Department of Mathematics

Hardy Space Over Half Plane

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Approval

The undersigned hereby certify that they have read and recommend to the school of graduate studies for acceptance of thesis entitled on Hardy Space over the upper half plane by Gezahegn Feyisa in partial fulfillment of the requirements for the Degree of Master of Science.

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ABSTRACT

The main consideration of this thesisis Hardy Space over upper half-Plane. Normed linear space, harmonic functions, sub-harmonic functions, Hardy space on the unit disc, Blaschke product and composition operators on Hardy space of a half-plane are also introduced in different perspectives.

INTRODUCTION

The theory of hardy-space has its origin in discoveries made fifty years ago by such mathematicians as G.H.Hardy, J. Riesz, V. Sirnov, G. Szego and the like. Most of this early work is concerned with the properties of individual functions of H^p class. In recent years, the development of functional analysis has stimulated new interests in the H^p classes.

This thesis report contains three chapters: its first two chapters are preliminary parts and the last part is the main body part of the carried out work. In the beginning part, we try to visualize the definitions and some basic properties of L^p-spaces. In the second chapter, we introduce about harmonic functions including mean value property, the maximum principle and the poissonintegral. Lastly in the third chapter, we introduce about sub-harmonic functions, Hardy space on the unit disc, Blaschkeproduct ,Hardy space over upper half-plane and composition operators on Hardy space of upper half-plane with the definitions and some basic properties of the topics and sub-topics. There are also lemmas, theorems and corollaries stated and proved under each topic.

CHAPTER ONE

The L^p Space

Completeness of the real numbers may be formulated by asserting that if a sequence $\{a_n\}$of real numbers for $\lim_{n,m\to\infty}|a_n - a_m| = 0$, there is a real number a for which $\lim_{n\to\infty}(a_n - a) = 0$. There is a corresponding completeness property for the Lebesgue integral. If $\{f_n\}$ is a sequence of functions in $L^p(E)$ the collection of measurable function f, measurable set E and $1\leq p <\infty$ for which $\lim_{n,m\to\infty}\int_E |f_n - f_m|^p = 0$, there is a function f belonging to $L^p(E)$ for which $\lim_{n\to\infty}\int_E |f_n - f|^p = 0$.

A collection Ƒ of functions in $L^p(E)$ is said to be dense in $L^p(E)$ provided for each g in $L^p(E)$and $\varepsilon> 0$, there is a function f belonging to Ƒ for which $\int_E |g - f|^p < \varepsilon$. The proofs of the Riesz-Fischer theorem and the denseness results are framed in the context of normed linear spaces of functions.

1.1.Normed linear space

Definition 1.1.1: Let X be a real linear space. A function $\|\cdot\|$: $X\to[0,\infty)$ is called a norm if and only if, the following are satisfied.

- **i.** Non negativity ($\|x\| \geq 0$ and $\|x\| = 0 \Leftrightarrow x = 0,\ \forall x \in X$).
- **ii.** Positive homogeneity($||\alpha x|| = |\alpha|||x||, \forall\alpha \in\mathbb{R}, \forall x \in X$)
- **iii.** The triagular inequality ($\|x + y\| \leq \|x\| + ||y||, \forall x, y \in X$).

The pairs $(X,\|\cdot\|)$ is called a normed linear space.

Examplea. $(R, |\cdot|)$ is called a normed linear space·

b. Let C[a , b] denote the linear space of continuous real valued functions on [a , b].

Define$\|\cdot\|$: C[a , b]→ $[0,\infty)$ by $\|f\| = \max_{x\in[a,b]}|f(x)|$. This defines a norm that we call the maximum norm.

Let E be a measurable set of real numbers. Define Ƒ to be the collection of all measurable extended real valued functions on E that are finite almost everywhere on E. Define two functions f and g in Ƒ to be equivalent and write $f \cong g$ provided f(x) = g(x) for almost all x∈ E. This is an equivalence relation. Therefore it induces a partition of Ƒ in to a disjoint collection of equivalence classes, which we denote by Ƒ /≅ and Ƒ / ≅ is a linear space.

For $1\leq p <\infty$ and $f$is a measurable function on E, we define

$L^p(E)= \{f\colon \int_E |f|^p \, d\mu < \infty\}$.

Example: Let E=(0,1] and f define by f(x) =$x^{\frac{-1}{3}}$ for 0<x ≤ 1. Then

$\int_0^1 |f|^2$ =$\lim_{\delta\to 0^+} \int_0^1 x^{\frac{-1}{3}}dx$ = $\lim_{\delta\to 0^+} 3x^{\frac{1}{3}}|_{\delta}^{1}$ = 3<∞.

Therefore $f \in L^2(E)$.

For two numbers a and b, $|a+b| \leq |a| + |b| \leq 2\max\{|a|, |b|\}$.

This implies $|a+b|^p \leq 2^p(|a|^p + |b|^p)$. Thus if $f, g \in L^p(E)$, then

$\int_E |f+g|^p \leq 2^p(\int_E |f|^p + \int_E |g|^p) < \infty$ and hence $f + g \in L^p(E)$

For any $f \in L^p(E)$ and α ∈R,

$\int_E |\alpha f|^p = |\alpha|^p \int_E |f|^p < \infty$ and hence $\alpha f \in L^p(E)$.

Therefore, $L^p(E)$ is a linear space.

A measurable function f is called essentially boundedif there is an M≥0, called an essencially upper bound for f, such that $|f(x)| \leq M$ for almost all x ∈ E.

We define, $L^\infty(E) = \{f: E \to \mathbb{R}: f$ is measurable and $\|f\|_\infty = ess \sup_{x\in E} |f(x)| < \infty\}$

Note that $L^\infty(E)$ is a linear space.

Example: For a function f in $L^\infty(E)$, define

$\|f\|_\infty = \inf\{M \geq 0 : m\{x: |f(x)| > M\} = 0\}$. We call $\|f\|_\infty$ the essential supremum of f and

$\|\cdot\|_\infty$ is a norm on $L^\infty(E)$.

1.2. Young's, HÖlder's and Minkowski's Inequalities.

Definition1.2.1: The conjugate of a number p∈ $(1, \infty)$ is a unique number q ∈ $(1, \infty)$ such that $\frac{1}{p} + \frac{1}{q} = 1$. The conjugate of 1 is defined to be ∞ and the conjugate of ∞ is 1.

i. Young's inequality

For 1< p <∞ q is the conjugate of p, and any positive numbers a and b,

$ab \leq \frac{1}{p}a^p + \frac{1}{q}b^q$.

Proof: The function g, define by g(x) $=\frac{1}{p}x^p + \frac{1}{q} - x$ for x>0 has a positive derivative on (1,∞) and negative derivative on (0, 1) and takes the value 0 at x = 1. Therefore the function g is non-negative on

(0,∞), that is, $x \le \frac{1}{p}x^p + \frac{1}{q}$, if x > 0.

In particular, for $x = ab^{1-q}$ gives us

$ab^{1-q} \le \frac{1}{p}a^p b^{(1-q)p} + \frac{1}{q}$

$ab^{1-q} \le \frac{1}{p}a^p b^{-q} + \frac{1}{q}$

Multiplying both sides by b^q

$ab \le \frac{1}{p}a^p + \frac{1}{q}b^q$ (since q=(q-1)p).■

ii. Hölder's Inequality

Let $1 \le p \le \infty$ *and* q *the conjugate of* p. *If* $f \in L^p(E)$ *and* $g \in L^q(E)$, *then* $fg \in L^1(E)$ *and*

$$\int_E |fg| \le ||f||_p ||g||_q.$$

Proof.case1: for p=1, q=∞

$$\int_E |f.g| \le \int_E |f|.||g||_\infty = ||g||_\infty \int_E |f| = ||f||_1 ||g||_\infty$$

Hence $\int_E |fg| \le ||f||_1 ||g||_\infty$ (since $|g(x)| \le ||g||_\infty$ for almost all x ∈ E).

Case2: For 1<p<∞

Either f=0 a.e. on E or g=0 a.e. on E, then there is nothing to prove.

Assume $||f||_p \neq o$ and $||g||_q \neq 0$.

By Young's inequality

$$\frac{|f|}{||f||_p} \cdot \frac{|g|}{||g||_q} \le \frac{1}{p}\frac{|f|^p}{||f||_p^p} + \frac{1}{q}\frac{|g|^q}{||g||_q^q}$$

$$\frac{1}{||f||_p ||g||_q}\int_E |fg| \le \frac{1}{p||f||_p^p}\int_E |f|^p + \frac{1}{q||g||_q^q}\int_E |g|^q = \frac{1}{p||f||_p^p} \cdot ||f||_p^p + \frac{1}{q||g||_q^q} \cdot ||g||_q^q$$

$$=\frac{1}{p}+\frac{1}{q}=1$$

$\frac{1}{||f||_p||g||_q}\int_E|fg| \leq 1$and hence

$\int_E |fg| \leq ||f||_p||g||_q$.

Moreover, if $f \neq 0$, the function $f^* = ||f||_p^{1-p} \cdot \mathrm{sgn}(f)|f|^{p-1}$ belongs to $L^q(E)$ and

$$\int_E f.f^* = ||f||_p \text{ and } ||f^*||_q = 1.$$

Observe that $f.f^* = ||f||_p^{1-p} \cdot |f|^p$ a.e on E

Therefore, $\int_E f.f^* = ||f||_p^{1-p}\int_E|f|^p = ||f||_p^{1-p} \cdot ||f||_p^p = ||f||p$ and

$$||f^*||_q^q = \int_E ||f||_p^{(1-p)q} \cdot |f|^{q(p-1)} = ||f||_p^{q-pq}\int_E |f|^p = ||f||_p^{-p} \cdot ||f||_p^p = 1$$

For $f \neq 0$ and $f \in L^p(E)$, f^* is called the conjugate function of f.■

The special case of HÖlder's inequality when p = q = 2 is called the Cauchy-schwarz inequality.

iii. Minkowski's Inequality

If 1≤ $p \leq \infty$ and $f, g \in L^p(E)$, then $||f+g||_p \leq ||f||_p + ||g||_p$.

Proof: For cases p = 1 and q =∞ it holds true by triangular inequality. Assume that $f + g \neq 0$. Then by HÖlder's inequality

$$||f+g||_p = \int_E (f+g)(f+g)^* = \int_E f.(\mathrm{f}+g)^* + \int_E g.(f+g)^*$$

$$\leq ||f||_p.||(f+g)^*||_q + ||g||_p.||(f+g)^*||_q$$

$$= ||f||_p.1 + ||g||_p.1 = ||f||_p + ||g||_p.$$

Hence $\|f+g\|_p \leq ||f||_p + \|g\|_p$.■

Corollary1.2.1[1]: *Let E be a measurable set and 1< p <∞. Suppose F is a family of functions in $L^p(E)$ that is bounded in $L^p(E)$ in the sense that there is a constant M for which $||f||_p$ ≤M for all f ∈F. Then the family F is uniformly integrable over E .*

Proof: Let $\varepsilon > 0$. Let A be a measurable subset of E of finite measure. Consider $L^p(A)$and $L^q(A)$, where $\frac{1}{p} + \frac{1}{q} = 1$. Define g(x)=1 ∀x ∈ A. Then g ∈ L^q(A). By HÖlder's inequality

$\int_A |f| = \int_A |f|.\, g \leq [\int_E |f|^p]^{\frac{1}{p}}.\, [\int_E |g|^q]^{\frac{1}{q}} \leq [\int_E |f|^p]^{\frac{1}{p}} [m(A)]^{\frac{1}{q}} \leq$ M $[m(A)]^{\frac{1}{q}}$

Choose $\delta = (\frac{\varepsilon}{M})^q. Then if m(A) < \delta, then for each f \epsilon F,$

$$\int_A |f| \leq M[m(A)]^{\frac{1}{q}} \leq M.\frac{\varepsilon}{M} = \varepsilon$$

Therefore the family F is uniformly integrable over E.■

Corollary1.2.2[3]: *Let E be a measurable set of finite measure and 1≤ $p_1 < p_2 \leq \infty$. Then $L^{p_2}(E) \subseteq L^{p_1}(E)$.*

Furthermore, $||f||_{p_1} \leq c||f||_{p_2}$ $for all f \epsilon L^{p_2}(E), where c = [m(E)]^{p_2 - p_1 / p_1 p_2} if p_2 < \infty and$

C= $[m(E)]^{\frac{1}{q}}$ *if* $p_2 = \infty$.

Proof: **Case1**$p_2 < \infty$

Define p =$\frac{p_2}{p_1} > 1$ and let q be the conjugate of p. Let $f \in L^{p_2}$(E). Then,

$\int_E (|f|^{p_1})^p = \int_E |f|^{p_2} < \infty$ and hence$|f|^{p_1} \epsilon L^p(E)$. Note that $\chi_E \epsilon L^q(E)$ since m(E) <∞.

By HÖlder's inequality

$\int_E |f|^{p_1} = \int_E |f|^{p_1} \chi_E \leq (\int_E |f|^{p_1 p})^{\frac{1}{p}}.(\int_E {\chi_E}^q)^{\frac{1}{q}} = (\int_E |f|^{p_2})^{\frac{p_1}{p_2}}.[m(E)]^{\frac{1}{q}}$

$= \|f\|_{p_2}^{p_1} . [m(E)]^{\frac{1}{q}}$

Taking the $\frac{1}{p_1}$ power of each side to obtain

$$\|f\|_{p_1} \leq \|f\|_{p_2} [\mathrm{m(E)}]^{(1-\frac{1}{\mathrm{p}})\frac{1}{\mathrm{p}_1}} = \|f\|_{p_2}.(m(E)^{\frac{p_2 - p_1}{p_1 p_2}} = c\|f\|_{p_2}.$$

Case2: For $p_2 = \infty$

Let $f \in L^p(E)$. Then $|f|^{p_1} \leq \|f\|_\infty^{p_1}$ for almost all x∈ E and therefore,

$\|f\|_{p_1} = \int_E |f|^{p_1})^{\frac{1}{p_1}} \leq (\int_E \|f\|_\infty^{p_1})^{\frac{1}{p_1}} = \|f\|_\infty [m(E)]^{\frac{1}{q}} = \mathrm{c}\|f\|_\infty = c\|f\|_{p_2}$.

Therefore $\|f\|_{p_1} \leq c\|f\|_{p_2}$. ■

$1.3.$ L^p is Complete: The Riesz-Fischer Theorem

Definition1.3.1: Let X be a real normed space.

1. A sequence $\{x_n\}$ in X is said to converge to x ϵ X that can be written as $x_n \to x$ in X or $lim_{x\to\infty} x_n = x in X provided that \ lim_{n\to\infty}\left||x_n - x|\right| = 0.$
2. A sequence $\{x_n\}$in X is said to be Cauchy in X if for each $\varepsilon > 0$ there is an N∈ ℕ such that $\|x_n - x_m\| < \varepsilon$ for all m,n≥ N.
3. X is said to be complete if every Cauchy sequence in X converges to a point in X. A complete normed space is called a Banach space.
4. Let $\{x_n\}$ be a sequence in X. We say the series $\sum_{n=1}^{\infty} x_n$ is summable (convergent) to x ϵ Xif the sequence $\{\sum_{i=1}^{n} x_i\}_{n=1}^{\infty}$ of partial sums conveges to x. We call x the sum of the series and write x=$\sum_{n=1}^{\infty} x_n$. We say $\sum_{n=1}^{\infty} x_n$ is absolute summable(convergent) if $\sum_{n=1}^{\infty} ||x_n||<\infty$.

Theorem1.3.1[3]: *A normed linear space X is complete if and only if every absolutely summable series is summable. Let* $\varepsilon > 0$*. Then there exists N* ϵ $\mathbb{N} such that \sum_{n=N+1}^{\infty}\|x_n\| < \varepsilon$

Proof: (⇒) Assume that X is a Banach space and let $\sum_{n=1}^{\infty} x_n$ be an absolutely summable series.

Let $\varepsilon > 0$. Then there exists N ϵ $\mathbb{N} such that \sum_{n=N+1}^{\infty}\|x_n\| < \varepsilon$ (since$\sum_{n=1}^{\infty}\|x_n\|<\infty$)

Now for n>m≥N,$\|\sum_{k=1}^{n} x_k - \sum_{k=1}^{m} x_k\| = \|\sum_{k=m+1}^{n} x_k\| \leq \sum_{k=m+1}^{n}\|x_k\| \leq \sum_{k=N+1}^{\infty}\|x_k\|<\infty$.

Therefore, $\{\sum_{k=1}^{n} x_k\}_{n=1}^{\infty}$ is a Cauchy sequence in X. Since X is complete, there is an x ϵ Xsuch that

$\sum_{n=1}^{\infty} x_n = lim_{n\to\infty} \sum_{k=1}^{n} x_k$ = x.

(⇐) Assume that every absolutely summable series in X is summable. Let $\{x_n\}_{k=1}^{\infty}$ be a Cauchy sequence in X.Then there exists a sequence $\{x_{n_k}\}_{k=1}^{\infty}$of $\{x_n\}$ such that $\|x_{n_k} - x_{n_{k+1}}\| < \frac{1}{2^k}$ for k =1,2,….Hence $\sum_{k=1}^{\infty}\|x_{n_{k+1}} - x_{n_k}\| \leq \sum \frac{1}{2^k}$ =1<∞.

Thus the series $\sum_{k=1}^{\infty}(x_{n_{k+1}} - x_{n_k})$ is absolutely summable and hence by the assumption let there is z ϵ $X such that$

z = $\sum_{k=1}^{\infty}(x_{n_{k+1}} - x_{n_k})$ = $lim\sum_{k=1}^{\infty}(x_{n_{k+1}} - x_{n_k})$=$lim_{m\to\infty}(x_{n_{m+1}} - x_{n_1})$.

Then x_{n_k} → x as k → ∞. Since$\{x_n\}$ is a Cauchy sequence, $lim_{n\to\infty} x_n$ =x.

Therefore X is complete. ■

Theorem1.3.2$(\boldsymbol{Riesz - Fischer})$**[3]**: *Let E be a measurable set and 1≤p≤ ∞.Then*$L^p(E)$ *is a Banach space.*

Proof:**Case1**:Assume that 1≤p<∞. Let $\{f_n\}$ be a sequence in $L^p(E)$ such that$\sum_{n=1}^{\infty}\|f_n\|_p$<∞.

Let M=$\sum_{n=1}^{\infty}\|f_n\|_p$. Put g_n(x)= $\sum_{i=1}^{n} f_i$(x). Then $\{g_n\}$ is an increasing sequence.

Let g(x)= $lim_{n\to\infty} g_n$(x). Then g is measurable and $\|g_n\|_p \leq \sum_{k=1}^{n}\|f_k\|_p \leq \sum_{k=1}^{\infty}\|f_k\|_p$

$\leq M^p$<∞.

Therefore, $g_n \epsilon L^p$(E) for n=1,2,3,…. Since ${g_n}^p \to g^p$, by the Monotone convergence theorem

$\int_E g^p = lim_{n\to\infty}\int_E {g_n}^p \leq M^p$<∞.

Hence, $g\epsilon L^p$(E) and hence g is finite almost everywhere. Thus g(x) =$\sum_{n=1}^{\infty}|f_n(x)|$ is finite

almost everywhere on E.

Define f(x)=$\begin{cases} 0, if g(x) = \infty \\ \sum_{n=1}^{\infty} f_n(x), if g(x) is finite \end{cases}$

For almost all x, f(x)= $\sum_{n=1}^{\infty} f_n(x). Hence f$ measurable. Since $|f|^p \leq g^p$ a.e. on E, $f\epsilon L^p(E)$.

Note that for almost all ϵ E, $|f(x) - \sum_{i=1}^{n} f_i(x)|^p \leq 2^p g^p(x)$.

Since $2^p g^p$ is integrable over E, by the Dominated convergence theorem

$lim_{n\to\infty}\int_E |f - \sum_{i=1}^{n} f_i|^p$ =0.(since$f$-$\sum_{i=1}^{n} f_i \to 0\ a.e.$). Thus

$\lim_{n\to\infty}\|f - \sum_{i=1}^{n} f_i\|_p$ =0.Therefore, f= $\sum_{k=1}^{\infty} f_k$.

Case2: Assume that p= ∞. Let $\{f_n\}$be a sequence in $L^{\infty}(E)$ such that $\sum_{n=1}^{\infty}\|f_n\|_{\infty}$<∞.

Since $|f_n(x)| \leq \|f_n\|_{\infty}$ a.e., $\sum_{n=1}^{\infty}|f_n(x)| \leq \sum_{n=1}^{\infty}\|f_n\|_{\infty}$<∞ for almost all x ϵ E.

Put g(x)=$\sum_{n=1}^{\infty}|f_n(x)|$. Then g is finite a.e. on E.

Let f(x) =$\begin{cases} \sum_{n=1}^{\infty} f_n(x), if g(x) is finite. \\ 0, if g(x) = \infty . \end{cases}$

Hence f is measurable and $|f(x)| \leq g(x) \leq M$ for almost all $x \epsilon E$.

Therefore, $f \epsilon L^{\infty}(E)$. Note that $|f(x) - \sum_{i=1}^{n} f_i(x)| \leq \sum_{i=n+1}^{\infty} |f_i(x)| \leq \sum_{i=n+1}^{\infty} \|f_i\|_{\infty}$ a.e. $x \epsilon$ E.

Given $\varepsilon > 0$ $there exist$ N ϵ $\mathbb{N}$ such that $\sum_{k=N+1}^{\infty} \|f_k\|_{\infty} < \varepsilon$. For all n≥ N,

$$\left\| f - \sum_{i=1}^{n} f_i \right\|_{\infty} \leq \sum_{k=n+1}^{\infty} \|f_n\|_{\infty} \leq \sum_{k=N+1}^{\infty} \|f_k\|_{\infty} < \varepsilon.$$

Therefore, $f = lim_{n\to\infty} \sum_{i=1}^{n} f_i$ in L^{∞}(E).■

1.4.Denseness and Separablity of L^p Space

Definition1.4.1: Let X be a normed linear space with norm $\|\cdot\|$. Given two subsets Ƒ and G of X with Ƒ⊆ G, we say that Ƒ is dense in G, provided for each function g in G and $\varepsilon > 0$, there is a function f in Ƒ for which $\|f - g\| < \varepsilon$.

Definition 1.4.2: A normed linear space X is said to be separable provided there is a countable subset that is dense in X.

Theorem1.4.1[3]:*Let E be a measurable set and 1* $\leq p \leq \infty$.*Then the subspace of simple functions in* $L^p(E)$ *is dense in* $L^p(E)$.

Proof: Let $f \epsilon L^p(E)$.

Case1: p=∞. There is a subset E_0 of E of measure zero for which f is bounded on E\$E_0$. Then there is a sequence of simple functions $\{\varphi_n\}_{n=1}^{\infty}$ on E\$E_0$ that converges uniformly on E\$E_0$ to f. Let $\epsilon > 0$. Choose N such that n≥ N and x ϵ E\$E_0$ implies $|\varphi_n(x) - f(x)| < \epsilon$.

Therefore, for n $\geq N \|\varphi_n - f\|_{\infty} \leq \epsilon$. Thus simple functions are dense in $L^{\infty}(E)$.

Case2: 1≤p $< \infty$.

Since f is measurable, there is a sequence $\{\varphi_n\}$ of simple functions on E such that

i) $\varphi_n \to f$ pointwise on E, and
ii) $|\varphi_n| \leq |f|$ on E for all n.

Since $|f|^p$ is integrable, by the Lebesgue Dominated convergence theorem

$\lim_{n\to\infty} \int_E |\varphi_n|^p = \int_E |f|^p$. This implies that $\|\varphi_n\|_p^p = \|f\|_p^p$ and this follows that

$\lim\limits_{n\to\infty} \|\varphi_n - f\|_p = 0$

Therefore, $\varphi_n \to f$ in $L^p(E)$, and the subspace of simple functions in $L^p(E)$ is dense in $L^p(E)$. ■

Corollary1.4.2[3]: *If $1 \le p < \infty$, then the subspace of step functions on [a , b] is dense in $L^p[a, b]$.*

Proof: Let g= χ_A for a measurable subset A of [a , b]. Let ε>0. There is a finite disjoint collection of open intervals, $\{I_k\}_{k=1}^n$, for which, if we define U= $\bigcup_{k=1}^n I_k$, then m(AΔU) $<\frac{\varepsilon^p}{3^p}$ Then $\chi_U = \sum_{k=1}^n \chi_{I_k}$ is a step function. Moreover, $\|\chi_A - \chi_U\|_p = [m(A\Delta U)]^{\frac{1}{p}}$.

Therefore, $\|\chi_A - \chi_U\| < \frac{\varepsilon}{3}$. Let $\varphi = \sum_{k=1}^n a_k \chi_{E_k}$ be a measurable simple function on [a , b].

WLOG assume that $a_k \neq 0$ for k=1, 2,…,n and [a , b] = $\bigcup_{k=1}^n E_k$. Then for k =1,…,nthere is a step function χ_{U_k} such that $\|\chi_{E_k} - \chi_{U_k}\| < \frac{\varepsilon^p}{n3^p|a_k|^p}$, put $\psi = \sum_{k=1}^n \alpha_k \chi_{U_k}$. Then ψ is a step function and

$\|\varphi - \psi\|_p \le \sum_{k=1}^n |\alpha_k| . ||\chi_{E_k} - \chi_{U_k}|| < \frac{1}{n} \sum_{k=1}^n \frac{\varepsilon}{3} = \frac{\varepsilon}{3}$.

Let $\varepsilon > 0$ and f $\in$ $L^p[a, b]$. Since the simple function is dense in $L^p[a, b]$, there is a simple function $\varphi \in L^p[a, b]$such that $\|f - \varphi\|_p < \frac{\varepsilon}{2}$.

It follows from the facts that there is a simple function S on [a , b] such that $\|\varphi - S\|$p < ε/2.

Therefore, $\|f - s\|$p ≤ $\|f - \varphi\|$p + $\|\varphi - s\|$p < ε/2 + ε/2 =ε.■

Theorem1.4.2[3]: *Let E be a measurable set and $1 \le p < \infty$. Then L^p(E) is separable.*

Proof: Let S[a , b] be the collection of step function on [a ,b]. Define S'[a , b] to be sub-collection of S[a , b] comprising step functions on [a ,b] that take rational values and for which there is a Partition P=$\{x_0, \dots, x_n\}$ of [a , b] with constant on (x_{k-1}, x_k), for 1≤ k≤ n and x_k rational for 1≤ k ≤ n-1.

Since the collection of finite sequences from rational is countable, S'[a , b] is countable. Let ε>0 and φ ∈S[a , b]. Then $\varphi \in \sum \alpha_k \chi_{(x_{k-1}, x_k)}$ for partition p=$\{x_0, \dots, x_n\}$ of [a , b]. Choose rational $r_1, r_2, \dots, r_m, q_1, q_2, \dots, q_{m-1}$ such that for 1≤ k ≤ m-1

$$|\alpha_k - r_k| < \frac{\sqrt{\varepsilon}}{2m} \text{ and } |x_k - q_k| < \sqrt{\varepsilon}.$$

Then ψ= $\sum_{i=1}^m r_k \chi_{(q_{i-1} - q_i)}$, where $q_0 = a and q_m = b$, belongs to S'[a , b]. Then

$$\|\varphi - \psi\|_p \le \sum_{i=1}^m |\alpha_k - r_k| \chi_{(q_{i-1} - q_i)} (|q_{k-1} - x_{k-1}| + |q_k - x_k|) < \sum_{k=1}^m \frac{\sqrt{\varepsilon}}{2m} (\sqrt{\varepsilon} + \sqrt{\varepsilon}) = \varepsilon.$$

Therefore, S'[a, b] is dense in S[a ,b].

Since S[a , b] is dense in L^p[a ,b], S'[a , b] is dense in L^p[a , b]. For each n ∈ N, define f_n to be the functions on ℝthat vanish outside [-n, n] and whose restrictions to [-n, n] belongs to S'[-n , n].

Define f=$\cup_{n=1}^{\infty} f_n$. Then f is a countable collection of functions in $L^p(\mathbb{R})$. By the Monotone convergence theorem,

$lim_{n\to\infty} \int_{[-n,n]} |f|^p = \int_{\mathbb{R}} |f|^p$ for all $f \in L^p(\mathbb{R})$.

Let ε> 0 and ∈ $L^p(\mathbb{R})$. Then there exists $s_n \in S^{'}$[-n, n] such that

$\left\|S_n - f\chi_{[-n,n]}\right\|_p < \frac{\epsilon}{2}$.

There is an N ∈ℕ such that

$\left\|f\chi_{[-n,n]} - f\right\|_p < \frac{\epsilon}{2}$.

Thus $\|S_N - f\|_p \le \left\|S_N - f\chi_{[-N,N]}\right\|_p + \left\|f\chi_{[-N,N]} - f\right\|_p < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$.

Therefore, f is a countable collection of function that is dense in $L^p(\mathbb{R})$. Finally, let E be a generating measurable set. Then $f_{/E} = \{\varphi_{/E} : \varphi \in f\}$ is a countable dense subset of $L^p(E)$ and therefore $L^p(E)$ is separable.■

Theorem1.4.3[3]: *Let E be measurable set and 1 ≤ $p < \infty$. Then $L^\infty(E)$ is not separable.*

Proof: Consider the case when E =[a , b]. Suppose there is a countable set $\{f_n\}_{n=1}^{\infty}$ that is dense in L^∞[a, b]. For each $x \in$ [a, b], select $\eta(x) \in \mathbb{N}$ for which $\left\|\chi_{[a,x]} - f_{\eta(x)}\right\|_\infty < \frac{1}{2}$.

Observe that if a ≤ $x_1 < x_2 \le b$, then

$\left\|\chi_{[a,x_1]} - \chi_{[a,x_2]}\right\|_\infty$=1.

Therefore η is one-to-one mapping of [a, b] onto a set of natural number. This is a contradiction

Because, [a , b] is uncountable and the set of natural numbers is countable. Thus it must be the case that $L^\infty[a , b]$ is not separable.■

1.5. The Riesz Representation for the dual of L^p

Definition1.5.1: Let X be a linear space. A function f: X→ $\mathbb{R}$ is said to be a linear functional if for all x, y∈ X and for all $\alpha, \beta \in \mathbb{R}$

$f(\alpha$x+βy) = $\alpha f(x) + \beta f(y)$.

Example : Let E be a measurable set, $1 \leq p < \infty$, q the conjugate of p, and g ∈ L^q(E). Define the functional T on L^q(E) by T(f)= $\int_E g.f$ for all f ∈ $L^p(E)$.

Definition1.5.2: Let X be a normed linear space.

i. A linear functional T on X is said to be bounded if there is an M≥ 0 for which |T(x)|≤ M‖ x‖ for all x ∈ X. The norm of T, $\|T\|_*$, is defined by $\|T\|_* = \sup\left\{\frac{|T(x)|}{\|x\|} : x \in X, x \neq 0\right\}$.

ii. The collection bounded linear functional on X is a linear space on which $\|\cdot\|_*$ is a norm. This normed linear space is called the dual space of X and denoted by X^*.

Proposition1.5.1[3]: *Let E be a measurable set, 1 ≤ $p < \infty$, q the conjugate of p, and $g \in L^q(E)$. Define the functional T on $L^p(E)$ by T(f) =$\int_E g.f$ for all $f \in L^p(E\ )$. Then T is a bounded linear functional on $L^p(E)$ and $\|T\|_* = \|g\|_q$.*

Proof: By HÖlder's inequality

$|T(f)|$=| $\int_E g.f| \leq \int_E |g.f| \leq \|g\|_q \|f\|_p \forall f \in L^p(E)$. Hence T is bounded linear functional on $L^p(E)$ and $\|T\|_* \leq \|g\|_q$. Note that $g^* = \|g\|_q^{q-1}$.sgn(g).$|g|^{q-1}$belongs to $L^p(E)$.

Hence $|T(g^*)| = \left|\int_E g.g^*\right| = \|g\|_q \,.\, \|g^*\|_p = \|g\|_q$

Therefore, $\|T\|_* = \|g\|_q$.■

Lemma1.5.1[3]: *Let E be a measurable set and 1 ≤ $p < \infty$.Suppose the function g is integrable over E and there is an M ≥ 0 for which $\left|\int_E g.\varphi\right| \leq M\|\varphi\|_p$ for every simple function φ in L^p(E). Then $g \in L^p(E)$, where q is the conjugate of p. Moreover$\|g\|_q \leq M$.*

Proof: Assume 1<p<∞ and let $\{\varphi_n\}$be an increasing sequence of simple functions which converges to $|g|^q$. Set $\psi_n = {\varphi^{\frac{1}{p}}}_n$sgn(g). Then ψ_n is a simple function, and therefore, $\int_E \varphi_n \leq \int_E \psi_n g \leq$ M$\|\psi\|_p$ = M$(\int_E \varphi_n)^{\frac{1}{p}}$ and hence $\int_E \varphi_n \leq M^q$.

By the Monotone convergence theorem, $\int_E |g|^q \leq M^q < \infty$.

Therefore $g \in L^p(E)$ and $\|g\|_q \leq$ M.

Consider the case p=1 and q=∞. Let A= $\{x: |g(x)| > M\}$.

Put φ = M$\chi_A$sgn(g). By hypothesis $\int_E g\varphi \leq M\|\varphi\|_1$.

Thus $\int_E M|g| \leq$ M$\int_A M = M^2 m(A)$.

$0 \leq \int_A M(|g| - M) \leq 0$ or $\int_A M(|g| - M) = 0$.

Since $|g| - M > 0$ on A, it must be the case that m(A) =0.■

Lemma1.5.2[3]: *Let $1 \leq p < \infty$, q is the conjugate of p, $g \in L^q[a, b]$ and T is a bounded linear function on $L^p[a, b]$ for which T(f) =$\int_{[a,b]} g.f$ for all $f \in L^p[a, b]$. Then g is unique.*

Proof: Let $g_1 \in L^q[a$, b] such that T(f) =$\int_a^b g_1 f, \forall f \in L^p[a$, b].

Let $[x_1, x_2] \subseteq [a$, b] and $\varphi = \chi_{[x_1, x_2]}$. Then $\int_a^b (g - g_1)\, \chi_{[x_1, x_2]} = 0 \Rightarrow \int_{x_1}^{x_2} (g - g_1) = 0$.

Therefore, $g - g_1 = 0$ almost everywhere on [a , b].

This implies $g = g_1$ almost everywhere on [a , b]. Hence g is unique.■

CHAPTER TWO

Harmonic and Subharmonic Functions

Let G be domain (connected open set) in the set of complex numbers$\mathbb{C}$. We shall consider complex valued functions of z = $x+iy \in G \subseteq \mathbb{C}$. A complex valued function f is called holomorphic on a domain G if it is complex differentiable in a neigbourhood of any point in G. Every holomorphic function is completely determined on a domain by its values in a neighbourhood of a single point. A similar property holds for harmonic functions.

2.1. Definition and examples of harmonic function

Definition2.1.1:A real valued function u on an open connected subset G of$\mathbb{C}$ is said to be harmonic if it has continuous second partial derivatives and satisfies the partial differential equation,

Δu =$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$

which is knows as Laplace's equation.

Example: Let $f: \mathbb{C} \to \mathbb{R}$ by $f(x+iy) = 3xy^2 - x^3$. Then $f$has continuous second partial derivatives,

$\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2}$ = -6x + 6x = 0.

Therefore f is harmonic function on $\mathbb{C}$.

Theorem2.1.1[4]: *Let U be an open disc with center (a , b). Suppose f and g be differentiable on U and$\frac{\partial f(z)}{\partial y} = \frac{\partial g(z)}{\partial x}$ for all z. Then there is a function h which is in C^2(U) such that*

$\frac{\partial h}{\partial x} = f \; and \; \frac{\partial h}{\partial y} = g.$

Proof: For each z = x +iy, define h(x , y)=$\int_a^x f(t,b)dt + \int_b^y g(x,s)ds$. Then by fundamental theorem of calculus:

$\frac{\partial h(x,y)}{\partial y} = g(x,y) \; and \, \frac{\partial h(x,y)}{\partial x} = f(x,b)$ ... ...(*)

Since g$\in$ C', differentiation under the integral sign is possible. Thus

$$\frac{\partial}{\partial x}\int_b^y g(x,s)\, ds = \int_b^y \frac{\partial}{\partial x} g(x,s) ds = \int_b^y \frac{\partial}{\partial y} f(x,s) ds = f(x,y) - f(x,b)$$

Now ,$\frac{\partial h}{\partial x} = \frac{\partial}{\partial x}\int_b^x f(t,b)dt + \frac{\partial}{\partial x}\int_b^y g(x,s)ds = f(x,b) + f(x,y) - f(x,b)$

$=f(x,y)$ … … … … … … … … … … … … … … (∗∗)

From (*) and (**),$\frac{\partial h}{\partial x} = f and \frac{\partial h}{\partial y} = g but f, g \epsilon$C' and this implies that h ∈ C^2.

Hence h is the required function. ■

Corollary2.1.1[4]: *Let u be harmonic on unit disc D. Then there is an analytic h on D such that Reh = u.*

Proof: Let$f = \frac{\partial u}{\partial y} and g = \frac{-\partial u}{\partial x} \Rightarrow \frac{\partial f}{\partial y} = \frac{\partial^2 u}{\partial y^2} and \frac{\partial g}{\partial x} = -\frac{\partial^2 u}{\partial x^2}$

Since u is harmonic function$\frac{\partial^2 u}{\partial x^2}$+$\frac{\partial^2 u}{\partial y^2}$= 0, for this $\frac{\partial f}{\partial y} = \frac{\partial g}{\partial x}$.

Then by the theorem2.1.1 above there exist v∈ C^2 such that $\frac{\partial v}{\partial x} = f(x,y) and \frac{\partial v}{\partial y} = g(x,y)$ which is equivalent to $\frac{\partial v}{\partial x} = \frac{\partial u}{\partial y}$ and$\frac{\partial v}{\partial y} = \frac{-\partial u}{\partial x}$ andh(z)= u(z) + iv(z) satisfies the Cauchy Riemann equation on D. Therefore u is the real part of h which is analytic. The function v is called a harmonic conjugate for u on D. Thus if u is a harmonic function on D we can find an analytic function h such that u is the real part of h. ■

2.2. Mean Value Property and Maximum Principle

Definition2.2.1:A continuous functionu**=G** → ℝ is said to have the mean value property (MVP) whenever$\bar{B}(a$, r) ⊆ G,then

$$u(a) = \frac{1}{2\pi}\int_0^{2\pi} \mathrm{u}(a + re^{i\theta})d\theta \ .$$

This fact has a number of important consequences. One of these is a property of harmonic that is analogues to the Cauchy integral formula. It allows us to ascertain the value of harmonic function u at the center of a disc from its value on the boundary.

Theorem2.2.1[1](The mean value theorem)

Suppose that u : G → ℝ is harmonic on G ⊆ℂ and that $\bar{B}$(a,r)⊆ Gfor some r > 0 then,

$$u(a)=\frac{1}{2\pi}\int_0^{2\pi} u(a + re^{i\theta})d\theta$$

Proof: Let f be analytic function on $\bar{B}(a$,r) such that Ref =u by Cauchy integral formula

$f(a) = \frac{1}{2\pi i}\int_\gamma \frac{f(\varepsilon)}{\varepsilon - a} d\varepsilon, whenever\, \gamma$is the circle with |z- a|=r.

Forε= a +$re^{i\theta}$, $then$

$$f(a) = \frac{1}{2\pi i}\int_o^{2\pi} \frac{f(a+ire^{i\theta})}{a+re^{i\theta}-a} ire^{i\theta} \mathrm{d}\theta = \frac{1}{2\pi}\int_0^{2\pi} f(a+re^{i\theta})d\theta$$

= u(a) + iv(a)

$$= \frac{1}{2\pi}\int \mathrm{u}(\ a+re^{i\theta})d\theta + i\frac{1}{2\pi}\int \mathrm{v}(\ a+re^{i\theta})d\theta$$

Therefore u(a) $= \frac{1}{2\pi}\int u(\ a+re^{i\theta})\ d\theta.$■

Theorem2.2.2[1](Maximum Principle)

Let G be an open connected set and suppose that u is a continuous real valued function in G with MVP. If there is a point G such that u(a) ≥ u(z) for all z in G, then u is a constant function. i.e.u(z) = u(a) for all z in G

Proof: Let A={$z \epsilon G: u(a) = u(z)$}.

Claim A = G. From continuity of u,the set A is closed subset of G as

A= u$'$[(-∞, u(a)) ∪ (u(a), ∞)]

Let z_0 be in G. Since G is open set we can choose r>0 such that $\bar{B}(z_0$,r) is contained in G. Suppose there is a point b in B(z_0,r) such that u(b)≠ u(a) $then$ u(b) < u(a). This implies that by continuity taking $\varepsilon = \mathrm{u}(a) - \mathrm{u(b)}$ we get $\mathrm{u(z)} < u(a) = \mathrm{u}(\mathrm{z}_0)$ for all in neighbor hood of b. In particular if ρ=|z_0−b| and b=z_0 +$\rho e^{i\beta}$, $0 \le \beta < 2\pi$, then there is a proper interval I of [0,2π] such thatβ in I and $\mathrm{u}(\mathrm{z}_0 + \rho e^{i\beta}) < \mathrm{u}(\mathrm{z}_0)$ for all θ in I.
Hence by the MVP,
$\mathrm{u}(\mathrm{z}_0) = \frac{1}{2\pi}\int_0^{2\pi} \mathrm{u}(\mathrm{z}_0 + \rho e^{i\theta})d\theta < u(\mathrm{z}_0)$ which is a contradition. So $\mathrm{B}(\mathrm{z}_0, \mathrm{r}) \subseteq$ A and A is alsoopen. By connectedness of G, A=G.

According to MVP, if u(z) is analytic on a domain D, then $|f(z)|$ can not have a maximum any where in D unless $f(z)$ is constant and if $f(z)$ be analytic in a bounded region D and let $f(z)$ be continuous in the closed region $\bar{D}$ then $|f(\mathrm{z})|$assumes its maximum on the boundary of the region.■

2.3.Poisson Integrals

Definition2.3.1:P(r,t)=$\sum_{n=-\infty}^{\infty} r^{|n|} e^{int}$, $0 \le r < 1$and t in $\mathbb{R}$ is called the poisson kernel.

To give the definition of the Poisson integrals let us begin the discussion by stating and proving the following theorem.

Theorem2.3.1[4]: *For each real number r, 0* $\le r < 1$, *the two sided infinite series P(r, t) =* $\sum_{n=-\infty}^{\infty} r^{|n|} e^{int}$ *converges uniformly for t in* $\mathbb{R}$. *That is the sequence* S_N *is given by*

$S_N(t)$ $=\sum_{n=-N}^{N} r^{|n|} e^{int}$ *converges uniformly on* $\mathbb{R}$

Proof: To see this we write

$S_N(t) = 1+2 \sum_{n=1}^{N} r^{|n|}(e^{int} + e^{-int}) = 1+2\sum_{n=1}^{N} r^n \, cosnt$

But $|s_N(t)| = 1+2\sum_{n=1}^{N} 2r^{|n|}$ as $|r^n(e^{int} + e^{-int})| \le 2r^n$

Therefore this shows that it converges uniformly. But $\lim S_N(t) = 1+2 \sum r^n \left(e^{int} + e^{-int}\right)$which converge uniformly on $\mathbb{R}$.■

Note1. $P(r,t) = \sum_{n=1}^{\infty} r^{|n|} (e^{int} + e^{int}) = 1 + \sum_{n=1}^{\infty} 2r^n cosnt$ (since $\frac{e^{int}+e^{-int}}{2} = cosnt$)

2. Let z =$re^{i\theta}$. Then $P(r,\theta\text{-}t) = \sum_{n=-\infty}^{\infty} r^{|n|} e^{in(\theta-t)}$.

$$= \mathrm{Re}[\frac{e^{it}+re^{i\theta}}{e^{i\theta}-re^{i\theta}}] = \frac{1-r^2}{1-2rcos(\theta-t)+r^2}$$

But $P(r,\theta - t) = 1+\sum_{n=1}^{\infty} r^n \left[e^{in(\theta-t)} + e^{-in(\theta-t)}\right]$

$$= 1+\sum_{n=1}^{\infty} r^n e^{in(\theta-t)} + \sum_{n=1}^{\infty} r^n e^{-in(\theta-t)}$$

$$= 1 +\sum_{n=1}^{\infty}(re^{i(\theta-t)})^n + \sum_{n=1}^{\infty}(re^{-i(\theta-t)})^n$$

$= 1 + \frac{re^{i(\theta-t)}}{1-re^{i(\theta-t)}} + \frac{re^{-i(\theta-t)}}{1-re^{-i(\theta-t)}}$

$= \frac{1-r^2}{1-2rcos(\theta-t)+r^2} = P(r,\theta - t) = p_{z_0}(\theta - t).$

For each fixed t in$\mathbb{R}$,$f(z) = \frac{e^{it}+z}{e^{it}-z}$ is analytic function which satisfies Cauchy Reimman equation .

Notice that P(r,t) >0 and$\frac{1}{2\pi}\int_0^{2\pi} p(r,t)dt = 1$

Indeed $\frac{1}{2\pi}\int_0^{2\pi} p(r,t)dt = \frac{1}{2\pi}\int_0^{2\pi} dt + \frac{1}{2\pi}\int_0^{2\pi} 2\cos nt\, dt = 1 + 0 = 1$

Definition2.3.2: Let f be continuous on ∂D={z: $|z| = 1$}, $then$

P$f(z)$ = $\frac{1}{2\pi}\int_0^{2\pi} p(r,\theta - t)f(e^{it})$ dt is said to be the poisson integral of f.

From the fact that P(r , $\theta - t) = Ref(z)$and the analyticity of f(z) = $\frac{e^{it}+z}{e^{it}-z}$, we can conclude that Pf(z) is harmonic function.

2.4. Poisson integrals in disc and upper half plane

By the mean value theorem, if u is harmonic on B(a ,r), then u(a) is the mean value of u on the circle $|z - a| = r$.

Now, if a=0 and r=1 we get a unit disc D = {z:|z|=1} = B(0,1)

Hence if u is continuous on the closed disc D and is harmonic on the open disc D,$\frac{\partial^2 u}{\partial X^2}$+$\frac{\partial^2 u}{\partial y^2}$= 0 which satisfies Laplace's equation and then,

$$u(0)=\frac{\mathbf{1}}{\mathbf{2\pi}}\int u(e^{i\theta})d\theta.$$

Theorem2.4.1[4]:*suppose that u is continuous on the closed unit disc $\overline{D}$ and is harmonic on the open unit disc D. Let $z_o=re^{i\theta_0}$ and then $u(z_o)$ = $\frac{1}{2\pi}\int_0^{2\pi} p_{z_0}(\theta)u(e^{i\theta})d\theta$, where $P_{z_0}(\theta)$is the poisson kernel z_o.*

Proof: consider the map τ(z)=$\frac{z+Z_0}{1+\bar{z}z}$: D→ $\overline{D}$ which is mobius transformation and τ^{-1}(z) exists which we will be given by

τ^{-1} (z) =$\frac{Z+Zo}{1+\bar{z}Z}$,for z on ∂D we get

|τ(z)| = 1. i.e. if z=$re^{i\theta}$then $\left|\frac{e^{i\theta}-z_0}{1-\overline{z_0}e^{i\theta}}\right|$= 1.This implies that τ(z)$\epsilon\partial$D. Therefore the unit circle ∂D is invariant under τ and hance τ(z) =$e^{i\varphi}$for some φ in ℝ .Then

$e^{i\varphi}$=$\frac{e^{i\theta}-z_0}{1-\bar{z}_0e^{i\theta}}$. The differentiation of τ(z) gives us

$\frac{d\tau}{d\theta}$=$\frac{1-|zo|^2}{|e^{i\theta}-z_0|^2} = \frac{1-r^2}{1-2r\cos(\theta-\theta_0)+r^2} = P_{z_0}(\theta-\theta_0)$ = p(r, $\theta - \theta_0$)
which is known as poisson kernel of z_0 in D.

Now consider τ and τ^{-1}which are continuous on $\overline{D}$ and harmonic on D. The composition of uand τ^{-1}. i.e. u∘ τ^{-1} is continuous on $\overline{D}$ and harmonic on D. Then by MVP,

u∘ $\tau^{-1}(0)=u(z_0)=\frac{1}{2\pi}\int(u\circ\tau^{-1})(e^{i\varphi})d\varphi$

$=\frac{\mathbf{1}}{\mathbf{2\pi}}\int u(\tau^{-1}(e^{i\varphi}))\,d\varphi=\frac{1}{2\pi}\int u(e^{i\theta})\,P_{Z0}(\theta)d\theta$

$u(z_0)=\frac{1}{2\pi}\int_0^{2\pi}p_{z_0}(\theta)u(e^{i\theta})d\theta$ which is known as poisson integral formula.

From this we can conclude that if u is continuous on $\overline{D}$ and harmonic on D, for each Z_0 in D, $u(z_0)$ can be represented by

$u(z_0)=\frac{1}{2\pi}\int u(e^{i\theta})P_{zo}(\theta)d\theta$

Suppose the map z: D→H be given by $z(\omega)=\frac{i(1-\omega)}{1+\omega}$.Clearly z is a conformal map between D and H such that z(i)= 1, z(1) = 0, z (-i) = -1 and z(-1) =∞. If ω on ∂D and $\omega\neq-1, then z(\omega)$ = t in ℝ, from the disc, we get $P_z(\theta)=P_z(e^{i\theta})=Re[\frac{e^{i\theta}+z}{e^{i\theta}-z}]$

Hence, $P_w(e^{i\theta})=Re[\frac{e^{i\theta}+w}{e^{i\theta}-w}]$.Let $z_0=z(w_0)$, $z_0=x_0+iy_0$. Under the inverse map of z above, i.e. $z^{-1}:H\to D: z^{-1}(z_0)=\frac{i-z_0}{i+z_0}=w_0$, each t in ℝ maps to ∂D.

For t in ℝ we have, $z^{-1}(t)=\frac{i-t}{i+t}=e^{i\theta}$, for some θ.

$P_{wo}(e^{i\theta})=Re\left[\frac{e^{i\theta}+w_0}{e^{i\theta}-w_0}\right]=P_{\omega o}(z^{-1}(t))$

$=\mathrm{Re}\left[\frac{\frac{i-t}{i+t}+\frac{i-z_0}{i+z_0}}{\frac{i-t}{i+t}-\frac{i-z_0}{i+z_0}}\right]=\frac{y_o(1+r^2)}{(X_o-t)^2+y_0{}^2}$ and

Hence normalizing the above equation yields

$$p_{\omega o}(e^{i\theta})\frac{d\theta}{dt}=\frac{y_o(1+r^2)}{\pi(x_o-t)^2+y_0{}^2(1+t^2)}dt=\frac{y_o}{\pi(x_o-t)^2+y_0{}^2}dt$$

From this we get,

$$p_{\omega o}(e^{i\theta})\frac{d\theta}{dt}=\frac{1}{\pi}\frac{y_o(1+r^2)}{(x_o-t)^2+y_0{}^2}$$ which is called the poisson kernel of z_0 in the upper half plane.

Based on above information if u is continuous on H∪{∞} and harmonic on H,then u∘z^{-1} is continuous on the closed unit disc and harmonic on the open disk and hence

$u \circ \tau^{-1}(\omega) = \frac{1}{2\pi}\int_0^{2\pi}(u(\tau^{-1}(e^{i\theta}))P_\omega(e^{i\theta})d\theta$

$= \int_{-\infty}^{\infty} u(t)\frac{y}{(x-t)^2+y^2}dt = P_z(t)$ Which is called the poisson integral in the upper half plane.

When t in $\mathbb{R}$ is fixed the poisson kernel for the upper half plane is harmonic function of z because

$P_z(t) = \frac{1}{\pi}\lim\left(\frac{1}{t-z}\right) = P_\gamma(t)$.

From its defining formula we see that $P_z(t) \leq \frac{C_z}{1+t^2}$ where C_z is a constant depending on Z.

Consequently if $1 \leq P \leq \infty$ then $P_z(t) \in L^q(\mathbb{R})$ and the function $u(z) = \int P_z(t)f(t)dt$ is harmonic on H Whenever $f \in L^p(\mathbb{R})$, $1 \leq P \leq \infty$, moreover, since $P_z(t)$ is continuos function, the above integral will still produce a harmonic function u(z) if $f(t)dt$ is replaced by finite measure $d\mu(t)$ or by a positive measure $d\mu(t)$ such that

$\int \frac{1}{1+t_2} d\mu(t) < \infty$. So that $\int P_z(t)d\mu(t)$ converges.

Lemma2.4.1[4]: *Let* $\in L^p$, $1 \leq p \leq \infty$ *and* f *is continuous at* x_0 *and* $u(x,y) = \int p_\gamma(t)f(x-t)dt$, *then*

$lim_{(x,y)\to x_0} u(x,y) = f(x_0)$.

Proof: We are going to show that given $\boldsymbol{\varepsilon} > 0$ *there exist* $\delta > 0$ *such* that

$|x - x_0| < \delta \Rightarrow |u(x,y) - f(x_0)| < \varepsilon$. Indeed since f is continuous at x_0 there exist $\delta > 0$ such that $|x - x_0| < \delta \Rightarrow |f(x) - f(x_0)| < \varepsilon$, *if* $x - t \epsilon (x_0 - \delta, x_0 + \delta)$, *then* $|(x-t) - x_0| < \delta$

$\Rightarrow |f(x - t) - f(x_0)| < \varepsilon$, but

$|u(x,y) - f(x_0)| = \int_{|t|<\delta} p_y(t)|f(x-t) - f(x_0)|dt + \int_{|t|\geq\delta} p_y(t)|f(x-t) - f(x_0)|dt$

Hence for the fixed δ from the above

$\int_{|t|\geq\delta} p_y(t)\left|f(x-t) - f(x_0)\right|dt \to 0$ as $y \to 0$ and with δ small and $|x - x_0|$ small

$\int P_y(t)|f(x-t) - f(x_0)|dt$ is small. Therefore the limit converges to $f(x_0)$.■

Lemma2.4.2[4]: *If u(z) is harmonic on H and bounded and continuous on $\overline{H}$, then*

u(z) $=\int P_y(x-t)\, u(t)dt.$

Proof: If u is continuous to ∞, then it is a consequence of the definition. But u may not be continuous at ∞. Let u(z) = v(z) $-\ \int \mathrm{P_y(x-t)\ u(t)dt}$,

Claim u(z)= 0. Since u is harmonic on H and continuous on$\overline{\mathrm{H}}$, and by the above Lemma , for x in ℝ, u(z)=0.

Set v(z)=$\begin{cases} u(z), & if y \geq 0 \\ -u(z), & if y < o \end{cases}$for z=x+iy

v is bounded harmonic function on the complex plane and then by Lioville theorem which says a bounded analytic function on the complex plane is constant, v(z) = v(0) = u(0) = 0.

Hence v(z)= 0 for all z in ℂ. This implies that

u(z)=$\int \mathrm{P_y(x-t)\ u(t)dt}$.■

2.5. Subharmonic functions

There are two standard approaches to define subharmonic functions. One is to require that $u_{xx}+u_{yy} \geq 0$ in the sense of distribution theory, matching the characteristics property of harmonic functions while the other one is by submean property, matching the mean value property of harmonic functions.

Recall that harmonic functions are continuous. However, subharmonic are not required to be continuous. A function f on a metric space X is known as continuous at x if for any given positive ε we can find an open ball B(x, δ) of radius δ about x, B(x, δ) = {y: d(x, y) <δ} such that f(x) $-\varepsilon \leq f(y) \leq f(x)+\varepsilon$ for y ∈ B(x, δ). If this two-sided inequality becomes only one-sided, then the function is said to be semi-continuous. The upper semi-continuous of such functions is introduced below.

Definition2.5.1: A function f: G →[-∞ , ∞) is called upper semi-continuous(u.s.c) at x ∈ G if for any given ε we can find a positive δ such that

$f(y) \leq f(x)+\varepsilon$for$y \in B(x,\delta)$⊆ G ⊆ℂ.

This definition is equivalent to the conditions given x_0 in G and $f(x_0)$ < k there exist$\delta > 0$ such that if $|x$- $x_0|$ <δ and x in G then $f(x) \leq$ k. Indeed if f is upper sem-continuous and f(x_0) < k, then the set

A = $\{x: x \in G, f(x) < k\}$ is open and $x_0 \in A$. Hence there isδ> 0 such that the ball B(x_0, δ) $\subseteq$ A. i.e.

If $|x - x_0|$<δ and x in G then $f(x) \leq$ k.

Example:$f(x) = \sin\left(\frac{1}{x}\right)$= $\begin{cases} \sin\left(\frac{1}{x}\right), & x \neq 0 \\ -\infty & , x = 0. \end{cases}$

Theorem 2.5.1[1]:*Let f be upper semi-continuous. Then f is bounded above on compact sets and attains its upper bound in every compact set.*

Proof: Based on the Bolzano-Weierstrass theorem and is the same as continuous functions. Let

M = $sup_{z\in K} f(x)$, where M may be ∞. By the definition of sup, there exists a sequence (x_n) in K such that $f(x_n)$ → M as n→ ∞. If K is compact then (x_n) contains a subsequence converging to a point x∈ K. It follows from the definition that M ≤ $f(x)$, hence M is finite. Also since$f(x)$ ≤ M on K, one can conclude that $f(x)$= M.■

We can observe that if f is finite then f is continuous on G if and only if f and – f are u.s.c. We have seen that harmonic functions can be defined in terms of the MVP if we replace equality by inequality, we obtain subharmonic functions in this relation.

Definition2.5.2: Let G be a region (open and connected) and let u: G→ ℝbe a continuous function. u is subharmonic function if whenever $\bar{B}(a, r) \subseteq$ G,

$$u(a) \leq \frac{1}{2\pi}\int_0^{2\pi} \varphi(a + re^{i\theta})d\theta.$$

Example: Every harmonic function is subharmonic function.

Theorem2.5.2[1]: If f is analytic function on G, then,

u(z) =$\begin{cases} log|f(z)| iff(z) \neq 0 \\ -\infty \quad iff(z) = 0 \end{cases}$ is subharmonic.

Proof: It is immediate that u is u.s.c and

-∞ = u$(z_0) \leq \frac{1}{2\pi}\int u(z_0 + pr^{i\theta})\, d$θif$f$(z)=0 and

u(z) = Re log f(z) = log$|f(z)|$ is harmonic if f(z) ≠ 0.

Thus u is sub-harmonic in G.■

Theorem 2.5.3 (Maximum Principle)[1]: Let u be subharmonic on a domain G in ℂ. Then

(a) If u attains a global maximum in G then u is constant.
(b) If G is bounded and $\lim_{z\to w} sup u(z) \leq 0$ for all w ∈ ∂G, then u ≤ 0 on G.

Proof: Suppose that u attains a maximum value M on D. Define

A = {z ∈G : u(z) < M}, K = {z ∈ G : u(z) = M}.

The set A is open because u is upper semi-continuous. The set K is open too because of the local submean property for subharmonic functions (any sufficiently small circle about z ∈ K must be lie in K, for if not then there is a circle that intersects with A, and since A is open the intersection will contain a segment of finite length hence the mean value integral will be < $2\pi M$ in violation of the local submean property). By assumption, A and K partition G. Since G is connected, one of the two sets must be empty. The set K is non-empty by assumption, therefore A is empty, and part (a) is proved.

To prove part (b), let us extend u to the boundary of G by u(w) = $\lim_{z\to w} sup u(z)$, for w ∈ ∂D. Then u is upper sem-continuous on $\bar{G}$. Since $\bar{G}$ is compact by Theorem 3.1.1 above u attains a maximum on $\bar{G}$. If the maximum point is in D, then u = 0 on $\bar{G}$ by part (a). If the maximum point is at the boundary of G then u ≤ 0 on $\bar{G}$.■

<u>**Remarks**</u>:

(i) If u is upper semi-continuous on a set G, then there is a decreasing sequence $\{f_n\}$ of continuous function on G such that $f_n(x) \to f$(x) as n→ ∞.

(ii) If f is upper semi-continuous on a compact set G, then f attain its maximum on G.

Theorem2.5.4[4]:*Let v be a subharmonic function in G and let $\varphi(t)$ be increasing convex function on [-∞, ∞) Continuous at t = -∞ . Then $\varphi \circ v$ is subharmonic in G.*

Proof: Since every convex function is continuous on R, φ is continuous on [-∞, ∞). It follows that $\varphi \circ v$ is u.s.c on G, if $z_0 \in$ G and if r <r(z_0), then because φ is increasing,

$$\varphi(v(z_0)) \leq \frac{1}{2\pi}\int (\varphi o v)(z_0 + re^{i\theta})\, d\theta.$$

Therefore $\varphi \circ v$ is subharmonic in G.■

Corollary2.5.1[4]: *If u is subharmonic in G, then $e^{\lambda u}$ for $\lambda > 0$ is subharmonic function.*

Proof: Since e^x is convex and increasing in (-∞ , ∞), and for $\lambda > 0$, λu is subharmonic, then by the above theorem $e^{\lambda u}$ is subharmonic in G.

Corollary2.5.2[4]: *If f is analytic in G, then $|f|^{\lambda}$for $\lambda > 0$ is subharmonic.*

Proof: Since f is analytic u = log$|f|$ is subharmonic. Then by the above corollary $|f|^{\lambda}$ = $e^{\lambda u}$ is subharmonic in G. The semi-continuity of u in the definition of subharmonic guarantees that u is measurable and bounded above on any compact subset of G.

Therefore, the integral in the above definition either converges or diverges to $-\infty$.■

Definition2.5.3: The subharmonic v(z) in G is said to have a harmonic majorant if there is a harmonic function u(z) such that v(z) ≤ u(z) for all z in G and the least harmonic majorant$u_L(z)$ is a harmonic majorant such that $u_L(z)$ ≤ u(z) for every harmonic majorant u(z) of v(z).

Theorem2.5.5[4]: *Let v be a subharmonic function in the unit disc D. Assume v(z) $\not\equiv -\infty$ for* $0 < r < 1$.

$$Let\ v_r(z)\ = \begin{cases} v(z), & for\ |z| \geq r \\ \frac{1}{2\pi}\int p_{z\backslash r}(\theta)v(\ re^{i\theta})\,d\theta & for\ |z| < r. \end{cases}$$

$v_r(z)$is subharmonic on D and it is harmonic on $|z| < r$, $v(z) \leq v_r(z)$, $z \in D$ and v_r is an increasing function of r and $v_r(z) = \frac{1}{2\pi}\int v(\ re^{i\theta})\,d\theta$.

Proof: To see that $v_r(z)$ is upper semi-continuous at a point $z_0 \in \partial B(0\ ,r)$, we must show

$lim_{z\to z_0} v_r(z) \leq v(z_0)$. This follows from the approximate identity properties of the poisson kernel and from semi-continuity of v.

Write z_0= $re^{i\theta_0}$, for $\varepsilon > 0, there is \delta > 0\ such that v\left(re^{i\theta}\right) < v(z_0)$ + ε if $|\theta - \theta_0|$<δ. Then if$|z| < r$and $|z - z_0|$ is small,

$v_r(z) = \frac{1}{2\pi}\int_{|\theta-\theta_0|\,\leq\,\delta} p_{z\backslash r}(\theta)[v(\ z_0 + \varepsilon)]\,d\theta\ +\ \frac{1}{2\pi}[sup v\left(re^{i\theta}\right)]\int_{|\theta-\theta_0|>\,\delta} p_{z\backslash r}(\theta)d\theta$
$\leq v(z_0)$ + 2$\boldsymbol{\varepsilon}$**.**

Hence v_r is upper semi-continuous. If we again take continuous functions $u_n(z)$ decreasing to v(z) on $\partial B(0\ ,r)$, then (by the same proof of the corollary we have) v(z) $\leq v_r(z)$.

Because of v is subharmonic, this inequality shows the mean value inequality at each point z_0 with$|z_0|$ = r. i.e.

$v(\ z_0) \leq \frac{1}{2\pi}\int v(z_0 +\ pe^{i\theta})\,d\theta$for p< r.

Consequently $v_r(z)$ is a subharmonic function on D.■

Theorem2.5.6[4]: *Let v be a subharmonic in the unit disc D. Then v has a harmonic majorant if and only if*

$Sup\frac{1}{2\pi}\int v(\, re^{i\theta})\, d\theta$ = *sup* $v_r(0)<\infty$ *and the least harmonic majorant of v is then*

u(z) = $lim_{r\to 1}\frac{1}{2\pi}\int p_{z\backslash r}(\theta)v(\, re^{i\theta})\, d\theta \ = \lim_{r\to 1} v_r(z)$.

Proof: Suppose that sup$v_r(0)<\infty$. Then by Haranck's theorem the function $v_r(z)$ increases to a finite harmonic function u(z) on D. Then by the above theorem v(z) ≤ $v_r(z)$ and $v_r(z)$ ≤ u(z) and hence v (z) ≤ u(z) for all z.

Therefore u is a harmonic majorant of v.

Conversely let u be a harmonic on D and let v(z) ≤ u(z) ∀z in D. This implies that $v_r(z)$ ≤u(z), $\forall r$.

Consequently $sup_r v_r(0)<\infty$, and again u(z) = $\lim_{r\to 1} v_r(z)$ is finite and harmonic.

Since $v_r(z)$ ≤u(z), we have $u_L(z)$ ≤ u(z) and hence u_L is the least harmonic majorant.

From continuity u(z) = $\lim_{r\to 1}$ u(rz). The least harmonic majorant of v can also be written as

$u_L(z) = lim_{r\to 1}\frac{1}{2\pi}\int p_z(\theta)v(\, re^{i\theta})\, d\theta.$■

CHAPTER THREE

Hardy Space (H^p) over upper half plane

The classical theory of a hardy space (H^p) is a mixture of real and complex analysis. There are two H^p theories; one over the disc and the other over the half plane. These twin theories are introduced simultaneously.

3.1. Definition and some properties of Hardy space on the unit Disc

We begin by presenting the main properties of Hardy spaces on the unit disc

$$D = \{z \in \mathbb{C}: |z| < 1\}.$$

We shall usually denote the boundary of D by $\partial D = \{z \in \mathbb{C} : |z| = 1\}$and the closure of D by $\overline{D}$ = D ∪ $\overline{D}$.

Definition3.1.1: Hardy Space $H^p(D)$ is a collection of all analytic functions$f: D \to \mathbb{C}$ such that

$$\underset{0\le r<1}{\boldsymbol{Sup}} \frac{1}{2\pi}\int_0^{2\pi}\left|f(\,re^{i\theta})\right|^p d\theta = \|f\|_{H^p}^p<\infty.$$

If p =∞ we say $f \in H^\infty$(D) if f(z) is a bounded analytic on D and we write

$$\underset{z\in D}{\boldsymbol{Sup}}\, |f(z)| = \|f\|_\infty.$$

Example: Let f(z) = z=$re^{i\theta}$, z in D, 0 ≤ r < 1. Then $|f(z)|$= r, 0<p<∞

$$\underset{0\le r<1}{\boldsymbol{Sup}} \frac{1}{2\pi}\int\left|f(\,re^{i\theta})\right|^p d\theta = \underset{0\le r<1}{\boldsymbol{Sup}} \frac{1}{2\pi}\int r^p\, d\theta \le 1.$$

Hence f ∈ H^p(D) and also for p = ∞,

$$\|f\|_\infty = \underset{z\in D}{\boldsymbol{Sup}}\, |f(z)| = \underset{z\in D}{\boldsymbol{Sup}}\, \mathrm{r} = 1<\infty.$$

Therefore $f \in H^\infty$(D).

Based on theorem 2.5.4 above the analytic function f(z) ∈ H^∞(D) if and only if the sub-harmonic function $|f(\,z)|^p$ has a harmonic majorant and that for p<∞, $\|f\|_{H^p}^p$ is the value of the leastharmonicmajorant at z=0 which is of the form:

$$u(z) = lim_{r\to 1}\frac{1}{2\pi}\int p_{z\backslash r}(\theta)\left|f(\,re^{i\theta})\right|^p d\theta.$$

Now if z=0, $p_{z\backslash r}(\theta)$ = 1, this implies that

$$u(0) = lim_{r\to 1}\frac{1}{2\pi}\int\left|f(\,re^{i\theta})\right|^p d\theta.$$

And it can be shown that,

$\underset{0\le r<1}{\boldsymbol{Sup}} \frac{1}{2\pi}\int\left|f(\,re^{i\theta})\right|^{p} d\theta = \ lim_{r\to 1} \frac{1}{2\pi}\int\left|f(\,re^{i\theta})\right|^{p} d\theta$ = $\|f\|_{H^p}^{p}$= u(0).

Theorem 3.1.1[8]:*H^p(D) is a Banach space.*

Proof: Let $\{f_n\}$ be a Cauchy sequence in H^p(D). By Lemma 3.1.1, for every r < 1 $\{f_n\}$ is a Cauchy sequence in C($\overline{D}$), with the uniform norm. By the completeness of C($\overline{D}$), there is g ∈C($\overline{D}$) such that $f_n \to g$ uniformly on $\overline{D}$.

Obviously, if $r_1 < r_2 < 1$, $g_{r_1} and\ g_{r_2}$ coincide on $\overline{D}_{r_1}$. It follows that the various g, with r < 1, are all restrictions of a unique function g continuous on D. We now prove that g is holomorphic in D. By Morera's theorem, this is true if and only if for every closed arc γ in D,

$\int_\gamma g(z)$dz = 0.

Let γ be such an arc. Since γ is contained in $\overline{D}$, for some r < 1, $f_n \to g$ uniformly on γ. Therefore

$\int_\gamma g(z)$dz = $\lim_{n\to\infty} f_n(z)dz,$

and each of this integrals is zero because the f_n are holomorphic.

We finally prove that $f_n \to g$ in H^p(D). Given ϵ> 0, let N be such that

$\|f_n - f_m\|_{H^p}$<∈for n, m ≥ N. Take r < 1. Since $f_m \to g$ uniformly on the circle

$|z|$ = r, if n ≥ N,

$\int_\gamma |f_{n-}g|^p = \lim_{m\to\infty} \int_\gamma |f_n - f_m|^p \le \lim_{m\to\infty} \|f_n - f_m\|_{H^p} \le \epsilon.$

Since this holds for every r <1, $\|f_n - g\|_{H^p} \le \epsilon$. Hence H^p(D) is a Banach space. ■

3.2. Blaschke Product

AnH^∞ function that has unit modulus almost everywhere on the boundary is called an inner function. A finite Blaschke product is one example of such functions. If we have an H^p function, we can use Blaschke product to factor out its zero, i.e. we can write $f(z) = B(z)g(z)$, where B is an infinite Blaschke product and $g \ne 0$.

Definition 3.2.1: A finite Blaschke product is a function of the form

B(z) =$e^{i\varphi}\prod_{j=1}^{n}\frac{z-z_j}{1-\overline{z_j}z}$,$|z_j|$< 1, j=1,2,…,n.

The function B has the properties:

(i) B is continuous on ∂D.

(ii) $|B(z)|$ = $\left|e^{i\varphi}\prod_{j=1}^{n}\frac{z-z_j}{1-\overline{z_j}z}\right|$ =$\left|e^{i\varphi}\right|\prod_{j=1}^{n}\left|\frac{z-z_j}{1-\overline{z_j}z}\right|$ = $\prod_{j=1}^{n}\frac{|z-z_j|}{|1-\overline{z_j}z|}$,

But $\frac{|z-z_j|}{|1-\overline{z_j}z|}$ = 1 ∀z ∈ ∂D and $|z_j|$< 1.

Therefore $|B(z)|$ = 1.

(iii) B has finitely many zeros in D.

The number of zeros of B is equal to its degree. A Blaschke product of degree zero is a constant function of absolute value 1. Then properties of B determine B up to a constant factor of modulus 1. Indeed, if an analytic function f(z) has these properties, and the same zeros with B, then by the maximum principle

$\left|\frac{f}{B}\right|$ ≤1 and $\left|\frac{B}{f}\right|$ ≤1 on D, and so f/B is constant. Now let us see {z_n} of a non zero $H^p(\mathbb{H})$ function on the disc satisfying Blaschke's product:

$\sum(1-|z_n|)$<∞.

Theorem 3.2.1[4]: *Let f be an analytic function on the disc, $f \neq 0$ and let {z_n} be the zeros of f(z). If $log|f(z)|$ has harmonic majorant, then*

$\sum(1-|z_n|)$<∞,

and if u(z) is the least harmonic majorant of $log|f(z)|$, then

$\sum(1-|z_n|)$ ≤*u(0)* - $log|f(0)|$.

Proof: Suppose $f(0) \neq 0$, then $\sup\limits_{r} \frac{1}{2\pi}\int \log|f(e^{i\theta})|\, d\theta$<∞. If u is the least harmonic majorant of log|f(z)|, then

u(0) = $\sup\limits_{r} \frac{1}{2\pi}\int \log|f(e^{i\theta})|\, d\theta = \lim\limits_{r\to 1} \frac{1}{2\pi}\int \log|f(e^{i\theta})|\, d\theta$.

Fixing r< 1 implies that $|z_n| \neq r$ for all n and let z_1, z_2, …, z_n be those zeros with $|z_j|$< 1. Then

F(rz) has zeros $\frac{z_1}{r}, \frac{z_2}{r}, \dots, \frac{z_n}{r}$. Let $B_r(z) = \prod_{j=1}^{n}\frac{z-z_j/r}{1-\overline{z_j}z/r}$, a finite Blaschke product with the same zeros as F(rz). And let g(z) = f(rz) / B_r(z), g is analytic and zero free on $\overline{D}$, so that

log|$g(0)$| = $\frac{1}{2\pi}\int \log|g(re^{i\theta})|\, d\theta$

Since $\log|g(e^{i\theta})| = \frac{|f(e^{i\theta})|}{|Br(z)|} = |f(e^{i\theta})|$. This gives Jensen formula

$\log|f(0)| + \sum_{|z_j|<1} \log\frac{r}{|z_j|} = \frac{1}{2\pi}\int \log|f(e^{i\theta})|\, d\theta$.

Letting r tends to 1 yields:

$\sum \mathrm{Log}\frac{1}{|z_j|} \leq \lim_{r\to 1}\frac{1}{2\pi}\int \log|f(re^{i\theta})|\, d\theta - \log|f(0)| = u(0) - \log|f(0)|$.

But $\log(1-|z_j|) \leq \log\frac{1}{|z_j|}\, \forall j$ and this implies that

$\sum_{j=1}^{n}(1-|z_j|) \leq \sum_{j=1}^{n}\frac{1}{|z_j|} \leq u(0) - \log|f(0)|$ and hence

$\sum_{j=1}^{n}(1-|z_j|) < \infty$.

If $f \in H^p(D)$ then $\log|f| \leq \frac{1}{p}|f|^p$ and $\log|f|$ has a harmonic majorant. Hence if $f \in H^p(D)$ or if

$f(w) = F(\tau(w))$, $F \in H^p(\mathbb{H})$, then $\sum(1-|z_n|) < \infty$. ■

Theorem 3.2.2[8]*(F. Riesz):* *Let $0 < p < \infty$, $f \in H^p(D)$, $f \neq 0$, $\{z_n\}$ be the zeros of $f(z)$ and $B(z)$ be the Blaschke product with zeros $\{z_n\}$. Then*

$g(z) = \frac{f(z)}{B(z)}$ is in $H^p(D)$ and $\|g\|_{H^p} = \|f\|_{H^p}$

Proof: If $f \in H^p$, then $B(z)$ converges. Let B_n be the Blaschke product with zeros $z_1, z_2, \ldots, z_n$ and let $g_n = f/B_n$, $g_n \to g$ as $n \to \infty$ and g_n is increasing function of r.

Fixing r<1 implies

$\int|g_n(re^{i\theta})|^p d\theta/2\pi \leq \lim_{r\to 1}\int\frac{|f(re^{i\theta})|^p}{|B_n(re^{i\theta})|^p}\frac{d\theta}{2\pi}$,but $\lim_{r\to 1}|B_n(re^{i\theta})| = |B_n(re^{i\theta})| = 1$.

If $1-r$ is small then $|B_n(re^{i\theta})| > 1-\varepsilon$ so that $\int|g_n(re^{i\theta})|^p d\theta/2\pi \leq \lim_{r\to 1}\int|f(re^{i\theta})|^p d\theta/2\pi$

$= \|f\|^p{}_{H^p}$.

This implies that $\|g\|^p{}_{H^p} = \lim_{n\to\infty}\int|g_n(re^{i\theta})|^p d\theta \leq \|f\|^p{}_{H^p}$ hence $\|g\| \leq \|f\|$…………(∗)

Since $|g_n|$ *is increasing to* $|g|$ *and* $|g| \geq |f|$ *we have* $\|f\|_{H^p} \leq \|g\|_{H^p}$…………………….(∗∗).

Therefore from (∗) and (∗∗) we get $\|f\|_{H^p} = \|g\|_{H^p}$. ■

Theorem3.2.3[4]: Let $f \in H^{\infty}(D)$, $\|f\| \leq 1$. Then the following are equivalent.

(i) f(z) = $\lambda B(z)$, where λ is a constant, $|\lambda|$ =1 and B(z) is a Blaschke product.

(ii) $\lim_{r\to 1} \int \left|\log f(re^{i\theta})\right| \frac{d\theta}{2\pi}$ = 0

(iii) Theleast harmonic majorant of log| f(z)| is 0.

Proof: Suppose that f(z) is a Blaschke product with zero {z_n}, and let $\varepsilon > 0$, we may divide f(z) by a finite Blaschke product B_n(z) so that $\left|\left(\frac{f}{B}\right)(0)\right|$> 1- ε. Since Bn is continuous on $\overline{D}$ and $\left|B_n(e^{i\theta})\right|$ =1,

$\lim_{r\to 1} \int \left|\log f(re^{i\theta})\right| d\theta$ = $\lim_{r\to 1} \int \left|\log \frac{f}{B_n}(re^{i\theta})\right| d\theta$. But since $\log|\frac{f}{B_n}|$ is sub-harmonic and

-log(1-ε) $\leq\int\log|\frac{f}{B_n}(re^{i\theta})|d\theta/2\pi \leq 0$.

As r→1 the limit of the integral becomes 0. Hence (ii) holds and equivalents to (iii).

Now to show (i) suppose (iii) holds. Let (z) =f(z)/B(z), where B(z) is the Blaschke product formed from the zeros of f(z). Then because $\|f\| \leq 1$, log| f(z)| $\leq$ log|g(z)| $\leq$ 0. Since log|g(z)| is the harmonic majorant of log| f(z)|, (iii) implies that log| g(z)|. Hence g(z) = λ where λ is constant and $|\lambda|$= 1and so (i) holds.■

Corollary 3.2.1[8]: Let$f$ $\in H^s(D)$, not identicallyzero, with $1 \leq s \leq \infty$, and let p, q ∈ $[1, \infty]$ be such that $\frac{1}{p} + \frac{1}{q} = \frac{1}{s}$. Then there exist $g \in H^s(D)$ and $h \in H^q(D)$ such that

$f = gh$,and$\|g\|_{H^p}^{p} = \|h\|_{H^q}^{q} = \|f\|_{H^s}^{s}$.

Proof. Let B be the Blaschke product of f and $\varphi = \frac{f}{B}$. Then $\varphi \in H^s(D)$ and has no zeroes in D. Let

$$g(z) = \varphi(z)^{\frac{s}{p}} = e^{\frac{s}{p}\log\varphi(z)}.$$

Then $|g(z)| = |\varphi(z)|^{\frac{s}{p}}$, so that

$\int_0^{2\pi}\left|g(re^{it})\right|^p dt = \int_0^{2\pi}\left|\varphi(re^{it})\right|^s dt = \|\varphi\|^s$.

Therefore $g \in H^p(D)$ and $\|g\|_{H^p}^{p} = \|\varphi\|_{H^s}^{s} = \|f\|_{H^s}^{s}$.

If we set h(z) = $\varphi(z)^{\frac{s}{q}}$B(z), then $g(z)h(z)$ = $\varphi(z)^{\frac{s}{q}+\frac{s}{q}}$B(z) = $f(z)$, and moreover

$\|h/B\|^q = \|\varphi\|^s$. Therefore, $\|h\|_{H^p}^p = \|h/B\|_{H^q}^q = \|\varphi\|_{H^s}^s = \|f\|_{H^s}^s$. ■

Lemma3.2.1[8]: *Let $f \in H^p(D)$ and 0<p ≤ ∞. Then, for every z ∈ D,*

$|f(z)| \leq C \frac{\|f\|_{H^p}}{(1-|z|)^{\frac{1}{p}}}$.

Proof:for p = ∞ the statement holds trivially since

$$|f(z)| \leq \|f\|_{H^\infty} = \sup_{z \in D} |f(z)|.$$

Suppose p < ∞. If $f = 0$ it holds clearly.

Assume $f \neq 0$. Then by F. Riesz's decomposition theorem(Let 0 < p < ∞, $f \in H^p(D)$,$\{z_n\}$ be the zeros of $f(z)$ *and* B(z) the Blaschke product with zeros $\{z_n\}$. Then $g(z) = \frac{f(z)}{B(z)} \in H^p(D)$)we can write

$f(z) = g(z)B(z)$, B is Blaschke factor,

$g \in H^p$and$\|f\|_{H^p} = \|g\|_{H^p}$.

Take r such that $|z| < r < 1$ and γ be circle centered at origin and radius r.

Since$g \in H^p(D) \Longrightarrow g^p \in H^1(D)$, by Cauchy integral formula

$(g(z))^p = \frac{1}{2\pi i} \int_\gamma \frac{(g(w))^p}{w-z} dw.$

Setting w = re^{it}, $dw = ire^{it}dt$, we get

$(g(z))^p = \frac{1}{2\pi} \int_0^{2\pi} \frac{(g(re^{it}))^p re^{it}}{re^{it}-z} dt.$

$\Longrightarrow |g(z)|^p \leq \frac{1}{2\pi} \left|\frac{(g(re^{it}))^p re^{it}}{re^{it}-z}\right|$dt

$\leq \frac{r}{2\pi(r-|z|)} \int_0^{2\pi} \left|g(re^{it})\right|^p$dt

$$\leq \frac{r}{r-|z|} \|g\|_{H^p}^p = \frac{r}{r-|z|} \|f\|_{H^p}^p$$

Then, letting r→ 1,

$$|f(z)|^p = (|g(z)||B(z)|)^p \leq |g(z)|^p \leq \frac{1}{1-|z|} \|f\|_{H^p}^p. \blacksquare$$

3.3. Definition and properties of Hardy Space over upper half-plane

We begin this section by introducing that we will use $\mathbb{H}$ for the upper half plane and $H^p(\mathbb{H})$ Hardy spaces of a upper half plane where 0 < p <∞, $\mathbb{H}$ = {z∈ $\mathbb{C}$: $Imz > 0$} which denotes the upper half plane. For each f: $\mathbb{H} \rightarrow \mathbb{C}$ holomophic on $\mathbb{H}$, $\|f\|_p$ is defined by

$$\|f\|_p = \underset{y>0}{\boldsymbol{Sup}} \left(\int_{\mathbb{R}} |f(\mathrm{x}+\mathrm{iy})|^{\mathrm{p}}\, dx\right)^{\frac{1}{p}}.$$

Definition 3.3.1: Let 0 <p<∞ and f be analytic function on the upper half plane $\mathbb{H}$. We say $f \in H^p(\mathbb{H})$ if

$$\underset{y>0}{\boldsymbol{Sup}} \int_{\mathbb{R}} |f(\mathrm{x}+\mathrm{iy})|^{\mathrm{p}}\, dx = \|f\|_{H^p}^p < \infty.$$

If p =∞ we say $f \in H^\infty(\mathbb{H})$ if f(z) is a bounded analytic on $\mathbb{H}$ and we write

$$\underset{y>0}{\boldsymbol{Sup}}\, |f(z)| = \|f\|_\infty.$$

Note that the definition of $H^p(\mathbb{H})$ involves all y, 0 <y<∞, instead of small values of y, like say $0 < y < 1$.

For example, if g(z) $=\frac{e^{-iz/p}}{(i+z)^{2/p}}$,

$$|g(\,x+iy)|^p = \frac{|e^{y-ix}|}{|x+(y+1)i|^2} = \frac{e^y}{(y+1)^2+x^2}.$$

This implies that $\int_{-\infty}^{\infty} \frac{e^y}{(y+1)^2+x^2}\, dx = \frac{e^y\pi}{y+1}$.

$\underset{\mathrm{y}>0}{\boldsymbol{Sup}} \frac{e^y\pi}{y+1}$is not finite.Therefore g ∉ $H^p(\mathbb{H})$ for p <∞.

Let z= $\tau(w)$ $=\frac{i(1-w)}{1+w}$ be the conformal mapping of D onto H. $f \circ \tau \in H^\infty(D)$ if and only if $f \in H^\infty(\mathbb{H})$, indeed, if $f \circ \tau \in H^\infty(D)$ then

$$\|f \circ \tau\|_\infty = \underset{z\in D}{\boldsymbol{Sup}}\, |f \circ \tau(w)| = \underset{z\in D}{\boldsymbol{Sup}}\, |f(\tau(w))| < \infty.$$

But for each z in H,there exists w in D such that $\tau(w)$= z, and hence

$\underset{z\in H}{\boldsymbol{Sup}}\, |f(z)| = \underset{w\in D}{\boldsymbol{Sup}}\, |f(\tau(w))| < \infty$. This implies that $\underset{z\in H}{\boldsymbol{Sup}}\, |f(z)| < \infty$.

Hence $f \in H^\infty(\mathbb{H})$.

Conversely suppose that ∈ $H^\infty(\mathbb{H})$, then for all w in D, sup$|(f \circ \tau)\,(w)| < \infty$, but $\tau(z)$ is in H.

This implies $\underset{z\in H}{\textbf{Sup}} |f(z)|<\infty$ and hence $f\circ\tau H^{\infty}(\mathbb{H})$.

In order to treat $H^p(D)$ and $H^p(\mathbb{H})$ together we are going to prove the following two lemmas.

If A and B are two connected, simply connected, proper open subsets of $\mathbb{C}$, there exists a conformal mapping φ from A onto B. For A = D, the unit disc, and B = $\mathbb{H}$, one can explicitely write such mappings. One of them is

$\varphi(z) = i\frac{1+z}{1-z}$, and it is called the Cayley transform.

Lemma 3.3.1[8]: The Cayley transform φ maps D onto $\mathbb{H}$, it is invertible, and

$\varphi^{-1}(\mathrm{w}) = \frac{w-i}{w+i}$.

Proof. By Cayley transform we have,

$\varphi(z) = i\frac{1+z}{1-z} = i\frac{(1+z)(1-\bar{z})}{|1-z|^2} = i\frac{1-|z|^2+2iImz}{|1-z|^2}$, so that

$\mathrm{Im}\varphi(z) = \frac{1-|z|^2}{|1-z|^2} > 0$, if z ∈ D.

Hence φ maps D into $\mathbb{H}$ the upper half plane.

It is easy to verify that φ is injective and that $\varphi^{-1}(\mathrm{w}) = \frac{w-i}{w-i}$ gives its inverse function. If $w \in \mathbb{H}$,

$|w-i| < |w+1|$by a simple geometric consideration, so that $|\varphi^{-1}(\mathrm{w})| < 1$.

This shows that φ is onto and hence we can use the Cayley transform to transfer harmonic functions from D to $\mathbb{H}$ and viceversa.■

Lemma 3.3.2[4]: *if 0 < p < ∞ and if $f \in H^p(\mathbb{H})$, then the subharmonic function $|f(z)|^p$ has harmonic majorant u(z) in $\mathbb{H}$ and*

u(i) $\leq \frac{1}{\pi}\|f\|_{H^p}^p$.

Proof: since$f \in H^p(\mathbb{H})$, then $|f|^p$ has harmonic majorant and theleast harmonic majorant is of the form u(z) =$\int p_y(x-t)\,|f(z)|^p$dt. Hence u(i) = $\int \frac{1}{\pi(t^2+1)}|f(z)|^p dt \leq \frac{1}{\pi}\int |f(t)|^p$dt.

This implies that u(i) $\leq \frac{1}{\pi}\int |f(t)|^p\, dt \leq \frac{1}{\pi}\int \|f\|^p$.

Therefore, u(i) $\leq \frac{1}{\pi}\|f\|_{H^p}^p$. ■

Theorem 3.3.1[4]: if *0 < p < ∞ and if* f *is analytic function in the upper half plane such that the subharmonic*$|f(z)|^p$ *has a harmonic majorant, then*

F(z) =$\frac{\pi^{-1/p}}{(z+i)^{2/p}}$*f(z) is in* $H^p(\mathbb{H})$ *and*

$\|f\|_{H^p}^p \le$ *u(i), where u is the least harmonic majorant of* $|f(z)|^p$.

Proof: let u be the least harmonic majorant of $|f(z)|^p$**.**The positive harmonic function has the form:

u(z) =cy + $\int p_y(x-t)$d$\mu(t)$, where c≥ 0 andμ is a positive measure on ℝ such that

$\int(1+t^2)^{-1}$d$\mu(t)$< ∞.

Consequently, $|F(z)|^p = \frac{1}{\pi(x^2+(y+1)^2)}|f(z)|^p \le \frac{1}{\pi(1+x^2)}$u(z) and hence

$|F(z)|^p \le \frac{cy}{\pi(1+x^2)} + \frac{1}{\pi(1+x^2)}\int p_y(x-t)\mathrm{d}\mu(t).$

Using Fubini theorem, $\int|F(x+iy)^p dx| \le$cy +$\frac{1}{\pi}\int(\int \frac{1}{1+x^2}\int p_y(x-t)\,dx)$ d$\mu(t)$.

But $\int\int\frac{1}{1+x^2}\int p_y(x-t)\,dx = \frac{y+1}{t^2+(y+1)^2} = p_{y+1}$(t). This implies that,

$\int|F(x+iy)|^p$dx ≤ cy + $\int p_{y+1}(t)d\,\mu(t)$ = -c + u((1+y)i) < ∞.

Taking the supermum we get $\|f\|_{H^p}^p < \infty$. But $\int|F(x+iy)|^p$dx is a decreasing function of y.

Hence $\|F\|_{H^p}^p = lim_{y\to o}\int|F(x+iy)|^p$dx.

From this We have:

$\|f\|_{H^p}^p \le$ u(i).■

Lemma 3.3.3[8]: *If* $f \in H^p(\mathbb{H})$, $z = x+iy \in \mathbb{H}$, *then*

$|f(x+iy)| \le C\frac{\|f\|_{H^p}}{y^{\frac{1}{p}}}$.

Proof: We assume that p < ∞, the other case being trivial. For every r < y, by the mean value property,

$f(z) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(z+re^{it})\,dt.$

Integrating in polar coordinates around z, we then have

$\frac{1}{|B(z,y)|}\int_{B(z,y)} f(w)dw = \frac{1}{\pi y^2}\int_0^y \int_{-\pi}^{\pi} f(z+re^{it})\, dtrdr$

$= \frac{2}{y^2}\int_0^y f(z)rdr$

$= f(z).$

Therefore, using Hӧlder's inequality and the inclusion B(z, y) ⊂ s_y= {$u + iv$:0 < 2y},

$$|f(x+iy)| \le \frac{1}{\pi y^2}\int_{B(z,y)} |f(w)|dw$$

$$\le \left(\frac{1}{\pi y^2}\int_{B(z,y)} |f(w)|^p dw\right)^{\frac{1}{p}}$$

$\le (\frac{1}{\pi y^2}\int_{s_y} |f(w)|^p dw)^{\frac{1}{p}} = (\frac{1}{\pi y^2}\int_{S_y} |f(u+iv)|^p dw)^{\frac{1}{p}}$

$\le \|f\|_{H^p}(\frac{1}{\pi y^2}\int_0^{2y} dv)^{\frac{1}{p}} = (\frac{2}{\pi y})^{\frac{1}{p}}\|f\|_{H^p}.$

Hence,$|f(z)| \le$ C$\frac{\|f\|_{H^p}}{y^{\frac{1}{p}}}$.■

Theorem 3.3.2 [8]: *For 0 <a< b, let $S_{a,b}$ = {$x + iy$: $a \le y \le b$}. If $f \in H^p(\mathbb{H})$, with p< ∞, then*

$$\lim_{z\to\infty, z\in S_{a,b}} f(z) = 0$$

Proof: For $z \in S_{a,b}$, let B_z be disc centered at z of radius a. By the concept of Lemma 3.3.3

$|f(z)| \le \frac{1}{\pi a^2}\int_{B_z} |f(w)|dw \le (\frac{1}{\pi a^2}\int_{B_z} |f(w)|^p dw)^{\frac{1}{p}}$………(*)

Now observe that $B_z \subset S_{0,b+a,}$ and that

$\int_{S_{0,b+a,}} |f(w)|^p dw = \int_0^{b+a}\int_{\mathbb{R}} |f(x+iy)|^p\, dxdy \le (b+a)\|f\|_{H^p} < \infty$.

Therefore, given $\varepsilon > 0$, there is M > 0 such that

$\int_0^{b+a}\int_{|x|>M} |f(x+iy)|^p\, dxdy < \pi a^2 \varepsilon^p$ ………………(**)

If $|Rez| > M + a, B_z \subset \{x + iy : |x| > M, 0 < y < b + a\}$. Putting (*) and (**) together, we obtain that $|f(z)| < \varepsilon$. ■

Theorem 3.3.3[4]: For 0 <p≤ ∞, H^p is complete.

Proof: Assume p <∞, we give the proof in the upper half plane; the reasoning for the disc is very similar. Due to the closed graph principle (Lemma 3.3.3 above), for each y>0, $f \in H^p(\mathbb{H})$ and z ∈ $\mathbb{H}$ we have the inequality

$|f(x+iy)| \leq (\frac{2}{\pi y})^{1/p} \|f\|_{H^p}^p.$

This shows that any H^pcauchy sequences $\{f_n\}$ converges point wise on $\mathbb{H}$ to analytic function f(z).

By Fatou's lemma:

$$\int |f(x+iy) - f_n(x+iy)|^p dx \leq \lim_{m\to\infty} \int |f_m(x+iy) - f_n(x+iy)|^p dx$$

$\leq \lim_{\mathrm{m}\to\infty} \|f_{\mathrm{m}} - f_{\mathrm{n}}\|_{\mathrm{H^p}}^{\mathrm{p}}.$

Hence $\|f - f_{\mathrm{n}}\|_{\mathrm{H^p}}^{\mathrm{p}} \leq \lim_{\mathrm{m}\to\infty} \|f_{\mathrm{m}} - f_{\mathrm{n}}\|_{\mathrm{H^p}}^{\mathrm{p}}.$

Therefore,$\mathrm{H^p}$ is a complete space.■

3.4. Composition Operators on Hardy Space

Definition 3.4.1: Composition operator induced by ψ, C_ψ is the linear operator on the space of all $\mathbb{C}$-valued functions on $\mathbb{H}$, $\mathbb{C}^{\mathbb{H}}$ is given by

$C_\psi f = f \circ \psi, f \in \mathbb{C}^{\mathbb{H}}.$

When for some p, 0< p <∞, $H^p(\mathbb{H})$ is left invariant by C_ψ, i.e.,

$C_\psi H^p(\mathbb{H}) \subseteq H^p(\mathbb{H}).$

Due to the closed graph principle and well-known inequality

$|f(w)| \leq (\frac{2}{\pi Imw})^{\frac{1}{p}} \|f\|_p, f \in H^p(\mathbb{H})$, w $\in \mathbb{H}$;

This means that the restriction operator will be equally denoted by C_ψ, and in such situation we say that ψ induces a bounded composition operator on$H^p(\mathbb{H})$.

Definition 3.4.2: Suppose that ψ is analytic on $\mathbb{H}$. For each fixed $y > 0$ the measurable $\psi_y(x) = \psi(x + iy)$, $x \in \mathbb{R}$ induces a Borel measure on $\mathbb{H}$, $m\psi_y{}^{-1}$ called the pull-back measure induced by ψ_y is given,

$$m\psi_y{}^{-1}(E) = |\{x \in \mathbb{R}: \psi(x + iy) \in E\}| \ldots\ldots\ldots\ldots\ldots \ (*) \text{ for each Borel subset } E \subseteq \mathbb{H}.$$

In $(*)$ $|\cdot|$ denotes the Lebesgue measure on $\mathbb{R}$. If ψ induces a bounded composition operator on $H^p(\mathbb{H})$ for some p, $0<p<\infty$, we can write $\int_{\mathbb{R}} |f(\psi(x + iy))|^p dx = \int_{\mathbb{R}} \left|f(\psi(f \circ \psi_y))\right|^p dx$

$$= \int_{\mathbb{R}} |f|^p d\, m\psi_y{}^{-1} \leq \left\|C_\psi f\,\right\|_p^p$$

$\leq \left\|C_\psi\right\|^p \|f\,\|_p^p, f \in H^p(\mathbb{H})$.

This implies that the family $\{\, m\psi_y{}^{-1}\}_{y>0}$ is a family of Carleson measures on $\mathbb{H}$ with common bound, i.e., there exist $c>0$ such that

$m\psi_y{}^{-1}(\{w \in \mathbb{H}: t<\text{Re}w<t+h,\ 0<\text{Im}w<h\}) \leq ch \ldots\ldots\ldots\ldots\ldots\ldots(**)$ and we shall denote by $Q_{t,h}$ for $h>0$, $t \in \mathbb{R}$, and $y>0$.

The main advantage of this characterization of boundedness is that it proves that an analytic map $\psi : \mathbb{H} \to \mathbb{H}$ simultaneously induces bounded composition operators on all $H^p(\mathbb{H})$ spaces.

Theorem 3.4.1[2]: *For any p, $0< p <\infty$ an analytic map: $\mathbb{H} \to \mathbb{H}$ induces a bounded composition operator on $H^p(\mathbb{H})$ if and only if the pull-back measures $m\psi_y{}^{-1}$, $y > 0$ are Carleson measures with common bound c, i.e., if and only if there is a $c > 0$ such that $(**)$ holds for any $h > 0$, $t \in \mathbb{R}$ and $y > 0$.*

Corollary 3.4.1[2]: *If ψ is bounded, analytic self mapping of $\mathbb{H}$, then ψ does not induce a bound composition operator on any of the spaces $H^p(\mathbb{H})$, $0 < p <\infty$.*

Proof: Choose $h>0$, such that $\psi(\mathbb{H}) \subseteq Q_{-h,2h}$, which is possible because ψ is bounded. We have that for any $y>0$,

$\psi_y{}^{-1}(Q_{-h,2h}) = \mathbb{R}$, so $m\psi_y{}^{-1}(Q_{-h,2h}) = \infty$, and therefore $(**)$ does not hold.■

Corollary 3.4.2[2]: *The only linear fractional mapping ψ which leave $\mathbb{H}$ invariant and induce bounded composition operators on the $H^p(\mathbb{H})$ spaces are those of the form $\psi(w) = \alpha w + \beta$ with $\alpha > 0$ and Im $\beta \geq 0$.*

Proof: We consider transformation of the form $\psi(w) = (aw + b) / (cw + d)$. By the previous corollary, we must have c=0, so ψ must be of the form $\psi(w) = \alpha w + \beta$. Indeed, if $c \neq 0$, and the zero of the denominator has negative imaginary part, then ψ is bounded on $\mathbb{H}$. If the denominator has its zero on the real line, sat at x, observe one can choose $t \in \mathbb{R}$ such that $t < x < x + h$. In that case, there is a neighborhood of ∞, i.e., a set N of the form $\mathbb{C}\backslash D$ where D is a closed disc such that $\psi(N) \underline{\subset} Q_{t,h}$. Therefore for some y>0 we shall have ${\psi_y}^{-1}(Q_{t,h}) = \mathbb{R}$, so$m{\psi_y}^{-1}(Q_{t,h}) = \infty$. We deduce ψ must be of the form $\psi(w) = \alpha w + \beta. \alpha$ can not be 0 because in the case ψ would be bound. Since ψ must leave $\mathbb{H}$ invariant, we must have α >0 and Imβ>0. Hence each such mapping satisfies condition $(**)$ is immediate.■

Corollary 3.4.3[2]:*The branch $\psi(w) = \sqrt{w}$ which maps $\mathbb{H}$ onto the first quadrant does not induce a bound composition operator on any of the $H^p(\mathbb{H})$-spaces.*

Proof: Fix any y_0>0 and set w=s + iy_0/2s. So $w^2 = s^2 - {y_0}^2/4s^2$ + iy_0. Therefore if h= y_0/2t, $\sqrt{w^2}$ = w $\in Q_{t,h}$ for each s, 0<t<s<t+ h. So $(s^2 - {y_0}^2/4s^2) \in {\psi_{y_0}}^{-1}(Q_{t,h})$ if0<t<s<t+ h and hence

$$\left|\left\{x \in \mathbb{R}: \sqrt{x + iy_0} \in Q_{t,h}\right\}\right| \geq (t+h)^2 - \frac{\frac{{y_0}^2}{4(t+h)^2} + \frac{{y_0}^2}{4t^2 - t^2}}{h} = \text{h+ 2t} + \frac{{y_0}^2(\text{h+ 2t})}{4t^2(t+h)^2} \to \infty \; if \; t \to \infty,$$

So, $(**)$ cannot hold.

The self mapping ϕ of the unit disc D induce bounded composition operators on all spaces H^p(D) if and only if they are holomorphic on D, their holomophybeing necessary because f(z)= z, z $\in D$ is an H^p(D)-functionfor each p, 0<p<∞, and $f \circ \phi = \phi$ must be in H^p(D). The Hardy spaces on H have worse properties, so far as composition operators are concerned. As we saw, there exist analytic self mappings on H, even linear fractional transforms which leave $\mathbb{H}$ invariant, and do not induces bounded composition operators on any of the $H^p(\mathbb{H})$-spaces, $0 < p < \infty$.■

Theorem 3.4.2[2]:*For any fixed p, 0 < p < ∞, if a continuous self mapping of$\mathbb{H}$, ψ: $\mathbb{H} \to \mathbb{H}$ induces a bounded composition operator on $H^p(\mathbb{H})$, then ψ must be holomophic on $\mathbb{H}$.*

***P*roof:** It will be to show that if $C_\psi H^p(\mathbb{H}) \subseteq H^p(\mathbb{H})$, then ψ is analytic on $\mathbb{H}$. Since f(w)=1/$(w + i)^{\frac{2}{p}}$, w $\in \mathbb{H}$ is an $H^p(\mathbb{H})$-function, hence$f \circ \psi \in H^p(\mathbb{H})$, so $f \circ \psi$ is analytic on $\mathbb{H}$.

f'(w) $\neq 0$, $\forall w \in \mathbb{H}$, so f is locally invertible with holomophic local inverses, i.e., for each arbitrary $w_0 \in \mathbb{H}$ we can choose an open neighborhood N of $f(\psi(w_0))$ where f has a holomorphic inverse f^{-1}, and an open neighborhood V of w_0, such that $(f \circ \psi)$(V) ⊆N so, for each w ∈ V we have

$f^{-1} \circ f \circ \psi$(w)= ψ(w), which shows that ψ must be holomorphic on V. Since w_0 was arbitrary chosen we deduce that ψ is analytic on $\mathbb{H}$. ■

Theorem 3.4.3[2]:*If ψ: $\mathbb{H} \to \mathbb{H}$ is analytic on $\mathbb{H}$, and the function $\eta(w, z)$=$|Re\psi'(w) + iIm\psi'(z)|$ z , w ∈ $\mathbb{H}$, is bounded below, then ψ induces bounded composition operator on the $H^p(\mathbb{H})$-spaces.*

Proof: Choose ε >0 such that $|\eta(\mathrm{w}, \mathrm{z})| \geq \varepsilon$ for all z, w ∈ $\mathbb{H}$.Forψ =u + iv, this means

$\left|\frac{\partial u(w)}{\partial x} + \frac{i\partial v(z)}{\partial x}\right| \geq \varepsilon$, for all z , w ∈ $\mathbb{H}$.

Take t ∈ $\mathbb{R}$, y >0, h>0 and observe that if$\psi(x_1 + iy)$, $\psi(x_2 + iy)$ ∈$Q_{t,h}$, then

$|\psi(x_1 + iy) - \psi(x_2 + iy)| \leq \sqrt{2}$h.

On the other hand, by the mean-value theorem, there exist s_1, s_2 $between$ x_1 and x_2 such that

$|\psi(x_1 + iy) - \psi(x_2 + iy)| = \left|\frac{\partial u(s_1 + iy)}{\partial x} + \frac{i\partial v(s_2 + iy)}{\partial x}\right| |x_1 - x_2|$

So, $|x_1 - x_2| \leq \sqrt{2}$h/$\varepsilon$, hence m${\psi_y}^{-1}(Q_{t,h}) \leq \sup\{|x_1 - x_2| : x_1, x_2 \in {\psi_y}^{-1}(Q_{t,h})\} \leq \sqrt{2}$h/$\varepsilon$ which proves that C_ψ is bounded. ■

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