ADDIS ABABA UNIVERSITY

COLLEGE OF NATURAL AND COMPITITIONAL SCIENCES

DEPARTMENT OF MATHEMATICS

APPLICATIONS OF GREEN'S FUNCTION TO DIRICHLET PROBLEMS

A Thesis summitted to department of Mathematics of Addis Ababa University in partial fulfillment of the reuirement of the Masters of Science Degree in Mathematics

By

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We, the undersined, hereby certfiy that we have read and examine this thesis on **Application of Green's function to Dirichlet problem** which is done by Chali Bekele Nemera in partial fuliffelemnt ot the requirements for MSc and recommend to the school of graduate studies for acceptance of this thesis.

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Declaration

I, the under signed, declare that this thesis is my original work and has not been presented for any other university and that all source of materials used for the thesis have been duly acknowledged.

Name: Chali Bekele

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Acknowledgement

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Abstract

The aim of this paper is to present a new definition of the Green function of the Dirichlet problem for the Laplace equation prompted by the theory of ordinary Differential equation and Partial Differential Equations. A Green function, a mathematical function that was introduced by George Green in 1793-1841. Green function use for solving Ordinary and Partial differential equations in different dimensions for both time dependent and time independent problems. Green function is used in many theories such as quantum field theory, Electrodynamics and statistical field theory to refer various types of functions.

Keywords: Laplace's equation, Poisson equation, Fundamental solution, Green's function, Green's theorem, Green's identities, Green's properties, Dirichlet boundary conditions.

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Introduction

In mathematics, Green's function is the impulse response of an inhomogeneous differential equation defined on a domain, with specified initial conditions or boundary conditions. Green's functions are named after the British Mathematician George Green, who first developed the concept in 1828 G.C.

Green's functions are a device used to solve difficult Ordinary and Partial Differential equations. One of the application of Green's function is to solve Dirichlet boundary value problems. Dirichlet problems asks for a function which solves a specified partial differential equation(PDE) in interior of a given region that takes prescribed values on the boundary of the region. For a given boundary value problem, Green's function is a fundamental solution satisfying a boundary conditions. One advantage of using Green's function is that it reduces the dimension of the problem by one.

The Dirichlet problem can be solved for many PDEs, although originally it was posed for Laplace's equations as well as the inhomogeneous part of Laplace's equation which is called Poisson's equations. The Dirichlet Green's is the unique fundamental solution that is identically zero at any given closed surface and there for eliminates the single layer potential. Green introduce the function U with three properties.

1. U must be zero on the surfaces.

2. U must be satisfies the potential equation in $\mathbb{R}$.

3. At a fixed undermined point p in the interior U, becomes infinite.

This paper concerned about application of Green's function to solve Dirichlet problems in two dimensions and the main objective of this paper is to solve Dirichlet problems of Laplace's equation and the equation of inhomogeneous part of Laplace's equations, Poisson's equations.

This paper has three main parts. In first part concerned with preliminaries, notations, definitions, Delta function and the properties Drac delta, basic properties of fundamental solutions of Laplace's equation, concepts and several results that are frequently used in the next parts.

The second part is concerned with Green's function, Green's theorems and Green's Identities those used to obtain properties of the solution of Laplace's equations and modifying this formula and introducing the Green's function.

The third part is the main part of the thesis concerned with application of Green's function to solve Dirichlet problems of Upper half plane, Disks and Rectangles.

CHAPTER 1

PERLIMINARIES

In this chapter we discuss basic definitions, properties of real valued functions, theories, several results and techniques of finding fundamental solutions os Laplace's equations and Poisson's equations that frequently used in later chapter.

Definitions and Notations

Notations

$\mathbb{R}$- the set of real numbers $\mathbb{R}$

Ω- denote region on plane

Δ- Laplace operator

∇- gradient

$\partial\Omega$- denote the boundary of the region

δ- denote Dirac delta function

$\mathbb{R}^n$- denote the set of real number in n-dimensions.

Acronymes

BVP- boundary value problem

ODE-Ordinary Differential Equation

PDE-Partial Differential Equation

BCs- Boundary Conditions

1.1.2 .Definitions

Definition 1.1 Let U be a function an open set which has second order continuous partial derivatives. Then the Laplacian of U in two dimension is defined by

$$\Delta U=\frac{\partial U}{\partial x^2}+\frac{\partial U}{\partial y^2}$$

Definition 1.2 Let U(x,y) be real valued function that has second order continuous partial derivatives in an open set O and satisfies the Laplace equation

$\Delta U= U_{xx} + U_{yy}=0$ in the region, then we say that U is harmonic in that region.

Definition 1.3 The n –dimension of Laplace equation in $\mathbb{R}^n$ is given by

$\Delta U=0$, $x\in \mathbb{R}^n$ and its inhomogeneous version $\Delta U=f$ is called Poisson's equation.

Definition 1.4 A function $U: \mathbb{R}^3 \rightarrow \mathbb{R}$ is said to be radially symmetric if there exists a real valued function of a single variable $f:[0, \infty] \rightarrow \mathbb{R}$ such that $U(x,y,z)=f(\sqrt{x^2+y^2+z^2})$ for all $x,y,z\in \mathbb{R}$

Definition 1.5 A Dirichlet problem is the problem of finding a function which solves specific PDE in an interior of a given region that takes prescribed value on the boundary of the region.

Definition 1.6 A Green's function is the impulse response of inhomogeneous linear differential operator defined on a domain with specified initial condition or boundary condition.

Definition 1.7 A function $\phi(x)$ is a test function if it has the following properties.

1. $\phi(x)$ and all its derivative exist and continuous at every points (ϕ is smooth).

2. The integrals of $\int_{-\infty}^{\infty}\phi(x)dx$ and all its derivative exist and are finite.

Delta Function

The Dirac delta function or impulse function δ ,which is not speaking function is used to us technical device in mathematical formulation of Quantum Mechanics. The Dirac delta function has the following three properties.

1. $\delta(\text{x})=\begin{cases}0\,, x\neq 0\\ \infty, x=0\end{cases}$ 2. $\int_{-\infty}^{\infty}\delta(x)dx=1$ 3. $\int_{-\infty}^{\infty}f(x)\delta(x-a)dx=f(a)$,

where f is continuous at x=a and which is called shifting property of delta function. The delta function has the following properties in 2D.

1. $\delta(\text{x}) = \begin{cases}0\,, (\,x,y)\neq 0\\ \infty, (x,y)=0\end{cases}$ (1)

2. $\int_{-\infty}^{\infty}\delta(x,y)dA=1$ (2)

3. $\int_{-\infty}^{\infty}f(y,Y)\delta(x-a,y-b)dA=f(a,b)$ (3)

And the same is true in 3D.To make proof with the delta function more rigorous ,we consider the sequence which is called the sequences of the delta functions that converges to the δ –function at least in the point wise sense. We can also define the delta function as the limit of various sequences of regular functions, for example;

Consider the sequence

$$\delta_n\,(x)=\frac{n}{\sqrt{\pi}}e^{(-nx)^2}$$

$$\int_{-\infty}^{\infty}\delta_n\,(x)dx=\int_{-\infty}^{\infty}\lim_{n\to\infty}G_n(x)dx$$

Let $I=\int_{-\infty}^{\infty}\lim_{n\to\infty}G_n(x)dx$

$$I=\int_{-\infty}^{\infty}\lim_{n\to\infty}\frac{n}{\sqrt{\pi}}e^{(-nx)^2}\,dx$$

$$I=\lim_{n\to\infty}\int_{-\infty}^{\infty}\frac{n}{\sqrt{\pi}}e^{(-t)^2}\frac{dt}{n}$$

$$I=\frac{1}{\sqrt{\pi}}\lim_{n\to\infty}\int_{-\infty}^{\infty}e^{(-t)^2}\,dt\quad ,$$

$$I=\frac{1}{\sqrt{\pi}}\pi$$

$$I=1=\int_{-\infty}^{\infty}\delta_n\,(x)dx$$

1.3 Laplace's Equations

1.3.1 Fundamental solution of Laplace's Equations

1.3.2 Derivative of fundamental solution

There are a lot of functions U which satisfy the Laplace's equation. I n particular any constant functions are harmonic functions. U(x,y)= x^2-y^2, U(x,y)= e^xcosy, … are solutions of Laplace's equations and of course we can list the number of others. Here, however, we are interested in finding a particular solutions of Laplace's equation which will allows us to solve Poisson's equations. Since Laplace's equation is invariant under translation and rotation, it is consequently seems advice able to search for radial solution. In n dimensions, let us seek solutions of $\Delta u(x,y)=0$ of the form $u(x,y)=v(r)$.That is we look for harmonic function U on set of real numbers such that

$$u(x)=v(|r|)$$

Where x= x_1, x_2, …,x_n and it reduces PDE into ODE which is generally *easier to solve*.

Letting $r=|x|$, *we see that* $U(x)=V(|X|)$ *is a radial solution of Lap*lace's equation in

$$u(x)=v(r)$$

Where $r=\sqrt{x^2+y^2}$ in 2D, $r=\sqrt{x^2+y^2+z^2}$ in 3D, … and $r=\sqrt{x_1^2+x_{2+\cdots+x_n^2}^2}$ in n-dimensions.

$$r=|x|$$

$$\frac{\partial r}{\partial x_i}=\frac{\partial |x|}{\partial x_i}$$

$$=\frac{\partial}{x_i}\sqrt{x_1^2+x_{2+\cdots+x_n^2}^2}$$

$$=\frac{2x_i}{2\sqrt{x_1^2+x_{2+\cdots+x_n^2}^2}}$$

$$=\frac{x_i}{r} \qquad (1.4)$$

Then $u(x) = v(x)$, by chain rule

$$\frac{\partial u}{\partial x_i} = \frac{\partial}{\partial x_i}[v(r)]$$

$$\frac{\partial u}{\partial x_i} = \frac{\partial r}{\partial x_i}[v'(r)]$$

$$ux_i = \frac{x_i}{r}[v'(r)]$$

Solving for ux_ix_i

$$ux_ix_i = \frac{\partial ux_i}{\partial x_i}$$

$$\frac{\partial^2 u}{\partial x_i^2} = \frac{\partial}{\partial x_i}\left[v'(r)\frac{\partial r}{\partial x_i}\right]$$

$$= \frac{x_i^2 v"(r)}{r^2} + v'(r)\left[\frac{\partial x_i}{\partial x_i r}\right]$$

$$= \frac{x_i^2 v"(r)}{r^2} + v'(r)\left[\frac{r\partial x_i}{\partial x_i} - \frac{r\partial x_i}{\partial x_i}\right]$$

$$= \frac{x_i^2 v"(r)}{r^2} + v'(r)\left[\frac{r}{r^2} - \frac{x_i x_i}{rr^2}\right]$$

$$ux_ix_i = \frac{x_i^2 v"(r)}{r^2} + v'(r)\left[\frac{1}{r} - \frac{x_i^2}{r^3}\right]$$

$$\sum_{i=1}^n ux_ix_i = \sum_{i=1}^n\left(\frac{x_i}{r^2}v"(r) + v'(r)\left(\frac{1}{r} - \frac{x_i^2}{r^3}\right)\right)$$

$$= v"(r)\sum_{i=1}^n \frac{x_i^2}{r^2} + v'(r)\sum_{i=1}^n \frac{1}{r} - v'(r)\sum_{i=1}^n \frac{x_i^2}{r^3}$$

$$\Delta u = v"(r)(1) + v'(r)\frac{n}{r} - v'(r)\frac{1}{r}$$

$$0 = v"(r)(1) + v'(r)\frac{n}{r} - v'(r)\frac{1}{r}$$

$$0 = v''(r) + v'(r)\left(\frac{n-1}{r}\right)$$

$$v''(r) = -v'(r)\left(\frac{n-1}{r}\right)$$

$$\int \frac{v''(r)}{v'(r)}dr = \int\left(\frac{1-n}{r}\right)dr$$

$$lnv'(r) = (1-n)lnr + A$$

$$v'(r) = \frac{A}{r^{n-1}} \qquad (1.5)$$

$$v(r) = Alnr + B$$

For n= 3

$$v(r) = \int Ar^{-(n-1)}dr$$

$$v(r) = \frac{Ar^{-(n-1)+1}}{-(n-1)+1} + B$$

$$v(r) = A\frac{r^{-(n-2)}}{-(n-2)} + B$$

$$v(r) = \frac{A}{(2-n)}r^{-(n-2)} + B$$

$$v(r) = \frac{A}{(2-n)r^{n-2}} + B$$

Therefore, from above calculation we have,

$$v(r) = \begin{cases} Alnr + B, n = 2 \\ \frac{A}{(2-n)r^{n-2}}, n \geq 3 \end{cases} \qquad (1.6)$$

We know that the function U defined ( ) above is satisfies

$\Delta u = 0, x \neq 0$ *and* $\Delta u(0) is\ undefined\ at\ x = 0$

Define the function v as follows for |x|≠0,

$$v(r) = k(|x|) = \begin{cases} \frac{1}{2\pi}\ln|x|, \ n = 2 \\ \frac{1}{n(2-n)\omega_n|x|^{n-2}}, \ n \geq 3 \end{cases} \qquad (1.7)$$

Where ω_n is the volume of the unit ball in $\mathbb{R}^n$and V satisfies Laplace's equation $\mathbb{R}^n - \{0\}$

Claim 1 V is a fundamental solution of Laplace's equation and satisfies

$-\Delta v = \delta(x), \quad n = 3$ .In the sense of distribution.

According to the definitional derivatives,

$$(\Delta k)(\varphi) = (-1)^n(\Delta\varphi)$$

$$(\Delta k)(\varphi) = (-1)^3(\Delta\varphi)$$

$$(\Delta k)(\varphi) = -k(\Delta\varphi)$$

$$(\Delta k)(\varphi) = \int_{\mathbb{R}^n} -k(\,(\Delta\varphi)$$

Where the last the last equality becomes from the fact that k(|x|) is locally integrable , we need only to show

$$\int_{\mathbb{R}^n} k\,(|x|)\varphi(x) = \varphi(0), \forall \epsilon c_0^\infty$$

Now we like to apply divergence theorem, but k(|x|) has singularity at x=0, we get around this by taking up the integral in two pieces.

$$\partial\Omega_\epsilon = \partial\Omega \cup \partial\Omega_\epsilon$$

$$\Omega = \mathrm{B_R} - \mathrm{B_\epsilon}$$

$$\int_{\partial\varepsilon} (\cup\,\Delta v - v\Delta u)dx = \int_{\partial\Omega\varepsilon}\left(u\frac{\partial v}{\partial n} - v\frac{\partial u}{\partial n}\right)ds$$

$$\int_{\partial\varepsilon}\left(\cup\,\Delta v dx - \int_{\Omega\varepsilon}\left(u\frac{\partial v}{\partial n} - v\frac{\partial u}{\partial n}\right)\right)ds + \int_{\partial\Omega\varepsilon}\left(u\frac{\partial v}{\partial n} - v\frac{\partial u}{\partial n}\right)ds$$

$$\int_{\Omega\varepsilon}\varphi\Delta k dx - \int_{\Omega\varepsilon} k(|x|)\Delta\varphi dx = \int_{\partial\Omega}\left(\varphi\frac{\partial k(|x|)}{\partial n} - \frac{k\partial\varphi}{\partial n}\right)ds + \int_{\partial B\varepsilon}\left(\varphi\frac{\partial K(|x|)}{\partial n} - k(|x|)\frac{\partial\varphi}{\partial n}\right)ds$$

$$-\int_{\Omega\varepsilon} k(|x|)\Delta\varphi dx = \int_{\partial B\varepsilon}\left(\varphi\frac{\partial k(|x|)}{\partial n} - k(|x|)\frac{\partial\varphi}{\partial n}\right)ds$$

$$-\int_{\Omega\varepsilon} k(|x|)\Delta\varphi dx = \int_{\partial B\varepsilon}\varphi\frac{\partial k}{\partial n}ds - \int_{\partial B\varepsilon} k\frac{\partial\varphi}{\partial n}ds$$

$$\int_{\Omega\varepsilon} k(|x|)\Delta\varphi dx = \lim_{\epsilon\to\infty} A - \lim_{\in\to\infty} B$$

$$\int_{\partial B\varepsilon(0)}\varphi\frac{\partial k}{\partial n}ds = \int_{\partial B\varepsilon(0)}\varphi\frac{\partial}{\partial\varepsilon}\left(\frac{\varepsilon^{2-n}}{(2-n)\omega_n}\right)ds$$

$$\int_{\partial B\varepsilon(0)}\varphi\frac{\partial k}{\partial n}ds = \int_{\partial B\varepsilon(0)}\varphi\frac{\partial}{\partial\varepsilon}(\varepsilon^{2-n})\left(\frac{1}{(2-n)\omega_n}\right)ds$$

$$\int_{\partial B\varepsilon(0)}\varphi\frac{\partial k}{\partial n}ds = \int_{\partial B\varepsilon(0)}\varphi(2-n)\varepsilon^{(2-n)-1}\left(\frac{1}{(2-n)\omega_n}\right)ds$$

$$\int_{\partial B\varepsilon(0)}\varphi\frac{\partial k}{\partial n}ds = \int_{\partial B\varepsilon(0)}\varphi\frac{-1}{\varepsilon^{n-1}\omega_n}\varphi(0)ds + \int_{\partial B\varepsilon(0)}\frac{-1}{\omega_n\varepsilon^{n-1}}[\varphi(x) - \varphi(0)]ds$$

$$\int_{\partial B\varepsilon(0)}\varphi\frac{\partial k}{\partial n}ds = \frac{-1}{\varepsilon^{n-1}}\varphi(0)\int_{\partial B\varepsilon(0)} ds + max_{x\epsilon\partial B\varepsilon(0)}[\varphi(x) - \varphi(0)]ds$$

$$\int_{\partial B\varepsilon(0)}\varphi\frac{\partial k}{\partial n}ds = \frac{-1}{\varepsilon^{n-1}\omega_n}\varphi(0)(\omega_n\varepsilon^{n-1})$$

$$A = -\varphi(0)$$

$$B = \int_{\partial B\varepsilon(0)} \varphi \frac{\partial k}{\partial n} ds$$

$$\left|\int_{\partial B\varepsilon(0)} k(|x|) \frac{\partial \varphi}{\partial n} ds\right| = |k(\varepsilon)| \int_{\partial B\varepsilon(0)} \left|\frac{\partial k}{\partial n}\right| ds$$

$$\leq |k(\varepsilon)| max|\nabla\varphi|(\omega_n \varepsilon^{n-1})$$

$$= |k(|\varepsilon|)|(\omega_n \varepsilon^{n-1}) max|\nabla\varphi|$$

$$B = \lim_{\varepsilon\to 0} \int_{\partial B\varepsilon(0)} k(|x|) \frac{\partial\varphi}{\partial n} ds = \lim_{\varepsilon\to 0}|k(|\varepsilon|)|(\omega_n \varepsilon^{n-1}) max|\nabla\varphi| = 0$$

Therefore,

$$-\int_\Omega k(|x|)\Delta\varphi(x)dx = A + B$$

$$-\int_\Omega k(|x|)\Delta\varphi(x)dx = -\varphi(0) + 0$$

$$-\int_\Omega k(|x|)\Delta\varphi(x)dx = -\varphi(0)$$

$$\int_\Omega k(|x|)\Delta\varphi(x)dx = \varphi(0) \qquad (1.8)$$

Well possedness and maximum principle.

Assuming that there are actually exists a solution of the Dirichlet problem for Laplace's equation, will prove that the solution is unique and is sensitive to small charge in the boundary condition. The proves make use of the maximum principle for Laplace's equation which we now state:

Maximum principle for Laplace's equation

Definition: Let Ω be a set that is bounded ,open and connected in their two or three dimensions .

Let U be a function that is harmonic in side Ω and continuous on $\Omega \cup \partial\Omega$. If U is attains its maximum or minimum value inside Ω, then U is constant.

Example: Let$u(x,y) = x^2 - y^2 + 8$. Find the maximum and minimum value of U on the disk $x^2 + y^2 \leq 1$

Solution

Since U is constant function, by maximum principle the extreme value of U occur on the boundary of the domain $u(x,y) = x^2 + y^2$

$y^2 = 1 - x^2$

substituting in to u get

$u(x, y) = x^2 - (1 - x^2) + 8$

$u(x, y) = 2x^2 + 8$

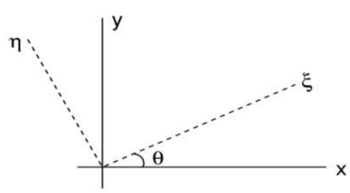
The maximum of $u(x, y) = 2x^2 + 8$ occurs when$x = 0 \Rightarrow y = \pm 1$

Minimum value $u(x, y) = u(0,1) = u(0, -1) = 7$

Maximum value $u(x, y) = u(1,0) = u(-1,0) = 9$

Translation and rotation invariance of Laplace's equation

1. Laplace's equation is a translation invariant of in two or three dimensions.



2.

Consider in two dimensional case:

By change of coordinates $\xi = x + \alpha$ and $\eta = y + \beta$ which has effect of shifting every points in the in plane in the direction of (α, β)

$u_{xx} + u_{yy} = 0$ can be written in terms of the new coordinate (ξ, η)

By using Chain rule,

$$u_\xi = u_x \frac{dx}{d\xi} + u_y \frac{dy}{d\xi}$$

$$u_\xi = u_x u_y$$

$$u_\xi = u_x \frac{dx}{d\eta} + u_y \frac{dy}{d\eta}$$

$$u_\xi = u_y \qquad (1.9)$$

And calculating the second derivative yields

$$u_{xx} = u_{\xi\xi} \quad \text{and} \quad u_{yy} = u_{\eta\eta}$$

Therefore, $u_{\xi\xi} = u_{\eta\eta}$ (1.10)

This demonstrating that Laplace equation is not affected by translating of the coordinates.

3. Laplace's equation is rotation invariant.

Making change of coordinate that rotates the plane through an angle θ doesn't affect Laplace's equation. In two dimensions such a change of coordinates makes the form

$$\begin{pmatrix} \xi \\ \eta \end{pmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

$$\begin{pmatrix} \xi \\ \eta \end{pmatrix} = \begin{bmatrix} x\cos\theta & -y\sin\theta \\ x\sin\theta & y\cos\theta \end{bmatrix}$$

This verifying that coordinate transformation really does rotate the plane through an angle θ is not difficult.

We should check that

1. The vector (ξ, η) and (x, y)have same length.
2. The angle between these two vectors is θ.

$$u_x = u_\xi \frac{d\xi}{dx} + u_y \frac{d\xi}{dx} = (cos\theta)u_\xi + (sin\theta)u_\xi \quad (1.11)$$

$$u_y = u_\xi \frac{d\xi}{dx} + u_y \frac{d\eta}{dy} = (cos\theta)u_\eta + (sin\theta)u_\eta \quad (1.12)$$

calculating the second derivative

$$u_{xx}+u_{yy} = (sin^2\theta + cos^2\theta)(u_{\xi\xi}+u_{\eta\eta}) \quad u_{xx}+u_{yy} = u_{\xi\xi}+u_{\eta\eta}$$

It follows that rotating our coordinate system has no effect on Laplace's equation.

CHAPTER 2

GREEN'S FUNCTION

2.1 GREEN'S THEOREM AND GREEN'S IDENTITIES

Green's theorem is mainly used for the integration of the line combined with a curved plane. This theorem shows the relationship between the line integral and surface integral. It is related to many theorems such as Gauss's and Stoke's theorem. Green's theorem is used to integrate the derivative in particular plane. If a line integral is given it converted to surface integral, double integral or vice versa by using Green's theorem. Green's theorem is a planner region in striking result from the calculus of several variables that relates as a line integral around a closed curve to double integral over the region bounded by the line.

2.1.1 Green's theorem

Theorem 2.1 Suppose that C is a positively oriented simple piecewise smooth and close curve with interior region Ω. If $u(x,y)$ and $v(x,y)$ be continuous functions with continuous first partial derivatives on Ω and $\partial\Omega$ then,

$$\int_c u(x,y)dx + v(x,y)dy = \iint_\Omega \left(\frac{\partial v}{\partial n} - \frac{\partial u}{\partial n} \right) dxdy \qquad (2.1)$$

where C is traversed so that Ω is always on the left side of C.

<u>poof:</u>

Let us prove in case of simple region Ω where the boundary curve C can be written as

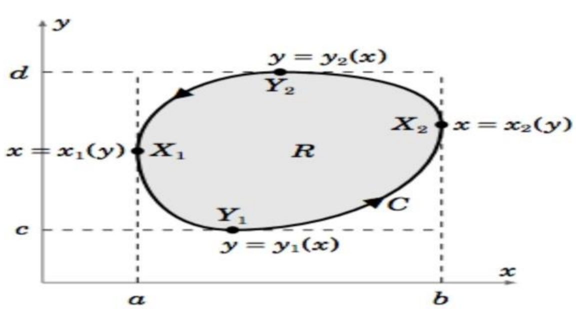


Fig 2.1

Let $C = C_1 \cup C_2$

C1 = the curve $y_1(x)$

C2 = the curve $y_2(x)$, x_1 and x_2 are points on the curve C farthest to left and right, y_1 and y_2 are lowest and highest points respectively.

$$\oint_c u(x,y)dx = \int_{C_1} u(x,y)dx + \int_{C_2} u(x,y)dx$$

$$\oint_c u(x,y)dx = \int_a^b u(x, y_1(x))dx + \int_b^a u(x, y_2(x))dx$$

$$\oint_c u(x,y)dx = \int_a^b u(x, y_1(x))dx - \int_a^b u(x, y_2(x))dx$$

$$\oint_c u(x,y)dx = -\int_a^b [u(x, y_1(x)) - u(x, y_2(x)]$$

$$\oint_c u(x,y)dx = -\int_a^b \int_{y_1(x)}^{y_2(x)} \frac{\partial u(x,y)}{\partial y} dxdy$$, by fundamental theorem of calculus

$$\oint_c u(x,y)dx = -\iint_\Omega \frac{\partial u}{\partial n} dA \qquad (2.2)$$

Again integrating $v(x,y)$ around **C** we've

$$\oint_c V(x,y)dy = \int_{C_1} V(x,y)dy + \int_{C_2} V(x,y)dy$$

$$\oint_c V(x,y)dy = \int_d^c V(x_1(y), y)dy + \int_c^d V(x_2(y), y)dy$$

$$\oint_c V(x,y)dy = -\int_c^d V(x_1(y), y)dy + \int_c^d V(x_2(y), y)dy$$

$$\oint_c V(x,y)dy = -\int_c^d [V(x_2(y), y) - V(x_1, y)]dy$$

$$\oint_c V(x,y)dy = -\int_c^d \int_{x_1(y)}^{x_2(y)} \frac{\partial V(x,y)}{\partial x} dxdy$$

$$\oint_c V(x,y)dy = -\iint_\Omega \frac{\partial V}{\partial x} dA$$

$$\int_c u(x,y)dx + v(x,y)dy = \iint_\Omega \left(\frac{\partial v}{\partial n} - \frac{\partial u}{\partial n}\right) dxdy \tag{2.3}$$

Theorem 2.2: Green's theorem for multiply connected regions.

Let Ω be a multiply connected region with boundary $\partial\Omega$.

Suppose that (x, y) *and* $v(x,y)$ continuous functions with continuous first partial derivatives on Ω and $\partial\Omega$. Then

$$\int_{\partial n} (u(x, y)dx + v(x, y)dy) = \iint_\Omega \left(\frac{\partial v}{\partial n} - \frac{\partial u}{\partial n}\right) dxdy \tag{2.4}$$

2.2.Green's identities

Theorem 2.3 (Divergence theorem)

Let Ω be a multiple connected region with boundary $\partial\Omega$ a bounded solid region with C^1 boundary curve and **n** be unit out ward vector on Ω and F be any C^1 vector field on then

$$\iint_\Omega div\, \vec{F} dA = \int_{\partial\Omega} (F.\vec{n})dS$$

$$\iint_\Omega div(\nabla u)dA = \int_{\partial\Omega} (\nabla u).ndS$$

$$\iint_\Omega \nabla(\nabla u)dA = \int_{\partial\Omega} \frac{\partial u}{\partial n} dS$$

$$\iint_\Omega \Delta u dA = \int_{\partial\Omega} \frac{\partial u}{\partial n} dS \tag{2.5}$$

Theorem 2.3 (Green's Identities)

Let Ω be a bounded region in the plane whose boundary $\partial\Omega$ is a closed and piecewise smooth curve.

Let $u(x,y)$and$v(x,y)$ be continuous , with continuous first and second order partial derivatives on Ω and$\partial\Omega$. Then

Green's first identity

$$\iint_{\Omega}(u\Delta v + \nabla u \nabla v)dA = \int_{\partial\Omega} U \frac{\partial v}{\partial n} dS \tag{2.6}$$

Green's second identity

$$\iint_{\Omega}(u\Delta v - \text{v}\Delta u)dA = \int_{\partial\Omega} U \frac{\partial v}{\partial n} dS - \int_{\partial\Omega} v \frac{\partial u}{\partial n} dS \tag{2.7}$$

Proof(1)

By definition (2 .8), we've

$$\frac{\partial v}{\partial n} dS = v_x dy - v_y dx$$

$$\frac{\partial v}{\partial n} dS = v_x dy - v_y dx$$

$$u \frac{\partial v}{\partial n} dS = u v_x dy - u v_y dx \tag{2.8}$$

Let $M(x, y) = -u v_y$ and $M(x, y) = -u v_x$

$$\int_{\partial\Omega} u \frac{\partial v}{\partial n} dS = \int_{\partial\Omega} \left(M(x, y)dx + N(x, y)dy \right)$$

$$\int_{\partial\Omega} u \frac{\partial v}{\partial n} dS = \iint_{\Omega} \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right) dxdy$$ by Green's theorem.

$$\int_{\partial\Omega} u \frac{\partial v}{\partial n} dS = \iint_{\Omega} \left((u_x v_x + u v_{xx}) - (u_y v_y - u v_{yy}) \right) dxdy$$

$$\int_{\partial\Omega} u \frac{\partial v}{\partial n} dS = \iint_{\Omega} \left((u_x v_x + u v_{xx} + u_y v_y + u v_{yy} \right) dxdy$$

$$\int_{\partial\Omega} u \frac{\partial v}{\partial n} dS = \iint_{\Omega} \left((u v_{xx} + u v_{yy}) + (u_x v_x + u_y v_y) \right) dxdy$$

$$\int_{\partial\Omega} u \frac{\partial v}{\partial n} dS = \iint_{\Omega} \left(u(v_{xx} + v_{yy}) + (v_x + v_y).(u_x + u_y) \right) dxdy$$

$$\int_{\partial\Omega} u \frac{\partial v}{\partial n} dS = \iint_{\Omega} \left(u\Delta v + \nabla v \nabla u \right) dxdy$$

Proof :- (2)

$$\frac{\partial u}{\partial n}dS = u_x\,dy - u_y\,dx$$

$$v\frac{\partial u}{\partial n}dS = v\,u_x\,dy - v\,u_y\,dx \qquad (2.9)$$

$$\int_{\partial\Omega}\left(u\frac{\partial v}{\partial n} - v\frac{\partial u}{\partial n}\right)dS = \int_{\partial\Omega}\left((v\,u_y - u\,v_y)dx + (u\,v_x - v\,u_x)dy\right)$$

Let $M(x,y) = v\,u_y - u\,v_y$ and $N(x,y) = U\,v_x - v\,u_x$

$$M_y(x,y) = (v_y u_y + v\,u_{yy}) - (u_y v_y + u\,v_{yy})$$

$$N_x(x,y) = (u_x v_x + u\,v_{xx}) - (u_x v_x + v_x u_{xx})$$

$$\int_{\partial\Omega}\left(u\frac{\partial v}{\partial n} - v\frac{\partial u}{\partial n}\right)dS = \int_{\partial\Omega}\left(u(v_{xx} + v_{yy}) - v(u_{xx} + u_{yy})\right)dxdy$$

$$\int_{\partial\Omega}\left(u\frac{\partial v}{\partial n} - v\frac{\partial u}{\partial n}\right)dS = \int_{\partial\Omega}\left(u(\Delta v) - v(\Delta u)\right)dxdy$$

2.3.Green's properties

Theorem 2.3 Let Ω be a region with boundary $\partial\Omega$ and G is Green's function. Then

1. $G(x,y;x_0,y_0)$ is harmonic function of (x,y) in Ω but not at (x_0,y_0) due to the logarithm part V.
2. $G(x,y;x_0,y_0) = 0$ on $\partial\Omega$, i. e G satisfies homogeneous condition. (2.10)
3. G is continuous at point of discontinuity.
4. Green's function is uniquely determined by the region Ω.
5. $G(x,y;x_{0,},y_0) = G(x_0,y_0;x,y)$ for all (x,y) and (x_0,y_0) $in\ \Omega$ (2.11)

2.4.Green's Representation Formula

2.4.1.Green's representation formula for 2D

Theorem 2.4 If $G(x, y_0)$ is a Green's function in domain Ω, then the solution of Laplace's equation in Ω is given by

$$u(x_0) = \int_{\partial\Omega} u(x)\frac{\partial G(x,y_0)}{\partial n}dS \qquad (2.12)$$

$$\iint_{\partial\Omega}(u\Delta G - G\Delta u)\,dS = \int_{\partial\Omega}\left(u\frac{\partial v}{\partial n} - G\frac{\partial u}{\partial n}\right)dS$$

$$\iint_{\partial\Omega}(u\Delta G - G\Delta u)\,dS = \int_{\partial\Omega}\left(u\frac{\partial v}{\partial n}\right)dS - \int_{\partial\Omega}\left(G\frac{\partial u}{\partial n}\right)dS$$

$$\iint_{\partial\Omega}(u\Delta G)dS - \iint_{\partial\Omega}(G\Delta u)dS = \int_{\partial\Omega}\left(u\frac{\partial v}{\partial n}\right)dS - \int_{\partial\Omega}\left(G\frac{\partial u}{\partial n}\right)dS\ , \quad \Delta U = 0 \ in \ and\ G = 0\,,\partial\Omega$$

$$\iint_{\partial\Omega}(u\Delta G)dS = \int_{\partial\Omega}\left(u\frac{\partial G}{\partial n}\right)dS$$

$$\iint_{\partial\Omega}(u\Delta G)dS = \int_{\partial\Omega}\left(u\frac{\partial G}{\partial n}\right)dS$$

$$\iint_{\partial\Omega}u(\delta(x - x_0)dS = \int_{\partial\Omega}\left(u\frac{\partial G}{\partial n}\right)dS$$

$$U(x_0) = \int_{\partial\Omega}\left(u\frac{\partial G}{\partial n}\right)dS$$

Theorem 2.5: Suppose U is harmonic in side Ω and continuous on its boundary $\partial\Omega$.Let (x_O, y_O)be a point in side Ω, then

$$u(x_o, y_o) = \frac{1}{2\pi}\left(u\frac{\partial v}{\partial n} - v\frac{\partial u}{\partial n}\right)ds \qquad 2.12$$

Proof :- Draw the negatively oriented circle c_ε around in Ω.

Let π_ϵ denotes the region that consists of Ω minus disk of radius ε around at. $(x_o, y_o) \Rightarrow \partial\Omega = \Omega + c_\varepsilon$ and take

$$v = \frac{1}{2\pi}ln(x - x_o)^2 + (y - y_0)^2$$

The both u and v are harmonic inΩ. Then apply Green's second identity

$$\iint_{\Omega}(u\Delta v - v\Delta u)dA = \int_{\partial\Omega}u\frac{\partial v}{\partial n} - v\frac{\partial u}{\partial n}ds + \int_{c\varepsilon}\left(u\frac{\partial v}{\partial n} - v\frac{\partial u}{\partial n}\right)ds$$

$$\iint_{\Omega}(u\Delta v - v\Delta u)dA = \int_{\partial\Omega}u\frac{\partial v}{\partial n} - \int_{\partial\Omega}v\frac{\partial u}{\partial n}ds + \int_{c\varepsilon}u\frac{\partial v}{\partial n}ds - \int_{c\varepsilon}v\frac{\partial u}{\partial n}ds$$

$$u(x_o, y_o) = \int_{c\varepsilon}u\frac{\partial v}{\partial n}ds - \int_{c\varepsilon}v\frac{\partial u}{\partial n}ds \quad , v = 0\ on\ \partial\Omega \Rightarrow \frac{\partial v}{\partial n} = 0\,,\partial\Omega$$

Since c_ε is negatively oriented we have c_ε positively oriented. Now parametrize c_ε as by,
$x(\theta) = x_o + rcos\theta, y(\theta) = y_o + rsin\theta, 0 \leq \theta \leq 2\pi$

$$\Rightarrow ds = rd\theta, \qquad v - \frac{1}{2\pi} lnr \Rightarrow \frac{\partial v}{\partial r} - \frac{1}{2\pi r}$$

$$u(x_o, y_o) = \int_{c\varepsilon} u \frac{\partial v}{\partial n} ds - \int_{c\varepsilon} v \frac{\partial u}{\partial n} ds \tag{2.13}$$

$$-\int_{c\varepsilon} u \frac{\partial u}{\partial n} - v \frac{\partial v}{\partial n} ds = \int_0^{2\pi} u(x,y) \frac{\partial v}{\partial n} ds - \int_0^{2r} v \frac{\partial u}{\partial n} ds$$

$$-\int_{c\varepsilon} u \frac{\partial u}{\partial n} - v \frac{\partial v}{\partial n} ds = \int_0^{2\pi} u(x,y) \frac{\partial v}{\partial n} ds$$

$$-\int_{c\varepsilon} u \frac{\partial u}{\partial n} - v \frac{\partial v}{\partial n} ds = \int_0^{2\pi} u(x,y) \left(\frac{1}{2\pi r}\right) rd\theta$$

$$-\int_{c\varepsilon} \left(u \frac{\partial u}{\partial n} - v \frac{\partial v}{\partial n}\right) ds = \int_0^{2\pi} u(x,y) \left(\frac{1}{2\pi}\right) d\theta$$

$$-\int_{c\varepsilon} \left(u \frac{\partial u}{\partial n} - v \frac{\partial v}{\partial n}\right) ds = \frac{1}{2\pi} \int_0^{2\pi} u(x,y) d\theta$$

$$-\int_{c\varepsilon} \left(u \frac{\partial u}{\partial n} - v \frac{\partial v}{\partial n}\right) ds = \frac{1}{2\pi} \int_0^{2\pi} u(x_o + rcos\theta,\ y_o + rsin\theta) d\theta$$

$$2\pi u(x_o, y_o) = \int_0^{2\pi} urcos\theta(x_o +, y_o + rsin\theta) d\theta$$

$$u(x_o, y_o) = \frac{1}{2\pi} \int \left(u \frac{\partial v}{\partial n} - v \frac{\partial u}{\partial n}\right) ds \tag{2.14}$$

Theorem 2.6 Let Ω be simple or multiplied regions with boundary $\partial\Omega$. Suppose that u is harmonic in Ω then $\int_{\partial\Omega} \frac{\partial u}{\partial n} ds = 0$

$$\int_{\partial\Omega} \left(u \frac{\partial v}{\partial n} - v \frac{\partial u}{\partial n}\right) ds = \iint_\Omega (u\Delta v - v\Delta u) dxdy$$

$$\int_{\partial\Omega} \left(u \frac{\partial(1)}{\partial n}\right) ds - \int_{\partial\Omega} (1) \frac{\partial u}{\partial n} ds = \iint_\Omega \big(u\Delta(1)\big) dxdy - \iint_\Omega \big((1)\Delta u\big) ds$$

$$\int_{\partial\Omega} \left(u \frac{0}{\partial n}\right) ds - \int_{\partial\Omega} \frac{\partial u}{\partial n} ds = \iint_\Omega \big(u(0)\big) dxdy - \iint_\Omega (\Delta u) ds$$

$$\int_{\partial\Omega} \frac{\partial u}{\partial n} ds = \iint_\Omega (\Delta u) ds \qquad , \Delta u = 0, \Omega$$

$$\int_{\partial\Omega} \frac{\partial u}{\partial n} ds = 0 \tag{2.15}$$

Now we modify the representation formula in theorem (   ) in such a way that the resulting line integral doesn't involve normal derivative of U. For this purpose, suppose that there is harmonic function H on   such that $H = -v$ on $\partial\Omega$, where V mentioned in theorem (2.6)

To construct Green's function for bounded regions by using fundamental solution V fo

$$\begin{cases} \Delta G = \delta(x - x_0) \ , \ x\epsilon\Omega \\ G(x, x_0) = 0 \quad for\ x\epsilon\Omega \end{cases} \tag{2.16}$$

Where $x_o \epsilon \Omega$ is a fixed poin

Now we construct Green's function as follows

$$G(x, x_0) = V(x - x_0) + H(x - x_0) \qquad (2.16)$$

In order to G satisfy (2.15) H must satisfy $\begin{cases} \Delta H(x, x_0) = 0 \ , \ \ x\epsilon\Omega \\ H(x, x_0) = -V(x, x_0) \end{cases}$ (2.16)

Theorem 2.7 Suppose U is harmonic in side Ω and continuous on its boundary $\partial\Omega$.Let (x_o, y_o) be a point in side **Ω** and $U(x_o, y_o) = \frac{1}{2\pi}\int\left(u\frac{\partial v}{\partial n} - v\frac{\partial u}{\partial n}\right) ds \ , then$

$$\frac{1}{4\pi}\int_{\partial\Omega}\frac{\partial}{\partial n} ln(x - x_o)^2 + (y - y_o)^2 dA = 1$$

$$G(x, x_0) = V(x - x_0) + H(x - x_o)$$

$$G(x, y; x_0, y_0) = \frac{1}{2\pi} ln\sqrt{(x - x_0)^2 + (y - y_{0)}}^2 - \frac{1}{2\pi} ln\sqrt{(x - x_0)^2 + (y - y_0)^2}$$

$$G(x, y \ ; x_0, y_0) = \frac{1}{4\pi}\ln((x - x_0)^2 + (y - y_0)^2 - \frac{1}{4\pi} ln(x - x_0)^2 + (y - y_0)^2)$$

$$U(x_0, y_{0)} = \frac{1}{2\pi}\int_{\partial\Omega} u(x, y)\frac{\partial G(x,y;x_o,y_o)}{\partial n} ds$$

$$U(x_0, y_{0)=} \qquad \frac{1}{4\pi}\int_{\partial\Omega} u(x, y)\frac{\partial}{\partial n}\left(\frac{1}{4\pi} ln(x - x_0)^2 + (y - y_o)^2\right) ds - \frac{1}{4\pi}\int_{\partial\Omega} u(x, y)\frac{\partial}{\partial n}(ln(x - x_o)^2) + (y - y_o)^2 ds$$

$$1 = \frac{1}{4\pi}\int_{\partial\Omega}(1)\frac{\partial}{\partial n}\left(\frac{1}{4\pi} ln(x - x_o)^2 + (y - y_o)^2 ds - \frac{1}{4\pi}\int_{\partial\Omega}(1)\frac{\partial}{\partial n}(ln(x - x_o)^2 + (y - y_o)^2)\right) ds$$

$$1 = \frac{1}{4\pi}\int_{\partial\Omega}\frac{\partial}{\partial n}\left(\frac{1}{4\pi} ln(x - x_o)^2 + (y - y_o)^2 ds - \frac{1}{4\pi}\int_{\partial\Omega}\frac{\partial}{\partial n}(ln(x - x_o)^2 + (y - y_o)^2)\right) ds$$

$$1 = \frac{1}{4\pi}\int_{\partial\Omega}(1)\frac{\partial}{\partial n}\left(\frac{1}{4\pi} ln(x - x_o)^2 + (y - y_o)^2 ds - \frac{1}{4\pi}\int_{\partial\Omega}(1)\frac{\partial}{\partial n}(ln(x - x_o)^2 + (y - y_o)^2)\right) ds$$

$$1 = \frac{1}{4\pi}\int_{\partial\Omega}(1)\frac{\partial}{\partial n}(ln(x - x_o)^2 + (y - y_o)^2) ds \qquad (2.17)$$

2.4.2 Representation formula in 3D.

Theorem 2.8 Suppose U is harmonic inside Ω and continuous on the boundary ∂Ω. Let (x_0, y_0, z_0) a point in side Ω. Then

$$\boldsymbol{U(x_0, y_0, z_0)} = \iint_{\Omega} \left(\boldsymbol{u \frac{\partial n}{\partial n} - v \frac{\partial u}{\partial n}} \right) \boldsymbol{d}V$$

Proof:-

Let $\varepsilon \in \Omega$ sufficiently small such that $\beta_\varepsilon(x_0) \subset \Omega$,where r is spherical coordinate that is a distance the point x from its centre x_0 of the Sphere. For simplicity

$\Omega_\varepsilon = \Omega - \beta_\varepsilon(x_0)$ and $\Omega_\varepsilon = \partial\Omega_\varepsilon \cup \partial\beta_\varepsilon$,Where $\partial\Omega_\varepsilon$ is the boundary of Ω_ε.

But $\partial\Omega_\varepsilon$ consists of two pieces $\partial\Omega_\varepsilon$ and $\partial\beta_\varepsilon$ coincide on their common boundary while those of $\partial\Omega_\varepsilon$ and $\partial\beta_\varepsilon$ have opposite in direction.

By applying Green's second identity

$$\iiint_\Omega (u\Delta v - v\Delta u) dV = \iint_{\partial\Omega_\varepsilon} \left(u \frac{\partial v}{\partial n} - v \frac{\partial u}{\partial n} \right) d$$

$$\iiint_\Omega (u\Delta v - v\Delta u) dV = \iint_{\partial\Omega \cup \partial\beta_\varepsilon} \left(u \frac{\partial v}{\partial n} - v \frac{\partial u}{\partial n} \right) dS$$

$$\iiint_\Omega (u\Delta v) dV - \iiint_\Omega (v\Delta u) dV = \iint_{\partial\Omega} \left(u \frac{\partial v}{\partial n} - v \frac{\partial u}{\partial n} \right) dS + \iint_{\partial\beta_\varepsilon} \left(u \frac{\partial v}{\partial n} - v \frac{\partial u}{\partial n} \right) dS$$

$$- \iiint_\Omega (v\Delta u) dV = \iint_{\partial\Omega} \left(u \frac{\partial v}{\partial n} - v \frac{\partial u}{\partial n} \right) dS + \iint_{\partial\beta_\varepsilon} \left(u \frac{\partial v}{\partial n} - v \frac{\partial u}{\partial n} \right) dS$$

$$0 = \iint_{\partial\Omega} \left(u \frac{\partial v}{\partial n} - v \frac{\partial u}{\partial n} \right) dS + \iint_{\partial\beta_\varepsilon} \left(u \frac{\partial v}{\partial n} - v \frac{\partial u}{\partial n} \right) dS$$

$$0 = \iint_{\partial\Omega} \left(u \frac{\partial v}{\partial n} - v \frac{\partial u}{\partial n} \right) dS + \lim_{x\to 0} \left[\frac{1}{4\pi} \iint_{\partial\beta_\varepsilon} \left(u \left(\frac{\partial}{\partial r} \right) \frac{1}{r} - \frac{1}{r} \left(\frac{\partial}{\partial r} u \right) \right) dS \right]$$

$$0 = \iint_{\partial\Omega} \left(u \frac{\partial v}{\partial n} - v \frac{\partial u}{\partial n} \right) dS + \lim_{\varepsilon\to 0} \left[\frac{1}{4\pi r^2} \iint_{\partial\beta_\varepsilon} u dS + \frac{1}{4\pi r} \iint_{\partial\beta_\varepsilon} \frac{\partial u}{\partial \beta_\varepsilon} dS \right]$$

$$0 = \iint_{\partial\Omega} \left(u \frac{\partial v}{\partial n} - v \frac{\partial u}{\partial n} \right) dS - \lim_{\varepsilon\to 0} \left[\frac{1}{4\pi r^2} (4\pi R^2) M_\varepsilon(u) + \frac{1}{4\pi r} \left(4\pi R^2 M_\varepsilon \left(\frac{\partial u}{\partial n} \right) \right) \right]$$

where (U) and $M_\varepsilon\left(\frac{\partial U}{\partial r}\right)$ are minimum values of u and $\frac{\partial U}{\partial r}$ respectively and dS is surface area of Sphere.

$$0 = \iint_{\partial\Omega}\left(u\frac{\partial \mathrm{v}}{\partial n} - v\frac{\partial u}{\partial n}\right)dS - \lim_{\mathcal{E}\to 0}\left[M_\varepsilon(u) + rM_\varepsilon\left(\frac{\partial u}{\partial n}\right)\right]$$

$$0 = \iint_{\partial\Omega}\left(u\frac{\partial \mathrm{v}}{\partial n} - v\frac{\partial u}{\partial n}\right)dS - \lim_{\mathcal{E}\to 0}\left[M_\varepsilon(u) + \mathcal{E}M_\varepsilon\left(\frac{\partial u}{\partial n}\right)\right]$$

$$0 = \iint_{\partial\Omega}\left(u\frac{\partial \mathrm{v}}{\partial n} - v\frac{\partial u}{\partial n}\right)dS - \lim_{\mathcal{E}\to 0}\left[M_\varepsilon(u) + \mathcal{E}M_\varepsilon\left(\frac{\partial u}{\partial n}\right)\right]$$

$$0 = \iint_{\partial\Omega}\left(u\frac{\partial \mathrm{v}}{\partial n} - v\frac{\partial u}{\partial n}\right)dS - \left[u(x_0) + (0)\frac{\partial u}{\partial n}(x_0)\right]$$

$$0 = \iint_{\partial\Omega}\left(u\frac{\partial \mathrm{v}}{\partial n} - v\frac{\partial u}{\partial n}\right)dS - U(x_0)$$

$$U(x_0) = \iint_{\partial\Omega}\left(u\frac{\partial \mathrm{v}}{\partial n} - v\frac{\partial u}{\partial n}\right)dS \qquad (2.19)$$

CHAPTER 3

APPLICATION OF GREEN'S FUNCTION FOR DIRICHLET PROBLEMS

3.1 Green's Function for Upper Half.

Let Ω the upper half region which is not bounded.How ever it can be shown the result still valid under added assumption such as boundedness of the harmonic function Ω . By method of image we construct a function such that $= 0 \ on\ \partial\Omega\ and\ y = 0$. We define G by introducing what is called image point $(x, -y)$ corresponding to (x, y).

Let r be a distance from $(x_0, y_0)\ to\ (x, y)$

$$r = \sqrt{(x_0 - x)^2 + (y_0 - y)^2} \quad (3.1)$$

$$r' = \sqrt{(x_0 - x)^2 + (y_0 + y)^2} \quad (3.2$$

$$v = \frac{1}{2\pi} lnr \ \Rightarrow \ v = \ \frac{1}{2\pi} ln\sqrt{(x_0 - x)^2 + (y_0 - y)^2}$$

$$v = \frac{1}{4\pi r} \ln((x_0 - x)^2 + (y_0 - y)^2)$$

$$H = -\frac{1}{2\pi} lnr \Rightarrow \ v = -\frac{1}{2\pi} ln\sqrt{(x_0 - x)^2 + (y_0 + y)^2}$$

$$H = \frac{1}{4\pi} ln((x_0 - x)^2 + (y_0 + y)^2)$$

$$G(x, y\ ; x_0, y_0) = V + H$$

Since the image point$(x, -y)$ is not in Ω and H is regular for all points $(x_0, y_0) \in \Omega$ and satisfy Laplace's equation

$$\Delta H = H_{xx} + H_{yy} = 0\ \ for(x_0, y_0) \in \ \Omega$$

$$G(x, y\ ; x_0, y_0) = \frac{1}{2\pi} lnr - \frac{1}{2\pi} lnr' \quad (3.3)$$

$$= \frac{1}{4\pi} \ln((x_0 - x)^2 + (y_0 - y)^2) - \frac{1}{4\pi} ln(x_0 - x)^2 + (y_0 + y)^2$$

$$= \frac{1}{4\pi} ln \frac{(x_0 - x)^2 + (y_0 - y)^2}{(x_0 - x)^2 + (y_0 + y)^2} \quad (3.4)$$

(3.4) is a Green's function for upper half where $x_0\ and\ y_0$ are arbitrary and $y, y_0 > 0$

The out ward unit normal to the boundary of the upper plane is the negative of y direction.

So calculate $\frac{\partial G}{\partial n}$ by using $\frac{\partial}{\partial n_0} = -\frac{\partial}{\partial y_0}$

$$\frac{\partial G(x,y\ ;x_0,0)}{\partial n_0} = -\frac{\partial G(x,y\ ;x_0,0)}{\partial y_0}$$

$$\frac{\partial G(x,y\ ;x_0,0)}{\partial n_0} = -\frac{\partial}{\partial y_0}\left(\frac{1}{4\pi} ln\frac{(x_0-x)^2+(y_0-y)^2}{(x_0-x)^2+(y_0+y)^2}\right)$$

$$= \frac{1}{4\pi}\left(\frac{2(0-y)}{(x_0-x)^2+(y-0)^2} - \frac{2(0+y)}{(x_0-x)^2+y+0)^2}\right)$$

$$= -\frac{1}{4\pi}\left(\frac{-4y}{(x_0-x)^2+(y)^2}\right)$$

$$= \frac{1}{\pi}\left(\frac{y}{(x_0-x)^2+(y)^2}\right) \qquad (3.5)$$

Theorem 3.1. (Poisson's Equation in Upper Half)

Consider the Poisson's equation in upper half plane

$$\Delta u = F \ in \ \Omega$$

$$u = \ f \ on \ \partial\Omega$$

Then show that the solution of upper half plane is

$$u\ (x_o, y_o) = \frac{1}{4\pi}\int_0^{\infty}\int_{-\infty}^{\infty} Fln\left(\frac{(x-x_o)^2+(y-y_o)^2}{(x-x_o)^2+(y+y_o)^2}\right)dxdy + \frac{1}{\pi}\int_0^{\infty} f\left(\frac{y}{(x-x_o)^2+y^2}\right)dy \qquad (3.6)$$

Proof

Using Green's 2nd identity

$$\iint_\Omega (u\Delta v - v\Delta u)dA = \int_{\partial\Omega}\left(u\frac{\partial v}{\partial n} - v\frac{\partial u}{\partial n}\right)ds$$

Now use $v = G(x, y; x_o, y_o)$

$$\Delta G = \delta(x - x_o)\delta(y - y_o)$$

$$G = 0 \ on \ \partial\Omega$$

$$\iint_\Omega (u\Delta G - G\Delta u)dA = \int_{\partial\Omega}\left(u\frac{\partial G}{\partial n} - G\frac{\partial u}{\partial n}\right)ds$$

$$\iint_\Omega u\Delta G dA - \iint_\Omega G\Delta u dA = \int_{\partial\Omega} u\frac{\partial G}{\partial n}ds - \int_{\partial\Omega} G\frac{\partial u}{\partial n}ds$$

$$\iint_{\Omega} u\Delta G dA - \iint_{\Omega} G\Delta u dA = \int_{\partial\Omega} u\frac{\partial G}{\partial n} ds$$

$$\iint_{\Omega} u\Delta G dA = \iint_{\Omega} G\Delta u dA + \int_{\partial\Omega} u\frac{\partial G}{\partial n} ds$$

$$\iint_{\partial\Omega} u\delta(x - x_o)\delta(y - y_o) dA = \iint_{\Omega} GF dA + \int_{\partial\Omega} f\frac{\partial G}{\partial n} ds$$

$$u(x_o - y_o) = \iint_{\Omega} \frac{1}{4\pi} ln\left(\frac{(x-x_o)^2+(y-y_o)^2}{(x-x_o)^2+(y+y_o)^2}\right) F dA + \int_0^{\infty} \frac{1}{\pi} \frac{y}{(x-x_o)^2+y_o} f ds$$

$$u(x_o - y_o) = \frac{1}{4\pi}\int_0^{\infty}\int_{-\infty}^{\infty} F(x,y)\, ln\left(\frac{(x-x_o)^2+(y-y_o)^2}{(x-x_o)^2+(y-y_o)^2}\right) dxdy + \frac{1}{\pi}\int_0^{\infty} f(x,y)\frac{y}{(x-x_o)^2+y^2} dx$$

Theorem 3.2:- solutions of Poisson's equation on the upper half plane

$$\Delta u = f, -\infty < x < \infty, y > 0$$

$$u = 0 \ on\ \partial\Omega$$

is given by: -

$$u(x_o, y_o) = \frac{1}{4\pi}\int_0^{\infty}\int_{-\infty}^{\infty} F(x,y)\left(\frac{(x-x_o)^2+(y-y_o)^2}{(x-x_o)^2+(y+y_o)^2}\right) dxdy$$

Proof

Put $f = 0\ in$ (3.6)

$$u(x_o - y_o) = \frac{1}{4\pi}\int_0^{\infty}\int_{-\infty}^{\infty} \Delta u G dxdy + \frac{1}{\pi}\int_0^{\infty} u\frac{\partial G}{\partial n} dx$$

$$u(x_o - y_o) = \frac{1}{4\pi}\int_0^{\infty}\int_{-\infty}^{\infty} F(x,y)\, ln\left(\frac{(x-x_o)^2+(y-y_o)^2}{(x-x_o)^2+(y-y_o)^2}\right) dxdy + \frac{1}{\pi}\int_0^{\infty} (0)\frac{y}{(x-x_o)^2+y^2} dx$$

$$u(x_o, y_o) = \frac{1}{4\pi}\int_0^{\infty}\int_{-\infty}^{\infty} F(x,y)\, ln\left(\frac{(x-x_o)^2+(y-y_o)^2}{(x-x_o)^2+(y+y_o)^2}\right) dxdy \tag{3.7}$$

Theorem 3.4 solutions of Dirichlet's problem on upper half plane

Suppose u is harmonic in half plane

$$\Delta u = 0\ in\ \Omega$$

$u(x_o, y_o) = f(x)\ on\ \partial\Omega$ then

$$u(x_o, y_o) = \frac{y}{\pi}\int_0^{\infty} \frac{f(x)}{(x-x_o)^2+y^2} dx$$

Proof

Put $\Delta u = 0\ in$ (2.5)

$$u(x_o, y_o) = \frac{1}{4\pi}\int_0^{\infty}\int_{-\infty}^{\infty} \Delta u\, ln\left(\frac{(x-x_o)^2+(y-y_o)^2}{(x-x_o)^2+(y-y_o)^2}\right) dxdy + \frac{1}{\pi}\int_0^{\infty} \frac{y}{(x-x_o)^2+y^2} u(x,y) dx$$

$$u(x_o, y_o) = \frac{1}{4\pi}\int_0^{\infty}\int_{-\infty}^{\infty}(0)\, ln\left(\frac{(x-x_o)^2+(y-y_o)^2}{(x-x_o)^2+(y-y_o)^2}\right)dxdy + \frac{y}{\pi}\int_0^{\infty}\frac{u(x,0)}{(x-x_o)^2+y^2}dx$$

$$u(x_o, y_o) = \frac{y}{\pi}\int_0^{\infty}\frac{f(x)}{(x-x_o)^2+y^2}dx \qquad (3.8)$$

3.2. Green's Function for Rectangular region

Consider Laplace's equation

$$\begin{cases}\Delta U = 0 \;, & 0 < x < a \;, 0 < y < b \\ U(x,b) = & f(x),\; x(0) = x(l) = y(0) = 0\end{cases} \qquad (3.9)$$

$$Let\ U(x,y) = X(x)Y(y)$$

$$X''Y + YX'' = 0$$

$$\frac{X''}{x} + \frac{Y''}{y} = 0 \Rightarrow \frac{-X''}{x} = \frac{Y''}{y} = \lambda$$

Consider the Dirichlet boundary condition.

$$\begin{cases}-x''(x) = \lambda X(x), 0 < x < l \\ \quad x(0) = x(l) = 0\end{cases}$$

$$-X'' = \lambda x$$

$$-X'' - \lambda x = 0$$

$$X'' + \lambda x = 0$$

Case1. $\lambda = \mu^2 > 0$

$$X'' + \lambda x = 0$$

$$X'' + \mu^2 x = 0$$

$$\Rightarrow m^2 + \mu^2 = 0$$

$$m^2 = -\mu^2$$

$$m = \pm i\mu$$

The corresponding characteristic polynomial

$$X(x) = Acos(\mu x) + Bsin(\mu x)$$

Plugging BCs in to solution

$$X(0) \Rightarrow Acos(\mu(0)) + Bsin(\mu(0)) = 0$$

$$Acos(0) + Bsin(0) = 0$$

$$A(1) = 0 \Rightarrow A = 0$$

$$X(L) = 0, \quad \Rightarrow Acos(\mu L) + Bsin(\mu L) = 0$$

$$(0)cos(\mu L) + Bsin(\mu L) = 0$$

$$Bsin(\mu L) = 0$$

$$sin(\mu L) = 0$$

$$\Rightarrow \mu L = n\pi$$

$$\mu = \frac{n\pi}{L} \quad , but\ \lambda = \mu^2 \Rightarrow \lambda = \left(\frac{n\pi}{L}\right)^2 \tag{3.10}$$

Since $\mu > 0$ we must take $n \geq 1$

To find eigen function,we now compute $\mu_n = \frac{n\pi}{L}$ for $n \geq 1$ back into the matrix

$$\begin{bmatrix} 1 & 0 \\ cos\frac{n\pi L}{L} & sin\frac{n\pi.L}{L} \end{bmatrix} \begin{bmatrix} A \\ B \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} (1) & 0 \\ (-1)^n & 0 \end{bmatrix} \begin{bmatrix} A \\ B \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{cases} (1)A + (0)B = 0 \\ (-1)^n A + (0)B = 0 \end{cases} \Rightarrow \begin{cases} A + 0 = 0 \\ (-1)^n A + 0 = 0 \end{cases} \Rightarrow A = 0 \text{ and B can be arbitrary}$$

$$X(x) = Acos\left(\frac{\pi nx}{L}\right) + Bsin\left(\frac{\pi nx}{L}\right)$$

$$X(x) = (0)cos\left(\frac{\pi nx}{L}\right) + Bsin\left(\frac{\pi nx}{L}\right)$$

$$X(x) = Bsin\left(\frac{\pi nx}{L}\right)$$

$X(x) = sin\left(\frac{\pi nx}{L}\right)$, since B is arbitrary by taking $B = 1$

Therefore, positive eigenvalues and eigenfunctions are respectively,

$$\lambda_n = (\mu_n)^2 = \left(\frac{n\pi}{L}\right)^2 \quad and \quad X(x) = sin\left(\frac{\pi nx}{L}\right) \tag{3.11}$$

Case2:- $\lambda = 0$

$$-X'' = 0 \Rightarrow X(x) = A + Bx$$

Plugging boundary conditions in to a solution ,

$$X(0) = 0 \Rightarrow A + B(0) = 0 \Rightarrow A = 0$$

$$X(L) = 0 \Rightarrow A + B(L) = 0$$

We can write the system of equation in matrix form

$$\begin{bmatrix} 1 & 0 \\ 1 & L \end{bmatrix} \begin{bmatrix} A \\ B \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Which only has trivial solution A = B = 0, because det$\begin{bmatrix} 1 & 0 \\ 1 & L \end{bmatrix} = L \neq 0$

$Therefore, X_0(x) = 0$ is the only solution to the B V P. So we have not zero eigenvalues.

Case3:- $\lambda = -\mu^2 < 0$

For $\mu > 0$ $we\ define\ \lambda = -\mu^2$

$$-X'' = -\mu^2 x$$

$$-X'' + \mu^2 x = 0$$

$$-m^2 + \mu^2 = 0$$

$$m = \pm\mu \tag{3.12}$$

So, $X(x) = Acosh(\mu x) + Bsinh(\mu x)$

$$X(0) \Rightarrow Acosh(\mu(0)) + Bsinh(\mu(0)) = 0$$

$$Acosh(0) + Bsinh(0) = 0$$

$$A(1) = 0 \Rightarrow A = 0$$

$$X(L) = 0 \Rightarrow Acosh(\mu L) + Bsinh(\mu L) = 0$$

We can write this system of equation by matrix form

$$\begin{bmatrix} 1 & 0 \\ \cosh(nL) & \sinh(nL) \end{bmatrix} \begin{bmatrix} A \\ B \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Which has a nontrivial solution when $det \begin{bmatrix} 1 & 0 \\ \cosh(\mu L) & \sinh(\mu L) \end{bmatrix} = 0$

$$\Rightarrow \sinh(\mu L) = 0 \Rightarrow \mu = 0$$

Since $\lambda > 0$, there are no choices of n that result in a non trival solution for A and B. we can concluded that there are no negative eigen values.

Remark:- we could have defined Xn = Bsin$X_n = Bsin\left(\frac{n\pi}{L}x\right)$ for $\forall B \neq 0$ to be an eigen function since all constant multiple of an eigen functions are eigenfunctions. Its standard to choose B = 1

Again turning to y for $\lambda = -\mu^2 < 0$

$$Y'' + \mu^2 y = 0$$

$$m^2 + \mu^2 = 0$$

$$m = \pm i\mu$$

$$Y(y) = Acosh(\mu y) + Bsinh(\mu y)$$

$$Y(0) = 0 \Rightarrow Acosh(\mu(0)) + Bsinh(\mu(0)) = 0$$

$$Acosh(0) + Bsinh(0) = 0$$

$$\Rightarrow A = 0$$

Superposing these solutions we get the general form of solution

$$U(x, y) = \sum_{i=0}^{\infty} X_n(x)Y_n(y)$$

$$U(x, y) = \sum_{i=0}^{\infty} B_n \sin(\frac{n\pi}{L}x)\sinh(\frac{n\pi}{L}y) \qquad (3.13)$$

Finally the boundary condition $U(x, b) = f(x)$ implies

$$f(x) = \sum_{i=1}^{\infty} Bsin(\frac{n\pi}{L}x)\sinh(\frac{n\pi}{L}b)$$

$$\text{F(x)} = f(x) = \sum_{n=1}^{\infty} B_n \sinh\left(\frac{n\pi}{L}b\right)\sin(\frac{n\pi}{L}x)$$

Choose the coefficients $B_n\sinh(\frac{n\pi b}{a})$ to be Fourier coefficient of f on $(0, L)$

$$B_n\sinh(\frac{n\pi b}{L}) = \frac{2}{L}\int_0^b f(x)\sin\left(\frac{n\pi}{L}x\right)dx\ ,$$

$$B_n = \frac{2}{Lsinh\left(\frac{n\pi b}{L}\right)}\int_0^L f(x)\sin\left(\frac{n\pi x}{L}\right)dx \qquad (3.14)$$

Consider the Poisson's equation

$$\begin{cases} \Delta u = f(x,y) \text{in the region} \\ u = 0 \text{, on the boundaries} \end{cases} \tag{3.15}$$

$$\Delta u = u_{xx} + u_{yy} = f(x,y)$$

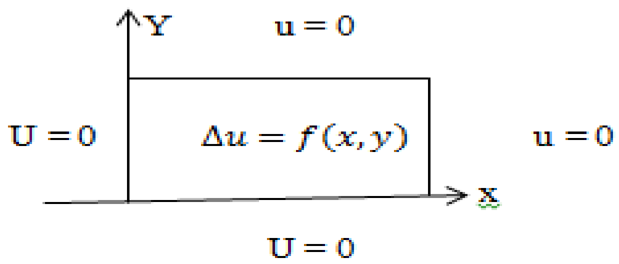


Fig.3.1

$$u(x,y) = X(x) + Y(y)$$

$$U_x(x,y) = X'(x)Y(y) \text{ and } U_y(x,y) = Y'(y)X(x)$$

$$U_{xx}(x,y) = X''(x)Y(y) \text{ and } U_{yy}(x,y) = Y''(y)X(x)$$

$$\Delta u = u_{xx} + u_{yy} = 0$$

$$X''y + Y''x = 0$$

$$\frac{X''y}{xy} + \frac{Y''x}{xy} = 0$$

$$\frac{-X''y}{xy} = \frac{Y''x}{xy} = k$$

$$\frac{-X''}{x} = \frac{Y''}{y} = k \tag{3.16}$$

$$\frac{-X''}{x} = \lambda \text{ and } \frac{Y''}{y} = \lambda$$

Case1. $\lambda = \mu^2 > 0$

$$X'' + \lambda x = 0$$

$$X'' + \mu^2 x = 0$$

$$\Rightarrow m^2 + \mu^2 = 0$$

$$m^2 = -\mu^2$$

$$m = \pm i\mu$$

$$X(x) = A\cos(\mu x) + B\sin(\mu x)$$

Plugging BCs in to solution

$$X(0) \Rightarrow A\cos(\mu(0)) + B\sin(\mu(0)) = 0$$

$$A\cos(0) + B\sin(0) = 0$$

$$A(1) = 0 \Rightarrow A = 0$$

$$X(L) = 0, \quad \Rightarrow A\cos(\mu L) + B\sin(\mu L) = 0$$

$$(0)\cos(\mu L) + B\sin(\mu L) = 0$$

$$B\sin(\mu L) = 0$$

$$\sin(\mu L) = 0$$

$$\Rightarrow \mu L = n\pi$$

$$\mu = \frac{n\pi}{L} \qquad , but\ \lambda = \mu^2 \Rightarrow \lambda = \left(\frac{n\pi}{L}\right)^2$$

Therefore, positive eigenvalues and Eigenfunctions are respectively,

$$\lambda_n = (\mu_n)^2 = \left(\frac{n\pi}{L}\right)^2 \quad and \quad X(x) = \sin\left(\frac{\pi n x}{L}\right)$$

Case2:- $\lambda = 0$

$$-X'' = 0 \Rightarrow X(x) = A + Bx$$

Plugging boundary conditions in to a solution ,

$$X(0) = 0 \Rightarrow A + B(0) = 0 \Rightarrow A = 0$$

$$X(L) = 0 \Rightarrow A + B(L) = 0 \Rightarrow B = 0$$

Therefore, $X_0(x) = 0$ is the only solution to the B V P. So we have not zero eigenvalues.

Case3:- $\lambda = -\mu^2 < 0$

For $\mu > 0\ we\ define\ \lambda = -\mu^2$

$$-X'' = -\mu^2 x$$

$$-X'' + \mu^2 x = 0$$

$$-m^2 + \mu^2 = 0$$

$$m = \pm\mu$$

So, $X(x) = Acosh(\mu x) + Bsinh(\mu x)$

$$X(0) \Rightarrow Acosh(\mu(0)) + Bsinh(\mu(0)) = 0$$

$$Acosh(0) + Bsinh(0) = 0$$

$$A(1) = 0 \Rightarrow A = 0$$

$$X(L) = 0 \Rightarrow Acosh(\mu L) + Bsinh(\mu L) = 0$$

$$\Rightarrow \sinh(\mu L) = 0 \Rightarrow \mu = 0$$

Since $\lambda > 0$, there are no choices of n that result in a non trival solution for A and B. we can concluded that there are no negative eigen values.

Again turning to y for $\boldsymbol{\lambda = -\mu^2 < 0}$

$$Y'' + \mu^2 y = 0$$

$$m^2 + \mu^2 = 0$$

$$m = \pm i\mu$$

$$Y(y) = Ccosh(\mu y) + Dsinh(\mu y)$$

$$Y(0) = 0 \ \Rightarrow Ccosh(\mu(0)) + Dsinh(\mu(0)) = 0$$

$$Ccosh(0) + Dsinh(0) = 0$$

$$\Rightarrow C = 0$$

$$Y(L) = 0 \Rightarrow Ccosh(\mu L) + Dsinh(\mu L) = 0$$

$$sinh(\mu L) = 0$$

$$\mu L = n\pi, \quad n = 1,2,3, \ldots ..$$

$$\mu = \frac{n\pi}{L} \Rightarrow \lambda_n = \frac{n^2\pi^2}{L^2} \ \ is\ eigenvalue\ and\ Y(y) = sin\left(\frac{\pi n x}{L}\right)\ eigen\ vector$$

Another solution approach method of 2D Eigenfunction expansion consider the function

$$\Phi_{mn}(x,y) = sin\left(\frac{n\pi}{a}x\right) sin\left(\frac{n\pi}{b}y\right) \tag{3.17}$$

Now $\left.\begin{matrix} \phi mn(0,y) = 0 \\ \phi mn(a,y) = 0 \\ \phi mn(x,0) = 0 \\ \phi mn(x,b) = 0 \end{matrix}\right\} \phi mn\ satisfies\ the\ BCs\ for\ each\ problems$

$$\Delta U = \Delta\phi mn$$

$$\Delta\phi mn(x,y) = sin\left(\frac{n\pi}{b}y\right)\left(sin\left(\frac{m\pi}{a}x\right)xx\right) + sin\left(\frac{m\pi}{a}x\right)\left(sin\left(\frac{n\pi}{b}y\right)yy\right)$$

$$= \left(\frac{m\pi}{a}\right)^2 sin\left(\frac{n\pi}{b}y\right) sin\left(\frac{m\pi}{a}x\right) - \left(\frac{n\pi}{b}\right)^2 sin\left(\frac{m\pi}{a}x\right) sin\left(\frac{n\pi}{b}y\right)$$

$$= -\left[\left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2\right] \sin\left(\frac{m\pi}{a}x\right) \sin\left(\frac{n\pi}{b}y\right) \tag{3.18}$$

$$\Delta\phi mn(x,y) = -\lambda mn\phi mn \ \rightarrow Helmholtz\ Equation$$

Again since $f(x,y)$ is defined for all $0 < x < a$ $,0 < y < b$ by double Fourier expansion

$$f(x,y) = \sum_{n=1}^{n\infty}\sum_{n=1}^{\infty} BmnSin\left(\frac{m\pi}{a}x\right) sin\left(\frac{n\pi}{b}y\right) \tag{3.19}$$

$$B_{mn} = \frac{4}{ab}\int_0^b\int_0^a U(x,y)\sin\left(\frac{m\pi}{a}x\right) sin\left(\frac{n\pi}{b}y\right) dxdy \tag{3.20}$$

$$\int_0^b\int_0^a \sin\left(\frac{m\pi}{a}x\right) sin\left(\frac{n\pi}{b}y\right) sin\left(\frac{\hat{m}\pi}{a}x\right) sin\left(\frac{\hat{n}\pi}{b}y\right) d_x d_y = \begin{cases} 0, & if\ \ (m,n) \neq (\hat{m},\hat{n}) \\ \frac{ab}{4}, & if\ (m,n) = (m,n) \end{cases} \tag{3.21}$$

Apply double Fourier series expansion to as a solution

$$\mathrm{u}(x,y) = \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} Emn\sin\left(\frac{m\pi}{a}x\right)\sin\left(\frac{n\pi}{b}y\right)$$

$$\Delta U = f(x,y)$$

$$\Delta\phi mn = f(x,y) \tag{3.22}$$

$$-\lambda_{mn}E_{mn} = \frac{4}{ab}\int_0^b\int_0^a f(x,y)\sin\left(\frac{m\pi}{a}x\right) sin\left(\frac{n\pi}{b}y\right) dxdy \tag{3.23}$$

$$E_{mn} = \frac{\frac{4}{ab}\int_0^b\int_0^a f(x,y)\sin\left(\frac{m\pi}{a}x\right)sin\left(\frac{n\pi}{b}y\right)dxdy}{-\lambda mn}$$

$$E_{mn} = \frac{\frac{4}{ab}\int_0^b \int_0^a f(x,y) sin\left(\frac{m\pi}{a}x\right) sin\left(\frac{n\pi}{b}y\right)dxdy}{-\left[\left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2\right]}$$

$$E_{mn} = \frac{-4}{\left[\left(\frac{m\pi}{a}\right)^2 \left(\frac{n\pi}{b}\right)^2\right]} \int_0^b \int_0^a f(x,y) \sin\left(\frac{m\pi}{a}x\right) \sin\left(\frac{n\pi}{b}y\right) dxdy \qquad (3.24)$$

Now by considering the poison equation $\Delta U = f(x,y)$ on an $a \times b$ rectangle R with zero boundary values we have for any $(x_0, y_0)\epsilon\mathbb{R}$

$$U(x_0, y_0) = \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} E_{mn} \sin\left(\frac{m\pi}{a}x_0\right) \sin\left(\frac{n\pi}{b}y_0\right) \qquad (3.25)$$

Then by using theorem 2.12

$$u(x_0, y_0) = \frac{1}{2\pi} \iint f(x,y) G(x,y; x_0, y_0) dxdy$$

$$\Rightarrow \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} Emn \sin\left(\frac{m\pi}{a}x_o\right) sin\left(\frac{n\pi}{b}y_0\right) = \frac{1}{2\pi} \int_0^b \int_0^a f(x,y) G(x,y; x_0, y_0) dxdy$$

$$\sum^{\infty}{}_{n=1} \sum_{m=1}^{\infty} \frac{-4}{\left[\left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2 .ab\right]} \int_0^b \int_0^a f(x,y) \sin\left(\frac{m\pi}{a}x\right) sin\left(\frac{n\pi}{b}y\right) dxdy. \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} sin\left(\frac{m\pi}{a}x_0\right) sin\left(\frac{n\pi}{b}y_0\right)$$

$$= \frac{1}{2\pi} \int_0^a \int_0^b f(x,y) G(x,y; x_0, y_0) dxdy$$

$$\frac{-4}{\left[\left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2 .ab\right]} \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} sin\left(\frac{m\pi}{a}x\right) sin\left(\frac{n\pi}{b}y\right) sin\left(\frac{m\pi}{a}xo\right) sin\left(\frac{n\pi}{b}yo\right) \int_0^b \int_0^a f(x,y) dxdy$$

$$= \frac{1}{2\pi} \int_0^a \int_0^b f(x,y) G(x,y; x_0, y_0) dxdy$$

$$(2\pi) \frac{-4}{\left[\left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2 .ab\right]} \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} sin\left(\frac{m\pi}{a}x\right) sin\left(\frac{n\pi}{b}y\right) sin\left(\frac{m\pi}{a}xo\right) sin\left(\frac{n\pi}{b}yo\right) \int_0^b \int_0^a f(x,y) dxdy$$

$$= \int_0^a \int_0^b f(x,y) G(x,y; x_0, y_0) dxdy$$

$$G(x,y; x_0, y_0) = \quad (2\pi) \frac{-4}{\left[\left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2 .ab\right]} \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} sin\left(\frac{m\pi}{a}x\right) sin\left(\frac{n\pi}{b}y\right) sin\left(\frac{m\pi}{a}\boldsymbol{x_0}\right) sin\left(\frac{n\pi}{b}\boldsymbol{y_0}\right)$$

$$= \frac{-8\pi}{\pi^2 \left[\left(\frac{m}{a}\right)^2 + \left(\frac{n}{b}\right)^2 .ab\right]} \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} sin\left(\frac{m\pi}{a}x\right) sin\left(\frac{n\pi}{b}y\right) sin\left(\frac{m\pi}{a}\boldsymbol{x_0}\right) sin\left(\frac{n\pi}{b}\boldsymbol{y_0}\right)$$

$$G(x,y;x_0,y_0) = \frac{-8}{\pi\left[\left(\frac{m}{a}\right)^2+\left(\frac{n}{a}\right)^2 .ab\right]} \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} sin\left(\frac{m\pi}{a}x\right) sin\left(\frac{n\pi}{b}y\right) sin\left(\frac{m\pi}{a}x_0\right) sin\left(\frac{n\pi}{b}y_0\right)$$

(3.25) is called the Green's function for rectangle.

3.3 .Green's Function for Disk

We can find Greens function for the disk by applying method of image and Steiner inversion of the circle by using the basic facts from plane Geometry about a circle.

In this section also we use Cosine law for triangles in order to find the distance between two points.

Throughout this section $\partial\Omega$ is denoted the disk centered at the origin with radius R > 0 and $\partial\Omega$ is positively oriented boundary.

From (3.4) previous section we have seen Greens function of the form of:-

$$G(x,y;x_0,y_0) = \frac{1}{2\pi} lnr + H(x,y;x_0,y_0) \quad (3.26)$$

$$G(x,y;x_0,y_0) \frac{1}{2\pi} \ln\sqrt{(x-x_0)^2 + (y-y_0)^2} + H(x,y;x_0,y_0)$$

Where H is harmonic on $\partial\Omega$ and $H(x,y;x_0,y_0) = -V(x,y;x_0,y_0)$, $\forall(x,y)$ $on\ \partial\Omega$

Suppose $A = (x_0,y_0)$be a point inside of $\partial\Omega$ other than the original and $A' = (x'_0, y'_0)$ is a point outside of the circle which lies on the line OA' .

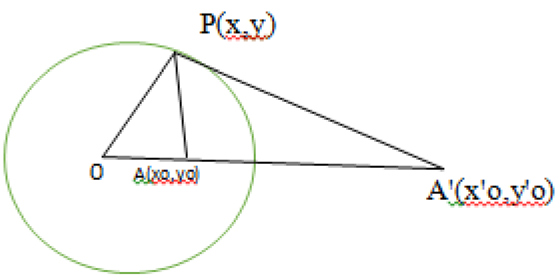


fig.3.1

$$|A.P \quad = K|P-A'||$$

$$\sqrt{(x-x_0)^2+(y-y_0)^2} = \text{k}\sqrt{(x-x_o')^2+(y-y_o')^2}$$

$$\ln\sqrt{(x-x_0)^2+(y-y_0)^2} = \ln\left(\sqrt{(x-x_0')^2+(y-y_0')^2}\right)$$

$$\frac{1}{2\pi} ln\sqrt{(x-x_0)^2+(y-y_0)^2} = \frac{1}{2\pi}\ln\left(k\sqrt{(x-x_0')^2+(y-y_0')^2}\right)$$

$$= ln\frac{1}{4\pi}(x-x_0)^2+(y-y_0)^2$$

$$= \frac{1}{4\pi}\ln\,(k)(x-x_0')^2+(y-y_0')^2)$$

$$\Rightarrow H = -\frac{1}{4\pi}\ln(k(x-x_0')^2+(y-y_0')^2)$$

$$= -\frac{1}{4\pi}\ln\,(x - x_0{'})^2 + \;(y - y_0{'})^2 - lnK \qquad (3.27)$$

Then H is harmonic for all $(x, y) \neq (x_0, y_0)$

In particular H is harmonic in $\partial\Omega\ \forall(x, y) \in\ \partial\Omega$. Then the Greens function for $\partial\Omega$ is:-

$$G(x, y; x_0, y_0) = \frac{1}{4\pi}\ln\frac{(x-xo)^2+(y-yo)^2}{(x-x_0{'})^2+(y-y_0{'})^2} - lnk \qquad (3.28)$$

Stainer Inversion

Let $\partial\Omega$ be a circle of radius R centered at the origin.

Let $A = (x_0, y_0)$is any point in $\partial\Omega$ other than the center.

Then the point A' = (x',y') is the inversion of a if

1) A lies on the ray of $\overline{OA'}$
2) $|oA|.|OA'| = r^2$
3) $for\ any\ point\ P(x, y) on\ \partial\Omega\ we\ have$

$$|A - P| = \frac{|A|}{R}|A' - P|$$

$$|A - P|\ = k|A' - P|$$

$$\Rightarrow k = \frac{|A-P|}{|A'-P|}\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

Since $k = \frac{|A|}{R}$

$$= \frac{\sqrt{(x_0-0)^2+(y-0)^2}}{R}$$

$$k = \frac{\sqrt{{x_0}^2+{y_0}^2}}{R}$$

$$K = \sqrt{\frac{{X_0}^2+{y_0}^2}{R^2}} \qquad (3.29)$$

From (3.4)

$$G(x, y; x_0, y_0) = \frac{1}{2\pi}\ln\frac{(x-x_0)^2+(y-y_0)^2}{(x-x')^2+(y-y')^2} - lnk \qquad (3.30)$$

$$G(x, y; x_0, y_0) = \frac{1}{4\pi}\,ln\left(\frac{(x-x_0)^2+(y-y_0)^2}{(x-x')^2+(y-y')^2}\right)\text{-}ln\sqrt{\frac{{x_0}^2+{y_0}^2}{R^2}} \qquad (3.31)$$

Since our aim is to solve the Poisson's equation in a disk with radius R that is subject to Dirichlet boundary condition it will be useful to have an expression for Greens function in polar coordinate and then use method of image, Stainer inversion and low of cosine for triangles.

If A, A' and P are polar coordinate then:-

A = $(r_0 cos\theta, r_0 sin\theta), A'\left(\frac{R^2}{ro}\right)co\beta$, $\frac{R^2}{ro}\,sinB)$ and P = (rcosθ,rsinθ)

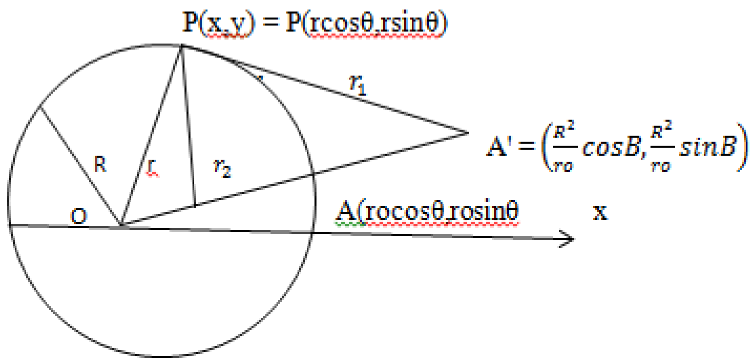


Fig.3.3

Since $\Delta AOP \sim \Delta OPA'$

$$\frac{|OA|}{|OP|} = \frac{|OP|}{|OA'|}$$

$$|OP|^2 = |OA|\,.\,|OA'|$$

$$R^2 = b|oA'|$$

$$R^2 = br_0$$

$$\Rightarrow b = \frac{R^2}{r_0}$$

By using cosine law for triangles

$$r_1^2 = r^2 + {r_0}^2 - 2rr_o cos(\theta - \beta) \tag{3.32}$$

$$|A - P|^2 = r^2 + {r_0}^2 - 2rr_o cos(\theta - \beta)$$

$$\Rightarrow |A - P| = \sqrt{r^2 + {r_o}^2 - 2rr_o cos(\theta - \beta)}$$

$$r^2 = r^2 + b^2 - 2rbcos(\theta - \beta) \tag{3.33}$$

$$= r^2 + \left(\frac{R^2}{r_o}\right)^2 2r\left(\frac{R^2}{r_o}\right)cos(\theta - \beta)$$

$$= r^2 + \frac{R^4}{{r_o}^2} - 2r\frac{R^2}{r_o}cos(\theta - \beta) \tag{3.34}$$

$$\Rightarrow |A' - P|^2 = r^2 + \frac{R^4}{{r_0}^2} - 2r\frac{r^2}{r_0}cos(\theta - \beta) \tag{3.35}$$

$$|A' - P| = \sqrt{r^2 + \frac{R^4}{{r_0}^2} - 2r\frac{R^2}{r_0}cos(\theta - \beta)} \tag{3.36}$$

To find Greens function G from (3.30) and (3.32)

$$G(r',\theta;r_0,\beta)=\frac{1}{4\pi}ln\left(\frac{r^2+r_0{}^2-2rr_0cos(\theta-\beta)}{r^2+\frac{R^4}{r_0{}^2}-2r\frac{R^2}{r_0}cos(\theta-\beta)}\right)$$

$$G(r,\theta;r_0,\beta)=(\frac{1}{4\pi}ln\left(R^2|\frac{r^2+r_0{}^2-2rr_0cos(\theta-B)}{r^2r_0{}^2+R^4-2R^2rr_0cos(\theta-B)}\right) \quad (3.37)$$

To drive a concrete formula for the solution of Dirichlet problems. So we need to find normal derivative $\frac{2G(r,\theta;r_0,\beta)}{\partial r_0}$ at the point on the circle $\partial\Omega$ and we note that $R=r$ on $\partial\Omega$

$$\frac{2G}{2r_0}\bigg|=\frac{1}{4\pi}\left(\frac{r^2r_0{}^2-2R^2rr_0cos(\theta-\beta)+R^2}{R^2[r^2+r_0{}^2.2rr_0cos(\theta-\beta)]}\right).\frac{\partial}{\partial r_0}\left(\frac{R^2[r^2r_0{}^2-2rr_0cos(\theta-\beta)]}{r^2r_0{}^2-2R^2rr_0cos(\theta-\beta)+R^2}\right)$$

$$=$$

$$\frac{1}{4\pi}\left(\frac{r^2r_0{}^2-2R^2rcor_0s(\theta-\beta)+R^2}{R^2[r^2+r_0{}^2.2rr_0cos(\theta-\beta)]}\right)\left(\frac{R^2[2r_0-2rcos(\theta-\beta)]r^2ro^2-2R^2rr_0cos(\theta-\beta)+R^4-2r^2r_0-2R^2rcos(\theta-\beta)\left[R^2([2r_0-2rcos(\theta-\beta)]\right]}{r^2r_0{}^2-2R^2rr_0cos(\theta-B)+R^2}\right)$$

$$\frac{\partial G}{\partial r_0}\bigg|=\frac{1}{4\pi}\left(\frac{R^2[2r-2RCOS(\theta-\beta)]-\left[2r^2R-2R^2rcos(\theta-\beta)\right]\left[r^2R^2-2R^3rcos(\theta-\beta)+R^4\right]}{[r^2R^2-2R^3rcos(\theta-\beta)+R^4][r^2R^2-2R^3rcos(\theta-\beta)+R^4]}\right)$$

$$=\frac{1}{4\pi}\left(\frac{R^2[2r-2RCOS(\theta-\beta)]-\left[2r^2R-2R^2rcos(\theta-\beta)\right]\left[r^2R^2-2R^3rcos(\theta-\beta)+R^4\right]}{[r^2R^2-2R^3rcos(\theta-\beta)+R^4]}\right)$$

$$=\frac{1}{4\pi}\left(\frac{2R^3-2R^2rcos(\theta-\beta)-2r^2R+2R^2rcos(\theta-\beta)}{[r^2R^2-2R^3rcos(\theta-\beta)+R^4]}\right)$$

$$=\frac{1}{4\pi}\left(\frac{2R^3-2r^2R}{R^2(r^2-2Rrcos(\theta-\beta)+R^2)}\right)$$

$$=\frac{1}{4\pi}\left(\frac{R(2R^2-2r^2}{R^2(r^2-2Rrcos(\theta-\beta)+R^2)}\right)$$

$$=\frac{1}{4\pi}\left(\frac{2(R^2-2r^2)}{R(r^2-2Rrcos(\theta-\beta)+R^2)}\right) \quad (3.38)$$

$$\frac{\partial G(r,\theta;r_0,\beta)}{\partial r_0}|_{r_0=R}=\left[\frac{R^2-r^2}{r^2R-2R^2rcos(℃\theta-\beta)+R^3}\right]$$

$$\frac{\partial G(r,\theta;r_0,\beta)}{\partial r_0}|_{r_0=R}=\frac{1}{2\pi}\left[\frac{R^2-r^2}{r^2R-2R^2rcos(℃\theta-\beta)+R^3}\right] \quad (3.39)$$

3.3 Theorem (Solution of Dirichlet problem on disk)

Suppose U is harmonic in Ω and $f(r,\theta)$is a functionon the disk $\partial\Omega$.

$$\Delta U(r,\theta) = f(r,\theta), on\ \partial\Omega, 0\leq r \leq R, 0 \leq \theta \leq 2\pi \tag{3.40}$$

$u = 0$, on $\partial\Omega$ then

$$U(r,\beta) = \frac{1}{2\pi}\int_0^{2\pi} f(\theta)\frac{R^2-r^2}{[r^2+R^2-2Rr\cos(\theta-\beta)]}d\theta$$

Proof:-

Green's 2nd identity in 2D

$$\iint_{\Omega}(U\Delta V - V\Delta U)dA = \int_{\partial\Omega}\left(U\frac{\partial V}{\partial n} - V\frac{\partial U}{\partial n}\right)dS$$

Let V be Green's function

$$V = G(r,\theta;r_0,\beta)$$

$$\iint_{\Omega}(U\Delta V - V\Delta U)dA = \int_{\partial\Omega}\left(U\frac{\partial V}{\partial n} - V\frac{\partial U}{\partial n}\right)dS$$

$$\iint_{\Omega}(U\Delta G - G\Delta U)dA = \int_{\partial\Omega}\left(U\frac{\partial G}{\partial n} - G\frac{\partial U}{\partial n}\right)dS$$

If we require it satisfy

$$\Delta G = \delta(r-r_0) - \delta(\theta-\beta), \quad r \leq R, 0 \leq \theta \leq 2\pi$$

$$G = 0, \ R = r \tag{3.41}$$

$$\iint_{\Omega}\left[U(r,\theta)\delta(r-r_0)\delta(\theta-\beta) - G(r,\theta;r_0,\beta)f(r,\theta)\right]dA = \int_{\partial\Omega}\left(U\frac{\partial G}{\partial n}\right)dS - \int_{\partial\Omega}\left(G\frac{\partial U}{\partial n}\right)dS \quad G = 0, \partial\Omega$$

$$\iint_{\Omega}U(r,\theta)\delta(r-r_0)\delta(\theta-\beta)dA - \iint_{\Omega}G(r,\theta;r_0,\beta)dA = \int_{\partial\Omega}\left(U\frac{\partial G}{\partial n}\right)dS$$

$$U(r,\beta) - \iint_{\Omega}G(r,\theta;r_0,\beta)f(r,\theta)dA = \int_{\partial\Omega}\left(U\frac{\partial G}{\partial n}\right)dS$$

Since the domain is a disk centered at the origin, the out ward unit normal vector

$$\hat{n} = \hat{r} \Rightarrow \frac{\partial}{\partial n} = \frac{\partial}{\partial r} \tag{3.42}$$

$$U(r,\beta)-\iint_{\Omega} G(r,\theta;r_0,\beta)f(r,\theta)dA=\int_{\partial\Omega}\left(U\frac{\partial G}{\partial r}\right)dS$$

Since the point on the boundary depends only on θ, $dS=Rd\theta$, $dA=rdrd\theta$ and $U(R,\theta)=f(\theta)$. $U(r,\beta_0)-\int_0^{\partial\pi}\int_0^R G(r,\theta,r_o,\beta_0)f(r,\theta)(rdrd\theta)=\int_0^{2\pi}U(R,\theta)\frac{\partial G}{\partial r}\Big|_{r=R} Rd\theta$

$$U(r,\beta_0)-\int_0^{\partial\pi}\int_0^R G(r,\theta,r_o,\beta_0)f(r,\theta)(rdrd\theta)=\int_0^{2\pi}f(\theta)\frac{\partial G}{\partial r}\Big|_{r=R} Rd\theta$$

Since Green's function has a symmetric property switch to the roles of $r_0\ and\ \beta\ with\ those\ of\ r\ and\ \beta$ respectively.

$$U(r,\beta)=\int_0^{\partial\pi}\int_0^R G(r_0,\theta_0;r,\beta)f(r_0,\theta_0)(r_0dr_0d\theta_0)+\int_0^{2\pi}f(\theta_0)\frac{\partial G}{\partial r}\Big|_{r=R} Rd\theta$$

$$u(r,\beta)=\int_0^{\partial\pi}\int_0^R\frac{1}{4\pi}ln\left(\frac{r^2+r_o{}^2-2rr_o\cos(\theta-\beta)}{r^2+r_o{}^2-2rr_o\cos(\theta-\beta)+R^4}\right)r_0f(r_0,\theta_0)dr_0d\theta_0$$

$$+\quad R\int_0^{\partial\pi}\frac{1}{2\pi}\frac{R^2-r^2}{Rr^2-2R^2rcos(\theta-B)+R^3}f(\theta_0)d\theta_0$$

$$\mathrm{u}(r,\beta)=\int_0^{\partial\pi}\int_0^R\frac{1}{4\pi}ln\left(\frac{r^2+r_o{}^2-2rr_o\cos(\theta-\beta)}{r^2+r_o{}^2-2rr_o\cos(\theta-\beta)+R^4}\right)r_0f(r_0,\theta_0)dr_0d\theta_0$$

$$+\quad R\int_0^{\partial\pi}\frac{1}{2\pi}\frac{R^2-r^2}{Rr^2-2R^2rcos(\theta-\beta)+R^3}f(\theta_0)d\theta_0 \qquad (3.43)$$

(3. 41) which is called Poisson's equation for disk

The solution of Laplace's equation is found by setting **f=0**

$$\mathrm{u}(r,\beta)=\int_0^{\partial\pi}\int_0^R\frac{1}{4\pi}ln\left(\frac{r^2+r_o{}^2-2rr_o\cos(\theta-\beta)}{r^2+r_o{}^2-2rr_o\cos(\theta-\beta)+R^4}\right)r_0(0)dr_0d\theta_0$$

$$+\quad R\int_0^{\partial\pi}\frac{1}{2\pi}\frac{R^2-r^2}{Rr^2-2R^2rcos(\theta-\beta)+R^3}f(\theta_0)d\theta_0$$

$$U(r,\beta)=\frac{1}{\partial\pi}\int_0^{\partial\pi}f(\theta)\frac{R^2-r^2}{r^2-2R\,cos(\theta-\beta)+R^2}f(\theta_0)d\theta_0 \qquad (3.44)$$

Conclusion

The basic idea we have discussed in application of Green's function to solve Dirichlet problems to present a new definition of the Green's function of the Dirichlet problem for Laplace equation prompted by theory of Differential Equations and investigate correctly solvable boundary value problems for Poisson equations in a punctured domain.

We have considered Laplace's equation

$\Delta u(x) = 0,\ x \in \Re^n$ and its inhomogeneous version, Poisson's equation

$\Delta u(x) = f,\ x \in \Re^n$.

There are a lot of functions u which satisfy this equation and we have discussed a radial solution of Laplace's by using the symmetric nature of Laplace's equation. Due to symmetry of Laplace's equation a radial solutions are reducing a PDE to an ODE which is generally easier to solve. A Dirac delta function which is a mathematical tool used to present a point source with fundamental solution $k(x)$ for Laplace's operator is a distribution satisfying

$\Delta k(x) = \delta(x)$, where δ the delta function supported at $x = 0$

The Green's function for Laplace's on 2D domains is defined in terms of the corresponding fundamental solution.

$G(x, x_0) = V(x - x_0) + H(x - x_0),$

$\Delta G = \delta(x - x_0)\ \ ,\ x \epsilon \Omega$
$G(x, x_0) = 0 \quad\ for\ x \epsilon \Omega$

Green's function for the region Ω and the integral on the boundary solve the Dirichlet problem in side Ω. The Dirichlet problems; Dirichlet problem of a disks, upper half and rectangles are solved by using method of Green's function.

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