

Dirichlet Series with Functional Equations, Automorphic Integrals and Arithmetical Identities

By

Tewelde Gebreegziabher

Adviser

Abdulkadir Hassen (Prof.)
Rowan University, USA

Co-Adviser

Humduma Legesse (Ph.D.)
Addis Ababa University

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teweldega@gmail.com
July 14, 2020

Declaration

I, with student ID number *GSR/2597/08*, hereby declare that this thesis is my own work and that it has not been previously submitted for assessment or completion of any post graduate qualification to another university or for another qualification.

Tewelde Gebreegziabher.

Date: September, 2019

Certificate

I hereby certify that I have read this dissertation prepared by Tewelde Gebreegziabher under my supervision and recommended that, it should be accepted as fulfilling the dissertation requirement.

Abdulkadir Hassen, Professor

Date: September, 2019

Hunduma Legesse, Ph.D.

Date: Jun, 2019

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Date: September, 2019

Supervisor: Abdulkadir Hassen (Professor), Rowan University, USA

Signature:_____

Co-Supervisor: Hunduma Legesse (Ph.D.), Addis Ababa University, Ethiopia

Signature:_____

External Examiner:_____

Internal Examiner:

Chairperson:

Signature:_____

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Tewelde Gebreegiabiher

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Dedication

This Thesis is Dedicated To:

My aunt **Kebelesh Tesfay,**
&
My daughter **Efrata Tewelde**

Abstract

Chanrasekharan and Narasimhan in [2] have shown that the functional equation $\Gamma(s)\varphi(s) = \Gamma(\delta-s)\psi(\delta-s)$ is equivalent to two arithmetical identities. In [5] Hawkins and Knopp proved a Hecke correspondence theorem for modular integrals with rational period function on Γ_θ , a sub group of the full modular group $\Gamma(1)$.

Sister Ann M. Heath in [1] considered the functional equation in the Hawkins and Knopp context. Analogous to Chandrasekharan and Narasimhan she showed its equivalence to two arithmetical identities associated with entire modular cusp integrals involving rational period functions for the full modular group.

In this dissertation we extend the results of Sister Ann M. Heath to entire automorphic integrals involving rational period functions on discrete Hecke group $G(\lambda)$, $\lambda > 0$. Moreover, we establish equivalence of two arithmetical identities with a functional equation associated with automorphic integrals involving log-polynomial-period functions on the Hecke groups.

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Chapter 1

Introduction

1.1 Preliminaries

Let $\{\lambda_n\}$ and $\{\mu_n\}$ be two strictly increasing sequences of positive real numbers diverging to ∞ , as $n \rightarrow \infty$ and let $\{a_n\}$ and $\{b_n\}$ be two sequences of complex numbers not identically zero. Consider the Dirichlet series φ and ψ defined by

$$\varphi(s) = \sum_{n=1}^{\infty} \frac{a_n}{\lambda_n^s} \quad \text{and} \quad \psi(s) = \sum_{n=1}^{\infty} \frac{b_n}{\mu_n^s}$$

with finite abscissas of absolute convergence σ_a and σ_b , respectively. Suppose that φ and ψ satisfy the functional equation

$$\Gamma(s)\varphi(s) = \Gamma(\delta - s)\psi(\delta - s), \quad (1.1.1)$$

where $\delta > 0$. Chandrasekharan and Narasimhan in [2] have shown that the functional equation (1.1.1) is equivalent to the following arithmetical identities:

$$\frac{1}{\Gamma(\rho + 1)} \sum'_{\lambda_n \leq x} a_n (x - \lambda_n)^\rho = \left(\frac{1}{2\pi}\right)^\rho \sum_{n=1}^{\infty} b_n \left(\frac{x}{\mu_n}\right)^{\frac{\delta+\rho}{2}} J_{\delta+\rho}\{4\pi\sqrt{\mu_n x}\} + Q_\rho(x), \quad (1.1.2)$$

and

$$\left(-\frac{1}{s} \frac{d}{ds}\right)^\rho \left[\frac{1}{s} \sum_{n=1}^{\infty} a_n e^{-s\sqrt{\lambda_n}} \right] = 2^{3\delta+\rho} \Gamma(\delta+\rho+\frac{1}{2}) \pi^{\delta-\frac{1}{2}} \sum_{n=1}^{\infty} \frac{b_n}{(s^2 + 16\pi^2 \mu_n)^{\delta+\rho+\frac{1}{2}}} + R_\rho(s). \quad (1.1.3)$$

In (1.1.2), we are assuming $x > 0$, $\rho \geq 2\beta - \delta - \frac{1}{2}$, $Q_\rho(x)$ is given by

$$Q_\rho(x) = \frac{1}{2\pi i} \oint_c \frac{\Phi(s)(2\pi)^s x^{s+\rho}}{\Gamma(s+\rho+1)} ds,$$

$\sum_{n=1}^{\infty} |b_n| \mu_n^{-\beta} < \infty$, and $J_\nu(z)$ denotes the usual Bessel function of the first kind of order ν .

In (1.1.3), we have are assuming $\operatorname{Re} s > 0$, ρ is a nonnegative integer satisfying $\rho > -\delta - \frac{1}{2}$, $\rho \geq \beta - \delta - \frac{1}{2}$, and

$$R_\rho(s) = \frac{1}{2\pi i} \oint_c \frac{\Phi(z)(2\pi)^z \Gamma(2z + 2\rho + 1) 2^{-\rho}}{\Gamma(z + \rho + 1)} s^{-2z-2\rho-1} dz.$$

Note that if $\beta > 0$, then identity (1.1.3) holds for ρ satisfying $\rho \geq \beta - \delta - \frac{1}{2}$, $\rho \in \mathbb{Z}_{\geq 0}$.

In [5] Hawkins and Knopp proved a Hecke correspondence theorem for modular integrals with rational period functions on Γ_θ , (generated by $Sz = z + 2$ and $Tz = \frac{-1}{z}$), a subgroup of the full modular group $\Gamma(1)$. In their work, the functional equation takes the form

$$\Phi(2k - s) - i^{2k} \Phi(s) = R_k(s), \tag{1.1.4}$$

where

$$\Phi(s) = \left(\frac{2\pi}{\lambda_n} \right)^{-s} \Gamma(s) \sum_{n=1}^{\infty} a_n n^{-s}$$

associated with a modular relation involving rational period functions of the form

$$F(z + \lambda_n) = F(z)$$

,

$$z^{-2k} F\left(\frac{-1}{z}\right) = F(z) + q(z),$$

where $\lambda_n = 2 \cos\left(\frac{\pi}{n}\right)$, with $3 \leq n \in \mathbb{N} \cup \{\infty\}$ and $2k \in \mathbb{Z}$.

Analogous to Chandrasekharan and Narasimhan, Sister Ann M. Heath in [1] showed that the functional equation in the Hawkins and Knopp context (1.1.4), and two type of arithmetical identities are equivalent. Sister Ann M. Heath considered the functional equation (1.1.4) and proved its equivalence to two arithmetical identities associated with entire modular cusp integrals forms involving rational period function for the full modular group $\Gamma(1)$. (See the next section for the definition.) In this dissertation, we shall use techniques of Chandrasekharan and Narasimhan to extend the results of Sister Ann M. Heath to entire automorphic integrals involving rational period function on discrete Hecke groups $G(\lambda)$, $\lambda > 0$. We shall discuss the Hecke group in the next section.

1.2 Hecke groups

In this section, we shall give the definition of the Hecke group and state some of its properties.

Definition: For $\lambda \in \mathbb{R}^+$, the Hecke group $G(\lambda)$ is defined as the subgroup of $SL_2(\mathbb{R})$ given by

$$G(\lambda) = \left\langle \begin{pmatrix} 1 & \lambda \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \right\rangle.$$

Equivalently $G(\lambda)$ is generated by the linear fractional transformations $S(z) = z + \lambda$ and $T(z) = -\frac{1}{z}$.

Elements of this group act on the Riemann sphere as linear fractional transformations, that is $Mz = \frac{az+b}{cz+d}$ for $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in G(\lambda)$, and $z \in \mathbb{C} \cup \{\infty\}$, thus M and $-M$ can be identified with the same matrices.

Remark: E. Hecke [17] showed the groups $G(\lambda)$ is discrete (operate discontinuously) as linear fractional transformations on the upper half plane $\mathcal{H} = \{z = x + iy : y > 0\}$ if and only if either $\lambda > 2$ or $\lambda = \lambda_p := 2\cos(\frac{\pi}{p})$, with $3 \leq p \in \mathbb{N} \cup \{\infty\}$.

For $\lambda > 2$, the Hecke group has one relation:

$$T^2 = -I,$$

while for $\lambda = \lambda_p$, there is a second relation:

$$T^2 = (S_{\lambda_p}T)^p = -I.$$

Special cases: $G(\lambda_3) = G(1)$ is called the *full modular group* and is denoted by $\Gamma(1)$. Also $G(\lambda_\infty)$ denoted by Γ_θ , is called the *theta group*. Note that Γ_θ is subgroup of $\Gamma(1)$.

1.3 List of basic functions and known functional equations

In this section, we list some important infinite series and integral representation of some basic functions. We also list some known functional equations which are important for our discussion later in this thesis.

We recall that the Gamma function $\Gamma(s)$ may be defined by the integral:

$$\Gamma(s) = \int_0^\infty e^{-t} t^{s-1} dt, \quad \operatorname{Re}(s) > 0.$$

We note that the Gamma functions satisfies the equation

$$\int_0^\infty e^{-pt} t^{z-1} dt = \frac{\Gamma(z)}{p^z}, \text{ for } \operatorname{Re} p > 0, \text{ and } \operatorname{Re} z > 0, \quad (1.3.1)$$

$$\frac{1}{2\pi i} \int_{\gamma-i\infty}^{\gamma+i\infty} \Gamma(-s) \Gamma(\beta + s) t^s ds = \Gamma(\beta) (1+t)^{-\beta}, \quad (1.3.2)$$

where $0 > \gamma > \operatorname{Re}(1 - \beta)$ and $|\arg t| < \pi$.

$$\frac{1}{2\pi i} \int_{\sigma-i\infty}^{\sigma+i\infty} e^{tt^{-z}} dt = \frac{1}{\Gamma(z)}, \text{ where } \operatorname{Re} z > 0, \sigma > 0. \quad (1.3.3)$$

Also note the *Legendere duplication formula*

$$\sqrt{\pi} \Gamma(2z) = 2^{2z-1} \Gamma(z) \Gamma(z + \frac{1}{2}), \quad (1.3.4)$$

and the formula

$$\Gamma(a - n) = (-1)^n \frac{\Gamma(a) \Gamma(1 - a)}{\Gamma(n - a + 1)}. \quad (1.3.5)$$

For $a \in \mathbb{R}, a \notin \mathbb{Z}$ and $n \in \mathbb{Z}, n \geq 0$.

The Gauss hypergeometric function is represented by the series

$${}_2F_1(\alpha, \beta; \gamma; z) = \sum_{n=0}^{\infty} \frac{(\alpha)_n (\beta)_n z^n}{(\gamma)_n n!}; \gamma \notin \mathbb{Z}_{\leq 0}.$$

${}_2F_1$ has the following property we shall use:

$$\int_1^\infty \frac{y^s}{(c+y)^r} \frac{dy}{y} = \frac{(c+1)^{-r}}{r-s} {}_2F_1\left[1, r; 1+r-s; \frac{c}{c+1}\right], \quad (1.3.6)$$

where $c \in \mathbb{C} \setminus (-\infty, -1]$, and $\sigma < r$ for $s = \sigma + it$.

The Bessel function of the first kind $J_\nu(s)$ of arbitrary order ν has the series representation

$$J_\nu(s) = \sum_{m=0}^{\infty} \frac{(-1)^m \left(\frac{s}{2}\right)^{\nu+2m}}{\Gamma(m+1) \Gamma(m+\nu+1)},$$

where $|s| < \infty, |\arg s| < \pi$, and ν can be real or complex.

We shall make use of the following identity and relations

$$\frac{1}{2\pi i} \int_{C-i\infty}^{C+i\infty} \frac{2^{s-\nu-1} \Gamma(\frac{s}{2})}{\Gamma(\nu - \frac{1}{2}s + 1)} x^{-s} ds = \frac{J_\nu(x)}{x^\nu}, \quad (1.3.7)$$

provided that $0 < C \leq \nu + 1, \nu > 0$. Also,

$$\int_0^{\infty} e^{-xy} J_{\nu}(a\sqrt{x}) x^{\frac{\nu}{2}} dx = \frac{2a^{\nu}}{(2y)^{\nu+1}} e^{-\frac{a^2}{4y}} \quad (1.3.8)$$

provided that $y > 0, \operatorname{Re}(\nu) > -1, a > 0$.

$$\frac{1}{2\pi i} \int_{w-i\infty}^{w+i\infty} \frac{e^{bs}}{(a^2 + s^2)^{\nu+\frac{1}{2}}} ds = J_{\nu}(ab) \left(\frac{b}{a}\right)^{\nu} \frac{\sqrt{\pi} 2^{-\nu}}{\Gamma(\nu + \frac{1}{2})}, \quad (1.3.9)$$

where $w > 0, \nu > -\frac{1}{2}, a > 0, b > 0$.

The confluent hypergeometric series of the first kind $\phi(\alpha, \gamma; z)$ represented by the series

$$\phi(\alpha, \gamma; z) = \sum_{n=0}^{\infty} \frac{(\alpha)_n z^n}{(\gamma)_n n!},$$

where $|z| < \infty, \gamma \neq 0, -1, \dots$ and $(\alpha)_n = \frac{\Gamma(\alpha+n)}{\Gamma(\alpha)}$.

The confluent hypergeometric function of the second kind is defined by

$$\Psi(\alpha, \gamma; z) = \frac{\Gamma(1-\gamma)}{\Gamma(1+\alpha-\gamma)} \Phi(\alpha, \gamma; z) + \frac{\Gamma(\gamma-1)}{\Gamma(\alpha)} z^{1-\gamma} \Phi(1+\alpha-\gamma, 2-\gamma; z), \quad (1.3.10)$$

where $|\arg z| < \pi, \gamma \neq 0, \pm 1, \pm 2, \dots$,

Integral representation of $\Psi(\alpha, \gamma; z)$ is given by

$$\Psi(\alpha, \gamma; z) = \frac{1}{2\pi i} \int_{\lambda-i\infty}^{\lambda+i\infty} \frac{\Gamma(\alpha+s)\Gamma(-s)\Gamma(1-\gamma-s)}{\Gamma(\alpha)\Gamma(\alpha-\gamma+1)} z^s dz, \quad (1.3.11)$$

where $-\operatorname{Re}(\alpha) < \lambda < \min(0, 1 - \operatorname{Re}\gamma)$.

The integral representation of beta function is given by

$$B(\alpha, \beta) = \int_0^1 t^{\alpha-1} (1-t)^{\beta-1} dt, \quad (1.3.12)$$

for $\operatorname{Re}(\alpha) > 0, \operatorname{Re}(\beta) > 0$.

Chapter 2

Automorphic Integrals with Rational period Functions

In this chapter, we discuss the definitions of automorphic integrals involving rational period functions on the Hecke groups.

Suppose $F(z)$ is a meromorphic function in the upper half plane \mathcal{H} that satisfies

$$F(z + \lambda) = F(z) \tag{2.0.1}$$

and

$$z^{-2k} F\left(\frac{-1}{z}\right) = F(z) + q(z), \tag{2.0.2}$$

where q is a rational function and k is in \mathbb{R} . Further assume that F has Fourier series expansion of the form

$$F(z) = \sum_{m=\nu}^{\infty} a_m e^{2\pi i m z / \lambda}, \tag{2.0.3}$$

where $\text{Im } z = y > y_o \geq 0$ and $\nu \in \mathbb{Z}$. The function F is called an *automorphic integral of weight $2k$ for the Hecke group $G(\lambda)$, with rational period function (RPF) $q(z)$* . If $q \equiv 0$ then F is an *automorphic form of weight $2k$ on $G(\lambda)$* .

If F is an automorphic integral and holomorphic in \mathcal{H} (that is, $\nu \geq 0$) and satisfies the growth condition

$$|F(z)| \leq C (|z|^\alpha + y^{-\beta}), \text{Im}(z) = y > 0,$$

for some constants $C, \alpha, \beta > 0$, and $z \in \mathcal{H}$, one can show that the coefficients a_m in (2.0.3) satisfy

$$a_m = \mathcal{O}(m^\beta), \quad n \rightarrow \infty.$$

In this case, F is called an *entire automorphic integral of weight $2k$ on $G(\lambda)$ with RPF q* .

For $M = \begin{pmatrix} * & * \\ c & d \end{pmatrix} \in G(\lambda)$ we define the stroke operator by

$$F|M := F|_{2k}^M = (cz + d)^{-2k} F(Mz).$$

The automorphic relation (2.0.2) can be expressed as $F|T = F + q$. In general, for a period function q_M , we have $F|M = F + q_M$. A straightforward calculation shows that $F|M_1M_2 = (F|M_1)|M_2$ for $M_1, M_2 \in G(\lambda)$, so that

$$q_{M_1M_2} = q_{M_1}|M_2 + q_{M_2}, M_1, M_2 \in G(\lambda). \quad (2.0.4)$$

Recall that if $\lambda \geq 2$, the Hecke group $G(\lambda)$ has only one relation $T^2 = -I$. In this case, (2.0.4) imposes a relation on the RPF q :

$$q|T + q = 0. \quad (2.0.5)$$

For the case $\lambda = \lambda_p = 2 \cos\left(\frac{\pi}{p}\right)$, $p \in \mathbb{Z}$, $p \geq 3$, the second relation $(S_{\lambda_p}T)^p = -I$ imposes another condition on q , namely

$$q | (S_{\lambda_p}T)^{p-1} + q | (S_{\lambda_p}T)^{p-2} + \cdots + q | (S_{\lambda_p}T) + q = 0. \quad (2.0.6)$$

Marvin Knopp [13] proved that the finite poles of a rational period function on $\Gamma(1)$ are only at 0 or real quadratic irrationals. He also showed that if q is an RPF of weight $2k > 0$ with poles in \mathbb{Q} , then

$$q(z) = \begin{cases} \alpha_0 \left(1 - \frac{1}{z^{2k}}\right) & \text{if } k > 1, \\ \alpha_0 \left(1 - \frac{1}{z^2}\right) + \frac{\alpha_1}{z} & \text{if } k = 1, \end{cases}$$

for some constants $\alpha_0, \alpha_1 \in \mathbb{C}$.

Observe that if $F(z) \equiv -\alpha_0$, then $(F|T)(z) = F(z) + q(z)$ implies that $z^{-2k}F\left(\frac{-1}{z}\right) - F(z) = q(z)$ and hence $q(z) = \alpha_0(1 - z^{-2k})$. Thus we consider $q(z) = \alpha_0(1 - z^{-2k})$ a trivial period function of weight $2k \in \mathbb{R}$. The following lemma is stated in the work of J. Hawkins and M. Knopp in [5], where their underlying group is the theta group Γ_θ . In this group we only have the relation (2.0.5). For the current context we state the lemma by extending to the Hecke groups $G(\lambda)$ which has both relations (2.0.5) and (2.0.6) and we present the detailed proof of the lemma.

Lemma 2.1. *Nontrivial rational period functions on the Hecke groups satisfying (2.0.5) and (2.0.6) exist only if the weight $2k$ is an integer.*

Proof. Let q be a nontrivial RPF on $G(\lambda)$ satisfying (2.0.5) and (2.0.6). For $\lambda \geq 2$, we have the relation $T^2 = -I$ and for $\lambda = 2 \cos\left(\frac{\pi}{p}\right)$, $p \in \mathbb{Z}$, $p \geq 3$, we have $(S_\lambda T)^p = -I$. Applying (2.0.4) there exists a holomorphic function F on the upper half plane \mathcal{H} such that $F|(-I) = F + q + q|T$, for $\lambda \geq 2$. And $F|(-I) = F + q + q|(S_\lambda T) + \cdots + q|(S_\lambda T)^{p-1}$, for $\lambda = 2 \cos\left(\frac{\pi}{p}\right)$, $p \in \mathbb{Z}$, $p \geq 3$. But by (2.0.5) and (2.0.6) we have $z^{-2k}q\left(\frac{-1}{z}\right) + q(z) = 0$ and $q|(S_\lambda T)^{p-1}(z) + \cdots + q(z) = 0$ respectively, then $(-1)^{-2k}F(z) = F(z)$. Now since $q(z)$ is a nontrivial RPF, $F(z)$ is nonconstant and holomorphic in \mathcal{H} . Then $(-1)^{-2k}F(z) = F(z)$. Since $F(z) \neq 0$ we have $(-1)^{-2k} = 1$ which implies $e^{2\pi ik} = 1$. Therefore. $k \in \mathbb{Z}$. \square

Wendell-Culp-Ressler in [[15], Lemma 3] showed that the poles of any rational periodic function q of weight $2k$, $k \in \mathbb{Z}^+$ on $G(\lambda)$ are real numbers. He also proved that the order of a nonzero pole of an RPF of weight $2k$ on $G(\lambda)$ is $[k]$. With appropriate modifications to fit for the current context of functions on $G(\lambda)$, the work of John. Hawkins and Marvin I. Knopp [5] can be used to state a special form *RPF* for the solution of (2.0.5). This form is given by the following lemma.

Lemma 2.2. *Let $r \in \mathbb{Z}$, $\alpha_j \in \mathbb{R} \setminus \{0\}$, $C_r, C_{rj} \in \mathbb{C}$ for $j = 1, 2, \dots, p$. Define*

$$f_r(z, 0) = z^{-r} - (-1)^r z^{-2k+r},$$

and

$$f_r(z, \alpha_j) = (z - \alpha_j)^{-r} - (-1)^r \alpha_j^{-r} z^{-2k+r} \left(z + \frac{1}{\alpha_j}\right)^{-r}.$$

Then

$$q(z) = \sum_{k \leq r \leq L} C_r f_r(z, 0) + \sum_{j=1}^p \sum_{r=1}^{M_j} C_{rj} f_r(z, \alpha_j) \quad (2.0.7)$$

satisfies

$$q|T + q = 0.$$

Theorem 2.3. *Suppose F is an entire automorphic integral function of weight $2k, k \in \mathbb{Z}^+$ for $G(\lambda)$ with RPF $q(z)$, where q has the form described by Lemma 2.2. Suppose further that F has a Fourier series expansion of the form*

$$F(z) = \sum_{m=0}^{\infty} a_m e^{2\pi i m z / \lambda}, \quad \text{with} \quad a_m = \mathcal{O}(m^\beta) \quad \beta > 0, \quad m \rightarrow \infty. \quad (2.0.8)$$

For $s = \sigma + it$, define

$$\varphi(s) = \sum_{m=1}^{\infty} a_m m^{-s} \quad \text{and} \quad \Phi(s) = \left(\frac{2\pi}{\lambda} \right)^{-s} \Gamma(s) \varphi(s). \quad (2.0.9)$$

Then $\Phi(s)$ has a meromorphic continuation to the whole complex plane and can be expressed in the form

$$\Phi(s) = D(s) + D^0(s) + E^0(s) + E^H(s) + E^B(s),$$

where

$$D(s) = \int_1^{\infty} (F(iy) - a_0) \left\{ y^s + i^{2k} y^{2k-s} \right\} \frac{dy}{y}, \quad (2.0.10)$$

$$D^0(s) = -a_0 \left\{ \frac{1}{s} - \frac{i^{2k}}{s-2k} \right\}, \quad (2.0.11)$$

$$E^0 = \sum_{k \leq r \leq L} C_r (-i)^r \left[\frac{1}{r-s} + \frac{i^{2k}}{r-(2k-s)} \right], \quad (2.0.12)$$

$$\begin{aligned} E^H(s) = & - \sum_{j=1}^p \sum_{r=1}^{M_j} C_{rj} \frac{(-i)^r}{(i\alpha_j + 1)^r} \left\{ \frac{1}{s} {}_2F_1 \left[1, r; 1+s; \frac{1}{(i\alpha_j + 1)} \right] \right. \\ & \left. + \frac{i^{2k}}{(2k-s)} {}_2F_1 \left[1, r; 1+(2k-s); \frac{1}{(i\alpha_j + 1)} \right] \right\} \end{aligned} \quad (2.0.13)$$

and

$$E^B(s) = i^{2k} \sum_{j=1}^p \sum_{r=1}^k C_{rj} \left(\frac{-1}{\alpha_j} \right)^r B(2k-s; r-(2k-s)) (i\alpha_j)^{2k-s}. \quad (2.0.14)$$

Moreover, $\Phi(s)$ satisfies the functional equation

$$\Phi(2k-s) - i^{2k} \Phi(s) = R(s), \quad (2.0.15)$$

where

$$R(s) = E^B(2k-s) - i^{2k} E^B(s).$$

Here ${}_2F_1[a, b, c; z]$ is the hypergeometric function and $B(a, b)$ is the Beta function.

Remark. The growth condition on a_m guarantees that $F(z)$ is holomorphic in \mathcal{H} . Furthermore $\varphi(s)$ converges absolutely for $\sigma > 1 + \beta$ and $\Phi(s) = \int_0^\infty (F(iy) - a_0) y^s \frac{dy}{y}$, for $\sigma > \beta$, where β is given by $a_m = \mathcal{O}(m^\beta)$. The proof we present below is given by Hawkins and Knopp in [5]. However, their underlying group is the theta group

$$\Gamma_\theta = \langle T, S_2 \rangle.$$

In Γ_θ , we only have the relation $T^2 = -I$. We shall give a modified proof that extends their proof to the Hecke group

$$G(\lambda_p) = \langle T, S_{\lambda_p} \rangle = \left\langle \begin{pmatrix} 1 & -1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 2 \cos\left(\frac{\pi}{p}\right) \\ 0 & 1 \end{pmatrix} \right\rangle, \quad p \in \mathbb{Z}, \quad p \geq 3.$$

Proof. Suppose F is as given in the theorem and q is as in Lemma 2.2 an RPF of F . Then, we apply the inverse Mellin transform to get

$$\begin{aligned} \Phi(s) &= \int_0^\infty (F(iy) - a_0) y^s \frac{dy}{y} \\ &= \int_1^\infty (F(iy) - a_0) y^s \frac{dy}{y} + \int_0^1 (F(iy) - a_0) y^s \frac{dy}{y}. \end{aligned}$$

Using (2.0.2) and change of variable $y \rightarrow \frac{1}{y}$ in the second integral, we get

$$\int_0^1 \{F(iy) - a_0\} y^s \frac{dy}{y} = \int_1^\infty \{i^{2k} y^{2k} F(iy) - a_0\} y^{-s} \frac{dy}{y} + i^{2k} \int_1^\infty y^{2k-s} q(iy) \frac{dy}{y}$$

and hence

$$\Phi(s) = \int_1^\infty \{F(iy) - a_0\} y^s \frac{dy}{y} + \int_1^\infty \{i^{2k} y^{2k} F(iy) - a_0\} y^{-s} \frac{dy}{y} + i^{2k} \int_1^\infty y^{2k-s} q(iy) \frac{dy}{y}.$$

We now evaluate and simplify each integral to obtain

$$\Phi(s) = D^0(s) + D(s) + E^0(s) + E^H(s) + E^B(s),$$

where

$$\begin{aligned} D^0(s) &= a_0 \left(\frac{i^{2k}}{s-2k} - \frac{1}{s} \right), \\ D(s) &= \int_1^\infty (F(iy) - a_0) \{y^s + i^{2k} y^{2k-s}\} \frac{dy}{y}, \end{aligned}$$

$$E^0(s) + E^H(s) + E^B(s) = i^{2k} \int_1^\infty y^{2k-s} q(iy) \frac{dy}{y}.$$

Moreover,

$$\Phi(2k - s) - i^{2k} \Phi(s) = R(s),$$

where

$$R(s) = i^{2k} \sum_{j=1}^P \sum_{r=1}^{M_j} C_{rj} \left(\frac{-1}{\alpha_j} \right)^r \left\{ (i\alpha_j)^s B(s, r - s) - i^{-s} \alpha_j^{2k-s} B(2k - s, r - (2k - s)) \right\}.$$

□

Corollary 2.4. *Suppose $\Phi(s)$, $D^0(s)$, $D(s)$, $E^0(s)$, $E^H(s)$ and $E^B(s)$ are given as in Theorem 2.3. Then*

(a) $\Phi(s)$ is bounded uniformly in σ in lacunary vertical strips of the form

$$S = \{s = \sigma + it : 2k - \delta \leq \sigma \leq \delta; |t| \geq t_0 > 0\}.$$

(b) δ in (a) can be chosen so that the poles of $\Phi(s)$ lying within the lines $s = (2k - \delta) + it$ and $s = \delta + it$ are listed below in the sets:

$$S_0 = \{0, 2k\},$$

$$S_{E_0} = \{2k - L, 2k - L + 1, \dots, k - 1, k, k + 1, \dots, 2k, \dots, L\},$$

$$S_H = \{[2k - \delta], \dots, 0\},$$

$$S_B = \{[2k - \delta], \dots, 2k - L, \dots, 2k - 1\}.$$

The poles of $\Phi(s)$ in each set arise from $D^0(s)$, $E^0(s)$, $E^H(s)$ and $E^B(s)$ respectively.

(c) The residues of $\Phi(s)$ are given by the formulas:

$$Res_{s \in S_0} [D^0(s)] = a_0 (i^{2k} - 1), \quad (2.0.16)$$

$$Res_{s \in S_{E^0}} [E^0(s)] = \sum_{m=k}^L C_m \{ -(-i)^m + i^{2k-m} \}, \quad (2.0.17)$$

$$Res_{s \in S_H} [E^H(s)] = - \sum_{j=1}^p \sum_{r=1}^k C_{rj} \left\{ \sum_{m=0}^{[\delta]-2k} \frac{\Gamma(r+m)}{\Gamma(r)} \frac{(-1)^r}{m!} i^m \alpha_j^{-r-m} \right\}, \quad (2.0.18)$$

$$Res_{s \in S_B} [E^B(s)] = \sum_{j=1}^p \sum_{r=1}^k C_{rj} \left\{ \sum_{m=0}^{[\delta]-r} \frac{\Gamma(r+m)}{\Gamma(r)} \frac{(-1)^{r+m}}{m!} i^{m+r-2k} \alpha_j^m \right\}. \quad (2.0.19)$$

Chapter 3

Equivalence of a Functional Equation with Arithmetical Identities

Four equivalences

In the following chapters we begin to prove our main results. Sister Ann M. Heath in [1] considered the functional equation developed by Hawkins and Knopp in [5]. She has shown that the functional equation of Hawkins and Knopp [5], $\Phi(2k-s) - i^{2k}\Phi(s) = R(s)$, is equivalent to two types of arithmetical identities associated with entire modular cusp integrals involving rational period function for the full modular group $\Gamma(1)$.

In this chapter we use techniques of Chandrasekharan and Narasimhan in [2] and we extend the results of Sister Ann M. Heath to entire automorphic integrals involving an *RPF* on discrete Hecke groups $G(\lambda)$, $\lambda > 0$.

In Chapter Four we consider entire automorphic integral functions F , with log-polynomial-period function on the discrete Hecke groups. For fixed real number k the set $\{\nu(M) : M \in G(\lambda)\}$ multiplier system for $G(\lambda)$ weight $2k$. We note in [3] the Riemann-Hecke-Bochner correspondence applying to automorphic integrals on the discrete Hecke groups. There is a relationship between the Fourier expansion of F which satisfies the automorphic transformation law and Dirichlet series with a functional equation for the Mellin transform of an automorphic integrals.

Our main goal in this chapter is to establish the equivalence of two arithmetical identities with a functional equation associated with entire automorphic integrals involving log-polynomial-period functions on the discrete Hecke groups. With ν a multiplier system for $G(\lambda)$ and weight $2k$.

3.1 First equivalence

In this section we begin by stating and proving the first equivalence of the four equivalence theorems we shall prove. This will be our main result in this section.

Theorem 3.1. *Let $\Phi(s)$ and $R(s)$ be as in Theorem 2.3. Then the functional equation*

$$\Phi(2k - s) - i^{2k}\Phi(s) = R(s) \quad (3.1.1)$$

is equivalent to the identity

$$\frac{1}{\Gamma(\rho + 1)} \sum_{0 \leq m \leq x}' a_m (x - m)^\rho = \Lambda_1(s) + \Lambda_2(s) + \Lambda_3(s) + \Lambda_4(s) + \Lambda_5(s) \quad (3.1.2)$$

where

$$\begin{aligned} \Lambda_1(s) &= i^{-2k} \left(\frac{2\pi}{\lambda} \right)^{-\rho} \sum_{m=1}^{\infty} a_m \left(\frac{x}{m} \right)^{\frac{\rho+2k}{2}} J_{\rho+2k} \left(\frac{4\pi\sqrt{mx}}{\lambda} \right) \\ \Lambda_2(s) &= i^{2k} \left(\frac{2\pi}{\lambda} \right)^{2k} \frac{a_0}{\Gamma(2k + \rho + 1)} x^{2k+\rho} \\ \Lambda_3(s) &= - \sum_{j=1}^p \sum_{r=1}^{M_j} C_{rj} \left(\frac{-1}{\alpha_j} \right)^r \frac{(i\alpha_j)^r \left(\frac{2\pi}{\lambda} \right)^r x^{r+\rho}}{\Gamma(r + \rho + 1)} {}_1F_1 \left(r, r + \rho + 1; \frac{-i\alpha_j 2\pi x}{\lambda} \right) \\ \Lambda_4(s) &= \sum_{j=1}^p \sum_{r=1}^{M_j} C_{rj} \left(\frac{-1}{\alpha_j} \right)^r \frac{(i)^{-2k} \left(\frac{2\pi}{\lambda} \right)^{2k} x^{2k+\rho}}{\Gamma(2k + \rho + 1)} {}_1F_1 \left(r, 2k + \rho + 1; \frac{-2\pi x}{i\alpha_j \lambda} \right) \\ \Lambda_5(s) &= - \sum_{m=k}^L \left\{ \frac{(-i)^m \left(\frac{2\pi}{\lambda} \right)^m x^{m+\rho}}{\Gamma(m + \rho + 1)} - \frac{i^{2k-m} \left(\frac{2\pi}{\lambda} \right)^{2k-m} x^{2k-m+\rho}}{\Gamma(2k - m + \rho + 1)} \right\}, \end{aligned}$$

$x > 0$, $\rho \geq 2\beta - 2k - \frac{1}{2}$, and β is a number for which $\sum_{m=1}^{\infty} \frac{|a_m|}{m^\beta} < \infty$.

Before we give the proof of Theorem 3.1, we shall state Perron's Formula. (See [2] for details.) We also remark that in this work, we shall use the convention

that $\int_{(b)}$ denotes $\int_{b-i\infty}^{b+i\infty}$.

Lemma 3.2. *Let σ_0 be the abscissa of absolute convergence for $\varphi(s) = \sum_{m=1}^{\infty} a_m \lambda_m^{-s}$ and $\{\lambda_m\}$ be a sequence of positive real numbers tending to ∞ as $m \rightarrow \infty$.*

Then for $k \geq 0, \sigma > 0$ and $\sigma > \sigma_0$,

$$\frac{1}{\Gamma(k+1)} \sum'_{\lambda_m \leq x} a_m (x - \lambda_m)^k = \frac{1}{2\pi i} \int_{(\sigma)} \frac{\Gamma(s)\varphi(s)x^{s+k}}{\Gamma(s+k+1)} ds, \quad (3.1.3)$$

where the prime ' on the summation sign indicates that if $k = 0$ and $x = \lambda_m$ for some positive integer m , then we count only $\frac{1}{2}a_m$.

Here we have the following lemmas which are used in the proof of the present theorem.

Lemma 3.3. The function $A_1(s)$ defined by

$$A_1(s) = \frac{1}{2\pi i} \int_{(2k-\sigma)} \frac{\left(\frac{2\pi}{\lambda}\right)^s \Phi(s)x^{s+\rho}}{\Gamma(s+\rho+1)} ds \quad (3.1.4)$$

satisfies

$$\begin{aligned} A_1(s) &= i^{-2k} \left(\frac{2\pi}{\lambda}\right)^{-\rho} \sum_{m=1}^{\infty} a_m \left(\frac{x}{m}\right)^{\frac{2k+\rho}{2}} J_{2k+\rho} \left(\frac{4\pi\sqrt{mx}}{\lambda}\right) \\ &+ \sum_{j=1}^p \sum_{r=1}^{M_j} C_{rj} \left(\frac{-1}{\alpha_j}\right)^r \sum_{m=0}^{[\delta-2k]} \frac{(-1)^m \Gamma(m+r)}{m! \Gamma(r)} \frac{\left(\frac{2\pi i \alpha_j}{\lambda x}\right)^{-m} x^\rho}{\Gamma(-m+\rho+1)} \\ &- \sum_{j=1}^p \sum_{r=1}^{M_j} C_{rj} \left(\frac{-1}{\alpha_j}\right)^r \frac{\left(\frac{-2\pi x i}{\lambda}\right)^r x^\rho}{\Gamma(r+\rho+1)} {}_1F_1 \left(r, \rho+r+1; \frac{-2\pi \alpha_j x}{\lambda}\right) \\ &- \sum_{j=1}^p \sum_{r=1}^{M_j} C_{rj} \left(\frac{-1}{\alpha_j}\right)^r \sum_{m=0}^{[\delta-r]} \frac{(-1)^m \Gamma(m+r)}{m! \Gamma(r)} \frac{i^{m+r-2k} \alpha_j^{m+r} \left(\frac{2\pi x}{\lambda}\right)^{2k-m-r} x^\rho}{\Gamma(2k-m-r+\rho+1)} \\ &+ \sum_{j=1}^p \sum_{r=1}^{M_j} C_{rj} \left(\frac{-1}{\alpha_j}\right)^r \frac{\left(\frac{2\pi x}{\lambda i}\right)^{2k} x^\rho}{\Gamma(2k+\rho+1)} {}_1F_1 \left(r, \rho+2k+1; \frac{-2\pi x}{i\lambda \alpha_j}\right). \end{aligned} \quad (3.1.5)$$

Lemma 3.4. The function $A_2(s)$ defined by

$$A_2(s) = \sum \operatorname{Res} \left\{ \frac{\left(\frac{2\pi}{\lambda}\right)^s \Phi(s)x^{s+\rho}}{\Gamma(s+\rho+1)} \right\} \quad (3.1.6)$$

satisfies

$$\begin{aligned} A_2(s) &= \frac{a_0 i^{2k} x^{2k+\rho} \left(\frac{2\pi}{\lambda}\right)^{2k}}{\Gamma(2k+\rho+1)} - \frac{a_0 x^\rho}{\Gamma(\rho+1)} \\ &+ \sum_{m=k}^L C_m \left\{ \frac{-(-i)^m \left(\frac{2\pi}{\lambda}\right)^m x^{m+\rho}}{\Gamma(m+\rho+1)} + \frac{(i)^{2k-m} \left(\frac{2\pi}{\lambda}\right)^{2k-m} x^{2k-m+\rho}}{\Gamma(2k-m+\rho+1)} \right\} \end{aligned}$$

$$\begin{aligned}
 & - \sum_{j=1}^p \sum_{r=1}^{M_j} C_{rj} \left\{ \sum_{m=0}^{[\delta]-2k} \frac{\Gamma(m+r)}{\Gamma(r)} \frac{(-1)^m}{m!} i^m \alpha_j^{-r-m} \frac{\left(\frac{2\pi}{\lambda}\right)^{-m} x^{-m+\rho}}{\Gamma(-m+\rho+1)} \right\} \\
 & + \sum_{j=1}^p \sum_{r=1}^{M_j} C_{rj} \left\{ \sum_{m=0}^{[\delta]-r} \frac{\Gamma(m+r)}{\Gamma(r)} \frac{(-1)^{m+r}}{m!} i^{m+r-2k} \alpha_j^m \frac{\left(\frac{2\pi}{\lambda}\right)^{2k-m-r} x^{2k-m-r+\rho}}{\Gamma(2k-m-r+\rho+1)} \right\}.
 \end{aligned} \tag{3.1.7}$$

Lemma 3.5. *The function $A_3(s)$ defined by*

$$A_3(s) = \frac{1}{2\pi i} \int_{(2k-\sigma)} \frac{\left(\frac{2\pi i \alpha_j}{\lambda}\right)^s B(s, r-s) x^{s+\rho}}{\Gamma(s+\rho+1)} ds$$

satisfies

$$\begin{aligned}
 A_3(s) &= - \sum_{m=0}^{[\sigma]-2k} \frac{(-1)^m \left(\frac{2\pi i \alpha_j}{\lambda}\right)^{-m} x^{-m+\rho} \Gamma(m+r)}{m! \Gamma(-m+\rho+1) \Gamma(r)} \\
 & - \sum_{m=0}^{\infty} \frac{(-1)^m \left(\frac{2\pi i \alpha_j}{\lambda}\right)^{m+r} x^{m+r+\rho} \Gamma(m+r)}{m! \Gamma(m+r+\rho+1) \Gamma(r)} \\
 &= - \sum_{m=0}^{[\sigma]-2k} \frac{(-1)^m \left(\frac{2\pi i \alpha_j}{\lambda}\right)^{-m} x^{-m+\rho} \Gamma(m+r)}{m! \Gamma(-m+\rho+1) \Gamma(r)} \\
 & - \frac{\left(\frac{2\pi i \alpha_j x}{\lambda}\right)^r x^\rho}{\Gamma(\rho+r+1)} {}_1F_1\left(r, \rho+r+1; -\frac{2\pi i \alpha_j x}{\lambda}\right).
 \end{aligned}$$

Proof. To prove Lemma 3.5 we consider a rectangular contour with vertices $[(2k-\sigma) - iT, N + \frac{1}{2} - iT], [N + \frac{1}{2} - iT, N + \frac{1}{2} + iT], [N + \frac{1}{2} + iT, (2k-\sigma) + iT], [(2k-\sigma) + iT, (2k-\sigma) - iT]$, where N and T s chosen so the line of integration is positively oriented. For fixed large integer N we have

$$\begin{aligned}
 & \frac{1}{2\pi i} \int_{(2k-\sigma)} \frac{\left(\frac{2\pi i \alpha_j}{\lambda}\right)^s B(s, r-s) x^{s+\rho}}{\Gamma(s+\rho+1)} ds \\
 &= \lim_{T \rightarrow \infty} \left\{ \int_{2k-\sigma-iT}^{N+\frac{1}{2}-iT} + \int_{N+\frac{1}{2}-iT}^{N+\frac{1}{2}+iT} + \int_{N+\frac{1}{2}+iT}^{2k-\sigma+iT} \right\} \frac{\left(\frac{2\pi i \alpha_j}{\lambda}\right)^s B(s, r-s) x^{s+\rho}}{\Gamma(s+\rho+1)} ds \\
 & - \sum_{s \in \text{pole set}} \text{Res} \frac{\left(\frac{2\pi i \alpha_j}{\lambda}\right)^s B(s, r-s) x^{s+\rho}}{\Gamma(s+\rho+1)}.
 \end{aligned} \tag{3.1.8}$$

We label the integrals in (3.1.8) m_1, m_2, m_3 respectively. Then

$$\begin{aligned} & \frac{1}{2\pi i} \int_{(2k-\sigma)} \frac{\left(\frac{2\pi i \alpha_j}{\lambda}\right)^s B(s, r-s) x^{s+\rho}}{\Gamma(s+\rho+1)} ds \\ &= \lim_{T \rightarrow \infty} \left\{ m_1 + m_2 + m_3 \right\} - \sum \operatorname{Res} \left\{ \frac{\left(\frac{2\pi i \alpha_j}{\lambda}\right)^s B(s, r-s) x^{s+\rho}}{\Gamma(s+\rho+1)} \right\}. \end{aligned}$$

First we calculate the sum of the residues involved in evaluating $A_3(s)$. That is

$$\sum_{s \in \text{PoleSet}} \operatorname{Res} \left\{ \frac{\left(\frac{2\pi i \alpha_j}{\lambda}\right)^s B(s, r-s) x^{s+\rho}}{\Gamma(s+\rho+1)} \right\}.$$

Since $\frac{\left(\frac{2\pi i \alpha_j}{\lambda}\right)^s x^{s+\rho}}{\Gamma(s+\rho+1)}$ is entire the pole set of the integrand arise from the beta function

$$B(s, r-s) = \frac{\Gamma(s)\Gamma(r-s)}{\Gamma(r)}.$$

Thus the integrand has simple poles at $s = -m$ for $\Gamma(s)$ and at $s = m+r$ for $\Gamma(r-s)$ with $m = 0, 1, 2, \dots$. The residue at $s = -m$ is

$$\frac{(-1)^m \left(\frac{2\pi i \alpha_j}{\lambda}\right)^{-m} x^{-m+\rho}}{m! \Gamma(-m+\rho+1)}$$

and the residue at $s = r+m$ is equal to

$$\frac{(-1)^{m+r} \left(\frac{2\pi i \alpha_j}{\lambda}\right)^{r+m} x^{r+m+\rho}}{m! \Gamma(m+r+\rho+1)}.$$

Hence

$$\begin{aligned} \sum_{s \in \text{PoleSet}} \operatorname{Res} \left\{ \frac{\left(\frac{2\pi i \alpha_j}{\lambda}\right)^s B(s, r-s) x^{s+\rho}}{\Gamma(s+\rho+1)} \right\} &= \sum_{m=0}^{[\sigma]-2k} \frac{(-1)^m \left(\frac{2\pi i \alpha_j}{\lambda}\right)^{-m} x^{-m+\rho} \Gamma(m+r)}{m! \Gamma(-m+\rho+1) \Gamma(r)} \\ &+ \sum_{m=0}^{N-r} \frac{(-1)^m \left(\frac{2\pi i \alpha_j}{\lambda}\right)^{m+r} x^{m+r+\rho} \Gamma(m+r)}{m! \Gamma(m+r+\rho+1) \Gamma(r)}. \end{aligned}$$

It remains to show that m_1, m_2 and m_3 converges to 0 as $N, T \rightarrow \infty$. We first m_1 and m_3 is treated in the same way using Stirling's formula and standard

branch conventions.

Remarks: By Stirling's formula we show $\left| \frac{B(s, r-s)(2\pi i \alpha_j x)^s x^\rho}{\Gamma(1+\rho+s)} \right| = \mathcal{O}\left(|t|^{-\rho-\sigma-\frac{3}{2}}\right) = \mathbf{o}(1)$ as $t \rightarrow \infty$, for $\rho > -\sigma - \frac{3}{2}$. Thus $|m_3| \rightarrow 0$ as $|T| \rightarrow \infty$. Similarly $|m_1| \rightarrow 0$ as $|T| \rightarrow \infty$. It remains to show the integral on the right converges to 0 as $N \rightarrow \infty$. To show this put

$$\begin{aligned} m_2(x, \lambda) &= \lim_{T \rightarrow \infty} \int_{N+\frac{1}{2}-iT}^{N+\frac{1}{2}+iT} \frac{\left(\frac{2\pi i \alpha_j}{\lambda}\right)^s B(s, r-s)x^{s+\rho}}{\Gamma(s+\rho+1)} ds \\ &= \int_{-\infty}^{\infty} \frac{\left(\frac{2\pi i \alpha_j}{\lambda}\right)^{N+\frac{1}{2}+it} \Gamma(N+\frac{1}{2}+it)\Gamma(r-N-\frac{1}{2}-it)x^{N+\frac{1}{2}+\rho+it}}{\Gamma(N+\frac{1}{2}+\rho+1+it)} dt \\ &= \mathcal{O}\left(\int_{-\infty}^{\infty} \left(\frac{2\pi \alpha_j x}{\lambda t}\right)^N \frac{dt}{e^{\frac{1}{2}\pi|t|}|t|^{2+\rho-r}}\right) \\ &= \mathbf{o}(1), \end{aligned}$$

as $N \rightarrow \infty$ provided that $\lambda t > 2\pi \alpha_j x$. By rewriting the infinite series representation for the confluent hypergeometric function as

$$-\frac{\left(\frac{2\pi i \alpha_j x}{\lambda}\right)^r x^\rho}{\Gamma(\rho+r+1)} {}_1F_1\left(r, \rho+r+1; -\frac{2\pi i \alpha_j x}{\lambda}\right),$$

proof of the lemma would be complete. \square

Lemma 3.6. *The function $A_4(s)$ defined by*

$$A_4(s) = \frac{1}{2\pi i} \int_{(2k-\sigma)} \frac{\left(\frac{2\pi}{\lambda i}\right)^s B(2k-s, r-(2k-s))x^{s+\rho}}{\Gamma(s+\rho+1)} ds$$

satisfies

$$\begin{aligned} A_4(s) &= -\sum_{m=0}^{[\delta-r]} \left(\frac{2\pi}{i\lambda}\right)^{2k-m-r} \frac{\alpha_j^{m+r}}{\Gamma(2k-m-r+\rho+1)} \frac{(-1)^m \Gamma(m+r)}{m! \Gamma(r)} x^{2k-m-r+\rho} \\ &\quad - \sum_{m=0}^{\infty} \left(\frac{2\pi}{i\lambda}\right)^{2k+m} \frac{\alpha_j^{-m}}{\Gamma(2k+m+\rho+1)} \frac{(-1)^m \Gamma(m+r)}{m! \Gamma(r)} x^{2k+m+\rho}. \end{aligned}$$

Proof. To show Lemma 3.6 we proceed in a similar manner of Lemma 3.5 on the rectangular contour positively oriented of the form:

$$[2k-\delta-iT, N+\frac{1}{2}-iT], [N+\frac{1}{2}-iT, N+\frac{1}{2}+iT], [N+\frac{1}{2}+iT, 2k-\delta+iT], [2k-$$

$\delta + iT, 2k - \delta - iT]$, where T and N chosen large. For fixed N , the integrals along the horizontal paths approach to zero as $T \rightarrow \infty$. Thus we have

$$A_4(s) = \frac{1}{2\pi i} \int_{(N+\frac{1}{2})} \left(\frac{2\pi}{\lambda i} \right)^s \alpha_j^{2k-s} \frac{B(2k-s, r-(2k-s))x^{s+\rho}}{\Gamma(s+\rho+1)} - \sum \text{Res} \left\{ \left(\frac{2\pi}{\lambda i} \right)^s \alpha_j^{2k-s} \frac{B(2k-s, r-(2k-s))x^{s+\rho}}{\Gamma(s+\rho+1)} \right\}.$$

To evaluate the sum of the residues involved in $A_4(s)$, the pole set of the integrand arise from the beta function

$$B(2k-s, r-(2k-s)) = \frac{\Gamma(2k-s)\Gamma(r-2k+s)}{\Gamma(r)},$$

as $\left(\frac{2\pi}{\lambda i} \right)^s \frac{\alpha_j^{2k-s} x^{s+\rho}}{\Gamma(s+\rho+1)}$ is entire function. Thus the integrand has simple poles at $s = 2k + m$ for $\Gamma(2k-s)$ and at $s = 2k - m - r$ for $\Gamma(r-2k+s)$ with $m = 0, 1, 2 \dots$. As a result we obtain

$$A_4(s) = - \sum_{m=0}^{[\delta-r]} \left(\frac{2\pi}{i\lambda} \right)^{2k-m-r} \frac{\alpha_j^{m+r}}{\Gamma(2k-m-r+\rho+1)} \frac{(-1)^m \Gamma(m+r)}{m! \Gamma(r)} x^{2k-m-r+\rho} - \sum_{m=0}^{N-2k} \left(\frac{2\pi}{i\lambda} \right)^{2k+m} \frac{\alpha_j^{-m}}{\Gamma(2k+m+\rho+1)} \frac{(-1)^m \Gamma(m+r)}{m! \Gamma(r)} x^{2k+m+\rho} + \frac{1}{2\pi i} \int_{(N+\frac{1}{2})} \left(\frac{2\pi}{\lambda i} \right)^s \alpha_j^{2k-s} \frac{B(2k-s, r-(2k-s))x^{s+\rho}}{\Gamma(s+\rho+1)}.$$

To show the integral on the right of the equation goes to zero as $N \rightarrow \infty$ we partition the interval $(-\infty, \infty)$ as $(-\infty, T] \cup [-T, T] \cup [T, \infty)$, and by choosing some large value of T and using Stirling's formula we may estimate the infinite strips. For $s \in [-T, T]$ we have $\frac{B(2k-s, r-(2k-s))x^{s+\rho}}{\Gamma(s+\rho+1)} = \frac{\Gamma(2k-s)\Gamma(r-2k+s)x^{s+\rho}}{\Gamma(r)\Gamma(s+\rho+1)}$. By Stirling's formula and the estimation $\Gamma(-s) = e^{-(s+\frac{1}{2})(\log s - \pi i) + s + \frac{1}{2} \log 2\pi} \left(1 + \mathcal{O}\left(\frac{1}{|s|}\right) \right)$ (see[7],page12) we obtain

$$\left| \frac{\Gamma(2k-s)\Gamma(r-2k+s)x^{s+\rho}}{\Gamma(r)\Gamma(s+\rho+1)} \right| = \mathcal{O} \left(\frac{e^N x^N}{(N-2k)^{N+1-2k}} \right) = \mathbf{o}(1),$$

as $N \rightarrow \infty$. Therefore Lemma 3.6 follows. \square

We now proceed the prove of our main result.

Proof. By Lemma 3.2 we have

$$\frac{1}{\Gamma(\rho+1)} \sum'_{0 \leq m \leq x} a_m (x-m)^\rho = \frac{1}{2\pi i} \int_{(\sigma)} \frac{\Gamma(s)\varphi(s)x^{s+\rho}}{\Gamma(s+\rho+1)} ds. \quad (3.1.9)$$

To evaluate the integral in (3.1.9) consider the positively oriented rectangular contour formed by $[(2k - \sigma) - iT, \sigma - iT], [\sigma - iT, \sigma + iT], [\sigma + iT, (2k - \sigma) + iT], [(2k - \sigma) + iT, (2k - \sigma) - iT]$, where T is chosen to be a large positive so that all of s (to enclose all poles of the function in the integral) is contained within the rectangle. Now applying the Cauchy Residue Theorem we have

$$\begin{aligned} & \frac{1}{2\pi i} \int_{(\sigma)} \frac{\Gamma(s)\varphi(s)x^{s+\rho}}{\Gamma(s+\rho+1)} ds \\ &= \frac{1}{2\pi i} \left\{ \int_{2k-\sigma-iT}^{\sigma-iT} + \int_{\sigma-iT}^{\sigma+iT} + \int_{\sigma+iT}^{2k-\sigma+iT} + \int_{2k-\sigma+iT}^{\sigma-iT} \right\} \frac{\Gamma(s)\varphi(s)x^{s+\rho}}{\Gamma(s+\rho+1)} ds \\ &= \sum_{s \in \text{Pole Set}} \text{Res} \left\{ \frac{\Gamma(s)\varphi(s)x^{s+\rho}}{\Gamma(s+\rho+1)} \right\}, \end{aligned} \quad (3.1.10)$$

where we have used $\Phi(s) = \left(\frac{2\pi}{\lambda}\right)^{-s} \Gamma(s)\varphi(s)$ in the last equality.

We now show that as $T \rightarrow \infty$, the integrals along the horizontal segments $[(2k - \sigma) - iT, \sigma - iT]$ and $[\sigma + iT, (2k - \sigma) + iT]$ tends to zero. To show this, we note that for $s = \sigma + it$ with $\sigma > \beta$, $\varphi(s) = \mathcal{O}(1)$ and by Stirling's formula we have $\frac{\Gamma(s)}{\Gamma(s+\rho+1)} = \mathcal{O}(t^{-\rho-1})$ for $\rho > -1$. Thus $\frac{\Gamma(s)\varphi(s)x^{s+\rho}}{\Gamma(s+\rho+1)} = \mathcal{O}(t^{-\rho-1}) = o(1)$ and hence the integrals along the horizontal lines converge to 0 as $T \rightarrow \infty$. Hence (3.1.10) can now be written as

$$\frac{1}{2\pi i} \int_{(\sigma)} \frac{\left(\frac{2\pi}{\lambda}\right)^s \Phi(s)x^{s+\rho}}{\Gamma(s+\rho+1)} ds = A_1(s) + A_2(s), \quad (3.1.11)$$

where

$$A_1(s) = \frac{1}{2\pi i} \int_{(2k-\sigma)} \frac{\left(\frac{2\pi}{\lambda}\right)^s \Phi(s)x^{s+\rho}}{\Gamma(s+\rho+1)} ds \quad (3.1.12)$$

$$A_2(s) = \sum \text{Res} \left\{ \frac{\left(\frac{2\pi}{\lambda}\right)^s \Phi(s)x^{s+\rho}}{\Gamma(s+\rho+1)} \right\}. \quad (3.1.13)$$

We now compute $A_1(s)$. For this we recall that Φ satisfies the functional equation

$$\Phi(2k - s) - i^{2k}\Phi(s) = R(s).$$

Thus $A_1(s)$ can be written as

$$A_1(s) = I(s) - H(s), \quad (3.1.14)$$

where

$$I(s) = \frac{1}{2\pi i} \int_{(2k-\sigma)} \frac{\left(\frac{2\pi}{\lambda}\right)^s i^{-2k} \Phi(2k-s)x^{s+\rho}}{\Gamma(s+\rho+1)} ds \quad (3.1.15)$$

and

$$H(s) = \frac{1}{2\pi i} \int_{(2k-\sigma)} \frac{\left(\frac{2\pi}{\lambda}\right)^s i^{-2k} R(s)x^{s+\rho}}{\Gamma(s+\rho+1)} ds. \quad (3.1.16)$$

Next, we use $\Phi(s) = \left(\frac{2\pi}{\lambda}\right)^{-s} \Gamma(s)\varphi(s)$ in (3.1.15) to get

$$I(s) = \frac{1}{2\pi i} \int_{(2k-\sigma)} \frac{\left(\frac{2\pi}{\lambda}\right)^{2s-2k} i^{-2k} \Gamma(2k-s)\varphi(2k-s)x^{s+\rho}}{\Gamma(s+\rho+1)} ds.$$

By changing of variable s to $2k-s$, in the integral we see that

$$\begin{aligned} I(s) &= \frac{1}{2\pi i} \int_{(\sigma)} \frac{\left(\frac{2\pi}{\lambda}\right)^{-2s+2k} i^{-2k} \Gamma(s)\varphi(s)x^{-s+\rho+2k}}{\Gamma(2k-s+\rho+1)} ds \\ &= \frac{i^{-2k} \left(\frac{4\pi^2}{\lambda^2}\right)^k x^{2k+\rho}}{2\pi i} \int_{(\sigma)} \frac{\left(\frac{4\pi^2 x}{\lambda^2}\right)^{-s} \Gamma(s)\varphi(s)}{\Gamma(2k-s+\rho+1)} ds \\ &= \frac{i^{-2k} \left(\frac{4\pi^2}{\lambda^2}\right)^k x^{2k+\rho}}{2\pi i} \int_{(\sigma)} \frac{\left(\frac{4\pi^2 x}{\lambda^2}\right)^{-s} \Gamma(s) \sum_{m=1}^{\infty} a_m m^{-s}}{\Gamma(2k-s+\rho+1)} ds \\ &= i^{-2k} \left(\frac{2\pi}{\lambda}\right)^{2k} x^{2k+\rho} \frac{1}{2\pi i} \sum_{m=1}^{\infty} a_m \int_{(\sigma)} \frac{\Gamma(s) \left(\frac{4\pi^2 xm}{\lambda^2}\right)^{-s}}{\Gamma(2k-s+\rho+1)} ds. \end{aligned}$$

Letting $\frac{w}{2} = s, \nu = 2k + \rho$, and by simplifying we obtain

$$I(s) = i^{-2k} \left(\frac{2\pi}{\lambda}\right)^{\nu-\rho} 2^\nu x^\nu \sum_{m=1}^{\infty} a_m \frac{1}{2\pi i} \int_{(2\sigma)} \frac{\Gamma\left(\frac{w}{2}\right) \left(\frac{4\pi\sqrt{mx}}{\lambda}\right)^{-w}}{\Gamma\left(\nu - \frac{w}{2} + 1\right)} 2^{w-\nu-1} dw.$$

Applying (1.3.9), we next replace ν by $2k + \rho$. After simplifying we have

$$I(s) = i^{-2k} \left(\frac{2\pi}{\lambda}\right)^{-\rho} \sum_{m=1}^{\infty} a_m \left(\frac{x}{m}\right)^{\frac{2k+\rho}{2}} J_{2k+\rho} \left(\frac{4\pi\sqrt{mx}}{\lambda}\right), \quad (3.1.17)$$

provided that $\rho > \sigma - 2k$ and $\sigma > 2k, k \in \mathbb{Z}$.

To evaluate

$$H(s) = \frac{1}{2\pi i} \int_{(2k-\sigma)} \frac{\left(\frac{2\pi}{\lambda}\right)^s i^{-2k} R(s)x^{s+\rho}}{\Gamma(s+\rho+1)} ds,$$

where

$$R(s) = i^{2k} \sum_{j=1}^P \sum_{r=1}^{M_j} C_{ri} \left(\frac{-1}{\alpha_j} \right)^r \left\{ (i\alpha_j)^s B(s, r-s) - i^{-s} \alpha_j^{2k-s} B(2k-s, r-(2k-s)) \right\}.$$

Then

$$H(s) = \sum_{j=1}^P \sum_{r=1}^{M_j} C_{ri} \left(\frac{-1}{\alpha_j} \right)^r \left\{ \frac{1}{2\pi i} \int_{(2k-\sigma)} \frac{\left(\frac{2\pi i \alpha_j}{\lambda} \right)^s B(s, r-s) x^{s+\rho}}{\Gamma(s+\rho+1)} ds - \alpha_j^{2k-s} \frac{1}{2\pi i} \int_{(2k-\sigma)} \frac{\left(\frac{2\pi}{\lambda i} \right)^s B(2k-s, r-(2k-s)) x^{s+\rho}}{\Gamma(s+\rho+1)} ds \right\}.$$

Let us rewrite $H(s)$ in terms of $A_3(s)$ and $A_4(s)$ as

$$H(s) = \sum_{j=1}^P \sum_{r=1}^{M_j} C_{ri} \left(\frac{-1}{\alpha_j} \right)^r \left\{ A_3(s) - \alpha_j^{2k-s} A_4(s) \right\}. \quad (3.1.18)$$

Using the series representation for the confluent hypergeometric functions we write the infinite series in $A_4(s)$ as

$$\frac{i^{-2k} x^{2k+\rho} \left(\frac{2\pi}{\lambda} \right)^{2k}}{\Gamma(2k+\rho+1)} {}_1F_1 \left(r, 2k+\rho+1; \frac{-2\pi x}{i\lambda\alpha_j} \right).$$

With simplification of the expressions in $A_3(s)$ and $A_4(s)$ we get

$$\begin{aligned} H(s) = & - \sum_{j=1}^p \sum_{r=1}^{M_j} C_{rj} \left(\frac{-1}{\alpha_j} \right)^r \sum_{m=0}^{[\delta-2k]} \frac{(-1)^m \Gamma(m+r)}{m! \Gamma(r)} \frac{\left(\frac{2\pi i \alpha_j}{\lambda x} \right)^{-m} x^\rho}{\Gamma(-m+\rho+1)} \\ & + \sum_{j=1}^p \sum_{r=1}^{M_j} C_{rj} \left(\frac{-1}{\alpha_j} \right)^r \frac{\left(\frac{-2\pi x i}{\lambda} \right)^r x^\rho}{\Gamma(r+\rho+1)} {}_1F_1 \left(r, \rho+r+1; \frac{-2\pi \alpha_j x}{\lambda} \right) \\ & + \sum_{j=1}^p \sum_{r=1}^{M_j} C_{rj} \left(\frac{-1}{\alpha_j} \right)^r \sum_{m=0}^{[\delta-r]} \frac{(-1)^m \Gamma(m+r)}{m! \Gamma(r)} \frac{i^{m+r-2k} \alpha_j^{m+r} \left(\frac{2\pi x}{\lambda} \right)^{2k-m-r} x^\rho}{\Gamma(2k-m-r+\rho+1)} \\ & - \sum_{j=1}^p \sum_{r=1}^{M_j} C_{rj} \left(\frac{-1}{\alpha_j} \right)^r \frac{\left(\frac{2\pi x}{\lambda i} \right)^{2k} x^\rho}{\Gamma(2k+\rho+1)} {}_1F_1 \left(r, \rho+2k+1; \frac{-2\pi x}{i\lambda\alpha_j} \right). \end{aligned}$$

Now since $A_1(s) = I(s) - H(s)$. So that (3.1.5) holds and proof of Lemma 3.3 is complete.

We now compute the sum of residues

$$A_2(s) = \sum_{s \in \text{pol set of } \Phi(s)} \text{Res} \left\{ \frac{\left(\frac{2\pi}{\lambda} \right)^s \Phi(s) x^{s+\rho}}{\Gamma(s+\rho+1)} \right\}.$$

Recall that the meromorphic continuation of $\Phi(s)$ has been expressed in terms of $D^0(s)$, $E^0(s)$, $E^H(s)$, and $E^B(s)$ as in Theorem 2.3 and hence

$$A_2(s) = \left\{ Res[D^0(s)] + Res[E^0(s)] + Res[E^H(s)] + Res[E^B(s)] \right\} \frac{\left(\frac{2\pi}{\lambda}\right)^s x^{s+\rho}}{\Gamma(s + \rho + 1)}.$$

Using the formulas (2.0.16), (2.0.17), (2.0.18), and (2.0.19) we find

$$\begin{aligned} A_2(s) &= \frac{a_0 i^{2k} x^{2k+\rho} \left(\frac{2\pi}{\lambda}\right)^{2k}}{\Gamma(2k + \rho + 1)} - \frac{a_0 x^\rho}{\Gamma(\rho + 1)} \\ &+ \sum_{m=k}^L C_m \left\{ \frac{-(-i)^m \left(\frac{2\pi}{\lambda}\right)^m x^{m+\rho}}{\Gamma(m + \rho + 1)} + \frac{(i)^{2k-m} \left(\frac{2\pi}{\lambda}\right)^{2k-m} x^{2k-m+\rho}}{\Gamma(2k - m + \rho + 1)} \right\} \\ &- \sum_{j=1}^p \sum_{r=1}^{M_j} C_{rj} \left\{ \sum_{m=0}^{[\delta]-2k} \frac{\Gamma(m+r)}{\Gamma(r)} \frac{(-1)^m}{m!} i^m \alpha_j^{-r-m} \frac{\left(\frac{2\pi}{\lambda}\right)^{-m} x^{-m+\rho}}{\Gamma(-m + \rho + 1)} \right\} \\ &+ \sum_{j=1}^p \sum_{r=1}^{M_j} C_{rj} \left\{ \sum_{m=0}^{[\delta]-r} \frac{\Gamma(m+r)}{\Gamma(r)} \frac{(-1)^{m+r}}{m!} i^{m+r-2k} \alpha_j^m \frac{\left(\frac{2\pi}{\lambda}\right)^{2k-m-r} x^{2k-m-r+\rho}}{\Gamma(2k - m - r + \rho + 1)} \right\}. \end{aligned} \quad (3.1.19)$$

Thus Lemma(3.4) holds. Hence we have expressed the integral in the right-hand side of (3.1.9) as

$$\frac{1}{2\pi i} \int_{(\delta)} \frac{\Gamma(s)\varphi(s)x^{s+\rho}}{\Gamma(s + \rho + 1)} ds = A_1(s) + A_2(s) = (I(s) - H(s)) + A_2(s),$$

and determine the expressions for $A_1(s)$ and $A_2(s)$. [See (3.1.5) and (3.1.19) respectively]. By comparison and cancellation of the corresponding sums of expressions in $A_1(s)$ and $A_2(s)$ we obtain

$$\begin{aligned} \frac{1}{2\pi i} \int_{(\delta)} \frac{\Gamma(s)\varphi(s)x^{s+\rho}}{\Gamma(s + \rho + 1)} ds &= i^{-2k} \left(\frac{2\pi}{\lambda}\right)^{-\rho} \sum_{m=1}^{\infty} a_m \left(\frac{x}{m}\right)^{\frac{\rho+2k}{2}} J_{\rho+2k} \left(\frac{4\pi\sqrt{mx}}{\lambda}\right) \\ &+ i^{2k} \left(\frac{2\pi}{\lambda}\right)^{2k} \frac{a_0}{\Gamma(2k + \rho + 1)} x^{2k+\rho} - \frac{a_0 x^\rho}{\Gamma(\rho + 1)} \\ &- \sum_{j=1}^p \sum_{r=1}^{M_j} C_{rj} \left(\frac{-1}{\alpha_j}\right)^r \frac{(i\alpha_j)^r \left(\frac{2\pi}{\lambda}\right)^r x^{r+\rho}}{\Gamma(r + \rho + 1)} {}_1F_1 \left(r, r + \rho + 1; \frac{-i\alpha_j 2\pi x}{\lambda}\right) \\ &+ \sum_{j=1}^p \sum_{r=1}^{M_j} C_{rj} \left(\frac{-1}{\alpha_j}\right)^r \frac{(i)^{-2k} \left(\frac{2\pi}{\lambda}\right)^{2k} x^{2k+\rho}}{\Gamma(2k + \rho + 1)} {}_1F_1 \left(r, 2k + \rho + 1; \frac{-2\pi x}{i\alpha_j \lambda}\right) \\ &- \sum_{m=k}^L \left\{ \frac{(-i)^m \left(\frac{2\pi}{\lambda}\right)^m x^{m+\rho}}{\Gamma(m + \rho + 1)} - \frac{i^{2k-m} \left(\frac{2\pi}{\lambda}\right)^{2k-m} x^{2k-m+\rho}}{\Gamma(2k - m + \rho + 1)} \right\}. \end{aligned} \quad (3.1.20)$$

Therefore, for $\rho \geq 0$, $\frac{\rho+2k}{2} > \delta$ we get the identity in (3.1.2). \square

Proof of converse

Let F be an entire automorphic integral and let $F(z)$ have a Fourier series expansion of the form

$$F(z) = \sum_{m=0}^{\infty} a_m e^{2\pi imz/\lambda},$$

for $z \in \mathcal{H}$ and F satisfies the relation

$$z^{-2k} F\left(\frac{-1}{z}\right) = F(z) + q(z), \quad (3.1.21)$$

where $q(z)$ is the rational periodic function given by Lemma (2.2). Then by (3.1.21) and the Fourier expansion of F we have

$$z^{-2k} \sum_{m=0}^{\infty} a_m e^{-2\pi im/\lambda z} = \sum_{m=0}^{\infty} a_m e^{2\pi imz/\lambda} + q(z).$$

Letting $z = \frac{iy\lambda}{2\pi}$, $y > 0$, then we get

$$\left(\frac{2\pi}{iy\lambda}\right)^{2k} \sum_{m=0}^{\infty} a_m e^{-\frac{4\pi^2 m}{y\lambda^2}} = \sum_{m=0}^{\infty} a_m e^{-my} + q\left(\frac{iy\lambda}{2\pi}\right). \quad (3.1.22)$$

To prove the converse it suffices to show that (3.1.20) implies (3.1.22). To this end we consider six integrals defining $L_1(y) \cdots, L_6(y)$, corresponding to the six expressions occurring in (3.1.20). To evaluate $L_1(y)$ for $\rho \geq 0$, the interchange of integration and summation is justified. Then we have

$$\begin{aligned} L_1(y) &= \int_0^{\infty} \left\{ \frac{1}{\Gamma(\rho+1)} \sum_{0 \leq m \leq x} a_m (x-m)^\rho \right\} y^{\rho+1} e^{-xy} dx, \\ &= \frac{y^{\rho+1}}{\Gamma(\rho+1)} \int_0^{\infty} \sum_{0 \leq m \leq x} a_m (x-m)^\rho e^{-xy} dx, \\ &= \frac{a_0 y^{\rho+1}}{\Gamma(\rho+1)} \int_0^{\infty} e^{-xy} x^\rho dx + \frac{y^{\rho+1}}{\Gamma(\rho+1)} \sum_{1 \leq m \leq x} a_m \int_0^{\infty} (x-m)^\rho e^{-xy} dx, \\ &= a_0 + \sum_{m=1}^{\infty} a_m e^{-my}, \\ &= \sum_{m=0}^{\infty} a_m e^{-my}. \end{aligned}$$

Similarly on the right side of $L_2(y)$ the interchange of integration and summation is justified for $\frac{\rho+2k}{2} \leq \beta$ and applying formula (1.3.8) we have

$$\begin{aligned} L_2(y) &= \int_0^\infty i^{-2k} \left(\frac{2\pi}{\lambda}\right)^{-\rho} \sum_{m=0}^\infty a_m \left(\frac{x}{m}\right)^{\frac{2k+\rho}{2}} J_{2k+\rho} \left(\frac{4\pi\sqrt{mx}}{\lambda}\right) y^{\rho+1} e^{-xy} dx, \\ &= i^{-2k} \left(\frac{2\pi}{\lambda}\right)^{-\rho} y^{\rho+1} \sum_{m=0}^\infty a_m m^{-\frac{(2k+\rho)}{2}} \int_0^\infty J_{2k+\rho} \left(\frac{4\pi\sqrt{mx}}{\lambda}\right) x^{\frac{2k+\rho}{2}} e^{-xy} dx, \\ &= i^{-2k} \left(\frac{2\pi}{\lambda}\right)^{-\rho} y^{\rho+1} \sum_{m=0}^\infty a_m m^{-\frac{(2k+\rho)}{2}} \left\{ \frac{2(4\pi\sqrt{m}/\lambda)^{2k+\rho}}{(2y)^{2k+\rho+1}} e^{-\frac{(4\pi\sqrt{m})^2}{4y\lambda^2}} \right\}. \end{aligned}$$

Thus with simple algebraic computations we obtain

$$L_2(y) = i^{-2k} \left(\frac{2\pi}{\lambda y}\right)^{2k} \sum_{m=0}^\infty a_m e^{-\frac{4\pi^2 m}{y\lambda^2}}.$$

Next we evaluate $L_3(y)$, where

$$L_3(y) = \int_0^\infty e^{-xy} y^{\rho+1} \left\{ i^{2k} \left(\frac{2\pi}{\lambda y}\right)^{2k} \frac{a_0}{\Gamma(2k + \rho + 1)} x^{2k+\rho} - \frac{a_0 x^\rho}{\Gamma(\rho + 1)} \right\} dx.$$

Using integration by substitution and the standard integral representation of $\Gamma(s)$ after some steps we get

$$L_3(y) = a_0 i^{2k} \left(\frac{2\pi}{\lambda y}\right)^{2k} - a_0.$$

We begin evaluation of $L_4(y)$ by using integral representation for $\Gamma(s)$ where,

$$L_4(y) = \int_0^\infty \sum_{m=k}^L C_m y^{\rho+1} e^{-xy} \left\{ \frac{(-i)^m (2\pi/\lambda)^m x^{m+\rho}}{\Gamma(m + \rho + 1)} - \frac{(i)^{2k-m} (2\pi/\lambda)^{2k-m} x^{2k-m+\rho}}{\Gamma(2k - m + \rho + 1)} \right\} dx.$$

Interchanging of the integration and the finite sum is justified. Thus by simplification of the above that yields

$$L_4(y) = \sum_{m=k}^L C_m \left\{ \left(\frac{2\pi}{i\lambda y}\right)^m - (-1)^m \left(\frac{2\pi}{i\lambda y}\right)^{2k-m} \right\}.$$

To evaluate the expression in $L_5(y)$ where,

$$L_5(y) = \int_0^\infty {}_1F_1 \left(r, \rho + r + 1; \frac{-2\pi i \alpha_j x}{\lambda} \right) \frac{x^{\rho+r} y^{\rho+1} e^{-xy}}{\Gamma(r + \rho + 1)} dx.$$

We begin by substituting the series representation

$${}_1F_1\left(r, \rho + r + 1; \frac{-2\pi i \alpha_j x}{\lambda}\right) = \sum_{m=0}^{\infty} \frac{\Gamma(m+r)}{\Gamma(r)} \frac{\Gamma(\rho+r+1)}{\Gamma(m+r+\rho+1)} \frac{(-i2\pi x/\lambda)^m}{m!}$$

into the integrand. The interchange of integration and summation is justified for $\lambda y > 2\pi\alpha_j$. Hence, we obtained

$$L_5(y) = \sum_{m=0}^{\infty} \frac{\Gamma(m+r)}{\Gamma(r)} \frac{(-1)^m}{m!} \left(\frac{i\alpha_j 2\pi}{\lambda y}\right)^m y^{-r}.$$

Which converges absolutely for $\lambda y > 2\pi\alpha_j$. Similar to evaluation of L_5 we compute

$$L_6(y) = \int_0^{\infty} {}_1F_1\left(r, \rho + 2k + 1; \frac{-2\pi x}{i\alpha_j \lambda}\right) \frac{x^{\rho+2k} y^{\rho+1} e^{-xy}}{\Gamma(2k+\rho+1)} dx.$$

We obtain

$$L_6(y) = \sum_{m=0}^{\infty} \frac{\Gamma(m+r)}{\Gamma(r)} \frac{(-1)^m}{m!} \left(\frac{2\pi}{i\lambda y \alpha_j}\right)^m y^{-2k},$$

which converges absolutely for $y > \frac{2\pi}{\lambda \alpha_j}$. Combining the results of the integrals in $L_1(y), L_2(y), \dots, L_6(y)$ respectively we have

$$\begin{aligned} L_1(y) &= L_2(y) + L_3(y) - L_4(y) - \sum_{j=1}^p \sum_{r=1}^{M_j} C_{rj} \left(\frac{-1}{\alpha_j}\right)^r (i\alpha_j)^r \left(\frac{2\pi}{\lambda}\right)^r L_5(y) \\ &\quad + \sum_{j=1}^p \sum_{r=1}^{M_j} C_{rj} \left(\frac{-1}{\alpha_j}\right)^r (i)^{-2k} \left(\frac{2\pi}{\lambda}\right)^{2k} L_6(y). \end{aligned}$$

Thus applying simple algebraic calculation, we conclude that the identity in (3.1.20) implies

$$\begin{aligned} \sum_{m=0}^{\infty} a_m e^{-my} &= \left(\frac{2\pi}{iy\lambda}\right)^{2k} \sum_{m=0}^{\infty} a_m e^{-\frac{4\pi^2 m}{y\lambda^2}} + a_0 i^{2k} \left(\frac{2\pi}{\lambda y}\right)^{2k} - a_0, \\ &\quad - \sum_{m=k}^L C_m \left\{ \left(\frac{2\pi}{i\lambda y}\right)^m - (-1)^m \left(\frac{2\pi}{i\lambda y}\right)^{2k-m} \right\}, \\ &\quad - \sum_{j=1}^p \sum_{r=1}^{M_j} C_{rj} \left(\frac{-1}{\alpha_j}\right)^r \left\{ \alpha_j^r \left(\frac{2\pi i}{y\lambda}\right)^r \sum_{m=0}^{\infty} \frac{\Gamma(m+r)}{\Gamma(r)} \frac{(-1)^m}{m!} \left(\frac{2\pi i \alpha_j}{\lambda y}\right)^m \right. \\ &\quad \left. - \left(\frac{2\pi}{iy\lambda}\right)^{2k} \sum_{m=0}^{\infty} \frac{\Gamma(m+r)}{\Gamma(r)} \frac{(-1)^m}{m!} \left(\frac{2\pi}{i\alpha_j \lambda y}\right)^m \right\}, \end{aligned} \quad (3.1.23)$$

provided that $\rho + 2k + \frac{1}{2} \geq 2\beta$ and $y > \max_{1 \leq j \leq p} \left\{ \frac{2\pi\alpha_j}{\lambda}, \frac{2\pi}{\alpha_j\lambda} \right\}$.

Recall that the rational periodic function in Lemma 2.1.

$$q(z) = \sum_{k \leq r \leq L} C_r f_r(z, 0) + \sum_{j=1}^p \sum_{r=1}^{M_j} C_{rj} f_r(z, \alpha_j),$$

where $f_r(z, 0) = z^{-r} - (-1)^r z^{-2k+r}$ and $f_r(z, \alpha_j) = (z - \alpha_j)^{-r} - (-1)^r \alpha_j^{-r} z^{-2k+r} \left(z + \frac{1}{\alpha_j} \right)^{-r}$.

Letting $z = \frac{iy\lambda}{2\pi}$ and applying the binomial expansion to $f_r(\frac{iy\lambda}{2\pi}, \alpha_j)$ after simplifying some steps we obtain

$$\begin{aligned} q\left(\frac{iy\lambda}{2\pi}\right) &= \sum_{r=1}^L C_r \left\{ \left(\frac{2\pi}{iy\lambda}\right)^r - (-1)^r \left(\frac{2\pi}{iy\lambda}\right)^{2k-r} \right\} \\ &\quad + \sum_{j=1}^p \sum_{r=1}^{M_j} C_{rj} \left(\frac{-1}{\alpha_j}\right)^r \left\{ \alpha_j^r \left(\frac{2\pi i}{y\lambda}\right)^r \sum_{m=0}^{\infty} \frac{\Gamma(m+r)}{\Gamma(r)} \frac{(-1)^m}{m!} \left(\frac{2\pi i\alpha_j}{\lambda y}\right)^m \right. \\ &\quad \left. - \left(\frac{2\pi}{iy\lambda}\right)^{2k} \sum_{m=0}^{\infty} \frac{\Gamma(m+r)}{\Gamma(r)} \frac{(-1)^m}{m!} \left(\frac{2\pi}{i\alpha_j\lambda y}\right)^m \right\}. \end{aligned} \quad (3.1.24)$$

Since $F(z)$ has a Fourier expansion of the form $\sum_{m=0}^{\infty} a_m e^{2\pi imz/\lambda}$ and $q(\frac{iy\lambda}{2\pi})$ is represented (3.1.24), (3.1.23) may be written as

$$F\left(\frac{iy\lambda}{2\pi}\right) = \left(\frac{iy\lambda}{2\pi}\right)^{-2k} F\left(\frac{-1}{\frac{iy\lambda}{2\pi}}\right) - q\left(\frac{iy\lambda}{2\pi}\right).$$

Hence by the identity theorem the automorphic transformation

$$z^{-2k} F\left(\frac{-1}{z}\right) = F(z) + q(z),$$

follows for $z \in \mathcal{H}$. This concludes the proof of the equivalence of the functional equation to the identity (3.1.20).

3.2 Second equivalence

Theorem 3.7. *Let $\Phi(s)$ and $R(s)$ as in Theorem 2.3 then the functional equation*

$$\Phi(2k - s) - i^{2k}\Phi(s) = R(s) \quad (3.2.1)$$

is equivalent to the identity

$$\begin{aligned} & \left(\frac{-1}{y} \frac{d}{dy}\right)^\rho \left(\frac{1}{y} \sum_{m=1}^{\infty} a_m e^{-y\sqrt{m}}\right) = -\frac{2^\rho}{\sqrt{\pi}y^{2\rho+1}} a_0 \Gamma\left(\rho + \frac{1}{2}\right) \\ & + \frac{i^{-2k}}{\sqrt{\pi}} 2^{4k+2\rho+1} \Gamma\left(2k + \rho + \frac{1}{2}\right) \left(\frac{2\pi}{\lambda}\right)^{2k} \sum_{m=0}^{\infty} \frac{a_m}{\left(y^2 + 4\left(\frac{2\pi}{\lambda}\right)^2 m\right)^{2k+\rho+\frac{1}{2}}} \\ & - \frac{2^\rho}{\sqrt{\pi}y^{2\rho+1}} \sum_{j=1}^p \sum_{r=1}^{M_j} C_{rj} \left(\frac{-1}{\alpha_j}\right)^r \Gamma\left(r + \rho + \frac{1}{2}\right) \Psi\left(r, -\rho + \frac{1}{2}; \frac{\lambda y^2}{8\pi i \alpha_j}\right) \\ & + \frac{2^\rho}{\sqrt{\pi}y^{2\rho+1}} \sum_{j=1}^p \sum_{r=1}^{M_j} C_{rj} \left(\frac{-1}{\alpha_j}\right)^r \left(\frac{i\alpha_j \lambda y^2}{8\pi}\right)^{r-2k} \alpha_j^{2k} \Gamma\left(2k + \rho + \frac{1}{2}\right) \\ & \times \Psi\left(r, r - 2k - \rho + \frac{1}{2}; \frac{i\alpha_j \lambda y^2}{8\pi}\right) \\ & - \frac{2^\rho}{\sqrt{\pi}y^{2\rho+1}} \sum_{m=k}^L \left\{ \left(\frac{8\pi i}{\lambda y^2}\right)^{2k-m} \Gamma(2k - m + \rho + 1) - \left(\frac{-8\pi i}{\lambda y^2}\right)^m \Gamma\left(m + \rho + \frac{1}{2}\right) \right\}. \end{aligned} \quad (3.2.2)$$

Provided that $\operatorname{Re}(y) > 0$, $\rho \in \mathbb{Z}$, $\rho \geq 0$ and $\rho + 2k \geq \beta + \frac{1}{2}$, where $\sum_{m=1}^{\infty} \frac{|a_m|}{m^\beta} < \infty$.

To prove Theorem 3.7 we note that since the identity in (3.1.2) is equivalent to the functional equation (3.2.1) (by Theorem 3.1), it suffices to show that the functional equation (3.2.1) implies the identity in (3.2.2) and that (3.2.2) in turn implies the identity (3.1.2). We begin by proving the first implication.

Proof. To show (3.2.1) implies (3.2.2), let $\varphi(s) = \sum_{n=1}^{\infty} \frac{a_n}{n^s}$ with $\sum_{n=1}^{\infty} \frac{|a_n|}{n^\beta} < \infty$.

Then for $\rho \geq 0$, $\delta > 0$, $\delta \geq \beta$ by Lemma 3.2 we have

$$\frac{1}{\Gamma(\rho + 1)} \sum'_{0 \leq m \leq x} a_m (x - m)^\rho = \frac{1}{2\pi i} \int_{\delta - i\infty}^{\delta + i\infty} \frac{\Gamma(s)\varphi(s)x^{s+\rho}}{\Gamma(s + \rho + 1)} ds. \quad (3.2.3)$$

If $x = m$ and $\rho = 0$, then the integral is defined by its primary formula. The dash on the summation is used to indicate that the term with $x = m$ must be

multiplied by $\frac{1}{2}$ in this case. As in Chandrasekharan and Narasimhan [[2] page 9], we multiply (3.2.3) by $e^{-y\sqrt{x}}x^{-\frac{1}{2}}$ and integrating relative to x from $x = 0$ to ∞ ; how ever, we further require $\delta > 2k$, where $\text{Res} = \delta$ is the vertical path of integration. Now choose $\delta = \beta + P$, where $P \in \mathbb{Z}$, and P is large enough to guarantee $\delta > 2k$ and $\delta \notin \mathbb{Z}$. Thus

$$\int_0^\infty e^{-y\sqrt{x}}x^{-\frac{1}{2}} \left\{ \frac{1}{\Gamma(\rho+1)} \sum_{0 \leq m \leq x} a_m(x-m)^\rho \right\} dx = \int_0^\infty e^{-y\sqrt{x}}x^{-\frac{1}{2}} \left\{ \frac{1}{2\pi i} \int_{(\delta)} \frac{\varphi(s)\Gamma(s)x^{s+\rho}}{\Gamma(s+\rho+1)} ds \right\} dx \quad (3.2.4)$$

with $\rho + 2k \geq \delta + \frac{1}{2}$, and $y \in \mathbb{R}^+$. We used the notation \int_δ for $\int_{\delta-i\infty}^{\delta+i\infty}$. Chandrasekharan and Narasimhanin [2] for $\rho + 2k + \frac{1}{2} \geq \beta$ and λ_n sequence of positive real numbers; $\lambda_n \rightarrow \infty$, showed the identity

$$\sum_{n=1}^\infty a_n \int_{\lambda_n}^\infty \frac{(x-\lambda_n)^\rho}{\Gamma(\rho+1)} e^{-y\sqrt{x}}x^{-\frac{1}{2}} dx = 2(-2)^\rho \left(\frac{1}{y} \frac{d}{dy} \right)^\rho \left[\frac{1}{y} \sum_{n=1}^\infty a_n e^{-y\sqrt{\lambda_n}} \right]. \quad (3.2.5)$$

Then by the identity in (3.2.5) with $\lambda_n = m$ we have

$$\sum_{m=1}^\infty a_m \int_m^\infty \frac{(x-m)^\rho}{\Gamma(\rho+1)} e^{-y\sqrt{x}}x^{-\frac{1}{2}} dx = 2(-2)^\rho \left(\frac{1}{y} \frac{d}{dy} \right)^\rho \left[\frac{1}{y} \sum_{m=1}^\infty a_m e^{-y\sqrt{m}} \right].$$

Now assuming that $\Phi(s) = \Gamma(s)\varphi(s) \left(\frac{2\pi}{\lambda}\right)^{-s}$, where $\text{Res} > \beta$ and $\sum_{m=1}^\infty \frac{|a_m|}{m^\beta} < \infty$, write the right-hand side of (3.2.3) as

$$U(s, \lambda) = \int_0^\infty e^{-y\sqrt{x}}x^{-\frac{1}{2}} \left\{ \frac{1}{2\pi i} \int_{(\delta)} \left(\frac{2\pi}{\lambda}\right)^s \frac{\Phi(s)x^{s+\rho}}{\Gamma(s+\rho+1)} ds \right\} dx,$$

for $\delta \geq \beta$. The order of integration may be interchange for $\rho \geq 0$. Then

$$\begin{aligned} U(s, \lambda) &= \frac{1}{2\pi i} \int_{(\delta)} \left(\frac{2\pi}{\lambda}\right)^s \frac{\Phi(s)}{\Gamma(s+\rho+1)} ds \int_0^\infty e^{-y\sqrt{x}}x^{s+\rho-\frac{1}{2}} dx, \\ &= \frac{2}{2\pi i} \int_{\delta-i\infty}^{\delta+i\infty} \left(\frac{2\pi}{\lambda}\right)^s \Phi(s) \frac{\Gamma(2s+2\rho+1)}{\Gamma(s+\rho+1)} \frac{1}{y^{2s+2\rho+1}} ds. \end{aligned}$$

Applying the formula (1.3.4) for $\Gamma(2s+2\rho+1)$ we have

$$U(s, \lambda) = \frac{1}{2\pi i} \int_{(\delta)} \left(\frac{2\pi}{\lambda}\right)^s \Phi(s) \frac{\Gamma(s+\rho+\frac{1}{2})}{\sqrt{\pi}} \frac{2^{2s+2\rho+1}}{y^{2s+2\rho+1}} ds.$$

To evaluate $U(s, \lambda)$ consider a rectangular region $[2k - \delta - iT, \delta - iT], [\delta - iT, \delta + iT], [\delta + iT, 2k - \delta + iT], [2k - \delta + iT, 2k - \delta - iT]$ oriented positively. $\text{Re}(s) = 2k - \delta$. If $\rho + 2k \geq \delta + \frac{1}{2}$ the poles of $\Gamma(s + \rho + \frac{1}{2})$ lie out side of the interval $[2k - \delta, \delta]$, then the poles of the integrand between $\text{Re}(s) = 2k - \delta$ and $\text{Re}(s) = \delta$ arise from $\Phi(s)$. The integral along the horizontal segments tends to zero as $T \rightarrow \infty$. Thus $U(s, \lambda)$ may be written as

$$U(s, \lambda) = \frac{1}{2\pi i} \int_{(2k-\delta)} \left(\frac{2\pi}{\lambda}\right)^s \Phi(s) \frac{\Gamma(s + \rho + \frac{1}{2}) 2^{2s+2\rho+1}}{\sqrt{\pi} y^{2s+2\rho+1}} ds + \sum_{s \in \text{polset}\Phi(s)} \text{Res} \left(\frac{2\pi}{\lambda}\right)^s \Phi(s) \frac{\Gamma(s + \rho + \frac{1}{2}) 2^{2s+2\rho+1}}{\sqrt{\pi} y^{2s+2\rho+1}}. \quad (3.2.6)$$

Write $U(s, \lambda)$ as $U(s, \lambda) = T_1(s, \lambda) + T_2(s, \lambda)$, where

$$T_1(s, \lambda) = \frac{1}{2\pi i} \int_{(2k-\delta)} \left(\frac{2\pi}{\lambda}\right)^s \Phi(s) \frac{\Gamma(s + \rho + \frac{1}{2}) 2^{2s+2\rho+1}}{\sqrt{\pi} y^{2s+2\rho+1}} ds,$$

and

$$T_2(s, \lambda) = \sum_{s \in \text{polset}\Phi(s)} \text{Res} \left(\frac{2\pi}{\lambda}\right)^s \Phi(s) \frac{\Gamma(s + \rho + \frac{1}{2}) 2^{2s+2\rho+1}}{\sqrt{\pi} y^{2s+2\rho+1}}. \quad (3.2.7)$$

From the functional equation (3.2.2) we have $\Phi(s) = i^{-2k} \left\{ \Phi(2k - s) - R(s) \right\}$, and substituting in to the integrand of $T_1(s, \lambda)$, we rewrite

$$T_1(s, \lambda) = V_1(s, \lambda) - V_2(s, \lambda),$$

where

$$V_1(s, \lambda) = \frac{1}{2\pi i} \int_{(2k-\delta)} \left(\frac{2\pi}{\lambda}\right)^s i^{-2k} \Phi(2k - s) \frac{\Gamma(s + \rho + \frac{1}{2}) 2^{2s+2\rho+1}}{\sqrt{\pi} y^{2s+2\rho+1}} ds,$$

and

$$V_2(s, \lambda) = \frac{1}{2\pi i} \int_{(2k-\delta)} \left(\frac{2\pi}{\lambda}\right)^s i^{-2k} R(s) \frac{\Gamma(s + \rho + \frac{1}{2}) 2^{2s+2\rho+1}}{\sqrt{\pi} y^{2s+2\rho+1}} ds.$$

To evaluate $V_1(s, \lambda)$ take $\vartheta = 2k - s$ and $\Phi(\vartheta) = \left(\frac{2\pi}{\lambda}\right)^{-\vartheta} \Gamma(\vartheta) \varphi(\vartheta)$, where $\varphi(\vartheta) = \sum_{m=1}^{\infty} \frac{a_m}{m^\vartheta}$, replace ϑ by $-\vartheta$. Then using the integral formula in (1.3.2) and employing simple algebraic manipulation we obtain

$$V_1(s, \lambda) = \frac{i^{-2k}}{\sqrt{\pi}} 2^{4k+2\rho+1} \Gamma\left(2k + \rho + \frac{1}{2}\right) \left(\frac{2\pi}{\lambda}\right)^{2k} \sum_{m=1}^{\infty} \frac{a_m}{\left(y^2 + \left(\frac{4\pi}{\lambda}\right)^2 m\right)^{2k+\rho+\frac{1}{2}}}.$$

The series converges for $y > 0$, provided $2k + \rho - \frac{1}{2} > \beta$. Since $\delta \geq \beta$ the series converges absolutely for $\rho \geq \delta - 2k + \frac{1}{2}$.

The evaluation of $V_2(s, \lambda)$ proceeds by substituting for $R(s)$ and rewriting the Beta function in terms of Γ function . Thus $V_2(s, \lambda)$ may be written as

$$V_2(s, \lambda) = \frac{1}{\sqrt{2\pi}} \frac{2^{2\rho+1}}{y^{2\rho+1}} \sum_{j=1}^p \sum_{r=1}^{M_j} C_{rj} \left(\frac{-1}{\alpha_j} \right)^r \left\{ Q_1(s, \lambda) - \alpha_j^{2k} Q_2(s, \lambda) \right\}, \quad (3.2.8)$$

where

$$Q_1(s, \lambda) = \frac{1}{2\pi i} \int_{(2k-\delta)} \left(\frac{8\pi i \alpha_j}{\lambda y^2} \right)^s \frac{\Gamma(s)\Gamma(r-s)}{\Gamma(r)} \Gamma\left(s + \rho + \frac{1}{2}\right) ds$$

and

$$Q_2(s, \lambda) = \frac{1}{2\pi i} \int_{(2k-\delta)} \left(\frac{8\pi}{i\lambda y^2 \alpha_j} \right)^s \frac{\Gamma(2k-s)\Gamma(r+s-2k)}{\Gamma(r)} \Gamma\left(s + \rho + \frac{1}{2}\right) ds.$$

To evaluate $Q_1(s, \lambda)$ replace s by $-s$ and applying the Cauchy's Residue Theorem on the rectangular region $[2k - \delta - iT, N + \frac{1}{4} - iT]$, $[N + \frac{1}{4} - iT, N + \frac{1}{4} + iT]$, $[N + \frac{1}{4} + iT, 2k - \delta + iT]$, $[2k - \delta + iT, 2k - \delta - iT]$ with positive orientation for large integer N and positive number T . The poles of the integrand arising from the gamma functions of $\Gamma(\rho + \frac{1}{2} - s)$ and $\Gamma(-s)$. The integral along the horizontal paths tends to 0 as $T \rightarrow \infty$. Then we obtained

$$\begin{aligned} & \frac{1}{2\pi i} \int_{(2k-\delta)} \left(\frac{8\pi i \alpha_j}{\lambda y^2} \right)^{-s} \frac{\Gamma(-s)\Gamma(r+s)}{\Gamma(r)} \Gamma\left(\rho + \frac{1}{2} - s\right) ds \\ &= \frac{1}{2\pi i} \int_{(N+\frac{1}{4})} \left(\frac{8\pi i \alpha_j}{\lambda y^2} \right)^{-s} \frac{\Gamma(-s)\Gamma(r+s)}{\Gamma(r)} \Gamma\left(\rho + \frac{1}{2} - s\right) ds \\ &+ \sum_{s \in \text{poleset}} \text{Res} \left\{ \left(\frac{8\pi i \alpha_j}{\lambda y^2} \right)^{-s} \frac{\Gamma(-s)\Gamma(r+s)}{\Gamma(r)} \Gamma\left(\rho + \frac{1}{2} - s\right) \right\}. \end{aligned}$$

Claim:

$$\frac{1}{2\pi i} \int_{(N+\frac{1}{4})} \left(\frac{8\pi i \alpha_j}{\lambda y^2} \right)^{-s} \frac{\Gamma(-s)\Gamma(r+s)}{\Gamma(r)} \Gamma\left(\rho + \frac{1}{2} - s\right) ds \rightarrow 0,$$

as $N \rightarrow \infty$. To proof the claim split $(-\infty, \infty)$ in to the intervals $(-\infty, -T]$, $[-T, T]$, $[T, \infty)$, for some large value of T . Let

$$M(s, \lambda) = \frac{1}{2\pi i} \int_{N+\frac{1}{4}+iT}^{N+\frac{1}{4}+i\infty} \left(\frac{8\pi i \alpha_j}{\lambda y^2} \right)^{-s} \frac{\Gamma(-s)\Gamma(r+s)}{\Gamma(r)} \Gamma\left(\rho + \frac{1}{2} - s\right) ds$$

the integral on $[T, \infty)$. Then by Stirling's formula we have

$$\begin{aligned} |M(s, \lambda)| &\leq \left(\frac{y^2\lambda}{8\pi\alpha_j}\right)^{\frac{1}{4}} \frac{2\pi\sqrt{2\pi}}{\Gamma(r)} \int_T^\infty \left(\frac{y^2\lambda}{8\pi\alpha_j}\right)^N |t|^{-N+\rho+r-\frac{3}{4}} e^{-\frac{3}{2}\pi t} dt \\ &\leq \left(\frac{y^2\lambda}{8\pi\alpha_j}\right)^{\frac{1}{4}} \frac{2\pi\sqrt{2\pi}}{\Gamma(r)} \left(\frac{y^2\lambda}{8\pi\alpha_j}\right)^N |T|^{-N+\rho+r-\frac{3}{4}} \int_T^\infty e^{-\frac{3}{2}\pi t} dt \\ &= \left(\frac{y^2\lambda}{8\pi\alpha_j}\right)^{\frac{1}{4}} \frac{4\pi\sqrt{2\pi}}{3\Gamma(r)} |T|^{\rho+r-\frac{3}{4}} e^{-\frac{3}{2}\pi|T|} \left(\frac{y^2\lambda}{8\pi\alpha_j T}\right)^N. \end{aligned}$$

By choosing for appropriate large T , we see that the last expression tends to zero, as $N \rightarrow \infty$. A similar approach can be applied along the path $\text{Res} = N + \frac{1}{4}$, for $\text{Im } s \in (-\infty, -T]$. Therefore by computing the residue of the poles we have

$$\begin{aligned} Q_1(s, \lambda) &= \sum_{[\delta]-2k+1}^{\infty} \left(\frac{8\pi i\alpha_j}{\lambda y^2}\right)^{-m} \frac{(-1)^m \Gamma(m+r)}{m! \Gamma(r)} \Gamma\left(\rho - m + \frac{1}{2}\right) \\ &\quad + \sum_{m=0}^{\infty} \left(\frac{8\pi i\alpha_j}{\lambda y^2}\right)^{-m-\rho-\frac{1}{2}} \frac{(-1)^m \Gamma(m+\rho+r+\frac{1}{2})}{m! \Gamma(r)} \Gamma\left(-\rho - m - \frac{1}{2}\right). \end{aligned}$$

Applying formula (1.3.5) and by simplifying, we obtain

$$\begin{aligned} Q_1(s, \lambda) &= \sum_{m=0}^{\infty} \left(\frac{8\pi i\alpha_j}{\lambda y^2}\right)^{-m} \frac{(-1)^m \Gamma(m+r)}{m! \Gamma(r)} (-1)^m \frac{\Gamma\left(\rho + \frac{1}{2}\right) \Gamma\left(-\rho + \frac{1}{2}\right)}{\Gamma\left(m - \rho + \frac{1}{2}\right)} \\ &\quad - \sum_{m=0}^{[\delta]-2k} \left(\frac{8\pi i\alpha_j}{\lambda y^2}\right)^{-m} \frac{(-1)^m \Gamma(m+r)}{m! \Gamma(r)} \Gamma\left(\rho - m + \frac{1}{2}\right) \\ &\quad + \sum_{m=0}^{\infty} \left(\frac{8\pi i\alpha_j}{\lambda y^2}\right)^{-m-\rho-\frac{1}{2}} \frac{(-1)^m}{m!} (-1)^m \frac{\Gamma\left(-\rho - \frac{1}{2}\right) \Gamma\left(\rho + \frac{3}{2}\right) \Gamma(m+\rho+r+\frac{1}{2})}{\Gamma\left(m+\rho+\frac{3}{2}\right) \Gamma(r)}. \end{aligned}$$

By rewriting the series representation for the confluent hypergeometric function of the form ${}_1F_1(a, b; z)$. We may rewrite $Q_1(s, \lambda)$ as

$$\begin{aligned} Q_1(s, \lambda) &= \Gamma\left(\rho + \frac{1}{2}\right) {}_1F_1\left(r, -\rho + \frac{1}{2}; \frac{\lambda y^2}{8\pi i\alpha_j}\right) \\ &\quad + \frac{\Gamma\left(\rho + r + \frac{1}{2}\right) \Gamma\left(-\rho - \frac{1}{2}\right)}{\Gamma(r)} \left(\frac{\lambda y^2}{8\pi i\alpha_j}\right)^{\rho+\frac{1}{2}} {}_1F_1\left(\rho + r + \frac{1}{2}, \rho + \frac{3}{2}; \frac{\lambda y^2}{8\pi i\alpha_j}\right) \\ &\quad - \sum_{m=0}^{[\delta]-2k} \left(\frac{8\pi i\alpha_j}{y^2\lambda}\right)^{-m} \frac{(-1)^m \Gamma(m+r)}{m! \Gamma(r)} \Gamma\left(\rho - m + \frac{1}{2}\right). \end{aligned}$$

Using the definition of the confluent hypergeometric function of second kind given in (1.3.10) we have

$$Q_1(y, \lambda) = \Gamma\left(\rho + r + \frac{1}{2}\right) \Psi\left(r, -\rho + \frac{1}{2}; \frac{\lambda y^2}{8\pi i \alpha_j}\right) - \sum_{m=0}^{[\delta]-2k} \left(\frac{8\pi i \alpha_j}{y^2 \lambda}\right)^{-m} \frac{(-1)^m \Gamma(m+r)}{m! \Gamma(r)} \Gamma\left(\rho - m + \frac{1}{2}\right). \quad (3.2.9)$$

To evaluate $Q_2(s, \lambda)$ applying the same analysis as $Q_1(y, \lambda)$ to the integral

$$Q_2(s, \lambda) = \frac{1}{2\pi i} \int_{(2k-\delta)} \left(\frac{8\pi}{i\lambda y^2 \alpha_j}\right)^s \frac{\Gamma(2k-s)\Gamma(r+s-2k)}{\Gamma(r)} \Gamma\left(s + \rho + \frac{1}{2}\right) ds$$

and we deduced

$$\begin{aligned} Q_2(s, \lambda) &= \sum_{m=0}^{\infty} \left(\frac{8\pi}{i\lambda y^2 \alpha_j}\right)^{-m-\rho-\frac{1}{2}} \frac{(-1)^m \Gamma(2k+m+\rho+\frac{1}{2})\Gamma(r-2k-m-\rho-\frac{1}{2})}{m! \Gamma(r)} \\ &+ \sum_{m=[\delta]-r+1}^{\infty} \left(\frac{8\pi}{i\lambda y^2 \alpha_j}\right)^{2k-m-r} \frac{(-1)^m \Gamma(2k-m-r+\rho+\frac{1}{2})\Gamma(m_r)}{m! \Gamma(r)} \\ &= \sum_{m=0}^{\infty} \left(\frac{8\pi}{i\lambda y^2 \alpha_j}\right)^{-m-\rho-\frac{1}{2}} \frac{(-1)^m \Gamma(2k+m+\rho+\frac{1}{2})\Gamma(r-2k-m-\rho-\frac{1}{2})}{m! \Gamma(r)} \\ &+ \sum_{m=0}^{\infty} \left(\frac{8\pi}{i\lambda y^2 \alpha_j}\right)^{2k-m-r} \frac{(-1)^m \Gamma(2k-m-r+\rho+\frac{1}{2})\Gamma(m+r)}{m! \Gamma(r)} \\ &- \sum_{m=0}^{[\delta]-r} \left(\frac{8\pi}{i\lambda y^2 \alpha_j}\right)^{2k-m-r} \frac{(-1)^m \Gamma(2k-m-r+\rho+\frac{1}{2})\Gamma(m+r)}{m! \Gamma(r)}. \end{aligned}$$

Then applying the formula in (1.3.4) and simplifying, we obtain

$$\begin{aligned} Q_2(s, \lambda) &= \left(\frac{8\pi}{i\alpha_j \lambda y^2}\right)^{-\rho-\frac{1}{2}} \frac{\Gamma(r-2k-\rho-\frac{1}{2})\Gamma(2k+\rho+\frac{1}{2})}{\Gamma(r)} \\ &\times {}_1F_1\left(2k+\rho+\frac{1}{2}, 2k+\rho+\frac{3}{2}-r; \frac{i\alpha_j \lambda y^2}{8\pi}\right) \\ &+ \left(\frac{8\pi}{i\alpha_j \lambda y^2}\right)^{2k-r} \Gamma\left(2k+\rho+\frac{1}{2}-r\right) \times {}_1F_1\left(r, r-\rho-2k+\frac{1}{2}; \frac{i\alpha_j \lambda y^2}{8\pi}\right) \\ &- \sum_{m=0}^{[\delta]-r} \left(\frac{8\pi}{i\lambda y^2 \alpha_j}\right)^{2k-m-r} \frac{(-1)^m \Gamma(2k-m-r+\rho+\frac{1}{2})\Gamma(m+r)}{m! \Gamma(r)}. \end{aligned}$$

Again by the definition of confluent hypergeometric function of second kind in

(1.3.10) we have

$$\begin{aligned}
 Q_2(s, \lambda) &= \Gamma\left(2k + \rho + \frac{1}{2}\right) \left(\frac{i\alpha_j \lambda y^2}{8\pi}\right)^{r-2k} \Psi\left(r, r - \rho - 2k + \frac{1}{2}; \frac{i\alpha_j \lambda y^2}{8\pi}\right) \\
 &\quad - \sum_{m=0}^{[\delta]-r} \left(\frac{8\pi}{i\lambda y^2 \alpha_j}\right)^{2k-m-r} \frac{(-1)^m \Gamma(m+r)}{m!} \frac{\Gamma(m+r)}{\Gamma(r)} \Gamma\left(-m - r + \rho + \frac{1}{2}\right).
 \end{aligned} \tag{3.2.10}$$

Thus for $\rho \geq 0, \rho \in \mathbb{Z}, \rho + 2k \geq \delta + \frac{1}{2}, \delta > 2k$ and $y \in \mathbb{R}^+$ the functional equation in (3.2.1) implies

$$\begin{aligned}
 &\frac{2^{2\rho+1}}{\sqrt{\pi} y^{2\rho+1}} a_o \Gamma\left(\rho + \frac{1}{2}\right) + 2^{\rho+1} \left(-\frac{1}{y} \frac{d}{dy}\right)^\rho \left[\frac{1}{y} \sum_{m=1}^{\infty} a_m e^{-y\sqrt{m}}\right] = U(\lambda, s) \\
 &= T_1(s, \lambda) + T_2(s, \lambda) \\
 &= V_1(s, \lambda) - V_2(s, \lambda) + T_2(s, \lambda) \\
 &= \frac{i^{-2k}}{\sqrt{\pi}} 2^{4k+2\rho+1} \Gamma\left(2k + \rho + \frac{1}{2}\right) \left(\frac{2\pi}{\lambda}\right)^{2k} \sum_{m=1}^{\infty} \frac{a_m}{\left(y^2 + \left(\frac{4\pi}{\lambda}\right)^2 m\right)^{2k+\rho+\frac{1}{2}}} - V_2(s, \lambda) + T_2(s, \lambda).
 \end{aligned}$$

Hence

$$\begin{aligned}
 &2^{\rho+1} \left(-\frac{1}{y} \frac{d}{dy}\right)^\rho \left[\frac{1}{y} \sum_{m=1}^{\infty} a_m e^{-y\sqrt{m}}\right] = \\
 &\frac{i^{-2k}}{\sqrt{\pi}} 2^{4k+2\rho+1} \Gamma\left(2k + \rho + \frac{1}{2}\right) \left(\frac{2\pi}{\lambda}\right)^{2k} \sum_{m=1}^{\infty} \frac{a_m}{\left(y^2 + \left(\frac{4\pi}{\lambda}\right)^2 m\right)^{2k+\rho+\frac{1}{2}}} \\
 &\quad - V_2(s, \lambda) + T_2(s, \lambda) - \frac{2^{2\rho+1}}{\sqrt{\pi} y^{2\rho+1}} a_o \Gamma\left(\rho + \frac{1}{2}\right),
 \end{aligned} \tag{3.2.11}$$

where $V_2(s, \lambda)$ and $T_2(s, \lambda)$ given by (3.2.8) and (3.2.7) respectively. From (3.2.9) and (3.2.10), we obtain an explicit formula for $V_2(s, \lambda)$.

For explicit formula of $T_2(s, \lambda)$, we start by recalling that

$$T_2(s, \lambda) = \sum_{s \in \text{pole set of } \Phi(s)} \text{Res} \left\{ \left(\frac{2\pi}{\lambda}\right)^s \frac{2^{2s+2\rho+1} \Gamma\left(s + \rho + \frac{1}{2}\right)}{\sqrt{\pi} y^{2s+2\rho+1}} \Phi(s) \right\}.$$

Since $\rho + 2k \geq \delta + \frac{1}{2}$ the function $\left(\frac{2\pi}{\lambda}\right)^s \frac{2^{2s+2\rho+1} \Gamma\left(s + \rho + \frac{1}{2}\right)}{\sqrt{\pi} y^{2s+2\rho+1}}$ is analytic in the region bounded by the vertical lines $\text{Res} = 2k - \delta$ and $\text{Res} = \delta$. Recall that in Theorem (2.3) $\Phi(s) = D(s) + D^0(s) + E^0(s) + E^H(s) + E^B(s)$, where $D(s), D^0(s), E^0(s), E^H(s)$, and $E^B(s)$ are functions in (2.0.10), (2.0.11), (2.0.12), (2.0.13)

and (2.0.14) respectively. We use the residues of $\Phi(s)$ are the formulas in Corollary (2.4)[see (2.0.16), (2.0.17), (2.0.18)] and (2.0.19). Therefore, after simplifying we obtain

$$\begin{aligned}
 T_2(s, \lambda) = & \frac{2^{2\rho+1}}{\sqrt{\pi}y^{2\rho+1}} \left\{ \left(\frac{8\pi i}{\lambda y^2} \right)^{2k} \Gamma \left(2k + \rho + \frac{1}{2} \right) - a_0 \Gamma \left(\rho + \frac{1}{2} \right) \right. \\
 & + \sum_{m=k}^L C_m \left\{ \left(\frac{8\pi i}{\lambda y^2} \right)^{2k-m} \Gamma \left(2k - m + \rho + \frac{1}{2} \right) - \left(-\frac{8\pi i}{\lambda y^2} \right)^m \Gamma \left(m + \rho + \frac{1}{2} \right) \right\} \\
 & + \sum_{j=1}^p \sum_{r=1}^{M_j} C_{rj} \left[\sum_{m=0}^{[\delta-r]} \frac{\Gamma(m+r)}{\Gamma(r)} \frac{(-1)^m}{m!} \left(\frac{8\pi}{\lambda y^2} \right)^{2k-m-r} i^{m+r-2k} \alpha_j^m \times \right. \\
 & \quad \Gamma \left(2k - m - r + \rho + \frac{1}{2} \right) \\
 & \left. - \sum_{m=0}^{[\delta-2k]} \frac{\Gamma(m+r)}{\Gamma(r)} \frac{(-1)^m}{m!} \left(\frac{\lambda y^2}{8\pi} \right)^m i^m \Gamma \left(-m + \rho + \frac{1}{2} \right) \alpha_j^{-m-r} \right] \left. \right\}. \tag{3.2.12}
 \end{aligned}$$

Now substitute (3.2.12) in to (3.2.11) for $T_2(s, \lambda)$ and by substituting the respective expressions in (3.2.9), (3.2.10) for $V_2(s, \lambda)$, cancellation of terms take place. Therefore. by rearranging and combining expressions after some simplification, we obtain the identity (3.2.2).

Proof of the converse

To prove the converse, we need to show that (3.2.2) implies the functional equation (3.2.1). However, in Theorem (3.1) we prove that the identity (3.1.2) is equivalent to the functional equation (3.2.1) provided that $\rho \geq 2\beta - 2k - \frac{1}{2}$. Thus it suffices to show that the identity (3.2.2) implies (3.1.2). Indeed multiply (3.1.2) by $e^{y\sqrt{x}}$, with $\text{Re} y > 0$ and $x > 0$ and integrate the expression along vertical path $\text{Res} = \vartheta$, where $\vartheta > 0$. The left hand side (3.2.2) can be evaluated using the formula stated by Chandrasekharan and Narasimhan in [[2],page 9]

$$\sum_{m=1}^{\infty} a_m \frac{1}{2\pi i} \int_{(\vartheta)} e^{y\sqrt{x}} \left(-\frac{1}{y} \frac{d}{dy} \right)^{\rho} \left[\frac{1}{y} e^{-y\sqrt{m}} \right] dy = \frac{1}{\Gamma(\rho+1)} \sum'_{m \leq x} a_m (x-m)^{\rho} 2^{-\rho}, \tag{3.2.13}$$

while the right hand side of (3.2.2) we calculate the integral of each terms one by one. So put

$$P(\lambda, y) = \frac{i^{-2k}}{\sqrt{\pi}} 2^{4k+\rho} \Gamma\left(2k + \rho + \frac{1}{2}\right) \left(\frac{2\pi}{\lambda}\right)^{2k} \sum_{m=1}^{\infty} a_m \frac{1}{2\pi i} \int_{(\vartheta)} \frac{e^{y\sqrt{x}}}{[y^2 + \left(\frac{4\pi}{\lambda}\right)^2 m]^{2k+\rho+\frac{1}{2}}} dy.$$

By applying (1.3.9) after simplification we obtained

$$P(\lambda, y) = i^{-2k} \left(\frac{4\pi}{\lambda}\right)^{-\rho} \sum_{m=1}^{\infty} a_m \left(\frac{x}{m}\right)^{\frac{2k+\rho}{2}} J_{2k+\rho}\left(\frac{4\pi\sqrt{mx}}{\lambda}\right). \quad (3.2.14)$$

To evaluate the next term consider the integral $I_1(\lambda, x)$, where

$$I_1(\lambda, y) = \frac{1}{2\pi i} \int_{(\vartheta)} \frac{e^{y\sqrt{x}}}{y^{2\rho+1}} \Psi\left(r, -\rho + \frac{1}{2}; \frac{\lambda y^2}{8\pi i \alpha_j}\right) dy.$$

Using the integral in (1.3.11) we have

$$\begin{aligned} I_1(\lambda, y) &= \frac{1}{2\pi i} \int_{(\vartheta)} \frac{e^{y\sqrt{x}}}{y^{2\rho+1}} \left\{ \frac{1}{2\pi i} \int_{(\theta)} \frac{\Gamma(r+\tau)\Gamma(-\tau)\Gamma(\rho+\frac{1}{2}-\tau)}{\Gamma(r)\Gamma(r+\rho+\frac{1}{2})} \left(\frac{\lambda y^2}{8\pi i \alpha_j}\right)^{\tau} d\tau \right\} dy \\ &= \frac{1}{2\pi i} \int_{(\theta)} \left\{ \frac{\Gamma(r+\tau)\Gamma(-\tau)\Gamma(\rho+\frac{1}{2}-\tau)}{\Gamma(r)\Gamma(r+\rho+\frac{1}{2})} \left(\frac{\lambda}{8\pi i \alpha_j}\right)^{\tau} \frac{1}{2\pi i} \int_{(\vartheta)} e^{y\sqrt{x}} y^{2\tau-2\rho-1} dy \right\} d\tau \\ &= \frac{1}{2\pi i} \int_{(\theta)} \frac{\Gamma(r+\tau)\Gamma(-\tau)\Gamma(\rho+\frac{1}{2}-\tau)}{\Gamma(r)\Gamma(r+\rho+\frac{1}{2})} \left(\frac{\lambda}{8\pi i \alpha_j}\right)^{\tau} \frac{x^{\rho-\tau}}{\Gamma(2\rho-2\tau+1)} d\tau. \end{aligned}$$

Applying the formula in (1.3.4) to $\Gamma(2\rho-2\tau+1)$ we have

$$I_1(\lambda, y) = \frac{\sqrt{\pi} x^{\rho}}{2^{2\rho} \Gamma(r+\frac{1}{2}+\rho)\Gamma(r)} \left[\frac{1}{2\pi i} \int_{(\theta)} \frac{\Gamma(r+\tau)\Gamma(-\tau)}{\Gamma(\rho+1-\tau)} \left(\frac{\lambda}{2\pi i \alpha_j}\right)^{\tau} d\tau \right].$$

Let

$$M_1(\lambda, y) = \frac{1}{2\pi i} \int_{(\theta)} \frac{\Gamma(r+\tau)\Gamma(-\tau)}{\Gamma(\rho+1-\tau)} \left(\frac{\lambda}{2\pi i \alpha_j x}\right)^{\tau} d\tau.$$

The integral $M_1(\lambda, y)$ can be evaluated using Cauchy's Residue theorem on a rectangular region;

$[-N-\frac{1}{4}-iT, \theta-iT], [\theta-iT, \theta+iT], [\theta+iT, -N-\frac{1}{4}+iT], [-N-\frac{1}{4}+iT, -N-\frac{1}{4}-iT]$ positively oriented for N large integer. The integrals on the horizontal paths tends to zero as $T \rightarrow \infty$. Thus

$$\begin{aligned} M_1(y, \lambda) &= \sum \text{Res} \left\{ \frac{\Gamma(r+\tau)\Gamma(-\tau)}{\Gamma(\rho+1-\tau)} \left(\frac{\lambda}{2\pi i \alpha_j x}\right)^{\tau} \right\} \\ &\quad + \frac{1}{2\pi i} \int_{(-N-\frac{1}{4})} \frac{\Gamma(r+\tau)\Gamma(-\tau)}{\Gamma(\rho+1-\tau)} \left(\frac{\lambda}{2\pi i \alpha_j x}\right)^{\tau} d\tau, \end{aligned}$$

as $N \rightarrow \infty$, the integral along the path of $\text{Re}\tau = -N - \frac{1}{4}$ can be shown to tend to 0 by estimating the growth of the integrand. Hence by computing the residue we have

$$M_1(y, \lambda) = \sum_{m=0}^{\infty} \frac{(-1)^m}{m!} \frac{\Gamma(m+r)}{\Gamma(\rho+m+r+1)} \left(\frac{2\pi i \alpha_j x}{\lambda} \right)^{m+r}.$$

Hence

$$I_1(y, \lambda) = \frac{\sqrt{\pi} x^\rho}{2^{2\rho} \Gamma(r + \frac{1}{2} + \rho) \Gamma(r)} \sum_{m=0}^{\infty} \frac{(-1)^m}{m!} \frac{\Gamma(m+r)}{\Gamma(\rho+m+r+1)} \left(\frac{2\pi i \alpha_j x}{\lambda} \right)^{m+r}.$$

Therefore, we conclude that

$$\begin{aligned} & \frac{1}{2\pi i} \int_{(\vartheta)} \frac{e^{y\sqrt{x}}}{\sqrt{\pi} y^{2\rho+1}} 2^\rho \sum_{j=1}^p \sum_{r=1}^{M_j} C_{ri} \left(\frac{-1}{\alpha_j} \right)^r \Gamma(\rho+r+\frac{1}{2}) \Psi \left(r, -\rho+\frac{1}{2}; \frac{\lambda y^2}{8\pi i \alpha_j} \right) dy \\ &= \frac{2^\rho}{\sqrt{\pi}} \sum_{j=1}^p \sum_{r=1}^{M_j} C_{ri} \left(\frac{-1}{\alpha_j} \right)^r \Gamma(\rho+r+\frac{1}{2}) \frac{1}{2\pi i} \int_{(\vartheta)} \frac{e^{y\sqrt{x}}}{y^{2\rho+1}} \Psi \left(r, -\rho+\frac{1}{2}; \frac{\lambda y^2}{8\pi i \alpha_j} \right) dy \\ &= \frac{1}{2^\rho} \sum_{j=1}^p \sum_{r=1}^{M_j} C_{rj} \left(\frac{-1}{\alpha_j} \right)^r \left(\frac{2\pi i \alpha_j}{\lambda} \right)^r \frac{x^{\rho+r}}{\Gamma(\rho+r+1)} {}_1F_1 \left(r, r+\rho+1; \frac{-2\pi i \alpha_j x}{\lambda} \right). \end{aligned} \tag{3.2.15}$$

Next we analyze the integral $I_2(y, \lambda)$, where

$$I_2(y, \lambda) = \frac{1}{2\pi i} \int_{(\vartheta)} \frac{e^{y\sqrt{x}}}{y^{2\rho+1}} \left(\frac{\alpha_j i \lambda y^2}{8\pi} \right)^{r-2k} \Psi \left(r, r-2k-\rho+\frac{1}{2}; \frac{i \alpha_j \lambda^2}{8\pi} \right) dy.$$

For fixed r , $-r < \theta < -r + \frac{1}{2}$ and $-\frac{\pi}{2} \leq \arg \left(\frac{i \alpha_j \lambda^2}{8\pi} \right)$, applying the integral representation of confluent hypergeometric function of the second kind in (1.3.11) we have

$$\begin{aligned} I_2(y, \lambda) &= \frac{1}{2\pi i} \int_{(\vartheta)} \frac{e^{y\sqrt{x}}}{y^{2\rho+1}} \left(\frac{\alpha_j i \lambda y^2}{8\pi} \right)^{r-2k} \left\{ \frac{1}{2\pi i} \int_{(\theta)} \frac{\Gamma(r+\tau) \Gamma(-\tau) \Gamma(2k+\rho-\tau-r+\frac{1}{2})}{\Gamma(r) \Gamma(2k+\frac{1}{2}+\rho)} \right. \\ &\quad \left. \times \left(\frac{\alpha_j i \lambda y^2}{8\pi} \right)^\tau d\tau \right\} dy. \end{aligned}$$

By interchanging the order of integration we obtain the equation

$$I_2(y, \lambda) = \frac{1}{2\pi i} \int_{(\theta)} \left[\frac{\Gamma(r+\tau) \Gamma(-\tau) \Gamma(2k+\rho-\tau-r+\frac{1}{2})}{\Gamma(r) \Gamma(2k+\frac{1}{2}+\rho)} \left(\frac{\alpha_j i \lambda}{8\pi} \right)^{\tau+r-2k} \right]$$

$$\times \frac{1}{2\pi i} \int_{(\vartheta)} e^{y\sqrt{x}} y^{2\tau+2r-4k-2\rho-1} dy \Big] d\tau.$$

Put

$$G_1(y) = \frac{1}{2\pi i} \int_{(\vartheta)} e^{y\sqrt{x}} y^{2\tau+2r-4k-2\rho-1} dy,$$

and applying formula (1.3.3), we have

$$G_1(y) = \frac{x^{2k+\rho-r-\tau}}{\Gamma(4k+2\rho+1-2r-2\tau)}.$$

Thus

$$I_2(y, \lambda) = \frac{1}{2\pi i} \int_{(\vartheta)} \left[\frac{\Gamma(r+\tau)\Gamma(-\tau)\Gamma\left(2k+\rho-\tau-r+\frac{1}{2}\right)}{\Gamma(r)\Gamma\left(2k+\frac{1}{2}+\rho\right)} \left(\frac{\alpha_j i \lambda}{8\pi}\right)^{\tau+r-2k} \times \frac{x^{2k+\rho-r-\tau}}{\Gamma(4k+2\rho+1-2r-2\tau)} \right] d\tau,$$

and applying formula (1.3.4) to $\frac{1}{\Gamma(4k+2\rho-2r-2\tau+1)}$, so $I_2(y, \lambda)$ can be expressed

$$\text{as } I_2(y, \lambda) = \frac{\sqrt{\pi}}{\Gamma(r)\Gamma\left(2k+\rho+\frac{1}{2}\right)} \left(\frac{i\alpha_j \lambda}{8\pi}\right)^{r-2k} \left(\frac{x}{4}\right)^{2k+\rho-r} \times \frac{1}{2\pi i} \int_{(\vartheta)} \frac{\Gamma(r+\tau)\Gamma(-\tau)}{\Gamma(2k+\rho+1-r-\tau)} \left(\frac{i\alpha_j \lambda}{2\pi x}\right)^\tau d\tau.$$

Set

$$G_2(y) = \frac{1}{2\pi i} \int_{(\vartheta)} \frac{\Gamma(r+\tau)\Gamma(-\tau)}{\Gamma(2k+\rho+1-r-\tau)} \left(\frac{i\alpha_j \lambda}{2\pi x}\right)^\tau d\tau.$$

Now $G_2(y)$ can be evaluated using Cauchy's Residue theorem moving the line of integration to $\text{Re}\tau = -N - \frac{1}{4}$ for large integer N . As in evaluation of $I_1(y, \lambda)$, applying Stirlig's formula to estimate growth of Γ the integral along the path $\text{Re}\tau = -N - \frac{1}{4}$ can be shown to tend to 0 as $N \rightarrow \infty$, then we obtain

$$G_2(y) = \sum_{m=0}^{\infty} \frac{(-1)^m}{m!} \frac{m+r}{\Gamma(2k+\rho+1+m+1)} \left(\frac{i\alpha_j \lambda}{2\pi x}\right)^{-m-r}.$$

Hence

$$I_2(y, \lambda) = \frac{\sqrt{\pi}}{\Gamma(r)\Gamma\left(2k+\rho+\frac{1}{2}\right)} \left(\frac{i\alpha_j \lambda}{8\pi}\right)^{-2k} \frac{x^{2k+\rho}}{2^{4k+2\rho-2r}} \sum_{m=0}^{\infty} \frac{(-1)^m}{m!} \frac{\Gamma(m+r)}{\Gamma(2k+\rho+m+1)} \left(\frac{2\pi x}{i\alpha_j \lambda}\right)^m$$

Therefore. using this result for $I_2(y, \lambda)$ we find that

$$\frac{1}{2\pi i} \int_{(\vartheta)} \frac{2^\rho e^{y\sqrt{x}}}{\sqrt{\pi} y^{2\rho+1}} \sum_{j=1}^p \sum_{r=1}^{M_j} C_{ri} \left(\frac{-1}{\alpha_j}\right)^r \alpha_j^{2k} \Gamma\left(2k+\rho+\frac{1}{2}\right) \left(\frac{\alpha_j i \lambda y^2}{8\pi}\right)^{r-2k}$$

$$\begin{aligned}
 & \times \Psi \left(r, r - 2k - \rho + \frac{1}{2}; \frac{i\alpha_j \lambda^2}{8\pi} \right) dy \\
 &= \frac{2^\rho}{\sqrt{\pi}} \sum_{j=1}^p \sum_{r=1}^{M_j} C_{ri} \left(\frac{-1}{\alpha_j} \right)^r \alpha_j^{2k} \Gamma \left(2k + \rho + \frac{1}{2} \right) \frac{1}{2\pi i} \int_{(\vartheta)} \frac{e^{y\sqrt{x}}}{y^{2\rho+1}} \left(\frac{\alpha_j i \lambda y^2}{8\pi} \right)^{r-2k} \\
 & \quad \times \Psi \left(r, r - 2k - \rho + \frac{1}{2}; \frac{i\alpha_j \lambda^2}{8\pi} \right) dy \\
 &= \frac{1}{2^\rho} \sum_{j=1}^p \sum_{r=1}^{M_j} C_{ri} \left(\frac{-1}{\alpha_j} \right)^r \left(\frac{2\pi}{i\lambda} \right)^{2k} \frac{x^{2k+\rho}}{\Gamma(2k + \rho + 1)} \sum_{m=0}^{\infty} \frac{(r)_m}{(2k + \rho + 1)_m} \left(\frac{-2\pi x}{i\alpha_j \lambda} \right)^m \\
 &= \frac{1}{2^\rho} \sum_{j=1}^p \sum_{r=1}^{M_j} C_{ri} \left(\frac{-1}{\alpha_j} \right)^r \left(\frac{2\pi}{i\lambda} \right)^{2k} \frac{x^{2k+\rho}}{\Gamma(2k + \rho + 1)} {}_1F_1 \left(r, 2k + \rho + 1; \frac{-2\pi x}{i\alpha_j \lambda} \right).
 \end{aligned} \tag{3.2.16}$$

To analyze the remaining terms define

$$\begin{aligned}
 I_3(y, \lambda) = \sum_{m=k}^L C_m \frac{2^\rho}{\sqrt{\pi}} \left\{ \frac{1}{2\pi i} \int_{(\vartheta)} \frac{e^{y\sqrt{x}}}{y^{2\rho+1}} \left[\left(\frac{8\pi i}{\lambda y^2} \right)^{2k-m} \Gamma \left(2k - m + \rho + \frac{1}{2} \right) \right. \right. \\
 \left. \left. - \left(\frac{-8\pi i}{\lambda y^2} \right)^m \Gamma \left(m + \rho + \frac{1}{2} \right) \right] dy \right\}.
 \end{aligned}$$

Then evaluating each integral in $I_3(y, \lambda)$, applying formula (1.3.3) and simplifying, we get

$$\begin{aligned}
 I_3(y, \lambda) &= 2^\rho \sum_{m=k}^L C_m \left\{ \left(\frac{8\pi i}{\lambda} \right)^{2k-m} \frac{x^{k+m}}{2^{4k+2\rho-2m} \Gamma(2k + \rho - m + 1)} \right. \\
 & \quad \left. - \left(\frac{-8\pi i}{\lambda} \right)^m \frac{x^{2k+\rho-m}}{2^{2k+2m} \Gamma(\rho + m + 1)} \right\} \\
 &= \frac{1}{2^\rho} \sum_{m=k}^L C_m \left\{ \left(\frac{2\pi}{\lambda} \right)^{2k-m} \frac{i^{2k-m} x^{2k+\rho-m}}{\Gamma(2k + \rho - m + 1)} - \left(\frac{2\pi}{\lambda} \right)^m \frac{(-i)^m x^{\rho+m}}{\Gamma(\rho + m + 1)} \right\}.
 \end{aligned} \tag{3.2.17}$$

Finally to evaluate the last term in (3.2.2) define

$$I_4(y, \lambda) = -\frac{2^{\rho+1}}{\sqrt{\pi}} a_0 \Gamma \left(\rho + \frac{1}{2} \right) \frac{1}{2\pi i} \int_{(\vartheta)} \frac{e^{y\sqrt{x}}}{y^{2\rho+1}} dy = -\frac{a_0 x^\rho}{2^{\rho-1} \Gamma(\rho + 1)}. \tag{3.2.18}$$

Thus the expression on left of (3.2.13) is equal to the sum of each expressions evaluated separately as above. Thus combining the results in (3.2.14), (3.2.15),

(3.2.16), (3.2.17), and (3.2.18) respectively we have

$$\begin{aligned}
 \frac{1}{\Gamma(\rho+1)} \sum'_{m \leq x} a_m (x-m)^\rho 2^{-\rho} &= i^{-2k} \left(\frac{2\pi}{\lambda}\right)^{-\rho} 2^{-\rho} \sum_{m=1}^{\infty} a_m \left(\frac{x}{m}\right)^{\frac{2k+\rho}{2}} J_{2k+\rho} \left(\frac{4\pi\sqrt{mx}}{\lambda}\right) \\
 &- \frac{1}{2^\rho} \sum_{j=1}^p \sum_{r=1}^{M_j} C_{rj} \left(\frac{-1}{\alpha_j}\right)^r \left(\frac{2\pi i \alpha_j}{\lambda}\right)^r \frac{x^{\rho+r}}{\Gamma(\rho+r+1)} {}_1F_1 \left(r, r+\rho+1; \frac{-2\pi i \alpha_j x}{\lambda}\right) \\
 &+ \frac{1}{2^\rho} \sum_{j=1}^p \sum_{r=1}^{M_j} C_{ri} \left(\frac{-1}{\alpha_j}\right)^r \left(\frac{2\pi}{i\lambda}\right)^{2k} \frac{x^{2k+\rho}}{\Gamma(2k+\rho+1)} {}_1F_1 \left(r, 2k+\rho+1; \frac{-2\pi x}{i\alpha_j \lambda}\right) \\
 &+ \frac{1}{2^\rho} \sum_{m=k}^L C_m \left\{ \left(\frac{2\pi}{\lambda}\right)^{2k-m} \frac{i^{2k-m} x^{2k+\rho-m}}{\Gamma(2k+\rho-m+1)} - \left(\frac{2\pi}{\lambda}\right)^m \frac{(-i)^m x^{\rho+m}}{\Gamma(\rho+m+1)} \right\} \\
 &+ \frac{1}{2^\rho} i^{2k} \left(\frac{2\pi}{\lambda}\right)^{2k} \frac{x^{2k+\rho}}{\Gamma(2k+\rho+1)} - \frac{a_0 x^\rho}{2^{\rho-1} \Gamma(\rho+1)}.
 \end{aligned}$$

There for we conclude

$$\begin{aligned}
 \frac{1}{\Gamma(\rho+1)} \sum'_{0 \leq m \leq x} a_m (x-m)^\rho &= i^{-2k} \left(\frac{2\pi}{\lambda}\right)^{-\rho} \sum_{m=1}^{\infty} a_m \left(\frac{x}{m}\right)^{\frac{2k+\rho}{2}} J_{2k+\rho} \left(\frac{4\pi\sqrt{mx}}{\lambda}\right) \\
 &+ i^{2k} \left(\frac{2\pi}{\lambda}\right)^{2k} \frac{x^{2k+\rho} a_0}{\Gamma(2k+\rho+1)} \\
 &+ \sum_{m=k}^L C_m \left\{ \left(\frac{2\pi}{\lambda}\right)^{2k-m} \frac{i^{2k-m} x^{2k+\rho-m}}{\Gamma(2k+\rho-m+1)} - \left(\frac{2\pi}{\lambda}\right)^m \frac{(-i)^m x^{\rho+m}}{\Gamma(\rho+m+1)} \right\} \\
 &- \sum_{j=1}^p \sum_{r=1}^{M_j} C_{rj} \left(\frac{-1}{\alpha_j}\right)^r \left(\frac{2\pi i \alpha_j}{\lambda}\right)^r \frac{x^{\rho+r}}{\Gamma(\rho+r+1)} {}_1F_1 \left(r, r+\rho+1; \frac{-2\pi i \alpha_j x}{\lambda}\right) \\
 &+ \sum_{j=1}^p \sum_{r=1}^{M_j} C_{ri} \left(\frac{-1}{\alpha_j}\right)^r \left(\frac{2\pi}{i\lambda}\right)^{2k} \frac{x^{2k+\rho}}{\Gamma(2k+\rho+1)} {}_1F_1 \left(r, 2k+\rho+1; \frac{-2\pi x}{i\alpha_j \lambda}\right).
 \end{aligned}$$

Hence we showed that the identity in (3.2.2) implies (3.1.2). Since (3.1.2) equivalent to the functional equation (3.2.1). This completes the proof of the converse and we conclude proof of the theorem . \square

Chapter 4

Dirichlet Series, Automorphic Integrals with Log-Polynomial Period Functions and Arithmetical Identities

Our main goal in this chapter is to establish the equivalence of two arithmetical identities with a functional equation associated with entire automorphic integrals involving log-polynomial-period functions on the discrete Hecke groups.

4.1 Review of Preliminary concepts

Let ν be a complex function defined by $\nu : G(\lambda) \rightarrow \mathbb{C}$ for all $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in G(\lambda)$, $\nu(M)$ independent of z and $|\nu(M)| = 1$. If ν satisfies the consistency condition

$$\nu(M_3)(c_3z + d_3)^{2k} = \nu(M_1)\nu(M_2)(c_1M_2z + d_1)^{2k}(c_2z + d_2)^{2k}$$

for all $M_1, M_2 \in G(\lambda)$, $M_1M_2 = M_3$, $M_j = \begin{pmatrix} a_j & b_j \\ c_j & d_j \end{pmatrix}$, $j = 1, 2, 3$, $z \in \mathcal{H}$, where $k \in \mathbb{R}$ then ν is called a *Multiplier system* for $G(\lambda)$. $2k$ is the weight of ν on $G(\lambda)$.

Suppose F is a holomorphic function in the upper half plane \mathcal{H} and $k \in \mathbb{R}$. Let ν be a multiplier system on $G(\lambda)$ of real weight $2k$. Suppose that F satisfies the automorphic relations

$$\bar{\nu}(S_\lambda)F(z + \lambda) = F(z), \quad \nu(S_\lambda) = e^{2\pi i\kappa}, \quad 0 \leq \kappa < 1 \quad (4.1.1)$$

and

$$\bar{\nu}(T)z^{-2k}F\left(\frac{-1}{z}\right) = F(z) + q(z) \quad \forall z \in \mathcal{H}, \quad (4.1.2)$$

where $q(z) = \sum_{j=1}^N z^{\alpha_j} \sum_{t=0}^{M_j} \beta_{jt} (\log z)^t$, $\alpha_j, \beta_{jt} \in \mathbb{C}$. Furthermore, assume that

F has Fourier series expansion of the form

$$F(z) = \sum_{m=0}^{\infty} a_m e^{2\pi i m z / \lambda}, \quad (4.1.3)$$

where $\text{Im } z = y > 0$ and $a_m = \mathcal{O}(m^\delta)$, for some $\delta \in \mathbb{R}^+$. A function F satisfying (4.1.1), (4.1.2) and (4.1.3) is called an entire automorphic integral of weight $2k$ with multiplier system ν for $G(\lambda)$. The function $q(z)$ in (4.1.2) is called the log-polynomial period function of the entire automorphic integral function F .

Remark. Abdulkadir Hassen [3] has characterized completely the log-polynomial period functions for entire automorphic integrals weight $2k$, $k \in \mathbb{R}$ on $G(\lambda)$, the discrete Hecke group, in the cases:

- (1) $k \geq 1$, $\nu(S_\lambda) = 1$
- (2) $k > 0$, $\nu(S_\lambda) \neq 1$
- (3) $k \geq 0$, $2k \in \mathbb{Z}$, $\nu(S_\lambda) = 1$ and
- (4) $k \leq 0$, $\nu(S_\lambda) \neq 1$.

For the purpose of this work we consider only the multiplier system which satisfy $\nu(S_\lambda) = 1$. That is ($\kappa = 0$). In [4] Paul C. Pasles stated a Riemann-Hecke-Bochner correspondence for entire automorphic integrals on the Hecke groups. We modified and restate the theorem here for our context with multiplier system ν , and we give the main steps of the proof. (see [4], Theorem 3.1).

Theorem 4.1. *Let $k \in \mathbb{R}^+ \cup \{0\}$, $\lambda \geq 2$ or $\lambda = 2 \cos\left(\frac{\pi}{p}\right)$, $p \in \mathbb{Z}$, $p \geq 3$. Suppose that F is an entire automorphic integral of weight $2k$, multiplier system ν and $\nu(S_\lambda) = 1$. Let $q(z)$ be the log-polynomial period function of F on $G(\lambda)$ and let F have Fourier expansion of the form*

$$F(z) = \sum_{m=0}^{\infty} a_m e^{2\pi i m z / \lambda},$$

for $z \in \mathcal{H}$, where

$$a_m = \mathcal{O}(m^\beta) \text{ for some } \beta > 0, m \rightarrow \infty.$$

$$\text{Let } \varphi(s) = \sum_{m=1}^{\infty} a_m m^{-s}, \quad s = \sigma + it. \text{ And put}$$

$$\Phi_F(s) = \left(\frac{2\pi}{\lambda}\right)^{-s} \Gamma(s) \varphi(s), \text{ for } \sigma > \beta + 1.$$

Then

(A) $\Phi_F(s)$ has a meromorphic continuation to the whole complex plane with at most a finite number of poles given by:

$$\Phi_F(s) = D_k(s) + M_k(s) + L_k(s),$$

where

$$D_k(s) = \int_1^\infty \left\{ F(iy) - a_0 \right\} y^{s-1} ds + i^{2k} \nu(T) \int_1^\infty \left\{ F(iy) - a_0 \right\} y^{2k-s-1} ds, \quad (4.1.4)$$

$$M_k(s) = a_0 \left\{ \frac{i^{2k} \nu(T)}{s - 2k} - \frac{1}{s} \right\}, \quad (4.1.5)$$

and

$$L_k(s) = - \sum_{j=1}^N i^{2k+\alpha_j} \sum_{t=0}^{M_j} b_{jt} \sum_{l=0}^t \binom{t}{l} \left(\frac{i\pi}{2} \right)^{t-l} \frac{(-1)^{l+1} l!}{(2k + \alpha_j - s)^{l+1}}, \quad (4.1.6)$$

for $\text{Res} > 2k + \max_j |\text{Re} \alpha_j|$.

(B) $\Phi_F(s)$ is bounded uniformly and absolutely in each σ in a lacunary vertical strips of the form

$$S(\sigma_1, \sigma_2, t_0) = \{s : \sigma_1 \leq \sigma \leq \sigma_2, |\text{Im } s| = t \geq t_0\},$$

where $\sigma_1, \sigma_2 \in \mathbb{R}$ and $t_0 > \max_j |\text{Im } \alpha_j|$.

Moreover,

(C) $\Phi_F(s)$ satisfies the functional equation

$$\Phi_F(2k - s) = e^{\pi i k} \nu(T) \Phi_F(s). \quad (4.1.7)$$

Proof. Suppose F is an entire automorphic integral of weight $2k$, $k \in \mathbb{R}$, multiplier system ν and associated Lppf $q(z)$. For $\text{Res} > 2k + \max_j |\text{Re}\alpha_j|$. Applying Mellin transform of F , we have

$$\begin{aligned}\Phi_F(s) &= \int_0^\infty (F(iy) - a_0) y^s \frac{dy}{y} \\ &= \int_1^\infty (F(iy) - a_0) y^s \frac{dy}{y} + \int_0^1 (F(iy) - a_0) y^s \frac{dy}{y}.\end{aligned}$$

Taking $y \mapsto \frac{1}{y}$ in the latter integral and applying the transformation law in (4.1.2), we get

$$\begin{aligned}\int_0^1 (F(iy) - a_0) y^s \frac{dy}{y} &= \nu(T) i^{2k} \int_1^\infty (F(iy) - a_0) y^{2k-s-1} dy + \nu(T) i^{2k} a_0 \int_1^\infty y^{2k-s-1} dy \\ &\quad + \nu(T) i^{2k} \int_1^\infty q(iy) y^{2k-s-1} dy - \frac{a_0}{s} \\ &= \nu(T) i^{2k} \int_1^\infty (F(iy) - a_0) y^{2k-s-1} dy + a_0 \left[\frac{\nu(T) i^{2k}}{s-2k} - \frac{1}{s} \right] \\ &\quad + \nu(T) i^{2k} \int_1^\infty q(iy) y^{2k-s-1} dy.\end{aligned}$$

Thus

$$\Phi_F(s) = D_k(s) + M_k(s) + L_k(s),$$

where

$$D_k(s) = \int_1^\infty (F(iy) - a_0) y^{s-1} dy + \nu(T) i^{2k} \int_1^\infty (F(iy) - a_0) y^{2k-s-1} dy,$$

$$M_k = a_0 \left[\frac{\nu(T) i^{2k}}{s-2k} - \frac{1}{s} \right],$$

and

$$L_k(s) = \nu(T) i^{2k} \int_1^\infty q(iy) y^{2k-s-1} dy.$$

$D_k(s)$ is entire and converges uniformly (absolutely) on compact subset of \mathbb{C} , for large $\text{Re}(s)$, and $D_k(s)$ satisfies the functional equation $D_k(2k-s) = e^{\pi ik} \nu(T) D_k(s)$. Similarly $M_k(s)$ is meromorphic with simple poles at $(s = 2k, 0)$, and satisfies the functional equation $M_k(2k-s) = e^{\pi ik} \nu(T) M_k(s)$. In

order to show the meromorphic continuation of $\Phi_F(s)$ we need the meromorphic continuation of $L_k(s)$ to \mathbb{C} . Now by definition

$$\begin{aligned} L_k(s) &= \nu(T) i^{2k} \int_1^\infty q(iy) y^{2k-s-1} dy \\ &= \nu(T) i^{2k} \int_1^\infty \sum_{j=1}^N (iy)^{\alpha_j} \sum_{t=0}^{M_j} \beta_{jt} (\log iy)^t y^{2k-s-1} dy. \end{aligned}$$

The integral and the finite double sums may be interchanged, since each terms is absolutely integrable. Observe that $\log iy = \log y + i \arg(iy) = \log y + \frac{i\pi}{2}$, $y > 0$. Then applying the binomial theorem we get

$$\left(\frac{i\pi}{2} + \log y \right)^t = \sum_{l=0}^t \binom{t}{l} \left(\frac{i\pi}{2} \right)^{t-l} (\log y)^l.$$

To show the meromorphic continuation of $L_k(s)$ we require the following result.

Lemma 4.2. *Let $\eta \in \mathbb{C}$, $Re\eta < -1$ for $l \in \mathbb{Z}_{\geq 0}$. We have*

$$\int_1^\infty y^\eta (\log y)^l = \frac{(-1)^{l+1} l!}{(\eta + 1)^{l+1}}.$$

Proof. It is not hard to prove the lemma using induction and integration by parts. □

Applying the binomial representation for $\log iy$ we have

$$L_k(s) = \nu(T) \sum_{j=1}^N i^{2k+\alpha_j} \sum_{t=0}^{M_j} \beta_{jt} \sum_{l=0}^t \binom{t}{l} \left(\frac{i\pi}{2} \right)^{t-l} \int_1^\infty y^{2k+\alpha_j-s-1} (\log y)^l dy.$$

So that by Lemma 4.2 we have

$$L_k(s) = \nu(T) \sum_{j=1}^N i^{2k+\alpha_j} \sum_{t=0}^{M_j} \beta_{jt} \sum_{l=0}^t \binom{t}{l} \left(\frac{i\pi}{2} \right)^{t-l} \frac{l!}{(s - (\alpha_j + 2k))^{l+1}},$$

for $Re s > 2k + \max_j Re\alpha_j$. We assume always that $\beta_{jM_j} \neq 0$. Then $L_k(s)$ is meromorphic in \mathbb{C} with poles of order, $M_j + 1$ at $s = 2k + \alpha_j$. Thus

$$\Phi_F(s) = D_k(s) + M_k(s) + L_k(s)$$

is meromorphic on \mathbb{C} with finite simple poles at $s = 2k, 0$ of order one and $s = 2k + \alpha_j$ of order $M_j + 1$. This completes the prove of (A). To prove $\Phi_F(2k - s) = e^{\pi ik} \nu(T) \Phi(s)$ it is enough to show that $L_k(2k - s) = e^{\pi ik} \nu(T) L_k(s)$.

First we require the following lemma.

Lemma 4.3. *If F and q are as in Theorem 4.1, then*

$$q\left(\frac{-1}{z}\right) = -(-z)^{2k}\bar{\nu}(T)q(z), \forall z \in \mathcal{H}. \quad (4.1.8)$$

Remark: $(\nu(T)i^{2k})^2 = 1$.

Proof. By taking $z \rightarrow \frac{-1}{z}$ in (4.1.2), we have

$$F(z) = \nu(T)(-z)^{-2k} \left[\nu(T)z^{2k}F(z) + \nu(T)z^{2k}q(z) + q\left(\frac{-1}{z}\right) \right].$$

Which implies

$$\begin{aligned} F(z) - \nu^2(T)(-z)^{-2k}z^{2k}F(z) &= \nu^2(-z)^{-2k}z^{2k}q(z) + \nu(T)(-z)^{-2k}q\left(\frac{-1}{z}\right) \\ &= i^{4k}\nu^2(T)q(z) + \nu(T)(-z)^{-2k}q\left(\frac{-1}{z}\right) \\ &= (i^2\nu(T))^2q(z) + \nu(T)(-z)^{-2k}q\left(\frac{-1}{z}\right) \\ &= q(z) + \nu(T)(-z)^{-2k}q\left(\frac{-1}{z}\right). \end{aligned}$$

Therefore.

$$F(z) - \nu^2(T)(i^{2k})^2F(z) = q(z) + \nu(T)(-z)^{-2k}q\left(\frac{-1}{z}\right).$$

Then by the above remark we have

$$q(z) + \nu(T)(-z)^{-2k}q\left(\frac{-1}{z}\right) = 0, \forall z \in \mathcal{H}$$

and the proof of the lemma is completed. □

Now from $L_k(s) = i^{2k}\nu(T) \int_1^\infty y^{2k-s-1}q(iy)dy$ implies

$$L_k(2k-s) = i^{2k}\nu(T) \int_1^\infty y^{s-1}q(iy)dy.$$

Applying Lemma 4.3 and replace y by $\frac{1}{y}$ in the integral we obtain

$$i^{-2k}\bar{\nu}(T)L_k(2k-s) = -i^{2k}\nu(T) \int_0^1 y^{2k-s-1}q(iy)dy.$$

Then by definition of q we have

$$i^{-2k}\bar{\nu}(T)L_k(2k-s) = -i^{2k}\nu(T) \int_0^1 \sum_{j=1}^N (iy)^{\alpha_j} \sum_{t=0}^{M_j} \beta_{jt} (\log iy)^t y^{2k-s-1} dy.$$

From the relation $\log iy = \frac{i\pi}{2} + \log y$ and applying the binomial theorem we get

$$(\log iy)^t = \left(\frac{i\pi}{2} + \log y \right)^t = \sum_{l=0}^t \binom{t}{l} \left(\frac{i\pi}{2} \right)^{t-l} (\log y)^l.$$

By taking $y \mapsto \frac{1}{y}$ we see that

$$i^{-2k}\bar{\nu}(T)L_k(2k-s) = -i^{2k}\nu(T) \sum_{j=1}^N (i)^{\alpha_j} \sum_{t=0}^{M_j} \beta_{jt} \sum_{l=0}^t \binom{t}{l} \left(\frac{i\pi}{2} \right)^{t-l} \int_1^\infty y^{s-2k-\alpha_j-1} (\log y)^l (-1)^l dy.$$

Applying lemma4.2 and by simplifying we have

$$i^{-2k}\bar{\nu}(T)L_k(2k-s) = \nu(T) \sum_{j=1}^N (i)^{\alpha_j+2k} \sum_{t=0}^{M_j} \beta_{jt} \sum_{l=0}^t \binom{t}{l} \left(\frac{i\pi}{2} \right)^{t-l} \frac{l!}{[s - (2k + \alpha_j)]^{l+1}},$$

where

$$L_k(s) = \nu(T) \sum_{j=1}^N (i)^{\alpha_j+2k} \sum_{t=0}^{M_j} \beta_{jt} \sum_{l=0}^t \binom{t}{l} \left(\frac{i\pi}{2} \right)^{t-l} \frac{l!}{[s - (2k + \alpha_j)]^{l+1}}.$$

Therefore, $L_k(2k-s) = e^{\pi ik}\nu(T)L_k(s)$. This complete the prove of (C). Finally to show the boundedness condition (B). The rational functions M_k and L_k are bounded in lacunary vertical strips which do not contain poles of M_k and L_k . Thus to prove (B) it suffices to show the boundedness of D_k . In fact, by $a_m = \mathcal{O}(m^\beta)$ for $\sigma > 1 + \beta$,

$$\left| \int_1^\infty [F(iy) - a_0]y^{s-1} dy \right| = \mathcal{O}(\zeta(\sigma - \beta)\Gamma(\sigma)).$$

Hence (B). And complete the prove of the theorem. □

4.2 Third equivalence

In this section we prove our third and the fourth results. Sister Ann M. Heath in [1] considers the functional equation of J. Hawkins and M. Knopp in [5]

$$\Phi(2k - s) - i^{2k}\Phi(s) = R(s).$$

She has shown that the functional equation is equivalent to two types of arithmetical identities associated with entire modular integrals involving rational period functions on the full modular group $\Gamma(1)$. Here we establish two arithmetical identities associated with entire automorphic integrals involving log-polynomial period functions on the discrete Hecke group $G(\lambda)$, analogous to Sister Ann M. Heath and we show them to be equivalent to the functional equation (4.1.7),

$$\Phi_F(2k - s) = e^{\pi ik}\nu(T)\Phi_F(s).$$

Theorem 4.4. *Let $\Phi_F(s), q(z)$ and $F(z)$ be as in Theorem 4.1. Then the functional equation*

$$\Phi_F(2k - s) = e^{\pi ik}\nu(T)\Phi_F(s) \tag{4.2.1}$$

is equivalent to the arithmetical identity

$$\begin{aligned} \frac{1}{\Gamma(\rho + 1)} \sum_{0 \leq m \leq x} a_m(x - m)^\rho &= e^{-\pi ik}\bar{\nu}(T) \left(\frac{2\pi}{\lambda}\right)^{-\rho} \sum_{m=1}^{\infty} a_m \left(\frac{x}{m}\right)^{\frac{2k+\rho}{2}} J_{2k+\rho} \left(\frac{4\pi\sqrt{mx}}{\lambda}\right) \\ &+ \left(\frac{2\pi}{\lambda}\right)^{2k} \frac{\nu(T)e^{\pi ik}a_0x^{2k+\rho}}{\Gamma(2k + \rho + 1)} - \frac{a_0x^\rho}{\Gamma(\rho + 1)} \\ &+ \left(\frac{2\pi}{\lambda}\right)^{2k+\alpha_j} \frac{x^{2k+\rho+\alpha_j}}{\Gamma(2k + \alpha_j + \rho + 1)} \nu(T)e^{\pi ik} \sum_{j=1}^N i^{\alpha_j} \sum_{t=0}^{M_j} \beta_{jt} \left(\frac{i\pi}{2}\right)^t, \end{aligned} \tag{4.2.2}$$

for $x > 0$, $\rho \in \mathbb{Z}_{\geq 0}$, $\alpha_j, \beta_{jt} \in \mathbb{C}$, $\beta_{jM_j} \neq 0$ and $N, M_j \in \mathbb{Z}$, provided $\rho \geq 2\beta - 2k$, where $a_m = \mathcal{O}(m^\beta)$.

Proof of (4.2.1) implies (4.2.2)

Here we first show the functional equation (4.2.1) implies the identity (4.2.2).

Proof. Applying the formula in (3.2), for $\rho \geq 0, b \geq \beta$ and $a_m = \mathcal{O}(m^\beta)$ we have

$$\frac{1}{\Gamma(\rho + 1)} \sum'_{0 \leq m \leq x} a_m(x - m)^\rho = \frac{1}{2\pi i} \int_{(b)} \frac{\Gamma(s)\varphi(s)x^{s+\rho}}{\Gamma(s + \rho + 1)} ds. \tag{4.2.3}$$

Here the prime ' indicates that if $m = x, a_m$ is to be multiplied by $\frac{1}{2}$ and

$$\varphi(s) = \sum_{m=1}^{\infty} \frac{a_m}{m^s} \text{ with } \sum_{m=1}^{\infty} \frac{|a_m|}{m^\beta} < \infty.$$

To evaluate the integral on the right side of (4.2.3), for $2k - b < \sigma < b$, where $s = \sigma + it$. Consider rectangular region with vertices $2k - b \pm T$ and $b \pm T$, having positive orientation. Let T be positive large number so that Stirling's formula can be used in approximating $\Gamma(s)$ as $|s| \rightarrow \infty$. Using Stirling's formula and the Phragmen-Lindelof theorem we may show the integrals along the horizontal paths tend to 0 as $T \rightarrow \infty$. Hence with the substitution, $\left(\frac{2\pi}{\lambda}\right)^s \Phi_F(s) = \Gamma(s)\varphi(s)$, the right hand side of (4.2.3) may be written as

$$\begin{aligned} \frac{1}{2\pi i} \int_{(b)} \frac{\Gamma(s)\varphi(s)}{\Gamma(s + \rho + 1)} ds &= \frac{1}{2\pi i} \int_{(b)} \left(\frac{2\pi}{\lambda}\right)^s \frac{\Phi_F(s)x^{s+\rho}}{\Gamma(s + \rho + 1)} ds \\ &= \frac{1}{2\pi i} \int_{(2k-b)} \left(\frac{2\pi}{\lambda}\right)^s \frac{\Phi_F(s)x^{s+\rho}}{\Gamma(s + \rho + 1)} ds \\ &\quad + \sum_{s \in \text{Pole set of } \Phi_F(s)} \text{Res} \left\{ \left(\frac{2\pi}{\lambda}\right)^s \frac{\Phi_F(s)x^{s+\rho}}{\Gamma(s + \rho + 1)} \right\}. \end{aligned}$$

Put

$$\omega_1(s, \lambda) = \frac{1}{2\pi i} \int_{(2k-b)} \left(\frac{2\pi}{\lambda}\right)^s \frac{\Phi_F(s)x^{s+\rho}}{\Gamma(s + \rho + 1)} ds$$

and

$$\omega_2(s, \lambda) = \sum_{s \in \text{Pole set of } \Phi_F(s)} \text{Res} \left\{ \left(\frac{2\pi}{\lambda}\right)^s \frac{\Phi_F(s)x^{s+\rho}}{\Gamma(s + \rho + 1)} \right\}.$$

To evaluate $\omega_1(s, \lambda)$ we begin by using the functional equation

$$\Phi_F(2k - s) = e^{\pi i k} \nu(T) \Phi(s).$$

$\omega_1(s, \lambda)$ can be written as

$$\omega_1(s, \lambda) = \frac{1}{2\pi i} \int_{(2k-b)} \left(\frac{2\pi}{\lambda}\right)^s \frac{e^{-\pi i k} \bar{\nu}(T) \Phi_F(2k - s)x^{s+\rho}}{\Gamma(s + \rho + 1)} ds.$$

By substituting

$$\Phi_F(2k - s) = \left(\frac{2\pi}{\lambda}\right)^s \Gamma(2k - s) \sum_{m=1}^{\infty} \frac{a_m}{m^{2k-s}}$$

and replace s by $2k - s$. Now applying (1.3.7) with simple algebraic manipulation we obtain

$$\omega_1(s, \lambda) = \left(\frac{2\pi}{\lambda}\right)^{-\rho} e^{-\pi i k} \bar{\nu}(T) \sum_{m=1}^{\infty} a_m \left(\frac{x}{m}\right)^{\frac{2k+\rho}{2}} J_{2k+\rho} \left(\frac{4\pi\sqrt{mx}}{\lambda}\right), \quad (4.2.4)$$

provided that $\rho \geq 2\beta - 2k$.

To complete the evaluation of the integral in (4.2.3) we calculate the residues in $\omega_2(s, \lambda)$. The meromorphic continuation Φ_F expressed in terms of M_k (has simple poles at $s = 0, 2k$) and L_k which has a pole at $s = 2k + \alpha_j$ of order $M_j + 1$. Thus a straight forward calculation of the residues of the functions we obtain

$$\begin{aligned} \omega_2(s, \lambda) = & \left(\frac{2\pi}{\lambda}\right)^{2k} e^{\pi ik} \frac{x^{2k+\rho} a_0 \nu(T)}{\Gamma(2k + \rho + 1)} - \frac{a_0 x^\rho}{\Gamma(\rho + 1)} \\ & + \left(\frac{2\pi}{\lambda}\right)^{2k+\alpha_j} \frac{x^{2k+\rho+\alpha_j}}{\Gamma(2k + \rho + 1 + \alpha_j)} \nu(T) e^{\pi ik} \sum_{j=1}^N i^{\alpha_j} \sum_{t=0}^{M_j} \beta_{jt} \left(\frac{i\pi}{2}\right)^t, \end{aligned} \quad (4.2.5)$$

for $\rho \in \mathbb{Z}_{\geq 0}$, $\alpha_j, \beta_{jt} \in \mathbb{C}$ and $\beta_{jM_j} \neq 0$.

The integral in the right of (4.2.3) is equal to the combination of the expressions in (4.2.4) and (4.2.5) respectively, thus it follows the identity (4.2.2).

Proof of converse of the theorem

Suppose F is an entire automorphic integral function. $F(z)$ has Fourier series expansion of the form

$$F(z) = \sum_{m=0}^{\infty} a_m e^{2\pi im/\lambda}. \quad (4.2.6)$$

For $z \in \mathcal{H}$, where $a_m = \mathcal{O}(m^\gamma)$ as $m \rightarrow \infty$, for some $\gamma > 0$. Put

$$\Phi_F(s) = \left(\frac{2\pi}{\lambda}\right)^{-s} \Gamma(s) \sum_{m=1}^{\infty} a_m m^{-s}.$$

For $z \in \mathcal{H}$ F satisfies the relation

$$\bar{\nu}(T) z^{-2k} F\left(\frac{-1}{z}\right) = F(z) + q(z) \forall z \in \mathcal{H}, \quad (4.2.7)$$

where

$$q(z) = \sum_{j=1}^N z^{\alpha_j} \sum_{t=0}^{M_j} \beta_{jt} (\log z)^t, \quad \alpha_j, \beta_{jt} \in \mathbb{C}.$$

Now using (4.2.6) and (4.2.7) we have

$$z^{-2k} \bar{\nu}(T) \sum_{m=0}^{\infty} a_m e^{-2\pi im/\lambda z} = \sum_{m=0}^{\infty} a_m e^{2\pi imz/\lambda} + q(z).$$

Letting $z = \frac{iy\lambda}{2\pi}$, $y > c$, $c \in \mathbb{R}^+$. Then we have that

$$\left(\frac{2\pi}{iy\lambda}\right)^{2k} \bar{\nu}(T) \sum_{m=0}^{\infty} a_m e^{\frac{-4\pi^2 m}{y\lambda^2}} = \sum_{m=0}^{\infty} a_m e^{-my} + q\left(\frac{iy\lambda}{2\pi}\right). \quad (4.2.8)$$

In proving the converse, it is enough to show that the identity in (4.2.2) implies (4.2.8). Since $q(z)$ is entire in \mathcal{H} and the series expansion for F in (4.2.6) is uniformly convergent on compact subsets of \mathcal{H} , the automorphic relation in (4.2.7) will follow by analytic for $\forall z \in \mathcal{H}$. Following the method used in Chandraseharan and Narasimhan [2], we multiply the identity in (4.2.2) through out by $y^{\rho+1}e^{-xy}$ with $y > 0$, integrate relative to x from 0 to ∞ . To this end consider four integrals (separate) corresponding to the terms occurring in (4.2.2) as

$$\varpi_1(x, \lambda) = \int_0^{\infty} \left\{ \frac{1}{\Gamma(\rho+1)} \sum_{0 \leq m \leq x} a_m (x-m)^{\rho} \right\} y^{\rho+1} e^{-xy} dx.$$

$$\varpi_2(x, \lambda) = \int_0^{\infty} \left\{ e^{-\pi ik} \bar{\nu}(T) \left(\frac{2\pi}{\lambda}\right)^{-\rho} \sum_{m=1}^{\infty} a_m \left(\frac{x}{m}\right)^{\frac{2k+\rho}{2}} J_{2k+\rho}\left(\frac{4\pi\sqrt{mx}}{\lambda}\right) \right\} y^{\rho+1} e^{-xy} dx.$$

$$\varpi_3(x, \lambda) = \int_0^{\infty} \left\{ e^{\pi ik} \left(\frac{2\pi}{\lambda}\right)^{2k} \frac{a_0 \nu(T) x^{2k+\rho}}{\Gamma(2k+\rho+1)} - \frac{a_0 x^{\rho}}{\Gamma(\rho+1)} \right\} y^{\rho+1} e^{-xy} dx.$$

$$\varpi_4(x, \lambda) = \int_0^{\infty} \left\{ \left(\frac{2\pi}{\lambda}\right)^{2k+\alpha_j} \frac{x^{2k+\rho+\alpha_j}}{\Gamma(2k+\rho+\alpha_j+1)} \nu(T) e^{\pi ik} \sum_{j=1}^N i^{\alpha_j} \sum_{t=0}^{M_j} \beta_{jt} \left(\frac{i\pi}{2}\right)^t \right\} y^{\rho+1} e^{-xy} dx$$

The evaluation of ϖ_1 and ϖ_2 follows as Chandrasekharan and Narasimhan found in [2]. Inter change of summation and integration being permitted for $\rho \geq 2\beta - 2k - \frac{1}{2}$. Then

$$\varpi_1(x, \lambda) = \frac{y^{\rho+1}}{\Gamma(\rho+1)} \int_0^{\infty} x^{\rho} e^{-xy} dx + \frac{y^{\rho+1}}{\Gamma(\rho+1)} \sum_{1 \leq m \leq x} a_m \int_m^{\infty} (x-\rho)^{\rho} e^{-xy} dx.$$

Integration by substitution and using the standard integral representation for $\Gamma(s)$ we get

$$\varpi_1(x, \lambda) = \sum_{m=0}^{\infty} a_m e^{-my}. \quad (4.2.9)$$

To compute ϖ_2 we applying the formula in (1.3.8), and with the substitution $\frac{4\pi\sqrt{m}}{\lambda} = a$, $\nu = \rho + 2k$ after simplifying, we obtain

$$\varpi_2(x, \lambda) = \left(\frac{2\pi}{\lambda y}\right)^{2k} e^{-\pi ik} \bar{\nu}(T) \sum_{m=1}^{\infty} a_m e^{\frac{-4\pi^2 m}{y\lambda^2}}. \quad (4.2.10)$$

The integral in $\varpi_3(x, \lambda)$ is evaluated using the integral representation for $\Gamma(s)$. Therefore. after simplifying, we see that

$$\varpi_3(x, \lambda) = \left(\frac{2\pi}{\lambda y}\right)^{2k} a_0 e^{\pi i k} \nu(T) - a_0. \quad (4.2.11)$$

To evaluate $\varpi_4(x, \lambda)$ since each term is integrable the inter change of the integration and the finite double sum is valid . Then integration by substitution and using the integral representation of $\Gamma(z)$ we obtain

$$\begin{aligned} \varpi_4(x, \lambda) &= \left(\frac{2\pi i}{\lambda y}\right)^{2k} \nu(T) \sum_{j=1}^N \left(\frac{2\pi i}{\lambda y}\right)^{\alpha_j} \sum_{t=0}^{M_j} \beta_{jt} \left(\frac{i\pi}{2}\right)^t \\ &= \left(\frac{-1}{i\lambda y/2\pi}\right)^{2k} \nu(T) \sum_{j=1}^N \left(\frac{-1}{\lambda y i/2\pi}\right)^{\alpha_j} \sum_{t=0}^{M_j} \beta_{jt} \left(\frac{i\pi}{2}\right)^t. \end{aligned} \quad (4.2.12)$$

For $y > 0$, we see that $\log\left(\frac{-1}{iy}\right) + \log y = \frac{i\pi}{2}$. Then for $l \in \mathbb{N} \cup \{0\}$ and applying the binomial theorem we have

$$\left(\log\left(\frac{-1}{iy}\right) + \log y\right)^t = \sum_{l=0}^t \binom{t}{l} \left(\log\left(\frac{-1}{iy}\right)\right)^{t-l} (\log y)^l.$$

If $l = 0$ we have $\left(\log\left(\frac{-1}{iy}\right)\right)^t = \left(\log\left(\frac{-1}{iy}\right) + \log y\right)^t$. Now replacing $\frac{-1}{iy}$ by $\frac{-1}{i\lambda y/2\pi}$ and substituting for $\left(\frac{i\pi}{2}\right)^t$ in (4.2.12) we have

$$\varpi_4(x, \lambda) = \left(\frac{-1}{i\lambda y/2\pi}\right)^{2k} \nu(T) \sum_{j=1}^N \left(\frac{-1}{\lambda y i/2\pi}\right)^{\alpha_j} \sum_{t=0}^{M_j} \beta_{jt} \left(\log\left(\frac{-1}{i\lambda y/2\pi}\right)\right)^t. \quad (4.2.13)$$

Observe that $\varpi_4(x, \lambda) = \left(\frac{-1}{i\lambda y/2\pi}\right)^{2k} \nu(T) q\left(\frac{-1}{i\lambda y/2\pi}\right)$, where $q(z)$ is the log-polynomial period function in (4.2.7). Then applying lemma4.3 we have $\varpi_4(x, \lambda) = \left(\frac{-1}{i\lambda y/2\pi}\right)^{2k} \nu(T) q\left(\frac{-1}{i\lambda y/2\pi}\right) = -q\left(\frac{i\lambda}{2\pi}\right)$. Combining the results in (4.2.9), (4.2.10), (4.2.11) and (4.2.12) we showed that the identity in (4.2.2) implies

$$\begin{aligned} \sum_{m=0}^{\infty} a_m e^{-my} &= \left(\frac{2\pi}{\lambda y}\right)^{2k} e^{-\pi i k} \bar{\nu}(T) \sum_{m=1}^{\infty} a_m e^{\frac{-4\pi^2 m}{y\lambda^2}} + \left(\frac{2\pi}{\lambda y}\right)^{2k} a_0 e^{\pi i k} \nu(T) \\ &+ \left(\frac{-1}{i\lambda y/2\pi}\right)^{2k} \nu(T) \sum_{j=1}^N \left(\frac{-1}{\lambda y i/2\pi}\right)^{\alpha_j} \sum_{t=0}^{M_j} \beta_{jt} \left(\log\left(\frac{-1}{i\lambda y/2\pi}\right)\right)^t. \end{aligned} \quad (4.2.14)$$

Observe that

$$\left(\frac{-1}{i\lambda y/2\pi}\right)^{2k} \nu(T) \sum_{j=1}^N \left(\frac{-1}{\lambda y i/2\pi}\right)^{\alpha_j} \sum_{t=0}^{M_j} \beta_{jt} \left(\log\left(\frac{-1}{iy\lambda/2\pi}\right)\right)^t = -q\left(\frac{iy\lambda}{2\pi}\right).$$

Since $F(z)$ is holomorphic and has a Fourier series expansion $F(z) = \sum_{m=0}^{\infty} a_m e^{2\pi i m z/\lambda}$

in \mathcal{H} . Then with $z = \frac{iy\lambda}{2\pi}$ (4.2.14) we may be written as (4.2.8). By the identity theorem, then the automorphic relation (4.1.2) follows $\forall z \in \mathcal{H}$, and this completes proof of the theorem. \square

4.3 Fourth equivalence

Theorem 4.5. *Let $\Phi_F(s), F(z)$ and $q(z)$ be as in Theorem 4.1 . Then the functional equation*

$$\Phi_F(2k - s) = e^{\pi i k} \nu(T) \Phi_F(s) \quad (4.3.1)$$

is equivalent to the arithmetical identity

$$\begin{aligned} \left(\frac{-1}{y} \frac{d}{dy}\right)^{\rho} \left(\frac{1}{y} \sum_{m=1}^{\infty} a_m e^{-y\sqrt{m}}\right) &= \frac{2^{\rho}}{\sqrt{\pi}} \Gamma\left(2k + \rho + \frac{1}{2}\right) \left(\frac{8\pi}{\lambda}\right)^{2k} \bar{\nu}(T) e^{-\pi i k} \\ &\times \sum_{m=0}^{\infty} \frac{a_m}{\left(y^2 + 4\left(\frac{2\pi}{\lambda}\right)^2 m\right)^{2k + \rho + \frac{1}{2}}} \\ &+ \frac{2^{\rho}}{\sqrt{\pi}} \frac{\Gamma(2k + \rho + \alpha_j + \frac{1}{2})}{y^{2\rho+1}} \left(\frac{8\pi}{\lambda y^2}\right)^{2k} \nu(T) e^{\pi i k} \\ &\times \sum_{j=1}^N \left(\frac{8\pi i}{\lambda y^2}\right)^{\alpha_j} \sum_{t=0}^{M_j} \beta_{jt} \left(\frac{i\pi}{2}\right)^t \\ &- \frac{2^{\rho+1}}{\sqrt{\pi}} \frac{a_0}{y^{2\rho+1}} \Gamma\left(\rho + \frac{1}{2}\right), \end{aligned} \quad (4.3.2)$$

provided that $\operatorname{Re} y \in \mathbb{R}^+$, $\alpha_j, \beta_{jt} \in \mathbb{C}$, $\rho \in \mathbb{Z}_{\geq 0}$, and $\rho > \beta - 2k + \frac{1}{2}$, where $\sum_{m=1}^{\infty} \frac{|a_m|}{m^{\beta}} < \infty$.

Since by Theorem 4.4 the functional equation (4.3.1) is equivalent to (4.2.2), it would be enough to show that the functional equation (4.3.1) implies (4.3.2) and that (4.3.2) in turn implies the identity (4.2.2).

Proof of implication

Proof. First we begin by showing that (4.3.1) implies (4.3.2). Let ρ be an integer, $\rho \geq 0$ and $\varphi(s) = \sum_{m=1}^{\infty} a_m m^{-s}$ with $a_m = \mathcal{O}(m^\beta)$. Then for $\gamma > 0, \gamma \geq \beta$ and applying the version of Perron's Lemma in 3.2, we have

$$\frac{1}{\Gamma(\rho+1)} \sum_{0 \leq m \leq x}' a_m (x-m)^\rho = \frac{1}{2\pi i} \int_{(\gamma)} \frac{\varphi(s)\Gamma(s)x^{s+\rho}}{\Gamma(s+\rho+1)} ds. \quad (4.3.3)$$

As in Chandrasekharan and Narasimhan in [2], we multiply (4.3.3) by $e^{-y\sqrt{x}}x^{\frac{1}{2}}$ and integrating relative to x on $[0, \infty)$. For $\gamma > 2k$, $\text{Re } s = \gamma$ is the vertical path of integration. Thus, we need to prove the identity (4.3.2) follows from the equation

$$\begin{aligned} \int_0^\infty e^{-y\sqrt{x}}x^{-\frac{1}{2}} \left\{ \frac{1}{\Gamma(\rho+1)} \sum_{0 \leq m \leq x}' a_m (x-m)^\rho \right\} dx \\ = \int_0^\infty e^{-y\sqrt{x}}x^{-\frac{1}{2}} \left\{ \frac{1}{2\pi i} \int_{(\gamma)} \frac{\varphi(s)\Gamma(s)x^{s+\rho}}{\Gamma(s+\rho+1)} ds \right\} dx, \end{aligned} \quad (4.3.4)$$

where $\rho \in \mathbb{Z}_{\geq 0}, \rho \geq \gamma - 2k + \frac{1}{2}$, and $y \in \mathbb{R}^+$. Chandrasekharan and Narasimhan [2] for $\lambda_n \in \mathbb{R}^+, \lambda_n \rightarrow \infty$ as $n \rightarrow \infty$ showed the identity

$$\sum_{n=1}^{\infty} a_n \int_{\lambda_n}^{\infty} \frac{(x-n)^\rho}{\Gamma(\rho+1)} e^{-y\sqrt{x}}x^{-\frac{1}{2}} dx = 2(-2)^\rho \left(\frac{1}{y} \frac{d}{dy} \right)^\rho \left[\frac{1}{y} \sum_{n=1}^{\infty} a_n e^{-y\sqrt{\lambda_n}} \right]. \quad (4.3.5)$$

Then by the identity in (4.3.5) with $\lambda_n = m$, we have

$$\sum_{m=1}^{\infty} a_m \int_m^{\infty} \frac{(x-m)^\rho}{\Gamma(\rho+1)} e^{-y\sqrt{x}}x^{-\frac{1}{2}} dx = 2(2)^\rho \left(\frac{-1}{y} \frac{d}{dy} \right)^\rho \left[\frac{1}{y} \sum_{m=1}^{\infty} a_m e^{-y\sqrt{m}} \right].$$

Now on the left side of (4.3.4) the interchange of integration and summation is justified for $\rho \geq 0$. Then

$$\begin{aligned} \int_0^\infty e^{-y\sqrt{x}}x^{-\frac{1}{2}} \left\{ \frac{1}{\Gamma(\rho+1)} \sum_{0 \leq m \leq x}' a_m (x-m)^\rho \right\} dx &= \frac{a_0}{\Gamma(\rho+1)} \int_0^\infty e^{-y\sqrt{x}}x^{\rho-\frac{1}{2}} dx \\ &+ \frac{1}{\Gamma(\rho+1)} \times \\ &\sum_{1 \leq m \leq x} a_m \int_m^\infty e^{-y\sqrt{x}}x^{-\frac{1}{2}} (x-m)^\rho dx. \end{aligned}$$

Thus applying (4.3.5) we have

$$\int_0^\infty e^{-y\sqrt{x}} x^{-\frac{1}{2}} \left\{ \frac{1}{\Gamma(\rho+1)} \sum'_{0 \leq m \leq x} a_m (x-m)^\rho \right\} dx = \frac{2^{2\rho+1}}{y^{2\rho+1}} a_0 \frac{\Gamma(\rho + \frac{1}{2})}{\sqrt{\pi}} + 2^{\rho+1} \left(\frac{-1}{y} \frac{d}{dp} \right)^\rho \left[\frac{1}{y} \sum_{m=1}^\infty a_m e^{y\sqrt{m}} \right]. \quad (4.3.6)$$

Now assuming $\Phi_F(s) = \left(\frac{2\pi}{\lambda}\right)^{-s} \Gamma(s)\varphi(s)$, where $\text{Res} > \beta$ the right-hand side of (4.3.4) may be written as

$$W(s, \lambda) = \int_0^\infty e^{-y\sqrt{x}} x^{-\frac{1}{2}} \left\{ \frac{1}{2\pi i} \int_{(\gamma)} \left(\frac{2\pi}{\lambda}\right)^s \frac{\Phi_F(s)x^{s+\rho}}{\Gamma(s+\rho+1)} ds \right\} dx,$$

for $\gamma \geq \beta$. We may interchange the order of integration for $\rho \geq 0$. Then we have

$$W(s, \lambda) = \frac{1}{2\pi i} \int_{(\gamma)} \left\{ \left(\frac{2\pi}{\lambda}\right)^s \frac{\Phi_F(s)}{\Gamma(s+\rho+1)} \int_0^\infty e^{-y\sqrt{x}} x^{s+\rho-\frac{1}{2}} dx \right\} ds.$$

Using integration by substitution and applying the formula in (1.3.4) for $\Gamma(s)$, we obtain

$$W(s, \lambda) = \frac{1}{2\pi i} \int_{(\gamma)} \left(\frac{2\pi}{\lambda}\right)^s \Phi_F(s) \frac{2^{2\rho+2s+1}}{y^{2\rho+2s+1}} \frac{\Gamma(s+\rho+\frac{1}{2})}{\sqrt{\pi}}.$$

To evaluate this integral we shall consider a rectangular region with vertices $2k - \gamma \pm iT$ and $\gamma \pm iT$, $s = \sigma + it$ on its interior. Using Cauchy's residue theorem, the line of integration oriented with positively to $\text{Res} = 2k - \gamma$. If $\rho \geq \gamma - 2k + \frac{1}{2}$ the poles of $\Gamma(s + \rho + \frac{1}{2})$ lie to the left of the vertical path. Then the poles of the integrand in the interval $[2k - \gamma, \gamma]$ arise from the function $\Phi_F(s)$. Applying Stirling's formula and the Phragmen-Lindelof theorem the integrals along both horizontal paths tend to 0 as $T \rightarrow \infty$, and thus $W(s, \lambda)$ may be written as

$$W(s, \lambda) = w_1(s, \lambda) + w_2(s, \lambda),$$

where

$$w_1(s, \lambda) = \frac{1}{2\pi i} \int_{(2k-\gamma)} \left(\frac{2\pi}{\lambda}\right)^s \Phi_F(s) \frac{\Gamma(s+\rho+\frac{1}{2})}{\sqrt{\pi}} \frac{2^{2s+2\rho+1}}{y^{2s+2\rho+1}} ds,$$

and

$$w_2(s, \lambda) = \sum_{s \in \text{pol set } \Phi(s)} \text{Res} \left\{ \left(\frac{2\pi}{\lambda}\right)^s \Phi(s) \frac{\Gamma(s+\rho+\frac{1}{2})}{\sqrt{\pi}} \frac{2^{2s+2\rho+1}}{y^{2s+2\rho+1}} \right\}.$$

To evaluate $w_1(s, \lambda)$ recall the functional equation

$$\Phi_F(2k - s) = e^{\pi ik} \nu(T) \Phi_F(s)$$

and substitute in to the integral of $w_1(s, \lambda)$. Then we have

$$w_1(s, \lambda) = \bar{\nu}(T) e^{-\pi ik} \frac{2^{2\rho+1}}{y^{2\rho+1}} \left\{ \frac{1}{2\pi i} \int_{(2k-\gamma)} \left(\frac{2\pi}{\lambda} \right)^s \Phi_F(2k-s) \frac{2^{2s} \Gamma(s + \rho + \frac{1}{2})}{y^{2s} \sqrt{\pi}} ds \right\}.$$

Now replace θ by $2k-s$ and then setting $\Phi_F(\theta) = \left(\frac{2\pi}{\lambda}\right)^{-\theta} \Gamma(\theta) \sum_{m=1}^{\infty} a_m m^{-\theta}$. By organizing the expressions $w_1(s, \lambda)$ may be written as

$$w_1(s, \lambda) = \bar{\nu}(T) e^{-\pi ik} \frac{2^{2\rho+1}}{y^{2\rho+1}} \sum_{m=1}^{\infty} a_m \left\{ \frac{1}{2\pi i} \int_{(\gamma)} \Gamma(\theta) \Gamma(2k-\theta+\rho+\frac{1}{2}) \left(\frac{16m\pi^2}{\lambda^2 y^2} \right)^{-\theta} d\theta \right\}.$$

The interchange of the integration with the summation is justified for $\rho > \beta - 2k + \frac{1}{2}$.

Replace θ by $-\theta$ and using formula (1.3.2) after simplifying, we obtain

$$w_1(s, \lambda) = \frac{\bar{\nu}(T) e^{-\pi ik}}{\sqrt{\pi}} 2^{4k+2\rho+1} \Gamma\left(2k + \rho + \frac{1}{2}\right) \left(\frac{2\pi}{\lambda}\right)^{2k} \sum_{m=1}^{\infty} \frac{a_m}{\left(y^2 + \left(\frac{4\pi}{\lambda}\right)^2 m\right)^{2k+\rho+\frac{1}{2}}}. \quad (4.3.7)$$

The series on (4.3.7) converges absolutely for $\text{Re } y \in \mathbb{R}^+$, provided $\rho > \beta - 2k + \frac{1}{2}$.

To proceed with evaluation of $w_2(s, \lambda)$ recall that in Theorem 4.1 we have that $\Phi_F(s) = D_k(s) + M_k(s) + L_k(s)$, where $D_k(s)$, $M_k(s)$ and $L_k(s)$ are functions in (4.1.4), (4.1.5), and (4.1.6) respectively. For $\rho \geq \gamma - 2k + \frac{1}{2}$ the function $\left(\frac{2\pi}{\lambda}\right)^s \frac{\Gamma(s+\rho+\frac{1}{2})}{\sqrt{\pi}} \frac{2^{2s+2\rho+1}}{y^{2s+2\rho+1}}$ is analytic in the region bounded by $\text{Res} = 2k - \gamma$ and $\text{Res} = \gamma$, thus the poles of the integrand in the interval $[2k - \gamma, \gamma]$ arise from $\Phi_F(s)$. We also note that $D_k(s)$ is entire, $M_k(s)$ has simple poles at $(s = 2k, 0)$ and $L_k(s)$ has a pole at $s = 2k + \alpha_j$ of order $M_j + 1$, where $\alpha_j \in \mathbb{C}$, $M_j \in \mathbb{Z}_{\geq 0}$. Therefore by computing the residues of M_k and L_k after simplifying, we obtain

$$\begin{aligned} w_2(s, \lambda) &= \frac{2^{2\rho+1}}{\sqrt{\pi} y^{\rho+1}} a_0 \nu(T) e^{\pi ik} \left(\frac{8\pi}{\lambda y^2} \right)^{2k} \Gamma\left(2k + \rho + \frac{1}{2}\right) - \frac{2^{2\rho+1}}{\sqrt{\pi} y^{\rho+1}} a_0 \Gamma\left(\rho + \frac{1}{2}\right) \\ &+ \frac{2^{2\rho+1}}{\sqrt{\pi} y^{\rho+1}} \nu(T) e^{\pi ik} \left(\frac{8\pi}{\lambda y^2} \right)^{2k} \Gamma\left(2k + \rho + \alpha_j + \frac{1}{2}\right) \sum_{j=1}^N i^{\alpha_j} \sum_{t=0}^{M_j} \beta_{jt} \left(\frac{i\pi}{2} \right)^t. \end{aligned} \quad (4.3.8)$$

Therefore by rewriting $W(s, \lambda)$ as combination of (4.3.7) and (4.3.8) and substituting the respective expressions in to the right-hand side of (4.3.4), with simple rearrangement, we obtain the identity (4.3.2), provided that $\rho \in \mathbb{Z}_{\geq 0}$, $\rho \geq \beta - 2k + \frac{1}{2}$ and $y \in \mathbb{R}^+$, where $\sum_{m=1}^{\infty} \frac{|a_m|}{m^\beta} < \infty$.

Proof of converse of the theorem

To prove the converse it suffices to show that (4.3.2) implies (4.2.2). Following the technique of Chandrasekharan and Narasimhan in [2], multiply (4.3.2) by $e^{y\sqrt{x}}$, with $\text{Re } y > 0$, and $x > 0$ and integrate the expression along a vertical path $\text{Re } s = \theta$, where $\theta > 0$. The left hand side of the transformed equation can be evaluated using the formula

$$\sum_{m=1}^{\infty} a_m \frac{1}{2\pi i} \int_{(\theta)} e^{y\sqrt{x}} \left(\frac{-1}{y} \frac{d}{dy} \right)^{\rho} \left[\frac{1}{y} e^{-y\sqrt{m}} \right] dy = \frac{1}{\Gamma(\rho+1)} \sum'_{m \leq x} (x-m)^{\rho} 2^{-\rho}, \quad (4.3.9)$$

while the right hand side of (4.3.2) we calculate the integral of each terms one by one . So put

$$h_1(y, \lambda) = \frac{1}{2\pi i} \int_{(\theta)} e^{y\sqrt{x}} \frac{2^{\rho}}{\sqrt{\pi}} \bar{\nu}(T) i^{-2k} 2^{4k} \Gamma \left(2k + \rho + \frac{1}{2} \right) \left(\frac{2\pi}{\lambda} \right)^{2k} \sum_{m=1}^{\infty} \frac{a_m}{(y^2 + (\frac{4\pi}{\lambda})^2 m)^{2k+\rho+\frac{1}{2}}} dy,$$

$$h_2(y, \lambda) = \frac{1}{2\pi i} \int_{(\theta)} e^{y\sqrt{x}} \frac{2^{\rho}}{\sqrt{\pi}} \frac{a_0 \Gamma(2k + \rho + \frac{1}{2})}{y^{4k+2\rho+1}} \nu(T) e^{\pi i k} \left(\frac{8\pi}{\lambda} \right)^{2k} dy,$$

$$h_3(y, \lambda) = \frac{1}{2\pi i} \int_{(\theta)} e^{y\sqrt{x}} \frac{2^{\rho+1}}{\sqrt{\pi}} a_0 \frac{\Gamma(\rho + \frac{1}{2})}{y^{2\rho+1}} dy,$$

and

$$h_4(y, \lambda) = \frac{1}{2\pi i} \int_{(\theta)} e^{y\sqrt{x}} \frac{2^{\rho}}{\sqrt{\pi}} \frac{\Gamma(2k + \rho + \alpha_j + \frac{1}{2})}{y^{2\rho+1}} \left(\frac{8\pi}{\lambda y^2} \right)^{2k} \nu(T) e^{\pi i k} \sum_{j=1}^N \left(\frac{8\pi i}{\lambda y^2} \right)^{\alpha_j} \sum_{t=0}^{M_j} \beta_{jt} \left(\frac{i\pi}{2} \right)^t$$

The evaluation of $h_1(y, \lambda)$ follows as in [2]. The interchange of integration and summation being justified for $\rho > \beta - 2k + \frac{1}{2}$ in the right-hand side of $h_1(y, \lambda)$. Applying the formula in 1.3.9 and simplifying each expressions, we obtain

$$h_1(y, \lambda) = \bar{\nu}(T) e^{-\pi i k} \left(\frac{\lambda}{2\pi} \right)^{\rho} 2^{-\rho} \sum_{m=1}^{\infty} a_m \left(\frac{x}{m} \right)^{\frac{2k+\rho}{2}} J_{2k+\rho} \left(\frac{4\pi\sqrt{mx}}{\lambda} \right). \quad (4.3.10)$$

To evaluate $h_2(y, \lambda)$ we rewrite as

$$h_2(y, \lambda) = \frac{2^{\rho}}{\sqrt{\pi}} a_0 \nu(T) e^{\pi i k} \left(\frac{8\pi}{\lambda} \right)^{2k} \Gamma \left(2k + \rho + \frac{1}{2} \right) \frac{1}{2\pi i} \int_{(\theta)} \frac{e^{y\sqrt{x}}}{y^{4k+2\rho+1}} dy.$$

Put

$$g_1(y) = \frac{1}{2\pi i} \int_{(\theta)} \frac{e^{y\sqrt{x}}}{y^{4k+2\rho+1}} dy.$$

To compute $g_1(y)$ we use the formula in (1.3.3) and apply the formula in (1.3.4) for $\frac{1}{\Gamma(4k+2\rho+1)}$. After simplification we obtain

$$h_2(y, \lambda) = 2^{-\rho} a_0 \nu(T) e^{\pi i k} \left(\frac{2\pi}{\lambda} \right)^{2k} \frac{x^{2k+\rho}}{\Gamma(2k + \rho + 1)}. \quad (4.3.11)$$

We evaluate $h_3(y, \lambda)$ applying the techniques analogous to those used in evaluating $h_2(y, \lambda)$. Thus, we conclude that

$$h_3(y, \lambda) = \frac{2a_0}{2^\rho} \frac{x^\rho}{\Gamma(\rho + 1)}. \quad (4.3.12)$$

Next we consider $h_4(y, \lambda)$ > The interchange of the integration and the finite double sums is justified and we write as:

$$h_4(y, \lambda) = \frac{2^\rho}{\sqrt{\pi}} \nu(T) e^{\pi i k} \left(\frac{8\pi}{\lambda} \right)^{2k} \Gamma \left(2k + \rho + \alpha_j + \frac{1}{2} \right) \sum_{j=1}^N \left(\frac{8\pi i}{\lambda} \right)^{\alpha_j} \sum_{t=0}^{M_j} \beta_{jt} \left(\frac{i\pi}{2} \right)^t \times \frac{1}{2\pi i} \int_{(\theta)} \frac{e^{y\sqrt{x}}}{y^{4k+2\rho+\alpha_j+1}} dy.$$

Now put

$$g_2(y) = \frac{1}{2\pi i} \int_{(\theta)} \frac{e^{y\sqrt{x}}}{y^{4k+2\rho+\alpha_j+1}} dy.$$

Applying (1.3.3) in $g_2(y)$ and using formula (1.3.4) for $\frac{1}{\Gamma(4k+2\rho+1\alpha_j+1)}$. By substituting the expression derived for $g_2(y)$ and simplifying, we obtained

$$h_4(y, \lambda) = 2^{-\rho} \nu(T) e^{\pi i k} \frac{x^{2k+\rho}}{\Gamma(2k + \rho + \alpha_j + 1)} \left(\frac{2\pi}{\lambda} \right)^{2k} \sum_{j=1}^N \left(\frac{2\pi i x}{\lambda} \right)^{\alpha_j} \sum_{t=0}^{M_j} \beta_{jt} \left(\frac{i\pi}{2} \right)^t. \quad (4.3.13)$$

Therefore. we have

$$\frac{1}{\Gamma(\rho + 1)} \sum'_{m \leq x} a_m (x - m)^\rho 2^{-\rho} = h_1 + h_2 + h_3 + h_4. \quad (4.3.14)$$

By substituting the expressions (4.3.10), (4.3.11), (4.3.12), and (4.3.13) in to $h_1, h_2, h_3,$ and h_4 respectively and multiply both sides of the equation (4.3.14) by 2^ρ the identity in (4.2.2) holds for $\rho \geq 2\beta - 2k + \frac{1}{2}$. This completes the proof of the converse, since (4.2.2) is equivalent to (4.3.1) we showed in Theorem 4.4, and this concludes the proof of the theorem. \square

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