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THE MATHEMATICAL BASIS OF GAUGE THEORIES OF THE YANG-MILLS TYPE AND DIRAC MONOPOLES

by

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ABSTRACT

After a geometric and intuitive introduction to non-trivial fiber bundles, a precise presentation of principal fiber bundles with Lie group structure is given. The methods of differential geometry are set up. It is shown how a gauge potential can be regarded as a connection in some fiber bundle, and the corresponding gauge field as the associated curvature. As an example of a physical case which leads to the consideration of non-trivial bundles, the magnetic monopole is introduced in its Dirac form together with some notions of duality symmetry (self dual fields).

Monopoles in the recent Wu-Yang formulation, and electrodynamics with monopoles identified with non-trivial U(1) bundles are studied. The Yang-Mills action functional is defined.

The Yang-Mills field equation are derived in terms of the field (curvature). The dual symmetric counterparts of the field equations are established as a consequence of the Bianchi identity of differential geometry. It is shown how the global aspects of the theory (finiteness of the action and the boundary conditions on the gauge potentials) are encoded in the structure of a bundle. The asymptotically vanishing gauge potentials (or pure gauges) are classified using the notions of instanton number and homotopy.

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INTRODUCTION

The most satisfactory unifying theme for elementary particle interactions discovered during the last decade or two is that of gauge theory. Weak and electromagnetic interactions have already been unified in a gauge theoretic framework. The quark-gluon picture of strong interactions and the mechanism of quark confinement are also expected to be ultimately understood in the same framework.

Theorists are attempting to understand the Einstein theory of gravity as a gauge theory.

The characteristically geometric nature of the general theory of relativity is enough to make one pay attention to the potential use of geometric methods in formulating physical laws.

Herman Weyl{31} invented the basic idea of "gauge transformation" and extended it to a theory of the electromagnetic field interacting with particle fields.

F. London amended Weyl's work and deduced a more realistic notion of "gauge transformation" in its modern "minimal coupling" form.

The full quantum mechanical implication of a gauge transformation (the transformation of a field under an internal symmetry group with space-time dependent parameters) was not revealed until Schwinger{32} treated the electromagnetic field as arising as a consequence of a local U(1) gauge symmetry for the quantized Dirac field.

Example: For $G = Su(2)$, the invariants are integers K given by

$$K = {}^{1}/_{16\Pi^2}\int_{S^4} Tr\,(F_{\mu\nu}{}^{*}F^{\mu\nu})\,d^4x$$

For other groups, similar relations exists but with a different normalization, the integer K being replaced by a suitable multiple[10].

These ideas were used by Polyakov and coworkers[29], 't Hooft[33], Jackiw & Rebbi[34] to discuss the properties of the gauge theory "vacuums" and to classify these vacua according to characteristic classes of the appropriate bundles.

Assuming that the notions of differentiable manifold are understood, the first six chapters discuss in detail the differential geometry of fiber bundles and the geometric interpretations of non-abelial gauge fields. The Wu-Yang formalism is then discussed and the relevance of fiber bundle language clearly shown in the framework of U(1) magnetic monopoles. In fact the Wu-Yang treatment of the monopole problem[14] is one of the first works that pointed out the use of the differential geometry of fiber bundles to the bulk of the physics community. The last chapter discusses the intimate relationship between the asymptotic condition on the gauge potentials and the non-trivial topology of the associated bundles. Three appendices are added at the end to make the thesis self contained. Appendix A is a concise introduction to the problem of the Dirac monopole. Appendices B and C are just rudimentary descriptions of the minimum mathematical prerequisite to understanding the classification of pure gauges.

1. THE GEOMETRY OF FIBER BUNDLES {1} {2} {4}

1.1 INTRODUCTION

The concept of a fiber bundle is a generalization of a direct product. Some examples from physics and geometry will motivate the need for such a generalization.

(i) In the dynamics of antiquity both space and time are absolute. Every physical event is characterized by an instant of time and a location in space; i.e.; the space-time of antiquity is a direct product $E=T\times R^3$, T being the axis of time and R^3 the three dimensional Euclidean space.
In Galilean dynamics, time remains absolute but space becomes relative; in the sense that for a given event at a fixed instant of time, different spacial coordinates, related to each other by Galilean transformation equations can be ascribed to it. This can be described by saying that $\exists$ an onto mapping $\Pi:E\to T$ which associates to any $p\in E$ the corresponding time coordinate $t=\Pi(p)\in T$. The time axis T is called the base space and the set $\Pi^{-1}(t)$ of all events simultaneous with p is called the fiber over t. R^3 is called the typical fiber since each fiber $\Pi^{-1}(t)$ is isomorphic to it. The total space E of this fiber bundle is representable as the direct product $T\times R^3$ and E is said to be trivialisable. Formally, if E is trivialisable then there is a (trivialising) map $H:E\to T\times R^3$ such that $H(p) = \left(\Pi(p), \underline{r}(p)\right)$, where $\underline{r}(p)$ is the position vector of the event p relative to an inertial observer.
N.B.: There is no "natural" isomorphism between E & $T\times R^3$.

(ii). The following example is given in order to illustrate the nature of bundle spaces with structure group, i.e., principal fiber bundles. The exact definition will be presented soon in such a way that its intimacy with gauge field theory is easily discernible.

Take an orthonormal dyad $(\underset{=}{e}_1,\underset{=}{e}_2)$ of vectors tangent at some point to a two dimensional sphere S^2 of preferred orientation. One can regard the set P of all orthonormal dyads with orientiation compatible to that of S^2 as the total space of a bundle with a natural projection $\Pi:P\rightarrow S^2$*. If $(\underset{=}{e}_1,\underset{=}{e}_2)$ is a dyad at $x\varepsilon S^2$, then, so is the pair$(\underset{=}{e}_1{}'\,\underset{=}{e}_2{}')$ related to $(\underset{=}{e}_1,\underset{=}{e}_2)$ by an orthogonal transformation:

$$\begin{pmatrix}\underset{=}{e}_1{}' \\ \underset{=}{e}_2{}'\end{pmatrix} = \begin{pmatrix}\cos\phi & \sin\phi \\ -\sin\phi & \cos\phi\end{pmatrix}\begin{pmatrix}\underset{=}{e}_1 \\ \underset{=}{e}_2\end{pmatrix}$$

Then given a frame $(\underset{=}{e}_1,\underset{=}{e}_2)$ at $x\varepsilon S^2$, all the other frames at the same point can be generated in the same way. Then So(2) is the typical fiber of the bundle P, and the above matrix equation defines the structure group (So(2)) action on P. P is an example of a principal fiber bundle. N.B.: This bundle cannot be trivialized. Indeed if there were a diffeomorphism $H:S^2\times So(2)\rightarrow P$ such that $\Pi o H(x,a)=x$, then one could determine a smooth vector field on S^2 by defining a map $S:S^2\rightarrow P$ by $S(x) \equiv H(x,a_o)$, a_o being a constant element† in So(2). This would contradict the "no combing of S^{2n}".

* i.e. to any set P of dyads, there corresponds the point of tangency on S .

† This is Brouwer's theorem: "a continuous vector field cannot be defined on an even dimensional sphere."

1.2 PRINCIPAL FIBER BUNDLES [1] [3] [6]

In this section, an exact definition of fiber bundles with (Lie) group structure will be given. In gauge theory language, the structure group corresponds to the gauge symmetry group, and the non-trivial global topology of the bundle space is a manifestation of the asymptotic conditions imposed on the Euclidean gauge potentials, in the sense that these boundary conditions at infinity are encoded in the actual construction of the principal fiber bundle over an appropriate compactification of four-dimensional Euclidean space.

Definition (1.2.a)[1] [6]: Let P and M be manifolds and let G be a Lie group. A smooth principal fiber bundle over M with structure group G is a quadruple $\mathbb{P} = (P,M,\Pi,G)$ satisfying the following conditions:

a) There exists a free right action $T: P\times G \to P$ denoted by $(u,a)\in P\times G \to ua \in P$.

b) G induces an equivalence relation with $M = P/G$: $\forall u_1, u_2 \in P$, $u_1 \sim u_2$ if and only if $\exists a\in G$ such that $u_1 = u_2 a$.

c) The canonical projection mapping Π is onto, smooth and satisfies $\Pi(ua) = \Pi(u)$.

d) admits a local trivialization $\{V_\alpha, \psi_\alpha\}$, i.e. $\forall x\in M$, $\exists V_\alpha \subset M$ such that $x\in V_\alpha$ and the diffeomorphism $\psi_\alpha : \Pi^{-1}(V_\alpha) \to V_\alpha \times G$; defined by $\psi_i(u) = \left(\Pi(u), \phi(u)\right)$, makes $\Pi^{-1}(V_\alpha) \subset P$ isomorphic to a direct product.

$\phi: \Pi^{-1}(V_\alpha) \to G$ satisfies $\phi(ua) = \phi(u)a$, $\forall u\in P$, $\forall a\in G$. We write G_x for the fiber $\Pi^{-1}(x)$ at x.

Definition (1.2.b): A global cross-section of a principal fiber bundle is a map $\sigma: M \to P$, such that $\Pi \circ \sigma = Id_M$ (identity on M).

[a local section $V_\alpha \subset M$ is a map $\sigma_\alpha : V_\alpha \to P$ such that $\Pi \circ \sigma_\alpha = Id_{V_\alpha}$]

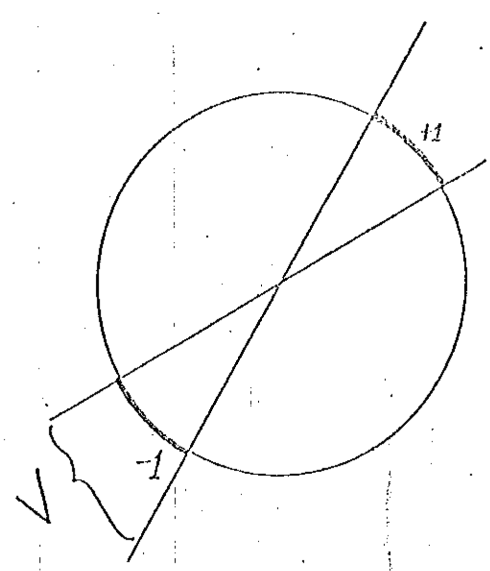
For a principal fiber bundle, the existence of a global section is equivalent to its triviality{1}.

1.3 EXAMPLES OF PRINCIPAL FIBER BUNDLES

Example (1.3.a): $\mathbb{P} = (R^4 \times G, \Pi, R^4, G)$ is just a trivial G bundle over R^4.

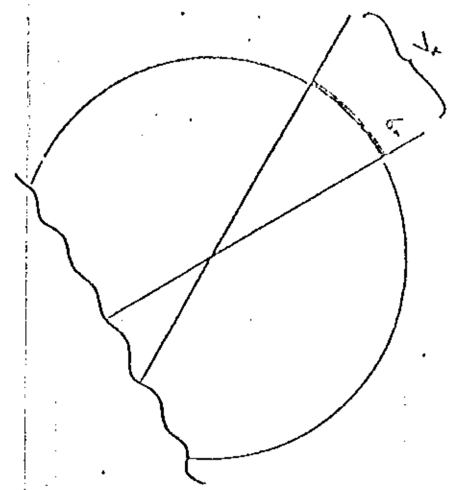
Example (1.3.b){1}: Take the one dimensional manifold S^1 and $Z_2 = \{-1,+1\}$ as a structure group with action on S^1 as follows: +1=Id, −1= antipodal map. The quotient space $M = S^1/G$ is the space of diameters.

$\forall p \in S^1$, $\Pi(p)$ = the diameter through p. Locally, for a fixed $x \in M$ i.e. a diameter, $\exists$ an open cone V containing the diameter with $\Pi^{-1}(V) = V \times Z_2$, i.e.; $(S^1, S^1/Z_2, \Pi, Z_2)$ is locally trival.



(Fig. 1.2.a)

If a finite union $\underset{\alpha}{\cup} V_\alpha$ covers the complete set of directions around the center, the images of the cross-sections $\sigma_\alpha : V_\alpha \to P$ are subsets of S^1 that are "bounded" by two remotely separated directions in V_α.



(Fig. 1.2.b)

N.B.: For a given V_α, both $\sigma_\alpha .1$ and $\sigma_\alpha .(-1)$ are local sections. $\sigma_\alpha .1$ is depicted in fig. (1.2.b).

1.4 TRANSITION FUNCTIONS {1} {3} {6}

The trivialising diffeomorphism $\psi_\alpha : \Pi^{-1}(V_\alpha) \to V_\alpha \times G$, $\psi_\alpha(u) = \left[\Pi(u), \phi_\alpha(u)\right]$, $u \in \Pi^{-1}(V_\alpha)$ defines a natural set of local cross-sections σ_α given by $\sigma_\alpha(x) = u\phi_\alpha(u)^{-1}$, $x \in V_\alpha$ & $\forall u \in \Pi^{-1}(x)$.

Explicitly, let us check that σ_α is indeed a locl cross-section.

Certainly, $\Pi\left[u(\phi_\alpha(u)^{-1}\right] = \Pi(u) = x$ since $\left[\phi_\alpha(u)\right]^{-1} \in G$. Also $\sigma_\alpha(x) \equiv u\left[\phi_\alpha(u)\right]^{-1}$ is well defined as $\forall u, v \in \Pi^{-1}(x)$, $\exists\, a \in G$ such that $v = ua$ and $v.\left[\phi_\alpha(v)\right]^{-1} = (ua)\left[\phi_\alpha(ua)\right]^{-1} = uaa^{-1}\left[\phi_\alpha(u)\right]^{-1} = u\left[\phi_\alpha(u)\right]^{-1}$.

Since $\phi_\alpha\left[\sigma_\alpha(x)\right] = \phi_\alpha\left[u(\phi_\alpha(u))^{-1}\right] = \phi_\alpha(u)\left[\phi_\alpha(u)\right]^{-1} = e$,

we have $\psi_\alpha\left[\sigma_\alpha(x)\right] = \left[\Pi(\sigma_\alpha(x)),\phi_\alpha(\sigma_\alpha(x))\right] = (x,e)$.
Thus under the diffeomorphism $\psi_\alpha:\Pi^{-1}(V_\beta)\to V_\alpha\times G$, the local cross-section σ_α corresponds to $V_\alpha\times\{e\}$.

This is certainly a priviledged set of local sections, and we investigate their transformation properties. If $\{V_\alpha\}\alpha\varepsilon A$ is an open covering of M, let $x\varepsilon V_\beta\cap V_\beta$. Then $\sigma_\beta(x) = u\left[\phi_\beta(u)\right]^{-1} = u\left[\phi_\alpha(u)\right]^{-1}\phi_\alpha(u)\left[\phi_\beta(u)\right]^{-1} =$ $\sigma_\alpha(x)\phi_\alpha(u)\left[\phi_\beta(u)\right]^{-1}$. The group valued functions $\phi_\alpha(u)\left[\phi_\beta(u)\right]^{-1}$ denoted by $\psi_{\alpha\beta}$ are called transition functions for the bundle. The transformation rule $\sigma_\beta(x) = \sigma_\alpha(x)\psi_{\alpha\beta}$ is consistant with the fact that $x\varepsilon M$ labels the equivalence classes in P/G and any point $u\varepsilon\Pi^{-1}(x)$ can be derived from another $u\varepsilon\Pi^{-1}(x)$ by an appropriate group action.

For $x\varepsilon V_\alpha\cap V_\beta\cap V_\gamma$, $\psi_{\alpha\beta}(x) = \phi_\alpha(u)\left[\phi_\beta(u)\right]^{-1}$

$$= \phi_\alpha(u)\left[\phi_\gamma(u)\right]^{-1}\phi_\gamma(u)\left[\phi_\beta(u)\right]^{-1} = \psi_{\alpha\gamma}\ \psi_{\gamma\beta}.$$

i.e. $\psi_{\alpha\beta}(x) = \psi_{\alpha\gamma}\ \psi_{\gamma\beta}$. This is the cocycle condition for transition functions. The transition functions are differentiable. The global topology of a bundle space is determined by the topologies of M & G and the type of transition functions.

In fact we have the following theorem:

Theorem (1.4.a)[1] [3] (Reconstruction theorem): Let M admit an open covering $\{V_\alpha\}\alpha\varepsilon A$ and let $\psi_{\alpha\beta}:V_\alpha\cap V_\beta\to G$ be a function satisfying the cocycle condition. Then a unique principal fiber bundle (P,M,Π,G) whose transition functions relative to $\{V_\alpha\}$ are the $\psi_{\alpha\beta}$'s, is determined.

We say that the quadruple $(M, G, \{V_\alpha\}, \psi_{\alpha\beta})$ constitutes a principal coordinate bundle. It's existence follows from the local triviality requirement of a principal fiber bundle.

However, the group structure G extends beyond just one variety $\psi_{\alpha\beta}$ the possible type of transition functions that can go with the open covering $\{V_\alpha\}$. Indeed another set $\sigma_\alpha'(x)$ of local sections given by $\sigma_\alpha'(x) = \sigma_\alpha(x) g_\alpha(x)$, $g_\alpha(x) \in G$, can be constructed.

By definition $\psi_{\alpha\beta}(x) = \phi_\alpha(u)\left[\phi_\beta(u)\right]^{-1}$, $\forall u \in \Pi^{-1}(x)$ and $\sigma_\alpha(x) = u\left[\phi_\alpha(u)\right]^{-1}$, $\sigma_\alpha'(x) = \sigma_\alpha(x) g_\alpha(x) = u\left[\phi_\alpha(u)\right]^{-1} g_\alpha(x)$. Since $x \in M$ is fixed, one can identify $\left[\phi_\alpha'(u)\right]^{-1} g_\alpha(x)$ with $\left[\phi_\alpha'(u)\right]^{-1}$. Hence $\psi_{\alpha\beta}'(x) = \phi_\alpha'(u)\left[\phi_\beta'(u)\right]^{-1} =$

$$\left[\left(\phi_\alpha(u)\right)^{-1} g_\alpha(x)\right]^{-1} \left(\phi_\beta(u)\right)^{-1} g_\beta(x) = g_\alpha^{-1}(x)\,\phi_\alpha(u)\left(\phi_\beta(u)\right)^{-1} g_\beta(x)$$

$$= g_\alpha^{-1}(x)\,\psi_{\alpha\beta}(x)\,g_\beta(x)$$

i.e., $\psi_{\alpha\beta}'(x) = g_\alpha^{-1}(x)\,\psi_{\alpha\beta}(x)\,g_\beta(x)$ and this is the transformation rule for transition functions.

$(M, G, \{V_\alpha\}, \psi_{\alpha\beta})$ & $(M, G, \{V_\alpha\}, \psi_{\alpha\beta}')$ are said to be equivalent coordinate bundles.

1.5 ASSOCIATED FIBER BUNDLE {1} {6}

In this section the relationship between a principal fiber bundle in which a generalization of the direct product is achieved for groups, and a bundle space "E" in which the typical fiber is a generalised space "F", will be examined. This bundle space is a generalization of the direct product M×F. If the structure group G of the principal fiber bundle has a fixed (left) action on F, F can be regarded as a representation space for G.

Using $\mathbb{P} = (P,M,\Pi,G)$, one can erect at points of M, copies of the space F and duplicate the non-trivial topology of $\mathbb{P}$.

A mapping $Q_a: P\times F \to P\times F$ such that $Q_a(U,\xi) = (ua, a^{-1}\xi)$, $u\varepsilon P$ & $\xi\varepsilon F$, establishes an equivalence relation "$\sim$" defined by $(u_1,\xi_1)\sim(u_2,\xi_2)$ if and only if $(u_2,\xi_2) = (u_1 a, a^{-1}\xi_1)$ for some fixed $a\varepsilon G$.

We define{1} the fiber bundle associated to $\mathbb{P}$ with standard fiber F, as the space $E(M,G,F,p) = p\times F/G$ equipped with the following differentiable structures:

a) Define a smooth onto mapping $\Pi_E: E\to M$ by Π_E (equivalence class of $(u,\xi)) = \Pi(u)$. Then, $\forall x\varepsilon M$, $\exists$ open V_x M, $(x\varepsilon V_x)$, such that $\Pi_E^{-1}(V_x)$ is an open submanifold of E.

b) $\Pi_E^{-1}(V_x)$ is isomorphic to $V_x\times F$.

A local section σ_α of E(M,G,F,P) is a mapping $\sigma_\alpha: V_\alpha \to E$ such that $\Pi_E o\sigma_\alpha = Id_{V_\alpha}$; $\{V_\alpha\}$ being an open covering of M.

Physically, if F is a linear representation space for G, for the description of matter fields (i.e. vector fields, spinor fields etc.), then a field in the trivial case is just a vector-valued (F-valued) function on space-time, or in other words a global section of M×F.

In the non-trivial case the field is described as a local section of some bundle E(M,G,F,P), the non-triviality arising from the actual pasting of copies of the representation space F at each space-time point.

<u>Example (1.5.a)</u>[1]: Suppose we have two strips of paper (1) & (2)

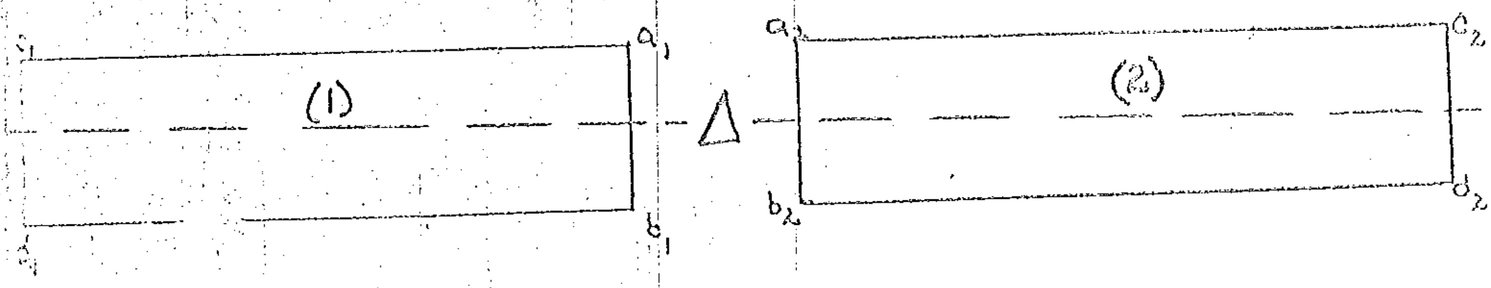


Fig. (1.5.a)

One can glue (1) & (2) together with a twist such that a_1 coincides with a_2, b_1 with b_2, c_1 with d_2 and d_1 with c_2, and get a möbius strip P (Fig. 1.5b). We can define an action of the

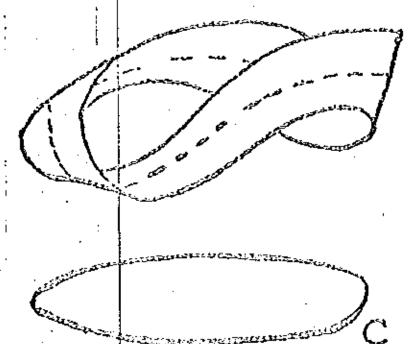


Fig. (1.5.b)

group G = {+1,-1} on the points of P, by referring to the original strips (1) & (2): +1 = Id

-1 = reflection with respect to the axis Δ.

If we identify the points of P that are related by the group action, we get a single, one-dimensional simple closed curve C.

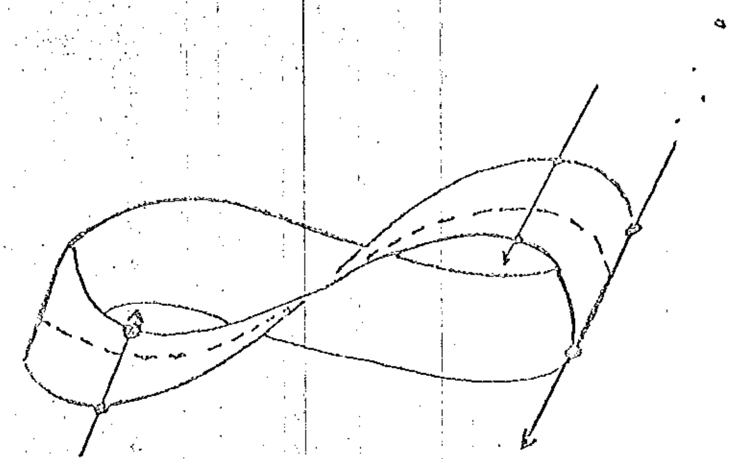


Fig. (1.5.c)

This constitutes a principal fiber bundle $\mathbb{P} = (P, C, \Pi, G)$. The associated bundle $E(C, G, R, P)$ is obtained by glueing a real line to all pairs of group related points (Fig. 1.5c).

2. METHODS OF DIFFERENTIAL GEOMETRY [1][3][4][8]

2.1 TANGENT & COTANGENT SPACES

Assuming the standard notion of a smooth i.e. C^∞ manifold M as a Hausdorff topological space, with local coordinates, (i.e., an open covering $\{V_\alpha\}$ of M and homeomorphisms $\phi_\alpha : V_\alpha \to U_\alpha$ where U_α is an open subset of R^n), we introduce the concept of tangent & cotangent spaces.

Let M be a C^∞ manifold. Let $p \varepsilon M$ and let $C_p^\infty(M)$ be the set of all real valued functions on M that are smooth at p. To any smooth curve through p, we can associate an operator X_p called the tangent vector at p such that $X_p f$ is the directional derivative of f at p, $f \varepsilon C_p^\infty(M)$. More explicitly, if $\lambda : [a,b] \to R^n$ is the curve in M passing through p associated to X_p, we have

$$X_p f = \lim_{\alpha \to 0} \frac{1}{\alpha}\{f\left(\lambda(t+\alpha)\right) - f\left(\lambda(t)\right)\} \qquad (2.1)$$

The tangent vectors X_p at a given point p form an n-dimensional real vector space, denoted by $T_p(M)$, and satisfy the following properties:

a) $X_p(\alpha f+\beta g) = \alpha X_p f+\beta X_p g$

b) $X_p(fg) = f(p)X_p(g)+g(p)X_p f$

$\forall f,g \varepsilon\ C_p^\infty(M)$, $\forall \alpha,\beta \varepsilon R$.

Let $x^1, \ldots, x^n$ be local coordinate basis in a neighborhood U of p. Then $\{\partial/\partial x^i\}$ form a basis of $T_p(M)$, and M and $T_p(M)$ have the same dimension.

A vector field X on a C^∞ manifold M is an assignment of a vector $X_p \varepsilon T_p(M)$ to each $p \varepsilon M$. In a given coordinate neighbourhood,

$$X = \sum_{j=1}^{n} \xi^j \, \partial/\partial x^j \ , \ \xi^j \text{ being real valued functions}$$

defined in that neighbourhood. Then $\forall f \in C_p^\infty(M)$,

$$Xf = \sum_{j=1}^{n} \xi^j \frac{\partial f}{\partial x^j} \ .$$

Given any two vector fields X, Y on M, we define a new vector field, the Lie bracket $[X,Y]$, by

$$[X,Y]f = X.Y(f) - Y.X(f) = (XY-YX)f \tag{2.2}$$

Locally, if $X = \sum_{i=1}^{n} \alpha^i \, \partial/\partial x^i$; $Y = \sum_{i=1}^{n} \beta^i \, \partial/\partial x^i$

then using $\left[\partial/\partial x^i, \ \partial/\partial x^j\right] = 0$, we get

$$[X,Y] = \sum_{i,j}^{n} \left(\alpha^j \frac{\partial \beta^i}{\partial x^j} - \beta^j \frac{\partial \alpha^i}{\partial x^j}\right) \frac{\partial}{\partial x^i} \tag{2.3}$$

we have the rules:

(a) $[X,Y] = -[Y,X]$

(b) $[fX,gY] = fg[X,Y] + f(Xg)Y - g(Yf)X$

(c) $\left[X,[Y,Z]\right] + \left[Y,[Z,X]\right] + \left[Z,[X,Y]\right] = 0$

(Jacobi's identity) (2.4)

The dual space to the tangent space, $T_p(M)$ is called the cotangent space $T_p{}^*(M)$. $T_p{}^*(M)$ is a space of linear functionals w_p on $T_p(M)$.

$\{dx^i\}$ is a basis for $T_p{}^*(M)$ dual to $\{\partial/\partial x^i\}$, where by definition

$$dx^i(\partial/\partial x^i) = \delta^i_j \tag{2.5}$$

A 1-form w on M is an assignment of a covector at each point p of M. Locally, a 1-form can be written as

$$w = \sum_j \xi_j dx^j \tag{2.6}$$

2.2 EXTERIOR ALGEBRA OF FORMS {1} {7} {8}

r-covariant and s-contravariant tensors at $p \varepsilon M$ are (r+s)-linear real valued functions on $\left[T_p(M)\right]^r \times \left[T_p{}^*(M)\right]^s$.

The vector space of these tensors is denoted by $T_r{}^s(M_p)$. Locally, any $\beta \varepsilon T_r{}^s(M_p)$ can be represented as

$$\beta = \sum_{i_1,\ldots i_{r+s}} \alpha_{i_1 \ldots i_r}^{i_{r+1} \ldots i_{r+s}} \left(dx^{i_1} \otimes \ldots \otimes dx^{i_r}\right) \left(\partial/\partial x^{i_{r+1}} \otimes \cdots \otimes \partial/\partial x^{i_{r+s}}\right) \tag{2.7}$$

In particular, any arbitrary $\alpha \varepsilon T_r{}^o(M_p)$ has the form

$$\alpha = \sum_{i_1,\ldots,i_r} \alpha_{i_1 \ldots i_r} \left(dx^{i_1} \otimes \ldots \otimes dx^{i_r}\right) \tag{2.8}$$

The action of the product space $T_r{}^o(M_p)$ is made more explicit as follows:

If $w_1, \ldots, w_r \varepsilon T_p{}^*(M)$, then $\Omega = w_1 \otimes \ldots \otimes w_r$ is a mapping $\Omega : T^r(M_p) \to R$ given by

$$\Omega(V^1, \ldots V^r) = w_1(V^1) \ldots w_r(V^r) \tag{2.9}$$

where $V^i \varepsilon T_p(M)$.

Generally, if $w\epsilon T_a{}^o(M_p)$, $\eta\epsilon T_b{}^o(M_p)$ then $w \otimes \eta\epsilon T_{a+b}{}^o(M_p)$ is defined by

$$(w \otimes \eta)(V^1,\ldots V^{a+b}) = w(V^1,\ldots V^a)\eta(V^{a+1},\ldots V^{a+b}) \tag{2.10}$$

$\forall\ V^1,\ldots V^{a+b}\epsilon T_p(M)$.

From $T_p{}^*(M)$ we can construct a useful subspace $\Lambda^r T_p{}^*(M)$ of $T_r{}^s(M_p)$. This consists of alternating tensors of the form

$$w = \sum_{i_1<\ldots<i_r} \alpha_{i_1\ldots i_r}\left(dx^{i_1} \wedge\ldots\wedge dx^{i_r}\right) \tag{2.11}$$

the coefficients $\alpha_{i_1\ldots i_r}$ being totally anti-symmetric.

The vector space $\Lambda^r T_p{}^*(M)$ is of dimension $\binom{n}{r}$.

If $w\epsilon T_r{}^o(M_p)$, we define

$$\mathrm{Alt}\,w = \frac{1}{r!}\sum_{\sigma\epsilon S_r}\epsilon_\sigma w_\sigma \tag{2.12}$$

where S_r is the permutation group of r objects, ϵ_σ the parity of σ and

$$w_\sigma(V^1,\ldots V^r) = w(V^{\sigma(1)},\ldots V^{\sigma(r)}) \tag{2.13}$$

Any form w is said to be alternating if $w_\sigma = \epsilon_\sigma w\ \forall\sigma\epsilon S_r$.

Let $w\epsilon\Lambda^a T_p{}^*(M)$ and $\eta\epsilon\Lambda^b T_p{}^*(M)$, then the wedge product $w\wedge\eta\epsilon\Lambda^{a+b}T_p{}^*(M)$ is defined by

$$w\wedge\eta = \frac{1}{a!b!}\mathrm{Alt}(w \otimes \eta). \tag{2.14}$$

The wedge product is associative and

$$w\wedge\eta = (-1)^{ab}\eta\wedge w \tag{2.15}$$

Generally, if $w^j \epsilon T_p{}^*(M)$, $j=1,2,\ldots,n$, and $V^1,\ldots V^p \epsilon T_p(M)$, then

$$(w^1 \wedge \ldots \wedge w^n)(V^1,\ldots V^n) = \mathrm{Det}(w^i(V_j)) \tag{2.16}$$

2.3 THE d-OPERATION[1] [7]

The "exterior derivative" is another useful operation on differential forms. It is defined in terms of a linear operator "d" acting on forms with the following properties.

a) It maps r-forms into (r+1)-forms.

b) It maps functions into their total derivatives.

c) For an r-form α, d has the Leibnitz property

$$d(\alpha\wedge\beta) = d\alpha\wedge\beta + (-1)^r \alpha\wedge d\beta \tag{2.17}$$

d) $d(dw) = d^2 w = 0 \quad \forall w \epsilon \Lambda^r T_p{}^*(M)$

The above four properties define the "d" operator in a coordinate free manner[1]. Locally, for an r-form

$$= \sum_{i_1<\ldots<i_r} \alpha_{i_1\ldots i_r}\, dx^{i_1}\wedge\ldots\wedge dx^{i_r}, \text{ we have}$$

$$dw = \sum_{i_1<\ldots<i_r} \frac{\partial \alpha_{i_1\ldots i_r}}{\partial x^{i_o}}\, dx^{i_o}\wedge dx^{i_1}\wedge\ldots\wedge dx^{i_r} \dagger \tag{2.18}$$

Then the components of dw in the basis $dx^{i_o}\wedge\ldots\wedge dx^{i_r}$, are

$$(dw)_{i_1\ldots i_{r+1}} = \sum_{s=1}^{r+1} (-1)^{s+1} \frac{\partial}{\partial x^s} \alpha_{i_1\ldots\hat{i}_s\ldots i_{r+1}} \tag{2.19}$$

$\hat{i}_s$ meaning that the suffix i_s is omitted.

$\dagger$We have adopted the convention that the new form is always inserted before any wedge product previously existing.

2.4 LINEAR DIFFERENTIALS AND PULL BACKS [1] [7]

Let M & N be smooth manifolds of dimension m & n respectively, and let $\phi: M \to N$ be smooth. The linear mapping $\phi_*: T_p(M) \to T_{\phi(p)}(N)$ defined by $(\phi_* X)f = X(f o \phi)$ is said to be the linear differential of ϕ at $p \varepsilon M$, where $X \varepsilon T_p(M)$ and $f: N \to R$ is smooth.

Given an r-form W on N, the r-form $\phi^* W$ on M, defined by

$$\phi^* w(x_1, \ldots x_n) = w(\phi_* x_1, \ldots \phi_* x_n), \tag{2.20}$$

$\forall\ x_1, \ldots x_n \varepsilon T_p(M)$, is said to be the pull back of w by ϕ.

$$\begin{tikzcd} T_p(M) \arrow[r, "\phi_*"] \arrow[dr, "\phi^*(\omega) = \omega \circ \phi_*"'] & T_{\phi(p)}(N) \arrow[d, "\omega"] \\ & R^1 \end{tikzcd}$$

The pull back is compatible with "d" & "Λ":

(a) $d(\phi^* w) = \phi^*(dw)$

(b) $\phi^*(w^1 \Lambda w^2) = \phi^*(w^1) \Lambda \phi^*(w^2)$ (2.21)

2.5 HODGE OPERATORS & FORM DUALITY [1] [4] [9]

Let M be an oriented n-dimensional Riemannian manifold with metric g_{ij}. Define an n-form τ called the Riemannian volume of M by

$$\tau \equiv \frac{1}{n!} \tau_{i_1 \ldots i_n} dx^{i_1} \wedge \ldots \wedge dx^{i_n}$$

$$\equiv \sqrt{|g|}\, dx^1 \wedge \ldots \wedge dx^n \qquad (2.22)$$

where $|g| = |\mathrm{Det}(g_{ij})|$†.

Then $\tau_{i_1 \ldots i_n} = \varepsilon^{1 \ldots n}_{i_1 \ldots i_n} \tau_{1 \ldots n} = \varepsilon^{1 \ldots n}_{i_1 \ldots i_n} \sqrt{|g|}$,

the generalized Kronecker symbol $\varepsilon^{j_1 \ldots j_p}_{i_1 \ldots i_p}$ $1 \le p \le n$ being defined by the following properties:

(a) $\varepsilon^{j_1 \ldots j_p}_{i_1 \ldots i_p}$ is totally anti-symmetric in its upper and lower indices.

(b) For $i_1 < \ldots < i_p$, $j_1 < \ldots < j_p$,

$$\varepsilon^{j_1 \ldots j_p}_{i_1 \ldots i_p} = \begin{cases} 1 & \text{for } i_k = j_k,\ k = 1, \ldots p \\ 0 & \text{for } i_k \neq j_k \end{cases}$$

The "raising" and "lowering" properties of the anti-symmetric indices of the coefficient functions of an n-form on M follow the rule

$$\alpha^{1 \ldots n} = g^{1 i_1} \ldots g^{n i_n} \alpha_{i_1 \ldots i_n} \qquad (2.23)$$

from which can be derived the rules[9]

$$\alpha^{1 \ldots n} = \frac{1}{|g|} \alpha_{1 \ldots n} \qquad (2.24)$$

$$\tau^{i_1 \ldots i_n} = \frac{1}{\sqrt{|g|}} \varepsilon^{1 \ldots n}_{i_1 \ldots i_n} \qquad (2.25)$$

†g_{ij} is an nxn matrix whose entries could be functions, defined on M.

The Hodge ($*$) (duality) operator is an isomorphism $\Lambda^r T_p^*(M) \to \Lambda^{n-r} T_p^*(M)$, $\beta \to *\beta$ defined locally as follows:

Let $\alpha \in \Lambda^r T_p^*(M)$ be given in terms of local coordinates as

$$\alpha = \sum_{i_1 < \ldots < i_r} \alpha_{i_1 \ldots i_r} \, dx^{i_1} \wedge \ldots \wedge dx^{i_r}$$

The form adjoint ($*\alpha$) of α is an $(n-r)$-form defined by

$$*\alpha = \sum_{j_1 < \ldots < j_{n-r}} (*\alpha)_{j_1 \ldots j_{n-r}} \, dx^{j_1} \wedge \ldots \wedge dx^{j_{n-r}}$$

such that

$$(*\alpha)_{j_1 \ldots j_{n-r}} = \sum_{i_1 \ldots i_r} \varepsilon^{1 \ldots n}_{i_1 \ldots i_r j_1 \ldots j_{n-r}} \sqrt{|g|} \, \alpha^{i_1 \ldots i_r} \tag{2.26}$$

"$*$" is an invertible operator with inverse $(*)^{-1} = (-1)^{r(n-r)} *$ and satisfies [1] [9]

$$\alpha \wedge *\beta = \beta \wedge *\alpha \text{ for any two forms } \alpha \text{ \& } \beta.$$

The global scalar product of two 1-forms α & β, denoted by $\langle\alpha|\beta\rangle$ is defined by

$$\langle\alpha|\beta\rangle = \int_M \alpha \wedge *\beta \; ^* \tag{2.27}$$

The scalar product satisfies the usual rules:

(a) $\langle\alpha|\beta\rangle = \langle\beta|\alpha\rangle$

(b) $\langle\alpha|\alpha\rangle \geq 0$ and $\langle\alpha|\alpha\rangle = 0 \iff \alpha = 0$

(c) $\langle *\alpha | *\beta\rangle = \langle\alpha|\beta\rangle$

*The integral is evaluated in terms of the integral over an open subset of R^2 of the pull back of the integrand [9].

3. GAUGE SYMMETRY GROUP AND ITS LIE ALGEBRA

3.1 LIE GROUP MANIFOLDS

We will use the differential geometric concepts discussed so far for the important case in which the manifold is a Lie group manifold G of gauge transformations: e.g. u(1) or a "global" phase transformation in quantum electrodynamics, or Su(2) as an example of a non-abelian gauge-field theory of the Yang-Mills type.

The group manifolds for u(1) and Su(2) are circle and 3-dimensional sphere respectively.

The G manifold is smooth and finite dimensional. Two natural group actions which are mappings from G to itself are defined by

$$L_s : G \to G, \quad R_s : G \to G; \; t, s \in G$$
$$t \to st \qquad t \to ts \; ;$$

the left and right translations, respectively.

The Lie algebra Γ is the tangent space $T_e(G)$ at the identity e of G, where the generators $a_i \in T_e(G)$ satisfy the structure equation and the Lie bracket operation is defined for any pair of vectors in Γ.

The space dual to Γ is denoted by $T_e{}^*(G) \equiv \Gamma^*$ and

$$\langle w^j, a_k \rangle = \delta^j_k \; ; \; \{w^j\} \; \& \; \{a_k\} \text{ being the bases of}$$

Γ^* and Γ respectively.

Under the action of the linear differential of L_s, $X_e \in T_e(G)$ generates a left-invariant* vector field

*λ is a left-invariant vector field if $L_{s*} \circ \lambda = \lambda \circ L_s$, $s \in G$.

$$X_s = L_{s*}(X_e)$$

defined on the group manifold.

Remarkably, the set of left-invariant vector fields is closed under the bracket operation{4}. Similarly, any 1-form $W_e \varepsilon T_e{}^*(G)$ induces a left invariant 1-form* w_s by means of the pull back.

$$L_s^* w_s = w_e \iff w_s = L^*_{s^{-1}} w_e$$

Thus, the 1-forms $\{w^j\}$ which are a basis of $T^*(G)$, give rise to linearly independent 1-forms $\{w^i_s\}$ defined at $s \varepsilon G$ via the pull back actions.

Now, the set $\{w^j \wedge w^k\}$ is a basis for 2-forms. So, in particular, we have

$$dw^i = \sum_{j,k} f^i_{jk} w^j \wedge w^k$$

where $f^i_{jk} = -f^i_{ki}$.

Evaluating the above 2-form on (a_m, a_n); $a_m, a_n \varepsilon T_e(G)$, we have

$$dw^i(a_m, a_n) = \sum_{j,k} f^i_{jk} w^j \wedge w^k \ (a_m, a_n)$$

$$= \sum_{j,k} f^i_{jk} \{w^j(a_m) w^k(a_n) - w^j(a_n) w^k(a_m)\}$$

$$= \sum_{j,k} f^i_{jk} \left(\delta^j_m \delta^k_n - \delta^j_n \delta^k_m\right) = 2f^i_{mn}$$

On the other hand, given a left-invariant 1-form* w, we have

$$dw(X,Y) = -w(\left[X,Y\right]) \quad \{3\}\ \{4\}$$

$\forall X, Y \varepsilon \Gamma$ (3.1)

*A differential form w on G is left invariant if $L^*_g w_{L_g h} = w_h$, $g, h \varepsilon G$

$$\Rightarrow dw^i(a_m,a_n) = -w^i([a_m,a_n]) = -w^i \sum_\rho C^\rho_{mn}\, a_\rho$$
$$= -\sum_\rho C^\rho_{mn} w^i(a_\rho) = -\sum_\rho C^\rho_{mn}\delta^i_\rho$$
$$= -C^i_{mn}$$

i.e. $dw^i(a_m,a_n) = 2f^i_{mn} = -C^i_{mn} \Leftrightarrow f^i_{mn} = -\tfrac{1}{2}C^i_{mn}$.

Hence the coefficient functions in the expansion of the 2-form dw^i are in fact the structure constants of the Lie group, i.e.,

$$dw^i = -\tfrac{1}{2}C^i_{jk}w^j \wedge w^k \tag{3.2}$$

This is the Maurer-Cartan structure equation.

An inner automorphism of G which is indispensable for the geometric interpretation of gauge transformations is the adjoint map $\mathrm{Int}_a: G \to G$ defined by
$\mathrm{Int}_a(g) = aga^{-1}$.

Its linear differential $\mathrm{Int}_{a*} \equiv \mathrm{Ad}_a$ is an isomorphism of the invariant vector fields on the G-manifold; in the sense that it maps a tangent space of the G-manifold (which is isomorphic to $\mathbb{R}^n$, n being the dimension of G) to another tangent space of the same manifold. Then the map

$\mathrm{Ad}: G \to \mathrm{Aut}\left(\bigcup_{a\in G} T_a(G)\right)$ constitutes what is known as the linear adjoint representation of G.

3.2 LIE ALGEBRA VALUED FORMS [1] [4] [10]

For our discussion of non-abelian gauge field theories and the theory of connections on principal fiber bundles, we need to extend the concept of real-valued r-forms defined on $\Lambda^r T_p(M)$ to vector valued, and in particular, Lie algebra valued r-forms.

Let $\Lambda^r T_p{}^*(M)$ denote the space of $\mathbb{R}^1$- valued r-forms at $p \varepsilon M$. Let V be a p-dimensional vector space. A V-valued differential r-form at p is an element of the product space

$$V \otimes \Lambda^r T_p{}^*(M)$$ i.e., an r-linear

anti-symmetric mapping from $\Lambda^r T_p(M)$ into V.

If $\{a_j\}$, $j=1, --- \wp$ is a basis for V, and $w^j \varepsilon \Lambda^r T_p{}^*(M)$, then a V-valued r-form Ω^r is uniquely represented as

$$\Omega^r = \sum_j a_j \otimes w^j, \text{ i.e. for } X_1, \ldots X_r \varepsilon T_p(M),$$

$$\Omega^r(X_1, \ldots X_r) = \sum_j w^j(X_1, \ldots X_r) a_j.$$

In terms of the natural coframe basis $\{dx^{i_1} \wedge \ldots \wedge dx^{i_\nu}\}$ of $\Lambda^r T_p{}^*(M)$,

$$\Omega^r = \frac{1}{r!} \sum_{i_1 < i_2 < \ldots < i_p} w^j_{i_1 i_2 \ldots i_\nu} a_j \otimes dx^{i_1} \wedge \ldots \wedge dx^{i_\nu},$$

$w^j_{i_1 \ldots i_p} a_j$ being V-valued anti-symmetric tensors.

We can extend the exterior derivative d to V-valued r-forms by setting $d(\Omega^p) = \sum_j a_j d(w^j)$. Since vector multiplication is not defined, we have no exterior (wedge) product of V-valued forms.

However consider a vector space Γ with a Lie algebra structure given by

(a) $\left[a_i, a_j\right] = \sum_k C^k_{ij}\, a_k$ ($\{a_i\}$ being a basis for Γ)

(structure equation)

(b) $\left[X, Y\right] = -\left[Y, X\right]$, $X, Y \epsilon \Gamma$

(c) $\left[X, \left[Y, Z\right]\right] + \left[Y, \left[Z, X\right]\right] + \left[Z, \left[X, Y\right]\right] = 0,$

$X, Y, Z \epsilon \Gamma$ (3.3)

We can define what is known as the graded-Lie bracket operation for a Γ-valued r-form, ψ & q-form, Φ

$$\psi = \sum_j a_j \psi^j \;, \; \Phi = \sum_k a_k \xi^k \;, \; j,k=1,\ldots r, q$$

by

$$\left[\psi, \Phi\right] = \sum_{j,k} \left[a_j, a_k\right] \otimes \psi^j \wedge \xi^k = \sum_{i,j,k} C^i_{jk}\, a_i \otimes \psi^j \wedge \xi^k \tag{3.4}$$

Using the anti-symmetry property $C^i_{jk} = -C^i_{kj}$ and eq(2.15), we have

$$\left[\psi, \Phi\right] = (-1)^{pq+1} \left[\Phi, \psi\right] \tag{3.5}$$

Using the Jacobi identity (3.3.c) for elements of Γ, we obtain, the graded Jacobi identity

$$(-1)^{ps}\left[\psi, \left[\Phi, \tau\right]\right] + (-1)^{qr}\left[\Phi, \left[\tau, \psi\right]\right] + (-1)^{sq} \left[\tau, \left[\psi, \Phi\right]\right] = 0 \tag{3.6}$$

where ψ, ϕ, τ are V-valued r, q, s forms, respectively. The expression for the exterior derivative of the graded Lie bracket is also an immediate consequence of the definition of $[\ ,\ ]$; the "d" for V-valued forms and the "d" for the wedge product of R^1-valued forms and is given by

$$d[\psi, \tau] = (-1)^r [\psi, d\tau] + [d\psi, \tau] \qquad (3.7)$$

4. CONNECTIONS ON A PRINCIPAL FIBER BUNDLE [1][4][6]

4.1 VECTOR FIELDS AND FORMS ON A FIBER BUNDLE

For $u \varepsilon P$, the map $L_u : G \to P$, $a \to ua$, is a smooth map. A smooth right translation mapping by a group action can be defined on the bundle space P:

$$R_a : P \to P, \quad u \to ua \text{ when } a \varepsilon G.$$

The linear differential $L_{u*} : T_e(G) \to T_u(P)$ at the group identity e is an isomorphism between the Lie algebra of G and the set of tangent spaces on P.

Let $A \varepsilon \Gamma$. The fundamental vector field on P associated to A and denoted by $\sum(A)$, is the vector field on P which is generated by the global one-parameter group of transformations

$$a_t = \varepsilon \exp(tA), \quad t \varepsilon R^1,$$

such that

$$\sum(A)_u of = \frac{d}{dt} f\left(R_{Exp(tA)}(u)\right)\Big|_{t=0} , \qquad (4.1)$$

f being a smooth function defined on P.

<u>N.B.</u>: Since $u(t) = R_{Exp(tA)}(u) = u\varepsilon\exp(tA)$ is a curve in P through u, $\sum(A)_u$ is a vector tangent to the fiber through u. The fiber being a sub-manifold of P, $\sum(A)_u$ is also tangent to P.

The set of fundamental vector fields on P is denoted by $\mathfrak{X}(P)$. The map $\sigma : \Gamma \to \mathfrak{X}(P)$, $A \to \sum(A)_u$, $u \varepsilon P$ is a Lie algebra homomorphism:

$$\sum\left(\left[A,B\right]\right)_u = \left[\sum(A)_u, \sum(B)_u\right] \qquad (4.2)$$

For $X\epsilon\mathfrak{X}(P)$, $a\epsilon G$; Xa is a unique vector field on P such that

$$\text{i) } Xa \equiv R_{a*}(X)$$

and $$\text{ii) } [X,Y]a = [Xa,Ya] \, , \quad X,Y\epsilon\mathfrak{X}(P) \qquad (4.3)$$

If a vector field X on P is such that

$R_{a*}(X) = Xa = X$, $\forall a\epsilon G$, then X is said to be an invariant vector field, and the set of such vectors is denoted by $\mathfrak{X}^I(P)$.

Let V_u be the subspace of $T_u(P)$ of vectors tangent to the fiber through $u\epsilon P$. A connection Γ in P is a direct sum decomposition

$$T_u(P) = V_u + H_u$$

i.e. a choice of a linear space H_u such that:

i) $H_{ua} = R_{a*}H_u$

ii) H_u depends differentiably on U.

Naturally enough H_u is called the horizontal subspace at u and $\dim H_u = \dim M$ {6}, $\forall u\epsilon P$.

Similarly, V_u is called the vertical subspace at u (the space of vectors tangent to the fiber).

N.B.: $X\epsilon\mathfrak{X}(P)$ is vertical if $X_u \epsilon V_u \; \forall u\epsilon P$. If X is vertical, $\Pi_*(x) = 0$. The set of vertical fields $\mathfrak{X}_V(P)$ is a subalgebra of $\mathfrak{X}(P)$ since $\Pi_*(X_1) = \Pi_*(X_2) = 0 \Rightarrow \Pi_*[X_1,X_2] = 0$ {6}.

Given a connection Γ in P, the unqiue direct sum decomposition of $X\epsilon T_u(P)$ into horizontal and vertical parts can be realized by introducing what is known as the connection

1-form W. W is a Lie-algebra valued 1-form defined by:{1}{3}{6}

i) $W\left[\sum(\Lambda)_u\right] = \Lambda$, $\forall u \varepsilon P$ & $\forall \Lambda \varepsilon \Gamma$

ii) The pull back $(R_a^*)(W) = Ad_a^{-1} W$, $\forall a \varepsilon G$, and X_u is horizontal if and only if $W(X_u) = 0$, i.e.,

$$Ker(W) = \bigcup_{u \varepsilon P} H_u.$$

Conversely, a Γ-valued 1-form W on P satisfying i) and ii) determines a unique connection Γ in P whose connection 1-form is W.

The connection 1-form W on P can be expressed in terms of local 1-forms defined on each open subset of the covering $\{V_\alpha\}$ of M.

Let $\psi_\alpha : \Pi^{-1}(V_\alpha) \to V_\alpha \times G$; $\psi_\alpha(u) = \left(\Pi(u), \phi_\alpha(u)\right)$ be a family of trivialising isomorphisms and let

$\psi_{\alpha\beta} : V_\alpha \cap V_\beta \to G$ be defined by

$\psi_{\alpha\beta}\left(\Pi(u)\right) \equiv \phi_\alpha(u)\phi_\beta(u)^{-1}$. †

Let $\sigma_\alpha : V_\alpha \to P$ be the preferred set of local sections defined by $\sigma_\alpha(x) = \psi_\alpha^{-1}(x,e)$; where $x \varepsilon V_\alpha$ and e is the identity of G. For each α define a Lie-algebra valued 1-form W_α on V_α by

$$W_\alpha \to \sigma_\alpha^*(W) \qquad (4.4)$$

†If $\Pi(u) = \Pi(v)$, then $\exists a \varepsilon G$ such that $v = ua$. Then $\psi_{\alpha\beta}\left(\Pi(v)\right) = \phi_\alpha(ua)\phi_\beta^{-1}(ua) = \phi_\alpha(u)aa^{-1}\phi_\beta^{-1}(u) = \phi_\alpha(u)\phi_\beta^{-1}(u)$, i.e. $\psi_{\alpha\beta}\left(\Pi(u)\right) = \psi_{\alpha\beta}\left(\Pi(v)\right)$ and $\psi_{\alpha\beta}$ is well defined.

Theorem (4.1.a):[1],[3] The local 1-forms W_α satisify the compatability conditions $W_\beta = Ad_{\psi_{\alpha\beta}^{-1}}(W_\alpha) + \psi_{\alpha\beta}^{-1} d\psi_{\alpha\beta}$ in $V_\alpha \cap V_\beta$. Conversely, given a set of local 1-forms $\{W_\alpha\}$ on M satisifying the above comptability conditions, $\exists$ a connection 1-form W on P such that

$$W_\alpha = \sigma_\alpha{}^*(w), \quad \forall\alpha.$$

Suppose the total fiber space is a trivial M×G bundle. We can then take the comptability relations to be such that the local open sets V_α and V_β coincides with M, and the corresponding sections are global sections $\sigma_\alpha(x)$, $\sigma_\beta(x)$ respectively.

The section transformation rule $\sigma_\beta(x) = \sigma_\alpha(x)\psi_{\alpha\beta}(x)$, $\psi_{\alpha\beta}\varepsilon G$ can be simply written as $\sigma_2(x) = \sigma_1(x)g(x)$. Then, the corresponding global 1-forms W_2 and W_1 are related on M, by

$$W_2 = Ad_{g^{-1}}(w_1) + g^{-1}dg. \tag{4.5}$$

This relation can be written in local coordinate form:

Let M be a four-dimensional manifold with local coordinates $X_\mu(\mu=1,2,3,4)$ and an open covering $\{V_\alpha\}$.

Then

$$W_\alpha = A^\mu_\alpha(x)dx_\mu,$$

where $A^\mu_\alpha(x)$ is a smooth, Lie-algebra-valued function on V_α.

If we transform the local section σ_α by a group action $g(x)$, $\sigma_\alpha{}'(x) = \sigma_\alpha(x)g(x)$, then there will be a corresponding transformation of the local 1-form:

$$W_\alpha{}' = \sigma_\alpha^{*\prime}(w) = A_\alpha'^{\mu}(x)dx_\mu$$

i.e. $$A_{\alpha\mu}'(x)dx^\mu = Ad_g^{-1}\left(A_{\alpha\mu}(x)dx^\mu\right) + g^{-1}dg$$

$$\doteq \left(g^{-1}(x)A_{\alpha\mu}(x)g(x) + g(x)^{-1}\partial_\mu g(x)\right)dx^\mu$$

By the linear independence of the basis $\{dx^\mu\}$ of 1-forms, we have

$$A_{\alpha\mu}'(x) = g(x)^{-1}A_{\alpha\mu}(x)g(x) + g^{-1}(x)\partial_\mu g(x) \qquad (4.6)$$

This is exactly the gauge transformation rule that can be derived by variational methods.[4]

4.2 GEOMETRY & GAUGE TRANSFORMATION

The transformation rule in eq (4.6) suggests the geometrization of gauge potentials:

The coordinates of a connection form in a principal fiber bundle P(M,G) can be identified with the potentials of a non-abelian gauge field theory with internal symmetry group G. Furthermore, the choice of a local section σ_α on P coordinatizes the fiber spaces locally and establishes a one-to-one correspondence between the connection form W and the local 1-forms $A_{\alpha\mu}dx^\mu$, i.e., the choice of a local section is equivalent to a choice of gauge.

Transformation of a local section under the action of the group induces a gauge field transformation:

Conversely, if the local gauge fields $A_{\alpha\mu}(x)$ are given, then $\exists$ a unique connection 1-form W on P satisfying

$$A_{\alpha\mu}(x)dx^\mu = \sigma_\alpha{}^*(W)$$

<u>N.B.</u>: Though W_α & $A_{\alpha\mu}(x)$ are defined locally, W is a global object, given once and for all for a fixed fiber space.

5. COVARIANT DERIVATIVE IN A PRINCIPAL FIBER BUNDLE[1]

5.1 D_μ AS THE HORIZONTAL LIFT OF ∂_μ

In a neighbourhood V_α, with local coordinates X^μ, the local section $\sigma_\alpha : V_\alpha \to P$ defines a local connection 1-form $W_\alpha = \sigma_\alpha{}^*(W) = \sum_\mu A_{\alpha\mu} dx^\mu$. The Γ-valued W_α acts on the set of tangent vectors on M. In particular, the image of the vector field $\partial_\mu \equiv \partial/\partial x^\mu$ under W_α is

$$W_\alpha(\partial_\mu) = \sigma_\alpha{}^*W(\partial_\mu) = W(\sigma_{\alpha*})(\partial_\mu) \;\dagger$$

$$= \sum_\nu A_{\alpha\nu}(x)\, dx^\nu(\partial_\mu) = \sum_\nu A_{\alpha\nu}(x)\, \delta^\nu_\mu$$

$$= A_{\alpha\mu}(x) = W\left(\sum(A_{\alpha\mu})\right) \;\dagger\dagger$$

$$\Leftrightarrow W(\sigma_{\alpha*}\partial_\mu) - W(\sum A_{\alpha\mu}) = W\left(\sigma_{\alpha*}\partial_\mu - \sum(A_{\alpha\mu})\right) = 0$$

i.e. the vector field $\sigma_{\alpha*}\partial_\mu - \sum(A_{\alpha\mu})$ is in the kernel of the connection form W. Hence $\sigma_{\alpha*}\partial_\mu - \sum(A_{\alpha\mu})$ is horizontal. Moreover its projection by Π_*, viz:

$$\Pi_*\left(\sigma_{\alpha*}\partial_\mu - \sum(A_{\alpha\mu})\right) = \Pi_*(\sigma_{\alpha*}\partial_\mu) - \Pi_*\left(\sum(A_{\alpha\mu})\right)$$

$$= \Pi_*(\sigma_{\alpha*}\partial_\mu) = \partial_\mu, \text{ since}$$

† This follows from the definition of linear differentials and pull backs.

†† This is a consequence of the definitions of $\sum(A_{\alpha\mu})$ and the connection 1-form W.

$\sum(A_{\alpha\mu})$ is vertical and $\Pi_*\sum(A_{\alpha\mu}) = 0$; i.e. $\sigma_{\alpha*}\partial_\mu - \sum(A_{\alpha\mu})$ is the horizontal lift † of the ordinary derivative ∂_μ.

We denote the horizontal lift by

$$\tilde{\partial}_\mu\big|_u = \sigma_{\alpha*}\partial_\mu - \sum(A_{\alpha\mu}) \qquad (5.1)$$

where $u = \sigma_\alpha(x)$. The local section $\sigma_\alpha:V_\alpha \to P$ is a trivialising map of P, since $\Pi^{-1}(V_\alpha) \subset P$ and locally $\Pi^{-1}(V_\alpha) \simeq V_\alpha \times G$.

Thus, $\sigma_\alpha(V_\alpha)$ being just $\{(x,e)|x \varepsilon V_\alpha\}$, $\sigma_{\alpha*}\partial_\mu$ can be identified with ∂_μ itself.

Similarly, to identify $\sum(A_{\alpha\mu})$ with something more basic, we consider the action of $\sum(A_{\alpha\mu})$ on a differentiable function f in P:

$\sum(A_{\alpha\mu})$ is a vector field at $u_o = \sigma_\alpha(x_o) = (x_o,e)$ (Fig. (5.1)). $u_t = u_o \exp(tA_{\alpha\mu})$ is a path in P starting from u_o and generated by the right action of a one-parameter subgroup of the structure group G. By definition,

$R_a:P\to P$, $R_a(x,g) = (x,ga)$ $\big((x,g)\varepsilon P,\ a\varepsilon G\big)$, hence $u_t = u_o\exp(tA_{\alpha\mu}) = \big(x_o, \exp(tA_{\alpha\mu})\big)$.

$\sum(A_{\alpha\mu})\big|_{u_o} of \equiv \frac{d}{dt} f(u_t)\big|_{t=0}$. As $u_o, u_t \varepsilon \Pi^{-1}(x_o)$ and $u_o = (x_o,e)$, $u_t = (x_o, \exp(tA_{\alpha\mu}))$, one just has to consider the restriction of f to $\Pi^{-1}(x_o)$.

Let $f\big|_{\Pi^{-1}(x_o)} \equiv F$.

† $\tilde{X}\varepsilon T_u(P)$ is said to be a horizontal lift of $X\varepsilon T_{\Pi(u)}(M)$, if $\Pi_*\tilde{X} = X$.

Then $\frac{d}{dt} f(u_t)\Big|_{t=0} = \sum(A_{\alpha\mu})\Big|_{u_o} \circ f$

$$\equiv \lim_{\alpha \to 0} \frac{F(x_o, \varepsilon\exp(tA_{\alpha\mu})) - F(x_o, e)}{t}$$

i.e., the directional derivative $\sum(A_{\alpha\mu})\Big|_{u_o}$ of of f along u_t for a fixed $x_o \varepsilon M$ is a differentiation on the group manifold.

By definition of a Lie algebra generator,

$$\sum(A_{\alpha\mu})\Big|_{u_o} \circ f = A_{\alpha\mu} f \quad ;$$

i.e. $\sum(A_{\alpha\mu})\Big|_{u_o}$ is locally identifiable with $A_{\alpha\mu}$

Hence

$$\tilde{\partial}_\mu\Big|_{u_o} \equiv D_\mu = \partial_\mu - A_\mu \,\dagger \qquad (5.2)$$

and D_μ is the horizontal lift of ∂_μ.

Fig. (5.1) depicts the relationship between the vector operators D_μ, $\sum(A_\mu)$ and $\sigma_{\alpha*}\partial_\mu$.

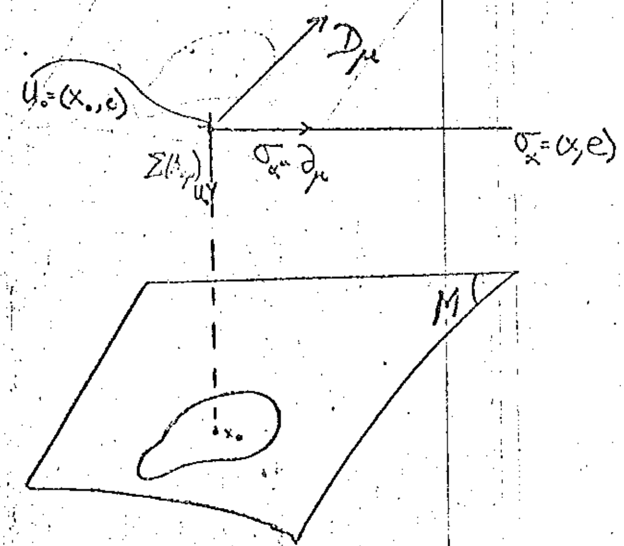


Fig. (5.1)

†Since the expression is a local expression, "α" is suppressed.

Like the connection 1-forms W_α, the operator D_μ is a section dependent object. If $D_\mu = \partial_\mu - A_\mu$ at $\sigma_\alpha(x)\epsilon\Pi^{-1}(x)$, then at another point $\sigma_\alpha'(x) = \sigma_\alpha(x)g(x)$ on the same fiber,

$$D_\mu' = \partial_\mu - g^{-1}A_\mu g + g^{-1}\partial_\mu g \tag{5.3}$$

By a lemma[1][6], if X & Y are two commuting vector fields on the base space, $[\tilde{X},\tilde{Y}]$ is vertical†. In particular since $[\partial_\mu,\partial_\nu] = 0$, the commutator $[\tilde{\partial}_\mu,\tilde{\partial}_\nu] \equiv [D_\mu,D_\nu]$ is a vertical field.

Using the local expression $D_\mu = \partial_\mu - A_\mu$,[1][4] one can show that

$$[D_\mu,D_\nu] = -\{(\partial_\mu A_\nu - \partial_\nu A_\mu) + [A_\mu,A_\nu]\} \tag{5.4}$$

The Lie algebra element $F_{\mu\nu} \equiv \partial_\mu A_\nu - \partial_\nu A_\mu + [A_\mu,A_\nu]$ will later be identified with the gauge field.

If local sections σ_α and σ_β are given on each coordinate patch V_α & V_β respectively, we have

$\sigma_\beta(x) = \sigma_\alpha(x)\psi_{\alpha\beta}(x)$, and the local transformation rule for the gauge field can be derived from the local transformation rule for the gauge potentials eq. (4.5) and eq. (5.4).

On each overlap we have[1]

$$(F_{\mu\nu})_\beta = Ad_{\psi_{\alpha\beta}^{-1}}(F_{\mu\nu})_\alpha \tag{5.5}$$

†For any vector field A on M, $\tilde{A}$ denotes its horizontal lift.

<u>N.B.</u>: The actual action of the D_μ operator depends fundamentally on the transformation properties of the functions on which it acts.

<u>Ex:</u> let $\psi: P \to G$ be a function defined on a principal fiber bundle $P(M, Su(n))$ satisfying $\psi(ua) = a^{-1}\psi(u)a$, $\forall a \varepsilon G$, $\forall u \varepsilon P$. Representing G and its Lie algebra by nxn matrices,

$$D_\mu \psi|_{u_o} \equiv \partial_\mu \psi - \lim_{t \to 0} \left[\frac{e^{-tA_\mu}\psi(u_o)e^{tA_\mu} - \psi(u_o)}{t} \right] . \quad (5.6)$$

Expanding $e^{-tA_\mu}\psi(u_o)e^{tA_\mu}$ in a neighborhood of $t=0$ and evaluating the limit we get an explicit way of writing the "covariant derivative" operator action

$$D_\mu \psi|_{u_o} = \partial_\mu \psi + \left[\bar{A}_\mu, \bar{\psi}\right], \; u_o \varepsilon P$$

Thus

$$D_\mu = \partial_\mu + \left[\bar{A}_\mu, *\right] \qquad (5.7)$$

If such a ψ is a section of an associated bundle E, then the action of D_μ is referred as the covariant differentiation in an associated bundle and eq. (5.7) is written as[1]

$$\nabla_\mu \equiv \partial_\mu + \left[\bar{A}_\mu, *\right]$$

6. CURVATURE ON A PRINCIPAL FIBER BUNDLE [1] [3]

6.1 THE CURVATURE FORM

In terms of the natural 2-form basis $\{dx^\mu \wedge dx^\nu\}$ and the local gauge fields $(F_{\mu\nu})_\alpha$, we can construct a Lie algebra valued 2-form Ω_α acting on an ordered pair of vector fields tangent to the base space given by

$$\Omega_\alpha = \tfrac{1}{2}(F_{\mu\nu})_\alpha \, dx^\mu \wedge dx^\nu \tag{6.1}$$

From the gauge field transformation rule (eq. 5.5), we have

$$\Omega_\beta = \mathrm{Ad}\,\psi_{\alpha\beta}^{-1}\Omega_\alpha \tag{6.2}$$

Since the coordinatizing map $\phi_\alpha : \Pi^{-1}(V_\alpha) \to G$ takes the constant value "e" on the image of the local section σ_α, we have

$$\forall u \in \Pi^{-1}(V_\alpha), \quad u = \sigma_\alpha(\Pi(u)) \cdot \phi_\alpha(u)$$

The curvature form Ω is a Lie algebra valued 2-form on the bundle space defined by

$$\Omega = \mathrm{Ad}_{\phi_\alpha^{-1}}(\Pi^*\Omega_\alpha) \tag{6.3}$$

N.B.: The transformation rule (eq. 6.2) guarantees that there is no discrepancy between possible definitions of Ω on any overlap:

$$\begin{aligned}
\mathrm{Ad}_{\phi_\beta^{-1}}(\Pi^*\Omega_\beta) &= \phi_\beta^{-1}(\Pi^*\Omega_\beta)\phi_\beta \\
&= \phi_\beta^{-1}(\Pi^*\mathrm{Ad}_{\psi_{\alpha\beta}^{-1}}\Omega_\alpha)\phi_\beta \\
&= \phi_\beta^{-1}(\Pi^*\phi_\beta(u)\phi_\alpha^{-1}(u)\Omega_\alpha\phi_\alpha(u)\phi_\beta^{-1}\phi_\beta)
\end{aligned}$$

$$= \phi_\alpha^{-1}(u)\Pi^*\Omega_\alpha \phi_\alpha(u)$$

$$= Ad_{\phi_\alpha^{-1}}\Pi^*\Omega_\alpha$$

i.e. if Y and Z are two vectors tangent to the bundle in $\Pi^{-1}(V_\alpha \cap V_\beta)$, the consistency we demonstrated reads as

$$Ad_{\phi_\alpha^{-1}}(\Pi^*\Omega_\alpha)(Y,Z) = Ad_{\phi_\beta^{-1}}(\Pi^*\Omega_\beta)(Y,Z) \tag{6.4}$$

Let (P,G) be a principal fiber bundle with a connection 1-form W. The curvature form Ω is globally related with the connection form W by the structure equation{1}{3}

$$\Omega = dW + \tfrac{1}{2}[W,W] \tag{6.5}$$

If $X,Y \epsilon T_u(P)$, $u\epsilon P$

$$\Omega(X,Y) = dW(X,Y) + \tfrac{1}{2}[W(X),W(Y)] \text{ \{1\}\{3\}} \tag{6.6}$$

eq. (6.5) reduces to

$$\Omega(X,Y) = dW(X,Y) \tag{6.7}$$

when X & Y are horizontal* vectors.

Decomposing X and Y in eq. (6.6) into their horizontal and vertical parts

$$(hX, hY, vX, vY)$$

and using the bilinearity property of Ω, we have

$$\Omega(X,Y) = dW(vX,vY) + dW(vX,hY) + dw(hX,hY) + dW(hX,vY) + \tfrac{1}{2}[W(vX),W(vY)]$$

Since the kernel of W is the horizontal subspace of $T_u(P)$, this reduces to

*This is because the kernel of W is the horizontal subspace of $T_u(P)$.

but $\left[A_\alpha \wedge A_\alpha , A_\alpha\right] = 0$

Hence

$$dF_\alpha + \left[A_\alpha , F_\alpha\right] = 0 \qquad (6.14)$$

This is known as the Bianchi identity. It is significantly less cumbersome to establish it from eq. (6.12) than eq. (6.13). N.B.: eq. (6.14) is not a conditional statement but an identity of differential geometry {10}{26}.

Eq. (6.14) is equivalent to the second set of Maxwell equations

$$\nabla_{\left[\mu,\right.} F_{\left.\nu\lambda\right]} \equiv \nabla_\mu F_{\nu\lambda} + \nabla_\nu F_{\lambda\mu} + \nabla_\lambda F_{\mu\nu} = 0 \;\dagger\dagger \qquad (6.15)$$

where ∇_μ is the covariant derivative associated with the adjoint representation of Γ (refer to section (5.1)).

If $\sigma_\alpha' : V_\alpha \to P$ is another section and $\sigma_\alpha' = \sigma_\alpha . g(x)$, with $g : V_\alpha \to G$ a gauge transformation, then after a lengthy but straightforward substitution and manipulation of wedge products it follows from eq. (6.12) and eq. (4.5) that

$$F_\alpha' = g^{-1}(x) F_\alpha g(x) \qquad (6.16)$$

The Yang-Mills field intensities F_α are not the same kinds of objects as the corresponding potentials, in that they do not have the inhomogeneous term $g^{-1}dg$ in their transformation rule. For this reason the fields are described as tensor types and the potentials as affine types.

† The local index "α" is suppressed for convenience.

7. THE WU-YANG MONOPOLE FORMALISM[13] [14] [15]

One of the many physical situations which suggest the use of the fiber bundle language is the magnetic monopole problem. (Appendix A)

Historically, the magnetic monopole is a theoretical artifice introduced by Dirac in order to make Maxwell's equations possess a symmetry known as duality symmetry. (Appendix A)

The equations of classical electrodyamics, i.e., Maxwell's equations' that read as

$$\text{a)}\ \underline{\nabla}\cdot\underline{E} = \rho \qquad \text{b)}\ \underline{\nabla}\times\underline{B} - \frac{\partial E}{\partial t} = j$$

$$\text{a)}\ \underline{\nabla}\cdot\underline{B} = 0 \qquad \text{b)}\ \underline{\nabla}\times\underline{E} + \frac{\partial B}{\partial t} = 0 \qquad (7.1)$$

may be written compactly in relativistic notation as

$$\text{a)}\ \partial^{\mu}F_{\mu\nu} = j_{\nu}$$

$$\text{b)}\ \partial^{\mu}{*F}_{\mu\nu} = 0 \qquad (7.2)$$

where $F^{oi} = -E^{i}$, $F^{ij} = -\epsilon_{ijk}B^{k}$ and ${*F}^{\mu\nu} = \frac{1}{2}\epsilon^{\mu\nu\rho\sigma}F_{\rho\sigma}$.

In free space where the current density j_{ν} vanishes, the electrodynamical equations exhibit a symmetry under the "duality" transformation:

$$F^{\mu\nu}\to{*F}^{\mu\nu}, \quad {*F}^{\mu\nu}\to F^{\mu\nu} \quad \text{or}$$

equivalently

$\underline{E}\to\underline{B}$, $\underline{B}\to E$. The "duality" transformation then interchanges electricity with magnetism. To sustain this symmetry between electricity and magnetism even in the presence of matter (charges), Dirac introduced a hypothetical "magnetic current" four vector $g_\nu = (\sigma,\underline{K})$ on the right-hand side of eq. (7.2b) (where σ is the "magnetic charge" or monopole density) giving the modified field equations:

$$\begin{aligned} &\text{a)}\ \partial^\mu F_{\mu\nu} = j_\nu \\ &\text{b)}\ \partial^\mu {*F}_{\mu\nu} = -g_\nu \end{aligned} \qquad (7.3)$$

The price the Dirac monopole formalism has to pay in order to preserve duality symmetry in electrodynamics is that it is now impossible to introduce an analytic vector potential $A = (A_o,\underline{A})$ as a canonical field variable via

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \qquad \text{(Appendix A)}$$

However Dirac characterized the physics of the situation using a singular potential and a string† along which the potential fails to be analytic. It can be shown that{13} gauge transformation of the singular potential yields a "new" potential with a string position shifted from the previous one, i.e., the string is something that cannot be detected by physical experiments†† and is devoid of actual physical meaning{13}.

However, one can accommodate the string mathematically, and even derive an important result - the Dirac quantisation condition - from the fact that the string is unphysical{13}.

In the four and a half decades since the publication of Dirac's paper, many theorists have studied the problem of extending classical and quantum electrodynamics to include

† The string is just a subset of R^3 in which $A=(A_o,\underline{A})$ fails to be regular.

†† See the "Bohm-Ahronov experiment" in {13} & {14}.

monopoles. Elimination of the Dirac string has been one of the main goals.

In order to avoid singularities, Wu and Yang[14][15] consider space about a monopole ($R^3 \setminus \{0\}$) to be divided into two overlapping regions.

The region without the origin, above the lower cone in Fig. (7.1) is denoted by R_a. Similarly, we denote the region without the origin, under the upper cone by R_b.

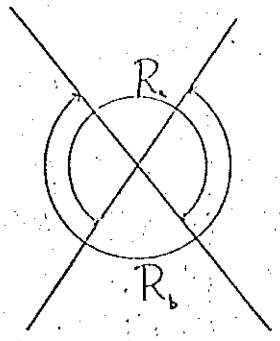


Fig. (7.1)

We have $R_a \cup R_b = R^3 \setminus \{0\}$ and in each region $R_a(R_b)$ a regular vector potential $(A_\mu)_a$ & $(A_\mu)_b$ is chosen such that in the region of overlap, the two vector potentials have the same curl. The regions can be parametrized as

$$R_a: \; 0 \leq \theta < \pi/2 + \delta \; , \; r > 0 \; ; \; 0 \leq \phi < 2\pi$$

$$R_b: \; \pi/2 - \delta < \theta \leq \pi \; , \; r > 0 \; ; \; 0 \leq \phi < 2\pi$$

with an overlap extending throughout $\pi/2 - \delta < \theta < \pi/2 + \delta$, (we assume $0 < \delta \leq \pi/2$). The spherical symmetry around a monopole suggests that we choose a potential

$$\underline{A}(\underline{r}) = A(\underline{r})\,\hat{\underline{e}}_\phi$$

$\hat{\underline{e}}_\phi$ being a unit vector in the ϕ direction. In R_a we choose a potential such that[15],

$$(A_r)_a = (A_\theta)_a = 0 \quad ; \quad (A_\phi)_a = {}^{g}/_{r\sin\theta}(1-\cos\theta) \qquad (7.4)$$

Similarly in R_b[15]

$$(A_r)_b = (A_\theta)_b = 0 \quad ; \quad (A_\phi)_b = {}^{-g}/_{r\sin\theta}(1+\cos\theta) \qquad (7.5)$$

Indeed using the expressions[17]

$$(\underline{\nabla}\times\underline{A})_r = \frac{1}{r\sin\theta}\left[\frac{\partial(\sin\theta A_\phi)}{\partial\theta} - \frac{\partial A_\theta}{\partial\phi}\right]$$

$$(\underline{\nabla}\times\underline{A})_\theta = \frac{1}{r\sin\theta}\left[\frac{\partial A_r}{\partial\phi} - \frac{1}{r}\frac{\partial}{\partial r}(rA_\phi)\right]$$

$$(\underline{\nabla}\times\underline{A})_\phi = \frac{1}{r}\left[\frac{\partial(rA_\theta)}{\partial r} - \frac{\partial A_r}{\partial\theta}\right]$$

for $\underline{\nabla}\times\underline{A}$ in spherical coordinates, one can easily demonstrate that

$$\underline{\nabla}\times\underline{A} = {}^{g}/_{r^2}\ \hat{\underline{e}}_r$$

in each region.

Both $(A_\phi)_a$ and $(A_\phi)_b$ are singularity free in R_a & R_b respectively. Since by construction

$$\underline{\nabla}\times\underline{A}_a - \underline{\nabla}\times\underline{A}_b = 0 \quad ,$$

$$\underline{\nabla}\times(\underline{A}_a - \underline{A}_b) = 0$$

there exists a scalar potential α such that

$$e(A_\mu)_a - e(A_\mu)_b = \partial_\mu\alpha\ \dagger \qquad (7.6)$$

Transforming eq. (7.6) into a spherical coordinate system and solving for α, one gets

$$\alpha = 2ge\phi \qquad (7.7)$$

†Since we are considering a static (time independent) situation, $\mu = 1,2,3$.

ϕ being the azimuthal angle.

The Schrödinger equation for an electron in a monopole field reads†

$$\frac{1}{2m}(-i\underline{\nabla}-e\underline{A}_a)^2\psi_a = E\psi_a \qquad \text{in } R_a$$

$$\frac{1}{2m}(-i\underline{\nabla}-e\underline{A}_b)^2\psi_b = E\psi_b \qquad \text{in } R_b$$

By the well-known gauge principle[18][27], since the two vector potentials differ by a gradient, the wave functions ψ_a and ψ_b are related by a phase factor transformation.

$$\psi_a = S_{ab}\psi_b \quad , \quad S_{ab} = \exp(2ieg\phi)$$

$$\text{i.e.} \quad \psi_a = \exp(2ieg\phi)\,\psi_b \tag{7.8}$$

Since ψ_a and ψ_b are both single valued on the equator, which lies in $R_a \cap R_b$, the phase factor must be single valued and so

$$2ge = n\epsilon Z \tag{7.9}$$

which is the Dirac quantisation condition.

In the overlap region, ψ_a & ψ_b are connected by a transition function $\exp(2ige\phi)$ which is an element of U(1); the group of unimodular complex numbers, or the displacement group on a unit circle. U(1) is the basic gauge transformation group of electromagnetism. Any two such ψ's, are called a wave section. Hence, around a magnetic monopole, the electron wave functions are wave sections and not ordinary functions.

The Wu-Yang formalism makes the monopole theory conceptually identical to a fiber bundle theory.

† This is in a system with $\hbar = 1$.

The space $M = R^3 \setminus \{0\}$, the group $G = U(1)$, the local covering $\{R_a, R_b\}$ and the transition functions $S_{ab} \equiv e^{2ge\phi i} = e^{in\phi}$, $n \in Z$ constitute a non-trivial coordinate bundle

$$\{R^3 \setminus \{0\}, U(1), \{R_a, R_b\}, e^{in\phi}\}$$

The wave sections ψ_a & ψ_b satisfying $\psi_b(x) = \psi_a(x) S_{ab}$, are local sections corresponding to R_a & R_b respectively.

The triviality of a bundle space is equivalent to being able to choose a global section. For the Wu-Yang monopole bundle, we have non-triviality because the existence of the monopole ($g \neq 0$) imples that there will always be a "twist of phase" ($S_{ab} \neq 1$) and so the bundle cannot admit a global section. This is the reason why electromagnetism without monopoles is a trivial U(1) bundle while with a monopole it is a non-trivial U(1) bundle.

This relationship between the Dirac magnetic monopole and non-trivial U(1) bundles can be made more concrete and manifest by identifying or constructing the "actual" principal fiber bundle to which the Dirac monopole problem corresponds. It turns out that this bundle is a classifying † U(1) bundle over S^2.{1}{26}

The topology of the space $R^3 \setminus \{0\}$ is equivalent to that of its deformation retract; a two dimensional solid

† Given $R^3 \setminus \{0\}$ and U(1) as a base manifold and fiber respectively, the various coordinate bundles one can construct are distinguished by their transition functions, which involve integer parameters. In turn, each $P(S^2, U(1))$ has integers associated with it; called Chern numbers.

sphere of radius $r^{(1)}$. Let S^2 be the spherical surface of this solid sphere. We will construct a U(1) bundle with S^2 as a base manifold that corresponds naturally with the Dirac monopole as follows (Fig. (7.2)):

A fiber $F = U(1) = S^1$; the fiber being parametrized by ψ as $e^{i\psi}$.

Base $M = S^2$; parametrized by coordinates (θ,ϕ); $0\leq\theta<\Pi$, $0\leq\phi<2\Pi$.

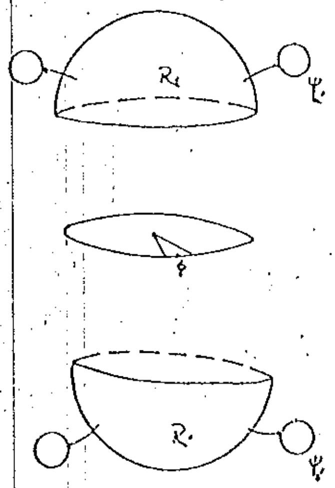


Fig. (7.2)

Two closed hemispherical patches R_a' & R_b' cover the base manifold where their common boundary "C" is parametrized by the equatorial angle. Like any bundle, locally, the bundle is a trivial product:

$R_a' \times U(1)$; parametrized as $(\theta,\phi;e^{i\psi_a'})$

$R_b' \times U(1)$; parametrized as $(\theta,\phi;e^{i\psi_b'})$

The fiber coordinates on R_a' and R_b' can be related along $R_a' \cap R_b'$ by regarding the transition functions as U(1) valued functions of ϕ. This will yield a principal U(1) bundle $P(S^2, U(1))$. We choose[26] transition functions $e^{in\phi}$ and we have

$$e^{i\psi_b'} = e^{in\phi}e^{i\psi_a'} \tag{7.10}$$

It is necessary that n be integral if the fibers are to fit together exactly during every complete rotation in ϕ around $R_a' \cap R_b'$.†

We can now define a connection on $P(S^2, U(1))$. In particular, if the gauge potentials satisfy Maxwell's equations, we have a principal fiber bundle $P(S^2, U(1))$ which corresponds to the Dirac magnetic monopole.

We have the usual gauge transformation eq. (4.5)

$$A_b' = A_a' + e^{-in\phi}de^{in\phi} \tag{7.11}$$

For U(1), by convention we factor out the complex factor (i). This corresponds to the convention in which Lie algebra elements are represented by antihermitian matrices[26]. Hence the second term in eq. (7.11) becomes,

$$e^{-in\phi}de^{in\phi} = ind\phi \rightarrow nd\phi .$$

Hence we have

$$A_b' = A_a' + nd\phi \tag{7.12}$$

Following the above convention for a U(1) group, the curvature form of $P(S^2, U(1))$, is pure imaginary and hence is given by

$$\Omega = iF \tag{7.13}$$

† Or else the fiber space will not have the structure of a manifold.

Thus the total Chern class (Appendix C) of $P(S^2,U(1))$ becomes

$$C(P) = \mathrm{Det}\left(1 + \frac{i\Omega}{2\Pi}\right) = \mathrm{Det}\left(1 + \frac{i^2F}{2\Pi}\right)$$

$$= 1 - F/_{2\Pi} \quad \dagger$$

So the first Chern class $C_1(P)$ is given by

$$C_1(P) = -F/_{2\Pi} \tag{7.14}$$

all other characteristic classes $C_i(P)(i\geq 2)$ being zero.

One can integrate $C_1(P)$ on S^2 to get the first Chern number

$$C_1 = \int_{S^2} C_1 \tag{7.15}$$

We will soon see that C_1 is the negative of the monopole quantum number "n" and is something given once and for all for a given $P(S^2,U(1))$, i.e. it depends only on the transition functions or the gauge transformation that relates A_a' to A_b', and not specifically to the actual potentials choosen. Two potentials A_a' and A_b' that need not satisfy the Maxwell's equation but which are related by eq. (7.2) yield the same Chern number. To see this, let $A_b'(x) = A_a'(x) + nd\phi;\ x\epsilon S^2$. Then

$$C_1 = \int_{S^2} C_1 = -\frac{1}{2\Pi}\int_{S^2} F = -\frac{1}{2\Pi}\int_{S^2} F_{\mu\nu}dx^\mu \Lambda dx^\nu$$

$$= -\frac{1}{2\Pi}\left(\int_{R_a'} F_{\mu\nu}dx^\mu\Lambda dx^\nu + \int_{R_b'} F_{\mu\nu}dx^\mu\Lambda dx^\nu\right)$$

Making use of eq. (6.12) for an Abelian gauge group U(1), we have

† Note that the matrix representation of the Lie algebra of U(1) is in terms of 1×1 matrices; this is because U(1) is a one dimensional Lie group.

$$C_1 = -\frac{1}{2\Pi}\left[\int_{R_a'} d\Lambda_a' + \int_{R_b'} d\Lambda_b'\right]$$

Using Stoke's theorem this becomes

$$C_1 = -\frac{1}{2\Pi}\int_{S^1}(\Lambda_{R_b'} - \Lambda_{R_a'})$$

where the negative sign occurs because the boundary of R_a' ; $\partial R_a' = S^1$ has the opposite orientation of $\partial R_b' = S^1$.

Using eq. (7.12)

$$C_1 = -\frac{1}{2\Pi}\int_{S^1} n d\phi = -\frac{1}{2\Pi} n(2\Pi) = -n$$

i.e. $C_1 = -n$ (7.16)

Since the monopole bundles are characterized by the transition functions, (given the base space S^2 and the structure group $U(1)$)†, the integral of the first Chern class or the monopole number n in the Dirac quantization condition are actually classifying parameters for the monopole bundles. This is further ramified by the technique of homotopic classification of bundles (Appendix B):

It turns out that the number of inequivalent coordinate bundles $P(S^2, U(1))$ is specified by the homotopy classes of maps[1][4][17]

$h: S^1 \to U(1)$

i.e. the set $\Pi_1(U_1)$, whence $\Pi_1(U(1)) \simeq Z$. (Appendix B)

† Refer to the last foot note.

8. THE EQUATIONS OF MOTION & THE BOUNDARY CONDITIONS

8.1 THE YANG-MILLS FIELD EQUATIONS

The equations of motion for the Yang-Mills potentials are derivable as Euler-Lagrange partial differential equations from an associated action functional $S(A)$, defined by

$$S(A) = \tfrac{1}{2}\int_{R^4} \mathrm{Tr}\left[F_{\mu\nu}F^{\mu\nu}\right] dx^4 \tag{8.1}$$

$S(A)$ can be shown to be a gauge invariant quantity, i.e., for any two gauge potentials A and A' related by eq. (4.5), we have

$$S(A) = S(A').$$

Varying A_μ, we obtain the Yang-Mills field equations[10]

$$\frac{\delta S(A)}{\delta A^\nu} = (\partial^\mu F_{\mu\nu} + \left[A^\mu, F_{\mu\nu}\right] = 0 \tag{8.2}$$

which after introducing the covariant derivative operator

$$D_\mu = \partial_\mu + \left[A_\mu, *\right] ,$$

reads as

$$D^\mu F_{\mu\nu} = 0 \tag{8.3}$$

In addition to eq. 8.3, any smooth potential satisfies the Bianchi identities[1][10], or equivalently the self-dual equations of motion

$$D^\mu {*F}_{\mu\nu} = 0 \tag{8.4}$$

where ${*F}_{\mu\nu} = \tfrac{1}{2}\, \varepsilon_{\mu\nu\rho\sigma} F^{\rho\sigma}$.

Eqs. (8.3) and (8.4) have explicit invariance and covariance properties.

In particular, if the field F satisfies one of the equations

a) $*F = F$ (self-duality)

b) $*F = -F$ (anti-self-duality) (8.5)

then the field equations (eq. (8.3)) are automatic consequences of the Bianchi identities (eq. (8.4)).

N.B.: Since "*" is only defined on an oriented manifold, (8.5a) and (8.5b) correspond to two opposite orientations, and there is essentially no difference btween them.

8.2 ASYMPTOTIC CONDITIONS AND GLOBAL PROPERTIES OF POTENTIALS

The action functional S(A) and the gauge potentials are defined on R^4. Any principal fiber bundle with a contractible base manifold such as R^4 is necessarily trivial[26]. What then is the relevance of non-trivial bundles to Yang-Mills gauge theories?

The non-trivial and global character of gauge field problems arises from the asymptotic conditions imposed on the gauge potentials and fields. In fact we shall see how out of asymptotic data we can construct a possibly non-trivial fiber bundle whose connection and curvature satisfy the asymptotic condition implied by the data.

The physically relevant potentials are restricted to that set of potentials for which S(A) is finite and minimal:

$$S(A) = \tfrac{1}{2}\int \mathrm{Tr} \left| \left[\partial_\mu A_\nu - \partial_\nu A_\mu + \left[A_\mu, A_\nu\right]\right]\right|^2 < \infty$$

Otherwise, the space of connection forms on $P=R^4 \times G$ is a convex space with trivial topology[1] viz: if w_0 and w_1

are connection forms on P, then the linear combination

$$w(t) = (1-t)w_0+tw_1$$

is also a connection form on P. The finiteness of the action makes the topology of the connections non-trivial{1}{10}. To ensure that S(A) be finite we assume that the field $F_{\mu\nu}$ falls off at infinity in R^4 sufficiently fast as $|X| \to\infty$. Because of gauge freedom, the corresponding assymptotic condition for the potentials reads as follows:

As $|X|\to\infty$, we can find a gauge transformation $g(x)$ such that the potential in the new gauge should decay, i.e.

$$A_\mu(x) \sim g^{-1}(x)\partial_\mu g(x) \quad † \tag{8.6}$$

as $|X|\to\infty$.

<u>N.B.</u>: Conventionally one describes the above condition as the asymptotic flatness of the connection, since a choice like $A_\mu(x) = g^{-1}(x)\partial_\mu g(x)$ reduces the equation

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + [A_\mu, A_\nu] = 0$$

to an identity.

Moreover, $g(x)$ need be defined only outside a finite sphere $S^3(\rho_O)$ of radius ρ_O i.e. in a subregion of R^4 that is topologically like $S^3\times R$.

Let $g_\rho(x):S^3(\rho)\to G$ be the restriction of $g(x)$ to a sphere of radius $\rho(\rho\geq\rho_O)$. Since $g(x)$ and hence $g_\rho(x)$ are continuous, any change in ρ amounts to a continuous "deformation" of g_ρ. (Appendix B) Hence $g_\rho(x)$ determines an element of a third homotopy group $\Pi_3(G)$ (Appendix B) that is independent

† "~" denotes asymptotic behaviour including first derivatives.

of ρ, i.e. the homotopy class of g_ρ is attributable to $\Lambda(x)$ and $g(x)$.

The effect of gauge transformation of $\Lambda(x)$ on the homotopy class it determines can be simply established:

For arbitrarily large but fixed radius ρ,

$$A_\mu(x) = g^{-1}(x)\partial_\mu g(x) + \beta_\mu(x) \dagger \qquad (8.7)$$

where $\beta_\mu(x)$ vanishes at large distances. Another section $\sigma'(x)$ of $\mathfrak{p}$ given by $\sigma'(x) = \sigma(x).\gamma(x)\dagger\dagger$, $\gamma(x)\varepsilon G$ determines a "new" gauge field component given by[1]

$$A_\mu'(x) = (g\gamma)^{-1}\partial_\mu(g\gamma) + \gamma^{-1}\beta_\mu(x)\gamma . \qquad (8.8)$$

Since $\gamma^{-1}\beta_\mu(x)\gamma$ also vanishes asymptotically (as $\beta_\mu(x)$ vanishes at large distances) and $g\gamma$ is continuous, we conclude that an asymptotically flat gauge potential and all other potentials related to it by a gauge transformation determine the same element of $\Pi_3(G)$.

Now, for a compact and semi-simple Lie group G, $\Pi_3(G) \sim Z$ (Appendix B), in which case we have succeeded in slicing the set of asymptotically flat gauge potentials into sectors each labelled by an integer.

In discussing the behaviour of the potentials vanishing at infinity, it is natural to describe them on the set

$$V_\infty = \{X \,\big|\, |X|>\tfrac{1}{\varepsilon}\}U\{\infty\}.$$

† This is a large distance approximation, otherwise there is not any canonical way of accomplishing this split.

†† $\sigma^*(w) = A$.

We understand ∞ as a "point" such that $\infty \notin R^4$, which however has open neighborhoods of the form $\{X \mid |X| \geq \frac{1}{\eta}\} \subset R^4$; η being an arbitrary positive infinitesimal [1].

Imposing a differentiable structure, $R^4 \cup \{\infty\}$ becomes a compact space which is equivalent to S^4[1]. Alternatively, we can say that S^4 is the conformal compactification of R^4 which is obtained by adding a point at infinity. Moreoever, one can construct a principal fiber bundle $P = (S^4, G)$ over S^4, using $g(x) \in G$ as a transition function on the overlap $V_\infty \cap R^4 = \{X \mid |X| > \frac{1}{\varepsilon}\}$. P is reconstructed from the coordinate bundle

$$\left(S^4, G, \{R^4, V_\infty\}, g(x)\right).$$

It turns out that the above coordinate bundle, whose construction is based on the asymptotic data, yields representatives for every one of the equivalence classes of G-bundles over S^4.[1]

On closed manifolds like S^4, there are wellknown theorems of global differential geometry which establish relationships between the characteristic topological invariants of each bundle $P(S^4, G)$ and integral expressions derived from the corresponding curvature. In particular, the bundles $P(S^4, SU(n))$ are characterized by the integral of the second Chern class, c_2, i.e. the second Chern number C_2, (Appendix C)† given by[1][10]

† $C_i=0$ for $i>2$ and also $c_1=0$ (Appendix C)

$$C_2 = \int_{S^4} c_2 \cdot dM$$

$$= \frac{1}{8\Pi^2} \int_{S^4} \mathrm{Tr}(F_s \wedge F_s).$$

where

$$F_s = \tfrac{1}{2} F_{\mu\nu} dx^\mu \wedge dx^\nu$$

In terms of $F_{\mu\nu}$, C_2 is given by

$$C_2 = \frac{1}{32\Pi^2} \int_{S^4} \mathrm{Tr}(F_{\mu\nu} F_{\alpha\beta}) dx^\mu \wedge dx^\nu \wedge dx^\alpha \wedge dx^\beta$$

$$= \frac{1}{32\Pi^2} \int_{S^4} \mathrm{Tr}(F_{\mu\nu} F_{\alpha\beta}) \varepsilon^{\mu\nu\alpha\beta} dx^1 \wedge dx^2 \wedge dx^3 \wedge dx^4$$

$$= \frac{1}{32\Pi^2} \int_{R^4} \mathrm{Tr}(F_{\mu\nu} F_{\alpha\beta}) \varepsilon^{\mu\nu\alpha\beta} \sqrt{|g|}\; d^4x$$

equivalently,

$$C_2 = \frac{1}{16\Pi^2} \int_{R^4} \mathrm{Tr}(F_{\mu\nu} \tilde{F}^{\mu\nu}) d^4x \qquad (8.9)$$

where

$$\tilde{F}^{\mu\nu} = \tfrac{1}{2} \varepsilon^{\mu\nu\alpha\beta} \sqrt{|g|}\; F_{\alpha\beta}$$

The quantity C_2 is integral valued,{1}{26}{27} and is usually referred to in the physics literature as the topological quantum number, or instanton number.

It is remarkable that the instanton numbers $C_2 \varepsilon Z$ coincide with the "winding numbers" $(\sim \Pi_3(SU(n)))$ associated with the equivalence classes of gauge inequivalent potentials which make the Euclidean Yang-Mills action converge by virtue of their asymptotic flatness property.{1}{4}{11}{29}

Writing $S(A)$ in the form[30]

$$S(A) = \frac{1}{4}\int_{R^4} d^4x \mathrm{Tr}\left[(F\pm{}^*F)^2 \pm 2F_{\mu\nu}{}^*F_{\mu\nu}\right]$$

and using eq. (8.9), we see that $S(A)$ is bounded below in each equivalence class of gauge-related asymptotic potentials by $8\Pi^2|C_2|$, when the self-duality or anti-self-duality condition

$$F_{\mu\nu} = \pm{}^*F_{\mu\nu} \tag{8.10}$$

is satisfied.

Finding all the gauge potentials which satisfy eq. (8.10) (i.e. instantons) for a given $C_2 \varepsilon Z$ and gauge group G, has been one of the most important problems in Yang-Mills theory at the classical level. The quantization scheme for non-abelian gauge theories is still in its infancy. Without any doubt a deeper understanding at the classical level will be invaluable when one tries to quantize such a theory.

APPENDIX A

THE DIRAC STRING SINGULARITY [13][20][21]

On empirical grounds, electromagnetic phenomena depend solely upon the dynamics of electric charges, and so far no magnetic counterpart to electric charge has been discovered. However, adopting a theoretical viewpoint, this does not account for the remarkable symmetry between electric and magnetic quantities shown by Maxwell's fundamental equations in the presence of electric and "magnetic charge currents" j_μ and g_μ respectively

$$\partial^\mu F_{\mu\nu} = j_\nu \tag{A.1}$$

$$\partial^\mu {*F}_{\mu\nu} = -g_\nu \tag{A.2}$$

where

$$F^{\mu\nu} = \begin{bmatrix} 0 & -E_x & -E_y & -E_z \\ E_x & 0 & -B_z & B_y \\ E_y & B_z & 0 & -B_x \\ E_z & -B_y & B_x & 0 \end{bmatrix} \tag{A.3}$$

and

$${*F}^{\mu\nu} = \tfrac{1}{2}\varepsilon^{\mu\nu\alpha\beta}F_{\alpha\beta} = \begin{bmatrix} 0 & B_x & B_y & B_z \\ -B_x & 0 & -E_z & E_y \\ -B_y & E_z & 0 & -E_x \\ -B_z & -E_y & E_x & 0 \end{bmatrix} \tag{A.4}$$

The equations (A.1) and (A.2) are symmetric under the interchange of electric and magnetic quantities

$\underline{E} \leftrightarrow \underline{B}$ $\quad j_\mu \leftrightarrow g_\mu$.

Without the artifice of magnetic charge (monopole) current, Maxwell's equations do not exhibit such a symmetry†.

It is because eq. (A.1) & (A.2) are aesthetically appealing that monopoles became worthy of study (Dirac 1931){21}.

Introducing a vector potential via

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \tag{A.5}$$

we see that

$$\partial^\mu {*F_{\mu\nu}} = -g_\nu \iff \partial^\mu \tfrac{1}{2}\varepsilon_{\mu\nu\alpha\beta}(\partial^\alpha A^\beta - \partial^\beta A^\alpha) = -g_\nu$$

$$\iff g_\nu = \varepsilon_{\nu\mu\alpha\beta}\partial^\mu \partial^\alpha A^\beta$$

Thus, for $g_\nu \neq 0$,

$$\partial_\mu \partial_\alpha A_\beta \neq \partial_\alpha \partial_\mu A_\beta \tag{A.6}$$

and hence the vector potential A_β cannot be regular††. In particular let us consider an outward succession of simply connected surfaces surrounding the monopole, with A_β singular at one point on each of these surfaces.

One can then imagine the locus of singular points to be a continuous line (string) from the monopole to infinity. A simple example of such a potential is

$$A_o = 0 \;;\; \underline{A} = -\frac{\hat{e}_\phi}{4\pi} g \sin\theta /_{\gamma(1-\cos\theta)} \tag{A.7}$$

(A.7) describes a stationary monopole at the origin, since $\underline{\nabla}\times\underline{A} = {}^{g\underline{r}}/_{4\pi\gamma^3}$. The corresponding string lies along the positive Z-axis.

† This symmetry is usually called duality symmetry.

†† If A_β is regular, then the partial derivatives ∂_α & ∂_β commute {9}.

The Dirac string can be visualized as an infinitely long, thin solenoid. Lines of magnetic flux diverge in all directions (except through the singularity) from the monopole and return back from infinity through the string (Fig. A.1).

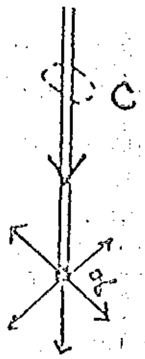


Fig. (A.1)

The string singularity however has no physical meaning, and in fact, one can shift the string to any position by an appropriate gauge transformation of the singular potential. Therefore any particle should not manifest unusual behaviour near the singularity, and when the particle is faraway from the monopole, its wave function ψ should satisfy the wave equation for a free particle. Furthermore ψ's phase should change by an integral multiple of 2Π when a small closed loop is described about the string.

Let

$$\psi = \phi \exp(i\beta) \tag{A.8}$$

ϕ being a function of definite phase at everypoint. Assuming that the particle is a free particle, it can easily be shown that [20] ϕ satisfies the wave equation for a particle in a potential given by

$$e\underline{A} = \underline{\nabla}\beta \tag{A.9}$$

For a non trivial situation, β will be non-integrable near a string[20][14]: around a small loop "C" enclosing the string (see Fig. (A.1))† the change in β is

$$\oint_C d\beta = \oint \underline{\nabla}\beta.\underline{dc} = e\oint \underline{A}.\underline{dc}.$$

By Stoke's theorem,

$$e\oint \underline{A}.\underline{dc} = e\int_S \underline{ds}.\underline{\nabla}\times\underline{A}, \text{ where}$$

S is an arbitrary surface bounded by the closed contour C. Thus

$$\oint d\beta = e\int_S \underline{ds}.\underline{\nabla}\times\underline{A} = e\int_S \underline{ds}.\underline{B} = e\Phi$$

where Φ is the magnetic flux through S.

Since the string carries all the inward flux (which balances the outward flux) we have, $\Phi = 4\Pi g$††, i.e.;

$$e\Phi = 4\Pi eg = 2\Pi\eta, \quad \eta\varepsilon Z$$

Thus we have the celebrated Dirac quantisation condition

$$2eg = \eta \tag{A.10}$$

† The size of the loop C in Fig. (A.1) is exaggerated, and the flux "trapped" by C is only that through the string.

†† The fact that the outward flux $\Phi = 4\Pi g$ follows from the inverse square law for the monopole field.

APPENDIX B

B.1 HOMOTOPY CLASSES [4][23][17]

Here we briefly discuss the notion of homotopic classification of maps. This is important in gauge theory as it turns out that gauges belonging to the same homotopy class are characterized by the same integer valued index. This index is known to be the integral of the highest characteristic class of $F_{\mu\nu}$ on S^4.

Two continuous maps $f_0:X\to Y$, $f_1:X\to Y$, X and Y being topological vector spaces are said to be homotopic (written $f_0 \sim f_1$) if they are continuously deformable into each other, i.e. $\exists$ a continuous map

$F:X\times I\to Y$, $(I \equiv [0,1])$ such that

$F(x,0) = f_0(x)$, $F(x,1) = f_1(x)$; $\forall x \varepsilon X$.

The mapping $F(x,t)$ which defines a continuous interpolation between f_0 and f_1 is called a homotopy. This defines an equivalence relation between homotpic maps. Such a class is called a homotopy class, and the set of all homotopy classes of maps from X to Y is denoted by $[X,Y]$.

The reflexivity and symmetry of homotopy equivalence are quite obvious; transitivity is established as follows:

Let f & $g:X\to Y$ be homotopic with F as a homotopy, Similarly Let g & $h:X\to Y$ be homotopic with G as a homotopy,

then f & $h:X\to Y$ are homotpic with the composite homotopy $H \equiv F.G$, where

$$H(x,t) = \begin{cases} F(x,2t), & 0 \le t \le \frac{1}{2} \\ G(x,2t-1); & \frac{1}{2} \le t \le 1 \end{cases}$$

Two topological spaces X & Y are said to be of the same homotopy type if there are two continuous maps $f:X\to Y$ and $g:Y\to X$ such that $g\circ f \sim Id_X$, $f\circ g \sim Id_Y$. This also defines an equivalence relation.

A space is said to be contractible if it is homotopic to a point. Equivalently, a space is contractible if $Id:X\to X$ is homotopic to a constant map $K_0:X\to x_0$, $x_0 \in X$. It can be shown that Euclidean space R^n is contractible, but S^n is not for any n.

Of particular interest is the homotopic classification of curves like $f:[0,1]\to Y$ in a topological space Y. Here, the number of homotopy classes in Y describes one of its topological properties. If Y is connected, a single homotopy class means that Y is simply connected, two classes means doubly connected etc.

<u>Example</u>: Let D be a region with a hole in it Fig. (B.1.a) i.e. D is not simply connected. Closed loops $f:S^1\to D$ with a fixed base point N fall into homotopy classes $[S^1,D]$, each class being designated by the number of times it winds around the hole. (or by the "winding number", negative "winding number"corresponding to winding in the opposite direction, and zero to a path which does not wind around the hole.) The homotopy classes constitute an obvious group structure corresponding to loop composition. They define the "1st homotopy group" $\Pi_1(D)$ of the region. Clearly we have $\Pi_1(D) \simeq Z$.

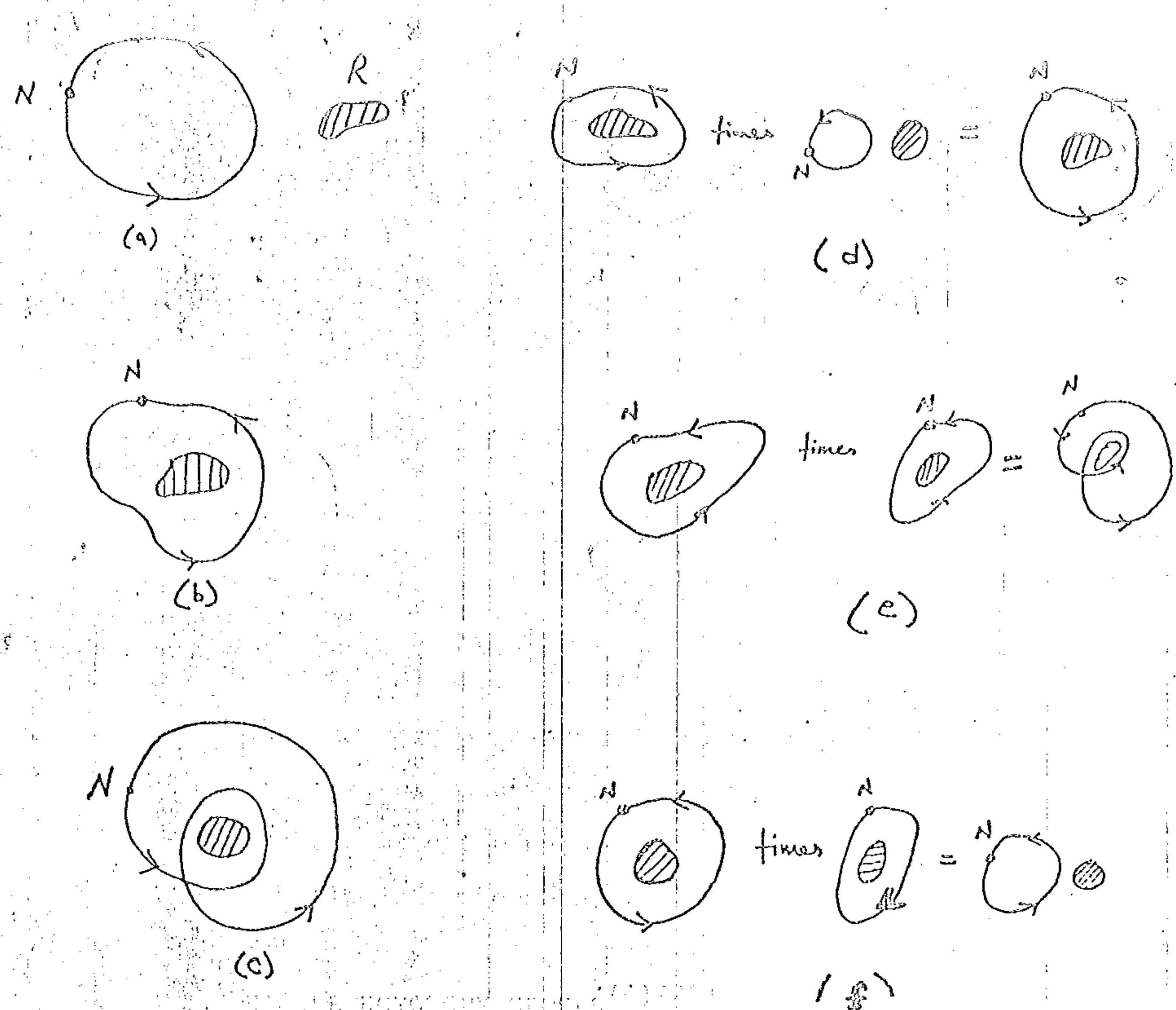


Fig. B.1[20]. a-c Represent mappings belonging to the elements of $\Pi_1(D)$ with winding number 0,1,2 respectively. d-f Instances of group multiplication. The shaded region R is the hole in the domain D.

B.2 HIGHER HOMOTOPY GROUPS[4][20]

Though it is not easy to understand intuitively the homotopy groups for $n>1$, their definition is quite similar to that of $\Pi_1(X)$.

The n-th homotopy group $\Pi_n(X)$ of a topological space X is just $[S^n, X]$, the homotopy classes of maps $f: S^n \to X$. $\forall n \geq 2$, $\Pi_n(X)$ is known to be abelian[23].

It is generally a very non-trivial exercise to calculate the higher homotopy groups. We quote the following homotopy groups of relevant manifolds and classical groups;[20][23]

$$\Pi_1(u_1) = \Pi_1(S^1) = Z \quad , \quad u_1 \sim S^1$$

$$\Pi_1(u_1 \times u_1) = \Pi_1(u_1) \times \Pi(u_1) = Z \times Z$$

$$\Pi_1(Su_2) = \Pi_1(S^3) = 0 \quad ; \; Su(2) \sim S^3$$

$$\Pi_1(u_2) = \Pi_1(S^3 \times S^1) = \Pi_1(S^3) \times \Pi_1(S^1) = Z$$

$$u_2 = S^3 \times S^1$$

For general higher homotopy groups

$$\Pi_n(A \times B) = \Pi_n(A) \times \Pi_n(B)$$

$$\Pi_n(S^m) = 0 \quad , \quad n < m$$

$$\Pi_n(S^n) = Z \quad , \quad \Pi_n(S^1) = 0 \quad , \quad n > 1$$

$$\Pi_2(S^n) = 0 \quad , \quad n \neq 2$$

$$\Pi_3(S^n) = 0 \quad , \quad n \neq 2,3$$

$$\Pi_3(S^2) = \Pi_3(S^3) = Z$$

In Yang-Mills theories the topological spaces of interest are non-abelian group manifolds G. The base point "N" in G is always chosen to be the identity element of the gauge group. We have the following useful results:[20]

$$\Pi_1\big(SO(n)\big) = Z_2 \quad , \quad n > 2$$

$$\Pi_1(U_n) = Z \quad , \quad n \geq 1$$

$$\Pi_1\big(SU(n)\big) = 0 \quad , \quad n \geq 2$$

$$\Pi_2(G) = 0 \quad , \quad G = \text{any Lie group}$$

$$\Pi_3(G) = Z \quad , \quad G = \text{any simple Lie group}$$

$$\Pi_3\big(SO(4)\big) = Z \times Z_2$$

$$\Pi_3\big(SO(n)\big) = Z \quad , \quad n > 4$$

$$\Pi_n(u_1) = 0 \quad , \quad n > 1$$

$$\Pi_n(u_n) = \begin{array}{ll} 0 & n \text{ even} \\ Z & n \text{ odd} \end{array}$$

APPENDIX C

THE CHERN-WEIL THEORY[1][10]

C.1 INTRODUCTION

Let us denote the real vector space of p-forms on M having vanishing exterior derivative, by $C^p(M,R)$. Such forms are said to be closed on M. Similarly, let $B^p(M,R)$ denote the real vector space of p-forms on M which can be expressed as exterior derivatives of (p-1) forms on M. These forms are known as exact forms. Using the fact that $d^2 = 0$, it is easy to see that $B^p(M,R)$ is a subspace of $C^p(M,R)$. Moreoever, we can form the quotient space

$$H^p(M,R) = C^p(M,R)/B^p(M,R)$$

$H^p(M,R)$ is thus the real vector space of equivalence classes of closed p-forms, two p-forms being equivalent if and only if they differ by an exact form. The elements of $H^p(M,R)$ are called the p-th cohomology classes of M.

If we set

$$H^*(M,R) \equiv H^0(M,R) \oplus H^1(M,R) \oplus ----,$$

then, the real cohomology classes, i.e. elements in $H^*(M,R)$ can be represented by closed forms on M. In terms of the connections defined on a given principal fiber bundle p(M,G), one can construct (using the Chern-Weil homomorphism) a set of closed forms whose cohomology classes are independent of the choice of the connection on p but are characteristic of p itself.

C.2 INVARIANT POLYNOMIALS

Let the structure group G of the principal fiber bundle p(M,G) be a closed subgroup of GL(n,c) †. Similarly let the lie algebra Γ of G be represented by a space of n×n complex hermitian matrices.

A function of $p^k: \Gamma \to R$ is a polynomial of degree k if, with respect to any basis of Γ, p^k is a polynomial function of degree k in the basis elements.

p^k is said to be an Ad_g invariant polynomial, or characteristic polynomial, if

$$p^k(A) = p^k(gAg^{-1}) \quad \forall g\varepsilon G \text{ and } \forall A\varepsilon\Gamma$$

If $\{\lambda_1, \lambda_2, \ldots \lambda_k\}$ are the eigenvalues of $A\varepsilon\ \Gamma$, then it turns out that $p^k(A)$ can be expressed as a symmetric function of the eigen values[26]: If $S_j(\lambda)$ is the j-th symmetric function given by

$$S_j(\lambda) = \sum_{i_1<i_2\ldots<i_j} \lambda_{i_1}\lambda_{i_2}\ldots\lambda_{i_j} \tag{C.1}$$

then $p^k(A)$ is a polynomial in the $S_j(\lambda)$:

$$p^k(A) = a + bS_1(\lambda) + CS_2(\lambda) + d\left[S_1(\lambda)\right]^2 + \ldots \tag{C.2}$$

Example

$$p^k(A) \equiv \mathrm{Det}(I+A) = \prod_{i=1}^{k} (1+\lambda_i)$$

$$= 1+S_1(\lambda) + S_2(\lambda) + \ldots S_k(\lambda) \tag{C.3}$$

† So $g\varepsilon G$ is represented by a unique n×n complex matrix.

Let $p(M,G)$ be a principal fiber bundle with connection Λ and curvature Ω. If the Lie algebra valued (and hence matrix valued) 2-form Ω is taken as argument for p^k, we have the following two remarkable results[1][10]:

i) For any invariant polynomial p^k, the 2k-form $p^k(\Omega)$ on p projects to a unique closed 2k-form $\bar{p}^k$ on M such that $p^k(\Omega) = \Pi^*(\bar{p}^k)$, Π being the projection defined in $p(M,G)$.

ii) The closed 2k-form $\bar{p}^k$ determines a unique de Rahm cohomology class in $H^{2k}(M,R)$ which is independent of the choice of the connection on p. This gives a unique map

$W: p(\Gamma) \to H^*(M,R)$† called the Chern-Weil homomorphism.

C.3 THE CHERN CLASSES OF p(M,G)

The total chern class c of the principal fiber bundle $p(M,G)$ is defined by[1][26]

$$c = \mathrm{Det}\left(I - \frac{X}{2\Pi i}\right) = \sum_{k=0}^{n} f_k(X), \quad X \varepsilon \Gamma \tag{C.4}$$

It is easy to see that†† the $f_k(X)$ are invariant polynomials of degree k defined on the Lie algebra. Furthermore, for any k, there exists a unique closed 2k-form c_k on M called the k-th Chern class of p such that

$\Pi^* c_k = f_k(\Omega)$.[1]

<u>N.B.</u>: If $n = \dim(M)$, $c_k = 0$ for $2k > n$, i.e. we have a finite number of Chern classes of P.

† $p(\Gamma)$ denotes the ring of invariant polynomials.
†† Compare (C.3) and (C.4).

The following theorem enables us to write the Chern classes directly in terms of the curvature.

Theorem (C.1)[1]: We can express c_k in terms of matrix-valued 2-form $\Omega^i{}_j$ as

$$\Pi^*(c_k) = \frac{(-1)^k}{(2i\Pi)^k . k!} \sum \varepsilon^{j_1 \ldots j_k}_{i_1 \ldots i_k} \, \Omega^{i_1}{}_{j_1} \wedge \ldots \wedge \Omega^{i_k}{}_{j_k} \qquad \text{(C.4)}$$

the summation being over all the ordered subsets $i_1 \ldots i_k$ of k elements in $\{1, \ldots n\}$, and all permutations $\begin{pmatrix} j_1 \ldots j_k \\ i_1 \ldots i_k \end{pmatrix}$.

N.B.: $\Pi^*(c_k) = f_k(\Omega)$. Since $f_k(g^{-1}\Omega g) = f_k(\Omega)$, the $2k$-forms c_k are gauge invariant quantities.

If V_α is a neighborhood in M,

$$\Pi^*\left(c_k \Big|_{V_\alpha}\right) = f_k\left(\Omega\Big|_{\Pi^{-1}(V_\alpha)}\right)$$

$$\Omega\Big|_{\Pi^{-1}(V_\alpha)} = Ad_{\phi_\alpha^{-1}}(\Pi^*\Omega_\alpha)$$

where $(\Omega_\alpha)^i_j = \frac{1}{2} F^a_{\mu\nu} (T_a)^i_j \, dx^\mu \wedge dx^\nu \, (T_a \varepsilon \Gamma\ \ )$

Hence, it follows from (C.4) that

$$c_k\Big|_{V_\alpha} = \frac{(-1)^k}{(2i\Pi)^k . k!} \sum \varepsilon^{j_1 \ldots j_k}_{i_1 \ldots i_k} (\Omega_\alpha)^{i_1}_{j_1} \wedge \ldots \wedge \Omega^{i_k}_{j_k} \ .$$

In particular, if the base manifold $M = S^4$, we have

$$c_1\Big|_\alpha = -{}^1\!/_{2\Pi i} \, Tr\Omega_\alpha$$

which vanishes if $G = Su(n)$.

$$c_2\Big|_\alpha = \frac{1}{8\Pi^2} \, Tr(\Omega_\alpha \wedge \Omega_\alpha)$$

$$c_k\Big|_\alpha = 0 \quad \forall \ k \geq 3$$

The second Chern class c_2 characterizes $p(S^4, Su(n))$. Furthermore the integral

$$C_2 = \int_{S^4} c_2 \cdot dM$$

is integral valued{1} and can be shown to be the winding number associated with the connection in $p(S^4, G)$ {1}{29}

$C_2 \varepsilon Z$ is usually referred as the Chern number or instanton number.

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