

GRADUATE SEMINAR REPORT

On

Fixed Point Theorems
and
Some applications of the Krein-Milman Theorem.

By

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PREFACE

This seminar report is the described report that I presented in *Jan. and June 2000*.
The report contains three chapters.

The first chapter contains most of the basic definitions, theorems and concepts, which
will be useful in the next chapters.

ACKNOWLEDGEMENT

In chapter two, we will give applications of *Robin-Riesz, Poincaré-
Birkhoff, Koenig-Liebniz and Darboux-Abraham Theorems*. In this chapter we stick
on the applications and we will not give the proofs of these theorems. In particular we will
apply these theorems to some conservative systems such as *Hamilton's system*.

I would like to thank my advisor and my instructor *Prof. Dr. Deumlich* for his help,
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so that I can write the report in a modern text programme.

(next part) theorems and their
applications such as the *Theorem of Poincaré-Birkhoff* that gives a sufficient condition for a
certain differential equation to be solvable. At the end of this chapter we will consider
non-linear integral equation called *Hammerstein Equation* and give a condition for the
existence of a solution.

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2.1. Applications of the Hahn-Banach Theorem	

PREFACE

This seminar report is the combined report that I presented in *Jan. and June 2000*. The report contains three chapters.

The first chapter contains most of the basic definitions, theorems and concepts, which will be useful in the next chapters.

In chapter two, we will generally consider the applications of *Hahn-Banach, Riesz-Representation, Krein-Milman and Banach-Alaoglu Theorems*. In this chapter we stick on the applications and we will not give the proofs of these theorems. In particular we will apply these theorems to prove approximation theorems such as *Bishop's (Generalization of the Stone-Weierstrass Theorem)* and an *Interpolation Theorems* and moreover we will apply the *Krein-Milman Theorem* intensively to prove a *Fixed-Point Theorem*.

Finally in the last chapter, we will consider some basic fixed-point theorems and their applications such as the *Theorem of Peano-Picard* that gives a sufficient condition for a certain differential equation to be solvable. At the end of this chapter we will consider non-linear integral equation called *Hammerstein Equation* and give a condition for the existence of a solution.

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1. Basic Concepts

This chapter contains most of the basic concepts, definitions and theorems which will be useful in proving many of the theorems in chapter two.

1.1: Topological vector spaces

Definition 1:

Let X be a vector space and \mathfrak{T} is a topology defined on X . Then the pair (X, \mathfrak{T}) is called a *topological vector space* if and only if

- For each $x \in X$, $\{x\}$ is closed set
- the vector space operations are continuous (i. e the addition and the scalar multiplication are continuous)

Here, \mathfrak{T} is called a *Vector topology* on X .

When we say that addition is continuous, it means that the function $\alpha: X \times X \rightarrow X$ defined by

$$\alpha(x, y) := x + y \text{ is continuous.}$$

i.e. for any neighbourhood V of $x + y$ in X , there exists neighbourhoods S and T of x and y respectively such that $S + T \subseteq V$.

Moreover, when we say that the scalar multiplication is continuous, it means that the function $\beta: K \times X \rightarrow X$ defined by

$$\beta(r, x) := rx \text{ is continuous.}$$

i. e for any neighbourhood W of rx in X , there exists a $\delta > 0$ and a neighbourhood M of x in X such that

$$\alpha \in (r - \delta, r + \delta) \text{ implies that } \alpha M \subseteq W.$$

Now we will characterize the open sets in X .

Property 1 :

Let X be a topological vector space and $a \in X, \lambda \in K, \lambda \neq 0$. Then

- $T: X \rightarrow X$ defined by $T(x) := a + x$ is a homeomorphism.
- $M_\lambda: X \rightarrow X$ defined by $M_\lambda(x) := \lambda x$ is a homeomorphism

Proof :

We will prove (a) and the proof of (b) can be done in a similar way.

Let $x, y \in X$ such that $T_a(x) = T_a(y)$. Then $a + x = a + y$. i. e. T is one to one.

To show that T_a is onto, let $z \in X$, then $z - a \in X$ and $T_a(z - a) = z$.

Then, T_a is bijective. //

Moreover since addition is continuous, T_a is continuous.

Now observe that T_a^{-1} exists and for $x \in X, T_a^{-1}(x) = T_{-a}(x) = x - a$.

Hence, by the continuity of addition,

we get that T_a^{-1} is continuous which implies that T_a is homeomorphism. //

Property 2:

Let X be a topological space and $E \subseteq X$.

Then E is open if and only if $a + E$ is open for $a \in X$.

Proof:

(\Rightarrow) Suppose E is open. By Property 1, we have $T_a^{-1} : X \rightarrow X$ is continuous, we have that

$$(T_a^{-1})^{-1}(E) = T_a(E) = a + E \text{ is open in } X.$$

(\Leftarrow) Suppose for each $a \in X$, $a + E$ is open.

Then $T_a^{-1} : X \rightarrow X$ is continuous and $(T_a^{-1})^{-1}(a + E) = E$ is open. //

Property 3:

Let X be a topological vector space. Then

(a) If V is a neighbourhood of 0, then $x + V$ is a neighbourhood of x for each $x \in X$.

(b) $B = \{x + V \mid x \in X, V \text{ is a neighbourhood of } 0\}$ is a basis of the vector topology on X .

$$(c) \tau = \left\{ \bigcup_{a \in X} (a + V) \mid V \text{ is a neighbourhood of } 0 \right\}.$$

The proof can be done using property 1, 2 and definition of a basis of a topology on X .

Now, we will give definition of some types of topological spaces.

Definition 2:

Let X be a topological vector space.

(a) a metric $d : X \times X \rightarrow \mathbf{R}$ is called *invariant* if and only if

$$d(x + y, y + z) = d(x, y) \text{ for all } x, y, z \in X.$$

Here, d is called *complete metric* on X if and only if every Cauchy sequence converges.

(b) X is called an *F-space* if and only if the vector topology on X is induced by a *complete invariant metric* on X .

(c) X is said to be *locally convex* if and only if every neighbourhood of 0 contains a convex neighbourhood

(d) a topological space is called *locally compact* if and only if 0 has a neighbourhood U such that \bar{U} is compact.

Remark:

Every normed space is locally convex.

proof:

Let X be a normed space. Then a metric $d: X \times X \rightarrow \mathbf{R}$ defined by

$$d(x,y) := \|x - y\| \quad \text{induces the topology on } X.$$

Let now V be a neighbourhood of 0 in X .

Then there exists $\varepsilon > 0$ such that

$$B(0, \varepsilon) \subseteq V.$$

We claim that $B(0, \varepsilon)$ is convex.

Let $x, y \in B(0, \varepsilon), \alpha \in (0,1)$. Then $d(0,x) < \varepsilon, d(0,y) < \varepsilon$. Hence, we get

$$\begin{aligned}
d(0, \alpha x + (1 - \alpha)y) &:= \|\alpha x + (1 - \alpha)y\| \\
&\leq \|\alpha x\| + \|(1 - \alpha)y\| \\
&= \alpha \|x\| + (1 - \alpha)\|y\| \\
&< \alpha \varepsilon + (1 - \alpha)\varepsilon = \varepsilon
\end{aligned}$$

Hence, $\alpha x + (1 - \alpha)y \in B(0, \varepsilon)$ i.e. $B(0, \varepsilon)$ is a convex neighbourhood of 0

Therefore, X is locally convex. //

Theorem 1:

Let V be a neighbourhood of 0 in a topological vector space

If $0 < r_n < r_{n+1}$ for $n \in \mathbf{N}$, $r_n \xrightarrow{n \rightarrow \infty} \infty$, Then $X = \bigcup_{n=1}^{\infty} r_n V$

Proof:

Clearly, since $r_n V \subseteq X$, we have that $\bigcup_{n=1}^{\infty} r_n V \subseteq X$. Now we show that $X \subseteq \bigcup_{n=1}^{\infty} r_n V$.

Let $x \in X$. Since a linear mapping defined on a finite dimensional space is continuous, we can see that the mapping $\alpha \rightarrow \alpha x$ is continuous.

Let V be a neighbourhood of 0. Then we put $H := \{r \in \mathbf{K} \mid r x \in V\}$. Then we can see

that H is an open set in \mathbf{K} . Moreover, $0 \in H$. Since $r_n \xrightarrow{n \rightarrow \infty} \infty$, we have $\frac{1}{r_n} \xrightarrow{n \rightarrow \infty} 0$.

Hence, for sufficiently great n , $\frac{1}{r_n} \in H$ and $\frac{1}{r_n} x \in V$.

Thus, $x \in r_n V$ for sufficiently great $n \in \mathbf{N}$. i.e. $X = \bigcup_{n=1}^{\infty} r_n V //$

1.2: *Hahn-Banach Extension and Separation Theorems*

We will state without proof the *Hahn-banach Extension Theorem* for any real vector space, *Continuous Extension Theorem* for normed spaces and *Hahn-Banach Separation Theorems*. The later will be used to prove *Approximation Theorems*.

Theorem 1: (Hahn-Banach Extension Theorem)

Suppose

- (a) M is a subspace of a real vector space X
- (b) $P: X \rightarrow \mathbf{R}$ such that (i) $P(x + y) \leq P(x) + P(y)$
(ii) $P(\alpha x) = \alpha P(x), \forall x, y \in X$ and $\alpha \geq 0$.
- (c) $f: M \rightarrow \mathbf{R}$ is linear and $f(x) \leq P(x)$ on M .

Then, there exists a linear functional $\lambda: X \rightarrow \mathbf{R}$ such that

$$\lambda(x) = f(x) \quad \forall x \in M$$

and

$$-P(-x) \leq \lambda(x) \leq P(x), \quad \forall x \in X.$$

The proof can be found in [1] and [6]

Theorem 2: (Separation Theorem)

Suppose A and B are disjoint, non empty convex sets in a topological vector space X .

- (a) If A is open, then there exists $\lambda \in X^*$ and $\gamma \in \mathbf{R}$ such that
$$\operatorname{Re} \lambda(x) \leq \gamma \leq \operatorname{Re} \lambda(y), \quad \forall x \in A \text{ and } \forall y \in B.$$
- (b) If A is compact and B is closed and X is locally convex, then
there exists $\lambda \in X^*, \gamma_1, \gamma_2 \in \mathbf{R}$ such that $\operatorname{Re} \lambda(x) < \gamma_1 < \gamma_2 < \operatorname{Re} \lambda(y), \forall x \in A, \forall y \in B.$

The proof can be found in [1] and [6]

Theorem 3:

Suppose M is a subspace of a locally convex space X and $x_0 \in X$.

If $x_0 \notin \overline{M}$, then there exists $\lambda \in X^*$ such that $\lambda(x) = 0$ on M .

Proof:

Let $A = \{x_0\}$ and $B = \overline{M}$. Then A is compact and B is closed. By part (b) of *Theorem 2* there exists $\lambda \in X^*$ such that $\{\lambda(x_0)\}$ and $\lambda(M)$ are disjoint.

Hence, $\lambda(M) \subseteq \mathbf{K}$ and since $\lambda(x_0) \notin \lambda(M)$, we have $\lambda(M)$ is a proper subspace of \mathbf{K} . Therefore, it is easy to show that $\lambda(M) = \{0\}$ and $\lambda(x_0) \neq 0$.

Now, define $\lambda_1(x) = \frac{\lambda(x)}{\lambda(x_0)}$. Then

$$\lambda_1 \in X^*, \lambda_1(x_0) = 1 \text{ and } \lambda_1 = 0 \text{ on } M. //$$

Remark:

Let M be a subspace of a locally convex space X and $x \in X$. We can use *Theorem 3* to show that $x \in \overline{M}$ by showing that $\lambda(x) = 0$ for all $\lambda \in X^*$ and $\lambda = 0$ on M .

We will use this concept to prove the *Stone-Weierstrass theorem*.

Now, we will give the theorem of *Hahn-Banach* for normed spaces.

Theorem 4: (Continuous Extension of Hahn-Banach Theorem)

Let X be a normed space and $U \subseteq X$ be a subspace of X and $f: U \rightarrow \mathbf{K}$ be continuous linear functional. Then there exists a continuous linear functional $g: X \rightarrow \mathbf{K}$ such that

(a) $g(x) = f(x), \forall x \in U$

(b) $\|g\| = \|f\|$.

Proof:

Define $P: X \rightarrow \mathbf{R}$ by $P(x) := \|f\| \|x\|$. Then it is easy to show that

(i) $P(x + y) = P(x) + P(y)$, and (ii) $P(\alpha x) = \alpha P(x), \forall x, y \in X, \alpha \geq 0$.

Then P satisfies the hypothesis of *Theorem 1*.

Thus there exists a linear functional

$$g: X \rightarrow \mathbf{R} \text{ such that } g = f \text{ on } U.$$

Moreover, $|g(x)| \leq P(x) := \|f\| \|x\|, \forall x \in X$

and

$$\|g\| \leq \|f\|.$$

Then g is continuous.

(1)

On the other hand for all $x \in U$, we have $|f(x)| = |g(x)| \leq \|g\| \|x\|$.

Therefore, we have that $\|f\| \leq \|g\|$

(2)

Hence, by (1) and (2), we have that $\|f\| = \|g\|$. //

Note :

This Theorem is important to prove an Approximation Theorem namely an *Interpolation Theorem*.

Corollary1:

If X is locally convex, then X^* separates points of X .

Proof:

Let $x, y \in X$ such that $x \neq y$. Then by part (b) of *Theorem 2* we have

$$\lambda(x) < \lambda(y) \text{ for some } \lambda \in X^*. //$$

Now, we will consider many applications of *Hahn-Banach, Banach Alaoglu and Krein-Milman Theorems*.

Definition:

$\text{Ext}(K)$ denotes the set of all extreme points of K .

Remark:

$x \in \text{Ext}(K)$ if and only if whenever $x = \alpha y + (1-\alpha)z$ for some $\alpha \in (0, 1)$ we have $x = y = z$.

2. Applications of Banach-Alaoglu and Krein-Milman theorems.

2.1: Weak* topology and the Banach Alaoglu Theorem.

Definition 1:

Let X be a topological vector space and X^* be the dual space. Then the weak topology of X^{**} is a topology on X^* such that it is generated by X^{**} i. e the weak topology \mathfrak{T}^* on X^* is given by

$$\mathfrak{T}^* = \left\{ \bigcup_{n=1}^{\infty} \left[\bigcap_{i=1}^n \lambda_i^{-1}(-r_i, r_i) \right] \mid \lambda_i \in X^{**}, r_i \in \mathbb{R}, 1 \leq i \leq n \right\}$$
 one can verify that \mathfrak{T}^* is a topology on X^* and is called the *weak* topology*.

Property 1:

- (a) \mathfrak{T}^* is the weakest topology on X^* such that $\lambda \in X^{**}$ is continuous.
- (b) $B^* = \{ \lambda \in X^* \mid |\lambda_i(\lambda)| \leq r_i, 1 \leq i \leq n, \lambda_i \in X^{**} \}$ is a basis for \mathfrak{T}^* .

The proof can be done by applying the definition of continuity and basis of a topology.

Now, we shall state the *Banach-Alaoglu Theorem* without proof.

Theorem 1: (Banach-Alaoglu Theorem)

Let X be a topological vector space and V be a neighbourhood of 0.

Let $K := \{ \lambda \in X^* \mid |\lambda(x)| \leq 1, \forall x \in V \}$. Then K is weak* -compact.

The proof is found in [1]

2.2 : The Krein Milman Theorem

2.2.1: Extreme Sets and Extreme Points:

Definition 1:

Let K be a subset of a vector spaces X and $S \subseteq K, S \neq \emptyset$.

Then S is said to be an *extreme set* if and only if for $x, y \in K$ and $\alpha \in (0, 1)$ we have $\alpha x + (1-\alpha)y \in S$ implies $x \in S$ and $y \in S$.

If $|S|=1$, the point $a \in S$ is called an *extreme point* of K .

Denotation:

$\text{Ext}(K)$ denotes the set of all extreme points of K .

Remark:

$a \in \text{Ext}(K)$ if and only if whenever $x, y \in S$ and $a = \alpha x + (1-\alpha)y$ for some $\alpha \in (0, 1)$, we have $x = y = a$.

Example 1:

Let K be a triangle in \mathbf{R}^2 with vertices a, b and c . Then,

- (a) $\text{Ext}(K) = \{a, b, c\}$
- (b) $[a, b], [b, c], [a, c]$ and $[a, b] \cup [b, c]$ are extreme sets of K .

Theorem 2: (Krein-Milman Theorem)

Suppose X is a locally convex topological vector spaces on which X^* separate points of X . If K is compact and convex set in X , then $K = \overline{\text{Conv}(\text{Ext}(K))}$.

The proof is found in [1].

Theorem 3: (Milman's Theorem)

If K is a compact convex set in a locally convex space and if $\overline{\text{Conv}(K)}$ is also compact, then every extreme point of $\overline{\text{Conv}(K)}$ lies in K .

The proof is found in [1].

2.3: Range of vector valued measure

We now give an application of the *Krein-Milman and Banach – Alaoglu Theorems*.

Definition 1:

Let X be any non- empty set and \mathfrak{R} be a σ - algebra of subsets of X .

Let μ be a finite signed measure (real valued) defined on \mathfrak{R} . Then μ is said to be *non-atomic* if and only if for any set $E \in \mathfrak{R}$ such that $|\mu|(E) > 0$, there exists a set $A \in \mathfrak{R}$ such that $0 < |\mu|(A) < |\mu|(E)$.

Theorem 3:

Let $\mu_1, \mu_2, \dots, \mu_n$ be real valued non-atomic signed measures on a σ - algebra \mathfrak{R} .

Define $\mu: \mathfrak{R} \rightarrow \mathbf{R}^n$ by $\mu(E) := (\mu_1(E), \mu_2(E), \dots, \mu_n(E))$.

Then $\mu(\mathfrak{R})$ is a compact convex subset of \mathbf{R}^n .

Proof:

Let B be the set of all bounded measurable real functions on X .

Define $\lambda: B \rightarrow \mathbf{R}^n$ by $\lambda(g) := (\int g d\mu_1, \int g d\mu_2, \dots, \int g d\mu_n)$.

Then λ is a linear mapping.

We put $\sigma := |\mu_1| + |\mu_2| + \dots + |\mu_n|$.

Then we can show that

$$B \subseteq L^\infty(\sigma).$$

Now, for $f, g \in L^\infty(\sigma)$, if $g = f$ σ -almost every where, then $\lambda(f) = \lambda(g)$.

Thus, λ can be considered as a linear mapping from $L^\infty(\sigma)$ in to \mathbf{R}^n .

Moreover, for $g \in L^\infty(\sigma)$ we have $\|\lambda(g)\| = \sqrt{\sum_{i=1}^n (\int g d\mu_i)^2}$.

But we know that $g(x) \leq |g(x)| \leq \|g\|$ σ -almost every where and we can see that

$$\|\lambda(g)\| \leq M\|g\|_{\infty} \quad \text{where} \quad M = \sqrt{\sum_{i=1}^n (\mu_i(X))^2}$$

Thus, since we know that $L^{\infty}(\sigma) = (L^1(\sigma))^*$,

$\lambda : L^{\infty}(\sigma) \rightarrow \mathbf{R}^n$ is weak*-continuous.

Now, Put $K := \{g \in L^{\infty}(\sigma) \mid 0 \leq g \leq 1\}$. Then K is convex.

We will show that K is weak*-compact.

Now from $\sigma := |\mu_1| + |\mu_2| + \dots + |\mu_n|$, we observe that for each $i \in \{1, 2, \dots, n\}$, μ_i is absolutely continuous with regard to σ . (see [8])

Hence, by Radon-Nikodym theorem (see [8]) for each $i \in \{1, 2, 3, \dots, n\}$ there exists a measurable function $f_i \geq 0$ such that

$$\mu_i(E) = \int_E f_i d\sigma. \quad (3)$$

From (3), we observe that $f_i \in L^1(\sigma)$ for all $i \in \{1, 2, \dots, n\}$.

Moreover, since $\sigma(X) < \infty$, we get $(L^{\infty}(\sigma))^* = L^1(\sigma)$.

Now for each $f \in L^1(\sigma)$ such that $f \geq 0$ we define a function F by

$$F: L^1(\sigma) \rightarrow \mathbf{R} \text{ by } F(g) := \int fg d\sigma. \quad (4)$$

Then, it can be easily shown that F is continuous linear functional.

(by using the Holder's inequality)

Now, we put $H := [0, \int f d\sigma]$.

Then H is a closed subset of \mathbf{R} and hence $K = F^{-1}(H)$ is weak*-closed.

Moreover, $K \subseteq B$, where $B := \{g \in L^{\infty}(\sigma) \mid \|g\| \leq 1\}$.

Let $V := \{f \in L^1(\sigma) \mid \|f\| \leq 1\}$. Then by Banach-Alaoglu Theorem we get B is weak*-compact and K is weak*-compact.

Therefore $\lambda(K)$ is compact and convex subset of \mathbf{R}^n .

We will show that $\lambda(K) = \mu(\mathfrak{R})$ which will complete the proof of the theorem.

Let $E \in \mathfrak{R}$. Then $\chi_E \in K$ and $\lambda(\chi_E) = \mu(E)$. i.e. $\mu(\mathfrak{R}) \subseteq \lambda(K)$. (5)

Now, we will show the opposite inclusion indirectly.

Suppose $\lambda(K) \not\subseteq \mu(\mathfrak{R})$. Then there exists a $p \neq 0$ such that

$$p \in \lambda(K) \text{ and } p \notin \mu(\mathfrak{R}).$$

We put $K_p := \{g \in K \mid \lambda(g) = p\}$. Then

K_p is convex and since λ is weak*-continuous, K_p is weak*-compact.

By the Krein-Milman Theorem the set K_p has an extreme point.

Since $p \notin \mu(\mathfrak{R})$, we can observe that K_p does not contain any characteristics function.

This implies that every extreme point of K_p is not a characteristics function

Let $g \in \text{Ext}(K_p)$ such that $g \neq \chi_E$ for all $E \in \mathfrak{R}$ in $L^{\infty}(\sigma)$. i.e.

- (a) $g \in L^\infty(\sigma)$
 (b) $g \neq \chi_E$ for all $E \in \mathfrak{R}$.

Then (a) implies that there exists $M > 0$ such that

$$|g(x)| \leq M \text{ a.e. on } X.$$

Choose E such that $\sigma(E) = 0$ and $|g(x)| > M$ for all $x \in E$.

Now, we put $A := X \setminus E$.

Then for all $x \in A$, we have that $|g(x)| \leq M$.

Case 1: $\sigma(X) = 0$. Then trivially we have $\mu(\mathfrak{R}) = \{0\} \subseteq \mathbf{R}^n$ is compact convex set.

Case 2: $\sigma(X) > 0$. Then

$$\sigma(X) = \sigma(A) + \sigma(X \setminus A) = \sigma(A) > 0.$$

and by (b) we can show that there exists $r > 0$ such that $r \leq g(x) \leq 1-r, x \in A$. (6)

Now we put $Y := \chi_A L^\infty(\sigma)$. Then we know that

$$L^\infty(\sigma) = (L^1(\sigma))^*$$

and

$$\chi_B \in L^1(\sigma) \text{ for all } B \in \mathfrak{R}.$$

Hence, $L^1(\sigma)$ contains non-zero element. Moreover, by Corollary 1 of Chapter 1 $L^1(\sigma)$ is locally convex and hence $(L^1(\sigma))^*$ separates points of $L^1(\sigma)$. Then both

$$(L^1(\sigma))^* \text{ and } L^\infty(\sigma) \text{ contain non-zero elements. (see [7])} \quad (7)$$

Therefore by (6) and (7) we get $Y \neq \{0\}$.

Let now $g_1 \in Y$ such that $g_1 \neq 0$. Then

$$g_1(x) = \begin{cases} 0 & \text{if } x \notin A \\ g_2(x) & \text{if } x \in A \end{cases} \quad \text{for some } g_2 \in L^\infty(\sigma).$$

Now since $g_2 \in L^\infty(\sigma)$, $g_2 \neq 0$.

Let $B \in \mathfrak{R}$ such that $g_2 \neq 0$ on B and $\sigma(B) = 0$.

We put $C := B \cup A^c$. Then

$$\sigma(C) = 0, B \subseteq C \text{ and } A^c \subseteq C.$$

Now we put $f := g_1 - g_2$. Then $f = 0$ on C^c and $f \neq 0$ on C . (8)

Hence it follows that $f = 0$ σ -almost everywhere and hence

$$\lambda(f) = 0.$$

From this it follows that $\lambda(g+f) = p$ and $\lambda(g-f) = p$ i.e. $g+f, g-f \in K_p$.

Moreover, $g = \frac{1}{2}(g+f) + \frac{1}{2}(g-f) \in K_p$.

Hence since g is an extreme point of K_p , we get that

$$g = g+f = g-f \text{ which implies } f = 0.$$

But this is a contradiction to (8).

Therefore, K_p contains a characteristic function. i.e. $\lambda(K) \subseteq \mu(\mathfrak{R})$. (9)

Hence, by (5) and (9) we get $\lambda(K) = \mu(\mathfrak{R})$.

Therefore, $\mu(\mathfrak{R})$ is compact and convex subset of \mathbf{R}^n .

Therefore, $\mu(\mathfrak{R})$ is compact and convex subset of \mathbf{R}^n . //

2.4: A fixed point theorem

This fixed point theorem is an application of the *Krein-Milman Theorem*.

Now we define some terms.

Definition 1:

Let $F: S \rightarrow S$ be a function where S is non-empty.

Then a point $x \in S$ is said to be a *fixed point* of F if and only if $F(x) = x$.

Definition 2:

Let Y be a vector space and K be a convex set. Then

A function $f: K \rightarrow Y$ is called *affine function* if and only if

$x, y \in K$ and $\alpha \in (0, 1)$ implies $f(\alpha x + (1-\alpha)y) = \alpha f(x) + (1-\alpha)f(y)$.

Definition 3:

Let G be a collection of mappings from K in to K . Then g is said to be *equicontinuous family* if and only if for any neighbourhood W of 0 in X , there exists a neighbourhood V of 0 in X such that if $x, y \in K$ and $x-y \in V$, then $T(x)-T(y) \in W$ for all $T \in G$.

Here, observe that each $T \in G$ is continuous.

Moreover, G is said to be *equicontinuous group of affine maps* if and only if G is a group under compositions of maps in G .

Definition 4:

Let X be a topological space and $K \subseteq X$. Let G be a collection of continuous maps from X in to X . Then K is said to be *invariant* under G if and only if $T(K) \subseteq K$ for all $T \in G$.

Sometimes we use the term *G-invariant*.

Now we prove the following

Lemma 1:

Let X be compact Hausdorff topological space and g be a group of *equicontinuous maps* from X in to X . Let $A \subseteq X$ be invariant under G . Then

(a) \overline{A} is G -invariant

(b) $\text{Conv}(\overline{A})$ is G -invariant

Proof:

We will prove only (a) the proof of (b) can be done in a similar way.

Let $T \in G$ and $x \in \overline{A}$. Then there exists a sequence (x_n) in A such that

$$x_n \xrightarrow{n \rightarrow \infty} x.$$

Moreover since T is continuous $T(x_n) \xrightarrow{n \rightarrow \infty} T(x)$. (8)

We will show that $T(x) \in \overline{A}$.

Observe that G separate points of X and since X is compact Hausdorff space, one can show that X is metrizable (For details see [1]).

Therefore, since $T(x_n) \in A$ for each $n \in \mathbb{N}$, and $T(x_n) \xrightarrow{n \rightarrow \infty} T(x)$.

Thus we get $T(x) \in \overline{A}$ //

Lemma 2:

Suppose X and Y are topological spaces such that Y is compact and $E \subseteq X \times Y$

Let $\pi: X \times Y \rightarrow X$ be the projection onto X .

If $p \in X \cap \overline{\pi(E)}$, then there exists $q \in Y$ such that $(p, q) \in \overline{E}$.

Proof:

We will prove by contrapositive.

Suppose there exists $p \in X$ such that $(p, q) \notin \overline{E}$.

Then there exists a neighbourhood W_q of $q \in Y$ such that for some neighbourhood V_q of p in X such that $(V_q \times W_q) \cap E = \emptyset$.

Observe that $Y \subseteq \bigcup_{q \in Y} W_q$.

Then since Y is compact there exists $q_1, \dots, q_n \in Y$ such that $Y \subseteq \bigcup_{i=1}^n W_{q_i}$.

Now, we put $V := \bigcap_{i=1}^n V_{q_i}$. (9)

Then V is a neighbourhood of p such that $V \cap \pi(E) = \emptyset$.

Hence, $p \notin \overline{\pi(E)}$ which is a contradiction.

Therefore, there exists $q \in Y$ such that $(p, q) \in \overline{E}$ //

Lemma 3:

Let X be a topological space and $x, y \in X$ such that $x \neq y$.

Then there exists a neighbourhood W of 0 such that $x - y \notin W$.

The proof is trivial.

Theorem 1:

Suppose that

- (a) K is a non-empty compact and convex set in a locally convex space X .
- (b) G is an equicontinuous group of affine maps from K into K

Then there exists $p \in K$ such that $T(p) = p, \forall T \in G$.

Proof:

Let $\Omega = \{H \subseteq K \mid H \neq \emptyset, H \text{ is compact convex}, T(H) \subseteq H, T \in G\}$

Observe that, since $K \in \Omega, \Omega \neq \emptyset$.

Now, We partially order Ω by inclusion.

Then by Hausdorff–Maximality Principle Ω contains a maximal totally ordered subcollection Ω_0 .

Now, we put $Y := \bigcap_{Q \in \Omega_0} Q$.

We will show that:

(i) Y is a minimal member of Ω .

(ii) $|Y| = 1$.

The proof for (i) is trivial except $Y \neq \emptyset$.

Now, we show (ii) and that $Y \neq \emptyset$.

Let Q_1 and Q_2 be in Ω_0 such that $Q_1 \cap Q_2 = \emptyset$.

Then Ω_0 is not totally ordered which is a contradiction.

So, by induction we can see that the intersection of finitely many elements of Ω_0 is non-empty. (10)

Thus since K is compact and Ω_0 is a collection of closed subsets of K satisfying the finite intersection property, $Y \neq \emptyset$. (For details see [5], finite intersection property)

Therefore (i) holds. (11)

Now, we prove (ii).

Suppose there are $x, y \in Y$ such that $x \neq y$. Then by Lemma 3 there exists a neighbourhood W of 0 such that $x - y \notin W$. (12)

Moreover, we will prove that

There exists a neighbourhood V of 0 such that $T(x) - T(y) \notin V$ for all $T \in G$. (13)

To prove (13) we assume that there exists $T \in G$ such that $T(x) - T(y) \in V$.

By equicontinuity we get

$$T^{-1}(T(x)) - T^{-1}(T(y)) = x - y \in W \text{ which is a contradiction to (12).}$$

Hence (13) holds true.

Now, define, $S := \{T(z) \mid T \in G\}$, where $z = \frac{1}{2}x + \frac{1}{2}y, x \neq y$

Then observe that $S \neq \emptyset, z \in Y$ and S is invariant under G .

Moreover, by Lemma 1, we have that \overline{S} and $\overline{\text{Conv}(S)}$ are G -invariant.

Then since $T(z) = \frac{1}{2}T(x) + \frac{1}{2}T(y) \in Y$, we get

$$S \subseteq Y, \overline{\text{Conv}(S)} \subseteq Y \text{ and } \overline{\text{Conv}(S)} \text{ is compact convex subset of } Y.$$

By the minimality of Y , we have that $Y = \overline{\text{Conv}(S)}$.

Moreover, by the Krein-Milman Theorem, we get $Y = \overline{\text{Conv}(\text{Ext}(Y))}$.

Let now $p \in Y$ be an extreme point.

Define $E := \{(T(z), T(x), T(y)) \mid T \in G\} \subseteq Y \times Y \times Y, x, y \in Y, x \neq y, z = \frac{1}{2}x + \frac{1}{2}y, x \neq y$.

Then, $\pi(E) = S$ (where π is projection to the 1st component).

Moreover, we have $p \in \overline{\pi(E)}$ and is an extreme point of Y .

Then, by Theorem 3, we see that $p \in \overline{S}$.

Now, by Lemma 2, there exists $(x^*, y^*) \in Y \times Y$ such that $(p, x^*, y^*) \in \overline{E}$.

Therefore, There exists a sequence $(T(z_n), T(x_n), T(y_n))$ in E such that

$$(T(z_n), T(x_n), T(y_n)) \xrightarrow{n \rightarrow \infty} (p, x^*, y^*)$$

(Here, $z_n = \frac{1}{2}x_n + \frac{1}{2}y_n, x_n \neq y_n, x_n, y_n \in Y, T(x_n) - T(y_n) \in V$).

Observe that $x^* - y^* \notin V$.

(14)

Moreover, since T is affine, we get $2p = x^* + y^*$ and

Since p is an extreme point of Y , we have $p = x^* = y^*$ i.e. $x^* - y^* = 0 \in V$ which is a contradiction to (14).

Therefore, Y contains exactly one point say p .

Since $T(Y) \subseteq Y$ for all $T \in G$, we get $T(p) = p$ for all $T \in G$.

Let $A: X \rightarrow X$ be a continuous linear mapping. Then a continuous linear mapping $A^*: X^* \rightarrow X^*$ is said to be adjoint of A if and only if

$$\langle Ax, y^* \rangle = \langle x, A^*y^* \rangle \quad \forall x \in X, \forall y^* \in X^*$$

Definition 3:

Let X be a normed space and X^* be its dual space, $M \subseteq X^*$.

Then the orthogonal space of M denoted by M^\perp is defined by

$$M^\perp = \{x \in X \mid \langle x, y^* \rangle = 0, \forall y^* \in M\}$$

Theorem 1:

Let X and Y be normed spaces and $A \in \mathcal{B}(X, Y)$.

Then there exists a unique $A^* \in \mathcal{B}(Y^*, X^*)$ such that

$$(a) \langle Ax, y^* \rangle = \langle x, A^*y^* \rangle, \quad \forall x \in X, \forall y^* \in Y^*$$

$$(b) \|A\| = \|A^*\|$$

The proof is found in [1] and [7].

Theorem 2:

Let X be a normed space and $B = \{x \in X \mid \|x\| \leq 1\}$, $B^* = \{y^* \in X^* \mid \|y^*\| \leq 1\}$. Then

$$(a) \text{For } x \in X, \|x\| = \sup_{y^* \in B^*} |\langle x, y^* \rangle|$$

(b) B^* is weak*-compact

Proof:

To prove (a), let $x \in X$ such that $x \neq 0$.

Then by Hahn-Banach Theorem, there exists $y^* \in X^*$ such that

$$\langle x, y^* \rangle = \|x\| \quad \text{and} \quad \|y^*\| = 1$$

2.5: Approximation theorems

2.5.1: Interpolation theorem:

The proof of this theorem involves *the Hahn-Banach, Riesz-Representation Theorem* on bounded linear functional on $C(X)$ where X is compact Hausdorff space. Moreover, it uses the adjoint of an operator and concepts from the duality theory in Banach spaces.

Thus we give some definitions and important theorems.

Denotations:

Let X and Y be normed spaces. Then

- (a) X^* denotes the set of all continuous linear mappings from X into \mathbf{K} .
- (b) $\mathcal{L}(X, Y)$ denotes the set of continuous linear mappings from X into Y .

Definition 2:

Let $A: X \rightarrow Y$ be a continuous linear mapping. Then a continuous linear mapping $A^*: X^* \rightarrow Y^*$ is said to be *adjoint of A* if and only if

$$\langle Ax, x^* \rangle = \langle x, A^*x^* \rangle \quad \forall x \in X, \forall x^* \in X^*.$$

Definition 3:

Let X be a normed space and X^* be its dual space, $M \subseteq X$.

Then the *orthogonal space of M* denoted by M^\perp is defined by

$$M^\perp = \{x^* \in X^* \mid \langle x, x^* \rangle = 0, \forall x \in M\}.$$

Theorem 1:

Let X and Y be normed spaces and $A \in \mathcal{L}(X, Y)$.

Then there exists a unique $A^* \in \mathcal{L}(Y^*, X^*)$ such that

- (a) $\langle Ax, y^* \rangle = \langle x, A^*y^* \rangle, \quad \forall x \in X, \forall y^* \in Y^*$
- (b) $\|A\| = \|A^*\|$

The proof is found in [1] and [7].

Theorem 2:

Let X be a normed space and $B = \{x \in X \mid \|x\| \leq 1\}$, $B^* = \{\lambda \in X^* \mid \|\lambda\| \leq 1\}$. Then

- (a) For $x \in X$, $\|x\| = \sup_{\lambda \in B^*} \langle x, \lambda \rangle$

- (b) B^* is weak* - compact

Proof:

To prove (a), let $x \in X$ such that $x \neq 0$.

Then by *Hahn-Banach Theorem*, there exists $\lambda \in X^*$ such that

$$\langle x, \lambda \rangle = \|x\| \quad \text{and} \quad \|\lambda\| = 1.$$

This implies that $\sup_{\lambda \in B^*} \langle x, \lambda \rangle \geq \|x\|$. (1)

On the other hand for all $\lambda \in B^*$ (i.e. λ is continuous) we get

$$|\langle x, \lambda \rangle| \leq \underbrace{\|\lambda\|}_{=1} \|x\| = \|x\|.$$

Therefore, we get $\sup_{\lambda \in B^*} \langle x, \lambda \rangle \leq \|x\|$. (2)

Thus, from (1) and (2), we have $\|x\| = \sup_{\lambda \in B^*} \langle x, \lambda \rangle$ //

To prove (b), let U be an open unit ball in X .

Then $\lambda \in B^*$ if and only if $|\langle x, \lambda \rangle| \leq 1$, for all x in U .

Hence, $B^* = \{ \lambda \in X^* \mid |\langle x, \lambda \rangle| \leq 1, \forall x \in U \}$.

Therefore, by *Banach-Alaoglu theorem*, we have, B^* is weak*-compact. //

Theorem 3 :

Let U and V be open unit balls in the Banach space X and Y respectively. If $A \in \mathcal{L}(X)$ and $\delta > 0$, then the following implications hold

(a) \Rightarrow (b) \Rightarrow (c) \Rightarrow (d) where,

(a) $\|A^* y^*\| \geq \delta \|y^*\|$

(b) $\delta V \subseteq A(U)$

(c) $\delta V \subseteq A(U)$

(d) $A(X) = Y$.

The proof is found in [1]

Theorem 4: (Riesz-Representation Theorem)

Let X be a compact Hausdorff space and $C(X)$ be the set of all complex continuous functions on X . Then for $\lambda \in C(X)^*$, there exists a unique complex Borel measure μ such that

(a) $\lambda(f) = \int f d\mu, f \in C(X)$.

(b) $\|\lambda\| = |\mu|(X) = \|\mu\|$.

The proof is found in [4] and [8]

Definition 4:

Let X and Y be normed spaces. Then X and Y are said to be *isometrically isomorphic* iff there is a bijective linear mapping λ from X to Y such that

$$\|\lambda(x)\| = \|x\| \text{ for each } x \in X.$$

Corollary 1:

Let B be the set of all borel signed measures on a compact hausdorff space X .
Then B and $C(X)^*$ are isometrically isomorphic.

Proof:

Define $F : C(X)^* \rightarrow B$ by $F(\lambda) = \mu_\lambda$ where μ_λ is the measure guaranted by the *Reisz-representation theorem*.

Clearly, F is bijective. We will Show that F is linear.

Let $\lambda_1, \lambda_2 \in C(X)^*$. then, there exists measures μ_1 and μ_2 satisfying

$$\lambda_1(f) = \int f d\mu_1 \quad \text{and} \quad \lambda_2(f) = \int f d\mu_2 = \int f d\mu_1 \quad \text{for all } f \in C(X).$$

Then,

$$(\lambda_1 + \lambda_2)(f) = \int f d\mu_1 + \int f d\mu_2 = \int f d(\mu_1 + \mu_2) = \int f d\mu_3 \quad \text{where } \mu_3 \text{ is the measure coressponding to } \lambda_1 + \lambda_2.$$

Moreover, $F(\lambda_1 + \lambda_2) = \mu_3$.

Thus by uniqueness of the measures we have $\mu_3 = \mu_1 + \mu_2 = F(\lambda_1 + \lambda_2)$.

Therefore, $F(\lambda_1 + \lambda_2) = F(\lambda_1) + F(\lambda_2)$ (i.e F is additive)

Similarly we can show that F is homogeneous.

Therefore, F is linear and bijective mapping from $C(X)^*$ onto B such that

$$\|\lambda\| = \|F(\lambda)\| = \|\mu_\lambda\|, \quad \text{for all } \lambda \in C(X)^*.$$

Hence ,we get that B and $C(X)^*$ are isometrically isomorphic.//

So from the above isometry, we can conclude many topological concepts like closedness, compactness etc. For instance $\{\mu \in B \mid \|\mu\| \leq 1\}$ is Weak*- compact. Moreover , we can identify the spaces $C(X)^*$ and B as identical spaces (i.e $C(X)^*$ and B are indistinguishable). Hence for the proof of the following theorem the complex borel measures play the role of bounded linear functionals on $C(X)$.

Theorem 5:(Interpolation Theorem)

Let Y be a closed subspace of $C(X)$, X is a compact hausdorff space. Let $K \subseteq X$ be compact and $\int \mu = 0$ for every $\mu \in Y^\perp$. If $g \in C(K)$ and $|g| < 1$, then there exists $f \in Y$ such that $f = g$ on K and $|f| < 1$ on X.

Proof:

$$\text{Define } \rho : Y \rightarrow C(K) \text{ by } \rho(f) := f|_K. \tag{1}$$

we put $B_Y = \{f \in Y \mid \|f\|_\infty < 1\}$ and $B_K = \{f \in C(K) \mid \|f\|_\infty < 1\}$. Then, we will show that $\rho(B_Y) = B_K$. From the definition of ρ it is easy to show that ρ is linear mapping.

$$\text{Now, } \|\rho(f)\|_\infty = \text{Max}_{x \in K} |\rho(f(x))| \leq \text{Max}_{x \in X} |\rho(f(x))| = \|f\|_\infty. \tag{2}$$

Thus, $\rho \in S(Y, C(K))$.

Now, we put $M(K) := C(K)^*$.

Then by *Theorem 1* of this section there exists a unique $\rho^* \in S(M(K), Y^*)$ such that

$$\begin{aligned} \text{(a)} \quad & \langle \rho f, \lambda \rangle = \langle f, \rho^* \lambda \rangle \quad \text{for all } f \in Y, \lambda \in M(K). \\ \text{(b)} \quad & \|\rho\| = \|\rho^*\|. \end{aligned} \tag{3}$$

Moreover, by the *Corollary 1* of this section, we observe that $C(K)^*$ is the banach space of all borel measures on K with total variation norm $\|\mu\| = |\mu|(K)$.

Let now, $\mu \in M(K)$. Then $\rho^* \mu \in Y^*$. Then by the *Continuous Extension Theorem of Hahn-Banach Theorem*, there exists $\sigma \in C(X)^*$ such that $\sigma = \rho^* \mu$ on Y and $\|\sigma\| = \|\rho^* \mu\|$.

Now, by *Reisz-Representation Theorem* and by (a) we have for all $f \in Y$,

$$\int f d\sigma = \langle f, \rho^* \mu \rangle = \langle \rho f, \mu \rangle = \int f d\mu \tag{4}$$

Moreover, observe that $C(X) \subseteq C(K)$, $K \subseteq X$ and $C(K)^* \subseteq C(X)^*$.

Now, since $\mu \in C(K)^*$, we have $\mu \in C(X)^*$ and we know that μ is a borel measure defined on a σ -algebra of subsets of K .

Hence we can consider $\mu \in C(X)^*$ with support in K .

Now, let $f \in Y$. Then from (4) we have

$$\int f d\sigma - \int f d\mu = 0 \quad \text{i.e. } \sigma - \mu \in Y^\perp.$$

Thus by assumption, we have $|\sigma - \mu|(K) = 0$ and for each set $E \subseteq K$, we get $\sigma(E) = \mu(E)$. Hence

$$\|\mu\| \leq \|\sigma\|$$

Or

$$\|\mu\| \leq \|\rho^* \mu\|. \tag{5}$$

Since Y is a closed subspace of $C(X)$, Y is a banach space and so does $C(K)$.

By *Theorem 3* of this section, where Y playing the role of X and $C(K)$ playing the role of Y and $\delta = 1$, we get

$$\|\mu\| \leq 1 \cdot \|\rho^* \mu\| \quad \text{for all } \mu \in C(K)^*.$$

and

$$\text{Part (a) of Theorem 3 implies that } B_K \subseteq \rho(B_Y). \tag{6}$$

Now, if $g \in C(K)$ such that $|g| < 1$ (i.e. $g \in C(K)^*$) then by (6) there exists $f \in Y$ such that $|f| < 1$ on X and $g = \rho(f) = \iota_K \cdot f$.

Remark:

By (d) of *Theorem 3*, we have $\rho(Y) = C(K)$. This implies that every continuous function g on K extends to a member of Y .

Now, we will give the proof of *Bishop's Theorem* which is a generalization of the *Stone-Weirestrass Theorem*.

2.5.2: Bishop's theorem :

Definition 1:

Let X be a compact hausdorff space and A be a subspace of $C(X)$. Then A is called an algebra if and only if $f.g \in A$ for all $f, g \in A$ where $(f.g)(t) := f(t) .g(t)$. for all $t \in S$.

Definition 2:

Let A be an algebra of $C(X)$ and $E \subseteq X$. Then E is said to be A -antisymmetric if and only if for all $f \in A$, $f|_E$ is real $\Rightarrow f(x) = C = \text{constant}$, for all $x \in E$.

Denotation:

$A_E = \{f|_E \mid E \text{ is an } A\text{-antisymmetric subset of } X\}$

Now, we define a relation \sim on X as follows.

For $x, y \in X$, $x \sim y$ if and only if

there exists an A - antisymmetric set E such that $\{x, y\} \subseteq E$.

Then,

(a) \sim define an equivalence relation on X .

(b) for each $x \in X$, \bar{x} is maximal A -antisymmetric set.

Proof:

We prove only (b).

Let $f \in A$ such that $f(h) \in \mathbf{R}$ for each $h \in \bar{x}$. Then there exists an A - antisymmetric set E_0 such that $\{h, x\} \subseteq E_0$.

Since, $h \in E_0$ and $f(h) \in \mathbf{R}$, we have $f(h) = k$ (constant). Hence, \bar{x} is A - antisymmetric subset of X . It can be easily shown that it is also maximal.//

Lemma 1:

Let (X, \mathfrak{R}, μ) be a measure space and $f: X \rightarrow [0, \infty]$ be a measurable function.

(a) Define $\sigma: \mathfrak{R} \rightarrow \mathbf{R}$ by

$$\sigma(E) := \int_E f d\mu. \text{ Then}$$

(b) σ is a measure defined on \mathfrak{R} .

(c) For each measurable function such that $g \geq 0$, we have, $\int g d\sigma = \int f g d\mu$.

Proof:

To prove (a) apply the *monotone convergence theorem* and to prove (b) use the fact that for each measurable function $f \geq 0$, there exists an increasing sequence of simple functions (φ_n) such that $\varphi_n(x) \leq f(x)$ for all $x \in X$. Hence the proof will be complete. (For more details refer [8]).

Theorem 1: (Bishop's Theorem)

Let A be a closed subalgebra of $C(X)$. Suppose $g \in C(X)$ and $g_E \in A_E$ for every A -antisymmetric set E . Then $g \in A$.

Proof:

We recall that $A^\perp = \{ \lambda \in C(X)^* \mid \langle f, \lambda \rangle = 0, \forall f \in A \}$. Then A^\perp consists of all borel measures μ on X such that $\int f d\mu = 0 \quad \forall f \in A$.

Now, we put $K = \{ \mu \in A^\perp \mid \|\mu\| \leq 1 \}$ and $\|\mu\| = |\mu|(X)$.

Then clearly K is convex and weak* - compact. Moreover, by the *Krein-Milman Theorem* we have

$$K = \overline{\text{Conv}(\text{Ext}(K))}.$$

Case 1: $K = \{0\}$

We will prove that $K = \{0\} \Rightarrow A^\perp = \{0\} \Rightarrow A = C(X)$.

Suppose $A^\perp \neq \{0\}$. Then there exists $\lambda \in C(X)^*$ such that

$$\lambda(g) \neq 0 \text{ for some } g \in C(X)$$

and

$$\lambda(f) = 0 \text{ for all } f \in A.$$

Then, by *Hahn-Banach Separation Theorem* $g \in \bar{A}$ and hence $g \in A$.

So, $\lambda(g) = 0$ which is a contradiction. i.e $A^\perp = \{0\}$.

Now, we show that $A^\perp = \{0\} \Rightarrow A = C(X)$.

Suppose there exists $g \in C(X) \setminus A$. Then again by *Hahn-Banach Separation Theorem* there exists $\lambda \in C(X)^*$ such that

$$\lambda(g) = 1$$

and

$$\lambda(f) = 0 \text{ for all } f \in A.$$

Hence, $\lambda \in A^\perp$ but $\lambda \neq 0$ i.e. $A^\perp \neq \{0\}$

Therefore, $A = C(X)$.

Case 2: $K \neq \{0\}$

Since, $K = \overline{\text{Conv}(\text{Ext}(K))}$, we can choose $\mu \in \text{Ext}(K)$ such that $\mu \neq 0$.

Then since K is convex μ is a boundary point of K i.e. $\|\mu\| = 1$

Now, Let E be the support of μ i.e $E = \overline{\{B \in \mathfrak{R} \mid \mu(B) \neq 0\}} \subseteq X$.

Then E is closed subset of a compact set X and hence compact.

Now, we will show that

(a) $\|\mu\| = |\mu|(E) = 1$.

and

(b) E is an A -antisymmetric set.

Now, let $\lambda \in C(X)^*$ such that $\lambda(f) = \int f d\mu$, for all $f \in C(X)$.
 Then by *Reisz-Representation Theorem* we get $\|\lambda\| = \|\mu\|$.
 Moreover since $\mu(B) = 0$ for all $B \notin E$, we have

$$\|\lambda\| = \|\mu\| = |\mu|(E) = 1 \quad (1)$$

Therefore, (a) holds.

Now, to show (b) let $f \in A$ such that $f|_E$ is real.
 Assume W.o.l.g that $|f| < 1$ on E .

Define measures σ and τ by

$$\sigma(B) = \frac{1}{2} \int_B (1+f) d\mu$$

and

$$\tau(B) = \frac{1}{2} \int_B (1-f) d\mu \quad (2)$$

Then by *Lemma 1* σ and τ are measures on X .
 Since A is a subspace of $C(X)$, we have

$$f^+ = \frac{f+|f|}{2} \quad \text{and} \quad f^- = \frac{|f|-f}{2} \quad \text{are in } A \text{ for all } f \in A.$$

We will show that σ and τ are in A^\perp .

To do this, let $f \in A$ and $f \geq 0$. Then by *Lemma 1* we have

$$\int f d\sigma = \frac{1}{2} \int f(1+f) d\mu = \underbrace{\frac{1}{2} \int f d\mu}_{=0} + \underbrace{\frac{1}{2} \int f^2 d\mu}_{=0} = 0. \quad (3)$$

Hence, $\int f d\sigma = 0$ for all $f \in A$ and $f \geq 0$.

Moreover, for arbitrary $f \in A$, we have $f = f^+ - f^-$
 and

$$\int f d\sigma = \int f^+ d\sigma - \int f^- d\sigma = 0 \text{ for all } f \in A. \quad (4)$$

Therefore, by (3) and (4) we get $\sigma \in A^\perp$. Similarly, we can show that $\tau \in A^\perp$.

Moreover, it is easy to see that

$$(i) \|\sigma\| > 0 \quad \text{and} \quad \|\tau\| > 0. \quad (5)$$

$$(ii) \|\sigma\| + \|\tau\| = \frac{1}{2} \int_E (1+f) d|\mu| + \frac{1}{2} \int_E (1-f) d|\mu| = |\mu|(E) = 1 \quad (6)$$

$$(iii) \mu = \sigma + \tau = \|\sigma\| \frac{\sigma}{\|\sigma\|} + \|\tau\| \frac{\tau}{\|\tau\|} \quad (7)$$

Moreover, $\sigma_1 = \frac{\sigma}{\|\sigma\|}$ and $\tau_1 = \frac{\tau}{\|\tau\|}$ are in K and since μ is an extreme point,
 we have

$$\mu = \sigma_1 = \tau_1.$$

Therefore, $\|\sigma\| \mu(E) = \sigma(E)$ and so we get

$$\frac{1}{2} \int_E (1+f) d\mu = \|\sigma\| \int_E d\mu.$$

Thus, it follows that $f(x) = 2\|\sigma\| - 1$ on E . i.e E is A -antisymmetric. (8)

Now, Let g satisfies the assumption of the theorem.

Then for the above extreme point μ in K , we have

$$\int g d\mu = \int f d\mu = \int f d\mu = 0. \quad (9)$$

Moreover since $K = \overline{\text{Conv}(\text{Ext}(K))}$,

$$\int g d\mu = 0 \text{ for all } \mu \in \text{Conv}(\text{Ext}(K)).$$

Now, for $\mu \in K$ there exists (μ_k) in $\text{Conv}(\text{Ext}(K))$ such that $\mu_k \xrightarrow{k \rightarrow \infty} \mu$.

Observe that $F : K \rightarrow \mathbf{R}$ defined by $F(\mu) := \int g d\mu$ is weak*-continuous function.

This implies that $\int g d\mu = F(\mu) = \lim_{k \rightarrow \infty} F(\mu_k) = \lim_{k \rightarrow \infty} \underbrace{\int g d\mu_k}_{=0} = 0.$ (10)

Therefore, for all $\mu \in K$, $\int g d\mu = 0.$ (11)

Moreover, by the assumption on g , we have $g = \chi_E f$ for some $f \in A$.

Then for $\mu \in A^\perp$ we get that

$$\int g d\mu = \int \chi_E f d\mu = \int f d\mu = \int f d\mu = 0. \quad (12)$$

So we get $\int g d\mu = 0$ for all $\mu \in A^\perp.$ (13)

Therefore by (13) we conclude that for all $\lambda \in C(X)^*$ such that $\lambda(f) = 0$ for all $f \in A$.

This implies that $\lambda(g) = 0$.

Since $C(X)$ is locally convex, by *Hahn-Banach Separation Theorem* we get

$$g \in \overline{A} \text{ and hence } g \in A. //$$

Now we state and prove a special case of the *Bishop's Theorem* called the *Stone-Weirestrass Theorem*

Theorem 2: (Stone- Weirestrass Theorem)

Let A be a closed subalgebra of $C(X)$ such that

- (a) A is self adjoint (i. e $f \in A \Rightarrow \bar{f} \in A$)
- (b) A separates points of X
- (c) at any $p \in X$ there exists $f \in A$ such that $f(p) \neq 0$.

Then, $A = C(X)$.

Proof:

Let $A_0 = \{Re(f) \mid f \in A\}$. Then by (a) $A_0 \subseteq A$.

Now, for $x, y \in X$ such that $x \neq y$, there exists $f \in A$ such that $f(x) \neq f(y)$.

Then, $Re f(x) \neq Re f(y)$. i. e A_0 separates points of X .

Now we will show that the only A - antisymmetric sets are singltons.

We use indirect proof.

Suppose there exists an A - antisymmetric set E such $|E| > 1$.

Let $x, y \in E$ such that $x \neq y$. Hence there exists $f \in A$ such that

$$Re f(x) \neq Re f(y) \tag{14}$$

Now we put $h(x) := Re f(x)$.

Then h is real valued on X and hence real on E .

By (14) observe that h is not constant on E which is a contradiction to the fact that E is an A - antisymmetric set in X .

Thus the only A -antisymmetric sets in X are singltons. (15)

Let now $g \in C(X)$ and $F = \{a\}$ is a maximal A - antisymmetric set in X .

Then $g_F = g(a)$ is constant.

Moreover, by © there exists $f \in A$ such that $f(a) \neq 0$.

Thus the constant function $\alpha(x) := 1 = \frac{1}{f(a)} f_1(x)$, $f_1 = f_F$ is in A .

Therefore A contains all constant functions.

This implies that $g_F \in A_F$. i. e g satisfies the assumption of the *Bishop's Theorem*.

Therefore, $g \in A$. i. e $A = C(X)$.//

3. Fixed point theorems

3.1: The Banach-Fixed Point Theorem

Definition 1 :

Let (X, d) be a metric space and $A : X \rightarrow X$ be a mapping. Then A is said to satisfy the *Lipschitz condition* if and only if there exists $\alpha > 0$ such that

$$d(A(x), A(y)) \leq \alpha d(x, y), \forall x, y \in X \quad (1)$$

If $\alpha < 1$ in (1), then A is called a *contraction mapping*.

Example:

Let $f : \mathbf{R} \rightarrow \mathbf{R}$ be differentiable and $|f'(x)| \leq \alpha \leq 1$ for all $x \in \mathbf{R}$.

Then f is a contraction.

The proof can be done easily by *The Mean value Theorem*.

Remark:

Every contraction is uniformly continuous

We shall now state and prove the most important property of contractions called the *Banach Fixed Point Theorem*.

Theorem 1:

Let (X, d) be a complete metric space and $A : X \rightarrow X$ be a contraction. Then A has unique fixed point.

Proof:

Since A is a contraction there exists $\alpha < 1$ such that

$$d(A(x), A(y)) \leq \alpha d(x, y), \forall x, y \in X.$$

Let x be an arbitrary point of X .

We put $x_1 := A(x)$, $x_2 := A(x_1)$, $x_{n+1} := A(x_n)$, $n \in \mathbf{N}$.

Then we have

$$x_n = A^n(x), n \in \mathbf{N}.$$

Let $n, m, r \in \mathbf{N}$ such that $m \geq n$.

Then, one can show by induction that

$$d(x_r, x_{r-1}) \leq \alpha^{r-1} d(x_1, x), \forall r \in \mathbf{N} \quad (2)$$

Now we will show that (x_n) is a Cauchy sequence in X .

For $n, m \in \mathbf{N}$ such that $m \geq n$, we have

$$d(x_m, x_n) \leq d(x_m, x_{m-1}) + d(x_{m-1}, x_n). \quad (3)$$

By repeated application of the triangle inequality we get

$$\begin{aligned} d(x_m, x_n) &\leq d(x_m, x_{m-1}) + d(x_{m-1}, x_n) \\ &\leq d(x, x_1) [\alpha^{m-1} + \alpha^{m-2} + \dots + \alpha^n] \\ &= \alpha^n d(x, x_1) [1 + \alpha + \alpha^2 + \dots + \alpha^{m-n}] \end{aligned}$$

$$\leq \alpha^n d(x, x_1) \sum_{n=0}^{\infty} \alpha^n = \frac{\alpha^n d(x, x_1)}{1 - \alpha} \quad (4)$$

Now, if $n \rightarrow \infty$, then $m \rightarrow \infty$.

Hence, from (4) we get

$$\frac{\alpha^n d(x, x_1)}{1 - \alpha} \xrightarrow{n \rightarrow \infty} 0. \quad (5)$$

Then (5) implies that

$$d(x_n, x_m) \xrightarrow{n, m \rightarrow \infty} 0. \quad (6)$$

So, (x_n) is a Cauchy sequence in X .

Moreover since X is complete, there exists $x \in X$ such that

$$x_n \xrightarrow{n \rightarrow \infty} x \in X.$$

Since A is continuous, we get

$$A(x_n) \xrightarrow{n \rightarrow \infty} A(x). \quad (7)$$

But by definition, we have $x_{n+1} = A(x_n)$. Therefore, we have

$$x = \lim_{n \rightarrow \infty} x_n = \lim_{n \rightarrow \infty} x_{n+1} = \lim_{n \rightarrow \infty} A(x_n) = A(x). \quad (8)$$

Hence by (8) we conclude that A has a fixed point.

Now we show that A has unique fixed point in X .

Let x and y be two different fixed points of A in X . Then,

$$d(x, y) = d(A(x), A(y)) \leq \alpha d(x, y)$$

$$\text{or} \quad (1 - \alpha)d(x, y) \leq 0. \quad (9)$$

So from (9) since $\alpha < 1$, we see that $d(x, y) = 0$ i.e. $x = y$.

Hence A has a unique fixed point in X . //

Corollary 1:

Let (X, d) be complete metric space and $A: X \rightarrow Y$.

If A^n is a contraction for some $n \in \mathbb{N}$, then the equation

$$A(x) = x$$

has a unique solution.

Proof:

We put $B = A^n$.

Then since B is a contraction, by the *Banach-Fixed Point Theorem* there exists $y \in X$ such that

$$B(y) = y.$$

Now, observe that

$$(A \circ B)(y) = A(y) \text{ and } A^{n+1} = A \circ B = B \circ A. \quad (10)$$

By assumption there exists a $c < 1$ such that

$$d(B(x), B(y)) \leq cd(x, y), \forall x, y \in X.$$

In particular for $x = A(y)$ we have that

$$d(B(y), (B \circ A)(y)) \leq cd(y, A(y)).$$

Now from (10) we see that

$$d(y, A(y)) = d(B(y), (B \circ A)(y)) \leq cd(y, A(y)). \quad (11)$$

So since $c < 1$, from (11) we conclude that $A(y) = y$.

One can easily show that y is unique.

Therefore, $A(x) = x$ has exactly one solution. //

Definition 2:

Let $k : [a, b] \times [a, b] \rightarrow \mathbf{K}$ be continuous. Then

$T : C[a, b] \rightarrow C[a, b]$ defined by

$$(Tx)(s) := \int_a^s k(s, t)x(t)dt$$

is called the *Volterra Integral Operator*.

Corollary 2:

The volterra integral operator has a unique fixed point.

Proof:

It suffice to show that T^n is a contraction for some $n \in \mathbf{N}$.

Let $\mu := \max_{a \leq t, s \leq b} |k(s, t)|$ and $x, y \in C[a, b]$.

and let $d(x, y) := \|x - y\|_\infty$ for all $x, y \in C[a, b]$. Then

$$\|T^n x - T^n y\| \leq \frac{\mu^n (b-a)^n}{n!} \|x - y\|. \quad (\text{See [6]}) \quad (12)$$

Then since $\frac{\mu^n (b-a)^n}{n!} \xrightarrow{n \rightarrow \infty} 0$, there exists $m \in \mathbf{N}$ such that

$$\frac{\mu^m (b-a)^m}{m!} < 1 \quad (13)$$

Now we put $\alpha := \frac{\mu^m (b-a)^m}{m!} < 1$. Then we get

$$d(T^m x, T^m y) \leq \alpha d(x, y) \text{ for all } x, y \in C[a, b]. \quad (14)$$

Therefore, from (14) we see that T^m is a contraction. //

3.1.1: Some applications

We now apply the *Banach Fixed Point Theorem* to infinite linear equation system.

Corollary 3:

$$\text{Let } \xi_i - \sum_{j=1}^{\infty} \alpha_{ij} \xi_j = \eta_i, \quad x = (\xi_1, \xi_2, \dots), \quad y = (\eta_1, \eta_2, \dots) \in l^{\infty}. \quad (1)$$

If $\alpha := \sup_i \sum_{j=1}^{\infty} |\alpha_{ij}| < 1$, then (1) has unique solution in l^{∞} .

Proof:

Let $x = (\xi_1, \xi_2, \dots), (\eta_1, \eta_2, \dots) \in l^{\infty}$. Then

$$d(x, y) = \sup_i |\xi_i - \eta_i|$$

can be shown to be a metric on l^{∞} and that l^{∞} is a complete metric space.

Now, define $T: l^{\infty} \rightarrow l^{\infty}$ by $T(x) = Ax$ where $A = (\alpha_{ij})$, $x = (\xi_1, \xi_2, \dots) \in l^{\infty}$.

Then we have

$$\begin{aligned} d(T(x), T(y)) &= d(Ax, Ay) \\ &\leq \sup_i \sum_{j=1}^{\infty} |\alpha_{ij}| |\xi_j - \eta_j| \\ &\leq \alpha \sup_i |\xi_i - \eta_i| = \alpha d(x, y) \end{aligned}$$

So, $d(T(x), T(y)) \leq \alpha d(x, y)$ for all $x, y \in l^{\infty}$ and since $\alpha < 1$, T is a contraction. Therefore, by the *Banach Fixed Point Theorem* T has unique fixed point in l^{∞} .

Hence, $\sum_{j=1}^{\infty} \alpha_{ij} \xi_j = \xi_i$, $i \in N$ has a unique solution in l^{∞} .

So, for any given $y = (\eta_1, \eta_2, \dots) \in l^{\infty}$, (1) has unique solution in l^{∞} . //

Now we give another application of the *Banach Fixed Point Theorem* to non linear integral equation.

Corollary 3:

Let $y \in C[a, b]$ and K be continuous on $[a, b] \times [a, b] \times \mathbf{R}$.

If K satisfies on $[a, b] \times [a, b]$ a *Lipschitz Condition* of the form

$$|K(s, t, x_1) - K(s, t, x_2)| \leq C|x_1 - x_2|, \text{ then}$$

$$x(s) - \mu \int_a^b K(s, t, x(t)) dt = y(s)$$

is unique solvable for any μ such that $|\mu| < \frac{1}{c(a-b)}$.

The proof is found in [9]

We shall now state and prove the famous *Theorem of Peano-Picard*.

We shall now state and prove the famous *Theorem of Peano-Picard*.

Theorem 1 (Theorem of Peano-Picard)

Let $f(x, y)$ and $\frac{\partial f}{\partial y}$ exist and be continuous in a closed rectangle $E = [a, b] \times [c, d]$ and

let (x_0, y_0) be an interior point of E .

Then the differential equation

$$\frac{dy}{dx} = f(x, y) \quad (2)$$

has unique solution $y = g(x)$ such that $y_0 = g(x_0)$.

Proof:

Since f and $\frac{\partial f}{\partial y}$ are continuous in E , there exist constants K and M such that

$$|f(x, y)| \leq K \quad \text{and} \quad \left| \frac{\partial f(x, y)}{\partial y} \right| \leq M \quad \text{for all } (x, y) \in E \quad (3)$$

Let now (x, y_1) and (x, y_2) be in E .

Then by *Mean value Theorem* there exist $p \in (y_1, y_2)$ such that

$$|f(x, y_1) - f(x, y_2)| = |y_1 - y_2| \left| \frac{\partial f(x, p)}{\partial y} \right| \quad (4)$$

Thus (3) and (4) implies

$$|f(x, y_1) - f(x, y_2)| \leq M |y_1 - y_2| \quad (5)$$

Now, we will replace the differential equation by an integral equation.

Let $y = g(x)$ be such that $\frac{dy}{dx} = f(x, y)$, $y_0 = g(x_0)$.

Then we have
$$\int_{x_0}^x \frac{dy}{dx} = \int_{x_0}^x f(u, g(u)) du \quad (6)$$

Or

$$g(x) = g(x_0) + \int_{x_0}^x f(u, g(u)) du \quad (7)$$

and conversely if $y = g(x)$ is a solution of (7), then it is also a solution of (2).

Hence it suffices to solve (7).

Now choose a number c such that $M c < 1$.

Let $F = \{ (x, y) \mid |x - x_0| \leq c, |y - y_0| \leq Kc \} \subseteq E$

and

$G = \{ g: B \rightarrow \mathbf{R} \mid |g(x) - g(x_0)| \leq Kc \}$ where $B = \{ x \in \mathbf{R} \mid |x - x_0| \leq c \}$.

Then $G \subseteq C[x_0 - c, x_0 + c]$ and hence since G is complete metric space where the metric on G is the metric induced by the norm in $C[x_0 - c, x_0 + c]$.

Now, define $T : G \rightarrow G$ by $T(g) = h$ where $h(x) = g(x_0) + \int_{x_0}^x f(u, g(u)) du$

Then

$$\begin{aligned} |h(x) - g(x_0)| &= \left| \int_{x_0}^x f(u, g(u)) du \right| \\ &\leq \int_{x_0}^x |f(u, g(u))| du \\ &\leq K(x - x_0) \leq Kc \end{aligned}$$

So, we have $|h(x) - g(x_0)| \leq Kc$ for all x in B .

Then $h \in G$.

(8)

Let now g_1, g_2 in G . Then $T(g_1) = h_1, T(g_2) = h_2$ for some h_1, h_2 .

Hence since $\left| \frac{\partial f(x, y)}{\partial y} \right| \leq M$, we get

$$\begin{aligned} |h_1(x) - h_2(x)| &= \left| \int_{x_0}^x [f(u, g_1(u)) - f(u, g_2(u))] du \right| \\ &\leq M \int_{x_0}^x |g_1(u) - g_2(u)| du \\ &\leq Mc \text{Sup} |g_1(x) - g_2(x)| = Mc d(g_1, g_2) \end{aligned}$$

Or

$$d(T(g_1), T(g_2)) \leq Mc d(g_1, g_2).$$

Then T is a contraction.

Hence by the *Banach Fixed Point Theorem* there exists $g \in G$ such that

$$T(g) = g.$$

i.e. $g(x) = g(x_0) + \int_{x_0}^x f(u, g(u)) du$ has unique solution.

Therefore, the theorem holds. //

3.2: Kakutani's Fixed Point Theorem.

Let X be a topological vector space and $K \subseteq X$ and F be a family of maps from K in to K . It is sometimes of interest to know when the members of F have a common fixed point in K . We shall now give sufficient conditions which guarantee a common fixed point in K . The theorem is called the *Kakutani's Fixed Point Theorem* proved in 1938.

Theorem 1: (Kakutani's Fixed Point Theorem)

Theorem 1: (Kakutani's Fixed Point Theorem)

Let X be a topological vector space and K be a non-empty compact and convex subset of X . Let F be a commuting family of continuous affine maps from K into K . Then there exists $p \in K$ such that $T(p) = p$ for all $T \in F$.

Proof:

Let $T \in F$ and V be a neighbourhood of 0 in X .

We put $T^1 := T$ and for each $n \in \mathbb{N}$, $T^{n+1} := T \circ T^n$.

$$\text{Let } T_n := \frac{1}{n}(I + T + T^2 + \dots + T^{n-1}).$$

Then we observe that (a) T_n is affine for each $n \in \mathbb{N}$

$$(b) T_m \circ T_n = T_n \circ T_m \text{ for } n, m \in \mathbb{N}$$

Now, define $F^* := \{T_1 \circ T_2 \circ \dots \circ T_n \mid n \in \mathbb{N}\}$. Then it can be easily shown that F^* is a semigroup. Then for $f, g \in F^*$, we get

$$h = f \circ g = g \circ f \in F^*.$$

Moreover, since $g(K) \subseteq K$ and $f(K) \subseteq K$, we get

$$(f \circ g)(K) \subseteq f(K) \text{ and } (g \circ f)(K) \subseteq K. \tag{9}$$

Now (9) implies that $h(K) \subseteq f(K) \cap g(K)$

$$\tag{10}$$

Moreover, from (10) we observe that $f(K) \cap g(K) \neq \emptyset$.

Thus by induction we can see that

$$\bigcap_{f \in F} f(K) \neq \emptyset \text{ where } F \text{ is a finite collection of members of } F^*.$$

Or

$\{f(K) \mid f \in F^*\}$ satisfies the finite intersection property. Moreover, since K is compact and $f(K) \subseteq K$ for all $f \in F^*$ and $f(K)$ is compact (i.e. closed), we have $\{f(K) \mid f \in F^*\}$ is a collection of closed subsets of K satisfying the finite intersection property.

$$\text{Thus, } \bigcap_{f \in F^*} f(K) \neq \emptyset. \tag{11}$$

Let $p \in \bigcap_{f \in F^*} f(K)$ for all $f \in F^*$. Since $T_n \in F^*$ for each $n \in \mathbb{N}$, we have $p \in T_n(K)$ i.e. $T_n(x_n) = p$ for some $x_n \in K$.

Or

$$p = \frac{1}{n}(x_n + T(x_n) + \dots + T^{n-1}(x_n))$$

Then since T is affine, we get

$$T(p) = \frac{1}{n}(T(x_n) + \dots + T^n(x_n)).$$

$$\text{Now } p - T(p) = \frac{1}{n}(x_n - T^n(x_n)) \in K - K \text{ for all } n \in \mathbb{N}. \tag{12}$$

Moreover, we observe that $K - K$ is compact and hence bounded. i.e. for sufficiently great $m \in \mathbb{N}$ $K - K \subseteq mV$.

So, $p - T(p) \in \frac{1}{m}(mV) = V$ for any neighbourhood V of 0.

Therefore, we conclude that $T(p) = p$ for all $T \in F$ //

Remark:

Any collection of continuous functions on a set K need not have a common fixed point in K .

Example:

Let $F = \{f, g\}$, $f(x) = \frac{x+1}{2}$, $g(x) = \frac{x-1}{2}$ and $K = [-1, 1]$.

Observe that 1 and -1 are the unique fixed points of f and g respectively. Thus, f and g do not have a common fixed point on K //

3.3: Brouwer's and Schauder-Tychonoff Fixed Point Theorems.

Brouwer's Fixed Point Theorem is one of the most important results in the fixed point theory. Here we will not prove the theorem, besides we will see the applications in proving the infinite version of the theorem called the *Schauder-Tychonoff Fixed Point Theorem* and its applications to non-linear integral equations.

Now, we give some definitions.

Definition 1:

- (a) Let $X = \mathbb{R}^n$ and $1 \leq k \leq n$, $x_1, x_2, \dots, x_k \in \mathbb{R}^n$. Then $H := \text{Conv}\{x_1, x_2, \dots, x_k\}$ is called a k -simplex and \overline{H} is called a closed k -simplex.
- (b) A subset H of a vector space is called an n -cell if and only if it is homeomorphic to a closed n -simplex.

Theorem 1: (Brouwer's Fixed Point Theorem)

Let H be a closed n -cell and $f: H \rightarrow H$ be continuous. Then there exists $x \in H$ such that $f(x) = x$.

The proof can be found in [2].

Lemma 1:

Let K be a non-empty compact convex subset of a finite dimensional space X . Then K is an n -cell for some $n \in \mathbb{N}$.

The proof can be found in [2].

Corollary 1:

Let K be a non- empty compact convex subset of a finite dimensional normed space X . If $f: K \rightarrow K$ is continuous, then f has a fixed point in K .

The proof follows from Theorem 1 and Lemma 1.

Remark:

Let $D = \{x \in \mathbb{R}^n \mid \|x\| \leq 1\}$. Then every continuous map from D in D has a fixed point.

Lemma 2:

Let $S = \{x_1, x_2, \dots, x_k\} \subseteq X$ and X be a normed space.

For $\epsilon > 0$, let $N(S, \epsilon) := \bigcup_{i=1}^k S(x_i, \epsilon)$.

Let $\varphi: N(S, \epsilon) \rightarrow \text{Conv } S$ be defined by $\varphi(x) := \sum_{i=1}^k \frac{\lambda_i(x)}{\lambda(x)} x_i$,

where $\lambda_i(x) := \text{Max}\{0, \epsilon - \|x - x_i\|\}$ and $\lambda(x) := \sum_{i=1}^k \lambda_i(x)$.

- Then (a) φ is continuous
- (b) $\|\varphi(x) - x\| < \epsilon, \forall x \in N(S, \epsilon)$

Proof:

Let $x \in N(S, \epsilon)$. Then $\|x - x_i\| < \epsilon$ for some i . Hence $\lambda_i(x) > 0$ and $\lambda(x) > 0$.

Moreover since λ and λ_i are continuous, we get φ is continuous.

This proves (a).

Now, to prove (b) it follows that

$$\begin{aligned} \varphi(x) - x &:= \sum_{i=1}^k \frac{\lambda_i(x)}{\lambda(x)} x_i - x \\ &= \sum_{j=1}^k \frac{\lambda_j(x)}{\lambda(x)} (x - x_j) \quad \left(\text{Since } \sum \frac{\lambda_j(x)}{\lambda(x)} = 1 \right) \end{aligned}$$

Then we get
$$\|\varphi(x) - x\| \leq \sum_{i=1}^k \frac{\lambda_i(x)}{\lambda(x)} \|x - x_i\|.$$

But, if $\lambda_i(x) > 0$, then $\|x - x_i\| < \epsilon$ and hence $\|\varphi(x) - x\| \leq \sum_{\lambda_i} \frac{\lambda_i(x)}{\lambda(x)} \|x - x_i\| < \epsilon$.

So, $\|\varphi(x) - x\| < \epsilon, \forall x \in N(S, \epsilon)$.

(The above function φ is called the *Schauder-projection*.)

Definition 2:

Let X be a topological space and $F \subseteq X$. Then F is said to be *relatively compact* if and only if every sequence of points of F has a convergent subsequence where the limit may not be in F .

Lemma 3:

Let E be a non-empty convex closed subset of a normed space X and F be relatively compact subset of E .

Then every continuous function from E into F has a fixed point.

Proof:

Since F is relatively compact, it is easy to see that \bar{F} is compact.

Moreover for each $n \in \mathbb{N}$, the inclusion $\bar{F} \subseteq \bigcup_{x \in \bar{F}} S(x, \frac{1}{n})$ holds.

So, since \bar{F} is compact, there exists $x_1, x_2, \dots, x_{k_n} \in \bar{F}$ such that

$$\bigcup_{i=1}^{k_n} S(x_i, \frac{1}{n}).$$

Now we put $K_n := \text{Conv}\{x_1, x_2, \dots, x_{k_n}\} := \text{Conv } S_n$, where $S_n := \{x_1, x_2, \dots, x_{k_n}\}$.

Let $\varphi_n : N(S_n, \frac{1}{n}) \rightarrow K_n$ be the schauder projection.

Then by Lemma 2 φ_n is continuous.

Moreover, we observe that $K_n \subseteq N(S_n, \frac{1}{n})$, $K_n \subseteq E$.

So, $\varphi_n \circ f : K_n \rightarrow K_n$ is continuous.

Then by Corollary 1 there exists $x_n \in K_n$ such that

$$(\varphi_n \circ f)(x_n) = x_n \text{ for each } n \in \mathbb{N}. \tag{1}$$

Moreover, by Lemma 2 we get

$$\|(\varphi_n \circ f)(x_n) - f(x_n)\| = \|x_n - f(x_n)\| < \frac{1}{n}. \tag{2}$$

Since $f(x_n) \in \bar{F}$ for each $n \in \mathbb{N}$ and \bar{F} is compact, there exists a subsequence (x_{n_k}) of (x_n) such that

$$f(x_{n_k}) \xrightarrow{k \rightarrow \infty} x \in \bar{F}.$$

Now from (2) we get

$$\|f(x_{n_k}) - x_{n_k}\| \leq \frac{1}{n_k} \text{ for sufficiently great } k \in \mathbb{N}.$$

Or

$$f(x_{n_k}) = x_{n_k} \text{ for sufficiently great } k \in \mathbb{N}. \tag{3}$$

Moreover since f is continuous and $x_{n_k} \xrightarrow{k \rightarrow \infty} x$, we get

$$f(x) = \lim_{k \rightarrow \infty} f(x_{n_k}) = \lim_{k \rightarrow \infty} x_{n_k} = x$$

i.e. $f(x) = x$.

Hence f has a fixed point. //

Now, we construct an example where the *Brouwer's Fixed Point Theorem* does not hold true in infinite dimensional space.

Example:

Let $X = \left\{ (\dots, \xi_{-1}, \xi_0, \xi_1, \dots) \mid \xi_i \in \mathbb{R}, \sum_{i=1}^{\infty} |\xi_i|^2 < \infty \right\}$ where for

$$x = (\dots, \xi_{-1}, \xi_0, \xi_1, \dots) \in X, \quad \|x\| = \sqrt{\sum_{i=-\infty}^{\infty} |\xi_i|^2}.$$

Let now $S = \{x \in X \mid \|x\| \leq 1\}$. Then S is the closed unit ball in X .

Now define $U: X \rightarrow X$ by $U(x) = y$, where

$$y := (\dots, \eta_{-1}, \eta_0, \eta_1, \eta_2, \dots) \in X \text{ Such that } \eta_n := \xi_{n-1}.$$

Observe that $\|U(x)\| = \|x\|$.

Now define $T: X \rightarrow X$ by $T(x) := U(x) + (1 - \|x\|)z$, where $z = (z_n)$, $z_0 = 1$ and $z_n = 0$ for all $n \neq 0$.

It is easy to show that T is continuous.

Now if $x \in X$ such that $\|x\| \leq 1$, then we have that

$$\begin{aligned} \|T(x)\| &= \|U(x) + (1 - \|x\|)z\| \\ &\leq \|U(x)\| + (1 - \|x\|) = 1 \end{aligned}$$

So, $T(S) \subseteq S$. Then $T: S \rightarrow S$ is continuous.

Suppose now there exists $x \in S$ such that $T(x) = x$.

Let $(T(x))_n$ denote the n^{th} co-ordinate of $T(x)$. Then

$$(T(x))_n = \xi_n.$$

But by the definition of T we have

$$\xi_n = \begin{cases} \xi_{n-1} & \text{if } n \neq 0 \\ \xi_{n-1} + (1 - \|x\|) & \text{if } n = 0 \end{cases} \quad (4)$$

Now since $\sum_{i=-\infty}^{\infty} |\xi_i|^2 < \infty$, we have that $\xi_n \xrightarrow{n \rightarrow \infty} 0$. (5)

Therefore, from (3), (4) and (5) we conclude that $x = 0$ i.e. $\|x\| = 0$, which is a contradiction to the definition of T as we get $0 = 1$.

Hence, T does not fix any point of S . //

Now, we shall state and prove an infinite version of the *Brouwer's Fixed Point Theorem* called the *Schauder-Tychonoff Fixed Point Theorem*.

Theorem 2(Schauder-Tychonoff Fixed Point Theorem)

Let X be a locally convex topological vector space. If K is a non-empty compact convex subset of X and $f : K \rightarrow K$ is continuous ,then $f(p) = p$ for some $p \in K$.

Proof:

We use indirect proof.

Suppose f fixes no point of K .

Let $G := \{(x, f(x)) \mid x \in K\} \subseteq X \times X$,

and

$$\Delta := \{(x, y) \mid y = x\}.$$

Then we observe that $F : K \rightarrow K$ defined by

$$F(x) := (x, f(x)) \text{ is continuous and } F(K) = G.$$

Since K is compact and F is continuous , we have G is also compact.

Now we will show that

(a) $G \cap \Delta = \emptyset$

(b) there exists a convex neighbourhood V of 0 such that $[G + (V \times V)] \cap \Delta = \emptyset$.

To show (a), suppose there exists $x \in K$ such that $(x, f(x)) \in G \cap \Delta$.

Then $f(x) = x$ which is a contradiction to our assumption.

Hence,(a) holds.

To show (b), suppose for any neighbourhood V of 0 there exists $x \in K, (c, d) \in V \times V$ such that $(x, f(x)) + (c, d) \in \Delta$. i.e.

$$x + c = d + f(x).$$

Since $(c, d) \in V \times V$ for any neighbourhood V of 0 , we conclude that $c = d = 0$ which implies that $f(x) = x$ which is also impossible.

Hence (b) holds .

Moreover, we can see from (b) that for any $x \in K$, there exists a neighbourhood V of 0 such that $f(x) \notin x + V$. (6)

Let now μ be the *Minkowski Functional* of V . Then

$$\mu \text{ is continuous and } \mu(x) < 1 \text{ if and only if } x \in V.$$

Define $\alpha : X \rightarrow X$ by $\alpha(x) := \text{Max}\{0, 1 - \mu(x)\}$. Then

$$\alpha(x) \geq 0, \forall x \in X \text{ and } \alpha(x) > 0, \forall x \in V. \tag{7}$$

Now observe that $K \subseteq \bigcup_{x \in K} (x + V)$.

Since K is compact, we choose $x_i \in K$ such that

$$K \subseteq \bigcup_{i=1}^n (x_i + V) \tag{8}$$

Now, we define $\alpha_i : X \rightarrow X$ by $\alpha_i(x) := \alpha(x - x_i)$, $i \in \{1, 2, 3, \dots, n\}$

and

$$\beta_i : K \rightarrow K \text{ by } \beta_i(x) := \frac{\alpha_i(x)}{\sum_{j=1}^n \alpha_j(x)}$$

Observe that if $x \in K$, $x - x_j \in V$ for some $j \in \{1, 2, 3, \dots, n\}$ and hence we have $\mu(x - x_j) < 1$ i.e. $\alpha_j(x) = 1 - \mu(x - x_j) > 0$

and

$$\sum_{j=1}^n \alpha_j(x) > 0, \quad \sum_{j=1}^n \beta_j(x) = 1, \quad \beta_j(x) > 0 \text{ for all } i=1, 2, 3, \dots, n$$

Thus β_i is well-defined function.

Now, we put $S := \{x_1, x_2, \dots, x_n\}$ and $H := \text{Conv}S$.

Then we see that $\dim H < \infty$ and H is convex.

Now we define $g : K \rightarrow K$ by $g(x) := \sum_{i=1}^n \beta_i(x) x_i$. Then

g is continuous, $g(K) = H$ and since K is compact, H is compact.

Moreover, $g \circ f : H \rightarrow H$ is continuous.

Then by applying the *Brouwer's Fixed Point Theorem*, there exists $x^* \in H$ such that

$$(g \circ f)(x^*) = x^* \tag{10}$$

Since $\beta_i(x) = 0, \forall x \notin x_i + V$, we get that

$$x - g(x) = \sum_{i=1}^n \beta_i(x)(x - x_i) \quad (x \in K)$$

is a convex combination of $x - x_i \in V$.

Then since V is convex, we have

$$x - g(x) \in V, \quad \forall x \in K. \tag{11}$$

In particular since $f(x^*) \in K$, we get $f(x^*) - g(f(x^*)) \in V$.

Or

$$f(x^*) \in g(f(x^*)) + V = x^* + V \text{ i.e. } f(x^*) \in x^* + V$$

which is a contradiction to the assertion proved in (b).

Therefore, there exists p in K such that $f(p) = p$ //

Now we give an application of the *Brouwer's Fixed Point Theorem*.

Theorem 3 (Theorem of Perron)

Let $A = (\alpha_{ij})$ be an $n \times n$ matrix such that $\alpha_{ij} > 0$ for all i and j .

Then, A has a positive eigenvalue.

Proof:

Let $T : R^n \rightarrow R^n$ be linear mapping corresponding to the matrix A with respect to the standard basis (e_i) of R^n .

We put $E := \{x \in l^1(n) \mid \|x\| = 1\}$ and $F := \text{Conv}\{e_1, e_2, \dots, e_n\}$.

Then we see that E is compact and also \bar{F} is compact.

Now, define $\alpha : F \rightarrow F$ $\alpha(x) := \frac{T(x)}{\|T(x)\|} = \frac{A(x)}{\|A(x)\|}$. Then,

Since α is linear and $\dim F < \infty$ we get
 α is continuous.

Now, for $x \in F$ ($x = \sum_{i=1}^n \lambda_i e_i$), $A(x) = \sum_{i=1}^n \left(\sum_{j=1}^n \alpha_{ij} \lambda_j \right) e_i$.

Thus, there exists $x \in F$ such that

$$\alpha(x) = x$$

or

$$A(x) = \|A(x)\|x.$$

Then $\|Ax\|$ is a positive eigenvalue of A .

3.3.1: Hammerstein Integral Equation:

Definition 1:

Let $k : [a, b] \times [a, b] \rightarrow \mathbb{R}$ and $f : [a, b] \times \mathbb{R} \rightarrow \mathbb{R}$ be continuous.

Then the equation
$$x(s) = \int_a^b k(s, t) f(t, x(t)) dt \quad (1)$$

is called *Hammerstein equation*.

Now, the following theorem will give a sufficient condition for the existence of a continuous solution for the above equation.

Theorem 4: (Existence theorem)

Let k be continuous on $[a, b] \times [a, b]$ and $f, \frac{\partial f}{\partial y}$ be continuous and bounded on $[a, b] \times \mathbb{R}$. Then the equation

$$x(s) = \int_a^b k(s, t) f(t, x(t)) dt$$

has a continuous solution.

Proof:

Define $T : C[a, b] \rightarrow C[a, b]$ by $(Tx)(s) := \int_a^b k(s, t) f(t, x(t)) dt$.

Since f is bounded, $|f(s, t)| \leq M$ for some $M \in \mathbb{R}_+$.

Moreover, since k is continuous on a compact set $[a, b] \times [a, b]$, we have

$$|k(s, t)| \leq L \quad \text{for some } L \in \mathbb{R}_+$$

and

k is uniformly continuous. Then we have

$$\begin{aligned} |(Tx)(s)| &= \left| \int_a^b k(s, t) f(t, x(t)) dt \right| \\ &\leq \int_a^b |k(s, t)| |f(t, x(t))| dt \\ &\leq LM(b-a) =: \rho \end{aligned}$$

Therefore, $|(Tx)(s)| \leq \rho, \forall s \in [a, b]$ (2)

Now we set $E := \{x \in C[a, b] \mid \|x\| \leq \rho\}$ and $F := \{Tx \mid x \in E\}$.

Then by (2) we see that F is uniformly bounded.

We will show that F is equicontinuous family.

Let $\varepsilon > 0$ be given.

Since k is uniformly continuous, there exists $\delta > 0$ and

$$|t_1 - t_2| < \delta \Rightarrow |k(t_1, s) - k(t_2, s)| < \frac{\varepsilon}{M(b-a)} \text{ for all } s \in [a, b]. \quad (3)$$

Now, $|t_1 - t_2| < \delta$ implies that

$$\begin{aligned} |(Tx)(t_1) - (Tx)(t_2)| &= \left| \int_a^b [k(t_1, s) - k(t_2, s)] f(t, x(t)) dt \right| \\ &\leq \int_a^b |k(t_1, s) - k(t_2, s)| |f(t, x(t))| dt \\ &\leq \frac{\varepsilon}{M(b-a)} (b-a)M = \varepsilon, \forall Tx \in F. \end{aligned}$$

Hence F is an equicontinuous family.

Thus by *Arzela-Ascoli Theorem* for any sequence $(Tx_n) \subseteq F$, there exists a subsequence (x_n) of (Tx_n) such that (Tx_n) is convergent.

Therefore, F is relatively compact subset of E .

Finally, we show that $T : E \rightarrow F$ is continuous.

Now, since f is continuous and differentiable with respect to the second component, by the *Mean value Theorem* there exists $c \in (z_1, z_2)$ such that

$$\begin{aligned} |f(s, z_1) - f(s, z_2)| &= |z_1 - z_2| \left| \frac{\partial f(s, c)}{\partial y} \right| \\ &\leq |z_1 - z_2| \underbrace{\text{Sup} \left| \frac{\partial f(s, c)}{\partial y} \right|}_{=N} \\ &= N|z_1 - z_2| \end{aligned}$$

Choose $\delta := \frac{\varepsilon}{LN(b-a)}$. Then

$$|z_1 - z_2| < \delta \Rightarrow |f(s, z_1) - f(s, z_2)| < \frac{\varepsilon}{L(b-a)} \quad (4)$$

Thus we get

$$\begin{aligned} \|Tx_1 - Tx_2\| &= \text{Max}_{a \leq t \leq b} |(Tx_1)(t) - (Tx_2)(t)| \\ &= \text{Max}_{a \leq t \leq b} \left| \int_a^b [k(s, t) f(t, x_1(t)) - k(s, t) f(t, x_2(t))] dt \right| \\ &\leq \text{Max}_{a \leq t \leq b} \int_a^b |k(s, t)| |f(t, x_1(t)) - f(t, x_2(t))| dt \\ &\leq \text{Max}_{a \leq t \leq b} [LN|x_1(t) - x_2(t)|(b-a)] \\ &\leq \|x_1 - x_2\| LN(b-a) \end{aligned}$$

So we get

$$\|x_1 - x_2\| < \delta = \frac{\varepsilon}{LN(b-a)} \quad \text{implies that} \quad \|Tx_1 - Tx_2\| < \varepsilon.$$

Hence T is uniformly continuous (i.e. T is continuous)

Therefore, by *Lemma 3* there exists $x_0 \in E \subseteq C[a, b]$ such that $Tx_0 = x_0$ on $[a, b]$.

Or

$$x_0(s) = \int_a^b k(s, t) f(t, x_0(t)) dt \quad (5)$$

Then (5) shows that

$$x(s) = \int_a^b k(s, t) f(t, x(t)) dt$$

has a continuous solution. //

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