+g)(s) = f(s)g(s) and (fg)(s) = f(g(s)). If we write maps on the right, we found that the function composition distributes on the left over point wise composition. So that f (g+h) =fg + fh for all f, g, h ϵ M (G). This endows the set M(G) with the structure of a near-ring (See [9]).

An algebraic structure $(S, +, .)$ is said to be a (left) **seminear-ring** if

1. (S, +) is semi group;
2. (S, .)is semi group and
3. $a(b+c) = ab + ac$, for all $a, b, c \in S$.

If (3) in the above definition is replaced by $(a+b)c = ac + bc$, for all $a, b, c \in S$, then $(S, +, .)$ is said to be a rightseminear-ring.

Example:

Every ring is both a left and right seminear-ring.

Within M(G) we have the subnear- ring $M_o(G)$ consisting of all functions G → G that maps the identity element of G to itself. Then M_0 (G) contains the set End (G) of endomorphism of G (asemigroup under composition of functions) and these are precisely the elements that always distribute on the right: (f+g) h = fh + gh for all f, g ϵ M_O (G) if and only if h ϵ End (G). We let E (G) be the subnear-ring of M_o (G) generated by the subset End(G). The fact that End(G) is a right distributive semigroup implies that E (G) is generated by End (G) as a group (using only the point wisecomposition).

An important result about this construction and motivation for the more general theorem of distributivelly generated near-ring (originating in [11] is Frohlichs' Theorem [3] that, a finite non-abelian simple group G we have E(G) = $(M_0(G))$. Further specific computations have been carried out for Dihedral groups [2] and General Linear groups [12].

If we replace the group G by a semi groups we attempt to generalize these ideas. The set M(S) of all functionsfrom S into S is now a **seminear-ring**: It is a semi-group under both composition of functions and point wise composition and left distributive holds .

We consider the subsemigroup of E^+(S) of M(S) generated by End(S) using point-wise composition. Since elements of End(S) are also right distributive in M(S), it follows that E^+ (S) is in fact a subseminear-ring of M(S).

The structure of E^+ (S) for a Brandt semi-group S was considered in[7].

In this project, we study the structure of E^+(S) when S is Clifford semi-group that is an inverse semi-group with a centralidempotent. The structure of Clifford semi-groups is well known: They are precisely the strong semi-lattice of groups.

The main result in this project which is based on [13], shows that S is a strong semi-lattice of groups in which all the linking maps are isomorphism ,then (E^+(S), +) has the same kind of structure and more over ,if $\wedge$ is the semi-lattice underlying S, the semi-lattice E^+(S) is End(S).

Chapter One

Preliminaries

In this chapter we consider definitions and examples of some preliminary concepts that will be used in our coming discussion

1.1 Groups and Rings

Definition 1.1

A semi group is a non-empty set S together with a binary operation$\star$ on S which is associative, i.e.$a \star (b \star c) = (a \star b) \star c$ for all $a, b, c \in S$.

Subsemigroup is any non-empty subset of a semigroup that is closed under the semigroup operation.

Definition 1.2

Let G be any non-empty set and$\star$ be a defined binary operation on G. Then the algebraic structure (G, $\star$) is said to be a group if

A. $\star$ is associative:i.e. for all a, b, c ϵ G, we have (a $\star$ b) $\star$ c = a $\star$ (b $\star$ c).

B. Existence of identity element;

There exists an element $e \in G$such that $a \star e = e \star a = a$ for all $a \epsilon G$

C. Existence of inverse;

For each a ϵ G, there exists an element $b \epsilon G$ such that a $\star$ b = b $\star$ a = e.

A group G is said to be **abelian** if for all a, b ϵ G, = a $\star$ b = b $\star$ a.

Definition 1.3

A non-empty subset H of a group G is said to be a **subgroup** of G if H itself is a group under the operation defined in G.

Definition 1.4

A subgroup H of a group G is called a **normal subgroup** of G if for every $g \in G$ and h ϵ H, $ghg^{-1} \epsilon$ H.

Definition 1.5

Let R be a non-empty set with two binary operations (usually calledaddition and multiplication), then (R, +, .) is said to be a **ring** if;

- **A.** (R, +) is an abelian group;
- **B.** (R, .)isasemigroup and
- **C.** Multiplication or (.)is distributive overaddition(+). That is, for all a, b, c ϵ R, we have $a.(b+c) = a.b + a.c$, (left distributive) and $(b+c).a, = b.a, + c.a,$ (right distributive)

Definition.1.6

A right **near-ring** is a nonempty set N together with two binary operations (+)and (.) such that;

- **A.** (N,+) is a group (not necessarily abelian);
- **B.** (N,.) is a semigroup and
- **C.** For all $a,b,c \in$ N, we have $(a+b).c = a.c + b.c$ (right distributive).

If C in the above is replaced by $a.(b+c) = a.b + a.c$, then, (N,+.) is called a left near-ring.

Note:

In near rings two ring axioms are missing. These are commutativity of addition and one side distributive law (either right distributive law or left distributive law)

Definition 1.7

A semi-ring is an algebraic structure consisting of a non-empty set R on which we have two operations, addition (usually denoted by +) and multiplication (usually denoted by.)such that the following hold:

1. Addition is associative, commutative and has aneutral element. That (R, +) is commutative monoid with identity element, denoted by 0. That is,
    - i. $(a+b)+c = a + (b + c)$;
    - ii. $a + b = b + a$, and
    - iii. $0 + a = a + 0$, for, $a,b,c \in$ R.

2. (R, .) ismonoid with icentity element, denoted by 1. That is,

   i. $(a.b).c = a.(b.c)$, and

   ii. $1.a = a.1$for all $a,b,c \in$ R.

3. Multiplicative is distributive over addition from either side. That is

   i. $a(b+c) = a\,b + a\,c$, and

   ii. $(a+b)c = a\,b + a\,c$, for all $a,b,c \in$ R.

4. The identity (neutral) element with respect to addition is multiplicatively absorbing (annihilates R).That is,0. $a = a.0 = 0$ for all $a \in$ R.

Note:

The difference between a ring and a semi-ring is that,in a semi-ring, with respect to additionthe set is a commutative monoid, not necessarily a commutative group; specifically elements in a semi-ring do not necessarily have an inverse with respect to addition.

1.2 Seminear-rings

Definition 1.8

An algebraic structure $(S,+,.)$ is said to be a (right)**seminear-ring** if;

A. **(S,+)** is a semigroup;

B. (S, .)is a semigroup and

C. $(a+b).c = a.c + b.c$ for all $a,b,c \in$ S.

Remark:

If (C) in the above definition is replaced by $c.(a+b) = c.a + c.b$,for all $a,b,c \in$ S, then $(S,+,.)$ is said to be aleftseminear-ring.

Thus, a left seminear-ring is a set L with two associative binary operationsthat satisfy the left distributive law. That is, for all $a,b,c \in$ L, $a.(b+c) = a.b = a.c$.

Definition1.9

Let $(L,+,.)$ be left seminear-ring. An element $d \in$ L is called distributive if it is also distributive on the right, that is, for $a,b \in L$,we have $(a+b)d = ad + bd$.

Theorem: 1.1

Let D be the set of distributive elements of a left seminear-ring (L,+,.). Then, D is a subsemigroup of (L,.).

Proof

Since (L,+, .) is a left semi near-ring, it suffices to consider the right distributively.

If D =∅, there is nothing to prove.
Suppose D≠ ∅ , and suppose c, dϵD, Then for a, b,ϵL, we have,

a+b) (c d) = [(a+b) c] d------------ ---associativity of ‘’ . ‘’

= [(a c + b c)] d ……….since c ϵ D

= (a c) d + (b c) d ……… since d ϵ D

= a (c d) + b (c d) ……….. associativity of ‘’ . ‘’

Therefore c, d ϵ DHence D is a subsemigroup.

Let S be a semi-group (written multiplicatively). The set M(S) of all functions from S into S is a seminear-ring under the multiplication operation given by the function composition, and the addition operation given by point-wise composition. That is, for all $f, g \epsilon M(S)$ and $s \in S$, we have,

i. $s(f+g) = (sf)(sg)$and

ii. $s(fg) = (sf)g$

Note: $\mathbf{s(f+g) = (sf)(sg)}$ is equivalent to $\mathbf{(f+g)(s) = f(s)\ g(s)}$and

$s(fg) = (sf)g$*is equivalent* to$(fg)(s) = f(g(s))$

Theorem 1.2
Let S be a semigroup (M(S),+,.) is a seminear-ring.

ProofLet f,g ϵ M(S) and s ϵ S ,

First we want to show well definedness of “ + “ and “ . “

i. (f+g)(s):= f(s)g(s)
and f(s).g(s) is a unique element in S for f(s), g(s) ϵ S ,since S is a semi group

Thus “ + “ is well defined

ii. (f.g)(s) = f(g(s)) and g(s) is a unique element since g is a function and f(g(s)) is a unique element since f is a function.
Thus “ . “ is well define.

Now it remains to show M(S),+ , .) is a right seminear-ring.Since S is a semi-group it is non-empty,

For f ,g and h∊ M(S) and s ∊ S , we have

i. [(f+g)+h](s) = (f+g)(s) . h(s) ………..definition
= [f(s) g(s)]h(s) ……… ..definition
= f(s)[g(s) h(s)] ………..Since the operation in S is associative
= f(s)[(g+h)(s)] ……… definition
= [f+(g+h)] (s) …………definition
This implies(f+g)+h = f+(g+h) and
hence (M(S),+) is a semigroup.

ii. .[(fg)h](s) = (fg) (h(s)) ………. ……..definition
= f(g(h(s))) ……… ………..definition
= f(g.h)(s)) …………………definition
= [f(gh)](s) …………………..definition
This implies (fg)h = f(gh) and
hence (M(S), .) is a semigroup

iii. [(f+g)h](s) = (f+g) (h(s)) ……. ………..definition of “ . “ on M(S)
= f(h(s)) . g(h(s)) ………….definition of “ + “ on M(S)
= (fh)(s).(gh)(s) … ……….definition of “ . “ on M(S)
= (fh+gh)(s) …………………definition of “ + “ on M(S) This implies (f+g)h = fh + gh
Hence (M(S),+,.) is a right seminear-ring

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Definition 1.10

Let S be a semi group. A function f ∊ M(S) is said to be morphism, if f(st) = f(s)f(t) for all s, t ∊ S

Theorem 1.3

Let S be a semigroup

End(S) = {f ϵ M(S): f is morphism} is a subsemi-group of (M(S),.).
Proof
Let α,β ϵ End(S) and s, t ϵ s .
Then (α.β)(s.t) = α(β(s.t))
= α(B(s)β(t))
= α(B(s))α(β(t))
= (α.β) (s)(α.β)(t)This implies)(α.β)(s.t) = (α.β)(s)(α.β)(t)
There for αβ ϵ End(S)
Thus End(S),.) is a semigroup and hence it is a subsemi-group of M(S) .

Definition1.11

Let S be a semi-group. An element a ϵ S is said to be an idempotent element, if a^2 = a. In a unary operation f, that is, a map from some set S into itself is called idempotent if, for all x in S,f(f(x)) = f(x).The set of all idempotent elements of a semi-group S is denoted by E(S).

Definition 1.12

Let S be the semi-group and $a \epsilon$ E(S).A mapping $C_a \in End(S)$ defined by $C_a(x) = a$, for all $x \in$ S is called a constant mapping determined by a.

Lemma 1.1

The semi-group of left distributive elements in M(S) is the semi-groupof endomorphisms of S.

Proof:

It is clear that an endomorphism is distributive, that is,

Let $f,\ g \in M(S)$ and φ ∈ End(S). Then for each s ϵ S, we have

$$\begin{aligned} s\,(f+g)\varphi &= ((sf)(sg))\,\varphi \\ &= (sf\,\varphi)\,(sg\,\varphi) \\ &= s\,(f\,\varphi + g\,\varphi) \end{aligned}$$

This implies, $(f+g)\varphi = f\varphi + g\varphi$ and

hence φ is a distributive element

Now,Suppose $d \in$ M(S) is distributive. For any $s \in$ S,let $c_s \in M(S)$ be constant function determined by $s \in$ S,defined by $xc_s = s$ for all $x \in$ S. Then for any $s, t, x \in S$ we have

$$
\begin{aligned}
(st)d = (xc_s)(xc_t)d &= x(c_s + c_t)d \\
&= x(c_s d + c_t d), \\
&= (xc_s d)(xc_t d) \\
&= (sd)(td)
\end{aligned}
$$

This implies $(st)d = (sd)(td)$.

That is d is endomorphism.

A semi-near-ring L is called distributivelly generated if (L,.) contains a multiplicative subsemigroup (S,.) of distributive elements which generates(L, +).

Theorem 1.4

If a semi group (L,+) is generated by (S,.) then L is the seminear-ring

Proof

Let L contains (S, .) which generates (L,+) distributively

Let a, b ϵ L and let f,g ,h ϵ M(L , +) be functions from (L , +) to (L , +)

Let d ϵ (S , .) is distributive, that is, (a+b) d = ad + bd .

Implies ad + bdϵ (L, +)

We want to show (L, +, .) is semi near-ring

I. ((f+g) +h) (ad+bd) = (f+g)(ad+bd) .h(ad+bd) …… …. ...Since (L, +) is a semi group
= f(ad+bd) . g (ad+bd) . h (ad+bd) … since (ad+bd) ϵ (L , +)
= f(ad+bd) . (g (ad+bd) . h (ad+bd))….associativity of “ . “
= f(ad+bd) . (g + h) (ad+bd) …………. since (ad+bd) ϵ (L , +)
= (f+ (g + h)) (ad+bd)

Implies, ((f+g) +h) =(f + (g + h))

II. ((f g)h) (ad+bd) = (fg)h(ad+bd)

= f(g.h(ad+bd))

= f(g.h) (ad+bd)

= (f(gh)) (ad+bd)

Implies,(f g) h = f (gh)

III. ((f+g)h) (ad+bd) = (f+g). h(ad+bd)
= f(h(ad+bd)).g(h(ad+bd))
=(fh)(ad+bd). (gh) (ad+bd)
= (fh + gh) (ad+bd)

Implies, (f+g) h =(fh + gh)

Hence (L, +, .) is (right) semi near-ring

We noticed that S need not be the semi-group of all distributive elements and such a distributivelly generated semi-near-ring (d.g.s.n.r) is denoted by (**L,S**)

If we consider the above seminear-rings M(S),then the setEnd(S), of all endomorphismsof S, is a distributive semigroup of M(S) ,and generates a distributivelly generated seminear-ring denoted by (E(s)),End(s))**.**Distributivellygenerated seminear-rings were first studied in [10].

Let $E^+(s)$ be the subsemigroup of (M(S), +) generated by End(S). It is clear that $E^+(s)$isthen a distributivelly generated seminear-ring called **endomorphism seminear-ring** of S.

Theorem 1.5

If S a commutative semigroup, then $E^+(S)$ = End(S) and ($E^+(S)$, +, .) is a **semi-ring**.

Proof

Let $E^+(S)$ = subsemigroup of (M(S), +) generated by End(S).

Let x_1 , x_2 and$x_3 \in$ S such that f(x) = x_1 , g(x) = x_2 and h(x) = x_3

Let (S,∗) be a commutative semigroup,i.e $x_1 * x_2 = x_2 * x_1$,$\forall x_1, x_2 \in S$

$E^+(s)$ is associative ……. since $E^+(s)$ is subsemi group of (M(S), +)

End(S) is associative …..since S is a seminear-ring.

Implies $E^+(s)$ = End(S)

Now let f,g,& h ϵE^+(s)

We want to show (E^+ (S),+, .) is a semi- ring

i. About (E^+ (S),+)

- (f(x) + g(x)) + h(x) = ($x_1 + x_2$) + x_3

= x_1 + (x_2 +x_3) ….. sincex_1 , x_2 &x_3 ϵ S

= f(x) +(g(x) +h(x))

- f(x) + g(x) = $x_1 + x_2$ = x_2 + x_1 …….. … since S is commutative.
  = g(x) + f(x)
- 0 + f(x) = 0 + x_1 = x_1 + o = x_1

Implies (E^+ (S),+) is associative , commutative and monoid with identity element 0

ii. .,About (E^+ (S), .)

- (f(x) . g(x)) . h(x) = ($x_1 . x_2$). X_3 = x_1 . (x_2 . x_3)
  = f(x).(g(x) . h(x)) …since S is commutative

- 1. f(x) = 1. x_1 = x_1
  Implies(E^+ (S), .)is associative and monoid with identity 1

iii. .Distributivelyof . on +

Since (E+ (S) = End(S) = seminear-ring

Let f(x) ϵ (E+ (S) be left distributive,

i,e, f(x) [g(x)+h(x)] = f(x)g(x) +f(x) h(x)

= x_1x_2 + x_1x_3

= x $_2x_1$ +x_3x_1

= (x_2 + x_3) x_1

= (g(x) + h(x)) f(x)

Implies (E+ (S) has both right and left distributive

iv. .About identity element (w.r.t.addition) 0,

0.f(x) = o . x_1 = x_1 . 0 = f(x) . 0 = 0

Implies 0 (identity element w.r.t addition) absorbs (annihilates) E^+(S) .

Therefore E^+ (S), + , .) is a semi-ring

1.3, Semilattice, Lattice, andSubsemilattice

Definition1.11

1. A**semilattice** is an algebraic structure $(S,\star)$satisfying the followingconditions.
   i. $x \star x = x;$
   ii. $x \star y = y \star x$ and
   iii. $x \star (y \star z) = (x \star y) \star z$, for all $x,y,z \in$ S.

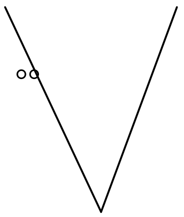
That is, a semilattice is an idempotent, commutativesemigroup.

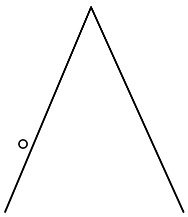
2. **Ajoin semilattice**(P, ≤, v) is a poset (P, ≤) ,(P is partially ordered set with respect to the relation≤) such that any two elements $x,y \in$ P have a least upper bound (suppermum) $x \vee y$ in S.
3. **Ameet semilattice (P, ≤∧)** is a poset such that any two elements x, y∈ P have a greatest lower bound (infumum) $x \wedge y$ in.
4. **Alattice** is an algebra L = (L, ʌ, v) satisfying the following conditions, For all x, y, z ϵ S
   a. $x \wedge x = x$ and $x \vee x = x;$
   b. $x \wedge y = y \wedge x$ and $x \vee y = y \vee x;$
   c. $x \wedge (y \wedge z) = (x \wedge y) \wedge z$ and $x \vee (y \vee z) = (x \vee y) \vee z;$
   d. $x \wedge (x \vee y) = x$ and $x \vee (x \wedge y) = x$

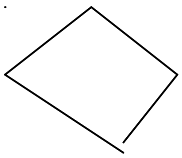
The first three points of the axioms say that L is both a meet and join semilattice and the fourth pair (called the absorption law) say that both operations induce the same order on L.

5. **A complete lattice** is an ordered set L in which every subset A has a greatest lower bound and a least upper bound.

The following diagram (figure.. 1)shows examples of meet-semilattice, join-semilattice and lattice respectively







°°°

Meet- semilatticejoin-semilatticelattice

Definition:1.12figure….1

Let (L, ≤, v∧) be a semilattice. A nonempty subsetS ⊆ Lis said to be asubsemilattice of L if for all $x,y \in$ S, $x \vee y \in S$and $x \wedge y \in$ S. **Theorem 1.6**

Let ∧ be a semilattice. Then the (End (∧), +) is also a semilattice.

Proof:

Given a semilattice (∧,∗)

Consider End(∧) = { f :(S,∧) → ~~(S,∧)~~: f is a morphism }

That is, f(a∗b) =f(a) ∗ f(b) for all a,b ϵ ∧.

Now for f,g and h ϵ End(∧) and sϵ ∧, we have the following.

i. (f+f) (s) = f(s) ∗f(s)= f(s) , because ∧ is a semilattice.

Thisimpliesf+f = f

ii. . (f+g) (s) = f(s) ∗ g(s) = g(s)∗ f(s)= (g+f)(s) ,, because ∧ is a semilattice

This implies f+g = g+f

iii. . [(f+g)+h](s = (f+g)(s) ∗h(s)

= [f(s)∗g(s)]∗h(s)

= f(s)∗[g(s)∗h(s)]

= f(s)∗ [(g+h)(s)]

= [f+(g+h)](s)

And this implies (f+g)+h = f+(g+h).

Hence (End(∧),+) is a semilattice

1.4.**Strong Semilattice of Groups**

Given a semilattice (L,∗)ice

Define≤ 0n L by, **β**≤ **α** if and only if **β**∗ **α** = **β**

Definition1.13

ACliffordsemigroup(strongsemilattice of groups) is a disjoint union of groups

$$S = \bigcup_{\alpha \epsilon \wedge} G_{\alpha},$$

indexed by a semilattice $\wedge$ together with a group homomorphism

$\boldsymbol{\phi_{\alpha,\beta}} : \boldsymbol{G_{\alpha}} \longrightarrow \mathbf{G}_{\beta}$whenever $\alpha \geq \beta$in $\wedge$, such that

i. for each $\boldsymbol{\alpha} \in \wedge$,the homomorphism$\boldsymbol{\phi_{\alpha,\alpha}}$ is the identity;

ii. if $\boldsymbol{\alpha} \geq \boldsymbol{\beta} \geq \boldsymbol{\gamma}$;then $\boldsymbol{\phi_{\alpha,\gamma}} = \boldsymbol{\phi_{\alpha,\beta}\phi_{\beta,\gamma}}$

The semigroup operation on$\boldsymbol{S} = \bigcup_{\alpha \epsilon \wedge} \boldsymbol{G_{\alpha}}$ is defined by:

$\boldsymbol{ab} = (\boldsymbol{\phi_{\alpha,\alpha\beta}}(\boldsymbol{a}).(\boldsymbol{\phi_{\beta,\alpha\beta}})$ **(b,)**if $a \in G_{\alpha}$and $b \in G_{\beta}$.

To illustrate that, leta,bϵ S , then we will have the following three cases

Case 1 .a,b ϵ G_{α} (i.e , they belong to the same groupG_{α}),

Then $\alpha = \beta$ and

ab $= (\phi_{\alpha,\alpha}(a).(\phi_{\alpha,\alpha}) (b)$ = a.b (. is the operation in G_{α})

Case 2.a ϵ G_{α} , b ϵ G_{β} with $\alpha \geq \beta$,

Then $\alpha\beta = \beta$ and$ab = (\phi_{\alpha,\alpha\beta}(a).(\phi_{\beta,\alpha\beta})$ (b) $= (\phi_{\alpha,\beta}(a).(\phi_{\beta,\beta} =$ $(\phi_{\alpha,\beta}(a).$ b ,that is$\phi_{\alpha,\beta}(a) \epsilon G_{\alpha}$, and . is the operation in G_{α})

Case 3.a ϵ G_{α} , b ϵ G_{α} with $\alpha \leq \beta$

Then , $\alpha\beta = \alpha$ and $ab = (\phi_{\alpha,\alpha}(a).(\phi_{\beta,\alpha})$ (b) = a .$\phi_{\beta,\alpha}$) (b)

That is $\phi_{\beta,\alpha}$ (b)ϵ G_{α} and (.) isthe operation in G_{α}

In all cases, the operation is associative and ,Hence, S is a semigroup. That is, let$\wedge$ be a semilattice and let G_{ε} be a semigroup for each $\varepsilon \epsilon \wedge$ with $G_{\alpha} \cap G_{\beta} = \emptyset$. If $\alpha \neq \beta$ & $\alpha,\beta \epsilon \wedge$, for each pair $\alpha,\beta \epsilon \wedge$ with $\alpha > \beta$, let $\phi_{\alpha,\beta} : G_{\alpha} \longrightarrow G_{\beta}$ be a semigroup homomorphism such that

i. $\phi_{\alpha,\alpha}$ is the identity

ii. if $\alpha \geq \beta \geq \Upsilon$ then $\phi_{\alpha,\Upsilon} = \phi_{\alpha,\beta}\,\phi_{\beta,\Upsilon}$

Consider$S = UG_{\alpha}$, $\alpha \in \wedge$with multiplication $ab = (a\phi_{\alpha,\, \alpha\beta})\ (b\phi_{\beta,\, \alpha\beta})$ if $a \in G_{\alpha}$& $b \in G_{\beta}$. See (3). Then the semigroup S is called **the strong semilattice of semigroups**G_{ε}. For α, $\beta \in \wedge$, we call **$\phi_{\alpha,\, \beta}$is the defining homomorphism** also called **structurehomomorphism**.

We denote a strong semilattice of semigroups S with defining homomorphism $\phi_{\alpha,\beta}$by $S = (\wedge: G_{\alpha}, \phi_{\alpha,\beta})$ for a semilattice $\wedge$. If we replace the semigroup by group, we call a**strong semilattice of groups** which is known as under the name of **Clifford semigroup**.

Chapter Two

2.1. Endomorphism of CliffordSemigroups

Definition 2.1

Let S and T be semigroups and $f: S \rightarrow T$ be a mapping. Then, f is called a **semigrouphomomorphismif $f(xy) = f(x)f(y)$, for all $x, y \in S$.**

The set of semigrouphomeomorphisms from S into T is denoted by Hom (S.T) and End(S) =Hom (S.S). An endomorphism ϕ of a semi group S is a mapping ϕ; $S \rightarrow S$ such that $\phi(xy) = \phi(x)\phi(y)$ for all $x, y \in S$.

The set End(S) forms a monoid with composition as a multiplicative and mappings are composed from the right to the left. For $f, g \in$ End(S), the composition of f and g is written as gof and $(gof)(x) = g(f(x))$ for all $x \in S$.

Lemma 2.1

Let $S = (\wedge, G_\alpha, \phi_{\alpha,\beta})$ be a strong semilattice of groups and let $f \in \text{End}(S)$, then the following holds:

a. f induces an endomorphism of the semilattice $\wedge$,

That is (End $(\wedge)$ = {f: f: $(S, \wedge) \longrightarrow (S, \wedge)$): f $(a \wedge b)$ = f (a) $\wedge$ f (b)

b. For each $\alpha \in G_\alpha$, we have $G_\alpha f \underline{c} G_{\alpha f}$

Proof:

Let e_α be the identity element of G_α. Now $G_\alpha f \subseteq G_\gamma$ for some $\Upsilon \in \wedge$ and since $e_\alpha f$ is idempotent, we have $e_\alpha f = e_\Upsilon$ we set $\alpha f = \Upsilon$. This implies $G_\alpha f \subseteq G_{\alpha f}$ since $e_\alpha e_\beta = e_{\alpha\beta}$, it follows that f is an endomorphism of $\wedge$.

The endomorphisms of Clifford semigroups were studied in detail in [3] under various restrictions on the properties of liking maps $\phi_{\alpha,\beta}$ .To pursue our study of the structure of E^+ (S),we shall assume the strongest of the conditions considered in [3],namely that the linking maps are all isomorphism . In this case we can simplify the description of S.

Lemma 2.2

Let S= $(\wedge, G_\alpha, \phi\alpha_{,\beta})$be a strong semilattice of groups in which all the linking maps $\phi_{\alpha,\beta}$ are isomorphism. For any $\lambda \in \wedge$, let S_λ be the strong semilattice of groups over $\wedge$

in which each group G_α, $\alpha \in \wedge$ is equal to G_λ and all the linking maps are the identity .Then S is isomorphic to S_λ.

Proof:

Given $S = (\wedge, G_{\alpha}, \phi_{\alpha,\beta})$

$\phi_{\alpha,\beta}$=isomorphism

$S_\lambda = (\wedge, G_\lambda, \phi_{\lambda,\lambda})$ and

$G_\alpha = G_\lambda$

We want to show that, $S \cong S_\lambda$

We define an isomorphism $\psi: S \to S_\lambda$ as follows. Its restriction ψ_α to $G\alpha$ is defined to be $\psi_\alpha = \phi_{\alpha,\alpha\lambda}\phi^{-1}_{\lambda,\alpha\lambda}$.. Then ψ is clearly bijective and we need only check that it is a homomorphism. To this end let $a \in G_\alpha$ and $b \in G_\beta$ so that in S we have

$ab = (a\phi_{\alpha,\alpha\beta})(b\phi_{\beta,\alpha\beta}) \in G_{\alpha\beta} = G\alpha G\beta$

In this proof ,we use the notation a_f for f(a)

We want to show $(ab)\psi = (a\psi)(b\psi)$

Then $(a\psi)(b\psi) = (a\psi_\alpha)(b\psi_\beta)$

$=(a\phi_{\alpha,\alpha\lambda}\phi^{-1}_{\lambda,\alpha\lambda})(b\phi_{\beta,\beta}\lambda\phi^{-1}\lambda_{,\beta\lambda)}$

whereasab $=((a\phi_{\alpha,\alpha\beta})(b\phi_{\beta,\beta\lambda}))\psi_{\alpha\beta}$

$= (a\phi_{\alpha,\alpha\beta})\psi_{\alpha\beta}(b\phi_{\beta,\beta\lambda})\psi_{\alpha\beta}$

Now $(a\phi_{\alpha,\alpha\beta})\psi_{\alpha\beta} = (a\phi_{\alpha,\alpha\beta})(\phi_{\alpha\beta,\alpha\beta\lambda}\phi^{-1}_{\lambda,\alpha\beta\lambda})$

$= a\phi_{\alpha,\alpha\beta\lambda}\phi^{-1}_{\lambda,\alpha\beta\lambda}$

$= a(\phi_{\alpha,\alpha\lambda}\phi_{\alpha\lambda,\alpha\beta\lambda})(\phi^{-1}_{\alpha\lambda,\alpha\beta\lambda}\phi^{-1}_{\lambda,\alpha\lambda})$

$=a\phi_{\alpha,\alpha\lambda}\phi^{-1}_{\lambda,\alpha\lambda}$

Similarly $(b\phi_{\beta.\alpha\beta})\psi\alpha\beta = (b\phi_{\beta,\alpha\beta})(\phi_{\alpha\beta,\alpha\beta\lambda}\phi^{-1}_{\lambda,\alpha\beta\lambda})$

$= b\phi_{\beta,\alpha\beta\lambda}\phi^{-1}_{\lambda,\alpha\beta\lambda}$

$= b(\phi_{\beta,\beta\lambda}\phi_{\beta\lambda,\alpha\beta\lambda})(\phi^{-1}_{\beta\lambda,\alpha\beta\lambda}\phi^{-1}_{\lambda,\beta\lambda})$

$=b\phi_{\beta,\beta\lambda}\phi^{-1}_{\lambda,\beta\lambda}$

Thisimplies $(ab)\psi = (a\psi)(b\psi) = (a\phi_{\alpha,\alpha\lambda}\phi^{-1}_{\lambda,\alpha\lambda})(b\phi_{\beta,\beta\lambda}\phi^{-1}_{\lambda,\beta\lambda})$

Hence ψ is a homomorphism and

therefore S is isomorphic to $S_{\lambda.}$

Now we can conclude from theorem 2.1, the results are similar for the linking maps being isomorphism or identity.

We may now assume that S is a strong semilattice of groups over $\wedge$ in which every group is equal to a fixed group G and with each linking map equal to the identity. Hence **S is the disjoint union of copies G_α of G** indexed by $\alpha \in \wedge$ .If $g \in G$, then we denote by $g^{(\alpha)}$ the copy of element g in $G_{\alpha.}$ In this notation the **multiplication in S** is given by

$g^{(\alpha)} h^{(\beta)} = gh^{(\alpha\beta)}$ --- (1)

Proposition 2.3

Any $\delta \in$ End (G) and $f \in$ End ($\wedge$) determine an endomorphism $\delta_f \in$ End (S) defined by $g^{(\alpha)}\delta_f = g\delta^{(\alpha f)}$ and every endomorphism of S arises in this way.Hence we have, **End (S) ≅ End (G) x End ($\wedge$),**as a semigroup of mappings

Proof:

To show that $\delta_f \in$ End(S), we have to check the preservation of the multiplication given in (1) above. That is almost trival. Take $g^{(\alpha)}, h^{(\beta)} \in$ S, then

$(g^{(\alpha)})\, \delta_f\, (h^{(\beta)})\delta_f = (g\delta)^{(\alpha f)}(h\delta)^{(\beta f)}$

$= (g\delta h\delta)^{((\alpha f)\,(\beta f)}$

$= ((gh)\,\delta)^{((\alpha\delta)\, f}$

$= (gh)^{(\alpha\beta)}\delta_f$

$= g^{(\alpha)})\, h^{(\beta)}\delta_f.$

Implies $(g^{(\alpha)})\, \delta_f\, (h^{(\beta)})\delta_f = g^{(\alpha)})\, h^{(\beta)}\delta_f$

Hence $\delta_f \in$ End(S)

Now let $\delta \in$ End(S) and let f be the induced endomorphism of $\wedge$ .For each $\alpha \in \wedge$, we have $\delta : G_\alpha \rightarrow G_{\alpha f}$ and since $G_\alpha = G = G_{\alpha f}$, the restriction of δ to G_α induces an endomorphism δ_α of G.

Now for any $g \in G$ and $\alpha, \beta, \in \wedge$, we have

$I_G^{(\alpha\beta)} = g^{(\alpha)} (g^{-1})^{(\beta)}$ by applying δ we obtain

$I_G^{(\alpha\beta) f} = g^{(\alpha)} \delta (g^{-1})^{(\beta)} \delta$

$= ((g \delta_\alpha) (g^{-1}\delta_\beta))^{((\alpha f) (\beta f))}$

$= ((g\delta_\alpha) (g^{-1} S_\beta))^{(\alpha\beta)f}$

Therefore $g\delta_\alpha = g\delta_\beta$ and $\delta \epsilon \operatorname{End}(S)$ induces the same endomorphism p on each group $G_{\alpha,}$ with $g^{(\alpha)}\delta = (gp)^{(\alpha f)}$. Therefore $\delta = p_f$. It is now clear that $(p,f) \to p_f$ is a bijection $\operatorname{End}(G) \times \operatorname{End}(\Lambda) \to \operatorname{End}(S)$ and since $g^{(\alpha)} p_f \delta_k = gp^{(\alpha f)}\delta_k = (gp\delta)^{(\alpha f K)}$ this bijection is a semigroup isomorphism .By reintroducing the isomorphic linking maps in to $S = (\Lambda, G_{\alpha,} \ \phi_{\alpha,\beta})$.For any $\lambda \epsilon \Lambda$,we may write an endomorphism Υ of S in the form $\Upsilon = \psi\delta_f\psi^{-1}$ where $\delta_f \epsilon \operatorname{End}(S_\lambda)$ and hence for $g \epsilon G_\alpha$, we have

$g\Upsilon = g\psi_\alpha\delta_f\psi^{-1}{}_{\alpha f}$

$$= g\phi_{\alpha,\ \alpha\lambda}\phi^{-1}{}_{\lambda,\alpha\lambda}\boldsymbol{\delta_f}\ \phi_{\alpha f,(\alpha f)\lambda}\ \phi^{-1}{}_{\lambda,(\alpha f)\lambda},$$

(which is the formula forΥ)

Chapter Three

3.1. Seminear-rings of Endomorphisms

In this chapter we assume that the group G is a finite.

Let G be a finite group. This implies that any mapping in the group E(G) is a positive combination of endomorphisms, and hence that the semigroup $E^+(G)$, generated by End (G) coincides with E(G).

For fixed $f \in$End (Λ), we have an embedding End(G) ⟶ End(S) by $\alpha \to \alpha_f$.We claim that this embedding induces a homomorphismγ_f: E (G) → $E^+(S)$

Suppose that $\varepsilon = \sigma_1 + \sigma_2 + \ldots + \sigma_m \in E\ (G)$ we define $\varepsilon_f = (\sigma_1)_f + (\sigma_2)_f + \ldots + (\sigma_m)_f$,Then for each $\alpha \in \Lambda$ and each $g^{(\alpha)} \in G_\alpha$,we have,

$g^{(\alpha)}\varepsilon_f = g^{(\alpha)}\ ((\sigma_1)_f + (\sigma_2)_f + \ldots + (\sigma_m)_f)$

$=(g^{(\alpha)}\ \sigma_{1f})(\ g^{(\alpha)}\ \sigma_{2f}\ )\ldots\ldots(\ g^{(\alpha)}\sigma_{mf})$

$= (g\sigma_1)\ (^{\alpha f)}\ (g\ \sigma_2)^{(\alpha f)}\ \ldots\ldots(g\ \sigma_2)^{(\alpha f)}$

$= \ ((g\ \sigma_1)\ (g\ \sigma_2)\ldots\ldots\ (g\ \sigma_2\ ))^{(\alpha f)}$

$= (g\ (\sigma_1 + \sigma_2 + \ldots\ldots + \sigma_m))^{(\alpha f)}$

$= (g\varepsilon)^{(\alpha f)}$

This implies $g^{(\alpha)}\varepsilon_f = (g\varepsilon)^{(\alpha f)}$.

Hence ε_f depends only on ε and f, andγ_f: $\varepsilon \to \varepsilon_f$ is a well defined embedding

$$E\,(G) \to E^+(S).$$

Moreover, if$\varepsilon_f = \eta_k$, then for all $g \in G$ and $\alpha \in \Lambda$, we have $(g\varepsilon)^{(\alpha f)} = (g\eta)^{(\alpha k)}$, hence f = k, and the images of distinct embeddingsγ_f ($f\in$ End (Λ)) are disjoint. We write **(G) $_{\mathbf{f}}$** for the image of **E (G)** under the embedding γ_f. For each $f\in$ End (Λ),E $(G)_f$ is a subgroup of $E^+(S)$ is isomorphic to E (G).

Now if $\theta \in E^+(S)$ we have $\theta = \theta_1 + \theta_2 \ldots. + \theta_m$ for some $\theta_j \in$ End(S) and hence there exists $\sigma_j \in$ End(G) and $f_j\in$End(Λ)such that $\varepsilon = (\sigma_1\ )f_1\ \ + (\sigma_2\ )f_2\ + \ldots. + (\sigma_m\ )f_m.$ Therefore $(E^+(s), +)$ is generated by the collection of disjoint subgroups E $(G)_f$ where $f \in$ End (Λ).

Now take $\varepsilon_1, \varepsilon_2 \in E(G)$ and $f_1, f_2 \in$ End(Λ) ,Then for all $g\in G$ and i = 0,1,2,..,n we have

$g^{(\alpha)}((\varepsilon_1\ )f_1 + (\ \varepsilon_2)f_2) = (\ g^{(\alpha)}(\varepsilon_1)f_1)(g^{(\alpha)}(\varepsilon_2)f_2)$

$= (g\varepsilon_1)^{(\alpha f_1)} (g\varepsilon_2) (^{\alpha f_2})$

$= ((g\varepsilon_1)(g\varepsilon_2))^{((\alpha f_1)(\alpha f_2))}$

$= (g(\varepsilon_1 + \varepsilon_2))^{((\alpha f_1)(\alpha f_2))}$

$= (g(\varepsilon_1 + \varepsilon_2))^{(\alpha (f_1 + f_2))}$

$= g^{(\alpha)}(\varepsilon_{1,} + \varepsilon_2)_{f1+f2}$

A straight forward induction argument then shows that

$(\varepsilon_1)\, f1 + (\varepsilon_2) f_2 + \ldots. + (\varepsilon_m)\, fm = (\varepsilon_1 + \varepsilon_2 + \ldots. + \varepsilon_m) f_1 + f_2 + \ldots + f_m$.

Therefore

$E^{+}(s) = \bigcup_{E\,(G)_f}$, $f \epsilon$ End $(\wedge)$.

And so **E^{+}(S) is a semilattice of groups**.

We first look at the **composition of maps** in (E^{+}(s).For $g^{(\alpha)} \epsilon G_\alpha$, we have

$g^{(\alpha)} \varepsilon_f \eta_k = (g\varepsilon)^{(\alpha f)} \eta_k$

$= (g\varepsilon\eta)^{(\alpha f k)}$

$= g^{(\alpha)} (\varepsilon\eta)\, f_k$

and hence **$\varepsilon_f \eta_k = (\varepsilon\eta)_{fk}$** --- (2)

Now we have linking homomorphism ϕ_{f_1, f_2}:E (G) $_{f_1} \rightarrow$ E (G) $_{f_2}$ whenever $f_1 \geq f_2$, defined by $\varepsilon_{f_1} \rightarrow \varepsilon_{f2}$. So the linking homomorphism are the identity maps between the indexed copies of E(G) in E^{+}(S), and for **the addition of $\varepsilon_{f1}, \varepsilon_{f2} \epsilon$ (E^{+}(s)**. we have

$\varepsilon_{f1} + \eta_{f2} = (\varepsilon + \eta)_{f1 + f2}$ -- (3)

$= (\varepsilon f_1)\, \phi_{f1, f1 + f2} + (\eta f_2) \phi_{f2, f1 + f2}$-------------------- (4)

and so **(E^{+}(s)** is the strong semilattice of its subgroups $E(G)_{f.}$ Now it is summarize the conclusions in the following theorem, returning to the case of a strongsemilattice of groups whose linking maps are isomorphism.

In the above discussion we have proved the following theorem which is the central part of this chapter

Theorem 3.1

Let S = ($\wedge$,G_α, $\phi\alpha$, $_\beta$) be a strong semilattice of finite groups in which all the linking maps $\phi_{\alpha,\beta}$ are isomorphism.

a. As a semigroup under composition of maps $E^+(S)$ is isomorphic toE (G) x E ($\wedge$) = E (G) xEnd ($\wedge$).

b. As a semigroup under addition of maps$E^+(S)$ is isomorphic to a strong semilattice of groups over the semilatticeEnd ($\wedge$), with each group isomorphic to E (G).

Chapter 4

Some Examples

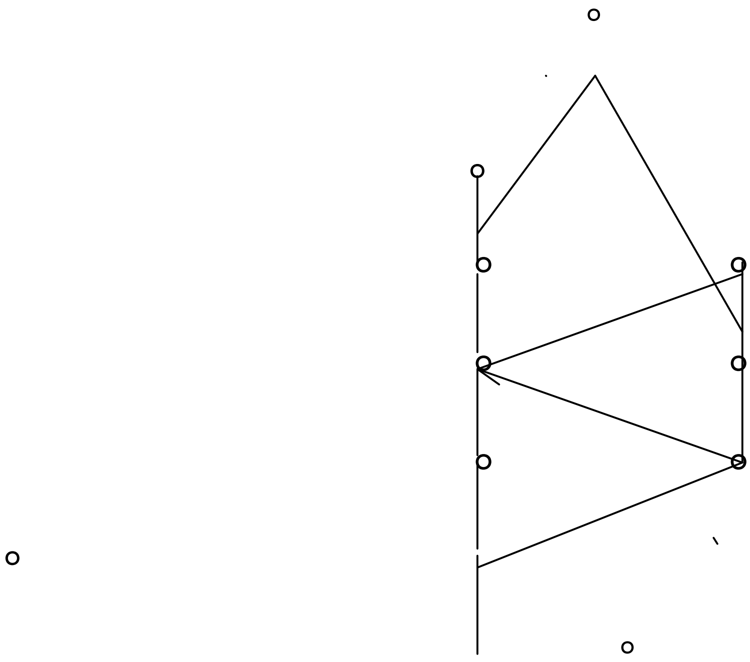
4.1. Finite Chains of Finite Groups

Let ∧ be the finite chain 0 < 1 < …< n. It is well-known that in this case

$$|\text{End}\,(\wedge)| = \binom{2n+1}{n}.$$

If n = 1 there are three endomorphism, and in this case (End (∧), +) is again a finite chain .Hence ifS = G_0U G_1 with an isomorphism $\phi: G_1 \rightarrow G_0$,

then**$E^+(S)$ = E (G) U E (G) U E (G).**For n > 1, the semilattice (End (∧), +) will not be a finite chain. For n = 2, it is the 10-element semilattice which is shown in figure 2 below.

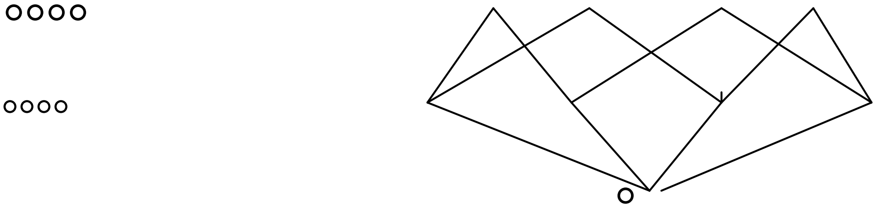


Figure……2

4.2. Finite Clifford semigroups over the free 2-generator semilattice

Let ∧ = (α, β, αβ) be the free 2- generator semilattice, with isomorphism:

$G_\alpha \rightarrow G_{\alpha\beta} \leftarrow G_\beta$.Then End(S) is the 9-element semilattice which is shown in figure 3 bellow



Figure…..3

The maximal element at the left and right hand end of this picture are the two automorphisms of ʌ. Thereare four endomorphism in the principal order ideal of End (ʌ) generated by the identity (id). In addition to id itself, we have a: $\alpha \rightarrow \alpha$, $\beta \rightarrow \alpha\beta$, b: $\alpha \rightarrow \alpha\beta$, $\beta \rightarrow \beta$ and the constant map $C = C\alpha\beta$ at $\alpha\beta$.

Consider the subseminear-ring

$$E_{1(id)}(S) = U\ {}_{E(G)_f} = E(G)_{id}\ U\ E(G)_a\ U\ E(G)_b\ U\ E(G)_c$$

A quick check reveals that the multiplication table for the subsemilattice {id, a, b, c} (under, +) coincides with its multiplication table under composition of maps. It follows that $E_{1\ (id)}$ (S) is astrong semilattice of near-rings.)

4.3. Non-isomorphic linking maps.

If the linking maps $\phi_{\alpha,\ \beta}$ in S are not isomorphisms, then further complications arise in the analysis of $E^{+}(S)$. As an illustration, consider the case n = 1 in example 4.1,so that $S = G_0\ U\ G_1$ with G_0& G_1finite, but with an arbitrary homomorphism ϕ: $G_1 \rightarrow G_0$. The three endomorphism of the chain 0 < 1 give rise to three types of endomorphism of s.

Let f ϵ End(S). If f induces the endomorphismc_0 which is a constant at 0 on the chain o < 1,then f is determined by f_0 ϵ End(G_0) , so that $f\,|_{G_0} = f_0$and $f\,|\,G_1 = \phi f_0$.

Similarly, if f induces the endomorphism c_1 which is a constant at 1 on the chain o < 1, then f is determined byf_0: $G_0 \rightarrow G_1$, and again we have $f\,|\,G_0 = f_0$ and $f\,|\,G_1 = \phi f_0$.

However if f induce the identity on the chain o < 1,then it is determined by two endomorphisms f_1 ϵ End(G_1) and f_0 ϵ End(G_0) such that $f_1\phi = \phi f_{0.}$ Hence as a set End(S) can be identified with the disjoint union,

End (G_0) U Φ U Hom (G_0,G_1), --- 5

Where$\Phi = \{(f_0,f_1) \epsilon \mathrm{End}\ (G_0) x \mathrm{End}\ (G_1): f_1\phi = \phi f_0\}$

Clearly, if $(f_0, f1) \epsilon$ **Φ,**then $(\mathrm{im}\phi)\ f_0 \subseteq \mathrm{im}\phi$ and $(\mathrm{Ker}\ \phi)\ f_1 \subseteq \mathrm{Ker}\ \phi$.

If ϕ is surjective, then the condition $(\mathrm{Ker}\phi)\ f_1 \subseteq \mathrm{Ker}\ \phi$ also imply that f_1 determines f_0 and so we may simplify the description ofΦ to$\Phi = \{f \epsilon \mathrm{End}\ (G_1): (\mathrm{Ker}\ \phi)\ f_1 \subseteq \mathrm{Ker}\ \phi\}$.

If ϕ is injective .then f_0 determines f_1 and so we may simplify Φ to $\Phi = \{f \epsilon \mathrm{End}(G_0) : (\mathrm{im}\ \phi) f_0 \subseteq \mathrm{im}\ \phi\}$.

Now each subset shown in the partition (5) is a subsemigroup of (E(S), .) : the **composition** in $\mathrm{End}(G_0)$ and in **Φ** is the obvious one in each case ,and If a, b ϵ Hom $(G_o, G1)$ then **$a.b = a \phi b$.**

We let $E_{C_0}(S)$ be thesubsemigroup of $(E^+(S), +)$ generated by End (G_o),$E_{id}(S)$ be the subsemigroup of $(E^+(S), +)$ generated by Φ, and $E_{C_1}(S)$ be the subsemigroup of $(E^+(S), +)$ generated by Hom (G_0, G_1). An element of $E_{C_1}(S)$ is represented by some function $\varepsilon: G_0 \rightarrow G_1$: ϵ acts on G_o and its action on S is given by defining $g\epsilon = g\phi\varepsilon$ if $g \epsilon G_1$. An element of $E_{id}(S)$ is represented by a pair maps (ϵ, η) with $\varepsilon: G_0 \rightarrow G_0$ and $\eta: G_1 \rightarrow G_1$ that satisfy $\phi\ \varepsilon = \eta\ \phi$. Finally an element of $E_{C_0}(S)$ is represented by $\varepsilon \epsilon E(G_0)$ acting on G_0 , and its action on S is again given by defining $g\varepsilon = g\ \phi\ \varepsilon$ if $g \epsilon G_1$ .Then we have a **decomposition**

$E^+(s) = E_{C_0}(S)\ U\ E_{id}(S)\ U\ E_{C_1}(S)$

(with $c_0 < id < c_1$ as endomorphism of the chain $o < 1$) of $(E^+(S), +)$ asa strongsemilattice of groups with linking maps.

$\phi_{C1,\ id}: E_{C1}(S) \rightarrow E_{id}(S),\ \varepsilon \rightarrow (\varepsilon\ \phi, \varepsilon)$

$\phi_{id,\ C0}: E_{id}(S) \rightarrow E_{C0}(S),\ (\varepsilon, \eta) \rightarrow \varepsilon$ and

$\phi_{C1,\ C0:}\ E_{C1}(S) \rightarrow E_{C0}(S),\ \varepsilon \rightarrow \varepsilon\phi)$

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