



**LOCALIZED BOUNDARY-DOMAIN INTEGRAL EQUATIONS
FOR DIRICHLET PROBLEM FOR SECOND ORDER ELLIPTIC
EQUATIONS WITH MATRIX VARIABLE COEFFICIENTS**
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Abstract

Employing a localized parametrix the Dirichlet boundary value problem for elliptic equations in the divergence form with general variable matrix coefficients is reduced to localized boundary domain integral equations (LBDIE) system. The *Levi function* given in the paper is checked. The equivalence between the Dirichlet problem and the LBDIE system is studied. It is established that the localized boundary domain integral operator obtained in the paper belongs to the Boutet De Monvel algebra and the operator Fredholm properties and invertibility is investigated by the Wiener-Hopf factorization method.

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1. INTRODUCTION

In this seminar it is considered that the Dirichlet boundary value problem (BVP) for second order elliptic partial differential equations in the divergence form with a general variable matrix of coefficients and develop the boundary-domain integral approach based on the *localized parametrix*. The BVP treated in the seminar is well investigated in the literature by the variational and when the corresponding fundamental solution is available in an explicit form also by the usual classical potential methods. The goal here is to show that solutions of the problem can be represented by *localized potentials* and that the *localized boundary-domain integral operator (LBDIEO)* corresponding to the Dirichlet problem is invertible.

In this case the localized parametrix $P_\chi(x, y)$ is represented as the product of the corresponding Levi function $P_1(y, x - y)$ of the differential operator under consideration by an appropriately chosen *cut-off function* $\chi(x, y)$ supported on some neighborhood of the origin. Clearly, the kernels of the corresponding localized potentials are supported in some neighborhood of the reference point y (assuming that x is an integration variable) and they do not solve the original differential equation, while the localized potentials preserve almost all mapping properties the usual non-localized ones.

By the direct approach we reduce the BVP to the localized boundary domain integral equation. First we establish the equivalence between the original boundary value problem and the corresponding LBDIEs system which proved to be a quite non-trivial problem and plays a crucial role in our analysis.

Afterwards establishing that the localized boundary domain integral operator obtained belongs to the Boutet de Monvel algebra of pseudo differential operators. With the help of Vishik-Eskin theory, based on the factorization method (Wiener-Hopf method), investigate Fredholm properties and invertibility of the corresponding localized boundary-domain operator in appropriate function spaces.

2. THE BOUNDARY VALUE PROBLEM, LOCALIZED POTENTIALS AND GREEN'S THIRD IDENTITY

Consider a uniformly elliptic second order scalar partial differential operator

$$A(x, \partial_x)u = \frac{\partial}{\partial x_k} (a_{kj}(x) \frac{\partial u}{\partial x_j})$$

Where $\partial x = (\partial_1, \partial_2, \partial_3)$, $\partial_j = \partial_{x_j} = \partial / \partial x_j$, $a_{kj} \in C^\infty$ and $a_{kj} = a_{jk}$, $j, k = 1, 2, 3$. Moreover, due to the uniform ellipticity, there are positive constants c_1 and c_2

$$c_1 |\xi|^2 \leq a_{kj}(x) \xi_k \xi_j \leq c_2 |\xi|^2 \quad \forall x \in \mathbb{R}^3, \forall \xi \in \mathbb{R}^3.$$

Here and in what follows summation is assumed from 1 to 3 over repeated indices if not otherwise stated.

Further, let Ω^+ be a bounded domain in \mathbb{R}^3 with a simply connected boundary $\partial\Omega = S \in C^\infty$, $\overline{\Omega^+} = \Omega^+ \cup S$. Throughout the paper $n = n_1, n_2, n_3$ denotes the unit normal vector to S directed outward the domain Ω^+ . Set $\Omega^- := \mathbb{R}^3 \setminus \overline{\Omega^+}$.

By $H^r(\Omega) = H_2^r$, $r \in \mathbb{R}$, we denote the Bessel Potential spaces on a domain Ω and on a closed manifold S with out boundary, while $\mathcal{D}(\mathbb{R}^3)$ stands for C^∞ functions in \mathbb{R}^3 . Recall that $H^0(\Omega) = L_2(\Omega)$ is a space of square integrable functions in Ω . Let us denote $u^\pm := \gamma^\pm u$, where $\gamma \equiv \gamma^+$ and γ^- are the trace operators on $\partial\Omega$ from the interior and exterior of Ω^+ respectively.

We also need the following subspace of $H^1(\Omega)$,

$$H^{1,0}(\Omega; A) := \{u \in H^1(\Omega) : A(x, \partial_x)u \in H^0(\Omega)\}$$

The Dirichlet boundary value problem reads as follows: Find a function $u \in H^{1,0}(\Omega; A)$ satisfying the differential equation

$$A(x, \partial_x)u = f \quad \text{in } \Omega^+ \tag{1}$$

and the boundary condition

$$u^+ = \varphi_0 \quad \text{on } S, \tag{2}$$

where $\varphi_0 \in H^{1/2}(S)$, $f \in H^0(\Omega^+)$.

Equation (1) is understood in the distributional sense, while the Dirichlet boundary condition (2) is understood in the usual trace sense.

Note that, the co-normal derivative is understood in the generalized functional trace sense defined by the following Green's identity for the function $u \in H^{1,0}(\Omega^+; A)$:

$$\langle T^+u, g \rangle_S := \int_{\Omega^+} A(x, \partial_x)u(x)v(x)dx + \int_{\Omega^+} a_{kj}(x) \partial_{x_j}u(x) \partial_{x_k}v(x)dx \quad (3)$$

Where $g \in H^{1/2}(S)$ is an arbitrary function and $v \in H^1(\Omega)$ is an extension of g from S on to the whole of Ω^+ , i.e., $v^+ = g$ on S , while $\langle \cdot, \cdot \rangle_S$ denotes the duality between the adjoint spaces $H^{-1/2}(S)$ and $H^{1/2}(S)$ which extends the usual bilinear $L_2(S)$ inner product.

Let us denote

$$P_1(x, y) = \frac{\alpha}{(\mathbf{a}^{-1}(y)x, x)^{\frac{1}{2}}} \quad \text{with} \quad \alpha = -\frac{1}{4\pi[(\det(\mathbf{a}(y)))^{\frac{1}{2}}]}, \quad \mathbf{a}(y) = [a_{kj}(y)]_{3 \times 3}$$

where \mathbf{a}^{-1} is the matrix inverse to \mathbf{a} and (\cdot, \cdot) denotes the usual scalar product in \mathbb{R}^3 . Also that $P_1(y, x - y)$ is the Levi function (parametrix) of the operator $A(x, \partial_x)$ and satisfies the equation

$$A(y, \partial_x)P_1(y, x - y) = \delta(x - y),$$

where $\delta(\cdot)$ is the Dirac delta function.

To show the above fact let's state the following Lemma.

Lemma 1 Let $u = u(x)$

and $x = Ay$ that is $u = u(Ay)$ for $A = (a_{ij})_{n \times n}$, with constant entries.

Then $\nabla_x = A^{-1}\nabla_y$.

Proof

For $u = u(Ay)$

$$\nabla_x u = \begin{pmatrix} u_{x_1} \\ u_{x_2} \\ \vdots \\ u_{x_n} \end{pmatrix} \quad \text{and} \quad \nabla_y u = \begin{pmatrix} u_{y_1} \\ u_{y_2} \\ \vdots \\ u_{y_n} \end{pmatrix}$$

But $\frac{\partial u}{\partial y_i} = \sum_{k=1}^n \frac{\partial u}{\partial x_k} \frac{\partial x_k}{\partial y_i}$, by Chain Rule

$$x_k = \sum_{j=1}^n a_{kj} y_j, \text{ so } \frac{\partial x_k}{\partial y_i} = a_{ki}$$

$$\text{Hence } \frac{\partial u}{\partial y_i} = \sum_{k=1}^n a_{ki} \frac{\partial u}{\partial x_k}$$

Which implies

$$\nabla_y = A \nabla_x \text{ i.e., } \nabla_x = A^{-1} \nabla_y \blacksquare$$

Now

$$\begin{aligned} A(y, \partial_x) P_1(y, x - y) &= \frac{\partial}{\partial x_k} \left(a_{kj}(y) \frac{\partial P_1(y, x - y)}{\partial x_j} \right) \\ &= \sum_{k=1}^3 \sum_{j=1}^3 \frac{\partial}{\partial x_k} \left(a_{kj}(y) \frac{\partial P_1(y, x - y)}{\partial x_j} \right) \\ &= \sum_{k=1}^3 \sum_{j=1}^3 \frac{\partial^2}{\partial x_k \partial x_j} P_1(y, x - y) \\ &= \alpha(y) \nabla_x^t \mathbf{a}(y) \nabla_x \frac{1}{(\mathbf{a}^{-1}(y)(x - y), x - y)^{\frac{1}{2}}} \end{aligned}$$

Since \mathbf{a}^{-1} is symmetric and positive definite, there is symmetric positive definite matrix \mathbf{d} such that $\mathbf{a}^{-1} = \mathbf{d}^2$ and $(\mathbf{a}^{-1}x - y, x - y) = (\mathbf{d}^2(x - y), x - y) = (x - y)^t \mathbf{d}^2(x - y)$

$$= |\mathbf{d}(x - y)|^2,$$

and $\det \mathbf{d} = [\det \mathbf{a}]^{\frac{1}{2}}$.

Hence

$$A(y, \partial_x) P_1(y, x - y) = \alpha(y) \nabla_x^t \mathbf{a}(y) \nabla_x |\mathbf{d}(x - y)|^{-1} \quad (*)$$

Now let $z = \mathbf{d}(x - y)$

implies that $x = \mathbf{d}^{-1}z + y$

then $\nabla_x = \mathbf{d} \nabla_z$.

Substituting this value in (*) we get

$$A(y, \partial_x) P_1(y, x - y) = \alpha(y) (\mathbf{d} \nabla_z)^t \mathbf{a}(y) (\mathbf{d} \nabla_z) |z|^{-1}$$

$$\begin{aligned}
&= \alpha(y) \nabla_z^t (\mathbf{d}^t \mathbf{a}(y) \mathbf{d}) \nabla_z |z|^{-1} \\
&= \alpha(y) \nabla_z^t \mathbf{I} \nabla_z |z|^{-1} \mathbf{d} \\
&= \alpha(y) \Delta_z |z|^{-1} \\
&= \alpha(y) (-4\pi \delta(z)) \\
&= \frac{-1}{4\pi \sqrt{\det \mathbf{a}(y)}} (-4\pi \delta(z)) \\
&= \det \mathbf{d}(y) \delta(z) \qquad \text{since } \det \mathbf{d} = [\det \mathbf{a}]^{-\frac{1}{2}} \\
&= \det \mathbf{d}(y) \delta(\mathbf{d}(x - y))
\end{aligned}$$

By linear transformation of distribution $\delta(\mathbf{d}(x - y)) = \frac{\delta(x - y)}{\det \mathbf{d}(y)}$

So we have

$$\begin{aligned}
A(y, \partial_x) P_1(y, x - y) &= \det \mathbf{d}(y) \delta(\mathbf{d}(x - y)) \\
&= \delta(x - y) \blacksquare
\end{aligned}$$

Let's define the following class of cut-off functions.

DEFINITION 1 We say $\chi_0 \in \mathcal{X}^\infty$ if $\chi_0(x) = \check{\chi}(|x|)$, $\check{\chi} \in C^\infty[0, \infty)$ and $\varrho^\alpha \check{\chi}^k(\varrho) \rightarrow 0$ as $\varrho \rightarrow 0$ for any real α and any non-negative integer k .

We say $\chi_0 \in \mathcal{X}_+^\infty$ if $\chi \in \mathcal{X}^\infty$, $\chi(0) = 1$ and $\sigma_{\chi_0}(\omega) > 0$ for all $\omega \in \mathbb{R}$,

where

$$\sigma_{\chi_0}(\omega) := \begin{cases} \frac{\hat{\chi}_s(\omega)}{\omega}, & \text{for } \omega \in \mathbb{R} \setminus \{0\} \\ \int_0^\infty \varrho \check{\chi}(\varrho) d\varrho, & \text{for } \omega = 0, \end{cases}$$

and $\hat{\chi}_s(\omega)$ denotes the sine-transform of the function $\check{\chi}$,

$$\hat{\chi}_s(\omega) := \int_0^\infty \check{\chi}(\varrho) \sin(\varrho\omega) d\varrho.$$

Note that \mathcal{X}^∞ and \mathcal{X}_+^∞ are subsets, respectively, of the classes X^∞ and X_+^∞ .

where X^k and X_+^k are defined as follows;

DEFINITION 2 We say $\chi \in X^k$, for $k \geq 0$, if $\chi(x) = \check{\chi}(|x|)$, $\check{\chi} \in W_1^k(0, \infty)$ and

$\varrho^\alpha \check{\chi}^k(\varrho) \in L_1(0, \infty)$. $W_1^k(a, b)$ denote the Sobolev space of function.

We say $\chi \in X_+^k$ for $k \geq 1$ if $\chi \in X^k$, $\chi(0) > 0$ and

$$\sigma_{\chi_0}(\omega) := \frac{\hat{\chi}_s(\omega)}{\omega} > 0, \forall \omega \in \mathbb{R}, \text{ where } \hat{\chi}_s(\omega) \text{ is as in Definition 1.}$$

Throughout the seminar we will consider functions χ of the form $\chi(y, z) = \chi_0(\mathbf{d}(y)z)$, where $\chi_0 \in \mathcal{X}^\infty$ if not otherwise stated.

Introduce the localized parametrix $P(y, x - y)$,

$$P(y, x - y) = P_\chi(y, x - y) = \chi(y, x - y)P_1(y, x - y).$$

And $A(x, \partial_x)P_1(y, x - y) = \delta(x - y) + R(x, y)$, where

$$\begin{aligned} R(x, y) = R_\chi(x, y) = & P_1(y, x - y)A(y, \partial_x)\chi(y, x - y) + 2a_{kj}(y) \frac{\partial \chi(y, x - y)}{\partial x_j} \frac{\partial P_1(y, x - y)}{\partial x_k} \\ & + [a_{kj}(x) - a_{kj}(y)] \frac{\partial^2 [\chi(y, x - y)P_1(y, x - y)]}{\partial x_j \partial x_k} + \frac{\partial a_{kj}(x)}{\partial x_k} \frac{\partial [\chi(y, x - y)P_1(y, x - y)]}{\partial x_j} \end{aligned}$$

The function $R(x, y)$ possesses a weak singularity of type

$$R(x, y) = \mathcal{O}(|x - y|)^{-2} \text{ as } x \rightarrow y$$

Let us introduce the, localized surface and volume potentials, based on the above defined localized parametrix P ,

$$Vg(y) := -\int_S P(y, x - y)g(x)dS_x, \quad Wg(x) := -\int_S [T(x, \partial_x)P(y, x - y)]g(x)dS_x, \quad (4)$$

$$\mathcal{P}h(y) := \int_{\Omega^+} P(y, x - y)h(x)dx, \quad \mathcal{R}h(y) := \int_{\Omega^+} R(y, x - y)h(x)dx. \quad (5)$$

If the domain of integration in the Newtonian volume potential is the whole space \mathbb{R}^3 , we employ the notation $\mathbf{P}h := \mathcal{P}h$.

Now we recall Green's second identity for the operator $A(x, \partial_x)$,

$$\int_{\Omega^+} [vA(x, \partial_x)u - uA(x, \partial_x)v]dx = \langle T^+u, v^+ \rangle_S - \langle T^+v, u^+ \rangle_S. \quad (6)$$

where $u, v \in H^{1,0}(\Omega^+; A)$.

Taking $v(x) = P(y, x - y)$, from (6) we arrive at the following localized third Green's identity for $u \in H^{1,0}(\Omega^+, A)$,

$$u + \mathcal{R}u - V(T^+u) + W(u^+) = \mathcal{P}(Au) \quad \text{in } \Omega^+. \quad (7)$$

3. LBDIE FORMULATION FOR DIRICHLET PROBLEM

To derive the equivalent LBDIE formulation of the Dirichlet problem we need some auxiliary material. Let us denote by E_0 the operator of extension of a function, defined in Ω^+ , by zero outside Ω^+ to a function defined in \mathbb{R}^3 . Therefore, for $f \in H^0(\Omega^+)$ we can rewrite the volume potential $\mathcal{P}f$ as a pseudodifferential operator

$$\begin{aligned} \mathcal{P}f(y) &= \int_{\Omega^+} P(y, x - y)f(x)dx = \int_{\mathbb{R}^3} P(y, x - y)E_0f(x)dx \\ &= \mathbf{P}(E_0f)(y) = \frac{1}{(2\pi)^3} \int_{\mathbb{R}^3} \mathfrak{S}(\mathbf{P})(y, \xi)e^{-i(y, \xi)}\mathcal{F}(E_0f)(\xi)d\xi, \end{aligned}$$

Where the symbol \mathcal{F} stands for the generalized Fourier transform and

$$\mathfrak{S}(\mathbf{P})(y, \xi) := \mathcal{F}_{z \rightarrow \xi}[P(y, z)] \quad \text{is the complete symbol of the operator } \mathbf{P}.$$

LEMMA2 Let $\chi_0 \in \mathcal{X}^\infty$. Then $\mathfrak{S}(\mathbf{P}) \in C^\infty(\mathbb{R}^3 \times \mathbb{R}^3)$, and the following expansion holds for $\xi \neq 0$ and any integer $k \geq 0$,

$$\begin{aligned} \mathfrak{S}(\mathbf{P})(y, \xi) &= \sum_{m=0}^{k^*} \frac{(-1)^{m+1}}{(\mathbf{a}(y)\xi, \xi)^{m+1}} \check{\chi}^{(2m)}(0) \\ &\quad - \frac{1}{(\mathbf{a}(y)\xi, \xi)^{(k+1)/2}} \int_0^\infty \sin((\mathbf{a}(y)\xi, \xi)^{1/2} \varrho + \frac{k\pi}{2}) \overline{\chi}_0^{(k)}(\varrho) d\varrho, \end{aligned} \quad (8)$$

Where k^* is the integer part of $(k - 1)/2$ and the sum disappears in (8) if $k^* < 0$, i.e., if $k = 0$.

For any multi indices α, β there are positive constants $c_{\alpha\beta}$ independent of ξ and y such that

$$\left| \partial_\xi^\alpha \partial_y^\beta \mathfrak{S}(\mathbf{P})(y, \xi) \right| < c_{\alpha\beta} (1 + |\xi|^2)^{-1-|\alpha|/2}$$

For any $\xi \in \mathbb{R}^3, y \in \mathbb{R}^3$.

We have the following corollary which follows from the above lemma 1.

COROLLARY 3 Let $\chi_0 \in \mathcal{X}^\infty$. Then $\mathfrak{S}(\mathbf{P})$ belongs to the *Hörmander* class of symbols $S^{-2}(\mathbb{R}^3 \times \mathbb{R}^3)$ and the operator $\mathbf{P}: H^s(\mathbb{R}^3) \rightarrow H^{s+2}(\mathbb{R}^3)$ is continuous for all $s \in \mathbb{R}$.

Before proving the above Corollary let us have the following two definitions and Theorem.

DEFINITION 3 Let m, ρ, δ be real numbers with $0 < \rho \leq 1$ and $0 \leq \delta < 1$. Then we denote by $S_{\rho, \delta}^m(X \times \mathbb{R}^N)$ the set of all $a \in C^\infty(X \times \mathbb{R}^N)$ such that for every compact set $K \subset X$ and all α, β the estimate

$$\left| D_x^\beta D_\theta^\alpha a(x, \theta) \right| \leq C_{\alpha, \beta} (1 + |\theta|)^{m - \rho|\alpha| + \delta|\beta|}, \quad x \in K, \theta \in \mathbb{R}^N$$

is valid for some constant $C_{\alpha, \beta}, K$.

The elements of $S_{\rho, \delta}^m$ are called *Symbols of order m and type ρ, δ* .

DEFINITION 4 If m is a real number then $S^m = S^m(\mathbb{R}^n \times \mathbb{R}^n)$ is the set of all $a \in C^\infty(\mathbb{R}^n \times \mathbb{R}^n)$ such that for all α, β the derivative of $a_{(\beta)}^{(\alpha)}(x, \xi) = \partial_\xi^\alpha \partial_x^\beta a(x, \xi)$ has the bound $\left| a_{(\beta)}^{(\alpha)}(x, \xi) \right| \leq C_{\alpha, \beta} (1 + |\xi|)^{m - |\alpha|}$, $\xi \in \mathbb{R}^n$.

S^m is called the space of symbols of order m .

Theorem3 If $a \in S^m$, then $a(x, D)$ is a continuous operator from H^s to H^{s-m} for every s .

Now the proof of Corollary3 proceeds as follows.

Proof:

For $\chi_0 \in \mathcal{X}^\infty$ and from Lemma1 $\mathfrak{S}(\mathbf{P}) \in C^\infty(\mathbb{R}^3 \times \mathbb{R}^3)$, then by Definition4 $\mathfrak{S}(\mathbf{P})$ belongs to S^{-2} . i.e., for the case $m = -2$ and $n = 3$.

Now from Theorem3 it is possible to deduce that $\mathbf{P}: H^s(\mathbb{R}^3) \rightarrow H^{s+2}(\mathbb{R}^3)$ is continuous for all $s \in \mathbb{R}$ ■

First we introduce the notation $A_\infty(\xi) := -|\xi|^2$ for the symbol of the Laplace operator and take in to account that $A(y, \xi) := -a_{kj}(y)\xi_k\xi_j$. Further, let

$$A'(y, \xi) := A(y, \xi) - A_\infty(\xi) = -a_{kj}(y)\xi_k\xi_j + |\xi|^2,$$

Let us denote $P_\infty(z) = -\chi_0(z)/4\pi|z|$ the localized parametrix of the Laplace operator, while $P'(y, x - y) := P(y, x - y) - P_\infty(x - y)$. Introducing the corresponding pseudodifferential operators

$$\begin{aligned}
\mathbf{P}_\infty h(y) &:= \int_{\mathbb{R}^3} P_\infty(x-y)h(x)dx, \quad \mathbf{P}'h(y) := \int_{\mathbb{R}^3} P'(y, x-y)h(x)dx, \\
\mathbf{P}'h(y) + \mathbf{P}_\infty h(y) &:= \int_{\mathbb{R}^3} P'(y, x-y)h(x)dx + \int_{\mathbb{R}^3} P_\infty(x-y)h(x)dx \\
&:= \int_{\mathbb{R}^3} (P'(y, x-y) + P_\infty(x-y))h(x)dx \\
&:= \int_{\mathbb{R}^3} P(y, x-y)h(x)dx \\
&:= \mathbf{P}h(y)
\end{aligned}$$

Then we arrive at the decomposition $\mathbf{P} = \mathbf{P}' + \mathbf{P}_\infty$. For the corresponding symbols we then obtain, $\mathfrak{S}(\mathbf{P}_\infty)(\xi) := \mathcal{F}_{z \rightarrow \xi}[P_\infty(z)] = \mathfrak{S}(\mathbf{P})(y, \xi) \setminus a_{ij} \equiv \delta_{ij}$,
 $\mathfrak{S}(\mathbf{P}')(y, \xi) := \mathfrak{S}(\mathbf{P})(y, \xi) - \mathfrak{S}(\mathbf{P}_\infty)(\xi)$.

Similar decomposition can be written for the layer potentials

$$\begin{aligned}
V\psi &= -\mathbf{P}(\gamma^*\psi) = -\mathbf{P}'(\gamma^*\psi) - \mathbf{P}_\infty(\gamma^*\psi) = V'\psi + V_\infty(\psi), \\
W\varphi &= -\mathbf{P}(T^*\varphi) = -\mathbf{P}'(T^*\varphi) - \mathbf{P}_\infty(T^*\varphi) = W'\varphi + W_\infty(\varphi),
\end{aligned}$$

Where

$$V'\psi := -\mathbf{P}'(\gamma^*\psi), V_\infty(\psi) := -\mathbf{P}_\infty(\gamma^*\psi), W'\varphi = -\mathbf{P}'(T^*\varphi), W_\infty(\varphi) = -\mathbf{P}_\infty(T^*\varphi) \quad (9)$$

and their explicit forms are given by expressions (4) after replacing there the kernel P with P' and P_∞ , respectively. here the operator $\gamma^*: H^{\frac{1}{2}-t}(S) \rightarrow H^{-t}(\mathbb{R}^3)$, $t > \frac{1}{2}$ is adjoint to the trace operator $\gamma: H^t(\mathbb{R}^3) \rightarrow H^{t-1/2}(S)$, $t > \frac{1}{2}$, i.e., is defined by the relation

$$\langle \gamma^*\psi, h \rangle_{\mathbb{R}^3} := \langle \psi, \gamma h \rangle_S, \quad \forall h \in H^t(\mathbb{R}^3), \quad \psi \in H^{\frac{1}{2}-t}, \quad t > \frac{1}{2} \quad (10)$$

Similarly, $T^*: H^{\frac{3}{2}-t}(S) \rightarrow H^{-t}(\mathbb{R}^3)$, $t > \frac{3}{2}$ is the operator adjoint to the classical (defined in terms of the trace) co-normal derivative operator $= a_{kj}n_k(x)\gamma\partial_{x_j}: H^t(\mathbb{R}^3) \rightarrow H^{t-\frac{3}{2}}(S)$, that is continuous for $t > \frac{3}{2}$ (for the infinitely smooth S), i.e.,

$$\langle T^*\varphi, h \rangle_{\mathbb{R}^3} := \langle \varphi, Th \rangle_S \text{ for any } h \in H^t(\mathbb{R}^3), \quad \varphi \in H^{\frac{3}{2}-t}(S), \quad t > \frac{3}{2}. \quad (11)$$

Now we can rewrite (7) in the form

$$\begin{aligned}
u + \mathcal{R}u - V(T^+u) + W(u^+) &= \mathcal{P}(Au) \text{ in } \Omega^+ \\
u + \mathcal{R}u - [V'(T^+u) + V_\infty(T^+u)] + W(u^+) &= \mathbf{P}'(E_0Au) + \mathbf{P}_\infty(E_0Au)
\end{aligned}$$

$$\begin{aligned}
u + \mathcal{R}u - V'(T^+u) - V_\infty(T^+u) - \mathbf{P}'(E_0Au) &= \mathbf{P}_\infty(E_0Au) - W(u^+) \\
u + \mathcal{R}u - V'(T^+u) - V_\infty(T^+u) - \mathbf{P}'(E_0Au) &= \mathbf{P}_\infty(E_0Au) - W(u^+) \text{ in } \Omega^+
\end{aligned} \tag{12}$$

Further, for $u \in H^{1,0}(\Omega^+, A)$ and $\mathcal{D}(\mathbb{R}^3)$ we can rewrite Green's second identity in Ω^+ as

$$\int_{\mathbb{R}^3} [vE_0Au - (E_0u)E_0Av] dx = \langle T^+u, \gamma v \rangle_s - \langle Tv, \gamma^+v \rangle_s,$$

which, in turn, in view of the notation (10)-(11) can be written in the form

$$\langle E_0Au, v \rangle_{\mathbb{R}^3} - \langle AE_0u, v \rangle_{\mathbb{R}^3} = \langle \gamma^*T^+u, v \rangle_{\mathbb{R}^3} = \langle T^*\gamma^+u, v \rangle_{\mathbb{R}^3}, \forall v \in \mathcal{D}(\mathbb{R}^3).$$

Whence we get $AE_0u = E_0Au - \gamma^*T^+u + T^*\gamma^+u$.

Therefore in view of (9), we get

$$\mathbf{P}'(AE_0u) = \mathbf{P}'(E_0Au) + V'(T^+u) - W'(u^+) \text{ in } \mathbb{R}^3. \tag{13}$$

which implies

$$\begin{aligned}
\{\mathbf{P}'(E_0Au)\}^+ + \mathcal{V}'(T^+u) &= \{\mathbf{P}'(AE_0u)\}^+ + \{W'(u^+)\}^+ \\
&= \{\mathbf{P}'(AE_0u)\}^+ + \{W(u^+)\}^+ - \{W_\infty(u^+)\}^+ \\
&= \{\mathbf{P}'(AE_0u)\}^+ - \frac{1}{2}u^+ + \mathcal{W}(u^+) + \frac{1}{2}\mu u^+ - \mathcal{W}_\infty(u^+) \\
&= \{\mathbf{P}'(AE_0u)\}^+ - \frac{1}{2}(1 - \mu)u^+ + \mathcal{W}(u^+) - \mathcal{W}_\infty(u^+),
\end{aligned} \tag{14}$$

where $\mu = a_{kj}n_kn_j$.

Due to (13), third Green's identity (12) can be also rewritten as

$$u + \mathcal{R}u - \mathbf{P}'A(E_0u) - V_\infty(T^+u) = \mathbf{P}_\infty(E_0Au) - W_\infty(u^+) \text{ in } \Omega^+. \tag{15}$$

Further we have

$$\begin{aligned}
\mathbf{P}'(AE_0u)(y) &= \langle P'(y, \cdot - y), AE_0u \rangle_{\mathbb{R}^3} = \langle AP'(y, \cdot - y), E_0u \rangle_{\mathbb{R}^3} = \langle AP'(y, \cdot - y), u \rangle_{\Omega^+} \\
&= \langle AP'(y, \cdot - y) - AP_\infty(\cdot - y), u \rangle_{\Omega^+} = u(y) + \mathcal{R}u(y) - \langle AP_\infty(\cdot - y), u \rangle_{\Omega^+} \\
&= u(y) + \mathcal{R}u(y) - \beta(y)u(y) - \mathcal{N}_\infty u(y)
\end{aligned} \tag{16}$$

where

$$\beta(y) = \frac{1}{3}[a_{11}(y) + a_{22}(y) + a_{33}(y)], \quad \mathcal{N}_\infty u(y) = \text{v.P.} \int N_\infty(x, y)u(x)dx,$$

$$N_\infty(x, y) := A(x, \partial_x)P_\infty(x - y) = \left[-\frac{a_{kj}(x)}{4\pi} \frac{\partial^2}{\partial x_k \partial x_j} \frac{1}{|x-y|} \right] + R_\infty(x, y), \quad x \neq y,$$

$$R_\infty(x, y) := -\frac{1}{4\pi} \left\{ \frac{\partial}{\partial x_k} \left[\frac{\partial \chi(x, y)}{\partial x_j} \frac{a_{kj}(x)}{|x-y|} \right] + \frac{\partial [a_{kj}(x)\chi(x, y)]}{\partial x_k} \frac{\partial}{\partial x_j} \frac{1}{|x-y|} + a_{kj}(x)\chi[\chi(x, y) - 1] \frac{\partial^2}{\partial x_k \partial x_j} \frac{1}{|x-y|} \right\}, \quad x \neq y,$$

Substituting (16) in to (15), we arrive at the equivalent form of the third Green's identity that does not involve the parametrix P ,

$$\beta u + \mathcal{N}_\infty u - \mathcal{V}_\infty(T^+u) + \mathcal{W}_\infty(\gamma^+u) = \mathcal{P}_\infty(Au) \quad \text{in } \Omega^+ \quad (17)$$

Taking the trace of (17) on S and using the jump properties of the layer potentials we get

$$\left(\beta - \frac{1}{2}\mu \right) u^+ + \gamma^+ \mathcal{N}_\infty u - \mathcal{V}_\infty(T^+u) + \mathcal{W}_\infty(\gamma^+u) = \gamma^+ \mathcal{P}_\infty(Au) \quad \text{on } S, \quad (18)$$

where \mathcal{V}_∞ and \mathcal{W}_∞ are direct values of the corresponding potentials V_∞ and W_∞ on the boundary S .

Denoting $\psi = T^+u$ and substituting equations (1) and (2) in (17) and (18) we obtain the LBDIE system with respect to the unknowns $u \in H^{1,0}(\Omega^+, A)$ and $\psi \in H^{-\frac{1}{2}}(S)$ considered as independent of each other (i.e. segregated),

$$\beta u + \mathcal{N}_\infty u - \mathcal{V}_\infty(\psi) = \mathcal{P}_\infty f - \mathcal{W}_\infty \varphi_0 \quad \text{in } \Omega^+, \quad (19)$$

$$\gamma^+ \mathcal{N}_\infty u - \mathcal{V}_\infty \psi = \gamma^+ \mathcal{P}_\infty(f) - \left(\beta - \frac{1}{2}\mu \right) \varphi_0 - \mathcal{W}_\infty \varphi_0 \quad \text{on } S, \quad (20)$$

Where $\varphi_0 \in H^{\frac{1}{2}}(S)$ and $f \in H^0(\Omega^+)$.

4. FORMULATION OF THE BASIC RESULTS

First, the following equivalence theorem can be proved.

Theorem 4 Let $\chi_0 \in \mathcal{X}_+^\infty$.

(i) If a function $u \in H^{1,0}(\Omega^+, A)$ solves the Dirichlet BVP (1)-(2), then it is unique and the pair $(u, \psi) \in H^{1,0}(\Omega^+, A) \times H^{-\frac{1}{2}}(S)$ with

$$\psi = T^+u \quad (21)$$

Solves the LBDIE system (19)-(20).

(ii) Vice versa, if a pair $(u, \psi) \in H^{1,0}(\Omega^+, A) \times H^{-\frac{1}{2}}(S)$ solves the LBDIE system (19)-(20), then it is unique and the function u solves the Dirichlet BVP (1)-(2) and the equality (21) holds.

From Theorem 4 it follows that LBDIE system (19)-(20), which has which has a special right hand side, is uniquely solvable in the class $H^{1,0}(\Omega^+, A) \times H^{-\frac{1}{2}}(S)$.

Now we investigate the LBDIO generated by the left hand side expression (19)-(20) in appropriate function spaces. The LBDIE system (19)-(20) with an arbitrary right hand side functions from the space $H^1(\Omega^+) \times H^{1/2}(S)$ can be written as

$$(\beta I + \mathcal{N}_\infty)u - \mathcal{V}_\infty(\psi) = F_1 \quad \text{in } \Omega^+ \quad (22)$$

$$\gamma^+ \mathcal{N}_\infty u - \mathcal{V}_\infty \psi = F_2 \quad \text{on } S, \quad (23)$$

where I stands for the identity operator and $F_1 \in H^1(\Omega^+)$, $F_2 \in H^{1/2}(S)$.

Denote by \mathcal{A} the LBDIO generated by the left hand side expressions in the LBDIE system (22)-(23) as

$$\mathcal{A} := \begin{bmatrix} r_{\Omega^+}(\beta I + \mathcal{N}_\infty) & -r_{\Omega^+} \mathcal{V}_\infty \\ \gamma^+ \mathcal{N}_\infty & -\mathcal{V}_\infty \end{bmatrix}.$$

The following assertion holds.

THEOREM 5 Let $\chi_0 \in \mathcal{X}_+^\infty$. Then the operator

$\mathcal{A}: H^{r+1}(\Omega^+) \times H^{r-\frac{1}{2}}(S) \rightarrow H^{r+1}(\Omega^+) \times H^{r+\frac{1}{2}}(S)$ is continuous and continuously invertible for any $r \geq 0$.

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