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ON

Weyl algebra and Modules over the Weyl algebra

(Submitted in partial fulfillment of M.Sc. Degree in mathematics)

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Introduction

In this seminar paper we study Weyl algebra and modules over the Weyl algebra. In the first chapter we give some basic concepts of rings and Modules with their properties as a preliminary.

In the second chapter we give definitions of Weyl algebra, modules over the Weyl algebra and also to describe a few important theorems, Lemma and proposition of modules over the Weyl algebra.

Chapter one

Preliminaries

1.1. Definitions and some properties on rings

Definition 1.1.1: A ring $(R, +, \cdot)$ consists of a non - empty set R and two binary operations "+" and "$\cdot$" satisfying:

i, $(R, +)$ is abelian group

ii, $(R, \cdot)$ is a semi group

iii, "+" and "$\cdot$" are related by the usual distributive laws.

(i.e. for all $a, b, c \in R$, $a \cdot (b+c) = a \cdot b + a \cdot c$ and $(a + b) \cdot c = a \cdot c + b \cdot c$)

Example: $(\mathbb{Z}, +, \cdot)$ is a ring, where $\mathbb{Z}$ is an integers, "+ "and "$\cdot$" usual addition and multiplication respectively.

Definition 1.1.2: A ring $(R, +, \cdot)$ is called

1) **Commutative** if "$\cdot$" is commutative (i.e. $ab = ba$, for all $a, b \in R$)
2) **Unitary (a ring with unity)** if $(R, \cdot)$ has an identity element, denoted by 1.
3) **Division ring** if it is a ring with unity and every non – zero element has a multiplicative inverse .
4) A **field** if it is a commutative division ring.

Example: $(\mathbb{R},+,\cdot)$ is a field, where $\mathbb{R}$ is the set of real numbers, "+ "and "$\cdot$" usual addition and multiplication respectively.

Definition 1.1.3: Let R be a ring with unity and let $R^* = R \setminus \{0\}$.

1) An element $a \in R^*$ is called **zero divisor** if $\exists\, b \in R^*$ such that $ab = ba = 0$.
2) R^* is said to be **integral domian** if R^* is commutative ring with unity and has no zero divisor.
3) An element $a \in R$ is called **unit or invertible** if $\exists\, b \in R$ such that $ab = ba = 1$.

Example: $(\mathbb{Z},+,\cdot)$ is integral domain.

1.2. Subrings and ideals

Definition 1.2.1: Let R be a ring. A subring of R is a non – empty subset $S \subseteq R$ and for all $a, b \in S$

i) $a - b \in S$

ii) $ab \in S$

Definition 1.2.2: Let R be a ring,

1) $I \subseteq R$ is called **left ideal** of R if

   i) I is an additive subgroup of R

   ii) $r \in R, x \in I \Rightarrow rx \in I$.

2) $I \subseteq R$ is called **right ideal** of R if

   i) I is an additive subgroup of R

   ii) $r \in R, x \in I \Rightarrow xr \in I$.

3) $I \subseteq R$ is called two sided **ideal** of R if I is both a left and right ideal of R.

Example: $(n\mathbb{Z},+,\cdot)$ is an ideal of $\mathbb{Z}$, where $n \in \mathbb{N}$

1.3. Characteristic of a ring

Definition 1.3.1: Let R be a ring. If there exists $n \in \mathbb{N}$ such that na = 0, for all $a \in R$, then the smallest such positive integer is called the characteristics of R and denoted by charR. If no such positive integer exists, then we say R has characteristic zero. (i.e. charR = 0)

Example: a, char $\mathbb{Z} = 0$

b, char $\mathbb{Z}_n = n$

Notation: K[x] is the set of polynomials in one indeterminate over a field K.

Addition and multiplication in K[x]

Definition 1.3.2: Let $f(x) = \sum_{i=0}^{\infty} a_i x^i$, $g(x) = \sum_{i=0}^{\infty} b_i x^i$ be two elements of K[x]. Then

i. $f(x) = g(x)$ if and only if $a_i = b_i$, for all i

ii. $f(x) + g(x) = \sum_{i=0}^{\infty} c_i x^i$, where $c_i = a_i + b_i$ for each i

iii. $f(x)g(x) = \sum_{i=0}^{\infty} d_i x^i$, where $d_i = \sum_{j=0}^{i} a_{i-j} b_j$

Observation 1: (K[x], +, $\cdot$) is a commutative ring.

Note: A ring R is said to be **simple** if it has no proper ideal.

1.4. Localization

Definition 1.4.1: Let R be a ring with unity. A subset $S \subseteq R$ is called a multiplicative set if

$$a, b \in S \Rightarrow ab \in S$$

Example: Let R be a ring with unity.

Let P be a prime ideal of R.

Let $S = R \setminus P$. Then S and P are a multiplicative sets.

Note: $RS^{-1} = R_s$ is called **localization of R at p** and usually denoted by R_p, where

$$R_s = \left\{ \frac{r}{s} : r \in R \text{ and } s \in S \right\}$$

Definition 1.4.2: $K[x]_f = \left\{ \frac{p}{f^r} : 0 \neq f \text{ and } p, f \in K[x], r \in \mathbb{N}_0 \right\}$ is localization of $K[x]$ at f, the set of rational functions of the form $\frac{p}{f^r}$, where $\mathbb{N}_0$ is set of natural number with zero.

Theorem 1.4.1: ($K[x]_f$, +) is abelian group.

Proof: Let f, h, p, q $\in$ K[x] and s, r, t $\in \mathbb{N}$ and $\frac{p}{f^r}, \frac{h}{f^t}, \frac{q}{f^s} \in K[x]_f$. Then

i. $\frac{p}{f^r} + \frac{q}{f^s} \in K[x]_f$ (since the sum of two rational functions is rational)

ii. $\frac{p}{f^r} + (\frac{q}{f^s} + \frac{h}{f^t}) = \frac{p}{f^r} + \frac{q f^t + h f^s}{f^s f^t}$

$$= \frac{p f^s f^t + (q f^t f^r + h f^s f^r)}{f^r f^s f^t}$$

$$= \frac{(p f^s f^t + q f^t f^r) + h f^s f^r}{f^r f^s f^t}$$

$$= \frac{(p f^s f^t + q f^t f^r)}{f^r f^s f^t} + \frac{h f^s f^r}{f^r f^s f^t}$$

$$= \left(\frac{p f^s f^t}{f^r f^s f^t} + \frac{q f^t f^r}{f^r f^s f^t}\right) + \frac{h}{f^t}$$

$$= \left(\frac{p f^s f^t}{f^r f^s f^t} + \frac{q f^t f^r}{f^r f^s f^t}\right) + \frac{h}{f^t}$$

$$= \left(\frac{p}{f^r} + \frac{q}{f^s}\right) + \frac{h}{f^t}$$

Hence "+" is associative

iii. since $0 + \frac{p}{f^r} = \frac{p}{f^r} + 0 = \frac{p}{f^r}$, $0 \in K[x]_f$ is additive identity.

iv. since $\frac{p}{f^r} + (\frac{-p}{f^r}) = (\frac{-p}{f^r}) + \frac{p}{f^r} = 0$, $\frac{-p}{f^r}$ is additive inverse for $\frac{p}{f^r}$.

v. since $\frac{p}{f^r} + \frac{q}{f^s} = \frac{p f^s + q f^r}{f^r f^s}$

$$= \frac{q f^r + p f^s}{f^r f^s}$$

$$= \frac{q f^r}{f^r f^s} + \frac{p f^s}{f^r f^s} = \frac{q}{f^s} + \frac{p}{f^r}$$

This implies "+" is commutative

Hence $(K[x]_f, +)$ is abelian group.

1.5. Definitions and some properties on modules

Definition 1.5.1: let (M, +) be abelian group and (R, +, $\cdot$) be a ring. M is called a left R-module or a left module if there exists a map $\because$ R x M $\rightarrow$ M, given that the following conditions are satisfied:

i) $a(x + y) = ax + ay$ whenever $a \in R$ and $x \in M$

ii) $(ab) x = a(bx)$ whenever $a, b \in R$ and $x \in M$

iii) $(a + b) x = ax + bx$ whenever $a, b \in R$ and $x \in M$

iv) $1x=x$ for all $x \in M$ and 1 denotes the identity element of R

Example: 1. ($\mathbb{Z}$, +, $\cdot$) is a module over itself.

2. (R, +, $\cdot$) is a module over itself.

1.5.1. Homomorphisms and isomorphisms over a module.

Let M and N be modules over a ring R.

Definition 1.5.1.1: A mapping $f:M \rightarrow N$ is said to be an R-homomorphism or a homomorphism of R-modules if it satisfies the following two conditions:

i. $f(x + y) = f(x) + f(y)$ whenever $x, y \in M$;

ii. $f(ax) = a f(x)$ whenever $a \in R$ and $x \in M$.

Definition 1.5.1.2: A homomorphism $f: M \rightarrow N$ of R-modules is called

i) a **monomorphism** if $x \neq y$ (where $x, y \in M$) implies that $f(x) \neq f(y)$.

ii) an **epimorphism** or **a surjection** if each element of N is the image of at least one element of M.

iii) **isomorphism** if f is both a monomorphism and an epimorphism

iv) **an endomorphism** if M = N

v) **an automorphism** if $f: M \rightarrow M$ is isomorphism.

Definition 1.5.1.3: Let $f : M \rightarrow N$ is a homomorphism of R-modules.

Then the set of elements of M which are mapped on to the zero element of N is called the **kernel** of f and is denoted by $Kerf$. i.e, $Kerf = \{x \in M / f(x) = 0_N\}$

Lemma 1.5.1.1: $f: M \rightarrow N$ of R-modules is a monomorphism if and only if $Kerf = \{0_M\}$.

Proof: $(\Rightarrow)$ Suppose f is monomorphism.

Let $x \in Kerf$. Then $f(x) = 0_N$ and $f(0) = 0_N$

Since f is a monomorphism, $x = 0_N$

Hence $Kerf = \{0_N\}$

$(\Leftarrow)$ Suppose $f(x) = f(y)$, for $x, y \in M$, we need to show that $x = y$.

$\Rightarrow f(x) - f(y) = 0$

$\Rightarrow f(x - y) = 0_N$,since f is homomorphism.

$\Rightarrow x - y \in Kerf$

$\Rightarrow x - y = 0_M$

$\Rightarrow x = y$

Hence f is monomorphism.

Note:-Two R–modules M and N are said to be isomorphic if there exists an isomorphism of M on to N and this is denoted by $M \cong N$

Note: If A is a subset of a set B, then we obtain a mapping $j: A \rightarrow B$ by putting $j(a) = a$ for every $a \in A$ .This mapping is called the inclusion mapping of A into B .

In the special case where A = B, this yields the identity mapping of B and denoted by i_B .

Definition 1.5.1.4: If $f: M \rightarrow N$ is a homomorphism, f (M) is called the **image** of the mapping f and the image of f denoted by imf.

Note: f is an epimorphism if and only if Imf = N.

1.5.2. Submodule

Definition 1.5.2.1: Let R be a ring and M be an R – module.

A non empty subset N of M is called **a submodule of M** if

i, $x, y \in N \Rightarrow x - y \in N$

ii, $a \in R, x \in N \Rightarrow ax \in N$

Example 1: Let f: $M \rightarrow N$ be a homomorphism, where M and N are modules.Then imf is a sub module of N.

Proof: i, since $f(0) = 0$,we have that $0 \in$ imf.

Hence imf$\neq \phi$.

ii, Let y_1 , $y_2 \in$ imf . Then $f(x_1) = y_1$ and $f(x_2) = y_2$ for some $x_1, x_2 \in$ M.

$y_1 - y_2 = f(x_1) - f(x_2)$

$= f(x_1 - x_2)$ ……………........... (since f is homomorphism)

$\Rightarrow y_1 - y_2 = f(x_1 - x_2) \in$ imf ……………… (since $x_1 - x_2 \in M$)

$\Rightarrow y_1 - y_2 \in$ imf

iii, Let a$\in$ R, $y_1 \in \operatorname{im} f$. Then $f(x_1) = y_1$, for some $x_1 \in$ M

Now, $a\, y_1 = a\, f(x_1) = f(a\, x_1)$ ……………. (since f is homomorphism)

$\Rightarrow a\, y_1 = f(a\, x_1) \in imf$

$\Rightarrow a\, y_1 \in imf$

Hence imf is a submodule of N.

Example 2: Let M be an R – module and $m \in M \setminus \{0\}$. Then Rm is submodule of M.

Proof: Let a, b $\in$Rm, a = r_1m, b = r_2m,for some $r_1, r_2 \in$ R

a-b = $r_1m - r_2m = (r_1 - r_2)m \in$ Rm

$ra = r(r_1m) = (rr_1)m \in$ Rm for $r \in$ R.

Hence Rm is submodule of M.

Definition 1.5.2.2: An R– module M is simple or irreducible if there is no submodule N such that $\{0\} \subset N \subset M$. $\left(i.e.\ if\ \{0\} \subset N \subset M\ ,then\ N = \{0\}\ or\ N = M\right)$

Or an R - module M is simple if and only if Rm = M,

for all m $\in M \setminus \{0\}$ and Rm is submodule of M.

Definition 1.5.2.3: Let M be an R– module, N be a submodule of M. Then $M/_N = \{x + N / x \in M\}$ is called quotient module or factor module. Define the operation "+" and "$\cdot$" on $M/_N$ as follows:

For x+N, y+N $\in M/N$, (x+N)+(y+N) = (x+y)+N and a(x+N) = ax +N for a $\in$R.

Note:-Let M be an R - module, N be a submodule of M. Then M/N is an R – module.

Let R be a commutative ring and let N, P be submodules of a module M, we define

(N: P) = $\{a \in R / aP \subseteq N\}$ is an ideal of R.

In particular, (0: M) = $\{a \in R / aM = 0\}$is an ideal of R. This ideal is called annihilator of M, denoted by Ann(M) or $\mathrm{Ann}_R^{(M)}$.

Definition 1.5.2.4: Let R be a ring and M be an R- module.

An element m$\in$ M is said to be a torsion element if there exist a non – zero r$\in$R such that rm = 0 (i.e. if Ann(m) is non – zero left ideal)

Note :- The set of all torsion elements of M is denoted by τ(M) .

- The module is called a torsion module if τ(M) = M.

1.5.3. Algebra

Definition 1.5.3.1: Let K be a commutative ring with unity .An algebra Over K

(or a K- algebra of M) is a ring M such that:

i, (M , +) is a unitary (left) K- module.

ii, a(mn) = (am)n = m(an) for all a$\in$K and m, n$\in$M.

Example:1. Let f : R $\rightarrow$ M be a ring homomorphism. Define ax = f(a).x , a$\in$R ,x$\in$M.

Then M is an R- algebra.

Proof: Let α , a $\in$ R and x, y$\in$M.Then,

i, a(x+y) = f(a)(x +y) = f(a)x + f(a)y = ax + ay

ii, (a + b)x = f(a +b)x = (f(a) + f(b))x ……….. (since f is homomorphism)

= f(a)x + f(b)x

= ax + bx

iii, (ab)x = f(ab)x = (f(a)f(b))x = f(a)(f(b)x) = a(bx)

iv, $1 \cdot x$ = x, for all x$\in$M. Hence (M , +) is unitary(left) R-module.

v, α (xy) = f(α)(xy) = (f(α)x)y = (α x)y and also $\alpha(xy) = f(\alpha)(xy) = (f(\alpha)x)y$

$= (xf(\alpha))y$, since M is a commutative ring.

$= x(f(\alpha)y)$

$= x(\alpha y)$.

Hence M is R- algebra.

2. If K is a commutative ring with identity, then the polynomial ring $K[x] = K[x_1, \dots, x_n]$ is a K- algebra with the respective K-module structure given in the usual way.

Proof : We need to show that i, (K[x],+) is a unitary (left) K-module.

ii, $a(fg) = (af)g = f(ag)$ for all $a \in K$ and $f, g \in K[x]$.

Now, since K[x] is a ring with unity (K[x], +) is abelian group.

Let a map $K \times K[x] \to K[x]$ be an action of K on K[x] by $(a, f) \mapsto af$, where $a \in K$ and $f \in K[x]$.

For $a, d \in K$ and $f, g \in K[x]$, where $f = (b_0, b_1, \dots)$ and $g = (c_0, c_1, \dots)$ and $b_0, b_1, \dots, c_0, c_1, \dots \in K$. We have,

1. $a(f+g) = a(b_0+c_0, b_1+c_1, \dots)$

$= (a(b_0+c_0), a(b_1+c_1), \dots)$

$= (ab_0+ac_0, ab_1+ac_1, \dots)$ ………(as "." is left distributive.)

$= (ab_0, ab_1, \dots) + (ac_0, ac_1, \dots)$

$= a(b_0, b_1, \dots) + a(c_0, c_1, \dots)$

$= af + ag$

2. $(a+d)f = (a+d)(b_0, b_1, \dots)$

$= ((a+d)b_0, (a+d)b_1, \dots)$ ……….. (as "." is left distributive.)

$= (ab_0+db_0, ab_1+db_1, \dots)$ ………….. (as "." is right distributive.)

$= (ab_0, ab_1, \dots) + (db_0, db_1, \dots)$

$= a(b_0, b_1, \dots) + d(b_0, b_1, \dots)$

$= af + df$

3. $(ad)f = (ad)\,(b_0, b_1, \ldots)$

$= ((ad)\,b_0, (ad)\,b_1, \ldots)$

$= (a(db_0), a(db_1), \ldots)$ ……… (as $(K[x], .)$ is semi-group)

$= a\,(db_0, db_1, \ldots)$

$= a\,(d\,(b_0, b_1, \ldots))$

$= a\,(df)$

4. $1.f = (1.b_0, 1.b_1, \ldots)$

$= (b_0, b_1, \ldots)$

$= f$

Therefore K[x] is a unitary left K-module.

Finally, we have; $a\,(fg) = a\,((b_0, b_1, \ldots)\,(c_0, c_1, \ldots))$

$= (a\,(b_0, b_1, \ldots))\,(c_0, c_1, \ldots)$

$= (ab_0, ab_1, \ldots)\,(c_0, c_1, \ldots)$

$= (b_0a, b_1a, \ldots)\,(c_0, c_1, \ldots)$,since $(K[x], +, \cdot)$ is commutative ring.

$= ((b_0, b_1, \ldots)\,a)\,(c_0, c_1, \ldots)$

$= (b_0, b_1, \ldots)\,(a\,(c_0, c_1, \ldots))$

$= f(ag)$

$\therefore$ K[x] is a K- algebra

Chapter Two

Weyl algebra

2.1. Definitions and some properties on Weyl algebra

In this section the Weyl algebra is introduced as a ring of operators on a vector space of infinite dimension.

Notation: K denotes a field of characteristic zero.

$K[X]$ is the ring of polynomials $K[x_1,...,x_n]$ in "n" indeterminates over K.

The ring $K[X]$ is a vector space of infinite dimension over K. Its algebra of linear operators is denoted by $End_K(K[X])$. Recall that the algebra operations in the endomorphism ring are the addition and composition of operators. i.e.

define: $\psi+\varphi : K[X]\to K[X]$ by $(\psi+\varphi)(f) = \psi(f)+\varphi(f)$ for all $f\in K[X]$,

$\psi,\varphi\in End_K(K[X])$ and "+" is addition of endomorphisms (2.1.1)

And $\psi\circ\varphi=\psi\varphi : K[X]\to K[X]$ by $\psi\varphi(f)=\psi(\varphi(f)), \forall f\in K[X]$, $\psi,\varphi\in End_K(K[X])$

"o" is composition of endomorphisms (2.1.2)

Theorem 2.1.1: $(End_K(K[X]),+,\circ)$ is a ring.

Proof: i, Let $f,g\in K[X]$, and $\psi,\varphi\in End_K(K[X])$, $\alpha\in K$

$$(\psi+\varphi)(f+g) = \psi(f+g)+\varphi(f+g) \quad \text{(by (2.1.1))}$$

$$= \psi(f)+\psi(g)+\varphi(f)+\varphi(g)$$

$$= (\psi+\varphi)(f)+(\psi+\varphi)(g)$$

$$(\psi+\varphi)(\alpha f)= \psi(\alpha f)+\varphi(\alpha f)$$

$$= \alpha\psi(f)+\alpha\varphi(f)$$

$$= \alpha(\psi+\varphi)(f)$$

$$\Rightarrow \psi+\varphi\in End_K(K[X])$$

And $\psi\varphi(f+g)=\psi(\varphi(f+g))$ (by (2.1.2))

$$= \psi(\varphi(f)+\varphi(f))$$

$$= \psi(\varphi(f))+\psi(\varphi(g))$$

$$= \psi\varphi(f)+\psi\,\varphi(g)$$

$$\Rightarrow \psi\varphi \in End_K(K[X])$$

ii, For any $\psi,\varphi,\phi \in End_K(K[X])$ *and* $f \in K[X]$,

$$\begin{aligned}[(\psi+\varphi)+\phi](f) &= (\psi+\varphi)(f)+\phi(f) && (by(2.1.1))\\ &=[\psi(f)+\varphi(f)]+\phi(f)\\ &=\psi(f)+[\varphi(f)+\phi(f)] && (since\, "+"\, is\ associative\ on\ K[X])\\ &=\psi(f)+(\varphi+\phi)(f) && (by(2.1.1))\\ &=[\psi+(\varphi+\phi)](f)\end{aligned}$$

$$\Rightarrow (\psi+\varphi)+\phi = \psi+(\varphi+\phi)$$

Hence "+" isassociative on $End_K(K[X])$

iii, $\overline{0}: K[X] \to K[X]$ defined by $\overline{0}(f)=0, \forall f \in K[X]$

is the zero element in $End_K(K[X])$. So, $\overline{0}$ is an identity element for "+" on $End_K(K[X])$

iv, For any $\psi \in End_K(K[X])$,

the map $-\psi: K[X] \to K[X]$ given by $(-\psi)(f) = -\psi(f), \forall f \in K[X]$ is the additive inverse of ψ.

i.e, every element in $End_K(K[X])$ has an inverse

v, For any $\psi,\varphi \in End_K(K[X])$, and $f \in K[X]$

$$\begin{aligned}(\psi+\varphi)(f) &= \psi(f)+\varphi(f) && (by(2.1.1))\\ &=\varphi(f)+\psi(f) && ("+"\ is\ commutative\ on\ K[X])\\ &=(\varphi+\psi)(f)\end{aligned}$$

Hence $\psi+\varphi=\varphi+\psi$

Thus "+" is commutative.

Therefore ($End_K(K[X])$, +) is abelian group.

vi, $\psi o(\varphi+\phi)(f) = \psi(\varphi+\phi)(f)$ (by (2.1.2))

$$
\begin{aligned}
&= \psi((\varphi+\phi)(f)) \\
&= \psi(\varphi(f)+\phi(f)) && (by(2.1.1)) \\
&= \psi(\varphi(f))+\psi(\phi(f)) && (by(2.1.2)) \\
&= (\psi o\varphi)(f)+(\psi o\phi)(f) \\
&= (\psi o\varphi + \psi o\phi)(f)
\end{aligned}
$$

$\Rightarrow \psi o(\varphi+\phi) = \psi o\varphi + \psi o\phi$

$$
\begin{aligned}
\textit{Similarly}, [(\psi+\varphi)o\phi](f) &= (\psi+\varphi)(\phi(f)) \\
&= \psi(\phi(f))+\varphi(\phi(f)) && (by(2.1.2)) \\
&= (\psi o\phi)(f)+(\varphi o\phi)(f) \\
&= [(\psi o\phi) + (\varphi o\phi)](f)
\end{aligned}
$$

$\Rightarrow (\psi+\varphi)o\phi = \psi o\phi + \varphi o\phi$

Thus "o" is distributive over "+".

vii, $[(\psi o\varphi)o\phi](f) = (\psi o\varphi)(\phi(f))$

$$
\begin{aligned}
&= \psi(\varphi(\phi(f))) \\
&= \psi((\varphi o\phi)(f)) \\
&= (\psi o(\varphi o\phi))(f)
\end{aligned}
$$

$\Rightarrow (\psi o\varphi)o\phi = \psi o(\varphi o\phi)$

Thus "o" is associative on $End_K(K[X])$

Therefore ($End_K(K[X])$, +, o) is a ring.

The mappings,

$\hat{x}_i : K[X] \to K[X]$ are the operators of $K[X]$ which are defined on a polynomial $f \in K[x]$ by the formulae $\hat{x}_i(f) = x_i \cdot f$, where i = 1, 2, … , n.

and also the mappings,

$\partial_i : K[X] \to K[X]$ are the operators of $K[X]$ which are defined on a polynomial $f \in K[x]$ by the formulae $\partial_i(f) = \dfrac{\partial f}{\partial x_i}$, where i = 1, 2, … , n.

Definition 2.1.1: The n^{th} Weyl algebra A_n is the K– sub algebra of $End_K(K[X])$ generated by the operators $\hat{x}_1, \hat{x}_2, ..., \hat{x}_n$ and $\partial_1, \partial_2, ..., \partial_n$.

Note: 1.The elements of A_n are linear combinations over K of monomials in the generators $\hat{x}_1, \hat{x}_2, ..., \hat{x}_n, \partial_1, \partial_2, ..., \partial_n$.

2. $\partial_i^j(f)$ is the j^{th} derivative of f with respect to x_i, where $f \in K[X]$ and $j = 0, 1, …, n$, $i = 1, 2, …, n$.

Example: $x_1 \partial_1, x_1^2 \partial_1^3$ are elements of A_1

Lemma 2.1.1: A Weyl algebra A_n is not commutative.

Proof: consider the operator $\partial_i \hat{x}_i$ and apply it to a polynomial $f \in K[X]$.

Now, $\partial_i \hat{x}_i(f) = \partial_i(x_i \cdot f)$ (by definition).

$= f \cdot \partial_i(x_i) + x_i \partial_i(f)$ (product rule of differentiation)

$= f \dfrac{\partial}{\partial x_i}(x_i) + x_i \partial_i(f)$ $\left(\text{since } \dfrac{\partial}{\partial x_i}(x_i) = 1\right)$

$= f + x_i \partial_i f$

$= (1 + x_i \partial_i) f$

$= (1 + \hat{x}_i \partial_i) f$

Thus, as operators we see that $\partial_i \hat{x}_i = (1 + \hat{x}_i \partial_i)$…………… (2.1.3)

where 1 stands for identity operator. It follows A_n is not commutative.

Note: If P, Q $\in A_n$, then their commutator is the operator

$$[P, Q] = PQ - QP$$

The formula (2.1.3) becomes $[\partial_i, \hat{x}_i] = 1$. …………………. (2.1.4)

A similar calculation allow us to obtain formulae for commutators of the other generators of A_n .

These are $\left[\partial_i, \hat{x}_j\right] = \delta_{ij}$, $\left[\partial_i, \partial_j\right] = \left[\hat{x}_i, \hat{x}_j\right] = 0$ . ... (2.1.5)

where $1 \le j, i \le n$ Kronecker delta symbol $\delta_{ij} = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{otherwise} \end{cases}$

2.2: Canonical form

In this section we construct a basis for the Weyl algebra as a K-vector space. This basis is known as the canonical basis. If an element of A_n is written as a linear combination of canonical basis, then we say that it is in canonical form.

The multi-index α is an element of $\mathbb{N}^n$, and $\alpha = (\alpha_1, \alpha_2, ..., \alpha_n)$

Now x^α means the monomial $x_1^{\alpha_1} x_2^{\alpha_2} \ ... \ x_n^{\alpha_n}$

The degree of this monomial is the length $|\alpha|$ of the multi-index α , namely

$|\alpha| = \alpha_1 + \alpha_2 + ... + \alpha_n$

Notice that a pair (α, β) of multi-indices in $\mathbb{N}^n$ is itself a multi-index in $\mathbb{N}^{2n}$,

so it makes sense to talk of its length .

Note: The factorial of a multi – index $\beta \in \mathbb{N}^n$ is defined by $\beta! = \beta_1! \beta_2! \ ... \ \beta_n!$

Lemma 2.2.1: Let $\delta, \beta \in \mathbb{N}^n$ and assume that $|\delta| \le |\beta|$. Then $\partial^\beta (x^\delta) = \beta!$ if $\delta = \beta$, and other wise zero .

Proof: if $\delta = \beta$. Then, $\partial^\beta (x^\delta) = \partial^\beta (x^\beta)$

$$
\begin{aligned}
&= \partial^{(\beta_1 \ldots \beta_n)} (x^{(\beta_1 \ldots \beta_n)}) \\
&= (\partial_1^{\beta_1} \partial_2^{\beta_2} \ldots \partial_n^{\beta_n})(x_1^{\beta_1} x_2^{\beta_2} \ldots x_n^{\beta_n}) \\
&= (\partial_1^{\beta_1} \partial_2^{\beta_2} \ldots \partial_{n-1}^{\beta_{n-1}})(x_1^{\beta_1} x_2^{\beta_2} \ldots x_{n-1}^{\beta_{n-1}}) \beta_n! \\
&= (\partial_1^{\beta_1} \partial_2^{\beta_2} \ldots \partial_{n-2}^{\beta_{n-2}})(x_1^{\beta_1} x_2^{\beta_2} \ldots x_{n-2}^{\beta_{n-2}}) \beta_{n-1}! \, \beta_n! \\
&= (\partial_1^{\beta_1} \partial_2^{\beta_2} \ldots \partial_{n-3}^{\beta_{n-3}})(x_1^{\beta_1} x_2^{\beta_2} \ldots x_{n-3}^{\beta_{n-3}}) \beta_{n-2}! \beta_{n-1}! \, \beta_n! \\
&\vdots \\
&= \beta_1! \, \beta_2! \, \beta_3! \ldots \beta_{n-2}! \, \beta_{n-1}! \, \beta_n!
\end{aligned}
$$

$= \beta!$

If $\delta \neq \beta$ and $|\delta| < |\beta|$, $\partial^{\beta}(x^{\delta}) = 0$.

Proposition 2.2.1: The set B = { $x^{\alpha}\partial^{\beta}$: $\alpha, \beta \in \mathbb{N}^n$ } is a basis of A_n as a vector space over K.

Proof: i, Let $c_{\alpha\beta} \in K$ for $\alpha, \beta \in \mathbb{N}^n$ such that $\sum c_{\alpha\beta}\, x^{\alpha}\partial^{\beta} = 0$, where 0 is the zero operator.

Let $f \in$ K[X] be a monomial of degree β,

i.e. $f = x^{\beta} = x_1^{\beta_1} x_2^{\beta_2} \ldots x_n^{\beta_n}$ where $|\beta| = \beta_1 + \beta_2 + \ldots + \beta_n$ and $x = (x_1, x_2, \ldots, x_n)$

Now, $\sum c_{\alpha\beta}\, x^{\alpha}\partial^{\beta}(f) = 0(f) = 0$

$\Rightarrow \beta! \sum c_{\alpha\beta}\, x^{\alpha} = 0$ (by Lemma 2.2.1)

$\Rightarrow \sum c_{\alpha\beta}\, x^{\alpha} = 0$

Since $x^{\alpha} \neq 0$, we have $c_{\alpha\beta} = 0,\ \forall \alpha, \beta \in \mathbb{N}^n$

Hence B is linearly independent.

ii, Let D$\in$ A_n and D = $\sum c_{\alpha\beta}\, x^{\alpha}\partial^{\beta}$ be a finite linear combination, $c_{\alpha\beta} \in K$.

We need to show that if some $c_{\alpha\beta} \neq 0$, then D $\neq$ 0.

But D is a linear operator of K[X].

- Suppose D = 0

  $\Rightarrow c_{\alpha\beta} = 0,\ \forall \alpha, \beta \in \mathbb{N}^n$, since B is linearly independent .

  This follows that $c_{\alpha\beta} \neq 0 \Rightarrow$ D $\neq$ 0.

- Suppose D $\neq$ 0

  $\Rightarrow \exists 0 \neq f \in K[X]$ such that D (f) $\neq$ 0 (otherwise D (f) = 0, $\forall f \in K[X] \Rightarrow$ D = 0)

Choose δ be the index such that $c_{\alpha\delta} \neq 0$ for some index α, but $c_{\alpha\beta} = 0$, for all indices β such that $|\beta| < |\delta|$.

Now, D (x^{δ}) = $\sum c_{\alpha\delta}\, x^{\alpha}\partial^{\delta}(x^{\delta}) = \delta! \sum c_{\alpha\delta}\, x^{\alpha}$ (by Lemma 2.2.1)

$\Rightarrow$ D (x^{δ}) $\neq$ 0, since at least one of the coefficients $c_{\alpha\delta} \neq 0$ by the choice of δ .

Hence D $\neq 0 \Rightarrow$ D = $\sum c_{\alpha\beta} x^{\alpha}\partial^{\beta}$ for some $c_{\alpha\beta} \neq 0$. $\rightarrow\leftarrow$

Hence B is a basis of A_n.

Examples: 1. Find the canonical form of $\partial_2^2 x_1 + \partial_1 x_1$ in A_3

Solution:- $\partial_2^2 x_1 + \partial_1 x_1 = \partial_2 \partial_2 x_1 + \partial_1 x_1$

$= \partial_2 x_1 \partial_2 + (1 + x_1 \partial_1)$ .... (by (2.1.3))

$= (x_1 \partial_2)\partial_2 + 1 + x_1 \partial_1$ .... (by (2.1.5))

$= 1 + x_1 \partial_1 + x_1 \partial_2^2$

Hence $1 + x_1 \partial_1 + x_1 \partial_2^2$ is a canonical form of $\partial_2^2 x_1 + \partial_1 x_1$ in A_3.

2. Find the canonical form of $\partial_1^2 x_1^2$ in A_3.

Solution:- $\partial_1^2 x_1^2 = \partial_1 \partial_1 x_1 x_1$

$= \partial_1 (1 + x_1 \partial_1) x_1$ (by (2.1.3))

$= \partial_1 x_1 + \partial_1 x_1 \partial_1 x_1$

$= 1 + x_1 \partial_1 + \partial_1 x_1 (1 + x_1 \partial_1)$ (by (2.1.3))

$= 1 + x_1 \partial_1 + \partial_1 x_1 + \partial_1 x_1 x_1 \partial_1$

$= 1 + x_1 \partial_1 + (1 + x_1 \partial_1) + (1 + x_1 \partial_1) x_1 \partial_1$ (by (2.1.3))

$= 2 + 2 x_1 \partial_1 + x_1 \partial_1 + x_1 \partial_1 x_1 \partial_1$

$= 2 + 3 x_1 \partial_1 + x_1 (1 + x_1 \partial_1)\partial_1$ (by (2.1.3))

$= 2 + 3 x_1 \partial_1 + x_1 \partial_1 + x_1 x_1 \partial_1 \partial_1$

$= 2 + 4 x_1 \partial_1 + x_1^2 \partial_1^2$

Hence $2 + 4 x_1 \partial_1 + x_1^2 \partial_1^2$ is a canonical form of $\partial_1^2 x_1^2$ in A_3.

2.3. The degree of an operator

Let D$\in$ A_n. The degree of D is the largest length of the multi-indices $(\alpha, \beta) \in \mathbb{N}^n x \mathbb{N}^n$ for which $x^{\alpha} \partial^{\beta}$ appears with non- zero coefficient in the canonical form of D. It is denoted by deg (D). As the degree of a polynomial, we use the convention that the zero polynomial has degree $-\infty$.

Example: - The degree of $2x_1\partial_2 + x_1x_2^2\partial_1\partial_2$ is 5.

Theorem 2.3.1: The degree satisfies the following properties; for $D, D' \in A_n$;

1. $\deg(D + D') \le \max\{\deg(D), \deg(D')\}$
2. $\deg(DD') = \deg(D) + \deg(D')$
3. $\deg[D, D'] \le \deg(D) + \deg(D') - 2$.

Proof:-

1. Case1: If either D or D' is non-zero constant.

Without loss of generality assume that D' is non-zero constant.

Now, $\deg(D + D') = \deg(D)$, for D $\neq$ 0
$= \max\{\deg(D), 0\}$
$= \max\{\deg(D), \deg(D')\}$

For D = 0, $\deg(D + D') = \deg(D') = \max\{\deg(D), \deg(D')\}$

Case 2: I f $D, D' \in A_n$ are written in canonical form, then so is $D + D'$, and one concludes that $\deg(D + D') \le \max\{\deg(D), \deg(D')\}$

2. Case 1: If either $\deg(D)$ or $\deg(D')$ is zero.

Without loss of generality assume that $\deg(D') = 0$

This implies D' is constant and $D' \neq 0$.

$\deg(DD') = \deg(D)$
$= \deg(D) + \deg(D')$

Hence $\deg(DD') = \deg(D) + \deg(D')$

Case 2: Suppose $D = \partial^\beta$ and $D' = x^\alpha$ with $|\alpha| + |\beta| = $ k.

If $\beta_i \neq 0$, then $[\partial^\beta, x^\alpha] = \partial_i[\partial^{\beta - e_i}, x^\alpha] + [\partial_i, x^\alpha]\partial^{\beta - e_i}$, where e_i denotes the multi-index all of whose entries are zero, except the i[th] entry which is 1.

Since for some i, $\beta_i \neq 0$, by induction we have ;

$\deg[\partial^{\beta - e_i}] \le |\alpha| + |\beta| - 3$ *and* $\deg[\partial_i, x^\alpha] \le |\alpha| - 1$, since $\deg(\partial^{\beta - e_i}) = |\beta| - 1$

$\Rightarrow \deg(\partial_i[\partial^{\beta - e_i}, x^\alpha]) \le 1 + |\alpha| + |\beta| - 3 = |\alpha| + |\beta| - 2$

And $\deg([\partial_i, x^\alpha]\partial^{\beta - e_i}) \le |\alpha| - 1 + |\beta| - 1 = |\alpha| + |\beta| - 2$

Then $\deg[\partial^\beta, x^\alpha] = \deg(\partial_i[\partial^{\beta - e_i}, x^\alpha] + [\partial_i, x^\alpha]\partial^{\beta - e_i})$

$$\le \max\left\{\deg(\partial_i[\partial^{\beta-e_i},x^\alpha]),\deg([\partial_i,x^\alpha]\partial^{\beta-e_i})\right\} = |\alpha|+|\beta|-2$$

$\Rightarrow \deg[\partial^\beta,x^\alpha] \le |\alpha|+|\beta|-2$. But $[\partial^\beta,x^\alpha] = \partial^\beta x^\alpha - x^\alpha\partial^\beta$

$\Rightarrow \partial^\beta x^\alpha = x^\alpha\partial^\beta + [\partial^\beta,x^\alpha]$

Since $\deg(x^\alpha\partial^\beta) = |\alpha|+|\beta|$ *and* $\deg[\partial^\beta,x^\alpha] \le |\alpha|+|\beta|-2$

We conclude that $\deg(DD') = \deg(\partial^\beta x^\alpha) = \deg(x^\alpha\partial^\beta) = |\alpha|+|\beta| = \deg(D)+\deg(D')$ Hence $\deg(DD') = \deg(D)+\deg(D')$.

Case 3: Let $D = x^\delta\partial^\beta$ and $D' = x^\alpha\partial^\eta$

If $|\alpha| = |\beta| = 0$, then $D = x^\delta$ and $D' = \partial^\eta$. Thus, the result is the same as case 2.

Suppose $|\alpha|\ne 0, |\beta|\ne 0$. Then $DD' = x^\delta\left(\partial^\beta x^\alpha\right)\partial^\eta$

$$= x^\delta\left(x^\alpha\partial^\beta + [\partial^\beta,x^\alpha]\right)\partial^\eta$$

$$= x^{\delta+\alpha}\partial^{\beta+\eta} + x^\delta[\partial^\beta,x^\alpha]\partial^\eta$$

Since $[\partial^\beta,x^\alpha] = \partial^\beta x^\alpha - x^\alpha\partial^\beta$ and $\deg[\partial^\beta,x^\alpha] \le |\alpha|+|\beta|-2$,

$\deg(x^\delta[\partial^\beta,x^\alpha]\partial^\eta) \le |\delta|+|\alpha|+|\beta|-2+|\eta| = \deg(D)+\deg(D')-2$

Then $\deg(DD') = \deg\left(x^{\delta+\alpha}\partial^{\beta+\eta} + x^\delta[\partial^\beta,x^\alpha]\partial^\eta\right)$

$$\le \max\left\{\deg\left(x^{\delta+\alpha}\partial^{\beta+\eta}\right),\deg\left(x^\delta[\partial^\beta,x^\alpha]\partial^\eta\right)\right\}$$

$$= \deg\left(x^{\delta+\alpha}\partial^{\beta+\eta}\right)$$

$$= \deg(D)+\deg(D')$$

Hence $\deg(DD') = \deg(D)+\deg(D')$

Example: Let $P = x_2\partial_1$ and $Q = x_1\partial_1$. Then deg (P) =2 and deg (Q) =2

a, $P + Q = x_2\partial_1 + x_1\partial_1$

$\Rightarrow$ deg (P + Q) = 2

Hence deg (P + Q) = max {deg (P), deg (Q) } = 2

b, $PQ = x_2\partial_1 x_1\partial_1 = x_2(1 + x_1\partial_1)\partial_1 = x_2\partial_1 + x_2x_1\partial_1^2$

$\Rightarrow$ deg (PQ) = 4 = deg (P) + deg(Q) c, $[P, Q] = PQ - QP$

$$= x_2\partial_1 x_1\partial_1 - x_1\partial_1\, x_2\partial_1$$

$$= x_2(1+ x_1\partial_1)\partial_1 - x_1\, x_2\partial_1\,\partial_1$$

$$= x_2\partial_1 + x_2x_1\partial_1^2 - x_1\, x_2\partial_1^2$$

$$= x_2\partial_1$$

$\Rightarrow \deg (P , Q) = 2.$

$\Rightarrow \deg [P , Q] = \deg (P) + \deg (Q) - 2$

Corollary 2.3.1: The algebra A_n is a domain.

Proof :- Let P, Q $\in A_n$ with PQ = 0 (0 is zero operator).Then we want to show P = 0 or Q = 0

Suppose not, (i.e, $P \neq 0$ and $Q \neq 0$).

$\Rightarrow$ deg(P) = n and deg(Q) = m , for some n, m ≥ 0

$\Rightarrow$ deg(P) + deg(Q) = n + m ≥ 0

$\Rightarrow$ deg(PQ) = s $\geq$ 0 , for some s = n + m

$\Rightarrow PQ \neq 0 \ \rightarrow\leftarrow$

Hence P = 0 or Q = 0

Hence A_n is a domain.

Corollary 2.3.2: The only elements of A_n that have an inverse are the constants.

$Proof$: *Let* $0 \neq p \in A_n$ *has an inverse*

$\Rightarrow \exists\, p' \in A_n$ *such that* $pp' = 1$

$\Rightarrow \deg(pp') = \deg(1) = 0$

$\Rightarrow \deg(p) + \deg(p') = 0$

$\Rightarrow \deg(p) = \deg(p') = 0$, (since the degree of a non – zero operators is always non – negative.)

$\Rightarrow$ p is constant .

2.4. Ideal structure

Theorem 2.4.1 : The Weyl algebra A_n is simple.

Proof : Let I be a non - zero two sided ideal of A_n .

Let $0 \neq p \in I$ with smallest degree in I .

Case 1. deg (p) = 0

$\Rightarrow$ p is constant.

$\Rightarrow$ p is a unit in I (by corollary 2.3.2)

$\Rightarrow$ there exist $p^{-1} \in I$ such that $p^{-1}p = p\,p^{-1} = 1 \in I$,(since I is two sided ideal of A_n).

Now, let $x \in A_n$,$1 \in I$

$\Rightarrow x.1 = 1.x = x \in I$

$\Rightarrow$ I can be generated by 1

$\Rightarrow A_n \subseteq I$

But $I \subseteq A_n$

Hence $I = A_n$

Case 2: $p = \sum c_{\alpha\beta} x^{\alpha} \partial^{\beta}$, where $0 \neq c_{\alpha\beta} \in K$ $for\, \alpha, \beta \in N^n$

$\Rightarrow \deg(p) = \alpha + \beta > 0$

Now $x \in A_n$,$p \in I \Rightarrow xp, px \in I$, since I is two sided ideal of A_n.

If $\beta \neq 0$, then $xp - px \in I$ (otherwise $\beta = 0$. Then $xp - px = 0 \in I$ $\rightarrow\leftarrow$ to $I \neq 0$)

But from (theorem 2.3.1, # 3) we have

$\deg(xp - px) = \deg[x, p] \leq \deg(x) + \deg(p) - 2$

$= 1 + \alpha + \beta - 2$

$= \alpha + \beta - 1$, which is a contradiction to the choice of p (i.e. p has smallest degree in I). Thus $\beta = 0$.

Now, since $\alpha+\beta>0$ and $\beta=0$, we have $\alpha \neq 0$.

$\Rightarrow 0 \neq \partial_x \text{p} - \text{p}\, \partial_x = [\partial_x, \text{p}] \in I$

Moreover deg $[\partial_x, \text{p}] \leq \deg(\partial_x) + \deg(\text{p})$ -2

$= 1 + \alpha+\beta-2$

$= \alpha+\beta-1$, again contradiction to the choice of p (i.e. p has the smallest degree in I). Hence $\alpha = 0$.

$\Rightarrow$ I contains only the constant element (s).

Hence $I = A_n$.

Thus A_n does not contain a two - sided proper ideal.

Therefore A_n is simple.

Corollary 2.4.1: Every endomorphism of A_n is injective.

Proof: Let $\phi: A_n \to A_n$ be an endomorphism

Let $p, p' \in A_n$ *and* $\phi(p) = \phi(p')$

$\Rightarrow \phi(p) - \phi(p') = 0$

$\Rightarrow \phi(p-p') = 0$, since ϕ is homomorphism .

$\Rightarrow p - p' \in Ker\,\phi$

$\Rightarrow p - p' = 0$ since $Ker\,\phi$ is two sided ideal of A_n which implies $Ker\,\phi = 0$.

$\Rightarrow p = p'$

Hence ϕ is injective.

2.5. Modules over the Weyl algebra

The Weyl algebra was constructed as a subring of an endomorphism ring. Writing $\mathbb{C}[X]$ is a polynomial ring $\mathbb{C}[x_1, x_2, ..., x_n]$, where $\mathbb{C}$ is a complex number as a field K. Then we have that A_n is a sub ring of $End_{\mathbb{C}}(\mathbb{C}[X])$.One deduces from this that the polynomial ring is a left A_n-module.Thus the action of $\hat{x}_i$ on $\mathbb{C}[X]$ is by straight forward multiplication where as ∂_i acts by differentiation with respect to x_i .

Note: Let M be a K - vector space. Then to say M is A_n - module, it suffices to show that :

i, $[\partial_i, \hat{x}_j]u = \delta_{ij}\, u$

ii, $[\hat{x}_i, \hat{x}_j]u = [\partial_i, \partial_j]u = 0$ (2.1.6)

for every $u \in M$ and $\hat{x}_i, \hat{x}_j, \partial_i \, and\, \partial_j$ are operators of $K[X]$.

Theorem 2.5.1: $\mathbb{C}[X]$ is is a left A_n - module.

Proof: a mappings, A_n x $\mathbb{C}[X] \to \mathbb{C}[X]$ defined by $(\hat{x}_i, f) \mapsto \hat{x}_i(f) = x_i \cdot f$

$(\partial_i, f) \mapsto \partial_i(f)$ for all $f \in \mathbb{C}[X]$

i, $[\partial_i, \hat{x}_j](f) = (\partial_i \hat{x}_j - \hat{x}_j \partial_i)(f) = \begin{cases} 1 \cdot f & if\; i=j \\ 0 & otherwise \end{cases}$ where $i, j = 1, 2, \ldots, n$ (by (2.1.5))

ii, $[\partial_i, \partial_j](f) = 0 \cdot f = 0$ (by (2.1.5))

iii, $[\hat{x}_i, \hat{x}_j](f) = 0 \cdot f = 0$ (by (2.1.5))

Hence $\mathbb{C}[X]$ is is a left A_n - module.

Example: $\mathbb{C}[x]$ is a left A_1 - module with the action of x and ∂_x on $\mathbb{C}[x]$. Defined

$x, \partial_x : \mathbb{C}[x] \to \mathbb{C}[x]$, by $f \mapsto xf$ and $f \to \partial_x(f)$

Proof: From theorem 2.1.1 $(\mathbb{C}[x], +, \cdot)$ is a ring, then we have that $(\mathbb{C}[x], +)$ is abelian group.

Now, let $P, Q \in A_1$ and $f, g \in \mathbb{C}[x]$. Consider the monomials $P = x^{\alpha}\, \partial_x^{\beta}$,

$Q = x^{\alpha_1}\, \partial_x^{\beta_1}$ and $\alpha, \beta, \alpha_1, \beta_1 \in \mathbb{N}_0$.

i, $P\,(f+g) = x^{\alpha}\, \partial_x^{\beta}(f+g)$

$= x^{\alpha}(\partial_x^{\beta}(f+g))$

$= x^{\alpha}(\partial_x^{\beta}(f) + \partial_x^{\beta}(g))$ .... (by linearity).

$= x^{\alpha}(\partial_x^{\beta}(f)) + x^{\alpha}(\partial_x^{\beta}(g))$ .... (by definition).

$= (x^{\alpha}\, \partial_x^{\beta})(f) + (x^{\alpha}\, \partial_x^{\beta})(g)$

$= P\, f + P\, g$

ii, $(P+Q)f = (x^{\alpha}\, \partial_x^{\beta} + x^{\alpha_1}\, \partial_x^{\beta_1})f = (x^{\alpha}\, \partial_x^{\beta})f + (x^{\alpha_1}\, \partial_x^{\beta_1})f = P\, f + Q\, f$

iii, $(P\,Q)f = (x^{\alpha}\, \partial_x^{\beta}\, x^{\alpha_1}\, \partial_x^{\beta_1})f = (x^{\alpha}\, \partial_x^{\beta})((x^{\alpha_1}\, \partial_x^{\beta_1})f) = P\,(Q\,f)$

iv, $1(f) = 1.f = f$

Moreover,

$[\partial_x, x]\, f = (\partial_x x - x\partial_x)\, f = \begin{cases} 1 \cdot f \text{ if } i=j \\ 0 \text{ otherwise} \end{cases}$, where $i, j = 1,2,\ldots,n$ (by (2.1.5))

$[\partial_x, \partial_x]\, f = (\partial_x^2 - \partial_x^2)\, f = 0.\ f = 0$ (by (2.1.5))

$[x, x]\, f = (x^2 - x^2)\, f = 0.\ f = 0$ (by (2.1.5))

Hence $\mathbb{C}[x]$ is a left A_1- module.

Theorem 2.5.2: $\mathbb{C}[x]$ as A_n- module is simple.

Proof: let $f \in \mathbb{C}[x], f \neq 0$ and $A_n\ f$ be a submodule of $\mathbb{C}[x]$.

Let $f = \sum c_\alpha x^\alpha$, $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n)$ and $|\alpha| = \alpha_1 + \alpha_2 + \ldots + \alpha_n$, $x = (x_1, x_2, \ldots, x_n)$

Then $\partial_i^\alpha(f) = c_\alpha \cdot \alpha!$. This implies $1 = \dfrac{1}{c_\alpha\, \alpha!}\ \partial_i^\alpha\ f \in A_n f$

$\Rightarrow A_n f = \mathbb{C}[x]$.

Hence $\mathbb{C}[x]$ as A_n- module is simple.

Theorem 2.5.3:Let $f \neq 0, f \in \mathbb{C}[x]$, $\mathbb{C}[x]_f$ is a left A_n- module.

Proof: Define a map $\cdot : A_n \times\ \mathbb{C}[x]_f \to \mathbb{C}[x]_f$, given by

$$\left(\hat{x}_i, \frac{p}{f^r}\right) \mapsto \hat{x}_i \frac{p}{f^r} = \frac{x_i\, p}{f^r} \text{ and}$$

$$\left(\partial_i, \frac{p}{f^r}\right) \mapsto \partial_i\left(\frac{p}{f^r}\right) = \frac{f^r \partial_i\, p - p\,\partial_i(f^r)}{(f^r)^2}$$

$$= \frac{f^r \partial_i\, p - p\, r\, f^{r-1} \partial_i(f)}{(f^r)^2}$$

$$= \frac{f\, \partial_i(p) - r\, p\, \partial_i(f)}{f^{r+1}} \in \mathbb{C}[x]_f \text{ , where, } i = 1, \ldots, n.$$

Since the set of rational functions of the form $\frac{p}{f^r}$ is abelian group with an operation "+".

i.e. $(\mathbb{C}[x]_f, +)$ is abelian group. (see theorem 1.4.1)

Now, let p, q, h, f $\in \mathbb{C}[x]$ and $\frac{p}{f^r}$, $\frac{q}{f^t}$, $\frac{h}{f^s}$ $\in \mathbb{C}[x]_f$ and $\hat{x}_i, \partial_i \in A_n$, r, t, s $\in \mathbb{N}$

and where i = 1, 2, …, n

i, $\hat{x}_i\ (\frac{p}{f^r} + \frac{q}{f^t}) = x_i\left(\frac{pf^t + qf^r}{f^r f^t}\right)$

$$= \frac{x_i\, p\, f^t}{f^r\, f^t} + \frac{x_i\, q\, f^r}{f^r\, f^t}$$

$$= \frac{x_i\, p}{f^r} + \frac{x_i\, q}{f^t}$$

$$= x_i(\frac{p}{f^r}) + x_i(\frac{q}{f^t})$$

$\partial_i\ (\frac{p}{f^r} + \frac{q}{f^t}) = \partial_i\left(\frac{pf^t + qf^r}{f^r f^t}\right)$

$$= \frac{f^r\, f^t\, \partial_i\left(p\, f^t + q\, f^r\right) - \left(p\, f^t + q\, f^r\right)\partial_i\left(f^r\, f^t\right)}{(f^r\, f^t)^2}$$

$$= \frac{f^r\, f^t\left[f^t\, \partial_i\, p + p\,\partial_i\, f^t + f^r\, \partial_i\, q + q\,\partial_i\, f^r\right] - \left[\left(p\, f^t + q\, f^r\right)\left(f^t\partial_i f^r + f^r\partial_i f^t\right)\right]}{(f^r\, f^t)^2}$$

$$= \frac{f^r\, f^t\left[f^t\, \partial_i\, p + p\,\partial_i\, f^t + f^r\, \partial_i\, q + q\,\partial_i\, f^r\right]}{(f^r\, f^t)^2} - \frac{\left[\left(p\, f^t + q\, f^r\right)\left(f^t\partial_i f^r + f^r\partial_i f^t\right)\right]}{(f^r\, f^t)^2}$$

$$= \partial_i\ (\frac{p}{f^r}) + \partial_i\ (\frac{q}{f^t})$$

ii, $(\hat{x}_i + \partial_i)\frac{p}{f^r} = \frac{(\hat{x}_i + \partial_i)\, p}{f^r}$

$$= \frac{x_i p}{f^r} + \frac{\partial_i p}{f^r}$$

$$= \hat{x}_i(\frac{p}{f^r}) + \partial_i\left(\frac{p}{f^r}\right)$$

iii, $(\partial_i \hat{x}_i)\frac{p}{f^r} = \frac{\partial}{\partial x_i} x_i \frac{p}{f^r} = \frac{\partial}{\partial x_i}\left(x_i \frac{p}{f^r}\right) = \partial_i\left(\hat{x}_i \frac{p}{f^r}\right)$

$$(\hat{x}_i \partial_i)\frac{p}{f^r} = x_i \frac{\partial}{\partial x_i}\frac{p}{f^r} = x_i\left(\partial_i \frac{p}{f^r}\right) = \hat{x}_i\left(\partial_i \frac{p}{f^r}\right)$$

iv, $1(\frac{p}{f^r}) = 1\frac{p}{f^r} = \frac{p}{f^r}$

Moreover,

$$[\partial_i, \hat{x}_j]\left(\frac{p}{f^r}\right) = (\partial_i \hat{x}_j - \hat{x}_j \partial_i)\left(\frac{p}{f^r}\right) = \delta_{ij}\left(\frac{p}{f^r}\right) = \begin{cases} \frac{p}{f^r} & \text{if } i = j \\ 0 & \text{otherwise} \end{cases}$$, where i, j = 1, 2, … , n.

$$[\partial_i, \partial_j]\left(\frac{p}{f^r}\right) = 0.\left(\frac{p}{f^r}\right) = 0$$

$$[\hat{x}_i, \hat{x}_j]\left(\frac{p}{f^r}\right) = 0.\left(\frac{p}{f^r}\right) = 0$$ (by (2.1.5))

Hence $\mathbb{C}[x]_f$ is a left A_n - module.

Example: Let $f = x+1$, and arbitrary $p \in \mathbb{C}[x]$. Then $\mathbb{C}[x]_f$ is A_1 - module.

$$x\left(\frac{p}{f^r}\right) = x(\frac{p}{(x+1)^r}) = \frac{xp}{(x+1)^r} \in \mathbb{C}[x]_f$$

$$\partial_x\left(\frac{p}{f^r}\right) = \frac{(x+1)^r \partial_x(p) - r(x+1)^{r-1}p}{((x+1)^r)^2} = \frac{\partial_x(p) - r(x+1)^{-1}p}{(x+1)^r} \in \mathbb{C}[x]_f$$

$$[\partial_x, x](\frac{p}{(x+1)^r}) = (\partial_x x - x\partial_x)(\frac{p}{(x+1)^r}) = \begin{cases} 1 \cdot \frac{p}{(x+1)^r} & \text{if } i = j \\ 0 & \text{otherwise} \end{cases}$$ (by (2.1.5))

$$[\partial_x, \partial_x]\,(\frac{p}{(x+1)^r}) = 0 \cdot \frac{p}{(x+1)^r} = 0$$ (by (2.1.5))

$$[x, x]\,(\frac{p}{(x+1)^r}) = 0 \cdot \frac{p}{(x+1)^r} = 0$$ (by (2.1.5))

Hence $\mathbb{C}[x]_f$ is A_1 - module.

Lemma 2.5.4: Let R be a ring and M be an irreducible left R – module.

1, If $0 \neq u \in$ M, then M $\cong$ $R/Ann(u)$

2. If R is not a division ring, then M is a torsion module.

Proof: Consider a map f: $R \to$ M, defined by f (1) = $1 \cdot$ u = u

Let a, b, c$\in$R,

i, f(a+b) = f(a+b $\cdot$ 1) = (a+b) $\cdot$ u

= au + bu

= f (a) + f (b)

ii, f(ca) = (ca) $\cdot$ u = c (au) = c f(a)

Hence f is homomorphism.

Since $0 \neq u \in M$ and M is irreducible (i.e. for any $0 \neq u \in M$ $\exists 1 \in$ R such that f(1) = u),

f is surjective.

Kerf = {b$\in$ R / f (b) = 0} = {b$\in$ R / b $\cdot$ u = 0 for 0 $\neq$ u $\in$ M } = $Ann(u)$

Hence M $\cong$ $R/Ann(u)$ ……... … (by fundamental theorem of homomorphism)

2. Suppose $Ann(u) = 0$ for some $0 \neq$ u$\in$ M.

It follows from (1) above, M $\cong$ R, since $R/Ann(u)$ = R / {0} = R

Since M is irreducible, this can happen only if the left ideals of R is it self.

But in this case R is a division ring $\to\leftarrow$to the hypothesis.

Thus $Ann(u) \neq 0$. Hence if R is not a division ring, then M is a torsion module

2.5.1. Twisting

Let R be a ring and M be a left R – module. Suppose σ be an automorphism of R.

We shall define a new left module M_σ , as follows: As an abelian group, $M_\sigma = M$.

The difference lies in the action of R on M_σ.

Let $a \in R$ and $u \in M$, define $a \cdot u = \sigma(a) u$ .Then M_σ is a left R – module.

To show this,

Define $R \times M_\sigma \to M_\sigma$ by $(a, u) \mapsto a \cdot u = \sigma(a) u$.

i, $(a+b)u = \sigma(a+b) u$, for $a,b \in R$, $u \in M_\sigma$

$= (\sigma(a) + \sigma(b)) u$, since σ is an automorphism.

$= \sigma(a) u + \sigma(b) u$

$= au + bu$

ii, $(ab)u = \sigma(ab) u$

$= \sigma(a)\, \sigma(b) u$

$= \sigma(a) (bu)$

$= a (bu)$.

iii, $a (u + v) = \sigma(a) (u + v)$, $u, v \in M_\sigma$ and $a \in R$.

$= \sigma(a) u + \sigma(a) v$

$= au + av$

Hence M_σ is a left R – module.

It is called **the twisted module** of M by σ .

Proposition 2.5.1.1: Let R be a ring, M be a left R- module and σ be an automorphism of R. Then,

a, M_σ is irreducible if and only if M is irreducible .

b, M_σ is a torsion module if and only if M is a torsion module .

c, If N is a submodule of M , then $(M/N)_\sigma \cong M_\sigma/N_\sigma$.

d, Let J be a left ideal of R .

Set σ (J) = { σ (r): r$\in$J}. Then,

σ (J) is a left ideal of A_n and $\left({}^{R}\!/\!_{J}\right)_{\sigma} \cong {}^{R}\!/\!_{\sigma^{-1}(J)}$

Proof : a, ($\Rightarrow$) Suppose a left R- module M_σ is irreducible .

$\Rightarrow$ For $0 \neq u, v \in M_\sigma$ there exists $a \in R$ such that $\sigma(a)\, u = v$

$\Rightarrow$ au = v in M

Hence M is irreducible

($\Leftarrow$) Suppose an R – module M is irreducible.

$\Rightarrow$ For $0 \neq u, v \in M$ there exists $a \in R$ such that au = v

$\Rightarrow \sigma(a)\, u = v$ in M_σ

Hence M_σ is irreducible

b, ($\Rightarrow$) Suppose M_σ is a torsion module

$\Rightarrow$ There exists $u \in M_\sigma$ such that $\sigma(a)\, u = 0$, for some $a \in R$

$\Rightarrow$ au = 0

Hence M is torsion module.

($\Leftarrow$) Suppose M is torsion module

$\Rightarrow$ there exists $u \in M$ such that au =0 , for some $a \in R$

$\Rightarrow \sigma(a)\, u = 0$

Hence M_σ is a torsion module.

c, define $f: M_\sigma \to \left({}^{M}\!/\!_{N}\right)_\sigma$ by $f(\sigma(a)\mathrm{u}) = \sigma(a)(\mathrm{u}+N) = \sigma(a)\mathrm{u} + N_\sigma$, for a$\in$R, u$\in M$.

i, Well definedness

Suppose $\sigma(a_1)\mathrm{u}$, $\sigma(a_2)\mathrm{u} \in M_\sigma$ such that $\sigma(a_1)\mathrm{u} = \sigma(a_2)\mathrm{u}$

$\Rightarrow \sigma(a_1) = \sigma(a_2)$

$\Rightarrow a_1 = a_2$

Now, $f(\sigma(a_1)\mathrm{u}) = \sigma(a_1)(\mathrm{u} + N)$

$= \sigma(a_2)(\mathrm{u} + N)$

$= f(\sigma(a_2)\mathrm{u})$

Hence f is well defined.

ii, For each $\sigma(a)\mathrm{u} + N_\sigma \in (M/N)_\sigma$ there exists $\sigma(a)\mathrm{u} \in M_\sigma$ such that

$f(\sigma(a)\mathrm{u}) = \sigma(a)\mathrm{u} + N_\sigma$.

Hence f is surjective.

iii, Ker $f = \{\sigma(a)\mathrm{u} \in M_\sigma / f(\sigma(a)\mathrm{u}) = \sigma(a)(\mathrm{u} + N) = N_\sigma\}$

$= \{\sigma(a)\mathrm{u} \in M_\sigma / \sigma(a)\mathrm{u} + N_\sigma = N_\sigma\}$

$= \{\sigma(a)\mathrm{u} \in M_\sigma / \sigma(a)\mathrm{u} \in N_\sigma\}$

$= \{\sigma(a)\mathrm{u} \in M_\sigma \cap N_\sigma\}$

$= N_\sigma$, since N is a sub module of M .

Hence f is injective.

Therefore $(M/N)_\sigma \cong M_\sigma/N_\sigma$.

d. i, To proof $\sigma(J)$ is left ideal of A_n

- Since $\sigma(0) = 0$, for $0 \in$ J, σ(J) $\neq \phi$
- Let x, y $\in \sigma$(J). Then $x = \sigma(r_1)$, $y = \sigma(r_2)$, for $r_1, r_2 \in J$

  $x - y = \sigma(r_1) - \sigma(r_2) = \sigma(r_1 - r_2)$, since σ is an automorphism.

  $\Rightarrow x - y \in \sigma(J)$.
- Let a $\in A_n$ and x $\in \sigma$(J).

  $ax = a\sigma(r_1) = \sigma(ar_1)$, for $r_1 \in J$

  $\Rightarrow ax \in \sigma(J)$

Therefore σ(J) is left ideal of A_n.

ii, To proof $(R/J)_\sigma \cong R/\sigma^{-1}(J)$

Define $f : R \to (R/J)_\sigma$ by $f(1) = 1 + J$ and $f(b) = b \cdot f(1) = \sigma(b) + J$

1, well definedness

Suppose a = b, for a, b $\in$ R.

Now, f(a) = a $\cdot$ f(1) = σ(a) + J = σ(b) + J = b $\cdot$ f(1) = f(b)

Hence f is well defined.

2, Let $a,b\in R$. Then $f(a+b)=(a+b)\cdot f(1)$

$= \sigma(a+b)+J$

$= \sigma(a)+\sigma(b)+J$, since σ is automorphism

$= (\sigma(a)+J)+(\sigma(b)+J)$

$= f(a)+f(b)$

$f(ab)=ab\cdot f(1) = \sigma(ab)+J$

$= \sigma(a)\sigma(b)+J$, since σ is automorphism

$= (\sigma(a)+J)(\sigma(b)+J)$

$=\sigma(a)\sigma(b)$

Hence f is homomorphism.

3, For each $\sigma(b)+J\in (R/J)_\sigma$, there exists $b\in R$ such that $f(b)=b\cdot f(1) = \sigma(b)+J$

Hence f is surjective.

4, $Ker\, f=\{b\in R\,/\,f(b)=J\} = \{b\in R\,/\,b\cdot f(1)=J\}$

$= \{b\in R\,/\,\sigma(b)+J=J\}$

$= \{b\in R\,/\,\sigma(b)\in J\}$

$=\{b\in R\,/\,b\in\sigma^{-1}(J)\}$

$= \sigma^{-1}(J)$

Hence f is injective.

Therefore $(R/J)_\sigma \cong R/\sigma^{-1}(J)$

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