INTERACTION OF TWO-LEVEL ATOMS WITH A CAVITY MODE

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Abstract

Applying the solutions of the quantum Langevin equations, we calculate the variances of the quadrature operators for the cavity mode, the squeezing spectrum, spectrum of intensity fluctuations and the power spectrum of the output mode produced by two-level atoms interacting with a single cavity mode, driven by a coherent light and coupled to a squeezed vacuum reservoir. Employing the same solutions, we obtain the antinormally ordered characteristic function defined in the Heisenberg picture. With the aid of the resulting characteristic function, we determine the Q function which is then used to calculate the mean and variance of the photon number as well as the photon number distribution of the output mode.

CONTENTS

1. Introduction

The quantum noise and photon statistics of the light mode produced by two-level atoms interacting with a single cavity mode coupled to an ordinary vacuum have been extensively studied [1-6]. Recently, with the advent of squeezed light sources, much attention has been directed at modifying the properties of the cavity mode by replacing the ordinary vacuum surrounding the port mirror by a squeezed light.

Applying the Glauber-Sudarshan P-function, several researchers have studied the radiation generated by two-level atoms interacting with a cavity mode and coupled to a squeezed vacuum [7-11]. The P function is usually obtained by solving the Fokker-Planck equation which is derived from the equation of evolution of the reduced density operator for the system. However, the task of deriving the master equation [1, 5] and solving the Fokker-Planck equation [12] involves a great deal of mathematical manipulations. On the other hand, the derivation of the quantum Langevin equations [13] appears to involve a relatively less mathematical task. In addition, this approach is better suited for the calculation of two-time correlation functions. Applying c-number Langevin equations, the quantum noise and photon statistics of a laser coupled to a squeezed vacuum have been studied in Ref. [14].

In this thesis, we wish to study applying quantum Langevin equations, the quantum fluctuations and photon statistics of the light mode produced by two-level atoms interacting with a cavity mode, driven by a coherent light and coupled to a squeezed vacuum reservoir.

The thesis is organized as follows. We first derive the quantum Langevin equations for the cavity mode. Then using the solutions of the resulting equations, we calculate the variance of the quadrature operators for the cavity mode, the squeezing spectrum, spectrum of intensity fluctuations and the power spectrum of the output mode. Moreover, applying the same solutions we determine the antinormally ordered characteristic function defined in the Heisenberg picture. With the aid of the resulting characteristic function, we obtain the Q function for the output mode which is used to calculate the mean and variance of the photon number as well as the photon number distribution of the output mode.

We also discuss the effects of the driving coherent light and the squeezed light which has phase dependent properties and nonvanishing mean photon number on certain quantities of interest.

2. Quantum Langevin Equations

In this chapter we seek to investigate the interaction of two-level atoms with a cavity mode driven by a coherent light and coupled to a squeezed vacuum reservoir through the lossy output mirror of the cavity. We here consider the case for which two-level atoms, initially prepared in the upper level, are injected into the cavity at a certain rate r_a. We also consider the atoms to be noninteracting and at resonance with the cavity mode. This system is describable in the interaction picture by the quantum Hamiltonian ($\hbar = 1$):

$$H = H_M + H_{AM} + H_{MR}, \tag{2.1}$$

where H_M, H_{AM} and H_{MR} represent the interactions of the driving mode, the two level atoms and the squeezed vacuum reservoir with the cavity mode, respectively.
With the driving mode treated classically, the Hamiltonian that describes the interaction between the cavity mode and the driving mode at exact resonance is expressible as

$$H_M = i\varepsilon(a^\dagger - a), \tag{2.2}$$

where ε is proportional to the amplitude of the driving mode and a represents the annihilation operator for the cavity mode. The interaction Hamiltonian H_{MR} is given by

$$H_{MR} = i\sum_k \lambda_k \left[a^\dagger b_k e^{i(\omega_0-\omega_k)t} - ab_k^\dagger e^{-i(\omega_0-\omega_k)t}\right], \tag{2.3}$$

in which λ_k is the coupling constant and b_k represents the annihilation operator for the reservoir mode. Furthermore, the Hamiltonian H_{AM} that describes the interaction of a system of two-level atoms with a single cavity mode at exact resonance is expressible by

$$H_{AM} = ig\sum_j f(t,t_j)\left(a^\dagger \sigma_-^j - a\sigma_+^j\right), \tag{2.4}$$

where $f(t, t_j)$ is a function that determines the interaction between the j^{th} atom which is injected at time t_j and the radiation field. It can be defined as

$$f(t, t_j) = \Theta(t - t_j) \exp\left[-\frac{\gamma}{\sqrt{2}}(t - t_j)\right], \tag{2.5}$$

where $\Theta(t)$ is the usual step function which accounts for the fact that the j^{th} atom starts the interaction with the cavity mode at the injection time t_j, γ the atomic decay rate, and σ_-^j and σ_+^j are atomic operators in which σ_-^j takes an atom from the upper state into the lower state whereas σ_+^j takes an atom from the lower state to the upper state.

The annihilation operator for the cavity mode evolves in time according to the Heisenberg equation of motion:

$$\frac{da}{dt} = -i\,[a, H]. \tag{2.6}$$

Applying the Hamiltonian (2.1), we have

$$\frac{da}{dt} = -i[a, H_M] - i[a, H_{AM}] - i[a, H_{MR}]. \tag{2.7}$$

Since the cavity mode $a(t)$ commutes with the atomic and reservoir operators , we see that

$$[a, H_M] = i\varepsilon, \tag{2.8}$$

$$[a, H_{AM}] = ig\sum_j f(t, t_j)\sigma_-^j(t), \tag{2.9}$$

$$[a, H_{MR}] = i\sum_k \lambda_k b_k(t) e^{i(\omega_0 - \omega_k)t}. \tag{2.10}$$

Employing these results in (2.7), we get

$$\frac{da}{dt} = \varepsilon + \sum_k \lambda_k b_k(t) e^{i(\omega_0 - \omega_k)t} + g\sum_j f(t, t_j)\sigma_-^j(t). \tag{2.11}$$

In addition, applying the Hamiltonian (2.3) and the commutation relations

$$\left[b_j, b_k^\dagger\right] = \delta_{jk}, \tag{2.12a}$$

$$[b_j, b_k] = \left[b_j^\dagger, b_k^\dagger\right] = 0, \tag{2.12b}$$

one readily obtains the time evolution of the annihilation operator for the reservoir modes to be

$$\frac{db_k}{dt} = -\lambda_k a(t) e^{-i(\omega_0 - \omega_k)t}. \tag{2.13}$$

A formal solution of this equation can be put in the form

$$b_k(t) = b_k(0) - \lambda_k \int_0^t a(t') e^{-i(\omega_0 - \omega_k)t'} dt'. \tag{2.14}$$

Upon substituting (2.14) into (2.11), there follows

$$\frac{da}{dt} = \varepsilon - \int_0^t a(t') \sum_k \lambda_k^2 e^{i(\omega_0 - \omega_k)(t-t')} dt' + F_k(t) + g \sum_j f(t, t_j) \sigma_-^j(t), \tag{2.15}$$

where

$$F_\kappa(t) = \sum_k \lambda_k b_k(0) e^{i(\omega_0 - \omega_k)t}, \tag{2.16}$$

is the noise operator associated with the squeezed vacuum reservoir.

It proves to be convenient to set

$$\Gamma = \sum_k \lambda_k^2 e^{i(\omega_0 - \omega_k)(t-t')}. \tag{2.17}$$

Assuming that the reservoir modes are closely spaced in frequency, one can go from a discrete representation to a continuous representation by replacing the summation over k by an integral:

$$\sum_k \longrightarrow \int_0^\infty D(\omega) d\omega, \tag{2.18}$$

where $D(\omega)$ is the density of states for the reservoir modes. We thus write

$$\Gamma = \int_0^\infty d\omega D(\omega) \lambda^2(\omega) e^{i(\omega_0 - \omega)(t-t')}. \tag{2.19}$$

Upon setting $\omega' = \omega - \omega_0$, we see that

$$\Gamma = \int_{-\omega_0}^{\infty} d\omega' D(\omega' + \omega_0)\lambda^2(\omega' + \omega_0)e^{-i\omega'(t-t')}. \tag{2.20}$$

Since the exponential factor is a rapidly oscillating function of ω' except near $\omega' = 0$, one can replace $D(\omega_0 + \omega')$ and $\lambda^2(\omega_0 + \omega')$ by $D(\omega_0)$ and $\lambda^2(\omega_0)$ and extend the lower limit of integration to $-\infty$. In view of this, we have

$$\Gamma = D(\omega_0)\lambda^2(\omega_0) \int_{-\infty}^{\infty} d\omega' e^{-i\omega'(t-t')}, \tag{2.21}$$

so that on carrying out the integration, there follows

$$\Gamma = \kappa\delta(t - t'), \tag{2.22}$$

where

$$\kappa = 2\pi D(\omega_0)\lambda^2(\omega_0), \tag{2.23}$$

is defined to be the cavity damping constant.

Furthermore substituting (2.22) into (2.15) and carrying out the integration, we find

$$\frac{da}{dt} = \varepsilon - \frac{1}{2}\kappa a(t) + F_\kappa(t) + g\sum_j f(t, t_j)\sigma_-^j(t). \tag{2.24}$$

We next consider the time evolution of the atomic operators for the two-level atoms which are in resonant with the cavity mode. For a two-level atom with lower level $|1\rangle$ and upper level $|2\rangle$, we define the atomic operators at the injection time t_j by

$$\sigma_+^j(t_j) = |2\rangle_{jj}\langle 1|, \tag{2.25a}$$

$$\sigma_-^j(t_j) = |1\rangle_{jj}\langle 2|, \tag{2.25b}$$

and satisfy the commutation relation

$$[\sigma_+^j(t_j), \sigma_-^j(t_j)] = \sigma_z^j(t_j), \tag{2.26}$$

in which $\sigma_z^j(t_j)$ is the population difference operator defined by

$$\sigma_z^j(t_j) = |2\rangle_{jj}\langle 2| - |1\rangle_{jj}\langle 1|. \tag{2.27}$$

Assuming the individual atoms to be completely independent of each other, the commutator of the atomic operators at any time has the form

$$\left[\sigma_+^j(t), \sigma_-^k(t)\right] = \sigma_z^j(t)\delta_{jk}, \tag{2.28a}$$

$$\left[\sigma_+^j(t), \sigma_+^k(t)\right] = \left[\sigma_-^j(t), \sigma_-^k(t)\right] = 0. \tag{2.28b}$$

Employing these commutation relations and the Hamiltonian (2.1), the Heisenberg equation of motion for the atomic operator $\sigma_-^j(t)$ turns out to be

$$\frac{d\sigma_-{}^j(t)}{dt} = gf(t,t_j)\sigma_z^j(t)a(t). \tag{2.29}$$

Formally integrating Eq. (2.29), we obtain

$$\sigma_-{}^j(t) = \sigma_-{}^j(t_j) + g\int_{t_j}^{t} f(t',t_j)\sigma_z^j(t')a(t')dt', \tag{2.30}$$

so that substitution of (2.30) into (2.24) yields

$$\frac{da}{dt} = \varepsilon - \frac{1}{2}\kappa a(t) + g^2\int_{t_j}^{t} dt' \sum_j f(t,t_j)f(t',t_j)\sigma_z^j(t')a(t') + F_\kappa(t) + F_a(t), \tag{2.31}$$

where

$$F_a(t) = g\sum_j f(t,t_j)\sigma_-^j(t_j), \tag{2.32}$$

is the noise operator associated with the interaction of the cavity mode with the two-level atoms.

In order to simplify Eq. (2.31) we adopt some approximations. We assume that the annihilation operator for the cavity mode $a(t)$ changes slowly during the atomic life time γ^{-1}. That is, the time evolution of the atomic operators is much faster than the time evolution of the cavity mode operators. One can then replace $a(t')$ in the integral of Eq. (2.31) by $a(t)$. We also restrict our analysis to linear terms in the cavity mode operator. To this end, we replace the population difference operator $\sigma_z^j(t')$ by its expectation value at the initial time. Since the two-level atoms are initially prepared in the upper state $|2\rangle$, the initial density operator for the individual atom has the form

$$\rho_j(0) = |2\rangle_{jj}\langle 2|. \tag{2.33}$$

With the aid of (2.27) and (2.33), one readily obtains

$$\langle \sigma_z^j(t_j) \rangle = 1. \tag{2.34}$$

Thus upon replacing $\sigma_z^j(t')$ by its expectation value (2.34) and $a(t')$ by $a(t)$, the quantum Langevin equation (2.31) for the cavity mode operator takes the form

$$\frac{da}{dt} = \varepsilon - \frac{1}{2}(\kappa - \mathcal{A})a(t) + F_\kappa(t) + F_a(t), \tag{2.35}$$

in which

$$\mathcal{A} = 2g^2 \int_{-\infty}^{t} dt' \sum_j f(t, t_j) f(t', t_j), \tag{2.36}$$

is the linear gain coefficient. The quantum Langevin equation (2.35) has a solution of the form

$$a(t) = a(0)e^{-\frac{1}{2}(\kappa-\mathcal{A})t} + e^{-\frac{1}{2}(\kappa-\mathcal{A})t} \int_0^t [\varepsilon + F_\kappa(t') + F_a(t')]\, e^{\frac{1}{2}(\kappa-\mathcal{A})t'} dt', \tag{2.37}$$

where finite solutions are obtained for $\kappa > \mathcal{A}$.

Assuming a constant atomic injection rate, the sum over j in (2.36) can be replaced by an integral over the injection times t_j as

$$\sum_j \longrightarrow r_a \int_{-\infty}^{t} dt_j.$$

It then follows that

$$\mathcal{A} = 2g^2 r_a \int_{-\infty}^{t} dt' \int_{-\infty}^{t} dt_j f(t, t_j) f(t', t_j). \tag{2.38}$$

Applying (2.5) in (2.38), we have

$$\mathcal{A} = 2g^2 r_a \int_{-\infty}^{t} dt_j \Theta(t - t_j) \exp[-\frac{\gamma}{\sqrt{2}}(t - 2t_j)] \int_{-\infty}^{t} dt' \Theta(t' - t_j) \exp[-\frac{\gamma}{\sqrt{2}} t']. \tag{2.39}$$

On account of the definition of the step function

$$\Theta(t' - t_j) = \begin{cases} 1 & \text{for } t' > t_j \\ 0 & \text{for } t' < t_j \end{cases}, \tag{2.40}$$

we see that

$$\int_{-\infty}^{t} dt' \Theta(t' - t_j) \exp[-\frac{\gamma}{\sqrt{2}} t'] = lim_{\epsilon \to 0} \int_{t_j + \epsilon}^{t} \exp[-\frac{\gamma}{\sqrt{2}} t'] dt', \tag{2.41}$$

so that carrying out the integration and taking the limit as $\epsilon \to 0$, one readily obtains

$$\int_{-\infty}^{t} dt' \Theta(t' - t_j) \exp[-\frac{\gamma}{\sqrt{2}} t'] = -\frac{\sqrt{2}}{\gamma} \left[e^{-\frac{\gamma}{\sqrt{2}} t} - e^{-\frac{\gamma}{\sqrt{2}} t_j} \right]. \tag{2.42}$$

Upon substituting (2.42) into (2.39), one gets

$$\mathcal{A} = \frac{2\sqrt{2} g^2 r_a}{\gamma} e^{-\frac{\gamma}{\sqrt{2}} t} \left[\int_{-\infty}^{t} dt_j \Theta(t - t_j) e^{\frac{\gamma}{\sqrt{2}} t_j} - e^{-\frac{\gamma}{\sqrt{2}} t} \int_{-\infty}^{t} dt_j \Theta(t - t_j) e^{\frac{2\gamma}{\sqrt{2}} t_j} \right]. \tag{2.43}$$

Applying (2.40) along with integration by parts, the first integral in (2.43) can be put in the form

$$\int_{-\infty}^{t} dt_j \Theta(t - t_j) e^{\frac{\gamma}{\sqrt{2}} t_j} = lim_{\epsilon \to 0} \left\{ \left[\frac{\sqrt{2}}{\gamma} \Theta(t - t_j) e^{\frac{\gamma}{\sqrt{2}} t_j} \right]_{-\infty}^{t - \epsilon} - \frac{\sqrt{2}}{\gamma} \int_{-\infty}^{t - \epsilon} dt_j \delta(t - t_j) e^{\frac{\gamma}{\sqrt{2}} t_j} \right\}. \tag{2.44}$$

From the property of the delta function, we note that

$$\int_{-\infty}^{t - \epsilon} dt_j \delta(t - t_j) e^{\frac{\gamma}{\sqrt{2}} t_j} = 0. \tag{2.45}$$

Hence

$$\int_{-\infty}^{t} dt_j \Theta(t-t_j) e^{\frac{\gamma}{\sqrt{2}}t_j} = \frac{\sqrt{2}}{\gamma} e^{\frac{\gamma}{\sqrt{2}}t}. \tag{2.46}$$

Similarly, one finds

$$\int_{-\infty}^{t} dt_j \Theta(t-t_j) e^{\frac{2\gamma}{\sqrt{2}}t_j} = \frac{\sqrt{2}}{2\gamma} e^{\frac{2\gamma}{\sqrt{2}}t}. \tag{2.47}$$

In view of these results, the linear gain coefficient Eq. (2.43) takes the form [5]

$$\mathcal{A} = \frac{2g^2 r_a}{\gamma^2}. \tag{2.48}$$

We now proceed to study the various correlation functions of the noise operators $F_\kappa(t)$ and $F_a(t)$. For a squeezed vacuum reservoir [5]

$$\langle b_k \rangle = \langle b_k^\dagger \rangle = 0, \tag{2.49a}$$

$$\langle b_k^\dagger b_{k'} \rangle = N\delta_{kk'}, \tag{2.49b}$$

$$\langle b_k b^\dagger{}_{k'} \rangle = (N+1)\delta_{kk'}, \tag{2.49c}$$

$$\langle b_k b_{k'} \rangle = M\delta_{k',2k_0-k}, \tag{2.49d}$$

$$\langle b^\dagger{}_k b^\dagger{}_{k'} \rangle = M^*\delta_{k',2k_0-k}, \tag{2.49e}$$

where

$$N = sinh^2(r), \tag{2.50}$$

$$M = -cosh(r) sinh(r) e^{i\phi}, \tag{2.51}$$

in which ϕ is the reference phase of the squeezed light relative to the cavity mode and r is the squeeze parameter assumed to be real and positive. On account of (2.49a), we see that

$$\langle F_\kappa(t) \rangle = \langle F_\kappa^\dagger(t) \rangle = 0. \tag{2.52}$$

In addition, with the aid of (2.16), we have

$$\langle F_\kappa^\dagger(t)F_\kappa(t')\rangle = N\sum_{k,k'}\lambda_k\lambda'_k\langle b_k^\dagger(0)b'_k(0)\rangle \exp\left[-i(\omega_0-\omega_k)t + i(\omega_0-\omega'_k)t'\right]. \tag{2.53}$$

Hence upon substituting (2.49b) into (2.53), there follows

$$\langle F_\kappa^\dagger(t)F_\kappa(t')\rangle = N\sum_{k}\lambda_k^2 \exp\left[-i(\omega_0-\omega_k)(t-t')\right]. \tag{2.54}$$

On account of (2.17) and the result (2.22), we see that

$$\langle F_\kappa^\dagger(t)F_\kappa(t')\rangle = \kappa N\delta(t-t'). \tag{2.55}$$

Following a similar procedure, one can easily verify that

$$\langle F_\kappa(t)F_\kappa^\dagger(t')\rangle = \kappa(N+1)\delta(t-t'), \tag{2.56}$$

$$\langle F_\kappa(t)F_\kappa(t')\rangle = \kappa M\delta(t-t'), \tag{2.57}$$

$$\langle F_\kappa^\dagger(t)F_\kappa^\dagger(t')\rangle = \kappa M^*\delta(t-t'). \tag{2.58}$$

Moreover, applying (2.32), we see that

$$\langle F_a(t)\rangle = g\sum_{j} f(t,t_j)\langle\sigma_-^j(t_j)\rangle. \tag{2.59}$$

On account of (2.33), we find

$$\langle\sigma_-^j(t_j)\rangle = 0, \tag{2.60}$$

so that

$$\langle F_a(t)\rangle = 0. \tag{2.61}$$

Employing (2.32), the normally ordered two-time correlation function of the noise operators $F_a(t)$ and $F_a^\dagger(t)$ can be put in the form

$$\langle F_a^\dagger(t)F_a(t')\rangle = g^2\sum_{j,k} f(t,t_j)f(t',t_k)\langle\sigma_+^j(t_j)\sigma_-^k(t_k)\rangle. \tag{2.62}$$

In view of (2.33), (2.60) and the fact that the atoms are completely independent of each other, we see that

$$\langle \sigma_+^j(t_j)\sigma_-^k(t_k)\rangle = \delta_{jk}. \tag{2.63}$$

It then follows that

$$\langle F_a^\dagger(t)F_a(t')\rangle = g^2 \sum_j f(t,t_j)f(t',t_j). \tag{2.64}$$

On integrating both sides with respect to t', we find

$$\int_{-\infty}^{t} dt' \langle F_a^\dagger(t)F_a(t')\rangle = g^2 \int_{-\infty}^{t} dt' \sum_j f(t,t_j)f(t',t_j). \tag{2.65}$$

Hence on account of (2.36), we have

$$\int_{-\infty}^{t} dt' \langle F_a^\dagger(t)F_a(t')\rangle = \frac{1}{2}\mathcal{A}, \tag{2.66}$$

from which follows

$$\langle F_a^\dagger(t)F_a(t')\rangle = \mathcal{A}\delta(t-t'). \tag{2.67}$$

With the aid of (2.32), one can write

$$\langle F_a(t)F_a^\dagger(t')\rangle = g^2 \sum_{j,k} f(t,t_j)f(t',t_k)\langle \sigma_-^j(t_j)\sigma_+^k(t_k)\rangle. \tag{2.68}$$

Since

$$\langle \sigma_-^j(t_j)\sigma_+^k(t_k)\rangle = 0, \tag{2.69}$$

we have

$$\langle F_a(t)F_a^\dagger(t')\rangle = 0. \tag{2.70}$$

It can be also established in a similar fashion that

$$\langle F_a(t)F_a(t')\rangle = 0, \tag{2.71a}$$

$$\langle F_a^\dagger(t)F_a^\dagger(t')\rangle = 0. \tag{2.71b}$$

We note that the vanishing of these correlation functions is due to the preparation of the atoms initially in the upper level and the restriction to linear analysis. Moreover, applying (2.32) once more and the commutation relation (2.28a), we obtain

$$[F_a^\dagger(t), F_a(t')] = g^2 \sum_j f(t, t_j) f(t', t_j) \sigma_z^j(t_j). \tag{2.72}$$

Replacing the population difference operator $\sigma_z^j(t_j)$ by its expectation value (2.34), we have

$$[F_a^\dagger(t), F_a(t')] = g^2 \sum_j f(t, t_j) f(t', t_j). \tag{2.73}$$

Following a similar procedure as the one leading to (2.67), one can establish that

$$[F_a^\dagger(t), F_a(t')] = \mathcal{A}\delta(t - t'). \tag{2.74}$$

3. Quantum fluctuations

Applying the solutions of the quantum Langevin equations, we wish here to calculate the quadrature fluctuations of the cavity mode, the squeezing spectrum and spectrum of intensity fluctuations of the output mode generated by two-level atoms interacting with a resonant cavity mode, driven by a coherent light and coupled to a squeezed vacuum reservoir. We also calculate the power spectrum of the output mode from which the line width of the radiation is determined.

3.1 Quadrature fluctuations

In this section we want to study the squeezing properties of the cavity mode. The squeezing properties of a single mode light are described by two Hermitian operators defined as

$$a_1 = a + a^\dagger, \tag{3.1}$$

$$a_2 = i(a^\dagger - a). \tag{3.2}$$

The variance of a quadrature operator $a_j(t)$ is expressible as

$$\Delta a_j^2(t) = \langle a_j^2(t)\rangle - \langle a_j(t)\rangle^2. \tag{3.3}$$

Employing (3.1) along with (2.37), one readily obtains

$$a_1(t) = a_1(0)e^{-\frac{1}{2}(\kappa-A)t} + e^{-\frac{1}{2}(\kappa-A)t}\int_0^t \left[2\varepsilon + F_\kappa(t') + F_a(t') + F_\kappa^\dagger(t') + F_a^\dagger(t')\right] e^{\frac{1}{2}(\kappa-A)t'}dt'. \tag{3.4}$$

Since the cavity mode is initially in the vacuum state, we have

$$\langle a_1(0)\rangle = 0. \tag{3.5}$$

In view of (2.52) and (2.61), one easily gets

$$\langle a_1(t)\rangle = \frac{4\varepsilon}{\kappa - \mathcal{A}}\left[1 - e^{\frac{-1}{2}(\kappa-\mathcal{A})t}\right]. \tag{3.6}$$

This reduces at steady state to

$$\langle a_1(t)\rangle_{ss} = \frac{4\varepsilon}{\kappa - \mathcal{A}}. \tag{3.7}$$

On account of the fact that a noise operator at later times does not affect system operators at earlier times, and atomic and field operators are uncorrelated at all times, we have

$$\langle a_1(0)F_\kappa(t')\rangle = \langle a_1(0)F_a(t')\rangle = 0 \tag{3.8}$$

and

$$\langle F_\kappa(t)F_a(t')\rangle = 0. \tag{3.9}$$

With the aid of (3.4), (3.8) and (3.9), one can write

$$\langle a_1^2(t)\rangle = \langle a_1^2(0)\rangle e^{-(\kappa-\mathcal{A})t} + e^{-(\kappa-\mathcal{A})t}\int_0^t\int_0^t dt'dt''\left[4\varepsilon^2 + \langle F_\kappa(t')F_\kappa(t'')\rangle + \langle F_\kappa(t')F_\kappa^\dagger(t'')\rangle\right.$$
$$\left. + \langle F_\kappa^\dagger(t')F_\kappa(t'')\rangle + \langle F_\kappa^\dagger(t')F_\kappa^\dagger(t'')\rangle + \langle F_a^\dagger(t')F_a(t'')\rangle\right] e^{\frac{1}{2}(\kappa-\mathcal{A})(t'+t'')}. \tag{3.10}$$

We note that

$$\langle a_1^2(0)\rangle = 1. \tag{3.11}$$

In view of (2.55-58), (3.11) and (2.67), Eq. (3.10) takes the form

$$\langle a_1^2(t)\rangle = e^{-(\kappa-\mathcal{A})t} + \frac{16\varepsilon^2}{[\kappa-\mathcal{A}]^2}\left[1 - e^{-\frac{1}{2}(\kappa-\mathcal{A})t}\right]^2 + \frac{\kappa(2N - 2|M|cos(\phi) + 1) + \mathcal{A}}{\kappa - \mathcal{A}}\left[1 - e^{-(\kappa-\mathcal{A})t}\right], \tag{3.12}$$

and at steady state

$$\langle a_1^2(t)\rangle_{ss} = \frac{16\varepsilon^2}{[\kappa-\mathcal{A}]^2} + \frac{\kappa(2N - 2|M|cos(\phi) + 1) + \mathcal{A}}{\kappa - \mathcal{A}}. \tag{3.13}$$

On account of (3.7) and (3.13), the fluctuations in the first quadrature operator takes the form

$$\Delta a_1^2 = \frac{\kappa(2N - 2|M|cos(\phi) + 1) + \mathcal{A}}{\kappa - \mathcal{A}}. \tag{3.14}$$

Following a similar line of reasoning, one obtains for the fluctuations in the second quadrature operator at steady state to be

$$\Delta a_2^2 = \frac{\kappa(2N + 2|M|cos(\phi) + 1) + \mathcal{A}}{\kappa - \mathcal{A}}. \tag{3.15}$$

When the relative phase between the squeezed vacuum and the cavity mode is $\pi/2$, we obtain

$$\Delta a_1^2 = \frac{\mathcal{A}}{\kappa - \mathcal{A}} + \frac{\kappa}{\kappa - \mathcal{A}}(2N + 1), \tag{3.16}$$

$$\Delta a_2^2 = \frac{\mathcal{A}}{\kappa - \mathcal{A}} + \frac{\kappa}{\kappa - \mathcal{A}}(2N + 1), \tag{3.17}$$

which shows that the cavity mode does not exhibit any squeezing. However for $\phi = 0$, that is, when the squeezed vacuum is in phase with the cavity mode, the quadrature fluctuations turn out to be

$$\Delta a_1^2 = \frac{\mathcal{A}}{\kappa - \mathcal{A}} + \frac{\kappa}{\kappa - \mathcal{A}}e^{-2r}, \tag{3.18}$$

$$\Delta a_2^2 = \frac{\mathcal{A}}{\kappa - \mathcal{A}} + \frac{\kappa}{\kappa - \mathcal{A}}e^{2r}. \tag{3.19}$$

It can be easily shown that the cavity mode will be in a squeezed state for the squeeze parameter satisfying the inequality

$$r > \frac{1}{2}ln\left[\frac{\kappa}{\kappa - 2\mathcal{A}}\right]. \tag{3.20}$$

We therefore see that the squeezing property of the cavity mode depends on the phase of the input squeezed light relative to the cavity mode as well as the squeeze parameter. (Hereafter we confine our analysis to the case for which $\phi = 0$ so that $M = M^*$).

3.2 The squeezing spectrum

The squeezing spectrum for the output mode is defined as the Fourier transform of the two-time correlation function $\langle a_j^{out}(t+\tau), a_j^{out}(t)\rangle$:

$$S_j^{out}(\omega) = \int_{-\infty}^{\infty} d\tau e^{i\omega\tau} \langle a_j^{out}(t+\tau), a_j^{out}(t)\rangle_{ss}, \tag{3.21}$$

in which the two-time correlation function is defined by

$$\langle a_j^{out}(t+\tau), a_j^{out}(t)\rangle = \langle a_j^{out}(t+\tau) a_j^{out}(t)\rangle - \langle a_j^{out}(t+\tau)\rangle\langle a_j^{out}(t)\rangle, \tag{3.22}$$

where $a_j^{out}(t)$ is the output quadrature operator.

The input-output relation for the quadrature operators is expressible as

$$a_j^{out}(t) = \sqrt{\kappa} a_j(t) - a_j^{in}(t), \tag{3.23}$$

where $a_j^{in}(t)$ is the input quadrature operator with the input operator defined as

$$a^{in}(t) = \frac{1}{\sqrt{\kappa}} F_\kappa(t). \tag{3.24}$$

On account of (3.1) and (3.23), the two-time correlation function for the first output quadrature operator takes the form

$$\langle a_1^{out}(t+\tau), a_1^{out}(t)\rangle = \kappa\langle a_1(t+\tau)a_1(t)\rangle - \sqrt{\kappa}\langle a_1(t+\tau)a_1^{in}(t)\rangle - \sqrt{\kappa}\langle a_1^{in}(t+\tau)a_1(t)\rangle$$

$$+\langle a_1^{in}(t+\tau)a_1^{in}(t)\rangle - \kappa\langle a_1(t+\tau)\rangle\langle a_1(t)\rangle. \tag{3.25}$$

Next we need to obtain the explicit forms of the two-time correlation functions involved in (3.25).

We note that

$$\langle a_1^{in}(t+\tau)a_1(t)\rangle = 0, \tag{3.26}$$

$$\langle a_1^{in}(t+\tau)a_1^{in}(t)\rangle = \langle a_{in}(t+\tau)a_{in}(t)\rangle + \langle a_{in}(t+\tau)a_{in}^{\dagger}(t)\rangle + \langle a_{in}^{\dagger}(t+\tau)a_{in}(t)\rangle + \langle a_{in}^{\dagger}(t+\tau)a_{in}^{\dagger}(t)\rangle. \tag{3.27}$$

Applying the two-time correlation functions (2.55-58) along with (3.24), we obtain

$$\langle a_1^{in}(t+\tau)a_1^{in}(t)\rangle = (2N+2M+1)\delta(\tau). \tag{3.28}$$

It then follows from (2.50-51) that

$$\langle a_1^{in}(t+\tau)a_1^{in}(t)\rangle = e^{-2r}\delta(\tau). \tag{3.29}$$

Employing (3.4) and (3.5), we find

$$\langle a_1(t+\tau)a_1(t)\rangle = \langle a_1(0)a_1(0)\rangle e^{-(\kappa-\mathcal{A})(t+\frac{1}{2}\tau)} + e^{-(\kappa-\mathcal{A})(t+\frac{1}{2}\tau)} \int_0^{t+\tau}\int_0^t dt'dt'' \left[4\varepsilon^2 + \kappa\langle a_1^{in}(t')a_1^{in}(t'')\rangle \right.$$

$$\left. + \langle F_a^{\dagger}(t')F_a(t'')\rangle\right] e^{\frac{1}{2}(\kappa-\mathcal{A})(t'+t'')}. \tag{3.30}$$

With the aid of (2.67) and (3.29), Eq. (3.30) becomes

$$\langle a_1(t+\tau)a_1(t)\rangle = \langle a_1(0)a_1(0)\rangle e^{-(\kappa-\mathcal{A})(t+\frac{1}{2}\tau)} + \frac{16\varepsilon^2}{[\kappa-\mathcal{A}]^2}\left[1-e^{-\frac{1}{2}(\kappa-\mathcal{A})(t+\tau)}\right]\left[1-e^{-(\kappa-\mathcal{A})t}\right]$$

$$+ \left[\frac{\mathcal{A}}{\kappa-\mathcal{A}} + \frac{\kappa}{\kappa-\mathcal{A}}e^{-2r}\right]\left[1-e^{-(\kappa-\mathcal{A})t}\right]e^{-\frac{1}{2}(\kappa-\mathcal{A})\tau}. \tag{3.31}$$

We see that expression (3.31) at steady state reduces to

$$\langle a_1(t+\tau)a_1(t)\rangle_{ss} = \frac{16\varepsilon^2}{[\kappa-\mathcal{A}]^2} + \left[\frac{\mathcal{A}}{\kappa-\mathcal{A}} + \frac{\kappa}{\kappa-\mathcal{A}}e^{-2r}\right]e^{-\frac{1}{2}(\kappa-\mathcal{A})\tau}. \tag{3.32}$$

From (3.4), one can easily verify that

$$\langle a_1(t+\tau)\rangle_{ss} = \langle a_1(t)\rangle_{ss}, \tag{3.33}$$

so that application of (3.7) yields

$$\langle a_1(t+\tau)\rangle_{ss}\langle a_1(t)\rangle_{ss} = \frac{16\varepsilon^2}{[\kappa-\mathcal{A}]^2}. \tag{3.34}$$

Applying (3.4), (3.5) and (3.24), one readily obtains

$$\langle a_1(t+\tau)a_1^{in}(t)\rangle = \sqrt{\kappa}e^{-\frac{1}{2}(\kappa-\mathcal{A})(t+\tau)}\int_0^{t+\tau}\langle a_1^{in}(t')a_1^{in}(t)\rangle e^{\frac{1}{2}(\kappa-\mathcal{A})t'}dt'. \tag{3.35}$$

On account of (3.29), one gets

$$\langle a_1(t+\tau)a_1^{in}(t)\rangle_{ss} = \sqrt{\kappa}e^{-[2r+\frac{1}{2}(\kappa-\mathcal{A})\tau]}. \tag{3.36}$$

On substituting the results described by (3.29), (3.32), (3.34) and (3.36) into expression (3.25), the two-time correlation function for the first output quadrature operator at steady state turns out to be

$$\langle a_1^{out}(t+\tau), a_1^{out}(t)\rangle_{ss} = [1+e^{-2r}]\frac{\kappa\mathcal{A}}{(\kappa-\mathcal{A})}e^{-\frac{1}{2}(\kappa-\mathcal{A})\tau} + e^{-2r}\delta(\tau). \tag{3.37}$$

Thus on account of this result, the squeezing spectrum (3.21) for the first output quadrature operator takes the form

$$S_1^{out}(\omega) = e^{-2r}\int_{-\infty}^{\infty}e^{i\omega\tau}\delta(\tau)d\tau + [e^{-2r}+1]\frac{\kappa\mathcal{A}}{(\kappa-\mathcal{A})}\int_{-\infty}^{\infty}e^{[i\omega-\frac{1}{2}(\kappa-\mathcal{A})]\tau}d\tau. \tag{3.38}$$

It then follows that

$$S_1^{out}(\omega) = e^{-2r} + [e^{-2r}+1]\frac{\kappa\mathcal{A}}{(\kappa-\mathcal{A})}\int_{-\infty}^{\infty}e^{[i\omega-\frac{1}{2}(\kappa-\mathcal{A})]\tau}d\tau. \tag{3.39}$$

Now applying the stationarity property

$$\int_{-\infty}^{\infty}e^{[i\omega-\frac{1}{2}(\kappa-\mathcal{A})]\tau}d\tau = \int_0^{\infty}e^{[i\omega-\frac{1}{2}(\kappa-\mathcal{A})]\tau}d\tau + \int_0^{\infty}e^{[-i\omega-\frac{1}{2}(\kappa-\mathcal{A})]\tau}d\tau, \tag{3.40}$$

and carrying out the integration, one finds

$$\int_{-\infty}^{\infty}e^{[i\omega-\frac{1}{2}(\kappa-\mathcal{A})]\tau}d\tau = \frac{(\kappa-\mathcal{A})}{(\frac{\kappa-\mathcal{A}}{2})^2+\omega^2}. \tag{3.41}$$

In view of this result, we obtain the squeezing spectrum for the first output quadrature operator to be

$$S_1^{out}(\omega) = e^{-2r} + \frac{\kappa\mathcal{A}}{(\frac{\kappa-\mathcal{A}}{2})^2+\omega^2}\left[e^{-2r}+1\right]. \tag{3.42}$$

In addition, following a similar line of argument, it can be established that the squeezing spectrum for the second output quadrature operator has the form

$$S_2^{out}(\omega) = e^{2r} + \frac{\kappa\mathcal{A}}{(\frac{\kappa-\mathcal{A}}{2})^2 + \omega^2}\left[e^{2r} + 1\right]. \tag{3.43}$$

We note that these spectra are defined in the frame rotating at the atomic transition frequency ω_0, so that $\omega = 0$ represents the resonance frequency. In both cases the squeezing spectra are Lorentzians with half width of $\frac{1}{2}(\kappa - \mathcal{A})$. We see that the squeezed vacuum reservoir enhances the peak of the spectrum for a_2 and decreases that of a_1. However, it does not have any effect on the width of the spectrum. We also observe that the driving coherent light has no effect on the squeezing spectrum.

3.3 The spectrum of intensity fluctuations

In this section we would like to study the intensity fluctuations of the output radiation which is directly observable by photoelectron detectors placed at the output of the cavity. The spectrum of intensity fluctuations is defined as

$$S_{out}(\omega) = \int d\tau e^{i\omega\tau}\left\langle I_{out}(t+\tau), I_{out}(t)\right\rangle_{ss}, \tag{3.44}$$

where

$$\left\langle I_{out}(t+\tau), I_{out}(t)\right\rangle_{ss} = \left\langle I_{out}(t+\tau)I_{out}(t)\right\rangle_{ss} - \left\langle I_{out}(t+\tau)\right\rangle_{ss}\left\langle I_{out}(t)\right\rangle_{ss}, \tag{3.45}$$

$$I_{out}(t) = a_{out}^{\dagger}(t)a_{out}(t). \tag{3.46}$$

Applying (3.46) along with the commutation relation

$$[a_{out}(t+\tau), a_{out}^{\dagger}(t)] = \delta(\tau), \tag{3.47}$$

one can put (3.45) in the form

$$\left\langle I_{out}(t+\tau), I_{out}(t)\right\rangle_{ss} = \left\langle a_{out}^{\dagger}(t)a_{out}^{\dagger}(t+\tau)a_{out}(t+\tau)a_{out}(t)\right\rangle_{ss} + \left\langle a_{out}^{\dagger}(t+\tau)a_{out}(t)\right\rangle_{ss}\delta(\tau)$$

$$-\left\langle a^{\dagger}_{out}(t+\tau)a_{out}(t+\tau)\right\rangle_{ss}\left\langle a^{\dagger}_{out}(t)a_{out}(t)\right\rangle_{ss}. \tag{3.48}$$

Employing the input-output relation

$$a_{out}(t) = \sqrt{\kappa}a(t) - a_{in}(t), \tag{3.49}$$

one finds

$$\begin{aligned}
\left\langle a^{\dagger}_{out}(t)a^{\dagger}_{out}(t+\tau)a_{out}(t+\tau)a_{out}(t)\right\rangle_{ss} &= \kappa^2\left\langle a^{\dagger}(t)a^{\dagger}(t+\tau)a(t+\tau)a(t)\right\rangle_{ss} \\
-\kappa^{\frac{3}{2}}\Big[\left\langle a^{\dagger}(t)a^{\dagger}_{in}(t+\tau)a(t+\tau)a(t)\right\rangle_{ss} &+ \left\langle a^{\dagger}(t)a^{\dagger}(t+\tau)a_{in}(t+\tau)a(t)\right\rangle_{ss} \\
+\left\langle a^{\dagger}_{in}(t)a^{\dagger}(t+\tau)a(t+\tau)a(t)\right\rangle_{ss} &+ \left\langle a^{\dagger}(t)a^{\dagger}(t+\tau)a(t+\tau)a_{in}(t)\right\rangle_{ss}\Big] \\
+\kappa\Big[\left\langle a^{\dagger}(t)a^{\dagger}_{in}(t+\tau)a_{in}(t+\tau)a(t)\right\rangle_{ss} &+ \left\langle a^{\dagger}_{in}(t)a^{\dagger}_{in}(t+\tau)a(t+\tau)a(t)\right\rangle_{ss} \\
+\left\langle a^{\dagger}_{in}(t)a^{\dagger}(t+\tau)a_{in}(t+\tau)a(t)\right\rangle_{ss} &+ \left\langle a^{\dagger}(t)a^{\dagger}_{in}(t+\tau)a(t+\tau)a_{in}(t)\right\rangle_{ss} \\
+\left\langle a^{\dagger}(t)a^{\dagger}(t+\tau)a_{in}(t+\tau)a_{in}(t)\right\rangle_{ss} &+ \left\langle a^{\dagger}_{in}(t)a^{\dagger}(t+\tau)a(t+\tau)a_{in}(t)\right\rangle_{ss}\Big] \\
-\sqrt{\kappa}\Big[\left\langle a^{\dagger}_{in}(t)a^{\dagger}_{in}(t+\tau)a_{in}(t+\tau)a(t)\right\rangle_{ss} &+ \left\langle a^{\dagger}(t)a^{\dagger}_{in}(t+\tau)a_{in}(t+\tau)a_{in}(t)\right\rangle_{ss} \\
+\left\langle a^{\dagger}_{in}(t)a^{\dagger}_{in}(t+\tau)a(t+\tau)a_{in}(t)\right\rangle_{ss} &+ \left\langle a^{\dagger}_{in}(t)a^{\dagger}(t+\tau)a_{in}(t+\tau)a_{in}(t)\right\rangle_{ss}\Big] \\
&+\left\langle a^{\dagger}_{in}(t)a^{\dagger}_{in}(t+\tau)a_{in}(t+\tau)a_{in}(t)\right\rangle_{ss}.
\end{aligned} \tag{3.50}$$

We now proceed to calculate the various correlation functions involved in (3.50). Applying (2.37) and assuming the cavity mode to be initially in the vacuum state, the second-order correlation function can be put in the form

$$\begin{aligned}
\left\langle a^{\dagger}(t)a^{\dagger}(t+\tau)a(t+\tau)a(t)\right\rangle &= e^{-(\kappa-\mathcal{A})(2t+\tau)}\int_0^t\int_0^{t+\tau}\int_0^{t+\tau}\int_0^t\Big\langle\left[\varepsilon+F^{\dagger}_{\kappa}(t_1)+F^{\dagger}_{a}(t_1)\right] \\
&\times\left[\varepsilon+F^{\dagger}_{\kappa}(t_2)+F^{\dagger}_{a}(t_2)\right]\left[\varepsilon+F_{\kappa}(t_3)+F_{a}(t_3)\right]\left[\varepsilon+F_{\kappa}(t_4)+F_{a}(t_4)\right]\Big\rangle \\
&\times\exp\left[\frac{1}{2}(\kappa-\mathcal{A})(t_1+t_2+t_3+t_4)\right]dt_1dt_2dt_3dt_4.
\end{aligned} \tag{3.51}$$

On account of (2.71a-b), (3.8) and (3.9), we have

$$\left\langle a^\dagger(t)a^\dagger(t+\tau)a(t+\tau)a(t)\right\rangle = e^{-(\kappa-\mathcal{A})(2t+\tau)}\int_0^t\int_0^{t+\tau}\int_0^{t+\tau}\int_0^t \big[\varepsilon^4+\varepsilon^2\langle F_\kappa(t_3)F_\kappa(t_4)\rangle$$
$$+\varepsilon^2\langle F_\kappa^\dagger(t_2)F_\kappa(t_4)\rangle+\varepsilon^2\langle F_\kappa^\dagger(t_2)F_\kappa(t_3)\rangle++\varepsilon^2\langle F_a^\dagger(t_2)F_a(t_4)\rangle$$
$$+\varepsilon^2\langle F_a^\dagger(t_2)F_a(t_3)\rangle+\varepsilon^2\langle F_\kappa^\dagger(t_1)F_\kappa(t_4)\rangle+\varepsilon^2\langle F_\kappa^\dagger(t_1)F_\kappa(t_3)\rangle$$
$$+\varepsilon^2\langle F_\kappa^\dagger(t_1)F_\kappa^\dagger(t_2)\rangle+\varepsilon^2\langle F_a^\dagger(t_1)F_a(t_4)\rangle+\varepsilon^2\langle F_a^\dagger(t_1)F_a(t_3)\rangle$$
$$+\langle F_\kappa^\dagger(t_1)F_\kappa^\dagger(t_2)F_\kappa(t_3)F_\kappa(t_4)\rangle+\langle F_\kappa^\dagger(t_1)F_a^\dagger(t_2)F_\kappa(t_3)F_a(t_4)\rangle$$
$$+\langle F_\kappa^\dagger(t_1)F_a^\dagger(t_2)F_a(t_3)F_\kappa(t_4)\rangle+\langle F_a^\dagger(t_1)F_\kappa^\dagger(t_2)F_\kappa(t_3)F_a(t_4)\rangle$$
$$+\langle F_a^\dagger(t_1)F_\kappa^\dagger(t_2)F_a(t_3)F_\kappa(t_4)\rangle+\langle F_a^\dagger(t_1)F_a^\dagger(t_2)F_a(t_3)F_a(t_4)\rangle\big]$$
$$\times\exp\left[\frac{1}{2}(\kappa-\mathcal{A})(t_1+t_2+t_3+t_4)\right]dt_1dt_2dt_3dt_4. \tag{3.52}$$

In view of (2.52) and (2.61), the fourth order correlation function of the noise operators can be calculated employing the method described by [15]. Hence

$$\langle F_\kappa^\dagger(t_1)F_\kappa^\dagger(t_2)F_\kappa(t_3)F_\kappa(t_4)\rangle = \langle F_\kappa^\dagger(t_1)F_\kappa^\dagger(t_2)\rangle\langle F_\kappa(t_3)F_\kappa(t_4)\rangle+\langle F_\kappa^\dagger(t_1)F_\kappa(t_3)\rangle\langle F_\kappa^\dagger(t_2)F_\kappa(t_4)\rangle$$
$$+\langle F_\kappa^\dagger(t_1)F_\kappa(t_4)\rangle\langle F_\kappa^\dagger(t_2)F_\kappa(t_3)\rangle,$$
$$=\kappa^2M^2\delta(t_1-t_2)\delta(t_3-t_4)+\kappa^2N^2[(\delta(t_1-t_3)\delta(t_2-t_4)$$
$$+\delta(t_1-t_4)\delta(t_2-t_3)]. \tag{3.53a}$$

Following a similar procedure, one can establish that

$$\langle F_\kappa^\dagger(t_1)F_a^\dagger(t_2)F_\kappa(t_3)F_a(t_4)\rangle=\kappa\mathcal{A}N\delta(t_1-t_3)\delta(t_2-t_4), \tag{3.53b}$$

$$\langle F_\kappa^\dagger(t_1)F_a^\dagger(t_2)F_a(t_3)F_\kappa(t_4)\rangle=\kappa\mathcal{A}N\delta(t_1-t_4)\delta(t_2-t_3), \tag{3.53c}$$

$$\langle F_a^\dagger(t_1)F_\kappa^\dagger(t_2)F_\kappa(t_3)F_a(t_4)\rangle=\kappa\mathcal{A}N\delta(t_1-t_4)\delta(t_2-t_3), \tag{3.53d}$$

$$\langle F_a^\dagger(t_1)F_\kappa^\dagger(t_2)F_a(t_3)F_\kappa(t_4)\rangle = \kappa\mathcal{A}N\delta(t_1-t_3)\delta(t_2-t_4), \tag{3.53e}$$

$$\langle F_a^\dagger(t_1)F_a^\dagger(t_2)F_a(t_3)F_a(t_4)\rangle = \mathcal{A}^2[\delta(t_1-t_3)\delta(t_2-t_4)+\delta(t_1-t_4)\delta(t_2-t_3)]. \tag{3.53f}$$

Applying these results and the two-time correlation functions of the noise operators (2.55-58) and (2.67) in (3.52) and carrying out the integrations, the second-order correlation function at steady state takes the form

$$\left\langle a^\dagger(t)a^\dagger(t+\tau)a(t+\tau)a(t)\right\rangle_{ss} = \frac{16\varepsilon^4}{[\kappa-\mathcal{A}]^4}+\frac{8\varepsilon^2(\mathcal{A}+\kappa N)}{[\kappa-\mathcal{A}]^3}+\frac{(\mathcal{A}+\kappa N)^2}{[\kappa-\mathcal{A}]^2}$$
$$+\frac{8\varepsilon^2}{[\kappa-\mathcal{A}]^3}[\kappa N+\kappa M+\mathcal{A}]e^{-\frac{1}{2}(\kappa-\mathcal{A})\tau}+\frac{(\mathcal{A}+\kappa N)^2+\kappa^2M^2}{[\kappa-\mathcal{A}]^2}e^{-(\kappa-\mathcal{A})\tau}. \tag{3.54}$$

In addition, with the aid of (2.37) and (3.24), the second term in (3.50) can be written as

$$\langle a^\dagger(t)a_{in}^\dagger(t+\tau)a(t+\tau)a(t)\rangle = e^{-\frac{1}{2}(\kappa-\mathcal{A})(3t+\tau)}\int_0^t\int_0^{t+\tau}\int_0^t\Big\langle[\varepsilon+F_\kappa^\dagger(t_1)+F_a^\dagger(t_1)][a_{in}^\dagger(t+\tau)]$$
$$[\varepsilon+F_\kappa(t_2)+F_a(t_2)][\varepsilon+F_\kappa(t_3)+F_a(t_3)]\Big\rangle\exp\left[\frac{1}{2}(\kappa-\mathcal{A})(t_1+t_2+t_3)\right]dt_1dt_2dt_3,$$
$$= e^{-\frac{1}{2}(\kappa-\mathcal{A})(3t+\tau)}\int_0^t\int_0^{t+\tau}\int_0^t\{\sqrt{\kappa}\varepsilon^2[N\delta(t_3-(t+\tau))+N\delta(t_2-(t+\tau))+M\delta(t_1-(t+\tau))]$$
$$+\kappa^{\frac{3}{2}}M^2\delta(t_1-(t+\tau))\delta(t_2-t_3)+\kappa^{\frac{3}{2}}N^2[\delta(t_1-t_2)\delta(t_3-(t+\tau))+\delta(t_1-t_3)\delta(t_2-(t+\tau))]$$
$$+\sqrt{\kappa}\mathcal{A}N[\delta(t_1-t_3)\delta(t_2-(t+\tau))+\delta(t_1-t_2)\delta(t_3-(t+\tau))]\}$$
$$\times\exp\left[\frac{1}{2}(\kappa-\mathcal{A})(t_1+t_2+t_3)\right]dt_1dt_2dt_3. \tag{3.55}$$

We note that

$$\int_0^t dt' f(t')\delta(t'-(t+\tau)) = 0, \tag{3.56}$$

for a non zero τ. On carrying out the integrations in (3.55) taking into account (3.56), one obtains

$$\langle a^\dagger(t)a_{in}^\dagger(t+\tau)a(t+\tau)a(t)\rangle_{ss} = \frac{2\sqrt{\kappa}\varepsilon^2N}{[\kappa-\mathcal{A}]^2}+\frac{\kappa^{\frac{3}{2}}N^2}{2[\kappa-\mathcal{A}]}+\frac{\sqrt{\kappa}\mathcal{A}N}{2[\kappa-\mathcal{A}]}. \tag{3.57}$$

Following a similar procedure, one can establish that

$$\langle a^\dagger(t)a^\dagger(t+\tau)a_{in}(t+\tau)a(t)\rangle_{ss} = \frac{2\sqrt{\kappa}\varepsilon^2 N}{[\kappa-\mathcal{A}]^2} + \frac{\kappa^{\frac{3}{2}}N^2}{2[\kappa-\mathcal{A}]} + \frac{\sqrt{\kappa}\mathcal{A}N}{2[\kappa-\mathcal{A}]}, \tag{3.58}$$

$$\langle a_{in}^\dagger(t)a^\dagger(t+\tau)a(t+\tau)a(t)\rangle_{ss} = \langle a^\dagger(t)a^\dagger(t+\tau)a(t+\tau)a_{in}(t)\rangle_{ss}$$

$$= \frac{2\sqrt{\kappa}\varepsilon^2 N}{[\kappa-\mathcal{A}]^2} + \frac{\kappa^{\frac{3}{2}}N^2}{2[\kappa-\mathcal{A}]} + \frac{\sqrt{\kappa}\mathcal{A}N}{2[\kappa-\mathcal{A}]} + \frac{4\sqrt{\kappa}\varepsilon^2}{[\kappa-\mathcal{A}]^2}(N+M)e^{-\frac{1}{2}(\kappa-\mathcal{A})\tau}$$
$$+\frac{\sqrt{\kappa}}{[\kappa-\mathcal{A}]}[\kappa N^2 + \kappa M^2 + \mathcal{A}N]e^{-(\kappa-\mathcal{A})\tau}. \tag{3.59}$$

We next calculate those correlation functions having two input operators $a_{in}(t)$ in Eq. (3.50). With the aid of (2.37), we have

$$\langle a^\dagger(t)a_{in}^\dagger(t+\tau)a_{in}(t+\tau)a(t)\rangle = e^{-(\kappa-\mathcal{A})t}\int_0^t\int_0^t \Big\langle \Big[\varepsilon a_{in}^\dagger(t+\tau) + F_\kappa^\dagger(t_1)a_{in}^\dagger(t+\tau)$$
$$+F_a^\dagger(t_1)a_{in}^\dagger(t+\tau)\Big]\left[\varepsilon a_{in}(t+\tau) + a_{in}(t+\tau)F_\kappa(t_2) + a_{in}(t+\tau)F_a(t_2)\right]\Big\rangle$$
$$\times \exp\left[\frac{1}{2}(\kappa-\mathcal{A})(t_1+t_2)\right]dt_1dt_2,$$
$$= e^{-(\kappa-\mathcal{A})t}\int_0^t\int_0^t \Big[\varepsilon^2\langle a_{in}^\dagger(t+\tau)a_{in}(t+\tau)\rangle + \langle F_\kappa^\dagger(t_1)a_{in}^\dagger(t+\tau)a_{in}(t+\tau)F_\kappa(t_2)\rangle$$
$$+\langle F_a^\dagger(t_1)a_{in}^\dagger(t+\tau)a_{in}(t+\tau)F_a(t_2)\rangle\Big]\exp\left[\frac{1}{2}(\kappa-\mathcal{A})(t_1+t_2)\right]dt_1dt_2,$$
$$= e^{-(\kappa-\mathcal{A})t}\int_0^t\int_0^t \left[\varepsilon^2 N + \kappa(N^2+M^2)\delta(t_1-(t+\tau))\delta(t_2-(t+\tau))\right.$$
$$\left.+(\mathcal{A}N+\kappa N^2)\delta(t_1-t_2)\right]\exp\left[\frac{1}{2}(\kappa-\mathcal{A})(t_1+t_2)\right]dt_1dt_2. \tag{3.60}$$

After integrating Eq. (3.60), the obtained result at steady state has the form

$$\langle a^\dagger(t)a_{in}^\dagger(t+\tau)a_{in}(t+\tau)a(t)\rangle_{ss} = \frac{4\varepsilon^2 N}{[\kappa-\mathcal{A}]^2} + \frac{(\mathcal{A}+\kappa N)N}{[\kappa-\mathcal{A}]}. \tag{3.61}$$

Similarly, one can find

$$\langle a_{in}^\dagger(t)a_{in}^\dagger(t+\tau)a(t+\tau)a(t)\rangle_{ss} = \langle a^\dagger(t)a^\dagger(t+\tau)a_{in}(t+\tau)a_{in}(t)\rangle_{ss}$$

$$= \left[\frac{4\varepsilon^2 M}{[\kappa-\mathcal{A}]^2} + \frac{\kappa M^2}{[\kappa-\mathcal{A}]}e^{-\frac{1}{2}(\kappa-\mathcal{A})\tau}\right]\delta(\tau) + \frac{1}{4}\kappa N^2, \tag{3.62}$$

$$\langle a_{in}^\dagger(t)a^\dagger(t+\tau)a_{in}(t+\tau)a(t)\rangle_{ss} = \langle a^\dagger(t)a_{in}^\dagger(t+\tau)a(t+\tau)a_{in}(t)\rangle_{ss}$$

$$= \left[\frac{4\varepsilon^2 N}{[\kappa-\mathcal{A}]^2} + \frac{(\mathcal{A}+\kappa N)N}{[\kappa-\mathcal{A}]}e^{-\frac{1}{2}(\kappa-\mathcal{A})\tau}\right]\delta(\tau) + \frac{1}{4}\kappa N^2, \qquad (3.63)$$

$$\langle a_{in}^\dagger(t)a^\dagger(t+\tau)a(t+\tau)a_{in}(t)\rangle_{ss} = \frac{4\varepsilon^2 N}{[\kappa-\mathcal{A}]^2} + \frac{(\mathcal{A}+\kappa N)N}{[\kappa-\mathcal{A}]} + \kappa(N^2+M^2)e^{-(\kappa-\mathcal{A})\tau}. \quad (3.64)$$

Applying (2.37) once more, we have

$$\langle a_{in}^\dagger(t)a_{in}^\dagger(t+\tau)a_{in}(t+\tau)a(t)\rangle_{ss} = e^{-\frac{1}{2}(\kappa-\mathcal{A})t}\int_0^t dt_1\langle a_{in}^\dagger(t)a_{in}^\dagger(t+\tau)a_{in}(t+\tau)F_\kappa(t_1)\rangle e^{\frac{1}{2}(\kappa-\mathcal{A})t_1},$$

$$= e^{-\frac{1}{2}(\kappa-\mathcal{A})t}\int_0^t dt_1\left[\sqrt{\kappa}(M^2+N^2)\delta(\tau)\delta(t_1-(t+\tau)) + \sqrt{\kappa}N^2\delta(t_1-t)\right] \times e^{\frac{1}{2}(\kappa-\mathcal{A})t_1},$$

$$(3.65a)$$

from which follows

$$\langle a_{in}^\dagger(t)a_{in}^\dagger(t+\tau)a_{in}(t+\tau)a(t)\rangle_{ss} = \frac{1}{2}\sqrt{\kappa}N^2. \qquad (3.65b)$$

One can also verify in a similar manner that

$$\langle a^\dagger(t)a_{in}^\dagger(t+\tau)a_{in}(t+\tau)a_{in}(t)\rangle_{ss} = \frac{1}{2}\sqrt{\kappa}N^2, \qquad (3.66),$$

$$\langle a_{in}^\dagger(t)a_{in}^\dagger(t+\tau)a(t+\tau)a_{in}(t)\rangle_{ss} = \langle a_{in}^\dagger(t)a^\dagger(t+\tau)a_{in}(t+\tau)a_{in}(t)\rangle_{ss}$$

$$= \left[\sqrt{\kappa}(M^2+N^2)e^{-\frac{1}{2}(\kappa-\mathcal{A})\tau}\right]\delta(\tau) + \frac{1}{2}\sqrt{\kappa}N^2, \qquad (3.67),$$

$$\langle a_{in}^\dagger(t)a_{in}^\dagger(t+\tau)a_{in}(t+\tau)a_{in}(t)\rangle_{ss} = (M^2+N^2)\delta(\tau) + N^2. \qquad (3.68)$$

Upon substituting the results described by (3.54), (3.57-59), (3.61-64) and (3.65b-68) into expression (3.50), there follows

$$\left\langle a_{out}^\dagger(t)a_{out}^\dagger(t+\tau)a_{out}(t+\tau)a_{out}(t)\right\rangle_{ss} = \Gamma_1 + [\Gamma_2+\Gamma_3 e^{-\frac{1}{2}(\kappa-\mathcal{A})\tau}]\delta(\tau) + \Gamma_4 e^{-\frac{1}{2}(\kappa-\mathcal{A})\tau} + \Gamma_5 e^{-(\kappa-\mathcal{A})\tau},$$

$$(3.69)$$

where

$$\Gamma_1 = \frac{16\varepsilon^4\kappa^2}{[\kappa-\mathcal{A}]^4} + \frac{8\varepsilon^2\kappa^2(\mathcal{A}+\kappa N)}{[\kappa-\mathcal{A}]^3} + \frac{\kappa^2(\mathcal{A}+\kappa N)^2}{[\kappa-\mathcal{A}]^2} - \frac{8\varepsilon^2\kappa^2 N}{[\kappa-\mathcal{A}]^2} - \frac{2\kappa^3 N^2}{[\kappa-\mathcal{A}]}$$

$$-\frac{2\kappa^2 \mathcal{A}N}{[\kappa-\mathcal{A}]}+\frac{8\varepsilon^2\kappa N}{[\kappa-\mathcal{A}]^2}+\frac{2\kappa N(\mathcal{A}+\kappa N)}{[\kappa-\mathcal{A}]}+\kappa^2N^2+N^2(1-2\kappa), \tag{3.70}$$

$$\Gamma_2=\frac{8\varepsilon^2\kappa}{[\kappa-\mathcal{A}]^2}(M+N)+(N^2+M^2), \tag{3.71}$$

$$\Gamma_3=\frac{2\kappa\mathcal{A}}{[\kappa-\mathcal{A}]}(N^2+M^2+N), \tag{3.72}$$

$$\Gamma_4=\frac{8\varepsilon^2\kappa^2\mathcal{A}}{[\kappa-\mathcal{A}]^3}(N+M+1), \tag{3.73}$$

$$\Gamma_5=\frac{\kappa^2\mathcal{A}^2}{[\kappa-\mathcal{A}]^2}[(N+1)^2+M^2]. \tag{3.74}$$

Applying the input-output relation (3.49), we obtain

$$\langle a_{out}^\dagger(t+\tau)a_{out}(t)\rangle=\kappa\langle a^\dagger(t+\tau)a(t)\rangle-\sqrt{\kappa}\langle a^\dagger(t+\tau)a_{in}(t)\rangle-\sqrt{\kappa}\langle a_{in}^\dagger(t+\tau)a(t)\rangle+\langle a_{in}^\dagger(t+\tau)a_{in}(t)\rangle. \tag{3.75}$$

We note that

$$\langle a_{in}^\dagger(t+\tau)a(t)\rangle=0, \tag{3.76}$$

$$\langle a_{in}^\dagger(t+\tau)a_{in}(t)\rangle=N\delta(\tau). \tag{3.77}$$

From (2.37), we see that

$$\langle a^\dagger(t+\tau)a(t)\rangle=e^{-(\kappa-\mathcal{A})(t+\frac{1}{2}\tau)}\int_0^t\int_0^{t+\tau}\left[\varepsilon^2+\langle F_\kappa^\dagger(t_1)F_\kappa(t_2)\rangle+\langle F_a^\dagger(t_1)F_a(t_2)\rangle\right]$$
$$\times\exp\left[\frac{1}{2}(\kappa-\mathcal{A})(t_1+t_2)\right]dt_1dt_2. \tag{3.78}$$

It then follows that

$$\langle a^\dagger(t+\tau)a(t)\rangle_{ss}=\frac{4\varepsilon^2}{[\kappa-\mathcal{A}]^2}+\frac{\mathcal{A}+\kappa N}{[\kappa-\mathcal{A}]}e^{-\frac{1}{2}(\kappa-\mathcal{A})\tau}. \tag{3.79}$$

Furthermore, one easily finds

$$\langle a^\dagger(t+\tau)a_{in}(t)\rangle_{ss}=\sqrt{\kappa}Ne^{-\frac{1}{2}(\kappa-\mathcal{A})\tau}, \tag{3.80}$$

so that application of the results (3.76-77) and (3.79-80) in (3.75) yields

$$\langle a_{out}^\dagger(t+\tau)a_{out}(t)\rangle_{ss}=\frac{4\kappa\varepsilon^2}{[\kappa-\mathcal{A}]^2}+\frac{\kappa(\mathcal{A}+\kappa N)}{[\kappa-\mathcal{A}]}e^{-\frac{1}{2}(\kappa-\mathcal{A})\tau}-\kappa Ne^{-\frac{1}{2}(\kappa-\mathcal{A})\tau}+N\delta(\tau). \tag{3.81}$$

In addition, using (3.49) once more, we have

$$\langle a^\dagger_{out}(t+\tau)a_{out}(t+\tau)\rangle = \kappa\langle a^\dagger(t+\tau)a(t+\tau)\rangle - \sqrt{\kappa}\langle a^\dagger(t+\tau)a_{in}(t+\tau)\rangle$$

$$-\sqrt{\kappa}\langle a^\dagger_{in}(t+\tau)a(t+\tau)\rangle + \langle a^\dagger_{in}(t+\tau)a_{in}(t+\tau)\rangle. \tag{3.82}$$

Employing (2.37), one can verify that

$$\langle a^\dagger(t+\tau)a_{in}(t+\tau)\rangle_{ss} = \langle a^\dagger_{in}(t+\tau)a(t+\tau)\rangle_{ss} = \frac{1}{2}\sqrt{\kappa}N, \tag{3.83}$$

$$\langle a^\dagger_{in}(t+\tau)a(t+\tau)\rangle = N. \tag{3.84}$$

Following similar procedures used in arriving at (3.79), we find

$$\langle a^\dagger(t+\tau)a(t+\tau)\rangle_{ss} = \frac{4\varepsilon^2}{[\kappa-\mathcal{A}]^2} + \frac{\mathcal{A}+\kappa N}{[\kappa-\mathcal{A}]}. \tag{3.85}$$

Thus upon substituting these results into (3.82), there follows

$$\langle a^\dagger_{out}(t+\tau)a_{out}(t+\tau)\rangle_{ss} = \frac{4\kappa\varepsilon^2}{[\kappa-\mathcal{A}]^2} + \frac{\kappa(\mathcal{A}+\kappa N)}{[\kappa-\mathcal{A}]} + N(1-\kappa). \tag{3.86}$$

Finally, one can establish the relation

$$\langle a^\dagger_{out}(t+\tau)a_{out}(t+\tau)\rangle_{ss} = \langle a^\dagger_{out}(t)a_{out}(t)\rangle_{ss}, \tag{3.87}$$

and verify employing Eq. (3.86) that

$$\Gamma_1 = \langle a^\dagger_{out}(t+\tau)a_{out}(t+\tau)\rangle_{ss}\langle a^\dagger_{out}(t)a_{out}(t)\rangle_{ss}. \tag{3.88}$$

Hence, with the aid of (3.69), (3.81) and (3.88) the two-time correlation function (3.48) reduces to

$$\Big\langle I_{out}(t+\tau), I_{out}(t)\Big\rangle_{ss} = [\Gamma'_2 + \Gamma'_3 e^{-\frac{1}{2}(\kappa-\mathcal{A})\tau}]\delta(\tau) + \Gamma'_4 e^{-\frac{1}{2}(\kappa-\mathcal{A})\tau} + \Gamma'_5 e^{-(\kappa-\mathcal{A})\tau}, \tag{3.89}$$

where

$$\Gamma'_2 = \frac{8\varepsilon^2\kappa}{[\kappa-\mathcal{A}]^2}(M+N+\frac{1}{2}) + (N^2+M^2+N), \tag{3.90a}$$

$$\Gamma_3' = \frac{\kappa\mathcal{A}}{[\kappa-\mathcal{A}]}[2M^2 + (2N+1)(N+1)], \tag{3.90b}$$

$$\Gamma_4' = \frac{8\varepsilon^2\kappa^2\mathcal{A}}{[\kappa-\mathcal{A}]^3}(N+M+1), \tag{3.90c}$$

$$\Gamma_5' = \frac{\kappa^2\mathcal{A}^2}{[\kappa-\mathcal{A}]^2}[(N+1)^2 + M^2]. \tag{3.90d}$$

Upon substituting (3.89) into (3.44), and carrying out the integrations, the spectrum of intensity fluctuations takes the form

$$S_{out}(\omega) = \Gamma_2' + \Gamma_3' + 2\Gamma_4' \frac{[\frac{\kappa-\mathcal{A}}{2}]}{[\frac{\kappa-\mathcal{A}}{2}]^2 + \omega^2} + 2\Gamma_5' \frac{[\kappa-\mathcal{A}]}{[\kappa-\mathcal{A}]^2+\omega^2}. \tag{3.91}$$

For $\varepsilon = 0$, we see that

$$\Gamma_2' = (N^2 + M^2 + N), \tag{3.92a}$$

$$\Gamma_3' = \frac{\kappa\mathcal{A}}{[\kappa-\mathcal{A}]}[2M^2 + (2N+1)(N+1)], \tag{3.92b}$$

$$\Gamma_4' = 0, \tag{3.92c}$$

$$\Gamma_5' = \frac{\kappa^2\mathcal{A}^2}{[\kappa-\mathcal{A}]^2}[(N+1)^2 + M^2], \tag{3.92d}$$

so that

$$S_{out}(\omega) = \Gamma_2' + \Gamma_3' + 2\Gamma_5' \frac{[\kappa-\mathcal{A}]}{[\kappa-\mathcal{A}]^2+\omega^2}, \tag{3.93}$$

which is in complete agreement with the result obtained by Tesfaye [14].

It is also worth to consider the power spectrum of the output mode which is expressible as

$$S_p^{out}(\omega) = \int_{-\infty}^{\infty} d\tau \langle a_{out}^\dagger(t+\tau) a_{out}(t)\rangle_{ss} e^{-i\omega\tau} \tag{3.94}$$

Application of (3.81) in (3.94) yields

$$S_p^{out}(\omega) = N\int_{-\infty}^{\infty} d\tau e^{-i\omega\tau}\delta(\tau) + \frac{4\varepsilon^2\kappa}{[\kappa-\mathcal{A}]^2}\int_{-\infty}^{\infty} d\tau e^{-i\omega\tau} + \frac{\mathcal{A}\kappa(N+1)}{[\kappa-\mathcal{A}]}\int_{-\infty}^{\infty} d\tau e^{-[i\omega+\frac{1}{2}(\kappa-\mathcal{A})]\tau}, \tag{3.95}$$

from which follows

$$S_p^{out}(\omega) = N + \frac{8\pi\varepsilon^2\kappa}{[\kappa - \mathcal{A}]^2}\delta(\omega) + \frac{\mathcal{A}\kappa(N+1)}{\omega^2 + [\frac{\kappa-\mathcal{A}}{2}]^2}. \tag{3.96}$$

The power spectrum consists of a Lorentzian with half-width $\frac{1}{2}(\kappa - \mathcal{A})$ and a delta function. Both spectra are centered at $\omega = 0$ and are independent of the squeeze parameter. We observe that the delta function is a direct consequence of the interaction of the two-level atoms with the single resonant driving mode corresponding to the elastic Rayleigh scattering. Upon setting $\varepsilon = 0$, we have

$$S_p^{out}(\omega) = N + \frac{\mathcal{A}\kappa(N+1)}{\omega^2 + [\frac{\kappa-\mathcal{A}}{2}]^2}, \tag{3.97}$$

which is consistent with the result obtained in Ref. [14].

4. Photon Statistics

In this chapter we wish to determine the Q function employing the antinormally ordered characteristic function defined in the Heisenberg picture. With the aid of the Q function, we calculate the mean and variance of the photon number as well as the photon number distribution of the output radiation generated by two-level atoms interacting with a single resonant cavity mode, driven by a coherent light and coupled to a squeezed vacuum reservoir.

4.1 The Q function

In this section we wish to calculate the Q function for the output mode which can be used to calculate various quantities of interest. The Q function in terms of the antinormally ordered characteristic function is expressible as [16].

$$Q(\alpha_{out}, \alpha^*_{out}, t) = \frac{1}{\pi^2} \int d^2z \, \Phi_{out}(z, z^*, t) \exp\left[z^* \alpha_{out} - \alpha^*_{out} z\right], \tag{4.1}$$

where

$$\Phi_{out}(z, z^*, t) = Tr\left(\chi(0) e^{-z^* a_{out}(t)} e^{z a^\dagger_{out}(t)}\right), \tag{4.2}$$

where $\chi(0)$ is the combined density operator of the system and reservoir. Applying the Baker-Hausdorff identity

$$e^A \, e^B = e^{A+B} \, e^{\frac{1}{2}[A,B]}, \tag{4.3}$$

Eq. (4.2) can be put in the form

$$\phi_{out}(z, z^*, t) = e^{-\frac{1}{2}zz^*} Tr\left(\chi(0) e^{z a^\dagger_{out}(t) - z^* a_{out}(t)}\right), \tag{4.4}$$

Employing (2.37) and (3.49), one can write

$$a_{out}(t) = f(t) a_{out}(0) + h(t) + g(t), \tag{4.5}$$

where

$$f(t) = \sqrt{\kappa} e^{-\frac{1}{2}(\kappa - \mathcal{A})t}, \tag{4.6a}$$

$$g(t) = \frac{2\varepsilon\sqrt{\kappa}}{(\kappa - \mathcal{A})}\left[1 - e^{-\frac{1}{2}(\kappa - \mathcal{A})t}\right], \tag{4.6b}$$

$$h(t) = f(t)\left[\int_0^t \left[F_\kappa(t') + F_a(t')\right] e^{\frac{1}{2}(\kappa - \mathcal{A})t'} dt'\right] - a_{in}(t). \tag{4.6c}$$

Since the density operators of the system and the reservoir are initially uncorrelated [4], we have

$$\chi(0) = \rho(0)R. \tag{4.7}$$

With the aid of (4.5) and (4.7), the characteristic function (4.4) can be put in the form

$$\Phi_{out}(z, z^*, t) = e^{-\frac{1}{2}zz^*} Tr\left[\rho(0)Re^{\{g(t)(z-z^*)+f(t)(za^\dagger_{out}(0)-z^* a_{out}(0))+(zh^\dagger(t)-z^* h(t))\}}\right]. \tag{4.8}$$

Since $a_{out}(0)$ and the noise operators $F_\kappa(t)$ and $F_a(t)$ are not correlated, one can write

$$\Phi_{out} = e^{-\frac{1}{2}zz^*+g(t)(z-z^*)} Tr\left(\rho(0)e^{f(t)(za^\dagger_{out}(0)-z^* a_{out}(0))}\right)\left\langle e^{zh^\dagger(t)-z^* h(t)}\right\rangle_R. \tag{4.9}$$

Applying (4.3) in (4.9) once more, we obtain

$$\Phi_{out}(z, z^*, t) = \exp\left[-\frac{1}{2}zz^*(1 + f^2(t)) + g(t)(z - z^*)\right]$$

$$\times Tr\left(\rho(0)e^{f(t)za^\dagger_{out}(0)}e^{-f(t)z^* a_{out}(0)}\right)\left\langle e^{zh^\dagger(t)-z^* h(t)}\right\rangle_R. \tag{4.10}$$

Assuming the cavity mode to be initially in the vacuum state, one can easily see that

$$Tr\left(|0\rangle\langle 0|e^{f(t)za^\dagger_{out}(0)}e^{-f(t)z^* a_{out}(0)}\right) = 1, \tag{4.11}$$

so that

$$\Phi_{out}(z, z^*, t) = \exp\left[-\frac{1}{2}zz^*(1 + f^2(t)) + g(t)(z - z^*)\right]\left\langle e^{zh^\dagger(t)-z^* h(t)}\right\rangle_R. \tag{4.12}$$

On account of (4.6c), we note that

$$\langle h(t)\rangle = 0. \tag{4.13}$$

We thus note that h(t) is a random gaussian variable. Hence, we can write [17]

$$\left\langle \exp\left[zh^{\dagger}(t) - z^{*}h(t)\right]\right\rangle_{R} = \exp\langle\frac{1}{2}[zh^{\dagger}(t) - z^{*}h(t)]^{2}\rangle. \tag{4.14}$$

It then follows that

$$\left\langle e^{zh^{\dagger}(t)-z^{*}h(t)}\right\rangle_{R} = \exp\left[\frac{1}{2}z^{2}\langle h^{\dagger 2}(t)\rangle + \frac{1}{2}(z^{*})^{2}\langle h^{2}(t)\rangle - \frac{1}{2}zz^{*}[\langle h^{\dagger}(t)h(t)\rangle + \langle h(t)h^{\dagger}(t)\rangle]\right]. \tag{4.15}$$

We now proceed to calculate the various expectation values involved in (4.15). Employing (4.6c), we see that

$$\langle h^{2}(t)\rangle = \left\langle \left[f(t)\left(\int_{0}^{t}[F_{\kappa}(t') + F_{a}(t')]\, e^{\frac{1}{2}(\kappa-\mathcal{A})t'}dt'\right) - a_{in}(t)\right]\right.$$
$$\left.\left[f(t)\left(\int_{0}^{t}[F_{\kappa}(t'') + F_{a}(t'')]\, e^{\frac{1}{2}(\kappa-\mathcal{A})t''}dt''\right) - a_{in}(t)\right]\right\rangle, \tag{4.16}$$

from which follows

$$\langle h^{2}(t)\rangle = f^{2}(t)\int_{0}^{t}\int_{0}^{t}\langle F_{\kappa}(t')F_{\kappa}(t'')\rangle^{\frac{1}{2}(\kappa-\mathcal{A})(t'+t'')}dt'dt''$$
$$-2f(t)\int_{0}^{t}e^{\frac{1}{2}(\kappa-\mathcal{A})t'}\langle F_{\kappa}(t')a_{in}(t)\rangle dt' + \langle a_{in}(t)a_{in}(t)\rangle. \tag{4.17}$$

On account of (2.57) and (3.24), we get

$$\langle h^{2}(t)\rangle = \frac{\kappa^{2}M}{\kappa-\mathcal{A}}\left[1 - e^{-(\kappa-\mathcal{A})t}\right] + M(1-\kappa). \tag{4.18}$$

It can also be established in a similar manner that

$$\langle h^{2}(t)\rangle = \langle(h^{\dagger 2}(t))\rangle. \tag{4.19}$$

In addition, with the aid of (4.6c), one can write

$$\langle h^\dagger(t)h(t)\rangle = \Big\langle \Big[\Big(f(t)\int_0^t \left[F_\kappa^\dagger(t') + F_a^\dagger(t')\right] e^{\frac{1}{2}(\kappa-\mathcal{A})t'}dt'\Big) - a_{in}^\dagger(t)\Big]$$

$$\Big[\Big(f(t)\int_0^t [F_\kappa(t'') + F_a(t'')]\, e^{\frac{1}{2}(\kappa-\mathcal{A})t''}dt''\Big) - a_{in}(t)\Big]\Big\rangle,$$

$$= f^2(t)\int_0^t\int_0^t \left[\langle F_\kappa^\dagger(t')F_\kappa(t'')\rangle + \langle F_a^\dagger(t')F_a(t'')\rangle\right] e^{\frac{1}{2}(\kappa-\mathcal{A})(t'+t'')}dt'dt''$$

$$-2f(t)\int_0^t e^{\frac{1}{2}(\kappa-\mathcal{A})t'}\langle F_\kappa^\dagger(t')a_{in}(t)\rangle dt' + \langle a_{in}^\dagger(t)a_{in}(t)\rangle, \tag{4.20a}$$

from which follows

$$\langle h^\dagger(t)h(t)\rangle = \frac{\kappa^2 N + \kappa\mathcal{A}}{\kappa-\mathcal{A}}\left[1 - e^{-(\kappa-\mathcal{A})t}\right] + N(1-\kappa). \tag{4.20b}$$

Similarly, one obtains

$$\langle h(t)h^\dagger(t)\rangle = \frac{\kappa^2(N+1)}{\kappa-\mathcal{A}}\left[1 - e^{-(\kappa-\mathcal{A})t}\right] + (N+1)(1-\kappa). \tag{4.21}$$

Upon substituting (4.18-19), (4.20b) and (4.21) into (4.15), one gets

$$\left\langle e^{zh^\dagger(t)-z^*h(t)}\right\rangle_R = \exp\left\{\frac{1}{2}(z^2+z^{*2})\left[\frac{\kappa^2 M}{\kappa-\mathcal{A}}[1-e^{-(\kappa-\mathcal{A})t}] + M(1-\kappa)\right]\right.$$

$$\left.-\frac{1}{2}zz^*\left[\frac{\kappa^2(2N+1)+\kappa\mathcal{A}}{\kappa-\mathcal{A}}[1-e^{-(\kappa-\mathcal{A})t}] + (2N+1)(1-\kappa)\right]\right\}. \tag{4.22}$$

On account of this result, the characteristic function (4.12) takes the form

$$\Phi_{out}(z,z^*,t) = \exp\left[-D(t)zz^* + g(t)(z-z^*) + \frac{1}{2}C(t)(z^2+(z^*)^2)\right], \tag{4.23a}$$

where

$$D(t) = (N+1)(1-\kappa) + \frac{\kappa^2(N+1)}{\kappa-\mathcal{A}} - \frac{\kappa^2 N + \kappa\mathcal{A}}{\kappa-\mathcal{A}}e^{-(\kappa-\mathcal{A})t}, \tag{4.23b}$$

$$C(t) = M(1-\kappa) + \frac{\kappa^2 M}{\kappa-\mathcal{A}}[1-e^{-(\kappa-\mathcal{A})t}]. \tag{4.23c}$$

Applying (4.1), the Q function for the output mode can be expressed as

$$Q(\alpha_{out},\alpha_{out}^*,t) = \frac{1}{\pi^2}\int d^2z\, \exp[-D(t)zz^* + (\alpha_{out}-g(t))z^* - (\alpha_{out}^*-g(t))z$$

$$+\frac{1}{2}C(t)(z^2+z^{*2})\Big] . \tag{4.24}$$

Thus on performing the integration employing the relation

$$\int \frac{d^2z}{\pi} \exp\left[-azz^* + bz + cz^* + Az^2 + Bz^{*2})\right]$$

$$= \left[\frac{1}{a^2-4AB}\right]^{\frac{1}{2}} \exp\left[\frac{abc + Ac^2 + Bb^2}{a^2-4AB}\right],$$

one readily obtains

$$Q(\alpha_{out}, \alpha^*_{out}, t) = \frac{\exp\left[\frac{-g^2(t)}{D(t)+C(t)}\right]}{\pi\sqrt{D^2(t)-C^2(t)}} \exp\left[-F(t)\alpha_{out}\alpha^*_{out} + G(t)(\alpha_{out}+\alpha^*_{out})\right.$$

$$\left. +\frac{H(t)}{2}(\alpha^2_{out} + \alpha^{*2}_{out})\right], \tag{4.25}$$

in which

$$F(t) = \frac{D(t)}{D^2(t)-C^2(t)}, \tag{4.26a}$$

$$G(t) = \frac{g(t)}{D(t)+C(t)}, \tag{4.26b}$$

$$H(t) = \frac{C(t)}{D^2(t)-C^2(t)}. \tag{4.26c}$$

4.2 Mean and Variance of the photon number

In this section we proceed to calculate the mean and variance of the output photon number employing the Q function (4.25).

4.2.1 The mean photon number

The mean photon number of the output mode is expressible as

$$\langle\hat{n}(t)\rangle_{out} = Tr\{\rho(t)a^\dagger_{out}a_{out}\}. \tag{4.27a}$$

Applying the commutation relation

$$[a_{out}, a^\dagger_{out}] = 1, \tag{4.27b}$$

along with the fact that

$$Tr\{\rho(t)\} = 1,$$

Eq. (4.27a) can be put in the form

$$\langle\hat{n}(t)\rangle_{out} = Tr\{\rho(t)a_{out}a_{out}^{\dagger}\} - 1. \tag{4.28}$$

Introducing the identity operator

$$I = \int \frac{d^2\alpha_{out}}{\pi}|\alpha_{out}\rangle\langle\alpha_{out}|,$$

into (4.28), there follows

$$\langle\hat{n}(t)\rangle_{out} = \int d^2\alpha_{out}\frac{1}{\pi}\langle\alpha_{out}|\rho(t)|\alpha_{out}\rangle\alpha_{out}\alpha_{out}^* - 1, \tag{4.29}$$

from which we write

$$\langle\hat{n}(t)\rangle_{out} = \int Q(\alpha_{out}, \alpha_{out}^*, t)\alpha_{out}^*\alpha_{out}d^2\alpha_{out} - 1. \tag{4.30}$$

Applying the Q function (4.25), we have

$$\langle\hat{n}(t)\rangle_{out} = \frac{\exp\left[\frac{-g^2(t)}{D(t)+C(t)}\right]}{\sqrt{D^2(t) - C^2(t)}}\int\frac{d^2\alpha_{out}}{\pi}\alpha_{out}\alpha_{out}^*\exp\Big[-F(t)\alpha_{out}\alpha_{out}^* + G(t)(\alpha_{out} + \alpha_{out}^*)$$
$$+\frac{H(t)}{2}(\alpha_{out}^2 + \alpha^{*2}_{out})\Big] - 1. \tag{4.31}$$

This can be rewritten as

$$\langle\hat{n}(t)\rangle_{out} = -\frac{\exp\left[\frac{-g^2(t)}{D(t)+C(t)}\right]}{\sqrt{D^2(t) - C^2(t)}}\frac{\partial}{\partial F}\int\frac{d^2\alpha_{out}}{\pi}\exp\Big[-F(t)\alpha_{out}\alpha_{out}^* + G(t)(\alpha_{out} + \alpha_{out}^*)$$
$$+\frac{H(t)}{2}(\alpha_{out}^2 + \alpha^{*2}_{out})\Big] - 1, \tag{4.32}$$

so that on carrying out the integration, one finds

$$\langle\hat{n}(t)\rangle_{out} = -\frac{\exp\left[\frac{-g^2(t)}{D(t)+C(t)}\right]}{\sqrt{D^2(t) - C^2(t)}}\frac{\partial}{\partial F}\left[\frac{\exp\left[\frac{G^2}{F+H}\right]}{\sqrt{F^2 - H^2}}\right] - 1. \tag{4.33}$$

Hence on performing the differentiation along with (4.26a-c), we get

$$\langle \hat{n}(t) \rangle_{out} = D(t) + g^2(t) - 1. \tag{4.34}$$

Upon substituting the explicit expressions for D(t) and $g^2(t)$, we find

$$\langle \hat{n}(t) \rangle_{out} = \frac{4\kappa\varepsilon^2}{[\kappa - \mathcal{A}]^2}\left[1 - e^{-\frac{1}{2}(\kappa-\mathcal{A})t}\right]^2 + \frac{\kappa^2 N}{[\kappa - \mathcal{A}]}\left[1 - e^{-[\kappa-\mathcal{A}]t}\right]$$
$$+ \frac{\kappa\mathcal{A}}{[\kappa - \mathcal{A}]}\left[1 - e^{-[\kappa-\mathcal{A}]t}\right] + N(1-\kappa). \tag{4.35a}$$

At steady state, the mean photon number has the form

$$\langle \hat{n} \rangle_{out} = \kappa\left[\frac{4\varepsilon^2}{[\kappa - \mathcal{A}]^2} + \frac{\kappa N}{[\kappa - \mathcal{A}]} + \frac{\mathcal{A}}{[\kappa - \mathcal{A}]}\right] + N(1-\kappa). \tag{4.35b}$$

We note that the term $N(1-\kappa)$ represents the mean photon number of the reflected part of the squeezed vacuum while the remaining term represents the transmitted radiation through the output mirror which is κ times the intracavity mean photon number. In addition, if one sets $N = \varepsilon = 0$, then the remaining term in (4.35b) represents the output mean photon number of a laser operating below threshold.

4.2.2 The variance of the photon number

The variance of the output photon number is expressible as

$$\Delta n^2_{out}(t) = \langle a^\dagger_{out}(t)a_{out}(t)a^\dagger_{out}(t)a_{out}(t)\rangle - \langle a^\dagger_{out}(t)a_{out}(t)\rangle^2. \tag{4.36}$$

Applying the commutation relation (4.27b), one cau write (4.36) as

$$\Delta n^2_{out}(t) = \langle a_{out}{}^2(t)a^{\dagger}_{out}{}^2(t)\rangle - \langle a^\dagger_{out}(t)a_{out}(t)\rangle^2 - 3\langle a^\dagger_{out}(t)a_{out}(t)\rangle - 2. \tag{4.37}$$

With the aid of the Q function (4.25), the first term in (4.37) can be expressed as

$$\langle a_{out}{}^2(t)a^{\dagger}_{out}{}^2(t)\rangle = \frac{\exp\left[\frac{-g^2(t)}{D(t)+C(t)}\right]}{\sqrt{D^2(t) - C^2(t)}}\int \frac{d^2\alpha_{out}}{\pi}\alpha^2_{out}\alpha^{*2}_{out}\exp\left[-F(t)\alpha_{out}\alpha^*_{out} + G(t)(\alpha_{out} + \alpha^*_{out})\right.$$

$$+\frac{H(t)}{2}(\alpha^2_{out}+\alpha^{*2}_{out})\Big] ,$$

$$= \frac{\exp\left[\frac{-g^2(t)}{D(t)+C(t)}\right]}{\sqrt{D^2(t)-C^2(t)}} \frac{\partial^2}{\partial F^2} \int \frac{d^2\alpha_{out}}{\pi} \exp\Big[-F(t)\alpha_{out}\alpha^*_{out} + G(t)(\alpha_{out}+\alpha^*_{out}) + \frac{H(t)}{2}(\alpha^2_{out}+\alpha^{*2}_{out})\Big] . \tag{4.38}$$

Upon carrying out the integration, we find

$$\langle a^2_{out}(t) a^{\dagger\,2}_{out}(t)\rangle = \frac{\exp\left[\frac{-g^2(t)}{D(t)+C(t)}\right]}{\sqrt{D^2(t)-C^2(t)}} \frac{\partial^2}{\partial F^2}\left[\frac{\exp\left[\frac{G^2}{F+H}\right]}{\sqrt{F^2-H^2}}\right] , \tag{4.39}$$

so that on performing the differentiations and taking into account (4.33), one obtains

$$\langle a^2_{out}(t) a^{\dagger\,2}_{out}(t)\rangle = [\langle \hat{n}(t)\rangle_{out}+1]\left[\frac{F}{F^2-H^2}+\frac{G^2}{(F-H)^2}\right]$$

$$+\frac{\exp\left[\frac{-g^2(t)}{D(t)+C(t)}\right]}{\sqrt{D^2(t)-C^2(t)}} \times \frac{\exp\left[\frac{G^2}{F+H}\right]}{\sqrt{F^2-H^2}}\left[\frac{2F^2}{(F^2-H^2)^2}-\frac{1}{F^2-H^2}+\frac{2G^2}{(F-H)^2}\right] . \tag{4.40}$$

In view of (4.26a-c), Eq. (4.40) reduces to

$$\langle a^2_{out}(t) a^{\dagger\,2}_{out}(t)\rangle = (\langle \hat{n}(t)\rangle_{out}+1)\left[D(t)+g^2(t)\right] + \left[D^2(t)+C^2(t)+2g^2(t)(D(t)+C(t))\right] . \tag{4.41}$$

In addition, applying the result described by (4.34) in (4.41), we find

$$\langle a^2_{out}(t) a^{\dagger\,2}_{out}(t)\rangle = 2\left[\langle \hat{n}(t)\rangle_{out}+1\right]^2 + 4g^2(t)\left[\langle \hat{n}(t)\rangle_{out}+1\right] + g^4(t) + 2g^2(t)C(t) + C^2(t), \tag{4.42}$$

so that the variance of the output photon number (4.37) turns out to be

$$\Delta n^2_{out}(t) = \langle \hat{n}(t)\rangle_{out}\left[\langle \hat{n}(t)\rangle_{out}+1\right] + 4g^2(t)\left[\langle \hat{n}(t)\rangle_{out}+1\right] + g^4(t) + 2g^2(t)C(t) + C^2(t). \tag{4.43}$$

We note that

$$\Delta n^2_{out}(t) > \langle \hat{n}(t)\rangle_{out}, \tag{4.44}$$

which shows that the photon statistics of the output mode is super-Poissonian.

For $\varepsilon = 0$, we have

$$g = 0, \tag{4.45}$$

so that at steady state

$$\Delta n_{out}^2 = \langle \hat{n} \rangle_{out} [\langle \hat{n} \rangle_{out} + 1] + C^2, \tag{4.46}$$

in which

$$\langle \hat{n} \rangle_{out} = \frac{\kappa^2 N}{[\kappa - \mathcal{A}]} + \frac{\kappa \mathcal{A}}{[\kappa - \mathcal{A}]} + N(1 - \kappa). \tag{4.47}$$

If we further set $N = M = 0$ in (4.46), we then obtain

$$C = 0, \tag{4.48}$$

and hence

$$\Delta n_{out}^2 = \langle \hat{n} \rangle_{out} [\langle \hat{n} \rangle_{out} + 1], \tag{4.49}$$

where

$$\langle \hat{n} \rangle_{out} = \kappa \left[\frac{\mathcal{A}}{\kappa - \mathcal{A}} \right]. \tag{4.50}$$

We observe that the output radiation generated by two-level atoms interacting with a cavity mode and coupled to an ordinary vacuum is in a chaotic state.

4.3 The photon number distribution

The photon number distribution for the output mode is expressible as [16]

$$P_{out}(n,t) = \frac{\pi}{n!} \frac{\partial^{2n}}{\partial \alpha_{out}^n \partial \alpha_{out}^{*}{}^n} \left[Q(\alpha_{out}, \alpha_{out}^*, t) e^{\alpha_{out} \alpha_{out}^*} \right]_{\alpha_{out} = \alpha_{out}^* = 0}$$

Then employing the Q function (4.25), we have

$$P_{out}(n,t) = \frac{\exp\left[\frac{-g^2(t)}{D(t)+C(t)}\right]}{n! \sqrt{D^2(t) - C^2(t)}} \frac{\partial^{2n}}{\partial \alpha_{out}^n \partial \alpha_{out}^{*}{}^n} \exp\left[(1 - F(t)) \alpha_{out} \alpha_{out}^*\right.$$

$$+G(t)(\alpha_{out} + \alpha^*_{out}) + \frac{H(t)}{2}(\alpha^2_{out} + \alpha^{*\,2}_{out})\Big]_{\alpha_{out}=\alpha^*_{out}=0}, \tag{4.51}$$

and expanding the exponentials in power series, one obtains

$$P_{out}(n,t) = \frac{\exp\left[\frac{-g^2(t)}{D(t)+C(t)}\right]}{n!\sqrt{D^2(t) - C^2(t)}} \frac{\partial^{2n}}{\partial\alpha^n_{out}\partial\alpha^{*\,n}_{out}}$$

$$\times \sum_{i,j,k,l,m} \left[\frac{[1-F(t)]^i[G(t)]^{j+k}[\frac{H(t)}{2}]^{l+m}}{i!j!k!l!m!}\alpha^{i+j+2l}_{out}\alpha^{*\,i+k+2m}_{out}\right]_{\alpha_{out}=\alpha^*_{out}=0}. \tag{4.52}$$

Applying the relations

$$\frac{\partial^n}{\partial\alpha^n_{out}}\alpha^p_{out} = \frac{p!}{(p-n)!}\alpha^{p-n}_{out} \tag{4.53a}$$

and

$$\left[\alpha^{p-n}_{out}\right]_{\alpha_{out}=0} = \delta_{p,n}. \tag{4.53b}$$

One gets

$$P_{out}(n,t) = n!\frac{\exp\left[\frac{-g^2(t)}{D(t)+C(t)}\right]}{\sqrt{D^2(t) - C^2(t)}} \sum_{i,j,k,l,m} \frac{[1-F(t)]^i[G(t)]^{j+k}[\frac{H(t)}{2}]^{l+m}}{i!j!k!l!m!}\delta_{i+j+2l,n}\delta_{i+k+2m,n}. \tag{4.54}$$

From the Kronecker delta symbols in (4.54), we find

$$l = \frac{n-(i+j)}{2}, \tag{4.55}$$

$$m = \frac{n-(i+k)}{2}. \tag{4.56}$$

In view of these results, we obtain

$$P_{out}(n,t) = n!\frac{\exp\left[\frac{-g^2(t)}{D(t)+C(t)}\right]}{\sqrt{D^2(t) - C^2(t)}} \sum_{k=0}^{k=n-i}\sum_{i=0}^{\infty}\sum_{j=0}^{\infty} \frac{[1-F(t)]^i[G(t)]^{j+k}[\frac{H(t)}{2}]^{n-i-\frac{1}{2}(j+k)}}{i!j!k![\frac{n-(i+j)}{2}]![\frac{n-(i+k)}{2}]!}. \tag{4.57}$$

This represents the photon number distribution of the output mode generated by two-level atoms in an optical cavity driven by a coherent light and coupled to a squeezed vacuum reservoir. We note that the steady state photon number distribution of the output mode can be obtained from (4.57) employiug

$$g^2(\infty) = \frac{4\kappa\varepsilon^2}{[\kappa - \mathcal{A}]^2}, \tag{4.58a}$$

$$D(\infty) = (N+1)(1-\kappa) + \frac{\kappa^2(N+1)}{[\kappa - \mathcal{A}]}, \tag{4.58b}$$

$$C(\infty) = M(1-\kappa) + \frac{\kappa^2 M}{\kappa - \mathcal{A}}. \tag{4.58c}$$

Moreover, it is possible to obtain the intracavity photon number distribution from that of the output photon number distribution by neglecting terms associated with $(1-\kappa)$ and taking $\frac{1}{\kappa}$ of the remaining terms in expressions (4.58a), (4.58b) and (4.58c). Consequently, the intracavity photon number distribution at steady state turns out to be

$$P(n) = n! \frac{\exp\left[\frac{-g'^2}{D'+C'}\right]}{\sqrt{D'^2 - C'^2}} \sum_{k=0}^{k=n-i} \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} \frac{[1-F']^i [G']^{j+k} [\frac{H'}{2}]^{n-i-\frac{1}{2}(j+k)}}{i!j!k![\frac{n-(i+j)}{2}]![\frac{n-(i+k)}{2}]!} \tag{4.59}$$

where

$$g'^2(\infty) = \frac{4\varepsilon^2}{[\kappa - \mathcal{A}]^2}, \tag{4.60a}$$

$$D'(\infty) = \frac{\kappa(N+1)}{[\kappa - \mathcal{A}]}, \tag{4.60b}$$

$$C'(\infty) = \frac{\kappa M}{[\kappa - \mathcal{A}]}, \tag{4.60c}$$

and F', G', H' are similarly defined as in $(4.26a-c)$ employing the above expressions .

Next we would like to consider some special cases of interest for the results described by (4.57) and (4.59).

A. For $\varepsilon = 0$, we have

$$G' = 0, \tag{4.61}$$

and hence

$$G'^{j+k} = \delta_{j+k,0}, \tag{4.62}$$

from which follows

$$j = k = 0. \tag{4.63}$$

In view of this result, expression (4.59) reduces to [14]

$$P(n) = \frac{n!}{\sqrt{D'^2 - C'^2}} \sum_{i=0}^{n} \frac{[1-F']^i[\frac{H'}{2}]^{n-i}}{i![\frac{[n-i]}{2}!]^2}. \tag{4.64}$$

If one sets $N = M = 0$ in (4.64), there follows [5]

$$P(n) = \left(1 - \frac{\mathcal{A}}{\kappa}\right)\left(\frac{\mathcal{A}}{\kappa}\right)^n. \tag{4.65}$$

which is the steady state photon number distribution for a laser operating below threshold and coupled to ordinary vacuum reservoir.

B. For $\mathcal{A} = M = 0$, we see from $(4.58a - c)$ that

$$g^2(\infty) = \frac{4\varepsilon^2}{\kappa}, \tag{4.66a}$$

$$D(\infty) = N + 1, \tag{4.66b}$$

$$C(\infty) = 0. \tag{4.66c}$$

Employing (4.66c), we have

$$H = 0,$$

from which follows

$$\left[\frac{H(t)}{2}\right]^{n-i-\frac{1}{2}(j+k)} = \delta_{2n-2i,j+k}. \tag{4.67}$$

Moreover, from (4.57) we see that

$$\frac{n-(i+j)}{2} \geq 0,$$

$$\frac{n-(i+k)}{2} \geq 0.$$

On account of these expressions and (4.67), we find

$$j = k = n - i. \tag{4.68}$$

Consequently, the output photon number distribution for a coherently driven cavity mode coupled to a thermal reservoir at steady state takes the form

$$P_{out}(n) = \frac{\exp\left[-\frac{4\varepsilon^2}{\kappa(N+1)}\right]}{[N+1]} \sum_{i=0}^{n} \left[\frac{N}{N+1}\right]^i \left[\frac{4\varepsilon^2}{\kappa(N+1)^2}\right]^{n-i} \frac{n!}{i![(n-i)!]^2}. \tag{4.69}$$

Now if we set $n - i = p$, we then obtain

$$P_{out}(n) = \frac{N^n}{(N+1)^{n+1}} \exp\left[-\frac{4\varepsilon^2}{\kappa(N+1)}\right] \sum_{p=0}^{n} (-1)^p \left[-\frac{4\varepsilon^2}{\kappa N(N+1)}\right]^p$$
$$\times \frac{n!}{[p!]^2(n-p)!}. \tag{4.70}$$

In terms of the Laguerre polynomial defined by

$$\mathcal{L}_n(x) = \sum_{p=0}^{n} \frac{(-1)^p x^p}{[p!]^2(n-p)!} n!,$$

expression (4.70) can be put in the form

$$P_{out}(n) = \frac{N^n}{(N+1)^{n+1}} \exp\left[-\frac{4\varepsilon^2}{\kappa(N+1)}\right] \mathcal{L}_n\left[-\frac{4\varepsilon^2}{\kappa N(N+1)}\right]. \tag{4.71}$$

This represents the output photon number distribution for a coherently driven cavity mode coupled to a thermal reservoir.

Moreover, employing

$$g'^2(\infty) = \frac{4\varepsilon^2}{\kappa^2}, \tag{4.72a}$$

$$D'(\infty) = N + 1, \tag{4.72b}$$

$$C'(\infty) = 0, \tag{4.72c}$$

in (4.59) and following a similar line of reasoning as the one leading to (4.71), the intra-cavity photon number distribution for the case described above takes the form

$$P(n) = \frac{N^n}{(N+1)^{n+1}} \exp\left[-\frac{4\varepsilon^2}{\kappa^2(N+1)}\right] \mathcal{L}_n\left[-\frac{4\varepsilon^2}{\kappa^2 N(N+1)}\right]. \tag{4.73}$$

This represents the photon number distribution for the superposition of coherent and thermal light [18, 19].

C. For N=M=0, we see that expressions ($4.58a-c$) reduce to

$$g^2(\infty) = \frac{4\kappa\varepsilon^2}{[\kappa - \mathcal{A}]^2}, \tag{4.74a}$$

$$D(\infty) = 1 + \frac{\kappa\mathcal{A}}{[\kappa - \mathcal{A}]}, \tag{4.74b}$$

$$C(\infty) = 0. \tag{4.74c}$$

On account of these results expression (4.57) takes at steady state the form

$$P_{out}(n) = \left[\frac{\kappa - \mathcal{A}}{\kappa - \mathcal{A} + \kappa\mathcal{A}}\right] \exp\left[-\frac{4\varepsilon^2\kappa}{[\kappa - \mathcal{A}][\kappa - \mathcal{A} + \kappa\mathcal{A}]}\right]$$

$$\times \sum_{k=0}^{k=n-i} \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} \left[\frac{\kappa\mathcal{A}}{\kappa - \mathcal{A} + \kappa\mathcal{A}}\right]^i \left[\frac{2\varepsilon\sqrt{\kappa}}{\kappa - \mathcal{A}}\right]^{j+k} \frac{n!\,\delta_{2n-2i,j+k}}{i!j!k![\frac{n-(i+j)}{2}]![\frac{n-(i+k)}{2}]!}. \tag{4.75}$$

Following a similar procedure used to arrive at (4.71), we find

$$P_{out}(n) = \left(\frac{\kappa - \mathcal{A}}{\kappa\mathcal{A}}\right) \left[\frac{\kappa\mathcal{A}}{\kappa - \mathcal{A} + \kappa\mathcal{A}}\right]^{n+1} \exp\left[-\frac{4\varepsilon^2\kappa}{(\kappa - \mathcal{A})\,[\kappa - \mathcal{A} + \kappa\mathcal{A}]}\right]$$

$$\times \mathcal{L}_n\left[-\frac{\kappa}{\mathcal{A}\,[\kappa - \mathcal{A} + \kappa\mathcal{A}]}\right]. \tag{4.76}$$

D. Finally, we want to obtain the intracavity photon number distribution for the case N=M=0. We then note that

$$g'^2(\infty) = \frac{4\varepsilon^2}{[\kappa - \mathcal{A}]^2}, \tag{4.77a}$$

$$D'(\infty) = \frac{\kappa}{[\kappa - \mathcal{A}]} \tag{4.77b}$$

and

$$C'(\infty) = 0. \tag{4.77c}$$

On account of these results, expression (4.59) takes the form

$$P(n) = \left(\frac{\kappa - \mathcal{A}}{\kappa\mathcal{A}}\right) \exp\left[\frac{-4\varepsilon^2}{\kappa[\kappa - \mathcal{A}]}\right] \sum_{k=0}^{k=n-i} \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} \left[\frac{\mathcal{A}}{\kappa}\right]^i \left[\frac{2\varepsilon}{\kappa}\right]^{j+k}$$
$$\times \frac{\delta_{2n-2i,j+k}}{i!j!k![\frac{n-(i+j)}{2}]![\frac{n-(i+k)}{2}]!}, \tag{4.78}$$

so that application of the result (4.68) in (4.78) yields

$$P(n) = \left(\frac{\kappa - \mathcal{A}}{\kappa\mathcal{A}}\right) \left[\frac{\mathcal{A}}{\kappa}\right]^n \exp\left[\frac{-4\varepsilon^2}{\kappa[\kappa - \mathcal{A}]}\right] \mathcal{L}_n\left[-\frac{4\varepsilon^2}{\kappa\mathcal{A}}\right]. \tag{4.79}$$

This is the intracavity photon number distribution of the light mode produced by two-level atoms interacting with a cavity mode which is driven by a coherent light of amplitude ε.

5. Conclusion

With the aid of the solutions of the linearized Langevin equations, we have calculated the quadrature fluctuations of the cavity mode, the squeezing spectrum, spectrum of intensity fluctuations and power spectrum of the output radiation generated by two-level atoms interacting with a single resonant cavity mode, driven by a coherent light and coupled to a squeezed vacuum reservoir.

We have found that the cavity mode will be in a squeezed state for values of the squeeze parameter given by (3.20) and when it is in phase with the squeezed vacuum. We have seen that the squeezed vacuum increases the peak of one of the squeezing spectra and decreases that of the other. We have also seen that the squeezed vacuum increases the peak of the spectrum of intensity fluctuations. In addition, it has been found that the squeezed vacuum does not have any effect on the linewidth of the spectra of squeezing and intensity fluctuations.

Although the driving coherent light has no effect on the squeezing properties of the cavity mode, it increases the peak of the spectrum of intensity fluctuations.

Moreover, applying the solutions of the quantum Langevin equations, we have determined the antinormally ordered characteristic function defined in the Heisenberg picture. Employing the resulting characteristic function, the Q function for the output mode has been calculated. With the aid of this Q function, we have calculated the mean and variance of the photon number as well as the photon number distribution of the output mode. We have seen that the photon statistics of the output mode to be super-Poissonian. In addition, we have found that the driving coherent light and the squeezed vacuum reservoir enhance the mean and variance of the output photon number.

Finally, the intracavity photon number distribution has been obtained from the photon number distribution of the output mode, and thereby the photon statistics of the cavity mode has been studied for some special cases of interest.

We strongly believe that the method introduced in this work to determine the antinormally ordered characteristic function provides a significant mathematical simplification.

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