ADDIS ABABA UNIVERSITY

SCHOOL OF GRADUATE STUDIES

Pattern Avoiding Permutations

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Declaration Letter

I, Chakiso Abuto, declare that this project has been composed by me and that no part of the project has formed the basis for the award of any Degree, Diploma, Associate ship, Fellowship or any other similar title to me.

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Permission Letter

This is to certify that this project is compiled by Mr. **Chakiso Abuto** in the department of Mathematics, College of Mathematics and Computational Sciences, Addis Ababa University, under my supervision.

Melkamu Zeleke (Prof.)

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Table of Contents

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By: Chakiso Abuto

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List of frequently used Notation

$n!$	n (n-1)$\cdot\cdot\cdot$ 1
$\binom{n}{k}$	$\frac{n!}{k!(n-k)!}$
$[n]$	$\{1,2,\cdots,n\}$
S_n	set of all n-permutations
$S_n(q)$	number of n-permutations avoiding the pattern q
p	permutation
p^r	reverse permutation
p^c	permutation complement
p^{rc}	reverse permutation complement
p^{-1}	inverse permutation
$A_p(i,j)$	permutation matrix, where *(i,j)*-entry of a permutation

List of figure

Figures **page**

Summary of the Project

We present in this report the methods that provide better upper bounds for the number $S_n(q)$ of permutations of length n avoiding the pattern q.

We consider permutations that avoid the pattern q of length three, four and monotone patterns only. We use Milkós Bóna result to show that $S_n(q) < 288^n$. Similarly, We obtain the generating function H(x) of all 1342-avoiding permutations of length n as well as an exact formula for their number $S_n(1342)$.While achieving this, we bijectively prove that the number of indecomposable 1342-avoiding permutations of length n equals that of labeled plane trees $\beta(0,1)$ of a certain type on n vertices recently enumerated by Cori, Jacquard and Schaeffer, which is in turn known to be equal to the number of rooted bicubic maps enumerated by Tutte in 1963.

Finally, we will see that the Stanley–Wilf conjecture which states that: $S_n(q) \leq c_q^n$ for all positive integers n, where c_q is a constant.

Chapter-One

1. Introduction

1.1 Preliminaries

This project is mainly focuses on pattern avoidance, when the pattern is a permutation of length two, three and four, that is, a permutation p of a set S is a linear order of the finite elements of the set. The collection of permutations of the set [n] will be denoted S_n.

One of the most important breakthroughs in the subject of pattern-avoiding permutations has been the proof by Marcus and Tardos [8] of the so-called Stanley-Wilf conjecture, which had been open for over a decade. This is a basic result regarding the asymptotic behavior of the number of permutations that avoid a given pattern. It states that for any pattern q there exists a constant c_q such that, if $S_n(q)$ denotes the number of q-avoiding permutations of size n, then $S_n(q) \le c_q^n$. The notion of pattern avoidance that this result is concerned with is the standard one, namely, where a permutation is said to avoid a pattern if it does not contain any subsequence which is order-isomorphic to it.

For some particular patterns we obtain the same asymptotic behavior, and for patterns of length 3 the problem is solved as well, that is, the number of $S_n(q)$ n-permutations that avoid the pattern q of length three is the n^{th} *Catalan number*. For the monotone pattern it is not that much complicated as other patterns.

For some patterns q of length 4 we will give asymptotic upper and lower bounds on $S_n(q)$ to bound this patterns we pass through ups and downs. For patterns of length four we have 24 different possible choices are there, but by using symmetries, that is, *reverse, complement, and inverse operation,* We can obtain the representative of these patterns that are 1234, 1324, and1342. In this report we will try to obtain an exponential upper bound for n-permutations that avoid this patterns, meaning that, $S_n(1234) < 9^n$, $S_n(1324) < 288^n$ and

$S_n(1342) < 8^n$,some of these can be doing by classifying permutations and defining it. The other is obtained by forming a bijection between combinatorial objects like rooted plane trees with n vertices satisfying certain conditions, that is,$\beta(0,1)$-tree with indecomposable 1342-avoiding permutation.

Finally, this report ends with states two related conjectures concerning more popular areas of pattern avoidance. To state the conjectures we define the term "avoiding" in several contexts.

- ✓ **Permutation matrix**

A (0, 1)-matrix A is called a permutation matrix if every row of A and every column of A has a single 1-entry. Note that there is a bijection between the permutations in S_n and the n × n permutation matrices, defined by

$$A_p(i\,,j) = \begin{cases} 1 & if\ p_i = j \\ 0 & otherwise \end{cases} \quad \text{, for all } p \in S_n$$

.

- ✓ **Permutation avoidance(as function f)**

Given p∈ S_n and q∈ S_k, we will say that p contains q if there is an order preserving function *f: [k] → [n]* such that $q_i < q_j$ if and only if $p_{f(i)} < p_{f(j)}$. If, on the other hand, no such *f* exists, then we will say that p ***avoids*** q.

For example, let p = (<u>2</u>, 1, <u>4</u>, 3, <u>5</u>) and q = (1, 2, 3). To see that p contains q, define f as f(1) = 2, f(2) = 4, f(3) = 5 (the underlined elements). Thus *f* is order preserving and satisfies the definition of containment. Otherwise p avoids q

- ✓ **(0, 1)-matrix avoidance**

Let A be an m × n (0, 1)-matrix and P =($p_{i,j}$) be a $k \times l$ (0, 1)-matrix. We say that A contains P if there exists a $k \times l$ sub matrix Q = ($Q_{i,j}$) of A such that $Q_{i,j}$= 1 whenever($p_{i,j}$) = 1. Otherwise we say that A avoids P. Notice that we can delete rows and columns of A and change 1-entries to 0-entries (to obtain P exactly) but we cannot permute the remaining rows and columns or change 0-entries to 1-entries. If A contains P we identify the 1-entries of the matrix A corresponding to the entries ($Q_{i,j}$) of Q with $p_{i,j}$= 1 and say that these entries of A represent P. For a (0, 1) matrix A let *f (n, A)* be the maximum number of 1-entries in an n × n (0, 1) matrix avoiding A.

One should notice that the three definitions of avoidance are intimately related. In particular, every permutation is a sequence and every sequence can be written as a (0, 1)-matrix (in an identical way that the bijection between permutations and permutation matrices was formed at the beginning of this section). The conjectures' are Füredi − Hajnal conjecture and Stanley – Wilf conjecture, the two arguments are the same. Thus, F$\ddot{u}redi - Hajnal\ conjecture$ implies Stanley – Wilf conjecture

This report is structured as follows: There are five chapters in this project. In chapter one, we shall introduce the History of pattern avoidance. In chapter two, we have two sections. In section one we introduce basic definitions of pattern avoidance. And section two we introduce notion of pattern avoidance.

. In chapter three, we introduce pattern avoiding permutation, this chapter has three sections. In section one we shall introduce patterns of length three and in detail analyze the symmetries of each pattern of length three. In section two we introduce monotone patterns. In Section three we will find the representative of patterns of length four, namely, 1234, 1324, and, 1342 and we will discuss in detail to obtain upper bound.

In chapter four, we introduce the proof Stanley – Wilf conjecture. This section has three subsections. In the first subsections we shall see F$\ddot{u}redi - Hajnal\ conjacture$. In the second subsection we shall see avoiding matrices vs. avoiding permutations. Finally, we proof F$\ddot{u}redi - Hajnal\ conjacture$.

The last chapter of this project is conclusion of the report and listing of avenues for further research.

1.2 History of pattern avoidance

Pattern avoidance was first studied for S_n, the set of permutations of $[n]$, avoiding a subsequence pattern $q \in S_3$. Kunth [9] found that the number of permutations of $[n]$ avoiding any $q \in S_3$ is given by the n^{th} Catalan number. Later, Simion and Schmidt [17] determined $|S_n(T)|$, the number of permutations of $[n]$ simultaneously avoiding any given set of subsequence patterns $T \in S_3$.

Pattern avoiding permutations have been subject to much attention since the pioneering work by Knuth's [9], where he used them for studying stack sortable permutations. For a thorough summary of the current status of research, see B$\acute{o}$nas book [1].

Research on avoiding (subsequence) permutation patterns has become an important area of study in enumerative Combinatorics, as evidenced by the publication of a special volume of [3]. This area of research has applications to computer science and algebraic geometry and has proved to be useful in a variety of seemingly unrelated topics, including the theory of Kazhdan-Luszting polynomials, Chebyshev polynomials and alike.

The earliest recorded attempts to find the asymptotic behavior of $S_n(q)$ date back to the 1990 Ph.D. thesis of Julian West [7] (though some claim the problem is even older [1]). He asks if $S_n(q)$ and $S_n(q')$ are asymptotically equal for k-permutations q and q'. Slightly afterward, Richard Stanley and Herbert Wilf offered the following somewhat more cautious conjecture:

Conjecture 1.1. For all permutations p there exists a constant c_q such that$S_n(q) \leq c_q^n$.

The conjecture, the Stanley–Wilf conjecture, quickly grew a strong following.

In 1997, Mikl$\acute{o}$s B$\acute{o}$na [2] showed that West's original conjecture was too strong by finding 4-permutations q and q' with $S_n(q)$ and $S_n(q')$ displaying different growth rates. However, little else had been established. Soon afterward, several special cases of Conjecture1.1 were shown to be true. Most notably was the result of B$\acute{o}$na in [5], where he proved the Stanley–Wilf conjecture for layered permutations p, that is, for permutations consisting of an arbitrary number of increasing blocks with all elements of a block smaller than the elements of the previous block. Finally, paper of Zoltan F$\ddot{u}$redi and P´eter Hajnal that began a systematic study of avoidance problems on (0, 1)-matrices. In [6], Klazar showed that the following question from [10], which he named the F$\ddot{u}$redi–Hajnal conjecture, implied the Stanley–Wilf conjectures.

Chapter-Two

2. The Basics of Pattern avoidance

2.1 Basic definitions

❖ **Permutation**

The type of combinatorial objects we concentrate on in this project is the permutations, which are linear orderings on a finite set of objects, for example 3412. Since it is not important what kind of objects we are permuting, we often use the set $\{1, 2, \ldots, n\}$ and with this set, the permutations are bijections, $\{1, 2, \ldots, n\} \rightarrow \{1, 2, \ldots, n\}$. The size of a permutation is the size of the set it acts on, and the number of permutations of size n is $n!$.

❖ **Left-to-Right Minimum**

An entry of a permutation which is smaller than all the entries that precede it is called a left to right minima. For example if a permutation p= 465312, then 4, 3, and 1 are left to right minimum of p.

❖ **Right -to- left maximum**

An entry of a permutation is right to left maximum if it is larger than all entries on its right.

❖ **Class**

Two n-permutations X and Y are said to be in the same class if the left-to-right minima of X are the same as those of Y and they are in the same positions.

Example 3.3.7**:** Permutations **X=51324** and **Y=51234** are in the same class,

But **Z=24<u>3</u>15** and **V=24<u>1</u>35** are not in the same class, since the third entry of Z is not

a left -to- right minima where as that of V.

- **Indecomposable permutation:**

A permutation p=$p_1p_2 \ldots . p_n$, i s i ndecomposable pe rmutation, If th ere e xists no k∈ $[n-1]$ *so that for all* $i \le k \le j$, *we have* $p_i > p_j$. In other words, p is indecomposable if it cannot be cut into two parts so that everything before the cut is larger than everything after the cut.

For instance, 3142 is indecomposable, but 43512 not, because we could choose k=3, that is, we could cut between the third and fourth entries. If a permutation is not indecomposable, then we will call it decomposable.

- **Permutation matrix**

Let p∈ S_n , with p=$p_1p_2 \ldots . p_n$. The permutation matrix of p is the $n \times n$ A_p defined by

$$A_p(i,j) = \begin{cases} 1 & \text{if } p_i = j \\ 0 & \text{other wise} \end{cases}$$

- **Wilf-equivalent**

Two sets of patterns, q and q', are Wilf-equivalent if $|S_n(q)| = |S_n(q')|$ for all $n \geq 0$. Wilf-equivalence divides the pattern sets into equivalence classes, which are called *Wilf-classes*. A large class of Wilf-equivalences can be found by using the symmetries of the permutations. If $p = p_1 \, p_2 \, \ldots . p_n$ is a permutation, we define the **reverse** of p to be $p^r = p_n \, p_{n-1} \ldots . p_2 p_1$ and the **complement** of p to be $p^c = (n+1-p_1)\,(n+1-p_2) \cdots (n+1-p_n)$. These operations correspond to reflecting the permutation matrix vertically and horizontally, respectively.

- **Bipartite graphs:**

Bipartite graph is a graph whose vertex set can be split into two sets in such a way that each edge of the graph joins a vertex in the first set to a vertex in the second set.

- **Subgraphs**

Let *G* be a graph with vertex set *V (G)* and edge set *E (G)*. A subgraph of G is a graph all of whose vertices belong to *V(G)* and all of whose edges belong to *E(G).*

2.2 The notion of pattern avoidance

The notion of pattern avoidance is the generalization from pairs of entries to k- tuples of entries. Let p be a permutation p= 25641387, and q= 132. Then the three underlined tuples entries (2, 6, 4) in p forms a pattern or subsequence of type 132 *because* the entries (2, 6, 4) of p relate to each other as the entries 132 of q do. That is the first one is the smallest, the middle one is the largest, and the last one is the medium size.

On the other hand, there is no pattern of type 12345 in p; therefore we say that p avoids 12345.

We can define pattern avoidance both using the word form and the matrix form of the permutation.

Definition2.2 (pattern avoidance)

Let $q = (q_1, q_2 \ldots q_k) \in S_k$ be a permutation, and let $k \leq n$. we say that the permutation $p = (p_1, p_2 \ldots p_n) \in S_n$ contains q as a pattern if there are k -entries $p_{i_1,} p_{i_2} \ldots p_{i_k}$ in p so that $i_1 < i_2 < \ldots < i_k$, *and* $p_{i_a,} < p_{i_b,}$ if and only if $q_a < q_b$. Otherwise we say that p avoids q.

Example 2.1: The permutation *p*= 3451267 avoids the pattern 321 as it does not contain a decreasing subsequence of length 3.

In other words, p contains q as a pattern if p has a subsequence of elements that relate to one another the same way the elements of q do.

Chapter Three

3. Permutations avoiding Patterns of length three and four

3.1 Patterns of length three

Finding the number of S_n(q) of n-permutations that avoid the pattern q of general length is too difficult. In our case we study only the pattern of length 3, 4 and the monotone pattern of length k and their results.

Avoiding pattern of length two is trivial, i.e. if q= {12, 21}, then $S_n(12) = S_n(21) = 1$.

The first non- trivial case is avoiding patterns of length three.

Theorem 3.1.1 (the case when k=3)

Let S_n(q) = the number of n –permutations that avoid the pattern q of length three. Then

$S_n(q) = C_n = \frac{\binom{2n}{n}}{n+1}$ n^{th} Catalan number for all $q \in S_3$, n is a positive integer.

Proof: the proof of this theorem can be broken in to three cases

Case I: $S_n\ (132) = S_n\ (231) = S_n\ (312) = S_n\ (213)$

Case II: $S_n\ (123) = S_n\ (132)$

Case III: $S_n\ (132) = C_n = \frac{\binom{2n}{n}}{n+1}$

Case I: To prove that, as q$\in S_3$ then, $S_3 = \{123,132,213,231,312,321\}$, these patterns have many symmetries between them. Namely reverse and complement.

If a permutation p=$p_1p_2 \cdots p_n$, then we define the reverse of p as the permutation p =$p_np_{n-1} \cdots p_1$.

And its complement of P as the permutation p^c whose first entry is n+1-p_1, whose second entry is n+1-p_1 and whose i^{th} entry is n+1-p_i

If a permutation avoids 123, then its reverse avoids 321, thus $S_n(123) = S_n(321)$

Similarly, if a permutation avoids 132, then its reverse avoids 231, thus $S_n(132) = S_n(231)$

If a permutation avoids 132, then its complement avoids 312, thus $S_n(132) = S_n(312)$ and the reverse of its complement avoids 213 thus $S_n(132) = S_n(213)$

Therefore: $S_n\ (132) = S_n\ (231) = S_n\ (312) = S_n\ (213)$

All these symmetries leave us with one last question that is

Case II: $S_n\ (123) = S_n\ (132)$

Proof: There are several ways to proof this first non trivial result of the subject R Simion and Schmidt,

Note that the left to right- minima form a deceasing subsequence.

We are going to construct a bijection *f* from the set of all 132-avoiding n-permutations to the set of all 123- avoiding n-permutations with leaves all left to right minimal fixed.

The map *f* is defined as follows:

Keep t he left to rig ht- minima of P f ixed a nd w rite (rearrange) the r emaining e ntries in deceasing order.

Call t he obt ained pe rmutation *f (p)* and i s a lways 123 -avoiding s ince i t i s the uni on of t wo deceasing s ubsequences, one of w hich i s the s equence of left to r ight- minima and the deceasing sequence into which we arranged the remaining entries

Example 3.1 If p=67341258, then the left to right- minima of P are 6, 3 and 1,

Therefore, *f (p)* = 68371542

Thus, *f (p)* =68371542

We claim that left to right- minima of *P* and *f (p)* are the same. Observe that to compute *f (p)*, we c an i magine s witching pa irs of non - left to rig ht- minima in *P* that ar e n ot i n deceasing order. Since this moves a smaller entry to the right and a larger entry to the left, *f* cannot create new left to right- minima (or destroy existing left to right- minima).

Note: *f (p)* is the only 123 avoiding permutation with the given set and permutation of left to right- minima. That means *f (p)* is the only 123 avoiding permutation since it is the union of two disjoint decreasing subsequences.

Indeed, if there were two entries x and y that are not left to right- minima and form 12 patterns, then the left to right-minimum z that is closest to x on the left, and the entries x and y would form an increasing sequence. So this is impossible.

Now we can prove that f is a bijection by showing that it has an inverse.

Let q be an n-permutation that avoids 123 patterns.

Keep the left to right- minima of q fixed, and fill in the remaining positions with the remaining entries, moving to left to right as follows:

At each step –place the smallest element not used yet placed which is larger than the closest left to right minima on the left of the given position and calls the obtained permutation *g (q)*.

Example 3.1.2 if q=68371542, then the left to right-minima of q are 6,3 and 1.

To find *g (q)* = 6 __3__1__ __ _ .To the empty slot between 6 and 3, put the smallest element not used which is the larger than 6, that is 7.To the empty slot between 3 and 1, we put the smallest entry not used yet that is larger than 1, that is 2. We finish this way, by placing 5 and 8 to the remaining slots.

We get *g (q)* = 67341258 the obtained permutation is always 132 avoiding. Indeed, if there were a 132 pattern in it, than there would be one which starts with left to right minimum, but that is impossible as entries larger than any given left to right minimum are written in increasing order.

Note**:** *g (q)* is the only 132 avoiding permutation that has the same set and position of left to right minima as *q*.

Indeed, if at any given instance, two entries u< v that are larger than a left to right- minimum a were in decreasing order than avu would be a 132 pattern.

This proves that *g (f (p)) = p* implies that *f* is a bijection.

Case III: For all positive integers n, we have $S_n(132) = Cn = \frac{\binom{2n}{n}}{n+1}$

Proof: Suppose w e ha ve a 132 a voiding n - permutations in w hich t he e ntry n i s in t he i^{th} position. For p t o a void 132, i t m ust be that each e ntry to t he left of n i s larger than e ach entry to the r ight of n. I ndeed, i f x and y violet this c ondition, th an xny is a 132 p attern. Therefore, the set of entries on the left of n must be *{n-i+1, n-i+2… n-1}* and the set of entries on the right must be *[n-i]*.

Moreover, there ar e C_{i-1} possibilities f or th e o rder o f e ntries to th e le ft o f n and C_{n-i} possibilities for the order of entries on the right of n. So there is exactly $C_{i-1}C_{n-i}$ 132-avoiding n - permutations in w hich n is in th e i^{th} Position. S umming for a ll i we get t he following recursion

$$C_n = \sum_{i=0}^{n-1} \mathrm{C}_{i-1}\mathrm{C}_{n-i}$$

C(x) $=\sum_{n\geq 0} C_n x^n$, and let $C_0 = 1$, multiply (1) both sides by x^n and sum overall $n \geq 1$

which gives us

$\sum_{n\geq 1} c_n x^n = \sum_{n\geq 1}(\sum_{n\geq 0} c_{i-1}\ c_{n-i}\ ) x^n$ this implies

$C(x) - C_0 = x\ (\sum_{n\geq 1}(\sum_{i=1} c_{i-1}\ c_{n-i}\ ) x^{n-1}$

Since $\sum_{i=1} c_{i-1}\ c_{n-i}$ is the coefficient of x^{n-1} , by product formula of ogf

$$C(x) - C_0 = x\,(\sum_{n\geq 0} c_n x^n)(\sum_{n\geq 0} c_n x^n)$$

$C(x) - 1 = xC^2(x)$ this implies $xC^2(x) - C(x) + 1 = 0$

$$C(x) = \frac{1 \pm \sqrt{1-4x}}{2x}$$

But, C(x) has a constant term 1, so we have to choose the solution which has a constant term 1.

$C(x) = \frac{1-\sqrt{1-4x}}{2x}$ This is ordinary generating function of C_n, Catalan number, that is,

$$C(x) = \sum_{n\geq 0} \frac{1}{n+1}\binom{2n}{n} x^n$$

Therefore, $S_n(132) = Cn = \frac{\binom{2n}{n}}{n+1}$

This implies that

S_n (231) = S_n (132) = S_n (312)= S_n (231) = S_n (123) = S_n (321)= $C_n = \frac{\binom{2n}{n}}{n+1}$

Results: 1) All patterns of length three are avoided by the same number of n- Permutations.

2) The number $S_n\ (q)$ does not depend on pattern q of length three.

3.2 Monotone patterns

Monotone pattern is pattern which is either increasing or decreasing order. For example *1234…* or *nn-1n-2…* where n is an integer.

Theorem 3.2.1: For all positive integers n, *k>2*, we have

$$S_n\ (123 \ldots\ldots k) \leq (k-1)^{2n}$$

Proof: Let us define a rank function on the entries of each permutation P ∈ S_n (123 ...... k).

Set the rank of an entry P_i to be the length of any maximal increasing subsequence $P_{j_1}P_{j_2} \ldots\ P_i$

If any entry P_i has rank, say t, then there exist some increasing subsequences $P_{j_1}P_{j_1} \ldots P_{j_{t-1}}\ P_i$ of length t,

So for any entry $P_k > P_i$ with *k>i,* the rank of P_k have at least *t+1* entry, since we have the increasing subsequence $P_{j_1}P_{j_1} \ldots P_{j_{t-1}}\ P_i P_k$.

Therefore, for each $1 \leq t \leq n$, we see that the entries of rank t form a decreasing sequence.

There are at most n such decreasing sequences, as sets of indices they form a set partition of [n]. Since p is assumed to be 123….k-avoiding permutation P has no entry of rank k or greater, and there are at most *k-1* blocks of our set partition. Assigning each entry of P to one of k-1 blocks, we may over count and see that there are at most $(\mathrm{k}-1)^{\mathrm{n}}$ possible assignments of the indices to the blocks and at most $(k-1)^n$ possible assignments of the values to the blocks.

Thus, by product rule we have, $(k-1)^n(k-1)^n = (k-1)^{2n}$

Therefore: S_n (123 ...... k) $\leq (k-1)^{2n}$

For instance for k=3, we get that, $S_n\ (123) < 4^n$, agreeing to pervious results as

$S_n\ (123) = C_n$, and $C_n < \binom{2n}{n} < 4^n$

3.3 Patterns of length four

Let S_n (q) =the number of n-permutations that avoid the patterns q of length four.

There are 24 patterns of length four. Those are

1234 1243 1324 1342 1432 1423

2134 2143 2314 2341 2431 2413

3124 3142 3412 3421 3214 3241

4123 4132 4213 4231 4123 4132

But there are much trivial and non- trivial equivalence between them.

To decrease the number of patterns, we can take reverses and complements, to restrict our attention to those patterns of length four in which

i. The first element is smaller than the last one and

ii. The first element 1 or 2.

This leaves us with 11 patterns,

1234 1243 1324 1342 1432 1423

2134 2143 2314 2341 2413

Example 3.3.2 Let q=1423 then the permutation matrix A_q is equal to

$$A_q=\begin{pmatrix}1000\\0001\\0100\\0010\end{pmatrix} \quad \text{the transpose of } A_q \text{ is } A_q^T = \begin{pmatrix}1000\\0010\\0001\\0100\end{pmatrix}$$

Remark: If P= $P_1\ P_2\ ...\ P_n\ \in S_n$ and A_p is the permutation matrix of P, then

$$A_q^{-1} = A_q^T$$

Proof: let A be a permutation matrix and A^T be its transpose then we have AA^T =I where I is identity matrix. Thus A^{-1} is equal to A^T

Note: If P contains q, then P^{-1} clearly contains q^{-1} (as the inverse of a permutation matrix is its transpose).

Thus, $S_n(q) = S_n(q^{-1})$

If q =1423, then q^{-1}= 1342

Hence S_n(1423) = S_n(1342)

We can drop 1423, too, as its inverse 1342 remains on the list.

Again, if q=2314 then $q^c = 3241$ and it's reverse $(q^c)^r$= 1423 so we can drop 2314 as 1423 exist in the list.

The following general theorem of Backelin, J.West and Xin has involved proof; some of its special cases are applicable here.

Theorem 3.3.3: Let k be any positive integer, and let q be a permutation of the set

$\{k+1, k+2, \ldots, k+r\}$ then for all positive integers n, we have

$$S_n(123 \ldots . kq) = S_n(k\ (k-1)\ . . .\ 1q)$$

For instance, if r=2, k=2 and q=34 then by the above theorem we have

$$S_n(1234) = S_n(2134)$$

Also if q = 43 and the other parameters do not change, then by the above theorem we have

$S_n(1243) = S_n(2143)$ since 2134 and 2143 are reverse complements of each other, the patterns 2134 and 2143 can all be removed from the list.

Again let r=3 and K=1 then q=4 so we have $S_n(1234) = S_n(3214)$ by taking complements of 3214, we can remove 2341 and its reverse 1432.

This all leaves us with the patterns 1234, 1324, 1342, and 2413.

The following result of Stankova eliminates one more pattern.

Lemma3.3.4: For all positive integers n, we have

$$S_n(1342) = S_n(2413)$$

We note that stankova proved her result in the equivalent form of $S_n(4132) = S_n(3142)$

So we left with three patterns 1234, 1342, and 1324.

Numerical evidence computed by J.west shows that as n grows starting at n=1.

N	1	2	3	4	5	6	7	8
$S_n(1342)$	1	2	6	23	103	512	2740	15485
$S_n(1234)$	1	2	6	23	103	513	2761	15767
$S_n(1324)$	1	2	6	23	103	513	2762	15793

Table 3.3.1: Numerical evidence computed by J.west

These datas are upsetting following two reasons:-

- The number $S_n(q)$ no longer independent of q. that is, there are patterns of length four that are easier to avoid than others.
- The monotone pattern 1234 is neither the easiest nor the hardest to avoid. The following result is the earliest the most basic example in which a pattern was shown to be more restrictive than another pattern of the same size.

Theorem 3.3.5: For all n≥7, we have

$$S_n(1234) < S_n(1324)$$

The first step in our proof is the following classification of all n-permutations.

***Example 3.3.7*:** Permutations **X=51324** and **Y=51234** are in the same class,

But **Z=24<u>3</u>15** and **V=24<u>1</u>35** are not in the same class, since the third entry of Z is not a left -to- right minima where as that of V.

The sketch of proof of theorem 3.3.5: we have to show that

(1) Each non-empty class contains exactly one 1234-avoiding permutation and,

(2)Each non-empty class contains at least one 1324-avoiding permutation.

***P*roof of (1)**: Suppose we have already picked a class C, that is, we fixed the positions and values of all left -to- right minima and right -to- left maxima.

-It is clear that if we put all the remaining entries into the remaining slots in decreasing order, then we gets a1234-avoiding permutation.

- In fact, the permutation obtained this way consists of three decreasing Subsequences, that is, left –to-right minima, right-to-left maxima, and the remaining entries.

- If there were a 1234-pattern, by the pigeonhole principle two of its entries would be in the same decreasing subsequence, this would be a contradiction. Note that if C is non empty, then we can indeed write the remaining entries in decreasing order without conflicting with the existing constraints. Otherwise C would be empty. Therefore, C contains exactly one 1234-avoiding permutation.

Proof of (2): Note-If a permutation contains a 1324-pattern, then we can choose such a pattern So that its first element is left-to-right minimum and its last element is a right- to- left maximum.

-Indeed, we can just take any existing pattern and replace its first (last) element by its closest left (right) neighbor which is left-to-right minimum (right- to- left maximum).Therefore, to show that a permutation avoids 1324, it is enough to show that it does not contain a 1324 pattern having a left-to-right minimum its first element and a right- to- left maximum for its last element such a pattern is called a bad pattern.

Note: a left to right-minimum (right to left –maximum) can only be the first (last) element of a1324-pattern.

Now, taking any 1324-containing permutation, by the above argument it has a bad pattern. Interchange its 2^{nd} and 3^{rd} element. Observe that we can do this without violating the existing constraints, that is, no element x goes on the left of a left to right--minimum that is larger than x, and no element y goes on the right of right to left –maximum that is smaller than y.

The resulting permutation is in the same class as the original. Because, the left to right-minima and the right to left-maxima have not been changed.

Repeat this procedure as many times as possible, that is, as long as 1324-patterns can be found.

Note: Crucially each step of the procedure decreases the number of inversions of our permutation by at least one. Therefore we will have to stop after at most $\binom{n}{2}$ steps.

Then the resulting permutation will be in the same class as the original one, but it will have no *bad pattern* and therefore no 1324-pattern, as we claimed.

Now this is the right time to prove the theorem.

Our claim is to show that if n ≥ 7, then there exists classes that contain more than one 1324-avoiding permutation.

Case I: Let n$=$ 7, then we have the classes $3 * 1 * 7 * 5$ contains two such permutations namely 3612745 and 3416725, where stars denote the positions of the remaining entries.

Case II: Let n$>$ 7, then put the entries n n-1 n-2 …8 $3* \ 1 * 7 * 5$ with no additional stars. The obtained class again contains two 1324-avoiding permutations, coming from the Subwords 624 and 462 of the remaining entries.

Therefore the class contains more than one 1324-avoiding permutation.

The next point is it really significantly larger? Meaning that $S_n(1234)$ and $S_n(1324)$ are asymptotically equal; the following proposition answers this question in the negative sense.

Proposition3.3.7 The sequences $S_n(1234)$ and $S_n(1324)$ are not asymptotically equal.

Proof: Note: Two sequences *f(n)and h(n)*are asymptotically equal if $\frac{f(n)}{h(n)} \to 1$.

the number of 1234-aviding n-permutations is equal to the number of strong classes of n-permutations. On the other hand, the number of 1324-avoiding n-permutations is asymptotically more than the number of strong classes of n-permutations. These classes contain at least two 1324-avoiding permutations.

This completes the proof.

Remark: $\sqrt[n]{S_n\ (1234)} \to 9$, by monotone theorem$S_n$ $(123 \dots \mathrm{k}) \leq (k-1)^{2n}$

The core point in this subsection is obtaining an exponential upper bound for $S_n(1324)$.

3.3.1 The pattern 1324

The crucial property of this pattern is it starts with its minimal entry and it ends with its maximal entry. If we remove either of these entries, we get a pattern 132 or 213 for which an exponential upper bound is known.

To achieve our goal let us start by introducing some definition as follows:

Definition 3.3.1.1: An n- permutation p = $p_1 p_2 \dots . p_n$ is orderly if $p_1 < p_n$. And P is dual orderly if the entry 1 of p precedes the maximal entry n of p.

It is clear that p is orderly if and only if p^{-1} is dual orderly.

The importance of these permutations for us is explained by the following lemma.

Lemma 3.3.1.2 The number of orderly (resp. dual orderly) 1324-avoiding n-permutations is less than $\frac{8^n}{4(n+1)}$.

Proof: it is enough to prove the statement for orderly permutation as we can take inverses after that to get the other statement.

Each entry p_i of p has at least one of the following two properties.

i) $p_i \geq p_1$

ii) $p_i \leq p_n$

In words, everything is either larger than the first entry, or smaller than the last, possibly both.

Define S= $\{i|p_i \geq p_1\}$ andT=$\{i|p_i < p_1\}$. Then S and T are disjoint, S$\cup T = [n]$ and crucially, if $i\in$T, then in particular, $p_i < p_n$. we know that for any pattern q of length three, we have

S_n(q) = $C_n = \frac{\binom{2n}{n}}{n+1}$.

Let $|S| = s$ and$|T| = t$. T hen w e h ave C_{s-1} possibilities f or th e s ubstring P s o f e ntries belonging to indices in S. $C_t = C_{n-s}$ Possibility for the substring P_T of entries belonging to indices in S.

Indeed, Ps starts with its smallest entry and the rest of it must avoid 213, otherwise, together withp_n, a 1324 pattern is formed. Finally, we have $\binom{n-2}{s-2}$ choices for the set of indices that we denoted by S. once S is known, we have no liberty in choosing the entries p_i, i$\in S$ as they must simply be the S largest entries .therefore the total number of possibilities is

$$\sum_{s=2}^{n}\binom{n-2}{s-2}C_{s-1}C_{n-s} < 2^{n-2}\sum_{s=2}^{n} C_{s-1}C_{n-s} < 2^{n-2}C_n < \frac{8^n}{4(n+1)}$$

We have seen that it helps in our efforts to limit the number of 1324-avoiding permutations if

a large element is preceded by a small one .

To make good use of this observation, look at all non -inversions of a generic permutation

P= $p_1\, p_2 \ldots \ldots p_n$, that is, pairs *(i, j)* so that *i<j* and $p_i < p_j$

Find the non -inversions *(i, j)* for which $\max_{i,j}(j - i, p_j - p_i)$ is maximal

If there are several such pairs, take one of them; say the one that is lexicographically first.

We call this pair *(i, j)* ***the critical pair*** of p.

Example3.3.1.3: Let *P*=5716243. Then the critical pair of p is (3, 4), as $p_3 = 1$ and $p_4 = 6$, so $p_j - p_i$ = 5 there is no other non –inversion for which *j-i* or $p_j - p_i$ would be so high.

The following obvious, but will be important in what follows.

Proposition3.3.1.4: for any permutation, the critical pair is always a pair *(i, j)* in which p_i is a left to right minimum, and p_j is a right to left maximum.

The following definition proved to be useful for treating 1324-avoiding permutations.

Definition3.3.1.5: two permutations are in the same class if they have the same left to right minima, and the same right to left maxima, and they are in the same positions.

Example 3.3.1.6: the permutation **3612745** and **3416725** are in the same class.

Proposition3.3.1.7: The number of non-empty classes of n-permutation is less than 9^n.

Proof: suppose C be a class which contains exactly one 1234- avoiding permutation, namely the one in which all entries that are not left to right minima or right to left maxima are written in decreasing order. As we know that $S_n(1234) < 9^n$. therefore the statement is proved.

To achieve our goal, it is sufficient to find a constant C so that each class contains at most C^n 1324- avoiding n-permutations.

Choose a class A .by above proposition 2.5.1. We see that the critical pair of any permutation p∈ A is the same and it depends only on the left to the right minima and the right to left maxima and those are the same for all permutations in A.

We will now find an upper bound for the number of1324-avoiding n-permutations in A.

For symmetry reasons, we can assume that in the critical pair of $p \in A$, we have $j - i \geq p_j - p_i$, In other words, the maximum is attained by $j - i$.

We will now reconstruct p from its critical pair. First, all entries that precede P_i must be larger thanP_j. Indeed, if there existed k$< i$ so that$P_k < P_j$, then the pair *(j, k)* would be a "longer" non-inversion than the pair *(i, j)*, contradicting the critical property of *(i, j)*. Similarity, all entries that are on the right of P_j must be smaller thanP_i.

These shows that all entries P_t for which $P_i < P_t < P_j$ must be positioned between P_i andP_j, than, *i< t<j* must hold for them. However, if *j-i=*P_j*-*P_i*+b*, where b is a positive integer, then we can select b additional entries that will be located between P_i andP_j. We will call them excess entries; that is an excess entry is an entry P_u that is located between P_i andP_j, but does not satisfy $P_i < P_u < P_j$. The good news is that we do not have too many choices for the excess entries. No excess entry can be smaller than P_i –b. Indeed, if we had $P_u < P_i$ –b for an excess entry, then for the pair (u, j) the value defined by$\max_{(i,j)} j - i, P_i - P_i)$ would be larger than for the pair *(i, j)*, contradicting the critical property of *(i, j)*. By the analogous argument, no excess entry can be larger than P_j +b. Therefore, the set of b excess entries must be a subset of the at most-(2b)-elements set $(\{P_i - b, P_i$-b+1$, P_i - 1\} \cup \{P_j + 1, P_j + 2, \ldots, P_j + b\}) \cap [n]$. Therefore, we have at most $\binom{2b}{b}$ choices for the set of *j-i-1+b* elements that are located between $P_i and\ P_j. as\ P_i < P_j$, the partial permutation $P_i P_{i+1} \ldots P_j$ is ordely, and certainty 1342-avoiding. Therefore, by lemma 4.5.1, we have less than $\frac{8^{j-i+1}}{4(j-i+1)}$ choices for it once the set of entries has been chosen.

This proves altogether, we have less than $4^b \frac{8^{j-i+1}}{4(j-i+1)} < 32^{j-i}$ possibilities for the string$P_i P_{i+1} \ldots P_j$. We used the fact that *b≤j-i-1* as b counts the excess entries between *i* and *j*.

Note that: We have some room to spare, so we can say that the above upper bound remains valid even if we includes the permutations in which the maximum was attained by (P_i,P_j) and not by *(i, j)*.

We can now remove the entries $P_{i+1} \ldots P_{j-1}$ from over permutations. This will spilt our permutations into two parts, P_L on the left, and P_R on the right. It is possible that one of them is empty. We know exactly what entries belong to P_L and what entries belong toP_R, indeed each entry of P_L is larger than each entry ofP_R. Therefore, we do not loose any information if we relable t he en tries i n each of P_L andP_R so that t hey both s tart a t 1 (we call th is th e standardization of the strings). This will not change the location and relative value of the left-to-right m inima a nd r ight-to-left maxima e ither. T he s tring $P_{i+1} \dots P_{j-1}$ should not be standardized, however, as that would result in loss of information.

See figure 1 for the diagram of generic permutation, its critical pair, and the string P_L andP_R.

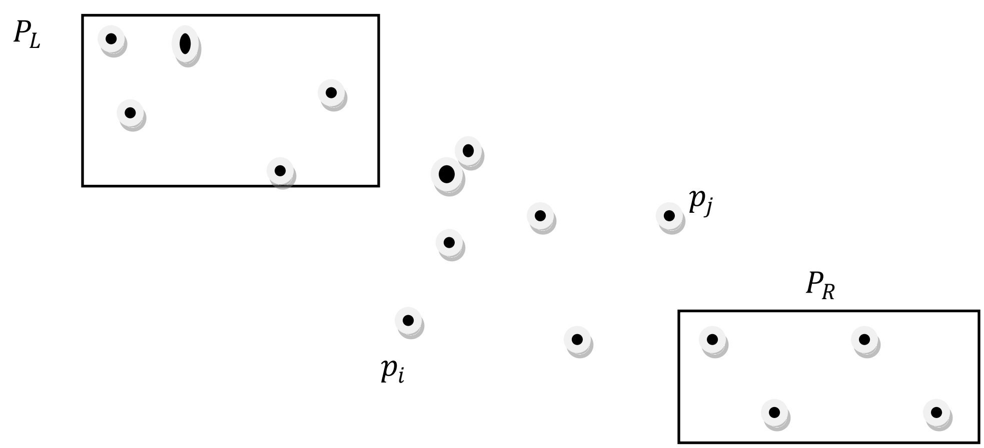


Figure 3.3.1.8: A generic permutation and its critical pair.

Then we iterate our procedure. That is, we find the critical pair of P_L andP_R, denote them by (i_L,j_L) and (i_R,j_R), and prove, just as above, there are at most$32^{j_k - i_k}$ possibilities for the string betweeni_R $and j_R$. T hen w e r emove t hese s trings again, c utting ou r pe rmutations i nto m ore parts, and soon, building a binary tree, like structure of strings. Note that the leaves of this tree will be orderly or dual orderly permutations. Note that this procedure of decomposing of our permutations is in jective. I ndeed, given t he s tandardized stringsP_L, th e p artial p ermuation $P_i \dots P_j$, and the standardized stringsP_R, we can easily recover P.

Iterating th is a lgorithm u ntil e ntries o f P th at are n ot le ft-to- right m inima or r ight-to-left maxima are removed, we prove the following.

Lemma 2.5.1.8: The number of 1324-avoiding n-permutations in any given class A is at most 32^n

Proof: the above description of the removal of entries by our method shows that the total number of 1324 –avoiding permutations in A is less than $32^{\sum_k j_k - i_k}$ where the summation ranges through all intervals(i_k, j_k) whose end points were critical pairs at some point. As these interiors of these intervals are all disjoint, $\sum_k j_k - i_k$= n-1

Now proving the upper bound for $S_n(1324)$ is a breeze.

Theorem 3.3.1.9: There exists an absolute constant C so that for all n, we have $S_n(1324) < C^n$.

Proof: As there are less than 9^n classes and less than 32^n n- permutations in each class that avoid 1324, therefore 9 ·32=288 so we do it.

3.3.2 The pattern 1342

In this section, we will provide an exact formula for$S_n(1342)$.

Theorem 3.3.2.1: For all positive integers n, we have

$$\boldsymbol{S_n(1342) = (-1)^{n-1} \cdot \frac{7n^2-3n-2}{2}} + 3\sum_{i=2}^{n} \boldsymbol{(-1)^{n-i} \cdot 2^{i+1} \cdot \frac{(2i-4)!}{i!(i-2)!} \cdot \binom{n-i+2}{2}},$$

This is a very surprising result. It is straightforward to prove from this formula that

$S_n(1342)$ <8^n For all n, and $\lim_{n\to\infty} \sqrt[n]{S_n(1342)} = 8$

The result itself is not the only interesting aspect of the facts surrounding the pattern 1342. We will see that permutations avoiding 1342 are in bijection with ***two different kinds of objects*** which at first look totally unrelated. The first, and for our purpose, more important, type of objects are specific kind of ***labeled trees***.

The following is the correspondence between trees and permutations.

Definition3.3.2.2: A rooted plane trees with non- negative integer labels *l (v)* on each of its vertices *V* is called a $\beta(0,1)$-tree if it satisfies the following conditions:

- If *V* is a leaf, then *l (V)* = 0, this explains the 0 in the name of $\beta(0,1)$- tree.
- If V is the root and $v_{1,}v_2$,...,v_k are its children, then *l(V)*=$\sum_{i=1}^{k} l\,(v_k)$
- If *V* is an internal node and $v_{1,}v_2$...v_k are its children, then *l (v)*$\leq 1 + \sum_{i=1}^{k} l\,(v_k)$ (this explains the 1 in the name of $\beta(0,1)$- tree.

Figure 1.Shows a $\beta(0,1)$-tree on 12 vertices.

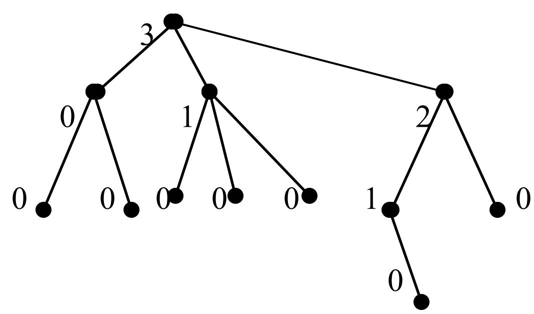


Figure1

A $\beta(0,1)$-tree

The importance of $\beta(0,1)$-trees for us is explained by the following theorem.

Theorem 3.3.2.3: For all positive integers n, there is a bijection F from the set of indecomposable 1342-avoiding n-permutations to the set $D_n^{\beta(0,1)}$ of $\beta(0,1)$-trees on n vertices.

Example 3.3.2.4: Let n =3. Then there are three indecomposable n-permutations, namely, 123, 132, and 213, and they all avoid 1342. ***Correspondingly***, there are three $\beta(0,1)$-trees on three vertices, as can be seen in figure 2

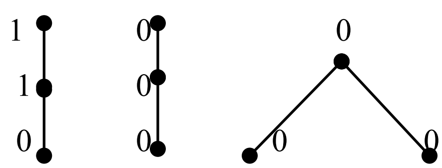


Figure4: the three $\beta(0,1)$-trees on three vertices.

A branch of rooted three is a tree whose top is one of the root's children. Some rooted trees may have one branch, which does not necessarily mean they consist of a single path.

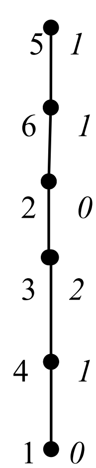
We start by treating two special types of $\beta(0,1)$- trees on n vertices. These cases are fairly simple-they will correspond to 231-avoiding (resp. 132-avoiding) permutations, but they will be our tools in dealing with the general case.

First we set up a bijection *f* from the set of all 1342-avoiding n-permutations starting with the entry 1 and the set of $\beta(0,1)$-trees on n vertices consisting of one single path. In other words, the domain of *f* is the set of 231-avoiding permutations of the set {2, 3... n}.

So let P=$p_1p_2\ldots p_n$ be a 1342-avoiding n-permutations so that p_1=1. Take unlabeled tree on n nodes consisting of a single path and give the label *l(i)* to its i^{th} node ($1\leq i \leq n-1$) by the following rule: *l (i)*= $\begin{cases} |\{j \leq i \; so \; that \; p_j > p_s for \; at \; least \; one \; s > i,\}| if \; i < n \\ l(n-1) \; if \; i = n. \end{cases}$

In words, *l (i) is the number of entries on the left of* p_i *(inclusive) which are larger than at least one of the entry on the right of* p_i.

Note that: This way we could define *f* on the set of *all n*-permutations starting with the entry 1, but in that case, as we will see, *f* would not be a bijection.(for example, the images of 1342 and 1432 would be identical). *Example***:** if *p*= 143265, then the labels of the nodes of *f (p)* are, from the leaf to the root,*0, 1, 2,0,1,1* . See figure 4.5.for easy reference, we wrote p_i to the i^{th} node of the path *f (p).* Figure 3 the $\beta(0,1)$-tree of p=143265



To avoid confusion in this figure, the entries of p will be written in, small Roman letters, and the labels of the nodes will be written in large italic letters.

Lemma 3.3.2.7: f is a bijection from the set of $D_n^{\beta(0,1)}$ of all $\beta(0,1)$ -trees on n-vertices consisting of one single path to the set of 1342-avoiding n-permutations starting with the entry 1.

Proof: it is easy to see that *f* indeed maps into the set of $\beta(0,1)$ –trees $l(i+1) \leq l(i) + 1$ for all i because there can be at most one entry counted by $l(i+1)$ and not counted by $l(i)$, namely the entry p_{i+1} . All labels are certainly non-negative and $l(1) = 0$

To prove that *f* is a bijection, if sufficient to show that it has an inverse, that is, for any $\beta(0,1)$-tree T consisting of a single path. We can find the only permutation *p* such that f *(p)* =T.

We claim that given T, we can recover the entry n of the pre image P.

First note that p was 1342-avoiding and started by 1.So any entry on the left of n must have been smaller than any entry on the right of n. In particular, the node preceding n must have label 0. Moreover, n is the left most entry p_i of p so that $l(j) > 0$ for all $j \geq i$ if there is such an entry at all, and n=p_n if there is none. That is, n corresponds to the node which starts the uninterrupted sequence of strictly positive labels which ends in the last node, if there is such a sequence, and corresponds to the last node otherwise. To see this, note that n is the largest of all entries. So in particular it is always larger than at least one entry in any non empty set of entries.

Once we located where n was in p, we simply delete the node corresponding to it from T and decrement all labels after it by 1. (If this means deleting the last node, we just change $l(n-1)$ so it is equal to $l(n-2)$ to satisfy the root condition. We can definitely do this because the node preceding *n* had label 0 and the node after *n* had a positive label (1 or 2), by our algorithm to locate *n*. Then we can proceed recursively, by finding the position of the entries *n-1, n-2…1* in p. This clearly defines the inverse of *f*, so we have proved that *f* is abjection. $\beta(0,1)$

Lemma 3.3.2.8 There is a bijection g from the set of 132-avoiding n-permutations ending with n to the set of $\beta(0,1)$-trees on n vertices with all labels equal to zero.

Proof: These $\beta(0,1)$-trees are in fact unlabeled plane trees. We prove that they are in one-to-one correspondence with the 132-avoiding permutations whose last entry is n.

Suppose we already know this for all positive integers $k<n$. Let T be a $\beta(0,1)$-tree on n vertices with all labels equal to 0 and root r.

Let r have t children, which are, from left-to- right, at the top of such unlabeled trees $T_1,T_2,\ldots,T_t$ on $n_{1,}\ n_{2,}\ldots n_t$ nodes.

Then by induction, each of the T_i corresponds to a 132 -avoiding n_i -permutation ending with n_i. Now add $\sum_{j=i+1}^{t} n_i$ to all entries of the permutation P_i associated with T_i, then concatenate all these strings and add n to the end get the permutation p associated with T.

This is clearly a bijection as the blocks of the first n-1 elements determine the branches of T.

Example 3.3.2.9: The permutation 45631278 corresponds to the $\beta(0,1)$ -tree with all labels equal to 0 shown in figure 4.

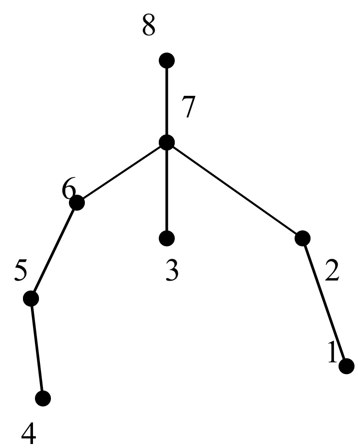


Figure 4 $\beta(0,1)$-trees of p=45631278

An easy way to interpret the corresponding permutation once we have its entry written to the corresponding nodes in the well-known pre order reading: for every node,

First write down the entries associated with its children from left to right, and then the entries associated with the node itself, and do this recursively for all the children of the node.

Note that: Such a $\beta(0,1)$- tree has only one branch if and only if the next-to-last element of the indecomposable 132-avoiding n-permutation corresponding to it is *n-1*.

Let us introduce some more notions before we attack the general case.

Proposition 3.3.2.10: Each non empty class C of n-permutations contains exactly one 132-avoiding permutation.

Proof: take all entries which are not left-to-right minima and fill all slots between the left-to-right minima with them as follows:

In each step place the smallest element which has not been placed yet which is larger than the previous left-to-right minimum.

The permutation obtained this way will be clearly 132-avoiding and it will be the only one in this class because any time we move away from this procedure, we create a 132-pattern.

Therefore the class contains exactly one 132-avoiding permutation.

Definition 3.3.2.11: The normalization *N (p)* of an n-permutation p is the only 132-avoiding permutation in the class C containing p.

Example 3.3.2.12 If p=356214, and *N (p)* = 345216

Defintion3.3.2.13: The normalization *N (T)* of $\beta(0,1)$ –tree T is the $\beta(0,1)$ –tree which is isomorphic to T as a plane tree, with all labels equal to zero.

It turns out that normalization preserves the indecomposable property.

Proposition 3.3.2.14: A permutation p is indecomposable if and only if *N (p)* is indecomposable.

Proof: let C be the class containing p, given by the set and position of its left-to-right minima. It is clear that if p $\in$ C is decomposable, then the only way to cut it into two parts (so that everything before the cut is larger than everything after the cut) is to cut it immediately before a left-to-right minimum *a* so that it is in the n+1$-a^{th}$ position, then all entries which are larger than *a* must be placed on the left of *a* and so all such permutations in C are decomposable. If there is no such a, then for all left-to-right minima m, there will be an entry b so that $m < b$ and b is on the right of m and so permutation in C will not be decomposable.

Corollary3.3.2.15 If p is an indecomposable n-permutation, then *N (p)* always ends with entry n.

Proof: Note that the only way for a 132 –avoiding n-permutation to be indecomposable is for it to end with n. Then the statement follows from corollary 3.3.2.15.

Now we are in a position to prove our theorem about the number of indecomposable 1342-avoiding permutations.

Theorem 3.3.2.16: The number of indecomposable n-permutations which avoid the pattern 1342 is I_n (1342) = $t_n = 3.2^{n-2} \cdot \frac{(2n-2)!}{(n+1)!(n-1)!}$

Proof: we are going to set up a bijection F between indecomposable n- permutations that avoid the pattern 1342 and $D_n^{\beta(0,1)}$

Let p be indecomposable 1342-avoiding n-permutations. Take *N (p)*. By corollary 2.5.2.15 its last elements is n. Define *f (N (p))* to be$\beta(0,1)$ –tree S associated to *N (p)* by the bijection of lemma 2. Now write the entries of p to the nodes of S so that for all i, the p_i is written to the node where $N(p)_i$ was written in S. In particular, the left -to- right minima remain unchanged.

Figure5. Shows how we associate the entries of permutation p=4621357 to the nodes of

N (p) =4521367.

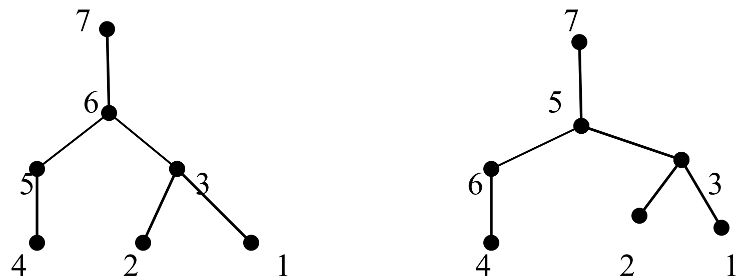


Figure 5: The unlabeled trees of N (p) =4521367 and P=4621357.

Now we are going to define the label of each node for this new β (0, 1)-tree T and obtain *F (p)* this way. As an unlabeled tree, T will be isomorphic to S, but its labels will be different.

Denote i the i^{th} node of T in the preorder reading, thus p_i is the i^{th} entry of p, (which is therefore associated to node *i*), while *l (i)* is the label of this node. We say that p_i beats p_j if there is an element p_h so that p_h, p_i, p_j are written in this order and they form a 132-pattern. Moreover, we say that p_i reaches p_k if there is a subsequence $\mathrm{p_i}, \mathrm{p}_{\mathrm{i+a_1}}, \dots\dots, \mathrm{p}_{\mathrm{i+a_t}}, \mathrm{p_k}$ of entries so that $i < i + a_1 < i + a_2 < \cdots < i + a_t < k$ and that any entry in this subsequence beats the next one.

In particular, if x beats y, then x also reaches y.

For example:

1) P=3716254, then 7 beats 6, 6 beats2, therefore 7 reaches 2.

2) P=361542, the entry 6 beats 5 and 4, 5 beats 4 and 2, and 4 beats 2, while 6 reaches 2

Each entry reaches all those elements it beats, too. Then let

l (i)= #{*j* descendants of *i* (including *i* itself) so that there is at least one *k*> i for which

p_j reaches p_k }, let F (p) be the $\beta(0,1)$-tree defined by these labels.

Descendant of *i* is an element of the tree whose top element is *i.*

First, it is easy to see that F indeed maps into the set of $\beta(0,1)$-tree: If v is an internal node and $v_1 v_2, \dots\dots, v_k$ are its children, then *l (v)*$\leq 1 + \sum_{i=1}^{k} l(v_k)$. Because there can be at most one entry counted by *l (v)* and not counted by any of its children's label, namely v itself. All labels are certainly no negative and all leaves that are the left to right minima have label zero.

Example 3.3.2.17: Let p= 58371624, then we have N (p) = 56341278, giving rise to the unlabeled tree shown on the left of figure 6.

We then compute the labels of *F (p)* to obtain the tree shown on the right of figure6

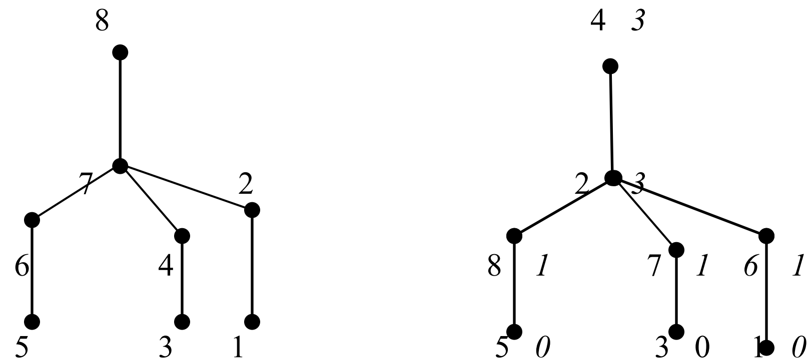


Figure 6: the tree S and F (p) for p = 58371624

To prove that F is bijection, it suffice to show that it has an inverse, that is, for any

$\beta(0,1)$-tree T∈ $D_n^{\beta(0,1)}$, we can find the only permutation p so that *F (p) =T.*

We again claim that given T, we can recover the position of n in the perimage.

Proposition3.3.2.19**:** Suppose p_n ≠n, that is, n is not associated to the root vertex. Then each ancestor of n, including n itself, has a positive label. If$p_n = n$, then *l (v)* =0 and thus there is no vertex with the above property.

Proof: If $p_n = n$, then nothing beats it, thus $p_n = 0$. suppose p_n is not the root vertex.

To prove our claim it is enough to show that for any node i which is the ancestor of j, there an entry p_k so that p_k is an ancestor of p_i and n= p_j reaches k. indeed this would imply that the entry p_j =n is counted by the label *l (i)* of. now let $a_m = p_1 > a_2 > \cdots >$ a_1 =1 be the left to right minimum of p so that n is located between a_r *and* a_{r+1} .then n certainly beats all elements located between a_r *and* a_{r+1} as a_r can play the role of 1 in the 132 – pattern. Clearly n must beat at least one entry y_1 on the right of a_{r+1} as well, otherwise p would be decomposable by cutting it right before a_{r+1}. If y_1 is on the right of i, we are done. If not, then y_1 must beat at least one entry y_2 which is on the other side ofa_{r_1+1}, were y is located between $a_{r_1}$and $a_{r_1+1}$for the same reason, and soon. This way we get a subsequence y_1 y_2, ….. so that n reaches each of the y_t and this subsequence eventually gets to the right of *i*, since each step we by pass at least one left to right minimum. Thus the proposition is proved.

***Proposition3.3.2.20*:** Suppose$p_n \neq n$.then n is the left most entry of p which has the property that all their ancestors has a positive label.

Proof: - suppose$p_k and\ n$ both have this property and that p_k is on the left of n. if there are several candidates for the role of p_k, choose the right most one .ifp_k beats an element y on the right of n by the participating in the 132 pattern x$p_k y$, then x$p_k ny$ is a 1342 pattern, which is contradiction. So p_k does not beat such an element before p_k . Still p_k must reach elements on the right of n, thus it beats some element v between p_k and n. this element v in turn beats some element w on the right of n by participating in some 132 -pattern tvw. However, this would imply that tvnw is 1342-pattern, a contradiction, which proves our claim. Therefore, we can recover the entry n of p from T. then we can proceed as in the proof of lemma1, that is, just delete n, subtract1 from the labels of its ancestors and iterate this procedure to get p.if any time during this procedure we find that the current root is associated to the maximal entry that has not been associated to other vertices yet, and the tree has more than one branch at this moment, then deleting the root vertex will split the tree in to smaller trees. Then we continue the same procedure on each of them. The set of the entries associated to each of these smaller trees is uniquely determined because T as unlabeled tree determines the left to right minima of p. therefore; we can always recover p in this way from T.

This proves that F is a bijection. Thus we have set up a bijection between the set of indecomposable 1342-avoiding n-permutations and $D_n^{\beta(0,1)}$.

$$\left|D_n^{\beta(0,1)}\right| = t_n = 3.2^{n-2}\frac{(2n-2)!}{(n-1)!(n+1)!}$$

Therefore the theorem is proved.

Note that in particular, F maps 132-avoiding permutations into $\beta(0,1)$ −trees with all labels equal to 0 and permutations starting with the entry 1 into $\beta(0,1)$ –trees consisting of a single path.

***Corollary3.3.2.21*:** the number of indecomposable 1342-avoiding n-permutations is

$$\left|D_n^{\beta(0,1)}\right| = t_n = 3.2^{n-2}\frac{(2n-2)!}{(n-1)!(n+1)!}$$

Proof: Since there is a bijection between the number of indecomposable 1342-avoiding n-permutations and the set of $D_n^{\beta(0,1)}$ -trees. So we will have

$$\left|D_n^{\beta(0,1)}\right| = t_n = 3.2^{n-2}\frac{(2n-2)!}{(n-1)!(n+1)!}$$

Now computing the numbers $S_n(1342)$ is now a breeze well by using generating functions.

Lemma 3.3.2.22:

Let S_n=$S_n(1342)$ and let H(x) =$\sum_{n=0}^{\infty} S_n x^n$. Furthermore,

Let F (x) = $\sum_{n=1}^{\infty} t_n x^n$, that is, let F (x) be the generating function of the numbers of indecomposable 1342-avoiding permutations. Then

$$H(x) = \sum_{i\geq 0} F^i(x) = \frac{1}{1-F(x)} = \frac{32x}{-8x^2+20x+1-(1-8x)^{3/2}}$$

Proof: - Tutte has obtained the numbers t_n by first computing a translate of their generating function

$$F(x) = \sum_{n=1}^{\infty} t_n\ x^n = \sum_{n=1}^{\infty} 3.2^{n-1}\frac{(2n)!}{(n+2)!n!}x^n$$

$$= \frac{8x^2+12x-1+(1-8x)^{3/2}}{32x}$$

The coefficients of F(x) are the numbers of indecomposable 1342-avoiding n-permutations. Any 1342 -avioding permutation has a unique decomposition into indecomposable permutations. This can consist of 1, 2, 3… blocks, implying that S_n=$\sum_{i=1}^{n} t_i S_{n-i}$

Therefore H(x) = $\frac{1}{1-F(x)}$ as claimed.

The other kinds of objects that are bijection with these permutations are rooted bicubic maps.

Rooted bicubic maps are planer maps in which each vertex has degree three, there is a distinguished –half edge (the root), and the underlying graph is bipartite.

Tutte has enumerated these maps when he obtained formula F(x) = $\sum_{n=1}^{\infty} t_n x^n$ as in the above mentioned a nd C ori, J acquard and S chaeffer t hen u sed the $\beta(0,1)$ −trees to f ind a m ore combinatorial proof of Tuttes result.

Note that we have the generating function of the numbers $S_n(1342)$, we are in a position to obtain an explicit formula for their number, so we have the following theorem.

Theorem 3.3.2.22:-for all n≥ 0, we have

$$S_n(1342)=(-1)^{n-1}.\frac{7n^2-3n-2}{2} + 3\sum_{i=2}^{n}(-1)^{n-i}.\,2^{i+1}.\frac{(2i-4)!}{i!(i-2)!}\binom{n-i+2}{2},$$

Proof: - multiply both the numerator and the denominator of H(x) by H(x)

$-8x^2 + 20x + 1 + (1-8x)^{3/2}$ after simplifying we get

$$H(x) = \frac{(1-8x)^{3/2} - 8x^2 + 20x + 1}{2(x+1)^3} \qquad (5)$$

As $(1-8x)^{3/2}$ =1- 12x+$\sum_{n\geq 2} 3.\,2^{n+2}x^n \frac{(2n-4)!}{n!(n-2)!}$ this implies our claim.

So the first few values of $S_n(1342)$ are 1,2,6,23,103,512,2740,15485,91245,555662…..

as $\frac{(2n-4)!}{n!(n-2)!} < \frac{8^{n-2}}{n^{2.5}}$ by Stirling’s formula.

We have then proved the following exponential upper bound for $S_n(1342)$.

***Corollary 2.5.2.23*:-** For all n, we have

$S_n(1342) < 8^n$

The number t_n satisfies the recurrence $t_n = \frac{(8n-12)t_{n-1}}{(n+1)}$

It is straightforward to check that the numbers I_n =t_n satisfy the following recurrence

$.t_n = \frac{8n-4t_{n-1}}{n+2}$. --*

In particular, $\sqrt[n]{t_n} \to 8$. Using this formula we can disprove a conjecture of Stanley claiming that for all permutation patterns q of length k, the sequence $\sqrt[n]{S_n(q)}$ converges to $(k-1)^2$.

Theorem4: $\mathbf{lim}_{n\to\infty} \sqrt[n]{S_n(\mathrm{q})} = \lim_{n\to\infty} \sqrt[n]{S_n(1342)} = 8$

Proof: - Clearly $t_n \leq S_n < 8^n$ by above corollary and we know from (*)

that, $\sqrt[n]{t_n} \to 8$ if $n \to \infty$ therefore, by Squeeze principle, we obtain the following corollary.

Corollary3.3.2.24:- we have $\lim_{n\to\infty} \sqrt[n]{S_n(12453)} = 8$

3.3.3 The Patterns of 1234

The pattern 1234 is the monotone pattern, therefore, by theorem 2.3 a very good upper bound for the numbers $S_n(123 \ldots\ldots k)$, applies to it. On the other hand, Ira Gessel by using certain techniques, obtained in Ira Gessel [16] obtained the following exact formula for these numbers

$$S_n(1234) = 2 \cdot \sum_{k=0}^{n} \binom{2k}{k} \binom{n}{k}^2 \frac{3k^2+2k+1-n-2nk}{(k+1)^2(k+2)(n-k+1)}$$

After a few years later Gessel obtained the following alternative form for his formula

$$S_n(1234) = \frac{1}{(n+1)^2(n+2)} \sum_{k=0}^{n} \binom{2k}{k} \binom{n+1}{k+1} \binom{n+2}{k+1}$$

In this new form, all term are non-negative, but there is still a division, suggesting that a direct combinatorial proof is probably difficult to find.

Result:-we can compute $\lim_{n\to\infty} \sqrt[n]{S_n(q)}$ for some pattern q. in fact we have seen that

$$lim_{n\to\infty} \sqrt[n]{S_n(q)} = \begin{cases} 4 \; if \; q \; is \; of \; length \; 3, \\ (k-1)^2 if \; q = 123 \cdots k, \\ 8 \; if q = 1342, \end{cases}$$

On the other hand, in every case when we saw an exact answer, it is an integer.

In general, M. Bóna [15] has recently proved that that

$\lim_{n\to\infty} \sqrt[n]{S_n(12453)} = 9+4\sqrt{2}$, so these limits are not even always rational.

By corollary 3.3.2.24 we have the following surprising results, that is,

$\lim_{n\to\infty} \sqrt[n]{S_n(1342)} \neq 8 \;\; \lim_{n\to\infty} \sqrt[n]{S_n(1234)} = 9$

That i s, ev en logarithmic sen se t he seq uence $S_n(1342)$ and $S_n(1234)$ are d ifferent. This observable fact is not well understood. If we could understand why the pattern 1342 is really so easy to avoid, then maybe we could use that information to find other, longer patterns that are easy to avoid.

Chapter-four

4. The proof of the Stanley-Wilf conjecture

The goal of this section is to present a proof of the Stanley- Wilf conjecture.

In order to do this, first we will introduce conjecture involving 0-1 matrices, show why it implies the Stanley-Wilf conjecture, and then prove the conjecture on the 0-1 matrices.

Conjecture 3.2.2*[Stanley-Wilf Conjecture, 1980]*

Let q be any pattern. Then there exists a constant c_q so that for all positive integers, we have

$$S_n(q) \leq c_q^n$$

Note: the Conjectured number c_q^n is very small compared to the number of all n-permutation, which is n!.

Conjecture 3.2.3*[Stanley-Wilf Conjecture, Alternative version]*

Let q be any pattern. Then the limit $\lim_{n\to\infty} \sqrt[n]{S_n(q)}$ exists.

Results: Stanley-Wilf Conjecture is true if q is of length three, with $c_q = 4$ and for monotone patterns $S_n(1234) \leq (4-1)^{2n} = 9^n$

4.1 The Füredi-Hajnal conjecture

Let us extend the notion of pattern avoidance to 0-1 matrices as follows.

Definition 4.1.1 Let A and P be matrices whose entries are either equal to 0 or to 1, and let P be of size k x l. We say that A ***contains*** P if A has $a\ k \times l$ sub matrix Q so that if $P_{i,j} = 1$, then $Q_{i,j} = 1$, for all i and j. If A does not contain P, then we say that A ***avoids*** P.

In other words, A contains P if we can delete some rows and some columns from A and obtain a matrix Q that has the same shape as P, and has a 1 in each position P does. Note that Q can have more one entry than P, but not less. Note that if all entries of A are equal to 1, then A contains all matrices P that have shorter side lengths than A.

Note: All matrices in this section will be 0-1 matrices, so we can not repeat that condition any more.

Example 4.1.2

Let A $=\begin{pmatrix}1001\\0110\\0011\\1000\end{pmatrix}$, and let P=$\begin{pmatrix}0 & 0\\1 & 1\end{pmatrix}$. Then A contains P as can be seen by deleting everything from A except the intersection of the first and third rows with the third and fourth columns. If Q=$\begin{pmatrix}1 & 1\\1 & 1\end{pmatrix}$, then A avoids Q

The famous Füredi-Hajnal conjecture, which was originally stated as a question, sounds similar to the Stanley-wilf conjecture.

Conjecture (4.1.3): [1] [Füredi-Hajnal conjecture, FHC]

Let P be any permutation matrix, and let $f(n, P)$ be the maximum number of 1 entry that a P-avoiding $n \times n$ matrix A can have. Then there exists a constant c_P so that

$$f(n, P) \leq cP^n.$$

4.2 Avoiding matrices vs. avoiding a permutations

In this section we are going to show the similarities of the two proof we sketch here Klazar's argument [4] that the Füredi-Hajnal conjecture implies the Stanley –Wilf conjecture.

In order to prove this connection between permutations and matrices, Klazar introduces another notion of avoidance and containment, that of bipartite graphs.

Definition 4.2.1

Let G ([a], $[a']$) and H ([b], $[b']$) be simple bipartite graphs. We say that G contains H as an ordered subgraph if there exist order preserving injections f: [b] → [a] and

f' : $[b']$ → $[a']$ so that if vv' an edge of H, then $f(v)f'(v')$ is an edge of G.

- Each n-permutation $p = p_1 p_2 \cdots p_n$ defines a bipartite graph G_p on $([n], [n])$ in the natural way. That is, G_p has n edges, given by (p_i, i), for $i \in [n]$.thus we have the following proposition.

Proposition4.2.2:

If the permutation p contains the permutation q as a pattern, then G_p contains G_q as an ordered subgraph.

Therefore, if $G_p(n)$ is the number of simple bipartite graphs on $([n], [n])$ avoiding the graphG_p, then we have$S_n(n) \leq G_p(n)$.

Note that if a bipartite graph G $([n], [n])$ avoidsG_p, then the adjacency matrix of A (G) of G must avoid the adjacency matrix of G_p. By assuming that for the rest of this section Conjecture 4.60 is true. Then A (G) can have at most cP^n entries equal to 1, that is, G can have at most cP^n edges. We are going to show that this leaves at most an exponential number of possibilities for G . let us contract G to the smaller bipartite graph G_1 that has vertex set $([\frac{n}{2}],[\frac{n}{2}]')$ as follows. If i and j' are two vertices of G_1 from different color classes, then let ij' be an edge of G_1 if there is at least one edge between the sets of vertices { *2i-1,2i*} and $\{(2j-1)', (2j)'\}$ in G. then it is clear thatG_1 inherits the G_p-avoiding property of G. on the other hand, there are at most $15^{c[n/2]}$ different graphs G that can lead to the same graphG_1 as there are 15 possible nonempty subsets of edges between {*2i-1,2i*} and $\{(2j-1)', (2j)'\}$ in G, and, by Conjecture 4.60, G_1 has at most c[n/2] edges. That, we have

$$G_p(n) \leq 15^{c[n/2]} G_p[\frac{n}{2}].$$

Iterating this argument until on the right hand side we have$G_p(1) = 2$, we get $G_p(n) \leq 15^{2cn}$.

So, by proposition 4.62, we see that

$$G_p(n) < 15^{2cn}.$$

Therefore, we have proved the following.

PROPOSITION 4.2.3

[6] If the F*ü*redi-Hajnal conjecture is true, then the Stanley-Wilf conjecture is also true.

4.3 The Proof of the F*ü*redi-Hajnal conjecture

We close this Chapter by presenting the recent spectacular proof of conjecture 3.1.3, given by Marcus and Tardos.

Let P be a k×k permutation matrix, and let A be an $n \times n$ matrix that avoids P and contains exactly $f(n, P)$ entries 1.

Assume for simplicity that n is divisible byk^2. The crucial idea of the proof is a decomposition of A into blocks. While the simple idea of decomposing a matrix into smaller matrices is not new, the novelty of the Marcus-Tardos method is that it decomposes A into $(\frac{n}{k^2} \cdot \frac{n}{k^2})$ blocks, which are each of size $k^2 \times k^2$

For $(i, j) \in \left[\frac{n}{k^2}\right] \times \left[\frac{n}{k^2}\right]$, let $S_{i,j}$ denote the sub matrix(block) of A that consists of the intersection of rows $(i\text{-}1)\, k^2{+}1, (i\text{-}1)\, k^2{+}2, \dots, i\, k^2$ and columns $(j\text{-}1)\, k^2{+}1, (j\text{-}1)\, k^2{+}2, \dots, j\, k^2$. we will now contract A into a much smaller matrix B as follows. Each entry of B will contain some information about a block of A. The matrix B= $(b_{i,j})$ is of size $\frac{n}{k^2} \times \frac{n}{k^2}$, and

$$b_{i,j} = \begin{cases} o & \textit{if all entries of } \mathrm{S_{i,j}} \text{ are zero,} \\ 1 & \textit{if not all entries of } \mathrm{S_{i,j}} \text{ are zero.} \end{cases}$$

PROPOSITION 4.3.1

The matrix B avoids P.

PROOF Assume not, and take a copy P_c of P in B. Then considering A, and using the fact that P is permutation matrix, we can take a 1 from each block of A that defined an entry of P_c, and get a copy of P in A.

DEFINITION 4.3.2 A block $S_{i,j}$ of A is called wide if it contains a 1 in at least k different columns. Similarly, a block is called tall if it contains a 1 in at least k rows.

Note: a block has k^2 columns, but it is called wide if at least k of these columns contains a 1.

LEMMA 4.3.3

For any fixed j, the set of blocks $C_j = \{ S_{i,j} | 1 \leq i \leq \frac{n}{k^2} \}$ of the matrix A contains less than

$(K-1)\binom{k^2}{k} + 1$ *wide blocks.*

PROOF we show that if the statements of the lemma were false, the A would contain a copy of P. Indeed, assuming that the number of wide blocks in C_j is at least $(k-1)\binom{k^2}{k} + 1$, by the pigeon-hole principle there would be a k-tuple of integers $1 \leq c_1 < c_2 < \ldots < c_k \leq k^2$ so that there are k blocks $S_{a_1,j}$, $S_{a_2,j}$, …, $S_{a_k,j}$ that all contain a 1 entry in column c_i, for $1 \leq i \leq k$. In that case, it is easy to find a copy of P in A, which is a contradiction. Indeed, if the single 1 in column i of P is in row $p(i)$, then choose a 1 from column c_i of $S_{a_{p(i)},j}$. As the blocks $S_{a_1,j}$, $S_{a_2,j}$, …, $S_{a_k,j}$ are positioned in a column, the n entries 1 chosen this way will prove that A contains P, which is a contradiction.

It goes without saying that the argument can be made for the array of blocks

$R_i = \{S_{i,j} \ |1 \leq j \leq \frac{n}{k^2}\}$, thereby giving the following lemma.

LEMMA 4.3.4

For any fixed i, the set of blocks $R_i = \{S_{i,j} \ |1 \leq i \leq \frac{n}{k^2}\}$ of the matrix A contains less than $(k-1)\binom{k^2}{k} + 1$ tall blocks.

We have seen that A cannot have too many wide or tall blocks, and proposition 4.64 seems to suggest that A cannot have too many nonzero blocks either. Putting together these observations, we get the following recursive estimate.

LEMMA 4.3.5

For any k×k permutation matrix P, and any positive multiples n of k^2, we have

$f(n, P) \leq (k\text{-}1)^2 f(\frac{n}{k^2}, P) + 2k^3 \binom{k^2}{k} n.$

PROOF by proposition 4.3.1 the number of non-zero blocks is at most $f(\frac{n}{k^2}, P)$. By lemma 4.3.3 and 4.3.4, there are at most $\frac{n}{k^2}$(k-1) $\binom{k^2}{k}$ wide blocks, and at most $\frac{n}{k^2}$(k-1) $\binom{k^2}{k}$ tall blocksLet us count how many 1 entries the various blocks of A can contribute to *f (n, P).* A wide (or tall) block can contribute at most k^4 entries 1, so the total contribution of these blocks is at most

$$2\frac{n}{k^2}(k-1)\binom{k^2}{k}k^4 < 2k^3\binom{k^2}{k}n$$

If a block is neither wide nor tall, by the pigeon-hole principle, it can contain at most $(k\text{-}1)^2$ entries 1. Multiplying this bound by the number of nonzero blocks yields that the contribution of all blocks that are not tall or wide is at most $(k\text{-}1)^2 \ f(\frac{n}{k^2}, P)$.Now add all the contributions of all this blocks gives us, $f(n, P) \leq (k\text{-}1)^2 f(\frac{n}{k^2}, P) + 2k^3 \binom{k^2}{k} n.$

Therefore the lemma is proved.

We know have all necessary tools to prove the Füredi-Hajnal conjecture.

Theorem 4.3.6

For all permutation matrices P of size k×k, we have

$f(n, P) \leq 2k^4 \binom{k^2}{k} n.$

Proof: we proof the statement by induction on n, the initial case of n=1 being obvious. (In fact, the statement is obvious when $n \leq k^2$), because then A has at most k^4 entries.)

Now assume the statement is true for all positive integers less than n, and prove it for n.

Let$n' = \lceil n/k^2 \rceil k^2$. Then by lemma 3.3.5, we have

$f(n, P) \leq f(n', P) + 2k^2 n$

$$\leq (k\text{-}1)^2 \ f(\frac{n'}{k^2}, P) + 2k^3 \binom{k^2}{k} n' + 2k^2 n$$

As in the worst case, we fill the part of A that cannot be partitioned $k^2 \times k^2$ blocks by entries 1. Applying the induction hypothesis to $f(\frac{n'}{k^2}, P)$, we get

$$f(n, P) \leq (k-1)^2[2k^4 \binom{k^2}{k} \frac{n'}{k^2}] + 2k^3 \binom{k^2}{k} n' + 2k^2 n$$

$$\leq 2k^2 ((k-1)^2 + k + 1) \binom{k^2}{k} n$$

$$\leq 2k^4 \binom{k^2}{k} n,$$

Since $k^2 > (k-1)^2 + k + 1$ whenever $k \geq 2$.

The proof of the Stanley –Wilf conjecture is now obvious. We include it because we want to show the numerical result obtained.

Corollary 4.3.7

For any permutation pattern q of length k, we have

$$S_n(q) \leq c_q^n, \quad \text{where} c_q = 15^{2k^4\binom{k^2}{k}}.$$

Proof Immediate from (4.14) and theorem 4.69.

The value $c_q = 15^{2k^4\binom{k^2}{k}}$ is certainly very high. For instance, if k=3, then we get $c_q = 15^{27216}$,

While we have seen that $c_3 = 4$ is the best possible result.

Therefore, there is a lot of room for further research here.

Chapter-five

5. Conclusion

In this project we have seen that pattern avoiding permutations for pattern of length three, monotone pattern and pattern of length four, the number of $S_n(q)$ n-permutations that avoid the pattern q is generally too difficult. But in this topic, we observed some arguments. Thus,

For any q$\in S_3$ it is known that $S_n(q) = \frac{1}{n+1}\binom{2n}{n}$ Bóna [11] calculated the precise value of $S_n(q)$ for q = 1342, and obtained exponential upper bounds for $S_n(q)$ for all q$\in S_4$ [12]. When q is the identity of S_k, $S_n(q)$ is the number of n-permutations with no increasing subsequence of length k. Such permutations can be partitioned into k-1 monotone subsequences, that is, in theorem 2.4 we have seen that it is less than$(k-1)^{2n}$.The exact asymptotic for this case is also known ([13]).

The following conjecture of Stanley and Wilf.

Conjecture: for every q there exists a constantc_q such that $S_n(q) \leq c_q^n$ for every n.

This Conjecture is known to be true in many special cases, as in this project we get some satisfied results. And see [14] and its references.

They also suggested a stronger conjecture, namely, that for every fixed q the limit, as n tends to infinity, $\lim_{n\to\infty} \sqrt[n]{S_n(q)}$ exists and is finite and positive. That is,

$\lim_{n\to\infty} \sqrt[n]{S_n(q)}$ For some pattern q. in fact we have seen that

$$\lim_{n\to\infty} \sqrt[n]{S_n(q)} = \begin{cases} 4 \quad , & if\ q\ is\ of\ length\ 3, \\ (k-1)^2 \quad , & if\ q = 123\cdots k, \\ 8 \quad , & if q = 1342, \end{cases}$$

As we observe in every case when we saw an exact answer, it is an integer.

In general, it is not true, B$o\acute{n}a$ [15] has recently proved that that

$\lim_{n\to\infty} \sqrt[n]{S_n(12453)}$ = 9+4$\sqrt{2}$, so these limits are not even always rational.

We conclude this project with a listing of questions for further study of this topic, the questions are:

1. For every permutation q $\in S_k$ there exist c_q such that $S_n(q) \le c_q^n$ for all n is a very large number. How do we reduce this number to smallest number?

2. What is the exact answer for $\lim_{n\to\infty} \sqrt[n]{S_n(q)}$?

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