



**Varational Formulation of Elliptic Partial Differential Equation**  
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## **Declaration**

I declare that this thesis has been developed by my own and that no portion of the thesis has served as the foundation for the awarding of any degree, diploma, associateship, fellowship, or other similar title to me.

**Ali Eshetu**

Signature \_\_\_\_\_ date \_\_\_\_\_

## **Permission letter**

This is to confirm that this thesis was completed by Mr. Ali Eshetu in the Department of Mathematics at Addis Abeba University, under my supervision.

I also confirm that the thesis can be submitted for evaluation by examiners and, eventually, for defense.

**Dr.Tsegaye Gedif**

Signature \_\_\_\_\_ Date\_\_\_\_\_

# Contents

Acknowledgment . . . . .	V
Abstract . . . . .	VI
Introduction . . . . .	1
Notation . . . . .	2
<b>1 Preliminaries</b>	<b>3</b>
1.1 Definitions . . . . .	3
1.1.1 Function Spaces . . . . .	5
1.1.2 Linear Operators on Normed Spaces . . . . .	7
1.1.3 Weak Derivatives . . . . .	16
1.1.4 Sobolev Spaces . . . . .	20
<b>2 Variational Formulations Of Elliptic Boundary Value Problems</b>	<b>25</b>
2.1 Review on Partial differential equation . . . . .	25
2.1.1 A model boundary value problem . . . . .	27
2.1.2 Some general results on existence and uniqueness . . . . .	30
2.1.3 The Lax-Milgram Lemma . . . . .	33
2.1.4 Weak formulations of linear elliptic boundary value problems . . . . .	35
<b>3 Summary</b>	<b>44</b>
Bibliography . . . . .	45

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## **Abstract**

The focus of this thesis is to examine weak or variational formulations of various elliptic boundary value problems and determine whether or not they are well-posed. The weak version of the homogeneous Dirichlet boundary value problem for the Poisson equation is first derived. A weak formulation can be thought of as an operator equation in its abstract form.

Additionally, we offer some broad conclusions on the existence and uniqueness of linear operator equations.

# Introduction

One of the most significant mathematical modeling instruments available at the moment is the elliptic partial differential equation (Elliptic PDE). But only a small number of partial differential equations allow for an analytical solution in some straightforward conditions. Therefore, it should come as no surprise that one of the main areas of computational mathematics research has been the numerical solution, or more accurately, the approximation of solutions to partial differential equations by numerical methods, ever since the development of computers made such computations possible for problems on large scales.

In this thesis, we consider variational (or weak) formulations of some elliptic boundary value problems and study the well-posedness of the variational problems. We begin with a derivation of a weak formulation of the homogeneous Dirichlet boundary value problem for the Poisson equation. In the abstract form, a weak formulation can be viewed as an operator equation. In the next section, we provide some general results on existence and uniqueness for linear operator equations.

The other section introduces and discusses the well-known Lax-Milgram Lemma, which is used to examine the well-posedness of variational formulations for a variety of linear elliptic boundary value problems in the section that follows.

Remember that  $\Omega$  denotes an open bounded set in  $\mathbb{R}^n$ , and we assume the boundary  $\Gamma = \partial\Omega$  is Lipschitz continuous. When necessary, we explicitly declare the assumption that  $\Omega$  is connected .

Elliptic partial differential equations are common in many sectors of science and engineering. These include equilibrium problems and steady-state phenomena. The most common example of such an equation is Poisson's equation.

**Key Characteristics:**

- **Second-Order:** Elliptic PDEs involve second-order partial derivatives, typically in the form of Laplacian or similar operators.
- **Smoothness:** The coefficients of the second-order derivatives must satisfy certain conditions.
- **Steady-State Behavior:** Elliptic PDEs often model steady-state or equilibrium conditions. They describe phenomena that are not changing with time.

## Notations

PDE	Partial Differential Equation.
$\mathbb{R}^n$	The Euclidean space of $n$ -dimension for $n > 1$ .
$\mathcal{S}(\mathbb{R}^n)$	the Schwartz space of all rapidly decreasing infinitely differentiable functions in $\mathbb{R}^n$ .
$x^T$	transpose of the vector $x \in \mathbb{R}^n, n \in \mathbb{N}$ .
$\alpha$	multi-index (an $n$ -tuple): $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_n)$
$ \alpha $	:= order (or length) of the multi-index $\alpha \in \mathbb{N}^n, n \in \mathbb{N}$ .
$\Omega$	open set in $\mathbb{R}^n$ .
$\partial\Omega = \Gamma$	the boundary of $\Omega$ .
$\bar{\Omega}$	Closure of $\Omega$ .
$\Gamma_D$	part of the boundary on which Dirichlet conditions are prescribed.
$\Gamma_N$	part of the boundary on which Neumann conditions are prescribed.
$\frac{\partial}{\partial\nu}$	normal derivative.
$D^\alpha$	partial derivative notation.
$D_w^\alpha$	weak derivative notation.
$H^1(\Omega)$	is the Sobolev space of function with square integrable first derivative.
$H_0^1(\Omega)$	the subspace of $H^1(\Omega)$ which vanishes at the boundary.
$H^{\frac{1}{2}}(\Gamma)$	is the Sobolev space defined on the boundary $\Gamma$ that have a "half" derivative in the $L^2$ sence.
$\mathbb{K}$	= real or complex number system

# Chapter 1

## Preliminaries

### 1.1 Definitions

**Definition 1.1.1.** A norm on a vector spaces  $V$  is a mapping  $\|\cdot\| : V \rightarrow \mathbb{R}$  that satisfies

1.  $\|u\| \geq 0$ , for any  $u \in V$ , and  $\|u\| = 0$  if and only if  $u = 0$ .
2.  $\|\lambda u\| = |\lambda|\|u\|$ , for any  $u \in V$ , and  $\lambda \in \mathbb{R}$ .
3.  $\|u + v\| \leq \|u\| + \|v\|$ , for all  $u, v \in V$ .

The space  $V$  is called a normed space.

**Definition 1.1.2.** Given a linear space  $V$ , a semi-norm  $|\cdot|$  is a function from  $V$  to  $\mathbb{R}$  with the properties of a norm except that  $|v| = 0$  does not necessarily imply  $v = 0$ .

**Definition 1.1.3.** Two norms  $\|\cdot\|_1$  and  $\|\cdot\|_2$  on a normed space  $V$  are said to be equivalent if there are positive constants  $c_1$  and  $c_2$  such that

$$c_1\|v\|_1 \leq \|v\|_2 \leq c_2\|v\|_1 \quad \forall v \in V.$$

**Definition 1.1.4.** Let  $V$  be a normed space. A sequence  $\{v_n\} \subset V$  is called a Cauchy sequence if for any  $\epsilon > 0$  there exists a number  $N(\epsilon)$  such that

$$\|v_m - v_n\| < \epsilon \quad \forall m, n > N(\epsilon)$$

Equivalently

$$\lim_{n,m \rightarrow \infty} \|v_m - v_n\| = 0$$

**Definition 1.1.5.** A normed space is said to be complete if every Cauchy sequence from the space converges to an element in the space. A complete normed space is called a Banach space.

Let  $A$  be a subset of a normed space  $V$ . Then,

- The set  $A$  is said to be closed in  $V$  if and only if  $\{v_n\} \subset A$  and  $v_n \rightarrow v$  imply that  $v \in A$  (A contains all its limit points).
- The closure  $\bar{A}$  of  $A$  is the smallest closed set in  $V$  containing  $A$ .
- The set  $A$  is dense in  $V$  if for every  $v \in V$  there exists a sequence  $\{v_n\}$  in  $A$  such that  $v_n \rightarrow v$ .

**Definition 1.1.6.** Let  $V$  be a vector space. A mapping  $(\cdot, \cdot) : V \times V \rightarrow \mathbb{K}$  is said to be an inner product on  $V$  if and only if for all vectors  $u, v$  and  $w$  and scalar  $\alpha$  :

1.  $(u + v, w) = (u, w) + (v, w)$
2.  $(\alpha u, v) = \alpha(u, v)$
3.  $(u, v) = \overline{(v, u)}$  (the bar denote complex conjugate)
4.  $(u, u) \geq 0$  and  $(u, u) = 0 \Leftrightarrow u = 0$

An inner product space is the space  $V$  plus the inner product  $(\cdot, \cdot)$ .  $V$  is an inner product space.  $V$  is referred to as a complex inner product space if  $\mathbb{K} = \mathbb{C}$ , and as a real inner product space when  $\mathbb{K} = \mathbb{R}$ . An inner product  $(\cdot, \cdot)$  induces a norm through the formula

$$\|v\| = \sqrt{(v, v)}, \quad v \in V$$

If all of the Cauchy sequences converge with respect to the induced norm, then the inner product is complete.

**Example 1.1.1.**

The inner product on  $\mathbb{R}^n$  given by  $(u, v) = \sum_{i=1}^n u_i v_i$  is complete. This is because the induced norm is the Euclidean norm, which is a complete norm.

**Definition 1.1.7.** A Hilbert space is an inner product space that is complete.

**Theorem 1.1.1.** (Cauchy-Schwarz Inequality)

Let  $V$  be an inner product space. Then

$$|(u, v)| \leq \|u\| \|v\| \quad \forall u, v \in V$$

and the equality holds if and only if  $u$  and  $v$  are linearly dependent.

### 1.1.1 Function Spaces

For  $n \in \mathbb{N}$  we call a vector  $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_n)$ ,  $\alpha_i \in \mathbb{N}$ , multi index with the absolute value  $|\alpha| = \alpha_1 + \dots + \alpha_n$  and with the factorial  $\alpha! = \alpha_1! \dots \alpha_n!$ . For  $x \in \mathbb{R}^n$ , we can therefore write

$$x^\alpha = x_1^{\alpha_1} \dots x_n^{\alpha_n}$$

If  $u$  is a sufficient smooth real valued function, then we can write partial derivatives as

$$D^\alpha u(x) := \left( \frac{\partial}{\partial x_1} \right)^{\alpha_1} \dots \left( \frac{\partial}{\partial x_n} \right)^{\alpha_n} u(x_1, \dots, x_n)$$

We denote by  $C(\Omega)$  the space of all real-valued functions that are continuous on  $\Omega$ . Since  $\Omega$  is open, a function from the space  $C(\Omega)$  is not necessarily bounded;

For example, consider, the continuous function  $u(x) = \ln x$  on  $(0, 1)$ . We denote further by  $C(\bar{\Omega})$  the space of functions that are bounded and uniformly continuous on  $\Omega$ . The notation  $C(\bar{\Omega})$  is consistent with the fact that a bounded and uniformly continuous function on  $\Omega$  has a unique continuous extension to  $\bar{\Omega}$ . The space  $C(\bar{\Omega})$  is a Banach space with the norm

$$\|u\|_{C(\bar{\Omega})} = \sup\{|u(x)| : x \in \Omega\} \equiv \max\{|u(x)| : x \in \bar{\Omega}\}$$

Let  $\Omega \subseteq \mathbb{R}^n$  be some open subset and assume  $k \in \mathbb{N}_0$ ,  $C^k(\Omega)$  is the space of functions which are bounded and  $k$  times continuously differentiable in  $\Omega$ . In particular, for  $u \in C^k(\Omega)$  the norm

$$\|u\|_{C^k(\Omega)} := \sum_{|\alpha| \leq k} \sup_{x \in \Omega} |D^\alpha u(x)|$$

is finite. Correspondingly,  $C^\infty(\Omega)$  is the space of functions which are bounded and infinitely often continuously differentiable.

$$\begin{aligned} C^\infty(\Omega) &= \bigcap_{k=0}^{\infty} C^k(\Omega) \equiv \{v \in C(\Omega) \mid v \in C^k(\Omega) \quad \forall k \in \mathbb{Z}^+\} \\ C^\infty(\bar{\Omega}) &= \bigcap_{k=0}^{\infty} C^k(\bar{\Omega}) \equiv \{v \in C(\bar{\Omega}) \mid v \in C^k(\bar{\Omega}) \quad \forall k \in \mathbb{Z}^+\} \end{aligned}$$

For a function  $u(x)$  defined for  $x \in \Omega$  we denote

$$\text{supp } u := \overline{\{x \in \Omega : u(x) \neq 0\}}.$$

Then,

$$C_0^\infty(\Omega) := \{u \in C^\infty(\Omega) : \text{supp } u \subset \Omega\}$$

is the space of  $C^\infty(\Omega)$  functions with compact support.

A function  $u$  defined on  $\Omega$  is said to be Lipschitz continuous if for some constant  $m$ , there holds the inequality

$$|u(x) - u(y)| \leq m\|x - y\| \quad \forall x, y \in \Omega$$

In this formula,  $\|x - y\|$  denotes the Euclidean distance between  $x$  and  $y$ . The smallest possible constant in the above inequality is called the Lipschitz constant of  $u$ , and is denoted by  $\text{Lip}(u)$ . The Lipschitz constant is characterized by the relation

$$\text{Lip}(u) = \sup \left\{ \frac{|u(x) - u(y)|}{\|x - y\|} \mid x, y \in \Omega, x \neq y \right\}$$

More generally, a function  $u$  is said to be Hölder continuous with exponent  $\beta \in (0, 1]$  if for some constant  $m$ ,

$$|u(x) - u(y)| \leq m\|x - y\|^\beta \quad \text{for } x, y \in \Omega$$

The Hölder space  $C^{0,\beta}(\bar{\Omega})$  is defined to be the subspace of  $C(\bar{\Omega})$  functions that are Hölder continuous with the exponent  $\beta$ . With the norm

$$\|u\|_{C^{0,\beta}(\bar{\Omega})} = \|u\|_{C(\bar{\Omega})} + \sup \left\{ \frac{|u(x) - u(y)|}{\|x - y\|^\beta} \mid x, y \in \Omega, x \neq y \right\}$$

the space  $C^{0,\beta}(\bar{\Omega})$  becomes a Banach space. When  $\beta = 1$ , the Hölder space  $C^{0,1}(\bar{\Omega})$  consists of all the Lipschitz continuous functions. For  $k \in \mathbb{Z}_+$  and  $\beta \in (0, 1]$ , we similarly define the Hölder space

$$C^{k,\beta}(\bar{\Omega}) = \{u \in C^k(\bar{\Omega}) \mid D^\alpha u \in C^{0,\beta}(\bar{\Omega}), \text{ for all } \alpha \text{ with } |\alpha| = k\}$$

this is a Banach space with the norm

$$\|u\|_{C^{k,\beta}(\bar{\Omega})} = \|u\|_{C^k(\bar{\Omega})} + \sum_{|\alpha|=k} \sup \left\{ \frac{|D^\alpha u(x) - D^\alpha u(y)|}{\|x - y\|^\beta} \mid x, y \in \Omega, x \neq y \right\}$$

The boundary of an open set  $\Omega \subset \mathbb{R}^n$  is defined as

$$\Gamma := \partial\Omega = \bar{\Omega} \cap (\mathbb{R}^n \setminus \Omega)$$

By  $L_p(\Omega)$  we denote the space of all equivalence classes of measurable func-

tions on whose powers of order  $p$  are integrable. The associated norm

$$\|u\|_{L_p(\Omega)} := \left\{ \int_{\Omega} |u(x)|^p dx \right\}^{\frac{1}{p}} \quad 1 \leq p < \infty$$

Two elements  $u, v \in L_p(\Omega)$  are identified with each other if they are different only on a set  $K$  of zero measure  $\mu(K) = 0$ .

In addition,  $L_{\infty}(\Omega)$  is the space of functions  $u$  which are measurable and bounded almost everywhere with the norm

$$\|u\|_{L_{\infty}(\Omega)} := \text{ess sup}\{|u(x)|\} := \inf_{k \subset \Omega, \mu(k)=0} \sup_{x \in \Omega \setminus k} |u(x)|$$

The spaces  $L_p(\Omega)$  are Banach spaces with respect to the norm  $\|\cdot\|_{L_p(\Omega)}$ . There holds the Minkowski inequality

$$\|u + v\|_{L_p(\Omega)} \leq \|u\|_{L_p(\Omega)} + \|v\|_{L_p(\Omega)} \quad \text{for all } u, v \in L_p(\Omega)$$

For  $u \in L_p(\Omega)$  and  $v \in L_q(\Omega)$  with conjugates parameters  $p$  and  $q$ , i.e.

$$\frac{1}{p} + \frac{1}{q} = 1$$

We further have Hölders inequality

$$\int_{\Omega} |u(x)v(x)| dx \leq \|u\|_{L_p(\Omega)} \|v\|_{L_q(\Omega)} \quad (1.1)$$

### 1.1.2 Linear Operators on Normed Spaces

Given two sets  $V$  and  $W$ , an operator  $T$  from  $V$  to  $W$  is a rule which assigns to each element in a subset of  $V$  a unique element in  $W$ . The domain  $\mathcal{D}(T)$  of  $T$  is the subset of  $V$  where  $T$  is defined,

$$\mathcal{D}(T) = \{v \in V \mid T(v) \text{ is defined}\}.$$

and the range  $\mathcal{R}(T)$  of  $T$  is the set of the elements in  $W$  generated by  $T$ ,

$$\mathcal{R}(T) = \{w \in W \mid w = T(v) \text{ for some } v \in \mathcal{D}(T)\}.$$

The null set of  $T$  (the set of the zeros of the operator) is defined as ,

$$\mathcal{N}(T) = \{v \in V \mid T(v) = 0\}.$$

**Definition 1.1.8.** Let  $V$  and  $W$  be two normed spaces. An operator  $T : V \rightarrow W$  is continuous at  $v \in \mathcal{D}(T)$  if

$$\{v_n\} \subset \mathcal{D}(T) \text{ and } v_n \rightarrow v \text{ in } V \text{ implies } T(v_n) \rightarrow T(v) \text{ in } W$$

as  $n \rightarrow \infty$

$T$  is said to be continuous if it is continuous over its domain  $\mathcal{D}(T)$ .

### Continuous linear operators

**Definition 1.1.9.** Let  $V$  and  $W$  be two linear spaces. An operator  $L : V \rightarrow W$  is said to be linear if

$$L(\alpha_1 v_1 + \alpha_2 v_2) = \alpha_1 L(v_1) + \alpha_2 L(v_2) \quad \forall v_1, v_2 \in V, \forall \alpha_1, \alpha_2 \in \mathbb{K}.$$

For a linear operator  $L$ , we usually write  $L(v)$  as  $Lv$ .

**Definition 1.1.10.** Let  $V$  and  $W$  be normed spaces.

A linear operator  $L$  is bounded if there exists a non-negative real number  $M$  such that:

$$\|L(v)\|_W \leq M\|v\|_V \quad \forall v \in V.$$

**Proposition 1.1.2.** Let  $V$  and  $W$  be normed spaces,

$L : V \rightarrow W$  a linear operator. Then  $L$  is bounded if and only if there exists a constant  $\gamma \geq 0$  such that

$$\|Lv\|_W \leq \gamma\|v\|_V \quad \forall v \in V \tag{1.2}$$

Proof. ( $\Leftarrow$ ) by definition (1.2) implies the boundedness.

( $\Rightarrow$ ), suppose  $L$  is bounded, then

$$\gamma := \sup_{v \in B_1} \|Lv\|_W < \infty$$

where  $B_1 = \{v \in V \mid \|v\|_V \leq 1\}$  is the unit ball centered at 0. Now for any  $v \neq 0$ ,  $v/\|v\|_V \in B_1$  and by the linearity of  $L$ ,  $\|Lv\|_W = \left\| L\left(v \frac{\|v\|_V}{\|v\|_V}\right) \right\|_W = \|v\|_V \|L(v/\|v\|_V)\|_W$  and  $\|L(v/\|v\|_V)\|_W \leq \gamma$  since  $L$  is bounded

$$\|Lv\|_W = \|v\|_V \|L(v/\|v\|_V)\|_W \leq \gamma\|v\|_V$$

i.e., (1.2) holds.

**Theorem 1.1.3.** Let  $V$  and  $W$  be normed spaces,  $L : V \rightarrow W$  a linear operator. Then  $L$  is continuous on  $V$  if and only if it is bounded on  $V$ .

( $\Rightarrow$ ) Suppose  $L : V \rightarrow W$  be a continuous linear operator. Thus we have to prove that  $L$  is bounded . We will prove it by contradiction . Assume  $L$  is not bounded , then

$$\begin{aligned} & \|Lv_n\| > n\|v_n\| \quad \forall v_n \in V \\ & \Rightarrow \frac{\|Lv_n\|}{n\|v_n\|} > 1 \Rightarrow \left| \frac{\|Lv_n\|}{n\|v_n\|} \right| > 1 \\ & \Rightarrow \left\| L\left(\frac{v_n}{n\|v_n\|}\right) \right\| > 1, \text{ since } L \text{ is linear} \\ & \Rightarrow \|L y_n\| > 1 \dots(i) , \text{ where } y_n = \frac{v_n}{n\|v_n\|} \\ & \|y_n\| = \left\| \frac{v_n}{n\|v_n\|} \right\| = \frac{1}{n} \\ & \Rightarrow \|y_n\| \rightarrow 0 , \text{ as } n \rightarrow \infty \\ & \Rightarrow y_n \rightarrow 0 , \text{ as } n \rightarrow \infty \\ & \Rightarrow Ly_n \rightarrow L(0) , \text{ as } n \rightarrow \infty \\ & \Rightarrow Ly_n \rightarrow 0 , \text{ as } n \rightarrow \infty \\ & \Rightarrow \|Ly_n\| \rightarrow 0 , \text{ as } n \rightarrow \infty \dots(ii). \end{aligned}$$

Relation (i) and(ii) contradict each other. Hence  $L$  is bounded .So  $L$  is continuous implies  $L$  is bounded.

( $\Leftarrow$ ) Suppose  $L$  be bounded  $\Rightarrow$  there exist a real number  $M > 0$  such that  $\|Lv\|_W \leq M\|v\|_V \quad \forall v \in V$

(iii) To prove that  $L$  is continuous.

Let  $v_n \rightarrow v$  in  $V \Rightarrow v_n - v \rightarrow 0$  in  $V$  , since  $v_n, v \in V \Rightarrow v_n - v \in V$ . Thus

$$\|Lv_n - Lv\|_W = \|L(v_n - v)\|_W \leq M\|v_n - v\|_V \rightarrow 0 \quad \text{as } n \rightarrow \infty$$

$$\|L(v_n - v)\| \rightarrow 0 \Rightarrow Lv_n \rightarrow Lv \text{ in } W$$

so  $v_n \rightarrow v$  in  $V \Rightarrow Lv_n \rightarrow Lv$  in  $W$ . Thus  $L$  is a continuous linear operator.

We use the notation  $\mathcal{L}(V, W)$  for the set of all the continuous linear operators from a normed space  $V$  to another normed space  $W$ . In the special case  $W = V$ , we use  $\mathcal{L}(V)$  to replace  $\mathcal{L}(V, V)$ . From the above theorem we see that for a linear operator, boundedness is equivalent to continuity. Thus if  $L \in \mathcal{L}(V, W)$ , it is meaningful to define

$$\|L\|_{V,W} = \sup_{0 \neq v \in V} \frac{\|Lv\|_W}{\|v\|_V} \quad (1.3)$$

**Theorem 1.1.4.** *The set  $\mathcal{L}(V, W)$  is a linear space, and (1.3) defines a norm over the space. The norm (1.3) is usually called the operator norm of  $L$ , which enjoys the following compatibility property*

$$\|Lv\|_W \leq \|L\|_{V,W} \|v\|_V \quad \forall v \in V \quad (1.4)$$

**Example 1.1.2.** *If  $V$  is a linear space. Then the identity operator  $I : V \rightarrow V$  belongs to  $\mathcal{L}(V)$ , and  $\|I\| = 1$ .*

**Theorem 1.1.5.** *Let  $V$  be a normed space, and  $W$  be a Banach space. Then  $\mathcal{L}(V, W)$  is a Banach space.*

Proof. Let  $\{L_n\}$  be a Cauchy sequence in  $\mathcal{L}(V, W)$ . This means

$$\epsilon_n \equiv \sup_{p \geq 1} \|L_{n+p} - L_n\| \rightarrow 0 \quad \text{as } n \rightarrow \infty$$

We must define a limit for  $\{L_n\}$  and show that it belongs to  $\mathcal{L}(V, W)$ . For each  $v \in V$ ,

$$\|L_{n+p}v - L_nv\|_W \leq \|L_{n+p} - L_n\|_W \|v\|_V \leq \epsilon_n \|v\|_V \rightarrow 0 \text{ as } n \rightarrow \infty. \quad (1.5)$$

Thus  $\{L_nv\}$  is a Cauchy sequence in  $W$ . Since  $W$  is complete, the sequence has a limit, denoted by  $L(v)$ . This defines an operator  $L : V \rightarrow W$ . Let us prove that  $L$  is linear, bounded, and  $\|L_n - L\|_{V,W} \rightarrow 0$  as  $n \rightarrow \infty$ .

For any  $v_1, v_2 \in V$  and  $\alpha_1, \alpha_2 \in \mathbb{K}$ ,

$$\begin{aligned} & \|L(\alpha_1v_1 + \alpha_2v_2) - (\alpha_1L(v_1) + \alpha_2L(v_2))\|_W \\ &= \|L(\alpha_1v_1 + \alpha_2v_2) - L_n(\alpha_1v_1 + \alpha_2v_2) + L_n(\alpha_1v_1 + \alpha_2v_2) - (\alpha_1L(v_1) + \alpha_2L(v_2))\|_W \\ &= \|L(\alpha_1v_1 + \alpha_2v_2) - L_n(\alpha_1v_1 + \alpha_2v_2) + \alpha_1L_n(v_1) + \alpha_2L_n(v_2) - \alpha_1L(v_1) - \alpha_2L(v_2)\|_W \\ &= \|(L - L_n)(\alpha_1v_1 + \alpha_2v_2) + \alpha_1(L_n - L)(v_1) + \alpha_2(L_n - L)(v_2)\|_W \\ &\leq \|(L - L_n)(\alpha_1v_1 + \alpha_2v_2)\|_W + |\alpha_1| \|(L_n - L)(v_1)\|_W + |\alpha_2| \|(L_n - L)(v_2)\|_W \end{aligned}$$

Take limit as  $n \rightarrow \infty$  both sides and assume  $\lim_{n \rightarrow \infty} \|L_n - L\|_W = 0$

$$\begin{aligned} & \Rightarrow \lim_{n \rightarrow \infty} \|L(\alpha_1v_1 + \alpha_2v_2) - (\alpha_1L(v_1) + \alpha_2L(v_2))\|_W \\ & \leq \lim_{n \rightarrow \infty} (\|(L - L_n)(\alpha_1v_1 + \alpha_2v_2)\|_W + |\alpha_1| \|(L_n - L)(v_1)\|_W + |\alpha_2| \|(L_n - L)(v_2)\|_W) \\ & = \lim_{n \rightarrow \infty} (\|(L - L_n)(\alpha_1v_1 + \alpha_2v_2)\|_W + |\alpha_1| \lim_{n \rightarrow \infty} \|(L_n - L)(v_1)\|_W \\ & \quad + |\alpha_2| \lim_{n \rightarrow \infty} \|(L_n - L)(v_2)\|_W) \\ & = 0 + |\alpha_1|(0) + |\alpha_2|(0) = 0 \end{aligned}$$

Hence

$$\begin{aligned} & \|L(\alpha_1v_1 + \alpha_2v_2) - (\alpha_1L(v_1) + \alpha_2L(v_2))\|_W \leq 0 \\ & \Rightarrow \|L(\alpha_1v_1 + \alpha_2v_2) - (\alpha_1L(v_1) + \alpha_2L(v_2))\|_W = 0 \\ & \Rightarrow L(\alpha_1v_1 + \alpha_2v_2) = \alpha_1L(v_1) + \alpha_2L(v_2) \end{aligned}$$

Thus  $L$  is linear.

Now for any  $v \in V$ , we take the limit  $p \rightarrow \infty$  in ((1.5)) to obtain

$$\|Lv - L_nv\|_W \leq \epsilon_n \|v\|_V$$

Thus

$$\|L - L_n\|_{V,W} = \sup_{\|v\|_V \leq 1} \|Lv - L_nv\|_W \leq \epsilon_n \rightarrow 0 \quad \text{as } n \rightarrow \infty$$

Hence  $L \in \mathcal{L}(V, W)$  and  $L_n \rightarrow L$  as  $n \rightarrow \infty$ .

### Linear functionals

The taking on of scalar values by linear operators is a significant specific case. Let  $W = \mathbb{K}$  be the set of scalars associated with  $V$ , and let  $V$  be a normed space. We refer to  $\mathcal{L}(V, \mathbb{K})$  and its elements as linear functionals. Given the completeness of  $\mathbb{K}$ ,  $\mathcal{L}(V, \mathbb{K})$  is a Banach space. This space is known as the dual space of  $V$  and is commonly represented as  $V'$ . For the most part, we use lower case letters to represent linear functions, like  $\ell$ . We refer to only bounded linear functionals as "linear functionals" in this thesis since we only use bounded linear functionals.

**Theorem 1.1.6.** *Let  $\Omega \subset \mathbb{R}^n$  be a bounded open set. It is a well-known result that for  $1 \leq p < \infty$ , the dual space of  $L_p(\Omega)$  can be identified with  $L_q(\Omega)$ . Here  $q$  is the conjugate exponent of  $p$ , defined by the relation*

$$\frac{1}{p} + \frac{1}{q} = 1$$

By convention,  $q = \infty$  when  $p = 1$ . In other words, given an  $\ell \in (L_p(\Omega))'$ , there is a function  $u \in L_q(\Omega)$ , uniquely determined a.e., such that

$$\ell(v) = \int_{\Omega} u(x)v(x)dx \quad \forall v \in L_p(\Omega) \quad (1.6)$$

Conversely, for any  $u \in L_q(\Omega)$ , the rule

$$u \mapsto \int_{\Omega} u(x)v(x)dx, \quad v \in L_p(\Omega)$$

defines a bounded linear functional on  $L_p(\Omega)$ . It is convenient to identify  $\ell \in (L_p(\Omega))'$  and  $u \in L_q(\Omega)$ , related as in (1.6). Then we write

$$(L_p(\Omega))' = L_q(\Omega), \quad 1 \leq p < \infty$$

Moreover,  $L_{\infty}(\Omega)$  is the dual space of  $L_1(\Omega)$ , but  $L_1(\Omega)$  is not the dual space of  $L_{\infty}(\Omega)$ . For  $p = 2$  we have  $L_2(\Omega)$  to be the space of all square integrable functions, and inequality (1.1) turns out to be the Cauchy Schwarz inequality

$$\int_{\Omega} |u(x)v(x)|dx \leq \|u\|_{L_2(\Omega)}\|v\|_{L_2(\Omega)}$$

Moreover, for  $u, v \in L_2(\Omega)$  we can define the inner product

$$(u, v)_{L_2(\Omega)} := \int_{\Omega} u(x)v(x)dx$$

and with

$$(u, u)_{L_2(\Omega)} = \|u\|_{L_2(\Omega)}^2 \quad \text{for all } u \in L_2(\Omega)$$

we conclude that  $L_2(\Omega)$  is Hilbert space.

**Definition 1.1.11.** Let  $V$  be a real or complex linear space,  $K \subset V$ . The set  $K$  is said to be convex if

$$u, v \in K \Rightarrow \lambda u + (1 - \lambda)v \in K \quad \forall \lambda \in (0, 1).$$

**Definition 1.1.12.** Let  $K$  be a convex set in a linear space  $V$ . A function  $f : K \rightarrow \mathbb{R}$  is said to be convex if

$$f(\lambda u + (1 - \lambda)v) \leq \lambda f(u) + (1 - \lambda)f(v) \quad \forall u, v \in K, \forall \lambda \in [0, 1]$$

The function  $f$  is strictly convex if the above inequality is strict for  $u \neq v$  and  $\lambda \in (0, 1)$ .

**Definition 1.1.13.** Let  $V$  be a Banach space. An operator  $P \in L(V)$  with the property  $P^2 = P$  is called a projection operator.

**Theorem 1.1.7.** (The Projection Theorem) Let  $M$  be a closed subspace of the Hilbert space  $H$ . Then  $M^\perp$  is also a closed subspace, and

$$H = M \oplus M^\perp$$

Where  $M^\perp := \{z \in H : (y, z) = 0 \quad \forall y \in M\}$ .

Further, in the decomposition  $x = y + z$  where  $y \in M, z \in M^\perp$ ,  $y$  is the element in  $M$  closest to  $x$ .

*Proof.* Let  $K = \{x - m | m \in M, x \in H\}$

$K$  is non empty since  $0 \in M$ , therefore  $x \in K$ . Now if  $k_1, k_2 \in K$  then there are  $m_1, m_2 \in M$  such that  $k_1 = x - m_1, k_2 = x - m_2$  and if  $0 < \lambda < 1$  then

$$\begin{aligned} \lambda k_1 + (1 - \lambda)k_2 &= \lambda(x - m_1) + (1 - \lambda)(x - m_2) = \lambda x - \lambda m_1 + x - m_2 + \\ &- \lambda(x - m_2) = \lambda x - \lambda m_1 + x - m_2 - \lambda x + \lambda m_2 \end{aligned}$$

$$= x - \lambda m_1 - m_2 + \lambda m_2$$

$$= x - \lambda m_1 - (1 - \lambda)m_2$$

$$= x - [\lambda m_1 + (1 - \lambda)m_2]$$

since  $\lambda m_1 + (1 - \lambda)m_2 \in M$  thus  $x - [\lambda m_1 + (1 - \lambda)m_2] \in K$ .

Similarly we can define  $k_n \rightarrow k$  as  $n \rightarrow \infty$

Since  $k_n \in K \therefore \exists m_n \in M$  such that  $k_n = x - m_n \Rightarrow k = x - m \in K$ ,  $k \in K$ . Thus,  $K$  is closed.

We know that in a Hilbert space if  $K$  is closed and convex in  $H$ , then  $H$  contains smallest norm in  $K$ . Suppose the element is  $z$ , now if  $m$  is an element in  $M$  of unit norm then  $z - (z, m)m$  is in  $K$  and

$$\begin{aligned}
& \|z\|^2 \leq \|z - (z, m)m\|^2, z \text{ is smallest in } K \\
& = (z - (z, m)m, z - (z, m)m) \\
& = (z, z) - (z, m)\overline{(m, z)} - \overline{(z, m)}(z, m) + (z, m)\overline{(z, m)}(m, m) \\
& = \|z\|^2 - (z, m)(z, m) - \overline{(z, m)}(z, m) + \|(z, m)\|^2\|m\|^2 \\
& = \|z\|^2 - 2\|(z, m)\|^2 + \|(z, m)\|^2\|m\|^2 \\
& = \|z\|^2 - 2\|(z, m)\|^2 + \|(z, m)\|^2, \because \|m\|^2 = 1 \\
& = \|z\|^2 - \|(z, m)\|^2 \\
& \Rightarrow \|z\|^2 \leq \|z\|^2 - \|(z, m)\|^2 \\
& \Rightarrow \|z\|^2 - \|z\|^2 \leq -\|(z, m)\|^2 \Rightarrow 0 \leq -\|(z, m)\|^2 \\
& \Rightarrow \|(z, m)\|^2 \leq 0 \Rightarrow \|z, m\| \leq 0 \\
& \Rightarrow \|(z, m)\| = 0 \Rightarrow (z, m) = 0 \Rightarrow z \in M^\perp
\end{aligned}$$

and

$$\because z \in K, \therefore \exists y \in M \text{ such that } z = x - y \Rightarrow x = y + z \quad y \in M, z \in M^\perp$$

For the uniqueness

suppose  $x = y_1 + z_1$  and  $x = y_2 + z_2$

$$y_1 + z_1 = y_2 + z_2 \Rightarrow y_1 - y_2 = z_2 - z_1 \Rightarrow y_1 - y_2, z_2 - z_1 \in MnM^\perp$$

$$\Rightarrow y_1 - y_2 = 0 \text{ and } z_2 - z_1 = 0$$

$$\Rightarrow y_1 = y_2, z_2 = z_1$$

i.e.  $x = y + z$  is unique. □

**Definition 1.1.14.** Let  $V$  be a real normed space, and  $A$  and  $B$  non-empty sets in  $V$ . The sets  $A$  and  $B$  are said to be separated if there is a non-zero linear continuous functional  $\ell$  on  $V$  and a number  $\alpha \in \mathbb{R}$  such that

$$\begin{aligned}\ell(u) &\leq \alpha \quad \forall u \in A \\ \ell(v) &\geq \alpha \quad \forall v \in B.\end{aligned}$$

If the inequalities are strict, then we say the sets  $A$  and  $B$  are strictly separated.

**Theorem 1.1.8.** Let  $V$  be a real normed space,  $A$  and  $B$  be two non-empty disjoint convex subsets of  $V$  such that one of them is compact, and the other is closed. Then the sets  $A$  and  $B$  can be strictly separated.

**Definition 1.1.15.** If  $V$  and  $W$  are vector spaces, a bilinear form  $a : V \times W \rightarrow \mathbb{R}$  is defined to be an operator with the properties,

$$\begin{aligned}a(\alpha u + \beta w, v) &= \alpha a(u, v) + \beta a(w, v) \quad u, w \in V, v \in W \\ a(u, \alpha v + \beta w) &= \alpha a(u, v) + \beta a(u, w) \quad u \in V, v, w \in W\end{aligned}$$

where  $\alpha$  and  $\beta$  are real numbers.

**Theorem 1.1.9.** (*Riesz representation theorem*) Let  $V$  be a real or complex Hilbert space,  $\ell \in V'$ . Then there is a unique  $u \in V$  for which

$$\ell(v) = (v, u) \quad \forall v \in V \tag{1.7}$$

In addition,

$$\|\ell\| = \|u\| \tag{1.8}$$

Proof. Assuming the existence of  $u$ , we first prove its uniqueness. Suppose  $\tilde{u} \in V$  satisfies

$$\ell(v) = (v, u) = (v, \tilde{u}) \quad \forall v \in V$$

Then

$$(v, u - \tilde{u}) = 0 \quad \forall v \in V$$

Take  $v = u - \tilde{u}$ . Then  $\|u - \tilde{u}\| = 0$ , which implies  $u = \tilde{u}$ .

Standard proof of existence. Denote

$$N = \mathcal{N}(\ell) = \{v \in V \mid \ell(v) = 0\}$$

which is a subspace of  $V$ . If  $N = V$ , then  $\|\ell\| = 0$ , and we may take  $u = 0$ .

Now suppose  $N \neq V$ . Then there exists at least one  $v_0 \in V$  such that  $\ell(v_0) \neq 0$ . It is possible to decompose  $V$  as the direct sum of  $N$  and  $N^\perp$  (By

The Projection Theorem). From this, we have the decomposition  $v_0 = v_1 + v_2$  with  $v_1 \in N$  and  $v_2 \in N^\perp$ . Then  $\ell(v_2) = \ell(v_0) \neq 0$ . For any  $v \in V$ , we have the property

$$\ell\left(v - \frac{\ell(v)}{\ell(v_2)}v_2\right) = 0$$

Thus

$$v - \frac{\ell(v)}{\ell(v_2)}v_2 \in N$$

and in particular, it is orthogonal to  $v_2$ ,  $\because v_2 \in N^\perp$  :

$$\begin{aligned} \left(v - \frac{\ell(v)}{\ell(v_2)}v_2, v_2\right) &= 0 \\ \Rightarrow (v, v_2) - \frac{\ell(v)}{\ell(v_2)}(v_2, v_2) &= 0 \\ \Rightarrow (v, v_2) &= \frac{\ell(v)}{\ell(v_2)}(v_2, v_2) \\ \Rightarrow (v, v_2) &= \frac{\ell(v)}{\ell(v_2)}\|v_2\|^2 \\ \Rightarrow \frac{(v, v_2)}{\|v_2\|^2}\ell(v_2) &= \ell(v) \\ \Rightarrow \left(v, \frac{\ell(v_2)}{\|v_2\|^2}v_2\right) &= \ell(v) \end{aligned}$$

i.e.,

$$\ell(v) = \left(v, \frac{\ell(v_2)}{\|v_2\|^2}v_2\right)$$

In other words, we may take  $u$  to be  $\left(\ell(v_2) / \|v_2\|^2\right)v_2$ .

We complete the proof of the theorem by showing (1.8). From (1.7) and the Schwarz inequality,

$$|\ell(v)| \leq \|u\|\|v\| \quad \forall v \in V$$

Hence

$$\|\ell\| \leq \|u\|$$

Let  $v = u$  in (1.7). Then

$$\ell(u) = \|u\|^2$$

and

$$\|\ell\| = \sup_{v \neq 0} \frac{|\ell(v)|}{\|v\|} \geq \frac{|\ell(u)|}{\|u\|} \geq \|u\|$$

Therefore, (1.8) holds.

This theorem is a fundamental tool in the solvability theory for elliptic partial differential equations, as we see later.

**Example 1.1.3.** Let  $\Omega \subset \mathbb{R}^n$  be open bounded.  $V = L^2(\Omega)$  is a Hilbert space. By the Riesz representation theorem, there is a one-to-one correspondence between  $V'$  and  $V$  by the relation (1.7). We can identify  $\ell \in V'$  with  $u \in V$  related by (1.7). In this sense,  $(L^2(\Omega))' = L^2(\Omega)$ .

### 1.1.3 Weak Derivatives

A sequence  $\{\varphi_j\}$  in  $C_0^\infty(\Omega)$  is said to converge to  $\varphi$  in  $C_0^\infty(\Omega)$  if

(a) There exists a compact set  $K$  in  $\Omega$  such that  $\varphi_j$  vanishes outside  $K$  for any  $j$ , and

(b) For each multi-index  $\alpha$ ,  $D^\alpha \varphi_j \rightarrow D^\alpha \varphi$  uniformly in  $\Omega$ . The space  $C_0^\infty(\Omega)$  endowed with this notion of convergence is called the space of test functions and is denoted by  $D(\Omega)$ .

**Definition 1.1.16.** A distribution on  $\Omega$  is a continuous linear functional on  $D(\Omega)$ . That is, a linear functional  $\ell$  on  $D(\Omega)$  is a distribution if and only if

$$\varphi_k \rightarrow \varphi \text{ in } D(\Omega) \text{ implies } \langle \ell, \varphi_k \rangle \rightarrow \langle \ell, \varphi \rangle.$$

The space of distributions is denoted by  $D'(\Omega)$ .

**Definition 1.1.17.** Let  $1 \leq p < \infty$ . A function  $v : \Omega \subset \mathbb{R}^n \rightarrow \mathbb{R}$  is said to be locally  $p$ -integrable,  $v \in L_{loc}^p(\Omega)$ , if for every  $x \in \Omega$ , there is an open neighborhood  $\Omega'$  of  $x$  such that  $\overline{\Omega'} \subset \Omega$  and  $v \in L_p(\Omega')$ .

**Lemma 1.1.10.** (Generalized Variational Lemma) Let  $v \in L_{loc}^1(\Omega)$  with  $\Omega$  a nonempty open set in  $\mathbb{R}^n$ . If

$$\int_{\Omega} v(x)\varphi(x)dx = 0 \quad \forall \varphi \in C_0^{\infty}(\Omega)$$

,then  $v = 0$  a.e. on  $\Omega$ .

Assume we are given a function  $f \in C^1(\Omega)$ ,then if  $\varphi \in C_0^{\infty}(\Omega)$ , we see from the integration by parts formula that

$$(i) \quad \int_{\Omega} f\varphi_{x_i}dx = - \int_{\Omega} f_{x_i}\varphi dx (i = 1, \dots, n).$$

There are no boundary terms, since  $\varphi$  has compact support in  $\Omega$  and then vanishes near  $\partial\Omega$ . More generally now,if  $k$  is a positive integer ,  $f \in C^k(\Omega)$ , and  $\alpha = (\alpha_1, \dots, \alpha_n)$  is a multi index of order  $|\alpha| = \alpha_1 + \dots + \alpha_n = k$ , then

$$(ii) \quad \int_{\Omega} fD^{\alpha}\varphi dx = (-1)^{|\alpha|} \int_{\Omega} D^{\alpha}f\varphi dx.$$

This equality holds since

$$D^{\alpha}u(x) := \left(\frac{\partial}{\partial x_1}\right)^{\alpha_1} \cdots \left(\frac{\partial}{\partial x_n}\right)^{\alpha_n} u(x_1, \dots, x_n)$$

and we can apply formula(i)  $|\alpha|$  times

We next examine formula (ii),valid for  $f \in C^k(\Omega)$ , and ask wether some variant of it might still be true even if  $f$  is not  $k$  times continuously differentiable. Now the left hand side of (ii) makes sense if  $f$  is only locally summable: the problem is rather that if  $f$  is not  $C^k$ , then the expression " $D^{\alpha}f$ " on the right hand side of (ii) has no obvious meaning. We resolve this difficulty by asking if there exists a locally summable function  $g$  for which formula (ii) is valid,with  $g$  replacing  $D^{\alpha}f$ :

**Definition 1.1.18.** Let  $f \in L_{loc}^1(\Omega)$  be locally integrable function on the open set  $\Omega \subseteq \mathbb{R}^n$ . Given a multi-indices  $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_n)$ , if there exist a locally integrable function  $g \in L_{loc}^1(\Omega)$  such that

$$\int_{\Omega} fD^{\alpha}\varphi dx = (-1)^{|\alpha|} \int_{\Omega} D^{\alpha}f\varphi dx = (-1)^{|\alpha|} \int_{\Omega} g\varphi dx$$

for all test functions, then we say that  $g$  is the weak  $\alpha$ -th derivatives of  $f$ , and write  $g = D^{\alpha}f$ .

**Example 1.1.4.** For  $n = 1, \Omega = \mathbb{R}, |x|'_w = \text{sgn } x$

Proof. for all  $\varphi \in D(\Omega)$

$$\begin{aligned}
\int_{-\infty}^{\infty} |x|\varphi'(x)dx &= \int_{-\infty}^0 -x\varphi'(x)dx + \int_0^{\infty} x\varphi'(x)dx \\
&= -x\varphi(x)|_{-\infty}^0 + x\varphi(x)|_0^{\infty} + \int_{-\infty}^0 \varphi(x)dx - \int_0^{\infty} \varphi(x)dx \\
&= \int_{-\infty}^0 \varphi(x)dx - \int_0^{\infty} \varphi(x)dx \\
&= (-1) \left[ -\int_{-\infty}^0 \varphi(x)dx + \int_0^{\infty} \varphi(x)dx \right] \\
&= (-1) \left[ \int_{-\infty}^0 -\varphi(x)dx + \int_0^{\infty} \varphi(x)dx \right] \\
&= (-1) \int_{-\infty}^{\infty} \text{sgn } x\varphi(x)dx \tag{1.9}
\end{aligned}$$

$$\int_{-\infty}^{\infty} |x|\varphi'(x)dx = (-1) \int_{-\infty}^{\infty} |x|'_w\varphi(x)dx \tag{1.10}$$

(1.10) follows from definition of weak derivative

Hence from (1.9) and (1.10) we can see that  $|x|'_w = \text{sgn } x$

**Example 1.1.5.** Let  $n=1, \Omega = (0, 2),$  and

$$f(x) = \begin{cases} x & 0 < x \leq 1 \\ 1 & 1 \leq x < 2 \end{cases}$$

Define

$$g(x) = \begin{cases} 1 & 0 < x \leq 1 \\ 0 & 1 < x < 2 \end{cases}$$

Let us show  $f' = g$  in the weak sense. To see this choose any  $\varphi \in C_0^\infty(\Omega)$  we must demonstrate

$$\int_0^2 f\varphi' dx = - \int_0^2 g\varphi dx$$

But we easily calculate

$$\int_0^2 f\varphi' dx = \int_0^1 x\varphi' dx + \int_1^2 \varphi' dx$$

$$= - \int_0^1 \varphi dx + \varphi(1) - \varphi(1) = - \int_0^1 \varphi dx = - \int_0^2 g\varphi dx$$

as required.

**Example 1.1.6.** Let  $n=1$ ,  $\Omega = (0, 2)$ , and

$$u(x) = \begin{cases} x & 0 < x \leq 1 \\ 2 & 1 < x < 2 \end{cases}$$

We assert  $u'$  does not exist in the weak sense. To check this, we must show there does not exist any function  $v \in C_{loc}^1(\Omega)$  satisfying

$$\int_0^2 u\phi' dx = - \int_0^2 v\phi dx \quad \text{for all } \phi \in C_0^\infty(\Omega). \quad (1.11)$$

Suppose, to the contrary, (1.11) were valid for some  $v$  and  $\phi$ . Then

$$- \int_0^2 v\phi dx = \int_0^2 u\phi' dx = \int_0^1 x\phi' dx + 2 \int_1^2 \phi' dx = - \int_0^1 \phi dx - \phi(1). \quad (1.12)$$

$$\Rightarrow - \int_0^2 v\phi dx = - \int_0^1 \phi dx - \phi(1).$$

$$\Rightarrow - \int_0^2 v\phi dx + \int_0^1 \phi dx = -\phi(1).$$

$$\Rightarrow \int_0^2 v\phi dx - \int_0^1 \phi dx = \phi(1). \quad (1.13)$$

Choose a sequence  $\{\phi_m\}_{m=1}^\infty$  of smooth function satisfying

$$0 \leq \phi_m \leq 1, \phi_m = 1, \phi_m(x) \rightarrow 0 \text{ for all } x \neq 1$$

Replacing  $\phi$  by  $\phi_m$  in (1.13) and sending  $m \rightarrow \infty$ , we discover

$$1 = \lim_{m \rightarrow \infty} \phi_m(1) = \lim_{m \rightarrow \infty} \left[ \int_0^2 v\phi_m dx - \int_0^1 \phi_m dx \right]$$

$$\Rightarrow 1 = \lim_{m \rightarrow \infty} \phi_m(1) = \int_0^2 v \lim_{m \rightarrow \infty} \phi_m dx - \int_0^1 \lim_{m \rightarrow \infty} \phi_m dx$$

$$\Rightarrow 1 = \lim_{m \rightarrow \infty} \phi_m(1) = \int_0^2 v 0 dx - \int_0^1 0 dx \Rightarrow 1 = \lim_{m \rightarrow \infty} \phi_m(1) = 0 - 0$$

$$\Rightarrow 1 = \lim_{m \rightarrow \infty} \phi_m(1) = 0$$

$$\left[ 1 = \lim_{m \rightarrow \infty} \phi_m(1) = \lim_{m \rightarrow \infty} \left[ \int_0^2 v\phi_m dx - \int_0^1 \phi_m dx \right] = 0 \right]$$

$$\Rightarrow 1 = 0 \text{ a contradiction.}$$

**Note :** Functions in Sobolev space are not necessarily smooth.

**Lemma 1.1.11.** (Uniqueness of weak derivatives). A weak  $\alpha^{th}$ -partial derivative of  $f$ , if it exists, is uniquely defined up to a set of measure zero.

*Proof.* Assume that  $g, \tilde{g} \in L_{loc}^1(\Omega)$  satisfy

$$\int_{\Omega} f D^{\alpha} \varphi dx = (-1)^{|\alpha|} \int_{\Omega} g \phi dx = (-1)^{|\alpha|} \int_{\Omega} \tilde{g} \phi dx$$

for all  $\varphi \in C_0^{\infty}(\Omega)$ . Then

$$\int_{\Omega} (g - \tilde{g}) \varphi dx = 0$$

for all  $\varphi \in C_0^{\infty}(\Omega)$ ;  $g - \tilde{g} = 0$  a.e.

$\Rightarrow g = \tilde{g}$  a.e.

Hence a weak  $\alpha^{th}$  -partial derivative of  $f$ , if it exists, is uniquely defined up to a set of measure zero.  $\square$

#### 1.1.4 Sobolev Spaces

**Definition 1.1.19.** Let  $\Omega \subset \mathbb{R}^n$  be an open set,  $1 \leq p \leq \infty$ , and  $k \geq 0$  be an integer. The Sobolev spaces  $W^{k,p}(\Omega)$  of order  $k$  is defined by

$W^{k,p}(\Omega) = \{f \in L_p(\Omega) : D^{\alpha} f \in L_p(\Omega), \text{ for all multi - indices } \alpha \text{ such that } |\alpha| \leq k\}$ .

The Sobolev space  $W_p^k(\Omega)$  consists of all locally summable functions  $f: \Omega \rightarrow \mathbb{R}$  such that for each multi-indices  $\alpha$  with  $|\alpha| \leq k$ ,  $D^{\alpha} f$  exists in the weak sense and belongs to  $L^p(\Omega)$ .

**Definition 1.1.20.** For  $f \in W^{k,p}(\Omega)$ , then  $W^{k,p}(\Omega)$  norm is

$$\|f\|_{W^{k,p}(\Omega)} = \left( \sum_{|\alpha| \leq k} \|D^{\alpha} f\|_{L_p(\Omega)}^p \right)^{\frac{1}{p}} \quad \text{if } 1 \leq p < \infty \quad (1.14)$$

and

$$\|f\|_{W^{k,p}(\Omega)} = \text{Sup}\{\|D^{\alpha} f\|_{L_{\infty}}\} \quad (p = \infty) \quad (1.15)$$

**Definition 1.1.21.** The standard seminorm over the space  $W^{k,p}(\Omega)$

$$|f|_{W^{k,p}(\Omega)} = \begin{cases} \left( \sum_{|\alpha|=k} \|D^{\alpha} f\|_{L_p(\Omega)}^p \right)^{\frac{1}{p}} & 1 \leq p < \infty \\ \text{Sup}\{\|D^{\alpha} f\|_{L_{\infty}}\} & p = \infty. \end{cases}$$

**Remark 1.1.1.** We can replace  $\|v\|_{W^{k,p}(\Omega)}$  by the simpler notation  $\|v\|_{k,p,\Omega}$  or even  $\|v\|_{k,p}$  when no confusion results.

**Theorem 1.1.12.** The space  $W^{k,p}(\Omega)$  is a Banach space.

*Proof.* Let  $f_1, f_2 \in W^{k,p}(\Omega)$ . For  $|\alpha| \leq k$ , call  $D^{\alpha} f_1, D^{\alpha} f_2$  their weak derivatives. Then, for any  $\lambda, \mu \in \mathbb{R}$ , the linear combination  $\lambda f_1 + \mu f_2$  is locally integrable function. Its weak derivatives are

$$D^{\alpha} (\lambda f_1 + \mu f_2) = \lambda D^{\alpha} f_1 + \mu D^{\alpha} f_2$$

Therefore,  $D^\alpha (\lambda f_1 + \mu f_2) \in L_p(\Omega)$  for every  $|\alpha| \leq k$ . This proves that  $W^{k,p}(\Omega)$  is a vector spaces. Next, we show that (1.7) and (1.8) are norm. Indeed, for  $\lambda \in \mathbb{R}$  and  $f \in W^{k,p}(\Omega)$  one has

$$\begin{aligned}\|\lambda f\|_{W^{k,p}(\Omega)} &= |\lambda| \|f\|_{W^{k,p}(\Omega)} \\ \|f\|_{W^{k,p}(\Omega)} &\geq \|f\|_{L(p)} \geq 0\end{aligned}$$

with equality holding if and only if  $f = 0$ . Moreover, if  $f_1, f_2 \in W^{k,p}(\Omega)$ , then  $1 \leq p < \infty$  Minkowski's inequality yields

$$\begin{aligned}\|f_1 + f_2\|_{W^{k,p}(\Omega)} &= \left( \sum_{|\alpha| \leq k} \|D^\alpha f_1 + D^\alpha f_2\|_{L_p(\Omega)}^p \right)^{\frac{1}{p}} \\ &\leq \left( \sum_{|\alpha| \leq k} \left( \|D^\alpha f_1\|_{L_p(\Omega)} + \|D^\alpha f_2\|_{L_p(\Omega)} \right)^p \right)^{\frac{1}{p}} \\ &\leq \left( \sum_{|\alpha| \leq k} \|D^\alpha f_1\|_{L_p(\Omega)}^p \right)^{\frac{1}{p}} + \left( \sum_{|\alpha| \leq k} \|D^\alpha f_2\|_{L_p(\Omega)}^p \right)^{\frac{1}{p}} \\ &= \|f_1\|_{W^{k,p}(\Omega)} + \|f_2\|_{W^{k,p}(\Omega)}\end{aligned}$$

In the case  $p = \infty$ , the above computation is repalced by

$$\begin{aligned}\|f_1 + f_2\|_{W^{k,p}(\Omega)} &= \text{Max}_{|\alpha| \leq k} \left( \|D^\alpha f_1 + D^\alpha f_2\|_{L_\infty(\Omega)} \right) \\ &\leq \text{Max}_{|\alpha| \leq k} \|D^\alpha f_1\|_{L_\infty(\Omega)} + \text{Max}_{|\alpha| \leq k} \|D^\alpha f_2\|_{L_\infty(\Omega)} \\ &= \|f_1\|_{W^{k,p}(\Omega)} + \|f_2\|_{W^{k,p}(\Omega)}\end{aligned}$$

To conclude the proof, we need to show that the spaces  $W^{k,p}(\Omega)$  is complete, hence is a Banach spaces. Let  $\{f_j\}_{j=1}^\infty$  be cauchy sequence in  $W^{k,p}(\mathbb{R}^n)$ . Then  $\{D^\alpha f_j\}_{j=1}^\infty$  are cauchy sequence in  $L_p(\mathbb{R}^n)$  for  $|\alpha| \leq k$ . Hence there are  $f^\alpha \in L_p(\mathbb{R}^n)$  with for  $|\alpha| \leq k$ . Hence there are  $f^\alpha \in L_p(\mathbb{R}^n)$  with

$$D^\alpha f_j \rightarrow f^\alpha \text{ in } f^\alpha \in L_p(\mathbb{R}^n), |\alpha| \leq k, \text{ and } f^0 = f.$$

It follows from

$$\int_{\mathbb{R}^n} D^\alpha f_j(x) \varphi(x) dx = (-1)^{|\alpha|} \int_{\mathbb{R}^n} f_j(x) D^\alpha \varphi(x) dx, \varphi \in \mathcal{S}(\mathbb{R}^n)$$

And Hölder's inequality applied to  $D^\alpha f_j \rightarrow f^\alpha, f_j \rightarrow f \in L_p(\mathbb{R}^n)$  and

$\varphi, D^\alpha \varphi \in \mathcal{S}(\mathbb{R}^n)$  that

$$\int_{\mathbb{R}^n} D^\alpha f^\alpha(x) \varphi(x) dx = (-1)^{|\alpha|} \int_{\mathbb{R}^n} f(x) D^\alpha \varphi(x) dx, \varphi \in \mathcal{S}(\mathbb{R}^n)$$

Then  $f^\alpha = D^\alpha f$ ,  $|\alpha| \leq k$ , and  $f \in W^{k,p}(\mathbb{R}^n)$  with  $f_j \rightarrow f$  in  $W^{k,p}(\mathbb{R}^n)$  for  $j \rightarrow \infty$ . Consequently,  $W^{k,p}(\mathbb{R}^n)$  is Banach space.  $\square$

**Definition 1.1.22.** We denote the  $k$ -th order Sobolev spaces in  $L_2(\Omega)$  by

$$H^k(\Omega) \equiv W^{k,2}(\Omega).$$

And the Sobolev spaces  $W^{k,2}(\Omega)$  equipped with the scalar product

$$(f, g)_{W^{k,2}(\Omega)} = \int_{\Omega} \sum_{|\alpha| \leq k} \partial^\alpha f(x) \overline{\partial^\alpha g(x)} dx$$

becomes Hilbert spaces.

**Definition 1.1.23.** The closure of  $C_0^\infty(\Omega)$  in the norm of  $W^{k,p}(\Omega)$  is denoted by  $W_0^{k,p}(\Omega)$ . So,  $W_0^{k,p}(\Omega)$  is subspace in the spaces  $W^{k,p}(\Omega)$ . When  $p = 2$ , we denote  $H_0^k(\Omega) \equiv W_0^{k,2}(\Omega)$ . With the spaces  $W_0^{k,p}(\Omega)$ , we can then define Sobolev spaces with negative order

We interpret  $W_0^{k,p}(\Omega)$  to be the space of all the functions  $v$  in  $W^{k,p}(\Omega)$  with the property that

$$D^\alpha v(x) = 0 \quad \text{on } \partial\Omega, \forall \alpha \text{ with } |\alpha| \leq k - 1.$$

**Definition 1.1.24.** For any  $1 \leq p \leq \infty$ , the Sobolev space  $W_0^{1,p}(\Omega)$  is defined as the closure of  $D(\Omega)$  with respect to the norm of  $W^{1,p}(\Omega)$ .

We set  $H_0^1(\Omega) = W_0^{1,2}(\Omega)$ . It is clear that  $H_0^1(\Omega) \subset H^1(\Omega)$ . So that  $H_0^1(\Omega)$  is a Hilbert space.

**Definition 1.1.25.** Let  $k \geq 0$ , either an integer or a non-integer. Let  $p \in [1, \infty)$  and denote its conjugate exponent  $p'$  defined by the relation  $\frac{1}{p} + \frac{1}{p'} = 1$ . Then we define  $W^{-k,p'}(\Omega)$  to be the dual space of  $W_0^{k,p}(\Omega)$ . In particular,  $H^{-k}(\Omega) \equiv W^{-k,2}(\Omega)$ .

On several occasions later, we need to use in particular the Sobolev space  $H^{-1}(\Omega)$ , defined as the dual of  $H_0^1(\Omega)$ . Thus, any  $\ell \in H^{-1}(\Omega)$  is a bounded linear functional on  $H_0^1(\Omega)$ :

$$|\ell(v)| \leq M \|v\|, \quad \forall v \in H_0^1(\Omega)$$

The norm of  $\ell$  is

$$\|\ell\|_{H^{-1}(\Omega)} = \sup_{v \in H_0^1(\Omega)} \frac{\|\ell(v)\|}{\|v\|_{H_0^1(\Omega)}}$$

Any function  $f \in L_2(\Omega)$  naturally induces a bounded linear functional  $f \in H^{-1}(\Omega)$  by the relation

$$\langle f, v \rangle = \int_{\Omega} f v dx \quad \forall v \in H_0^1(\Omega)$$

Sometimes even when  $f \in H^{-1}(\Omega) \setminus L_2(\Omega)$ , we write  $\int_{\Omega} f v dx$  for the duality pairing  $\langle f, v \rangle$  between  $H^{-1}(\Omega)$  and  $H_0^1(\Omega)$ , although integration in this situation does not make sense.

**Theorem 1.1.13.** *Assume  $\Omega$  is a Lipschitz domain in  $\mathbb{R}^n$ ,  $1 \leq p < \infty$ . Then there exists a continuous linear operator  $\gamma : W^{1,p}(\Omega) \rightarrow L_p(\partial\Omega)$  with the following properties.*

- (a)  $\gamma v = v | \partial\Omega$  if  $v \in W^{1,p}(\Omega) \cap C(\bar{\Omega})$
- (b) For some constant  $c > 0$ ,  $\|\gamma v\|_{L_p(\partial\Omega)} \leq c \|v\|_{W^{1,p}(\Omega)}$ ,  $\forall v \in W^{1,p}(\Omega)$
- (c) The mapping  $\gamma : W^{1,p}(\Omega) \rightarrow L_p(\partial\Omega)$  is compact; i.e., for any bounded sequence  $\{v_n\}$  in  $W^{1,p}(\Omega)$ , there is a subsequence  $\{v_{n'}\} \subset \{v_n\}$  such that  $\{\gamma v_{n'}\}$  is convergent in  $L_p(\partial\Omega)$ .

The operator  $\gamma$  is called the trace operator, and  $\gamma v$  can be called the generalized boundary value of  $v$ . The range  $\gamma(W^{1,p}(\Omega))$  is a space smaller than  $L_p(\partial\Omega)$ , namely  $W^{1-\frac{1}{p},p}(\partial\Omega)$ , a positive order Sobolev space over the boundary. When we discuss weak formulations of boundary value problems later in this thesis, we need to use traces of the  $H^1(\Omega)$  functions, that form the space  $H^{\frac{1}{2}}(\partial\Omega)$ ; in other words,

$$H^{\frac{1}{2}}(\partial\Omega) = \gamma(H^1(\Omega))$$

Correspondingly, we can use the following as the norm for  $H^{\frac{1}{2}}(\partial\Omega)$  :

$$\|g\|_{H^{\frac{1}{2}}(\partial\Omega)} = \inf_{\substack{v \in H^1(\Omega) \\ \gamma v = g}} \|v\|_{H^1(\Omega)}$$

**Remark 1.1.2.** 1. *The norm  $\sum_{|\alpha| \leq 1} \|\partial^{\alpha} u\|_{p,\Omega}$  is equivalent to the standard norm.*

2.  $W_p^0(\Omega) = L_p(\Omega)$ .

3.  $H^0(\mathbb{R}^n) = L^2(\mathbb{R}^n)$

**Proposition 1.1.14.** *(Poincaré inequality) There exists a constant  $C_{\Omega}$  such that*

$$\|u\|_{L_2(\Omega)} \leq C_{\Omega} \|\nabla u\|_{L_2(\Omega)} \quad \forall u \in H_0^1(\Omega)$$

where the constant  $C_{\Omega}$  is a constant depending on the diameter of  $\Omega$ .

**Theorem 1.1.15.** *(Divergence theorem) Let  $\Omega$  be a bounded Lipschitz domain in  $\mathbb{R}^n$  and  $Q : \Omega \rightarrow \mathbb{R}^n$  be a vector field whose components are in  $H^1(\Omega)$ . The following equality holds*

$$\int_{\Omega} \nabla \cdot Q dx = \int_{\partial\Omega} Q \cdot \nu ds$$

where  $\nu$  is the unit outward pointing normal to  $\partial\Omega$ .

**Proposition 1.1.16.** For  $Q \in H^1(\bar{\Omega}; \mathbb{R}^n)$ ,  $g \in H^1(\Omega)$ ,

$$\int_{\Omega} Q \cdot \nabla g dx = \int_{\partial\Omega} g Q \cdot \nu ds - \int_{\Omega} g \nabla \cdot Q dx$$

Using the divergence theorem, we see that

$$\int_{\Omega} \nabla \cdot (Qg) dx = \int_{\partial\Omega} Q \cdot \nu g ds$$

Alternatively,

$$\nabla \cdot (Qg) = g \cdot \nabla Q + Q \cdot \nabla g$$

By bringing together these equality, we obtain the desired formula.

For example, if  $Q = \nabla u$  and  $g = v$ , we have by integration by parts formula that

$$\int_{\Omega} \nabla u \cdot \nabla v dx = \int_{\partial\Omega} v \nabla u \cdot \nu ds - \int_{\Omega} v \Delta u dx$$

where  $\Delta$  is the Laplacian and noting that

$$\nabla u \cdot \nu = \frac{\partial u}{\partial \nu}$$

we obtain Green's formula

$$- \int_{\Omega} v \Delta u dx = - \int_{\partial\Omega} v \frac{\partial u}{\partial \nu} ds + \int_{\Omega} \nabla u \cdot \nabla v dx$$

## Chapter 2

# Variational Formulations Of Elliptic Boundary Value Problems

### 2.1 Review on Partial differential equation

**Definition 2.1.1.** Let  $u = u(x_1, \dots, x_n)$  be a function of  $n$  independent variables  $x_1, \dots, x_n$ . A Partial Differential Equation (PDE) is an equation that contains the independent variables  $x_1, \dots, x_n$ , the dependent variable or the unknown function  $u : \Omega \rightarrow \mathbb{R}$ , where  $\Omega$  is an open subset of  $\mathbb{R}^n$ ,  $n \geq 2$  and its partial derivatives up to some order. It has the form

$$F(x_1, \dots, x_n, u, u_{x_1}, \dots, u_{x_n}, u_{x_1x_1}, \dots, u_{x_ix_j}, \dots) = 0$$

where  $F$  is a given function and  $u_{x_j} = \partial u / \partial x_j$ ,  $u_{x_ix_j} = \partial^2 u / \partial x_i \partial x_j$ ,  $i, j = 1, \dots, n$  are the partial derivatives of  $u$ . Let  $\Omega \subseteq \mathbb{R}^n$  be an open set and  $s^{n \times n}$  be the set of  $n \times n$  real symmetric matrices. A second order partial differential equation on  $\Omega$  in an unknown  $u = u(x_1, \dots, x_n)$  is an equation of the form

$$F(x, u, Du, D^2u) = 0 \tag{2.1}$$

where  $F : \Omega \times \mathbb{R} \times \mathbb{R}^n \times s^{n \times n} \rightarrow \mathbb{R}$ . A typical point  $\varpi$  of  $\Gamma = \Omega \times \mathbb{R} \times \mathbb{R}^n \times s^{n \times n}$  is given by  $\varpi = (x, z, \xi, \eta)$  where  $x \in \Omega$ ,  $z \in \mathbb{R}$ ,  $\xi \in \mathbb{R}^n$ ,  $\eta \in s^{n \times n}$ .

**Definition 2.1.2.** The second order partial differential equation (2.1) is called linear if it is of the form

$$\sum_{i,j=1}^n a_{ij}(x) D_{ij}u + \sum_{i=1}^n b_i(x) D_iu + c(x)u + d(x) = 0$$

A partial differential equation (2.1) and assume that  $F$  is differentiable in the  $\eta$ . We extend  $F$  to the whole space of  $n \times n$  by say  $F = (x, z, \xi, \eta) = F(x, z, \xi, \frac{1}{2}(A + A^T))$ , where  $(x, z, \xi, A) \in \Omega \times \mathbb{R} \times \mathbb{R}^n \times s^{n \times n} \rightarrow \mathbb{R}$  the  $n \times n$  matrix  $[F_{ij}(\varpi)]_{n \times n}$  is symmetric where  $F_{ij} := \frac{\partial F}{\partial \eta_{ij}}$

**Definition 2.1.3.** *The equation (2.1) is said to be elliptic at a point  $\varpi = (x, z, \xi, \eta) \in \Gamma = \Omega \times \mathbb{R} \times \mathbb{R}^n \times s^{n \times n}$  if and only if the matrix  $[F_{ij}(\varpi)]_{n \times n}$  is positive definite, that is  $\sum_{i,j=1}^n \frac{\partial F}{\partial \eta_{ij}} \zeta_i \zeta_j > 0$  for all  $\zeta \in \mathbb{R}^n \setminus \{0\}$ .*

Equivalently, the partial differential equation (2.1) is elliptic at  $\varpi$  if and only if all the Eigen values (they depend on  $\varpi$ ) of  $[F_{ij}(\varpi)]_{n \times n}$  are positive.

**Example 2.1.1.** *The equation*

$$-\Delta u = f \quad \left( \text{where } \Delta = \sum_{i=1}^3 \frac{\partial^2}{\partial x_i^2} \quad \text{in } \mathbb{R}^3 \right)$$

is elliptic everywhere in  $\mathbb{R}^3$ . A partial differential equation subject to certain conditions in the form of initial or boundary conditions is known as an initial value problem (IVP) or boundary value problem (BVP). The Initial conditions, also known as Cauchy conditions, are the values of the unknown function  $u$  and a suitable number of its derivatives at the initial point, whereas the boundary conditions are the values on the boundary  $\partial\Omega$  of the domain  $\Omega$  in consideration.

The 3 predominant sorts of boundary situations are:

(i) Dirichlet conditions or boundary conditions of the first kind are the values of  $u$  prescribed at each point of the boundary  $\partial\Omega$ .

(ii) Neumann conditions or boundary conditions of the second kind are the values of the normal derivative of  $u$  prescribed at each point of the boundary  $\partial\Omega$ .

(iii) Robin conditions, also known as mixed boundary conditions or third-kind boundary conditions, are the values of a linear combination of  $u$  and its normal derivative imposed at each point of the boundary  $\partial\Omega$ .

**Remark 2.1.1.** *Dirichlet boundary conditions are frequently referred to as essential because they have a significant influence on the weak formulation: they determine the function space in which the solution is found. Neumann boundary conditions, on the other hand, have no effect on the function space and can therefore be naturally integrated into the boundary integral. Thus, they are known as natural.*

**Definition 2.1.4.** *(Hadamard's well-posedness) A problem is considered well-posed if*

1. *The solution is unique ,*
2. *The solution depends continuously on the given data.*

*otherwise the problem is ill-posed.*

### 2.1.1 A model boundary value problem

Let  $\Omega$  be a bounded domain in  $\mathbb{R}^n$  with a Lipschitz continuous boundary  $\Gamma$ . The unit outward normal vector  $n = (n_1, \dots, n_d)^T$  exists a.e. on  $\Gamma$ , and we will

use  $\partial u / \partial n$  to denote the normal derivative of  $u$  on  $\Gamma$ .

We use the following model boundary value problem as an illustration.

$$\begin{cases} -\Delta u = f, & \text{in } \Omega \\ u = 0, & \text{on } \Gamma. \end{cases} \quad (2.2)$$

Here  $\Delta$  denotes the Laplacian operator, defined by

$$\Delta u = \frac{\partial^2 u}{\partial x_i \partial x_i}$$

The differential equation in (2.2) is called the Poisson equation. The Poisson equation may be used to explain many physical processes, e.g., steady state heat conduction, electrostatics, deformation of a thin elastic membrane. A classical solution of the problem (2.2) is a smooth function  $u \in C^2(\Omega) \cap C(\bar{\Omega})$  which satisfies the differential equation (2.2)<sub>1</sub> and the boundary condition (2.2)<sub>2</sub> pointwise. Necessarily we have to assume  $f \in C(\Omega)$ , but this condition, or even the stronger condition  $f \in C(\bar{\Omega})$  does not guarantee the existence of a classical solution of the problem. One goal of introducing the weak formulation is to eliminate the high smoothness criterion from the solution, making it easier to find a (weak) solution.

To derive the weak formulation corresponding to (2.2), we temporarily assume it has a classical solution  $u \in C^2(\Omega) \cap C(\bar{\Omega})$ . We multiply the differential equation (2.2)<sub>1</sub> by an arbitrary function  $v \in C_0^\infty(\Omega)$ , and integrate the relation on  $\Omega$ ,

$$-\int_{\Omega} \Delta u v dx = \int_{\Omega} f v dx$$

An integration by parts for the integral on the left side yields

$$\int_{\Omega} \nabla u \cdot \nabla v dx - \int_{\Gamma} v \frac{\partial u}{\partial n} ds = \int_{\Omega} f v dx$$

recall that  $v = 0$  on  $\Gamma$

$$\Rightarrow \int_{\Omega} \nabla u \cdot \nabla v dx = \int_{\Omega} f v dx \quad (2.3)$$

This relation was proved under the assumptions  $u \in C^2(\Omega) \cap C(\bar{\Omega})$  and  $v \in C_0^\infty(\Omega)$ . However, for all the terms in the relation (2.3) to make sense, we only need to require the following regularities of  $u$  and  $v$  that is  $u, v \in H^1(\Omega)$ , assuming  $f \in L_2(\Omega)$ . Recalling the homogeneous Dirichlet boundary condition (2.2)<sub>2</sub>, we thus seek a solution  $u \in H_0^1(\Omega)$  satisfying the relation (2.3) for any  $v \in C_0^\infty(\Omega)$ . Since  $C_0^\infty(\Omega)$  is dense in  $H_0^1(\Omega)$ , the relation (2.3) is then valid for any  $v \in H_0^1(\Omega)$ . Therefore, the weak formulation of the boundary value problem (2.2) is

$$u \in H_0^1(\Omega), \quad \int_{\Omega} \nabla u \cdot \nabla v dx = \int_{\Omega} f v dx \quad \forall v \in H_0^1(\Omega) \quad (2.4)$$

Actually, we can even weaken the assumption,  $f \in L_2(\Omega)$ . It is enough for us to assume  $f \in H^{-1}(\Omega) = (H_0^1(\Omega))'$ , as long as we interpret the integral  $\int_{\Omega} f v dx$  as the duality pairing  $\langle f, v \rangle$  between  $H^{-1}(\Omega)$  and  $H_0^1(\Omega)$ . We adopt the convention of using  $\int_{\Omega} f v dx$  for  $\langle f, v \rangle$  when  $f \in H^{-1}(\Omega)$  and  $v \in H_0^1(\Omega)$ . We have shown that if  $u$  is a classical solution of (2.2), then it is also a solution of the weak formulation (2.4).

On the other hand, let  $u$  be a weak solution with the additional regularity  $u \in C^2(\Omega) \cap C(\bar{\Omega})$  and  $f \in C(\Omega)$ . Then for any  $v \in C_0^\infty(\Omega) \subset H_0^1(\Omega)$  from (2.4) we obtain

From integration by part

$$-\int_{\Omega} \Delta u v dx = \int_{\Omega} \nabla u \cdot \nabla v dx - \int_{\Gamma} v \frac{\partial u}{\partial n} ds$$

$$\Rightarrow -\int_{\Omega} \Delta u v dx = \int_{\Omega} \nabla u \cdot \nabla v dx$$

$$\Rightarrow - \int_{\Omega} \Delta u v dx = \int_{\Omega} f v dx \quad (\text{by (2.4)})$$

$$\int_{\Omega} (-\Delta u - f) v dx = 0$$

The differential equation (2.2)<sub>1</sub> is fulfilled when  $-\Delta u = f$  in  $\Omega$ . Also,  $u$  meets the homogeneous Dirichlet boundary condition pointwisely. Thus, we've demonstrated that the boundary value problem (2.2) and the variational problem (2.3) are technically equivalent. If the weak solution  $u$  doesn't have regularity,  $u \in C^2(\Omega) \cap C(\bar{\Omega})$ . We will state that  $u$  formally solves the boundary value problem (2.2).

We let  $V = H_0^1(\Omega)$ ,  $a(\cdot, \cdot) : V \times V \rightarrow \mathbb{R}$  the bilinear form defined by

$$a(u, v) = \int_{\Omega} \nabla u \cdot \nabla v dx \quad \text{for } u, v \in V$$

and  $\ell : V \rightarrow \mathbb{R}$  the linear functional defined by

$$\ell(v) = \int_{\Omega} f v dx \quad \text{for } v \in V$$

Then the weak formulation of the problem (2.2) is to find  $u \in V$  such that

$$a(u, v) = \ell(v) \quad \forall v \in V \quad (2.5)$$

We define a differential operator  $A$  associated with the boundary value problem (2.2) by

$$A : H_0^1(\Omega) \rightarrow H^{-1}(\Omega), \quad \langle Au, v \rangle = a(u, v) \quad \forall u, v \in H_0^1(\Omega).$$

Here,  $\langle \cdot, \cdot \rangle$  denotes the duality pairing between  $H_0^1(\Omega)$  and  $H^{-1}(\Omega)$ . Then the problem (2.5) can be viewed as a linear operator equation

$$Au = \ell \quad \text{in } H^{-1}(\Omega)$$

A formulation of the type (2.2) in the form of a partial differential equation and a set of boundary conditions is referred to as a classical formulation of a boundary value problem, whereas a formulation of the type (2.5) is known as a weak formulation. One advantage of weak formulations over classical formulations is that questions related to existence and uniqueness of solutions can be answered more satisfactorily.

### 2.1.2 Some general results on existence and uniqueness

We first present some general ideas and results on existence and uniqueness for a linear operator equation of the form

$$u \in V, \quad Lu = f \quad (2.6)$$

where  $L : \mathcal{D}(L) \subset V \rightarrow W$ ,  $V$  and  $W$  are Hilbert spaces, and  $f \in W$ . Notice that the solvability of the equation is equivalent to the condition  $\mathcal{R}(L) = W$ , whereas the uniqueness of a solution is equivalent to the condition  $\mathcal{N}(L) = \{0\}$ . A very basic existence result is the following theorem.

**Theorem 2.1.1.** *Let  $V$  and  $W$  be Hilbert spaces,  $L : \mathcal{D}(L) \subset V \rightarrow W$  a linear operator. Then  $\mathcal{R}(L) = W$  if and only if  $\mathcal{R}(L)$  is closed and  $\mathcal{R}(L)^\perp = \{0\}$ .*

*Proof.* If  $\mathcal{R}(L) = W$ , then obviously  $\mathcal{R}(L)$  is closed and  $\mathcal{R}(L)^\perp = \{0\}$ . Now assume  $\mathcal{R}(L)$  is closed and  $\mathcal{R}(L)^\perp = \{0\}$ , but  $\mathcal{R}(L) \neq W$ . Then  $\mathcal{R}(L)$  is a closed subspace of  $W$ . Let  $w \in W \setminus \mathcal{R}(L)$ . By theorem(1.1.8), the compact set  $\{w\}$  and the closed convex set  $\mathcal{R}(L)$  can be strictly separated by a closed hyperplane, i.e., there exists a  $w^* \in W'$  such that  $\langle w^*, w \rangle > 0$  and  $\langle w^*, Lv \rangle \leq 0$  for all  $v \in \mathcal{D}(L)$ . Since  $L$  is a linear operator,  $\mathcal{D}(L)$  is a subspace of  $V$ . Hence,  $\langle w^*, Lv \rangle = 0$  for all  $v \in \mathcal{D}(L)$ . Therefore,  $0 \neq w^* \in \mathcal{R}(L)^\perp$ . This is a contradiction.

Let us see under what conditions  $\mathcal{R}(L)$  is closed. We first introduce an important generalization of the notion of continuity.

**Definition 2.1.5.** *Let  $V$  and  $W$  be Banach spaces. An operator  $L : \mathcal{D}(L) \subset V \rightarrow W$  is said to be a closed operator if for any sequence  $\{v_n\} \subset \mathcal{D}(L)$ ,  $v_n \rightarrow v$  and  $L(v_n) \rightarrow w$  imply  $v \in \mathcal{D}(L)$  and  $w = L(v)$ .*

**Theorem 2.1.2.** *Let  $V$  and  $W$  be Hilbert spaces,  $L : \mathcal{D}(L) \subset V \rightarrow W$  a linear closed operator. Assume for some constant  $c > 0$ , the following a priori estimate holds:*

$$\|Lv\|_W \geq c\|v\|_V \quad \forall v \in \mathcal{D}(L) \quad (2.7)$$

which is usually called a stability estimate. Also assume  $\mathcal{R}(L)^\perp = \{0\}$ . Then for each  $f \in W$ , the equation (2.6) has a unique solution.

*Proof.* Let us verify that  $\mathcal{R}(L)$  is closed. Let  $\{f_n\}$  be a sequence in  $\mathcal{R}(L)$ , converging to  $f$ . Then there is a sequence  $\{v_n\} \subset \mathcal{D}(L)$  with  $f_n = Lv_n$ . By (2.7),

$$c\|v_n - v_m\|_V \leq \|f_n - f_m\|_W$$

Thus  $\{v_n\}$  is a Cauchy sequence in  $V$ . Since  $V$  is a Hilbert space, the sequence  $\{v_n\}$  converges :  $v_n \rightarrow v \in V$ . Now  $L$  is assumed to be closed,

we conclude that  $v \in \mathcal{D}(L)$  and  $f = Lv \in \mathcal{R}(L)$ . Hence  $\mathcal{R}(L)$  is closed. So we can invoke Theorem(2.1.1) to obtain the existence of a solution. Let  $v \in \mathcal{N}(L)$  implies  $Lv = 0$ , so that  $\|Lv\| = 0$  by stability estimate (2.7) we obtain  $c\|v\|_V \leq \|Lv\|_W = 0$ . Hence,  $v = 0$ . Thus  $\mathcal{N}(L) = \{0\}$ . Therefore there is a unique solution.

**Example 2.1.2.** Let  $V$  be a Hilbert space,  $L \in \mathcal{L}(V, V')$  be strongly monotone, i.e., for some constant  $c > 0$ ,

$$\langle Lv, v \rangle \geq c\|v\|_V^2 \quad \forall v \in V.$$

Then (2.7) holds because from the monotonicity,

$$|\langle Lv, v \rangle| \geq |c\|v\|_V^2| = c\|v\|_V^2 \quad \forall v \in V.$$

But

$$\begin{aligned} |\langle Lv, v \rangle| &\leq \|Lv\|_{V'} \|v\|_V \\ \Rightarrow \|Lv\|_{V'} \|v\|_V &\geq c\|v\|_V^2 \end{aligned}$$

which implies

$$\|Lv\|_{V'} \geq c\|v\|_V$$

Also  $\mathcal{R}(L)^\perp = \{0\}$ , since from  $v \perp \mathcal{R}(L)$  we have

$$c\|v\|_V^2 \leq \langle Lv, v \rangle = 0$$

and hence  $v = 0$ . Therefore from theorem (2.1.2) , under the stated assumptions, for any  $f \in V'$ , there is a unique solution  $u \in V$  to the equation  $Lu = f$  in  $V'$ .

**Example 2.1.3.** As a concrete example, we consider the weak formulation of the model elliptic boundary value problem (2.6). Here,  $\Omega \subset \mathbb{R}^n$  is an open bounded set with a Lipschitz boundary  $\partial\Omega$ ,  $V = H_0^1(\Omega)$  with the norm  $\|v\|_V = |v|_{H^1(\Omega)}$ , and  $V' = H^{-1}(\Omega)$ . Given  $f \in H^{-1}(\Omega)$ , consider the problem

$$\begin{cases} -\Delta u = f & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases} \quad (2.8)$$

We define the operator  $L : V \rightarrow V'$  by

$$\langle Lu, v \rangle = \int_{\Omega} \nabla u \cdot \nabla v dx \quad u, v \in V$$

Then  $L$  is linear, continuous, and strongly monotone; thus, we have

$$\langle Lv, v \rangle = \|v\|_V^2 \quad \forall v \in V$$

Proof.  $\forall u, v$  and  $w \in V$  and  $\forall \alpha, \beta \in \mathbb{R}$

$$\begin{aligned}
\langle L(\alpha u + \beta w), v \rangle &= \int_{\Omega} \nabla(\alpha u + \beta w) \cdot \nabla v dx \\
&= \int_{\Omega} (\alpha \nabla u + \beta \nabla w) \cdot \nabla v dx \\
&= \int_{\Omega} (\alpha \nabla u \cdot \nabla v + \beta \nabla w \cdot \nabla v) dx \\
&= \alpha \int_{\Omega} \nabla u \cdot \nabla v dx + \beta \int_{\Omega} \nabla w \cdot \nabla v dx \\
&= \alpha \langle Lu, v \rangle + \beta \langle Lw, v \rangle \\
&= \langle \alpha Lu, v \rangle + \langle \beta Lw, v \rangle \\
&= \langle \alpha Lu + \beta Lw, v \rangle
\end{aligned}$$

Thus,  $L(\alpha u + \beta w) = \alpha Lu + \beta Lw$ . Hence  $L$  is linear. Next, let us show continuity

$$\begin{aligned}
\langle Lu, v \rangle &= \int_{\Omega} \nabla u \cdot \nabla v dx \\
|\langle Lu, v \rangle| &\leq \int_{\Omega} |\nabla u \cdot \nabla v| dx \\
&\leq \int_{\Omega} \|\nabla u\| \|\nabla v\| dx \quad (\text{by Cauchy Schwarz inequality for } \mathbb{R}^2) \\
&\leq \|\nabla u\|_{L_2(\Omega)} \|\nabla v\|_{L_2(\Omega)} \quad (\text{by Cauchy Schwarz inequality for } L_2(\Omega))
\end{aligned}$$

Since,

$$\begin{aligned}
\|\nabla u\|_{L_2(\Omega)}^2 &= \int_{\Omega} \nabla u \cdot \nabla u \\
&\leq \int_{\Omega} (\nabla u \cdot \nabla u + u^2) \\
&= \|u\|_V^2
\end{aligned}$$

$\Rightarrow \|\nabla u\|_{L_2(\Omega)}^2 \leq \|u\|_V^2 \Rightarrow \|\nabla u\|_{L_2(\Omega)} \leq \|u\|_V \Rightarrow \langle Lu, u \rangle \leq \|u\|_V \|u\|_V$   
and similarly for  $v$ , it follows that

$$|\langle Lu, v \rangle| \leq \|u\|_V \|v\|_V \quad \text{for all } u, v \in V$$

implies,

$$\begin{aligned}
\frac{|\langle Lu, v \rangle|}{\|v\|_V} &\leq \|u\|_V \\
\sup_{v \neq 0} \frac{|\langle Lu, v \rangle|}{\|v\|_V} &\leq \|u\|_V
\end{aligned}$$

$$\Rightarrow \|Lu\| = \sup_{v \neq 0} \frac{|\langle Lu, v \rangle|}{\|v\|_V}$$

Thus,  $\|Lu\| \leq \|u\|_V$ . Hence  $L$  is bounded. Therefore  $L$  is continuous. Now let us show strong monotonicity of  $L$ . From Poincaré's inequality we get

$$\langle Lu, u \rangle \geq c^2 \|u\|_{H^1(\Omega)}^2 \quad \text{for all } u \in H_0^1(\Omega)$$

Therefore,  $L$  is strongly monotone.

Thus from example 2.1.3, for any  $f \in H^{-1}(\Omega)$ , there is a unique  $u \in H_0^1(\Omega)$  such that

$$\int_{\Omega} \nabla u \cdot \nabla v dx = \langle f, v \rangle \quad \forall v \in V$$

i.e., the boundary value problem (2.8) has a unique weak solution.

### 2.1.3 The Lax-Milgram Lemma

The Lax-Milgram Lemma is employed frequently in the study of the weak formulation of linear elliptic boundary value problems of the form (2.5).

For a real Banach space  $V$ , let us first explore the relation between a linear operator  $A : V \rightarrow V'$  and a bilinear form  $a : V \times V \rightarrow \mathbb{R}$  related by

$$\langle Au, v \rangle = a(u, v) \quad \forall u, v \in V \quad (2.9)$$

The bilinear form  $a(\cdot, \cdot)$  is continuous if and only if there exists  $M > 0$  such that

$$|a(u, v)| \leq M \|u\| \|v\| \quad \forall u, v \in V.$$

**Theorem 2.1.3.** *There exists a one-to-one correspondence between linear continuous operators  $A : V \rightarrow V'$  and continuous bilinear forms  $a : V \times V \rightarrow \mathbb{R}$ , given by the formula (2.9).*

*Proof.* If  $A \in \mathcal{L}(V, V')$ , then  $a : V \times V \rightarrow \mathbb{R}$  defined in (2.9) is bilinear and bounded:

$$|a(u, v)| \leq \|Au\| \|v\| \leq \|A\| \|u\| \|v\| \quad \forall u, v \in V.$$

Conversely, let  $a(\cdot, \cdot)$  be given as a continuous bilinear form on  $V$ . For any fixed  $u \in V$ , the map  $v \mapsto a(u, v)$  defines a linear continuous operator on  $V$ . Thus, there is an element  $Au \in V'$  such that (2.9) holds. From the bilinearity of  $a(\cdot, \cdot)$ , we obtain the linearity of  $A$ . From the boundedness of  $a(\cdot, \cdot)$ , we obtain the boundedness of  $A$ .  $\square$

With a linear operator  $A$  and a bilinear form  $a$  related through (2.9), many properties of the linear operator  $A$  can be defined through those of the bilinear form  $a$ , or vice versa. Some examples are (assuming  $V$  is a real Hilbert space):

- $a$  is bounded ( $|a(u, v)| \leq M\|u\|\|v\| \quad \forall u, v \in V$ ) if and only if  $A$  is bounded ( $\|Av\| \leq M\|v\| \quad \forall v \in V$ ).
- $a$  is positive ( $a(v, v) \geq 0 \quad \forall v \in V$ ) if and only if  $A$  is positive ( $\langle Av, v \rangle \geq 0 \quad \forall v \in V$ ).
- $a$  is strictly positive ( $a(v, v) > 0 \quad \forall 0 \neq v \in V$ ) if and only if  $A$  is strictly positive ( $\langle Av, v \rangle > 0 \quad 0 \neq v \in V$ ).
- $a$  is strongly positive or  $V$ -elliptic ( $a(v, v) \geq \alpha\|v\|^2 \quad \forall v \in V$ ) if and only if  $A$  is strongly positive ( $\langle Av, v \rangle \geq \alpha\|v\|^2 \quad \forall v \in V$ ).
- $a$  is symmetric ( $a(u, v) = a(v, u) \quad \forall u, v \in V$ ) if and only if  $A$  is symmetric ( $\langle Au, v \rangle = \langle Av, u \rangle \quad \forall u, v \in V$ ).

**Theorem 2.1.4.** (*Lax-Milgram Lemma*) Assume  $V$  is a Hilbert space,  $a(\cdot, \cdot)$  is a bounded,  $V$ -elliptic bilinear form on  $V$ ,  $\ell \in V'$ . Then there is a unique solution of the problem

$$u \in V, \quad a(u, v) = \ell(v) \quad \forall v \in V. \quad (2.10)$$

*Proof.* (Existence) Let  $A : V \rightarrow V'$  be the linear operator associated with the bilinear form  $a(\cdot, \cdot)$ ; see (2.9). Then  $A$  is bounded and strongly positive:  $\forall v \in V$ ,

$$\begin{aligned} \|Av\| &\leq M\|v\|, \\ \langle Av, v \rangle &\geq \alpha\|v\|^2 \end{aligned}$$

Denote  $\mathcal{J} : V' \rightarrow V$  the isometric dual mapping from the Riesz representation theorem. Then

$$a(u, v) = \langle Au, v \rangle = (\mathcal{J}Au, v) \quad \forall u, v \in V$$

and

$$\|\mathcal{J}Au\| = \|Au\| \quad \forall u \in V.$$

And, Let  $L = \mathcal{J}A : V \rightarrow V$ . We recall that  $\mathcal{R}(L) = V$  if and only if  $\mathcal{R}(L)$  is closed and  $\mathcal{R}(L)^\perp = \{0\}$ . To show  $\mathcal{R}(L)$  is closed, we let  $\{u_n\} \subset \mathcal{R}(L)$  be a sequence converging to  $u$ . Then  $u_n = Lw_n = \mathcal{J}Aw_n$  for some  $w_n \in V$ . We have

$$\|u_n - u_m\| = \|\mathcal{J}Aw_n - \mathcal{J}Aw_m\| = \|A(w_n - w_m)\| \geq \alpha\|w_n - w_m\|.$$

Hence  $\{w_n\}$  is a Cauchy sequence and so has a limit  $w \in V$ . Then

$$\|u_n - \mathcal{J}Aw\| = \|\mathcal{J}Aw_n - \mathcal{J}Aw\| = \|A(w_n - w)\| \leq M\|w_n - w\| \rightarrow 0.$$

Hence,  $u = \mathcal{J}Aw \in \mathcal{R}(L)$  and  $\mathcal{R}(L)$  is closed. Now suppose  $u \in \mathcal{R}(L)^\perp$ .

Then for any  $v \in V$ ,

$$0 = (\mathcal{J}Av, u) = a(v, u).$$

Taking  $v = u$  above, we have  $a(u, u) = 0$ . By the  $V$ -ellipticity of  $a(\cdot, \cdot)$ , we conclude  $u = 0$ .

$\Rightarrow R(L)^\perp = \{0\}$ . So we can invoke theorem (2.1.1) to obtain the existence of a solution.

(Uniqueness) Suppose  $u_1, u_2$  are two solutions of (2.10), i.e;

$$a(u_1, v) = \ell(v),$$

$$a(u_2, v) = \ell(v) \quad \forall v \in V$$

Subtraction and linearity give

$$a(u_2, v) - a(u_1, v) = a(u_2 - u_1, v) = 0 \quad \forall v \in V$$

In particular, choose  $v = u_2 - u_1$ , then we see

$$a(u_2 - u_1, u_2 - u_1) = 0.$$

By the  $V$ -ellipticity of  $a(\cdot, \cdot)$ ,

$$\begin{aligned} 0 &\geq \alpha \|u_2 - u_1\|_V^2 \\ \Rightarrow 0 &\geq \|u_2 - u_1\|_V \\ \Rightarrow u_2 &= u_1 \end{aligned}$$

□

**Example 2.1.4.** *Applying the Lax-Milgram Lemma, we conclude that the boundary value problem (2.8) has a unique weak solution  $u \in H_0^1(\Omega)$ .*

#### 2.1.4 Weak formulations of linear elliptic boundary value problems

In this section, we formulate and analyze weak formulations of some linear elliptic boundary value problems. To present the ideas clearly, we will frequently use boundary value problems associated with the Poisson equation

$$-\Delta u = f$$

and the Helmholtz equation

$$-\Delta u + u = f$$

as examples.

### 1. Problems with homogeneous Dirichlet boundary conditions

So far, we have studied the model elliptic boundary value problem corresponding to the Poisson equation with the homogeneous Dirichlet boundary condition

$$\begin{cases} -\Delta u = f & \text{in } \Gamma \\ u = 0 & \text{on } \partial\Omega \end{cases} \quad (2.11)$$

where  $f \in L_2(\Omega)$ . The weak formulation of the problem is

$$u \in V, \quad a(u, v) = \ell(v) \quad \forall v \in V. \quad (2.12)$$

Here

$$\begin{aligned} V &= H_0^1(\Omega) \\ a(u, v) &= \int_{\Omega} \nabla u \cdot \nabla v dx \quad \text{for } u, v \in V \\ \ell(v) &= \int_{\Omega} f v dx \quad \text{for } v \in V \end{aligned}$$

Now, we need to verify the condition of the Lax- Milgram Lemma

#### 1. Continuity of $a(u, v)$

By the Cauchy Schwarz inequality we've

$$|a(u, v)| \leq \|\nabla u\|_{L^2(\Omega)} \|\nabla v\|_{L^2(\Omega)} \leq \|\nabla u\|_{H_0^1(\Omega)} \|\nabla v\|_{H_0^1(\Omega)}$$

$\therefore a(u, v)$  continuous with  $m = 1$

$$\Rightarrow a(u, v) \text{ is bounded.}$$

#### 2. Coercivity of $a(u, v)$

By the Poincare inequality there exists a constant  $C_p$  such that

$$\|u\|_{L^2(\Omega)} \leq C_p \|\nabla u\|_{L^2(\Omega)}, \forall u \in H_0^1(\Omega)$$

$$\therefore a(u, u) = \|\nabla u\|_{L^2(\Omega)}^2 \geq \frac{1}{C_p^2} \|u\|_{L^2(\Omega)}^2 = \frac{1}{C_p^2} \|u\|_{H_0^1(\Omega)}^2$$

$$\Rightarrow a(u, u) \geq \frac{1}{C_p^2} \|u\|_{H_0^1(\Omega)}^2$$

Hence  $a(u, v)$  is coercive with  $\alpha = \frac{1}{C_p^2}$

#### 3. Bounded-ness of $\ell(v)$

By the Cauchy Schwarz inequality we've

$$|\ell(v)| \leq \|f\|_{L^2(\Omega)} \|v\|_{L^2(\Omega)} \leq \|f\|_{L^2(\Omega)} \|v\|_{H_0^1(\Omega)}$$

$\therefore \ell(v)$  is bounded with  $c = \|f\|_{L^2(\Omega)}$

Therefore all the conditions of Lax- Milgram Lemma are satisfied , then problem (2.12) has a unique solution  $u \in V$  by the Lax-Milgram Lemma.

## 2. Problems with non-homogeneous Dirichlet boundary conditions

Suppose that instead of (2.11)<sub>2</sub> the boundary condition is

$$u = g \quad \text{on } \Gamma. \quad (2.13)$$

To derive a weak formulation, we first assume the boundary value problem (2.11) and (2.13) has a classical solution  $u \in C^2(\Omega) \cap C(\bar{\Omega})$ . Multiplying the equation (2.11)<sub>1</sub> by a test function  $v$  with certain smoothness which validates the following calculations, and integrating over  $\Omega$ ,

we have

$$\int_{\Omega} -\Delta u v dx = \int_{\Omega} f v dx$$

Integrate by parts,

$$- \int_{\Gamma} \frac{\partial u}{\partial \nu} v ds + \int_{\Omega} \nabla u \cdot \nabla v dx = \int_{\Omega} f v dx$$

We now assume  $v = 0$  on  $\Gamma$  so that the boundary integral term vanishes; the boundary integral term would otherwise be difficult to deal with under the expected regularity condition  $u \in H^1(\Omega)$  on the weak solution. Thus we arrive at the relation

$$\int_{\Omega} \nabla u \cdot \nabla v dx = \int_{\Omega} f v dx$$

if  $v$  is smooth and  $v = 0$  on  $\Gamma$ . For each term in the above relation to make sense, we assume  $f \in L_2(\Omega)$ , and let  $u \in H^1(\Omega)$  and  $v \in H_0^1(\Omega)$ . Recall that the solution  $u$  should satisfy the boundary condition  $u = g$  on  $\Gamma$ . We observe that it is necessary to assume  $g \in H^{1/2}(\Omega)$ . Finally, we obtain the weak formulation for the boundary value problem (2.11) -(2.13):  $u \in H^1(\Omega)$ ,  $u = g$  on  $\Gamma$ ,

$$\int_{\Omega} \nabla u \cdot \nabla v dx = \int_{\Omega} f v dx \quad \forall v \in H_0^1(\Omega) \quad (2.14)$$

For the weak formulation (2.14), though, we cannot apply Lax-Milgram Lemma directly since the trial function  $u$  and the test function  $v$  do not lie in the same space. There is a standard way to get rid of this problem. Since  $g \in H^{1/2}(\Omega)$  and  $\gamma(H^1(\Omega)) = H^{1/2}(\Omega)$  (recall that  $\gamma$  is the trace operator), we have the existence of a function  $G \in H^1(\Omega)$  such that  $\gamma G = g$ . We remark

that finding the function  $G$  in practice may be nontrivial. Thus, setting

$$u = w + G$$

by substituting  $\nabla w + \nabla G$  in the place of  $\nabla u$  in problem (2.14) transformed into one of seeking  $w$  such that:  $w \in H_0^1(\Omega)$ ,

$$\int_{\Omega} (\nabla w + \nabla G) \cdot \nabla v dx = \int_{\Omega} f v dx \quad \forall v \in H_0^1(\Omega)$$

This implies

$$w \in H_0^1(\Omega), \quad \int_{\Omega} \nabla w \cdot \nabla v dx = \int_{\Omega} (f v - \nabla G \cdot \nabla v) dx \quad \forall v \in H_0^1(\Omega) \quad (2.15)$$

The classical form of the boundary value problem for  $w$  is

$$\begin{cases} -\Delta w & = f + \Delta G & \text{in } \Omega, \\ w & = 0 & \text{on } \Gamma. \end{cases}$$

Applying the Lax-Milgram Lemma, we have a unique solution  $w \in H_0^1(\Omega)$  of the problem (2.15). Then we set  $u = w + G$  to get a solution  $u$  of the problem (2.14). Notice that the choice of the function  $G$  is not unique, so the uniqueness of the solution  $u$  of the problem (2.14) does not follow from the above argument. Nevertheless we can show the uniqueness of  $u$  by a standard approach.

Assume both  $u_1$  and  $u_2$  are solution of the problem (2.14). Then the difference  $u_1 - u_2$  satisfies

$$u_1 - u_2 \in H_0^1(\Omega), \quad \int_{\Omega} \nabla (u_1 - u_2) \cdot \nabla v dx = 0 \quad \forall v \in H_0^1(\Omega)$$

Taking  $v = u_1 - u_2$ , we obtain

$$\begin{aligned} \int_{\Omega} \nabla (u_1 - u_2) \cdot \nabla (u_1 - u_2) dx &= 0 \\ \Rightarrow \int_{\Omega} |\nabla (u_1 - u_2)|^2 dx &= 0 \\ \Rightarrow |\nabla (u_1 - u_2)|^2 &= 0 \Rightarrow |\nabla (u_1 - u_2)| = 0 \end{aligned}$$

a.e in  $\Omega$  Thus,  $\nabla (u_1 - u_2) = 0$  a.e. in  $\Omega$ , and hence  $u_1 - u_2 = c$  a.e. in  $\Omega$ . Using the boundary condition  $u_1 - u_2 = g - g = 0$  a.e.  $\Gamma$ , we see that  $u_1 = u_2$  a.e. in  $\Omega$ .

### 3. Problems with Neumann boundary conditions

Consider the Neumann problem of determining  $u$  which satisfies

$$\begin{cases} -\Delta u + u = f & \text{in } \Omega \\ \partial u / \partial \nu = g & \text{on } \Gamma \end{cases} \quad (2.16)$$

Here  $f$  and  $g$  are given functions in  $\Omega$  and on  $\Gamma$ , respectively, and  $\partial/\partial\nu$  denotes the normal derivative on  $\Gamma$ . Again we first derive a weak formulation. Assume  $u \in C^2(\Omega) \cap C^1(\bar{\Omega})$  is a classical solution of the problem (2.16). Multiplying (2.16)<sub>1</sub> by an arbitrary test function  $v$  with certain smoothness for the following calculations to make sense, integrating over  $\Omega$  and performing an integration by parts, we obtain

$$\int_{\Omega} (\nabla u \cdot \nabla v + uv) dx = \int_{\Omega} f v dx + \int_{\Gamma} \frac{\partial u}{\partial \nu} v ds$$

Then, substitution of the Neumann boundary condition (2.16)<sub>2</sub> in the boundary term leads to the relation

$$\int_{\Omega} (\nabla u \cdot \nabla v + uv) dx = \int_{\Omega} f v dx + \int_{\Gamma} g v ds$$

Assume  $f \in L_2(\Omega)$ ,  $g \in L_2(\Gamma)$ . For each term in the above relation to make sense, it is natural to choose the space  $H^1(\Omega)$  for both the trial function  $u$  and the test function  $v$ . Thus, the weak formulation of the boundary value problem (2.16) is

$$u \in H^1(\Omega), \quad \int_{\Omega} (\nabla u \cdot \nabla v + uv) dx = \int_{\Omega} f v dx + \int_{\Gamma} g v ds \quad \forall v \in H^1(\Omega) \quad (2.17)$$

This problem has the form (2.12), where  $V = H^1(\Omega)$ ,  $a(\cdot, \cdot)$  and  $\ell(\cdot)$  are defined by

$$\begin{aligned} a(u, v) &= \int_{\Omega} (\nabla u \cdot \nabla v + uv) dx \\ \ell(v) &= \int_{\Omega} f v dx + \int_{\Gamma} g v ds, \end{aligned}$$

respectively. Applying the Lax-Milgram Lemma, we can show that the weak formulation (2.17) has a unique solution  $u \in H^1(\Omega)$ .

*Proof.* Now we need to show this weak formulation satisfies all the conditions of Lax-Milgram-Lemma

1. Boundedness of  $a(u, v)$

Using the Cauchy Schwartz inequality and the fact that the  $L_2$  norm is bounded by  $H^1(\Omega)$  norm, we have

Gradient term

$$\int_{\Omega} (\nabla u \cdot \nabla v) dx \leq \|\nabla u\|_{L^2(\Omega)} \|\nabla v\|_{L^2(\Omega)}$$

Function term

$$\int_{\Omega} uv dx \leq \|u\|_{L^2(\Omega)} \|v\|_{L^2(\Omega)}$$

Combining the inequalities. Therefore,

$$\begin{aligned} |a(u, v)| &= \left| \int_{\Omega} (\nabla u \cdot \nabla v) dx + \int_{\Omega} uv dx \right| \\ &\leq \|\nabla u\|_{L^2(\Omega)} \|\nabla v\|_{L^2(\Omega)} + \|u\|_{L^2(\Omega)} \|v\|_{L^2(\Omega)} \\ &\leq (\|\nabla u\|_{L^2(\Omega)}^2 + \|u\|_{L^2(\Omega)}^2)^{\frac{1}{2}} (\|\nabla v\|_{L^2(\Omega)}^2 + \|v\|_{L^2(\Omega)}^2)^{\frac{1}{2}} \\ &= \|u\|_{H^1(\Omega)} \|v\|_{H^1(\Omega)} \Rightarrow |a(u, v)| \leq \|u\|_{H^1(\Omega)} \|v\|_{H^1(\Omega)} \\ &\Rightarrow a(u, v) \text{ is bounded.} \end{aligned}$$

2. Coercivity of  $a(u, u)$

$$a(u, u) = \int_{\Omega} (\nabla u \cdot \nabla u + u^2) dx = \|\nabla u\|_{L^2(\Omega)}^2 + \|u\|_{L^2(\Omega)}^2.$$

Using the definition of the  $H^1$  norm:

Recall that the  $H^1$  norm is defined as:

$$\|u\|_{H^1(\Omega)}^2 = \|u\|_{L^2(\Omega)}^2 + \|\nabla u\|_{L^2(\Omega)}^2$$

Therefore, we have:  $a(u, u) = \|u\|_{L^2(\Omega)}^2 + \|\nabla u\|_{L^2(\Omega)}^2 \geq \frac{1}{2} (\|u\|_{L^2(\Omega)}^2 + \|\nabla u\|_{L^2(\Omega)}^2) = \frac{1}{2} \|u\|_{H^1(\Omega)}^2 \Rightarrow a(u, u)$  is coercive.

3. Continuity of  $\ell(v)$

$$\begin{aligned} |\ell(v)| &= \left| \int_{\Omega} f v dx + \int_{\Gamma} g v ds \right| \\ &\leq \int_{\Omega} |f| |v| dx + \int_{\Gamma} |g| |v| ds \\ &= \left( \int_{\Omega} (|f| |v| dx)^2 \right)^{\frac{1}{2}} + \left( \int_{\Gamma} (|g| |v| ds)^2 \right)^{\frac{1}{2}} \\ &= \|f\|_{L^2(\Omega)} \|v\|_{L^2(\Omega)} + \|g\|_{L^2(\Gamma)} \|v\|_{L^2(\Gamma)} \end{aligned}$$

Using the trace theorem we have

$$\|v\|_{L^2(\Gamma)} \leq c \|v\|_{H^1} \text{ for some constant } c.$$

$$\begin{aligned} |\ell(v)| &\leq \|f\|_{L^2(\Omega)} \|v\|_{L^2(\Omega)} + \|g\|_{L^2(\Gamma)} \|v\|_{L^2(\Gamma)} \\ &\leq \|f\|_{L^2(\Omega)} \|v\|_{L^2(\Omega)} + c \|g\|_{L^2(\Gamma)} \|v\|_{H^1} \\ &\Rightarrow \ell(v) \text{ is continuous. } \Rightarrow \ell(v) \text{ is bounded.} \end{aligned}$$

□

Hence the given Neumann boundary value problem for the Poisson equation satisfies all conditions of the Lax- Milgram- Lemma

Therefore the given Neumann boundary value problem for the Poisson equation has a unique solution by Lax- Milgram- Lemma

#### 4. Pure Neumann problem for the Poisson equation

The study of the pure Neumann problem for the Poisson equation is more delicate.

$$\begin{cases} -\Delta u = f & \text{in } \Omega \\ \partial u / \partial \nu = g & \text{on } \Gamma \end{cases} \quad (2.18)$$

where  $f \in L_2(\Omega)$  and  $g \in L_2(\Gamma)$  are given. In general, the problem (2.18) does not have a solution, and when the problem has a solution  $u$ , any function of the form  $u + c$ ,  $c \in \mathbb{R}$ , is a solution. This suggests that in formulating the weak version of this problem we should restrict ourselves to the subspace

$$V = \left\{ v \in H^1(\Omega) \mid \int_{\Omega} v dx = 0 \right\}$$

Formally, the corresponding weak formulation is

$$u \in H^1(\Omega) \quad \int_{\Omega} \nabla u \cdot \nabla v dx = \int_{\Omega} f v dx + \int_{\Gamma} g v ds \quad \forall v \in H^1(\Omega) \quad (2.19)$$

An application of equivalent norm theorem shows that over the space  $V$ ,  $|\cdot|_1$  is a norm equivalent to the norm  $\|\cdot\|_1$ .

The bilinear form  $a(u, v) = \int_{\Omega} \nabla u \cdot \nabla v dx$  is both continuous and  $V$ -elliptic. There is a unique solution to the problem

$$u \in V \quad \int_{\Omega} \nabla u \cdot \nabla v dx = \int_{\Omega} f v dx + \int_{\Gamma} g v ds \quad \forall v \in V$$

A necessary condition for (2.19) to have a solution is

$$\int_{\Omega} f dx + \int_{\Gamma} g ds = 0 \quad (2.20)$$

which is derived from (2.20) by taking the test function  $v = 1$ . Assume  $\Omega \subset \mathbb{R}^n$  is a Lipschitz domain, i.e.  $\Omega$  is open, bounded, connected, and its boundary is Lipschitz continuous. Let us show that the condition (2.20) is also a sufficient condition for the problem (2.19) to have a solution. Indeed, the problem (2.19) is most conveniently studied in the quotient space  $V = H^1(\Omega)/\mathbb{R}$ . The definition of a quotient space, where each element  $[v] \in V$  is an equivalence class  $[v] = \{v + \alpha \mid \alpha \in \mathbb{R}\}$ , and any  $v \in [v]$  is called a representative element of  $[v]$ .

## 5. Problems with mixed boundary conditions

It is also possible to provide several types of boundary conditions for different parts of the boundary. One such example is

$$\begin{cases} -\Delta u + u = f & \text{in } \Omega, \\ u = 0 & \text{on } \Gamma_D \\ \partial u / \partial \nu = g & \text{on } \Gamma_N \end{cases} \quad (2.21)$$

where  $\Gamma_D$  and  $\Gamma_N$  form a non-overlapping decomposition of the boundary  $\partial\Omega$  :  $\partial\Omega = \Gamma_D \cup \Gamma_N$ ,  $\Gamma_D$  is relatively closed,  $\Gamma_N$  is relatively open, and  $\Gamma_D \cap \Gamma_N = \emptyset$ . Assume  $\Omega$  is connected. The appropriate space in which to pose this problem in weak form is now

$$V = H_{\Gamma_D}^1(\Omega) = \{v \in H^1(\Omega) \mid v = 0 \text{ on } \Gamma_D\}.$$

Then the weak problem is again of the form (2.16) with

$$a(u, v) = \int_{\Omega} (\nabla u \cdot \nabla v + uv) dx$$

and

$$\ell(v) = \int_{\Omega} f v dx + \int_{\Gamma_N} g v ds$$

Under suitable assumptions, say  $f \in L^2(\Omega)$  and  $g \in L^2(\Gamma_N)$ , we can again apply the Lax-Milgram Lemma to conclude that the weak problem has a unique solution.

## 6. Problems with mixed boundary conditions

$$\begin{cases} -\Delta u = f & \text{in } \Omega, \\ u = \varphi_0 & \text{on } \Gamma_D \\ \partial u / \partial \nu = \psi_0 & \text{on } \Gamma_N \end{cases} \quad (2.22)$$

Proof.

1. Start with the Poisson Equation:

We have the Poisson equation:  $-\Delta u = f$  in  $\Omega$

2. Multiply by a Test Function:

Let  $v$  be a smooth test function that vanishes on the Dirichlet boundary  $\Gamma_D$  (i.e.,  $v = 0$  on  $\Gamma_D$ ). Multiply the Poisson equation by  $v$  and integrate over the domain  $\Omega$ :

$$\int_{\Omega} -\Delta u v dx = \int_{\Omega} f v dx$$

3. Apply Green's Identity:

Use Green's identity to rewrite the left-hand side. Green's identity is a generalization of integration by parts:

$$\begin{aligned}
\int_{\Omega} -\Delta u v dx &= \int_{\Omega} \nabla u \cdot \nabla v dx - \int_{\partial\Omega} \frac{\partial u}{\partial n} v dS = \int_{\Omega} f v dx \\
&\Rightarrow \int_{\Omega} \nabla u \cdot \nabla v dx - \int_{\partial\Omega} \frac{\partial u}{\partial n} v dS = \int_{\Omega} f v dx \\
&\Rightarrow \int_{\Omega} \nabla u \cdot \nabla v dx - \int_{\Gamma_N} \frac{\partial u}{\partial n} v dS = \int_{\Omega} f v dx \\
&\Rightarrow \int_{\Omega} \nabla u \cdot \nabla v dx = \int_{\Omega} f v dx + \int_{\Gamma_N} \frac{\partial u}{\partial n} v ds \\
&\Rightarrow \int_{\Omega} \nabla u \cdot \nabla v dx = \int_{\Omega} f v dx + \int_{\Gamma_N} \psi_0 v ds
\end{aligned}$$

Hence the weak formulation of the mixed boundary value problem is:

$$u \in H^1(\Omega) \quad \int_{\Omega} \nabla u \cdot \nabla v dx = \int_{\Omega} f v dx + \int_{\Gamma_N} \psi_0 v ds \quad \forall v \in H_{\Gamma_D}^1(\Omega)$$

$$a(u, v) = \int_{\Omega} \nabla u \cdot \nabla v dx \quad \text{and}$$

$$\ell(v) = \int_{\Omega} f v dx + \int_{\Gamma_N} \psi_0 v ds$$

## Chapter 3

### Summary

In this thesis, we have studied the Lax-Milgram lemma and its applications to the study of elliptic partial differential equations. We have provided a detailed proof of the lemma and discussed its implications for the existence and uniqueness of solutions to elliptic problems. We have also explored several applications of the lemma, including the study of the Dirichlet problem, the Neumann problem, and the mixed boundary value problem.

The Lax-Milgram lemma is a fundamental result in the theory of partial differential equations. It provides a powerful tool for proving the existence and uniqueness of solutions to a wide variety of elliptic problems. The lemma has also found applications in many other areas of mathematics and science, including fluid dynamics, elasticity, and image processing.

In the context of elliptic partial differential equations, the Lax-Milgram lemma can be used to prove the existence and uniqueness of solutions to a variety of boundary value problems. For example, the lemma can be used to prove the existence and uniqueness of solutions to the Dirichlet problem, the Neumann problem, and the mixed boundary value problem.

In conclusion, the Lax-Milgram lemma is a versatile and powerful tool that has had a profound impact on the study of elliptic partial differential equations. Its applications extend far beyond the realm of mathematics, and it continues to be a valuable tool for researchers in a wide variety of fields.

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