ADDIS ABABA UNIVERSITY
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GRADUTE SEMINAR REPORT

ON

Fredholm Alternative Theorem and Higher Order Ordinary Differential Equations

(Submitted in partial fulfillment of M.Sc.Degree in Mathematics)

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Preface

It is understood that modernization and civilization of our world, we live, are highly rely on scientific findings. These scientific findings are also either directly or indirectly interacted with the discipline of mathematics. Under the different streams of mathematics, higher order ordinary differential equations are applicable in many fields of area; example in radiative energy transfer, oscillation of string membrane or axle, engineering discipline,etc. Therefore, it is vital for one to study the concepts of higher order ordinary differential equations in depth. Among the basic concepts of higher order ordinary differential equations, the primary one is to determine whether a given equation possesses solution or not. If it has a solution, we need to determine whether it has a unique or infinite solution as much as possible. So as to know, the existence and uniqueness or infinite solutions of a higher order ordinary differential equation, the concept of Fredholm Alternative Theorem plays great role.

In this seminar paper, the concept of Fredholm Alternative theorem and its application on higher ordinary differential equations are dealt in depth.

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1.1 Introduction

Before coming to the main content, let us discuss on integral equation in which the unknown function appears under the integral sign. Here solving an integral equation means finding the unknown function satisfying the given integral equation.

Integral equations are classified according to three different dichotomies, creating six different kinds

- **Based on limit of integration**
  - **Both fixed**: Fredholm equation
  - **One variable**: Voltera equation
- **Placement of unknown function**
  - **Only inside integral:** first kind
  - **Both inside and outside integral**: second kind
- **Nature of known function**
  - **Identically zero**: homogeneous
  - **Not identically zero**: inhomogeneous (non homogeneous)

Voltera integral equation

It is an integral equation in which the upper limit of integration is variable.

Fredholm integral equation

An integral equation with a fixed domain of integration i.e. the limits of integrals are constants.

We classify fredholm integral equations as follow:

i) Fredholm integral equations of first kind

It is the most frequently appearing integral equation having the form,

$$\phi(x) = \lambda \int_a^b k(x,t) f(t)dt \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots.(1.1.1)$$

Here f is an unknown function, ϕ is known function and k is an other known function of two variables, often called the kernel function. The parameter λ is constant. An equation of the form (1.1.1) where the unknown function appears only under the integral sign is called Fredholm Integral equation of the fist kind.

ii) Fredholm Integral equation of the second kind

An equation of the form,

$$\phi(x) = f(x) + \lambda \int_b^a k(x,t)\, f(t)\, dt \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (1.1.2)$$

Where the unknown function appears under the integral sign and outside of it is called Fredholm integral equation of second kind.

In the equations (1.1.1) and (1.1.2) of the above, ϕ and k are two given functions and f is the unknown function, the function k is called the kernel or nucleus of the equation.

If the kernel k(x, t) is bounded and continuous then (1.1.1) and (1.1.2) are said to be non-singular.

However, if the range of integration is infinite, or if the kernel violates the above conditions (being bounded and continues), then (1.1.1) and (1.1.2) are said to be singular. If $\phi(x) = 0$, then the equations are called homogeneous, other wise inhomogeneous (nonhomogeneous).

1.2 The Fredholm Alternative

The Fredholm Alternative analyses the solutions of the non-homogeneous fredholm equation

$$y(x) = f(x) + \lambda \int_a^b k(x,t) y(t) dt \qquad x, t \in [a,b]$$

Where $f: [a,b] \to \Re$ and $k: [a,b]^2 \to \Re$ are continuous and λ is a constant. We shall establish it in the special and useful case when $k = k(x,t)$ is degenerate

(some authors say, of finite rank), that is when

$$K(x,t) = \sum_{j=1}^{n} g_j(x)\ h_j(t) \quad x, t \in [a,b]$$

and $g_j : [a,b] \to \Re$, $h_j : [a,b] \to \Re$ are continues $(j = 1, \ldots, n)$

Note that the summation is a finite one.

1.2.1 The simple case

To motivate both the statement and proof to the Fredholm Alternative Theorem, we now discuss an equation with perhaps the simplest degenerate kernel $K(x, t)$ of any interest;

namely,

$$K(x, t) = g(x)\, h(t), \quad x, t \in [a,b]$$

Where g and h are continuous on $[a, b]$ and neither of which is identically zero on $[a, b]$.

The equation is

$$y(x) = f(x) + \lambda \int_a^b g(x)\, h(t)\, y(t)\, dt \qquad \ldots\ldots\ldots\ldots\ldots\ (1.2.1)$$

$$= f(x) + \lambda\, M g(x) \quad \text{Where} \quad M = \int_a^b h(t)\, y(t)\, dt$$

We note immediately that the constant M depends on y. We discuss the equation in terms of its (non-homogeneous) transpose.

$$y(x) = f(x) + \lambda \int_a^b k(t, x)\, y(t)\, dt \qquad \ldots\ldots\ldots\ldots\ldots\ldots\ldots \qquad (1.2.2)$$

$$= f(x) + \lambda \int_a^b g(t) h(x) y(t) dt$$

$$= f(x) + \lambda N h(x) \ \textit{Where} \ \ N = \int_a^b y(t) g(t)\, dt$$

The corresponding homogeneous equation is

$$y(x) = \lambda \int_a^b g(x)\, h(t)\, y(t)\, dt \ = \lambda M\, g(x) \ \ldots\ldots\ldots\ldots\ \ldots\ldots\ldots\ldots\ldots (1.2.3)$$

And the corresponding (homogeneous) transpose is

$$y(x) = \lambda \int_a^b g(t)\, h(x)\, y(t)\, dt \ = \lambda\, Nh(x) \ldots\ldots\ldots \ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (1.2.4)$$

Multiplying (1.2.1) by h (x) and integrating with respect to x gives

$$y(x)\, h(x) = f(x)\, h(x) + \lambda\ M\, g(x)\, h(x)$$

$$\Rightarrow \quad y(x)\, h(x) - \lambda\ M\, g(x)\, h(x) = f(x)\, h(x)$$

Integrating with respect to x,

$$\int_a^b y(x)\, h(x)\, dx - \lambda M \int_a^b g(x)\, h(x)\, dx = \int_a^b f(x)\, h(x)\, dx$$

$$\Rightarrow \quad M - \lambda M \int_a^b g(x)\, h(x)\, dx = \int_a^b f(x)\, h(x)\, dx$$

$$\Rightarrow \quad (1 - \lambda \int_a^b g(x)\, h(x)\, dx)\, M = \int_a^b f(x)\, h(x)\, dx \; \ldots\ldots\ldots\ldots\ldots\ldots \; \ldots\ldots\ldots (1.2.5)$$

Again multiplying (1.2.2) by g(x) and integrating with respect to x gives

$$y(x) = f(x) + \lambda \int_a^b k(t,x)\, y(t)\, dt$$

$$\Rightarrow \quad y(x) = f(x) + \lambda N\, h(x), \quad N = \int_a^b g(t)\, y(t)\, dt$$

$$\Rightarrow \quad y(x)\, g(x) = f(x)\, g(x) + \lambda N\, g(x)\, h(x)$$

$$\Rightarrow \; y(x)\, g(x) - \lambda N\, g(x)\, h(x) = f(x)\, g(x)$$

Integrating with respect to x, we have

$$\int_a^b y(x)\, g(x)\, dx - \lambda N \int_a^b g(x)\, h(x)\, dx = \int_a^b f(x)\, g(x)\, dx$$

$$\Rightarrow (N - \lambda N \int_a^b g(x)\, h(x)\, dx) = \int_a^b f(x)\, g(x)\, dx$$

$$\Rightarrow (1 - \lambda \int_a^b g(x)\, h(x)\, dx)\, N = \int_a^b f(x)\, g(x)\, dx \; \ldots\ldots\ldots \; \ldots\ldots\ldots (1.2.6)$$

Hence from (1.2.5) and (1.2.6) we have that

$$(1 - \lambda \int_a^b g(x)\, h(x)\, dx)\, M = \int_a^b f(x)\, h(x)\, dx \quad , \quad M = \int_a^b h(t)\, y(t)\, dt$$

and

$$(1 - \lambda \int_a^b g(x)\, h(x)\, dx) N = \int_a^b f(x)\, g(x)\, dx \quad , \quad N = \int_a^b g(t)\, y(t)\, dt$$

If $\lambda \int_a^b g(x)h(x)dx \neq 1$, these equations allow the determination of (unique) M and N as

$$M = \frac{\int_a^b f(x)h(x)dx}{(1 - \lambda \int_a^b g(x)h(x)dx)}$$

and

$$N = \frac{\int_a^b f(x)g(x)dx}{(1-\lambda\int_a^b g(x)h(x)dx)}$$

Hence substituting these values in (1.2.1) and (1.2.2), we have

$$y(x) = f(x) + \lambda M g(x)$$

$$= f(x) + \frac{\lambda g(x)\int_a^b f(x)h(x)dx}{1-\lambda\int_a^b g(x)h(x)dx} \quad x \in [a,b]$$

and

$$y(x) = f(x) + \lambda N h(x)$$

$$= f(x) + \frac{\lambda h(x)\int_a^b f(x)g(x)dx}{1-\lambda\int_a^b g(x)h(x)dx}, \; x \in [a,b]$$

However, if $\lambda\int_a^b g(x)\, h(x)\, dx = 1$, then neither (1.2.1) nor (1.2.2) permits a unique solution. Indeed, from (1.2.5), (1.2.1) can not then have a solution unless

$$\int_a^b f(x)\, h(x)\, dx = 0 \; \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(1.2.7)$$

and from (1.2.6), the equation (1.2.2) can not have a solution unless

$$\int_a^b f(x)\, g(x)\, dx = 0$$

But every solution of (1.2.3) is of the form

$$y(x) = c\, g(x),\; x \in [a,b] \; \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(1.2.8)$$

where c is a constant, Because (1.2.3) is $y(x) = \lambda M g(x)$. Since λ and M are constants, we can let $c = \lambda M$ and then $y(x) = c\, g(x)$. Hence we can not have a solution unless the integral of f times the solution of (1.2.3) gives zero.

Further, if $y = cg$ on [a,b], for some constant c, and $\lambda\int_a^b g(x)\, h(x)\, dx = 1$, then

$\lambda M g(x) = c\, \lambda g(x) \int_a^b h(t)\, g(t)\, dt = c\, g(x) = y(x)$ for each x in $[a,b]$; and so (1.2.8) is the only solution of (1.2.3). Similarly, if $\lambda\int_a^b g(x)\, h(x)\, dx = 1$,

$$y(x) = d\, h(x),\; x \in [a,b]$$

where d is a constant, is the only solution of (1.2.4) .Thus, if there does not exist a unique solution to (1.2.1) (Which occurs here if $\lambda \int_a^b g(x)\, h(x)\, dx = 1$) then there are non-zero solutions of (1.2.3) and (1.2.4); and there can only be solutions of (1.2.1) provided 'the integral of f times the solution of (1.2.4)' is zero (here $\int_a^b f(x)\, h(x)\, dx = 0$).

In the case of this particularly simple kernel and when the 'consistency condition' $\int_a^b f(x)\, h(x)\, dx = 0$ is met, it is clear (by simple substitution) that $y = f(x)$ is a particular solution of (1.2.1). Further, if $y = Y$ is any solution of (2.1), then $y = Y - f$ is a solution of (1.2.3). Because $y(x) = f(x) + \lambda M g(x)$.

If Y is any solution of (1.2.1), then

$$Y = f + \lambda Mg$$

$$Y - f = \lambda\ Mg$$

Hence $Y - f$ is a solution of the homogeneous equation (1.2.3) i.e $y(x) = \lambda M g(x)$.

Therefore, if the consistency condition is met and there is no unique solution of (1.2.1), the complete solution of (1.2.1) is given, with arbitrary constant c by

$$y(x) = f(x) + cg(x),\ x \in [a, b]$$

(a particular solution plus the complete solution of (1.2.3))

Let us now consider the above idea using illustrative example

Example 1.2.1 Find the complete solution to each of the following Fredholm equations.

$$a)\ \ y(x) = x + \lambda \int_0^1 e^{x+t}\, y(t)\, dt\ \ ,\ x \in [0,1] \ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (1.2.9)$$

$$b)\ \ y(x) = x - \frac{2e^x}{e^2 - 1} + \lambda \int_0^1 e^{x+t}\, y(t)\, dt\ ,\ x \in [0,1] \ \ldots\ldots\ldots\ldots (1.2.10)$$

Solution:

(a)The kernel $e^{x+t} = e^x e^t$ is symmetric, and so (1.2.1) is the same as (1.2.2)

$$y(x) = x + \lambda Me^x \ \text{ where } \ M = \int_0^1 e^t\, y(t)\, dt$$

Where $g(x) = e^x$, $h(t) = e^t$, $f(x) = x$

Thus

$$\int_0^1 f(x)\, h(x)\, dx = \int_0^1 xe^x\, dx$$

$$= 1 \neq 0.$$

And also

$$\int_0^1 g(x)\, h(x)\, dx = \int_0^1 e^x e^x\, dx$$

$$= \int_0^1 e^{2x}\, dx$$

$$= ½\, (e^2 - 1)$$

We always find it useful to carry out such manipulation first. Multiplying (1.2.9) by $e^x = h(x)$ and integrating gives

$$(1 - \lambda \int_0^1 g(x)\, h(x)\, dx)\, M = \int_0^1 f(x)\, h(x)\, dx.$$

$$\Rightarrow \qquad (1 - \lambda \frac{(e^2-1)}{2})\, M = 1 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (1.2.11)$$

$$\Rightarrow \qquad M = \frac{2}{2-\lambda(e^2-1)}$$

Hence, if $\lambda \neq \dfrac{2}{e^2-1}$ and substituting for M, (1.2.9) has the unique continuous solution

$$y(x) = x + \frac{2\lambda}{2-(e^2-1)\lambda} e^x \text{ , } x \in [0, 1] \qquad (\text{using } y(x) = f(x) + \lambda\, M\, g(x))$$

When $\lambda = \dfrac{2}{e^2-1}$, (1.2.11) shows that there can be no solution to (1.2.9).

Now turning to (b), we have equation (1.2.1) and (1.2.2) to be the same.

Here in the equation,

$$y(x) = x - \frac{2e^x}{e^2-1} + \lambda \int_0^1 e^{x+t}\, y(t)\, dt$$

$$f(x) = x - \frac{2e^x}{e^2-1},\ g(x) = e^x,\ h(t) = e^t,\ M = \int_0^1 e^t\, y(t)\, dt$$

$$y(x) = x - \frac{2e^x}{e^2-1} + \lambda M e^x$$

Hence multiplying by $h(x) = e^x$ (1.2.10), and integrating with respect to x gives

$$(1 - \lambda \int_0^1 g(x)\, h(x)\, dx)\, M = \int_0^1 f(x)\, h(x)\, dx$$

Now $$\int_0^1 f(x)\, h(x)\, dx = \int_0^1 (x - \frac{2e^x}{e^2-1})\, e^x\, dx = \int_0^1 xe^x\, dx - \frac{2}{e^2-1}\int_0^1 e^{2x} dx$$

$$= 1 - \frac{2}{e^2-1} \cdot \frac{e^2-1}{2}$$

$$= 0$$

$$\int_0^1 g(x)\, h(x)\, d(x) = \int_0^1 e^x e^x dx = \int_0^1 e^{2x} dx = \frac{e^2-1}{2}$$

Hence $$(1 - \lambda \int_0^1 g(x)\, h(x)\, dx)\, M = \int_0^1 f(x)\, h\, (x)\, dx$$

$$\Rightarrow \quad (1 - \lambda\, (\frac{e^2-1}{2}))\, M = 0 \quad \text{……………………..……………… (1.2.12)}$$

and hence, if $\lambda \neq \frac{2}{e^2-1}$ has the unique continuous solution $y(x) = x - \frac{2e^x}{e^2-1}$, $x \in [0,1]$ since $M = 0$ .In the equation $(1 - \lambda\, (\frac{e^2-1}{2}))\, M = 0$ so long as $\lambda \neq \frac{2}{e^2-1}$.

When $\lambda = \frac{2}{e^2-1}$, (1.2.12) provides no restriction on M and the complete solution to (1.2.10) is given as

$$y(x) = x - \frac{2e^x}{e^2-1} + ce^x = x + c' e^x, \quad x \in [0,1]$$

Where c, c$'$ are arbitrary (though, of course, related) constants.

For both (a) and (b) the homogeneous/transposed homogeneous equation is $y(x) = \lambda Me^x$ having the unique solution $y = 0$ when $\lambda \neq \frac{2}{e^2-1}$, and when $\lambda = \frac{2}{e^2-1}$ the complete solution $y(x) = ce^x$ where c is an arbitrary constant. So, the consistency condition

$\int_a^b f(x)\, h(x)\, dx = 0$ in the above discussion is not met in case (a), as

$$\int_0^1 xe^x\, dx \neq 0$$

Where as it is met in case (b), where

$$\int_0^1 \left(x - \frac{2e^x}{e^2-1}\right) e^x\, dx = 0$$

These correspond to the 'right hand sides' of (1.2.11) and (1.2.12) being 1 and 0 respectively.

1.2.2 Some algebraic Preliminaries

In our proof of the Fredholm Alternative Theorem, we shall need to rely on some further elementary results from linear algebra.

Suppose A is the nxn matrix $A = (a_{ij})$ and that $A^T = (a_{ji})$ for i=1, …. n and j=1,…, n is the transpose of A. The system of linear equations

$$\begin{array}{ccccccccc}
a_{11}x_1 & + & a_{12}x_2 & + & \cdot\ \cdot\ \cdot & + & a_{1n}x_n & = & b_1 \\
a_{21}x_1 & + & a_{22}x_2 & + & \cdot\ \cdot\ \cdot & + & a_{2n}x_n & = & b_2 \\
a_{31}x_1 & + & a_{32}x_2 & + & \cdot\ \cdot\ \cdot & + & a_{3n}x_n & = & b_3 \\
 & & & & \cdot & & & & \\
 & & & & \cdot & & & & \\
 & & & & \cdot & & & & \\
a_{n1}x_1 & + & a_{n2}x_2 & + & \cdot\ \cdot\ \cdot & + & a_{nn}x_n & = & b_n
\end{array}$$

May by conveniently expressed as

$$\sum_{j=1}^{n} a_{ij}x_j = b_i, \qquad (i = 1 \ldots n)$$

Or more compactly as

$$AX = b$$

Where $x = (x_i)$ and $b = (b_i)$ are column vectors.

Proposition 1

a) $Rank\ A = Rank\ (A^T)$

b) If $Rank\ (A) = n$, then $det\ A \neq 0$ and the non-homogeneous systems

$$Ax = b,\ A^T y = b$$

have unique solutions $x = (x_i)$, $y = (y_i)$

c) If $1 \in Rank\ (A) = n - r \leq n - 1$, then $det\ A = 0$ and the systems of homogeneous equations $Ax = 0$, $A^T y = 0$ both have maximal linearly independent sets of solutions consisting of r elements. If $y^k = (y_i^k)$, $k = 1, \ldots, r$ is such a set for

$A^T y = 0$, then $Ax = b$ has solutions if and only if

$$b^T y^K = b_1 y_1^k + \ldots.. + b_n y_n^k = 0 \quad (k = 1, \ldots, r)$$

Proposition 2

The number of distinct solutions μ of the characteristic equation of A,

$$det\ (\mu I - A) = 0$$

is at most n.

1.2.3 The Fredholm Alternative Theorem

The theorem analyses the solution of the equation

$$y(x) = f(x) + \lambda \int_a^b k(x,t)\, y(t)\, dt,\ x,t \in [a,b] \quad \ldots\ldots\ldots\ldots(1.2.13)$$

in terms of the solutions of the corresponding equations

$$y(x) = f(x) + \lambda \int_a^b k(t,x)\, y(t)\, dt, \quad t,x \in [a,b] \quad \ldots\ldots\ldots\ldots (1.2.14)$$

$$y(x) = \lambda \int_a^b k(x,t)\, y(t)\, dt,, \quad t,x \in [a,b] \quad \ldots\ldots\ldots\ldots(1.2.15)$$

$$y(x) = \lambda \int_a^b k(t,x)\, y(t)\, dt, \quad t,x \in [a,b] \quad \ldots\ldots\ldots\ldots(1.2.16)$$

Where $f{:}[a,b] \rightarrow R$ and the kernel $k{:}\ [a,b]^2 \rightarrow R$ are continuous and λ is a constant.

Theorem1.2.1 (The Fredholm alternative)

For each (fixed) λ exactly one of the following statements is true.

1. The equation (1.2.13) possesses a unique continuous solution and, in particular, $f \equiv 0$ on [a,b] implies that $y \equiv 0$ on [a,b]. In this case (1.2.14) also possesses a unique continuous solution.
2. The equation (1.2.15) possesses a finite maximal linearly independent set of, say, r continues solutions $y_1, \ldots, y_r$

(for r>0). In this case, (1.2.16) also possesses a maximal linearly independent set of r continuous solutions $z_1 \ldots z_r$ and (1.2.13) has solutions if and only if the 'consistency conditions',

$$\int_a^b f(x)\, z_k(x)\, d_x = 0, \qquad (k = 1, \ldots, r).$$

are all met. When they are, the complete solution of (1.2.13) is given by

$$y(x) = g(x) + \sum_{i=1}^{n} c_i y_i(x), \quad x \in [a,b].$$

Where $c_1, \ldots, c_r$ are arbitrary constants and $g: [a,b] \to R$ is any continuous solution of (1.2.13). The above is true for general kernels, but when $k = k(x, t)$ is the degenerate kernel given by

$$k(x\,,\,t) = \sum_{j=1}^{n} g_j(x)\, h_j(t), \quad x,t \in [a,b] \ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (1.2.17)$$

There are at most n values of λ at which (2) occurs.

Proof

We may assume that each of the sets of functions $(g_1, \ldots, g_n)$, $(h_1, \ldots, h_n)$ is linearly independent on $[a,b]$; otherwise, we may express each of their elements interms of linearly independent subsets to reach the required form (For instance, if $(g_1, g_2, \ldots, g_n)$ is linearly independent but h_n is dependent on the linearly independent subset $(h_1, \ldots, h_{n-1})$ and

$$h_n(t) = \sum_{j=1}^{n-1} d_j\, h_j(t), \quad t \in [a,b]$$

Where d_j is a constant (j = 1, 2, …., n-1), then

$$k(x, t) = \sum_{j=1}^{n-1} g_j(x)\, h_j(t) + g_n(x) \sum_{j=1}^{n-1} d_j\, h_j(t)$$

$$= \sum_{j=1}^{n-1} (g_j(x) + d_j\, g_n(x))\, h_j(t)$$

for each x and each t in *[a , b]*. The reader may easily verify that each of the sets

$$(g_1 + d_1 g_n, \ldots, g_{n-1} + d_{n-1} g_n), \quad (h_1, \ldots, h_{n-1})$$

is linearly independent on [a ,b].

For k given by (1.2.17), the equations (1.2.13), (1.2.14), (1.2.15), (1.2.16) may be written as

$$y(x) = f(x) + \lambda \sum_{j=1}^{n} M_j g_j(x), \text{ where } M_j = \int_a^b h_j(t)\, y(t)\, dt,\ j = 1, \ldots, n\ . \quad (1.2.18)$$

$$y(x) = f(x) + \lambda \sum_{j=1}^{n} N_j h_j(x), \text{ where } N_j = \int_a^b g_j(t)\, y(t)\, dt,\ j = 1, \ldots, n \quad (1.2.19)$$

$$y(x) = \lambda \sum_{j=1}^{n} M_j g_j(x), \ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (1.2.20)$$

$$y(x) = \lambda \sum_{j=1}^{n} N_j h_j(x), \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (1.2.21)$$

For x in [a, b].So, a solution of (1.2.13) is determined once we know the values of M $_j$'s (which are, of course, defined in terms of the unknown quantity y).

First let us note that $\lambda = 0$ corresponds to the unique continuous solution $y = f$ of both (1.2.13) and (1.2.14): (1) occurs. From this point in the proof, we assume that $\lambda \neq 0$.

We now convert the problem to one of algebra, involving a system of simultaneous linear equations.

Multiplying (1.2.18) through by $h_i(x)$ and integrating with respect to x gives, on slight re-arrangement,

$$\mu M_i - \sum_{j=1}^{n} a_{ij} M_j = b_i \quad (i = 1, \ldots, n)$$

Once we define

$$\mu = \frac{1}{\lambda} \qquad a_{ij} = \int_a^b g_j(x)\, h_i(x)\, dx \qquad b_i = \mu \int_a^b f(x)\, h_i(x)\, dx.$$

For $i = 1, \ldots, n$ and j = 1, …n. These n equations in the M$_i$'s can be rewritten in the compact form

$$(\mu I - A)\, M = b \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (1.2.22)$$

Where I is identity matrix, A is the matrix

$A = (a_{ij})$ and $M = (M_i)$, $b = (b_i)$ are column vectors

Similarly, working in turn with (1.2.19), (1.2.20) and (1.2.21), we have

$$(\mu I - A)^T N = (\mu I - A^T)\, N = C \;\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (1.2.23)$$

Where $A^T = (a_{ji})$ is the transpose of A, $M = (m_i)$, $C = (c_i)$ are column vectors, and

$$c_i = \mu \int_a^b f(x)\, g_i(x)\, dx \quad (i = 1, \ldots., n).$$

$$(\mu I - A)\, M = 0 \;\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (1.2.24)$$

and

$$(\mu I - A)^T\; N = 0 \;\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (1.2.25)$$

By proposition 1, if $Rank\ (\mu I - A) = n$, (1.2.22) and (1.2.23) have unique solutions $M = (m_i)$ and $N = (N_i)$. Substitution of these values in (1.2.18) and (1.2.19) gives unique solutions to (1.2.13) and (1.2.14) in this case when (1) occurs.

We are left with establishing that when $1 \leq Rank\ (\mu I - A) = n - r \leq n - 1$, (2) must occur. In this case, both (1.2.24) and (1.2.25) have maximal linearly independent sets $M^k = (M_i^k)$ and $N^k = (N_ik)$, $K = 1, \ldots, r$ consisting of r elements.

Put

$$y_k(x) = \lambda \sum_{j=1}^{n} M_j^k\, g_j(x)$$

and

$$z_k(x) = \lambda \sum_{j=1}^{n} N_j^k\, h_j(x)$$

for each x in [a,b] and $k = 1, \ldots, r$. Each y_k is clearly a solution of (1.2.20) and each z_k a solution of (1.2.21)

We claim now that the set $(y_k : k = 1 \ldots r)$ is linearly independent on [a, b]. Suppose that

$$\sum_{k=1}^{r} c_k\, y_k = 0$$

For some constants $c_1, \ldots, c_r$. Then

$$\lambda \sum_{j=1}^{n} \Big(\sum_{k=1}^{r} c_k M_j^k\Big)\, g_j = 0$$

But, the g_j's are linearly independent; so

$$\sum_{k=1}^{r} c_k M_j^k = 0 \quad for\, j = 1, \ldots, n$$

That is

$$\sum_{k=1}^{r} c_k M^k = 0$$

Since the M^k's are also linearly independent, $c_k = 0$ for each $k = 1 \ldots r$ and the y_k's are thus independent also. Similarly, $(Z_k: k = 1, \ldots r)$ is linearly independent. These linearly independent sets (y_k), (z_k) can not be enlarged whilst remaining independent, for the argument above could then be reversed for the augmented set to contradict the maximality of the independent sefs (M^k) and (N^k). We have thus shown that maximal linearly independent sets of solutions of (1.2.15) and (1.2.16) have the same finite non-zero number of elements when (1) does not occur.

It remains to consider (1.2.13) in this (2) case. Proposition 1 asserts in this context that (1.2.22) has solutions if and only if

$$b^T N^k = 0, \text{ for } k = 1, \ldots, r.$$

that is, employing the definitions of b and $N^k = (N_j^k)$ and noting that N^k corresponds to the solution Z_k of (1.2.21), if and only if

$$\sum_{j=1}^{n} \left(\int_a^b f(x)\, h_j(x)\, dx\right) \left(\int_a^b g_j(t)\, Z_k(t)dt\right) = 0$$

For $k = 1 \ldots r$. This is equivalent to

$$\int_a^b \left[\int_a^b \left(\sum_{j=1}^{n} h_j(x)\, g_j(t)\right) Z_k(t)\, dt)\right] f(x)\, dx = 0$$

For $k = 1 \ldots r$.

Using again the fact that z_k is a solution of (1.2.16), the necessary and sufficient condition for (1.2.13) to have solutions reduces to $\int_a^b z_k(x) f(x) dx = 0$ for $k=1, \ldots r$

Suppose finally that g is a particular continuous solution of (1.2.18) (existing because the consistency conditions are met). Then, if y is any other solution of (1.2.18), y-g is a solution of

(1.2.20) and hence expressible as a linear combination of the elements of the maximal independent set of solutions (y_k):

$$y - g = \sum_{i=1}^{r} c_i y_i$$

For some constants c_1, …., c_r. The Fredholm Alternative Theorem is thus established for degenerate kernels.

<u>Note</u>:- The reader will want to remember to distinguish between the 'eigen values' λ corresponding to non-zero solutions of (1.2.20) in the case (2) and the eigen values

$\mu = \dfrac{1}{\lambda}$ which satisfy the matrix equation (1.2.24).

We now show how the method used in the proof of the Fredholm Theorem can be used in a practical example.

Example 1.2.2 Solve the integral equation

$$y(x) = f(x) + \lambda \int_0^{2\pi} \sin(x+t)\, y(t)\, dt \qquad t, x \in [0, 2\pi]$$

In two cases

$$(a)\ f(x) = 1,\ x \in [0, 2\pi] \qquad\qquad (b)\ f(x) = x,\ x \in [0, 2\pi]$$

Solution:

(a)The equation may be written as follow.Since (1.2.18) and (1.2.19) are the same.

$$y(x) = f(x) + \lambda M_1 \sin x + \lambda M_2 \cos x$$

where, $M_1 = \int_0^{2\pi} y(t) \cos t\, dt,$ $\qquad M_2 = \int_0^{2\pi} y(t) \sin t\, dt.$

Note that

$$\int_0^{2\pi} \cos^2 x\, dx = \int_0^{2\pi} \sin^2 x\, dx = \pi, \qquad \int_0^{2\pi} x \sin x\, dx = -2\pi$$

$$\int_0^{2\pi} x \cos x\, dx = \int_0^{2\pi} \cos x\, dx = \int_0^{2\pi} \sin x \cos x\, dx = 0$$

Multiplying (1.2.18) though by *cosx* and integrating with respect to *x* and carrying out the same operation with *sin x* give

$$M_1 - \lambda\, \pi\, M_2 = \int_0^{2\pi} f(x)\cos x\, dx$$

$$-\lambda\, \pi\, M_1 + M_2 = \int_0^{2\pi} f(x)\sin x\, dx \quad \text{..........................(1.2.26)}$$

When the determinant of the coefficients of the M_i 's is non-zero.

$$\begin{vmatrix} 1 & -\lambda\pi \\ -\lambda\pi & 1 \end{vmatrix} = 1-\lambda^2\pi^2 \neq 0$$

These equations have the unique solutions

$$M_1 = \frac{1}{1-\lambda^2\pi^2}\int_0^{2\pi} f(x)\,(\cos x + \lambda\pi \sin x)\, dx$$

$$M_2 = \frac{1}{1-\lambda^2\pi^2}\int_0^{2\pi} f(x)\,(\sin x + \lambda\pi \cos x)\, dx$$

In case (a), we thus have the solution

$$y(x) = 1 \qquad ,\ x \in [0,2\pi]$$

and in case (b), the solution

$$y(x) = x - \frac{2\pi\lambda}{1-\lambda^2\pi^2}(\lambda\pi \sin x + \cos x) \ \text{ provided } 1 - \lambda^2\pi^2 \neq 0.$$

We have dealt with (1)

The case (2) can only occur when $\lambda = \pm\, \pi^{-1}$. It is easy to check that the corresponding homogeneous equation

$$y(x) = \lambda\, M_1 \sin x + \lambda\, M_2 \cos x$$

Since (1.2.20) and (1.2.21) are the same

$$y(x) = \lambda\, M_1 \sin x + \lambda\, M_2 \cos x$$

only has the solutions

$$y(x) = c(\sin x + \cos x) \text{ when } \lambda = \pi^{-1}, x \in [0, 2\pi]$$

$$y(x) = d(\sin x - \cos x) \text{ when } \lambda = \pi^{-1}, x \in [0, 2\pi]$$

where *c, d* are constants. (To check this, consider the equation (1.2.26) with the right hand sides' zero, as is the case with the homogeneous equation). The consistency conditions which need to be met to allow solutions of (1.2.18) to exist are therefore

$$\int_0^{2\pi} (\sin x + \cos x) f(x)\, dx = 0, \text{ when } \lambda = \pi^{-1}$$

and

$$\int_0^{2\pi} (\sin x - \cos x) f(x)\, dx = 0, \text{ when } \lambda = -\pi^{-1}$$

In case (a), both conditions are clearly fulfilled. As $y(x) = f(x) = 1$, $x \in [0, 2\pi]$ is a particular solution when either $\lambda = \pi^{-1}$ or $\lambda = -\pi^{-1}$, the complete solution of (1.2.18) is when $\lambda = -\pi^{-1}$,

$$y(x) = 1 + c(\sin x + \cos x), \qquad x \in [0, 2\pi]$$

and when $\lambda = -\pi^{-1}$

$$y(x) = 1 + d(\sin x - \cos x), \qquad x \in [0, 2\pi]$$

where c, d are arbitrary constants.

In case (b), neither condition is met and (1.2.18) has, therefore, no solution when either $\lambda = \pi^{-1}$ or $\lambda = -\pi^{-1}$.

1.3. Higher Order Ordinary Differential Equations

1.3.1 Basic concepts and Definitions

A differential equation of order n is of the form $F(x, y, y', y'' \dots y^{(n)}) = 0$ or if solved for $y^{(n)}$ of the form $y^{(n)} = f(x, y, y', y'' \dots y^{(n-1)})$ where $y^{(n)} = \frac{d^n y}{dx^n}$.

Here such an ordinary differential equation is called linear if it can be written as

$$a_o(x)\, y^{(n)} + a_1(x)\, y^{(n-1)} + \dots + a_{n-1}(x)\, y' + a_n(x)\, y = f(x), \;\; a_o(x) \neq 0 \;\dots\dots\dots\dots(1.3.1)$$

The coefficients $a_o, a_1, a_2 \dots a_n$ and f are any given functions of x and y. If $y^{(n)}$ has coefficient 1, we call this the standard form.

An n^th - order ordinary differential equation that can not be written in the form (1.3.1) is called non -linear.

Example 1.3.1 $x^3 y^{(v)} + 3x^2 y^{(iv)} + x^2 y' + y = \sin x$ is linear ordinary differential equation

Example 1.3.2 $yy''' + xy'' + 3xy' = 4x + 2$ is higher order non-linear ordinary differential equation.

The right hand member f(x) is called the non homogeneous term. If f is identically zero equation (1.3.1) reduces to

$$a_o(x) y^{(n)} + a_1(x) y^{(n-1)} + \dots + a_{n-1}(x) y^{1} + a_n(x) y = 0 \;\dots\dots\dots\dots\dots\dots(1.3.2)$$

and is then called homogeneous.

1.3.2 Homogeneous Linear Differential Equations with Constant Coefficients

In this section we discuss the homogeneous linear n^th order differential equation.

$$a_o(x) y^{(n)} + a_1(x) y^{(n-1)} + \dots + a_{n-1}(x) y^{1} + a_n(x) y = 0 \;\dots\dots\dots\dots\dots\dots (1.3.3)$$

where $a_o(x) \neq 0, a_o, a_1, a_2, \dots, a_n$ are real constants.

To find the general solution of equation (1.3.3) We proceed as follows

1. We set up the characteristic equation for (1.3.3) as

$$a_o\lambda^n + a_1\lambda^{n-1}....+a_n = 0 \quad \text{.......................} \quad \text{.........................}(1.3.4)$$

2. Find the roots $\lambda_1, \lambda_2, ..., \lambda_n$ of the characteristic equation (1.3.4)

3. Depending on the roots of characteristic equation (1.3.4) , we write linearly independent particular solutions taking in to account the fact that
   (a) To each real single root λ of the characteristic equation, a particular solution $e^{\lambda x}$ is obtained
   (b) To each single pair of complex conjugate roots $\lambda_1 = \alpha + i\beta$, $\lambda_2 = \alpha - i\beta$. There are two linearly independent particular solutions $e^{\alpha x}\cos\beta x, e^{\alpha x}\sin\beta x$.
   (c) To each real root λ of multiplicity m, there are m linearly independent particular solutions $e^{\lambda x}, xe^{\lambda x}, x^2e,^{\lambda x}, ...x^{m-1}e^{\lambda x}$
   (d) To each pair of complex conjugate roots, $\lambda_1 = \alpha + i\beta$, $\lambda_2 = \alpha - i\beta$ of multiplicity m. There are 2m linearly independent particular solutions

$$e^{\alpha x}\cos\beta x, xe^{\alpha x}\cos\beta x, ..., x^{m-1}e^{\alpha x}\cos\beta x,$$

$$e^{\alpha x}\sin\beta x, xe^{\alpha x}\sin\beta x, ..., x^{m-1}\sin\beta x.$$

The number of particular solutions of the differential equation (1.3.3) constructed is equal to the order of the equation. Let us illustrate the above idea with examples.

Example1.3.3 Find the solution of the following differential equations.

i) $y^{(v)} - 2y^{(iv)} + 2y^{lll} - 4y^{ll} + y^{l} - 2y = 0$

ii) $y^{iv} - y = 0, y(0) = 1, \quad y^{l}(0) = y^{ll}(0) = y^{lll}(0) = 0$

Solutions:

i) The corresponding characteristic equation is

$$\lambda^5 - 2\lambda^4 + 2\lambda^3 - 4\lambda^2 + \lambda - 2 = 0$$

$$\Leftrightarrow \quad (\lambda - 2)(\lambda^2 + 1)^2 = 0$$

Here, the roots are $\lambda = 2$ a single one, and $\lambda = \pm i$, a pair of double imaginary roots.

Using (a) and (d) . The general solution is thus

$$y = c_1 e^{2x} + (c_2 + c_3 x)\cos x + (c_4 + c_5 x)\sin x.$$

ii) The corresponding characteristic equation is $\lambda^4 - 1 = 0$

$$\lambda^4 - 1 = 0 \Leftrightarrow (\lambda^2 - 1)(\lambda^2 + 1) = 0$$

Thus using (a) and (b), the general solution is

$$y(x) = c_1 e^{x} + c_2 e^{-x} + c_3 \cos x + c_4 \sin x$$

Now,

$$y(0) = 1 = c_1 + c_2 + c_3$$

$$y'(0) = 0 = c_1 - c_2 + c_4$$

$$y''(0) = 0 = c_1 + c_2 - c_3$$

$$y'''(0) = 0 = c_1 - c_2 - c_4$$

Hence solving simultaneously, we obtain $c_1 = c_2 = \frac{1}{4}$,

$C_3 = \frac{1}{2}, C_4 = 0$ and the desired solution is $y(x) = \frac{1}{4}(e^{x} + e^{-x} + 2\cos x)$

1.3.3 The Non homogeneous ordinary Differential Equation

In this section we consider the non homogeneous equation

$$a_0(x)y^{(n)}(x) + a_1(x)y^{(n-1)}(x) + ... + a_{n-1}(x)y'(x) + a_n(x)y = f(x) \quad .... (1.3.5)$$

Where $a_0, a_1, a_2, ..., a_n$ and f are functions of x with $a_0(x) \neq 0$ and $f(x) \neq 0$.

Using the operator notation, we can write (1.3.5) as

$$L_n = a_o(x)\frac{d^n}{dx^n} + a_1(x)\frac{d^{n-1}}{dx^{n-1}} + ... + a_{n-1}(x)\frac{d}{dx} + a_n(x)$$

$$\text{i.e. } L_n y = f(x) \quad .... (1.3.6)$$

Theorem 1.3.1

Let v be any solution of the non homogeneous equation and let u be any solution of the corresponding homogeneous equation.

$$L_n y = 0 \qquad \text{……………………………………} \qquad (1.3.7)$$

Then $u + v$ is also solution of the non homogeneous equation (1.3.5)

Proof

$$L_n[u(x)+v(x)] = a_o(x)\frac{d^n}{dx^n}[u(x)+v(x)] + a_1(x)\frac{d^{n-1}}{dx^{n-1}}[u(x)+v(x)] + \dots.$$

$$+a_{n-1}(x)\frac{d}{dx}[u(x)+v(x)] + a_n(x)[u(x)+v(x)]$$

$$= a_o(x)\frac{d^n}{dx^n}u(x) + a_1(x)\frac{d^{n-1}}{dx^{n-1}}u(x) + \dots.. + a_{n-1}(x)\frac{d}{dx}u(x) + a_n(x)u(x) +$$

$$a_o(x)\frac{d^n}{dx^n}v(x) + a_1(x)\frac{d^{n-1}}{dx^{n-1}}v(x) + \dots + a_{n-1}(x)\frac{d}{dx}v(x) + a_n(x)v(x)$$

$$= L_n[u(x)] + L_n[v(x)]$$

Now by hypothesis $L_n[v(x)] = F(x)$ and $L_n[u(x)] = 0$.

Thus $L_n[u(x)+v(x)] = F(x)$, that is $u + v$ is a solution of equation (1.3.5)

In particular, if $f_1, f_2 \dots f_n$ is a fundamental set of the homogenous equation (1.3.5), and v is any particular solution of the inhomogeneous equation (1.3.5) then,

$$c_1 f_1 + c_2 f_2 + \dots. + c_n f_n + v$$

is also a solution of the nonhomogeneous equation (1.3.5).

We now state and prove the so called superposition principle for particular solution of equation (1.3.5).

The Superposition Principle

Consider the following idea that describes the concept of the superposition principle

Theorem1.3.2

Let $L_n y = F_i(x), i = 1,2,..., m$ be m different non-homogeneous linear equation each of which has the same left member.

$$L_n y = a_o(x)\frac{d^n y}{dx^n} + a_1(x)\frac{d^{n-1}y}{dx^{n-1}} + ... + a_n(x)y$$

Let f_i be a particular solution of $L_n y = F_i(x), i = 1,2,...m$ then $\sum_{i=1}^{m} k_i f_i$ is a particular solution of the equation $L_n y = \sum_{i=1}^{m} k_i f_i(x)$ where $k_1, k_2, ..., k_m$ are constants.

Proof

$$L_n\left[\sum_{i=1}^{m} k_i f_i(x)\right] = a_0(x)\frac{d^n}{dx^n}\left[\sum_{i=1}^{m} k_i f_i(x)\right] + ... + a_n(x)\left[\sum_{i=1}^{m} k_i f_i(x)\right]$$

$$= k_1\left[a_o(x)\frac{d^n}{dx^n} f_1(x) + ... + a_n(x) f_1(x)\right] + ... + K_m\left[a_o(x)\frac{d^n}{dx^n} f(x) + ... + a_n(x) f_m(x)\right]$$

$$= K_1 L_n[f_1(x)] + ... + K_m L_n[f_m(x)]$$

By hypothesis, f_i satisfies $L_n y = F_i(x), i=1,2,...,m$.

Thus,

$$L_n[f_1(x)] = F_1(x),...., \ L_n[f_m(x)] = F_m(x)$$

and

$$L_n\left[\sum_{i=1}^{m} k_i f_i(x)\right] = K_1 F_1(x) + ... + K_m F_m(x).$$

That is $\sum_{i=1}^{m} k_i f_i$ is a solution of $L_n y = \sum_{i=1}^{m} k_i F_i(x)$.

Example 1.3.4 Solve the equation

$$y''' - 2y'' + 2y' = 2\cos 4x - \cos 2x + 3$$

Solution:

The general solution of the homogeneous equation using characteristic equation $\lambda^3 - 2\lambda^2 + 2\lambda = 0$ is $y_h = c_1 + (c_2 \cos x + c_3 \sin x)e^x$.

To find the particular solution, let us apply the superposition principle. Now let us find the particular solutions of the following three equations

$$y''' - 2y'' + 2y' = 2\cos 4x \quad \ldots\ldots\ldots\ldots\ldots\ldots \ldots\ldots\ldots\ldots\ldots\ldots (1.3.8)$$

$$y''' - 2y'' + 2y' = -\cos 2x \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (1.3.9)$$

$$y''' - 2y'' + 2y' = 3 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (1.3.10)$$

using the trial and error method, find particular solutions y_1, y_2 and y_3 of equation (1.3.8), (1.3.9) and (1.3.10) respectively

$$y_1 = \frac{1}{65}(\cos 4x - \frac{7}{4}\sin 4x)$$

$$y_2 = \frac{1}{10}(\frac{1}{2}\sin 2x - \cos 2x)$$

$$y_3 = \frac{3}{2}x.$$

By virtue of the superposition principle, the particular solution of the inhomogeneous equation is

$$y_p = \frac{1}{65}(\cos 4x - \frac{7}{4}\sin 4x) + \frac{1}{10}(\frac{\sin 2x}{2} - \cos 2x) + \frac{3}{2}x$$

The general solution of the original equation is

$$y = c_1 + (c_2 \cos x + c_3 \sin x)e^x + \frac{1}{65}(\cos 4x - \frac{7}{4}\sin 4x) + \frac{1}{10}(\frac{\sin 2x}{2} - \cos 2x) + \frac{3}{2}x$$

The Annihilator Method

This method helps us to find the particular solution of the non-homogeneous ordinary differential equation to some extent. But before let us discuss about the concept of an annihilator.

Definition 1.3.1

A linear differential operator A is said to annihilate a function f if

$$A[f](x) = 0,\ \forall\ x \ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (1.3.11)$$

that is, A annihilates f if f is a solution to the homogeneous linear differential equation (1.3.11) on $(-\infty, \infty)$

Example 1.3.5 $A = D-3$ annihilates $f(x) = e^{3x}$. Since $(D-3)(e^{3x}) = 3e^{3x} - 3e^{3x} = 0$

$A = D^2 - 4D + 20$ is an annihilator of $e^{2x} \sin 4x$

Since $(D^2 - 4D + 20)(e^{2x} \sin 4x) = 0$.

The differential operator $(D-r)^m$, m a positive integer, annihilates each of the functions.

$$e^{rx}, xe^{rx}, \ldots\ldots, x^{m-1}e^{rx}$$

Moreover, the differential operator $\left[(D-\alpha)^2 + \beta^2\right]^m$ annihilates each of the function

$$e^{\alpha x}\cos\beta x, xe^{\alpha x}\cos\beta x, \ldots\ldots, x^{m-1}e^{\alpha x}\cos\beta x$$

$$e^{\alpha x}\sin\beta x, xe^{\alpha x}\sin\beta x, \ldots\ldots, x^{m-1}e^{\alpha x}\sin\beta x$$

Since these are the 2m linearly independent solutions to $\left[(D-\alpha)^2 + \beta^2\right]^m y = 0$

Example 1.3.6: Find a differential operator that annihilates

$$6xe^{-4x} + 5e^{x}\sin 2x.$$

Solution

Let us consider the two functions whose sum appears as above. Observe that $(D+4)^2$ annihilates the function $f_1(x) = 6xe^{-4x}$.

Further $f_2(x) = 5e^x \sin 2x$ is annihilated by the operator $(D-1)^2 + 4$. Hence the composite operator $A := (D+4)^2[(D-1)^2 + 4]$ which is the same as the operator $\left[(D-1)^2 + 4\right](D+4)^2$, annihilates both f_1 and f_2. But then, by linearity, A also annihilates the sum $f_1 + f_2$.

We now show how annihilators can be used to determine particular solutions to certain non homogeneous equations.

Example 1.3.7 Find the form of a particular solution to

$$y'''(x) - 3y''(x) + 4y(x) = xe^{2x} - \cos x \ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (1.3.12)$$

Solution:

We first solve the corresponding homogeneous equation

$$y'''(x) - 3y''(x) + 4y(x) = 0 \quad\text{..(1.3.13)}$$

The characteristic equation is given by

$$r^3 - 3r^2 + 4 = (r+1)(r-2)^2 = 0 \quad\text{..(1.3.14)}$$

has a simple root at -1 and a double root at 2, and hence a general solution to (1.3.13) is

$$y_h(x) = c_1 e^{-x} + c_2 e^{2x} + c_3 x e^{2x} \quad\text{..................................(1.3.15)}$$

To determine y_p using the annihilator method, we observe that the term xe^{2x} is annihilated by the operator $(D-2)^2$ and the term $-\cos x$ is annihilated by $(D^2 + 1)$. Hence the composite operator

$$A := (D^2 + 1)(D - 2)^2 \quad\text{..(1.3.16)}$$

is an annihilator for the non homogeneous term $xe^{2x} - \cos x$.

If we now apply A to both sides of (1.3.13), we obtain

$$A\left[y''' - 3y'' + 4y\right] = A\left[xe^{2x} - \cos x\right].$$

$$(D^2 + 1)(D-2)^2(D+1)(D-2)^2[y] = 0 \quad\text{..............................(1.3.17)}$$

Regrouping the factors, we have

$$(D^2 + 1)(D-2)^4(D+1)[y] = 0 \quad\text{..(1.3.18)}$$

The characteristic equation associated with (1.3.18) is

$$(r^2 + 1)(r-2)^4(r+1) = 0 \quad\text{..(1.3.19)}$$

which has roots i, -i, 2, -1.

Hence a general homogeneous solution is

$$y(x) = c_1 \cos x + c_2 \sin x + (c_3 + c_4 x + c_5 x^2 + c_6 x^3)e^{2x} + c_7 e^{-x} \quad\text{.........................(1.3.20)}$$

Comparing (1.3.20) with the general solution to (1.3.13) given in equation (1.3.15),

we see that a particular solution for (1.3.13) is of the form.

$$y_p(x) = c_1 \cos x + c_2 \sin x + (c_5 x^2 + c_6 x^3)e^{2x} \quad \text{.....................................}(1.3.21)$$

substituting (1.3.21) for y_p in to (1.3.12) and solving the unknown constants yields,

$$y_p(x) = \frac{-7}{50}\cos x + \frac{1}{50}\sin x + \frac{1}{18}(x^3 - x^2)e^{2x}$$

The Method of Undetermined Coefficients

In this section, we give a procedure for finding a particular solution to a nonhomogeneous equation. For the right hand sides of special form in (1.3.5), the particular solution is easier to find by the so-called trial and error method. The general form of the right hand side *f(x)* of equation (1.3.5) allowing the use of the error method is as follows.

$$f(x) = e^{\alpha x}\left[p_\ell(x)\cos\beta x + Q_m(x)\sin\beta x\right]$$

$P_\ell(x)$ and $Q_m(x)$ being polynomials of degree ℓ and m respectively. In this case a particular solution y_p of equation (1.3.5) is sought in the form

$$y_p = x^s e^{\alpha x}\left[\widetilde{p}_k(x)\cos\beta x + \widetilde{Q}_k(x)\sin\beta x\right]$$

Where $k = max\ (m, \ell)$, $\widetilde{P}_k(x)$ and $\widetilde{Q}_k(x)$ are polynomials of the k^{th} degree of the general form with undetermined coefficients and s is the multiplicity of the root $\lambda = \alpha + i\beta$ of the characteristic equation (if $\alpha \pm i\beta$ is not a root of the characteristic equation, then $s = 0$)

A summary of the forms of particular solutions for various right hand sides is given below.

No	Right hand side of differential equation	Roots of characteristic equation	Forms of particular solutions
I	$P_m\ (x)$	1. Number 0 is not a root of characteristic equation	$\widetilde{P}_m(x)$
		2. Number 0 is a root of characteristic equation of multiplicity s	$x^s\widetilde{P}_m(x)$
II	$P_m(x)e^{\alpha x}$	1. Number α is not root of characteristic equation	$\widetilde{P}_m(x)\ e^{\alpha x}$
		2.Number α is a root of characteristic equation of multiplicity s	$x^s\widetilde{P}_m(x)e^{\alpha x}$
III	$P_n(x)\cos\beta x+Q_m(x)\sin\beta x$	1.Numbers $\pm i\beta$ are not roots of characteristic equation	$\widetilde{P}_k(x)\cos\beta x+$ $\widetilde{Q}_k(x)\sin\beta x$ $k=\max(m,n)$
		2.Numbers $\pm i\beta$ are roots of characteristic equation of multiplicity s	$x^s(\widetilde{P}_k(x)\cos\beta x+$ $\widetilde{Q}_k(x)\sin\beta x)$ $k=\max(m,n)$
IV	$e^{\alpha x}\,(P_n(x)\cos\beta x+Q_m(x)\sin\beta x)$	1.Numbers $\alpha\pm\ i\beta$ are not roots of characteristic equation	$(\widetilde{P}_k(x)\cos\beta x+$ $\widetilde{Q}_k(x)\sin\beta x)e^{\alpha x}$ $k=\max(m,n)$
		1.Numbers $\alpha\pm i\beta$ are roots of characteristic equation of multiplicity s	$x^s(\widetilde{P}_k(x)\cos\beta x+$ $\widetilde{Q}_k(x)\sin\beta x)e^{\alpha x}$ $k=\max(m,n)$

The first three forms of right-hand sides are particular case of form IV.

Now let us observe how the method of undeterminant helps us to solve some non-homogeneous differential equations.

Example 1.3.8 Find the general solution of the equation

$$y''' - y'' = 12x^2 + 6x.$$

Solution

The characteristic equation $\lambda^3 - \lambda^2 = 0$ has the roots $\lambda_1 = \lambda_2 = 0, \lambda_3 = 1$, and therefore the general solution of the corresponding homogeneous equation is

$$y_h = c_1 + c_2 x + c_3 e^x$$

Since the number 0 is a double root of the characteristic equation, the particular solution must be sought in the form (see Table 1, case I(2))

$$y_p = x^2(A_1x^2 + A_2x + A_3) = A_1x^4 + A_2x^3 + A_3x^2.$$

Substituting the expression for y_p in the given equation. We have,

$$-12A_1x^2 + (24A_1 - 6A_2)x + (6A_2 - 2A_3) = 12x^2 + 6x$$

$$\Rightarrow -12A_1 = 12$$

$$24A_1 - 6A_2 = 6$$

$$6A_2 - 2A_3 = 0$$

This system has solution $A_1 = -1, \quad A_2 = -5, \quad A_3 = -15$ and hence

$$y_p = -x^4 - 5x^3 - 15x^2$$

The general solution of the given equation is

$$y = c_1 + c_2x + c_3e^x - x^4 - 5x^3 - 15x^2$$

Example 1.3.9

Find the general solution of the equation

$$y'' + 2y' + 5y = e^{-x}\cos 2x.$$

Solution

The characteristic equation $\lambda^2 + 2\lambda + 5 = 0$ has the roots

$$\lambda_{1,2} = -1 \pm 2i$$

Hence,

$$y_h = (c_1 \cos 2x + c_2 \sin 2x)e^{-x}$$

Since the number $\alpha + i\beta = -1 + 2i$ is a simple root of the characteristic equation, y_p must be sought in the form (see Table 1, case IV. (2))

$$y_p = x(A\cos 2x + B\sin 2x)e^{-x},$$

then

$$y_p{}' = e^{-x}\left[(A - Ax + 2Bx)\cos 2x + (B - Bx - 2Ax)\sin 2x\right]$$

$$y_p{}'' = e^{-x}\left[(-2A - 3Ax + 4B - 4Bx)\cos 2x + (-2B - 3Bx - 4A + 4Ax)\sin 2x\right]$$

Substituting the expressions for y_p and its derivatives in the original equation and canceling e^{-x}. We have,

$$-4A \sin 2x + 4B \cos 2x = \cos 2x$$

Hence $A = 0,\ B = \dfrac{1}{4}$ and so

$$y_p = \frac{1}{4} x e^{-x} \sin 2x$$

The general solution of the given equation is

$$y(x) = (c_1 \cos 2x + c_2 \sin 2x)e^{-x} + \frac{1}{4} x e^{-x} \sin 2x$$

Example 1.3.10 Find the general solution of the equation

$$y'' + 10y' + 25y = 4e^{-5x}$$

Solution

The characteristic equation $\lambda^2 + 10\lambda + 25 = 0$ has a double root $\lambda_1 = \lambda_2 = -5$,

Therefore $y_h = (c_1 + c_2 x)e^{-5x}$.

Since $\alpha = -5$ is a root of the characteristic equation of multiplicity $s=2$, the particular solution y_p of the inhomogeneous equation is sought in the form (see Table 1, case II (2)).

$$y_p = Bx^2 e^{5x},$$

$$y'_p = B\,(2x - 5x^2)e^{-5x}$$

$$y''_p = B(2 - 20x + 25x^2)e^{-5x}.$$

Substituting the expressions for y_p, $y_p{}'$, $y_p{}''$ in the original equation we get

$2Be^{-5x} = 4e^{-5x}$ Hence $B=2$ and so $y_p = 2x^2e^{-5x}$.

Thus, the general solution of the given equation is

$$y(x) = (c_1 + c_2x)e^{-5x} + 2x^2e^{-5x}.$$

The Method of Variation of Parameters

In this section we see how the method of variation of parameters helps us to solve higher order linear equations with variable coefficients.

Our aim is to find a particular solution to

$$L[y](x) = g(x) \qquad \text{.........(1.3.22)}$$

Where $L[y] := y^{(n)} + a_1y^{n-1} + ... + a_ny$ and the coefficient functions $a_1, ..., a_n$ as well as g are continuous on (a,b). The method to be described requires that we already know a fundamental solution set $\{y_1, ..., y_n\}$ for the corresponding homogeneous equation

$$L[y](x) = 0 \qquad \text{.........(1.3.23)}$$

A general solution to (1.3.23) is then

$$y_h(x) = c_1y_1(x) + c_2y_2(x) + ... + c_ny_n(x). \qquad \text{.........(1.3.24)}$$

where $c_1 ..., c_n$ are arbitrary constants. In the method of variation of parameters, we assume that there exists a particular solution to (1.3.22) of the form

$$y_p(x) = \ell_1(x)y_1(x) + ... + \ell_n(x)y_n(x) \qquad \text{.........(1.3.25)}$$

and try to determine the functions $\ell_1, ..., \ell_n$.

Since there are n unknown functions, we will need n conditions (equations) to determine them.

These conditions are obtained as follows. Differentiating y_p in (1.3.25) gives

$$y'_p = (\ell_1 y'_1 + \dots\dots + \ell_n y'_n) + (\ell'_1 y_1 + \dots\dots + \ell'_n y_n) \dots\dots\dots\dots (1.3.26)$$

To prevent second and higher order derivatives of the unknown $\ell_1, \dots\dots, \ell_n$ from entering our later computations, we impose the condition

$$\ell'_1 y_1 + \dots\dots + \ell'_n y_n = 0$$

In a like manner, on computing $y_p^{//}, y_p^{///} \dots y_p^{(n-1)}$, we impose *(n-2)* additional conditions involving $\ell'_1, \dots\dots, \ell'_n$ namely

$$\ell'_1 y'_1 + \dots\dots + \ell'_n y'_n = 0, \dots\dots, \ell'_1 y_1^{(n-2)} + \dots\dots + \ell'_n y_n^{(n-2)} = 0$$

Finally, the n^{th} condition that we impose is that y_p satisfy the given equation (1.3.22). Using the previous conditions and the fact that $y_1, \dots\dots, y_n$ are solutions to the homogeneous equation,
Then $L\lfloor y_p \rfloor = g$ reduces to

$$\ell'_1 y_1^{(n-1)} + \dots\dots + \ell'_n y_n^{(n-1)} = g, \dots\dots\dots\dots\dots\dots (1.3.27)$$

We therefore, seek n functions $\ell_1^1, \dots\dots, \ell_n^1$ that satisfy the system

$$\begin{array}{ccccccccc}
y_1\ell'_1 & + & y_2\ell'_2 & + & . & . & . & +y_n\ell'_n & = & 0 \\
y'_1\ell'_1 & + & y'_2\ell'_2 & + & . & . & . & +y'_n\ell'_n & = & 0 \\
y''_1\ell'_1 & + & y''_2\ell'_2 & + & . & . & . & +y''_n\ell'_n & = & 0 \\
y'''_1\ell'_1 & + & y'''_2\ell'_2 & + & . & . & . & +y'''_n\ell'_n & = & 0 \\
 & & & . & & & & & & \\
 & & & . & & & & & & \\
 & & & . & & & & & & \\
y_1^{(n-1)}\ell'_1 & + & y_2^{(n-1)}\ell'_2 & . & . & . & . & +y_n^{(n-1)}\ell'_n & = & g
\end{array} \qquad (1.3.28)$$

That is

$$y_1^{(k)}\ell'_1 + y_2^{(k)}\ell'_2 + \dots\dots + y_n^{(k)}\ell'_n = 0 \qquad \text{for } k = 0, 1, \dots, n-2$$

One can note that 0^{th} derivative of a function is just the function itself.

This is because from the above,

$$\ell_l y_l^{(n)} + \ell_2 y_2^{(n)} + \ldots\ldots\ldots\ldots + \ell_n y_n^{(n)} + \ell_l' y_l^{(n-1)} + \ell_2' y_2^{\;(n-1)} + \ldots\ldots\ldots + \ell_n' y_n^{(n-1)} \quad (1.3.29)$$

Now we substitute (1.3.29) in to (1.3.22) and we get.

$$\ell_l y_l^{(n)} + \ell_2 y_2^{(n)} + \ldots\ldots\ldots\ldots + \ell_n y_n^{(n)} + \ell_l' y_l^{(n-1)} + \ell_2' y_2^{\;(n-1)} + \ldots\ldots\ldots + \ell_n' y_n^{(n-1)} +$$

$$a_1(x)\left[\ell_1 y_1^{(n-1)} + \ell_2 y_2^{(n-1)} + \ldots\ldots + \ell_n y_n^{(n-1)}\right] +$$

$$.$$

$$.$$

$$a_{n-1}(x)\left[\ell_1 y_l'^{1} + \ell_2 y_2' + \ldots\ldots + \ell_n y_n'\right] +$$

$$a_n(x)\left[\ell_1 y_1 + \ell_2 y_2 + \ldots\ldots + \ell_n y_n\right] +$$

$$\ell_l' y_l^{(n-1)} + \ell_2' y_2^{(n-1)} + \ldots\ldots + \ell_n' y_n^{(n-1)} = g$$

Recall that $y_l, y_2 \ldots, y_n$ are all solutions to the homogeneous differential equation and so all the quantities in the [ ] are zero and this reduces down to,

$$\ell_1{}' y_1^{\;(n-1)} + \ell_2' y_2^{(n-1)} + \ldots\ldots + \ell'_n y^{(n-1)} = g$$

So this equation, together with the above gives the determinant below.

A sufficient condition for the existence of a solution to system (1.3.28)

for x in *(a,b)* is that the determinant of the matrix made up of coefficients of $\ell_l', \ldots\ldots, \ell_n^1$ be different from zero for all x in *(a,b)*. But this determinant is the Wronskian

$$\begin{vmatrix} y_1 & y_2 & y_3 & \cdot & \cdot & \cdot & y_n \\ y_1' & y_2' & y_3' & \cdot & \cdot & \cdot & y_n' \\ & & \cdot & & & & \\ & & \cdot & & & & \\ & & \cdot & & & & \\ y_1^{(n-1)} & y_2^{(n-1)} & y_3^{(n-1)} & \cdot & \cdot & \cdot & y_n^{(n-1)} \end{vmatrix} = \mathrm{W}(\mathrm{y}_1, \ldots, \mathrm{y}_n) := \mathrm{W}(\mathrm{x})$$

Which is never zero on *(a,b)*, because $\{y_1, y_2, \ldots\ldots, y_n\}$ is a fundamental solution set. Solving (1.3.28) via Crammers rule, (1.3.30) is taken as the determinant of each solution of the coefficients. The numerators of the solution for u_k will be the determinant of the matrix of coefficients with k$^{\text{th}}$ column *(0, 0, 0... 0, g(x))*. For example the number for the first one,

$$u_1' = \begin{vmatrix} 0 & y_2 & y_3 & \cdots & y_n \\ 0 & y_2' & y_3' & \cdots & y_n' \\ 0 & y_2'' & y_3'' & \cdots & y_n'' \\ & & \vdots & & \\ g(t) & y_2^{(n-1)} & y_3^{(n-1)} & \cdots & y_n^{(n-1)} \end{vmatrix}$$

Now, by a nice property of determinant if we factor something out of one of the columns of a matrix then the determinant of the resulting matrix will be the factor times the determinant of new matrix. In other words if we factor *g(t)* out of this matrix we arrive at

$$\begin{vmatrix} 0 & y_2 & y_3 & \cdots & y_n \\ 0 & y_2' & y_3' & \cdots & y_n' \\ 0 & y_2'' & y_3'' & \cdots & y_n'' \\ & & \vdots & & \\ g(t) & y_2^{(n-1)} & y_3^{(n-1)} & \cdots & y_n^{(n-1)} \end{vmatrix} = g(t) \begin{vmatrix} 0 & y_2 & y_3 & \cdots & y_n \\ 0 & y_2' & y_3' & \cdots & y_n' \\ 0 & y_2'' & y_3'' & \cdots & y_n'' \\ & & \vdots & & \\ 1 & y_2^{(n-1)} & y_3^{(n-1)} & \cdots & y_n^{(n-1)} \end{vmatrix}$$

We did this only for the first one, but we could just as easily done this with any of the n solutions. So, let w_k represent the determinant we get by replacing the k$^{\text{th}}$ column of the Wronskian with column *(0,0,0,....,0, 1)* and the solution to the system can then be written as

$$\ell_1' = \frac{g(x)w_1(x)}{W(x)}, \quad \ell_2' = \frac{g(x)w_2(x)}{W(x)}, \ \ldots, \quad \ell_n' = \frac{g(x)w_n(x)}{W(x)},$$

We can now integrate each of these terms to determine just what the unknown functions

$\ell_1, \ell_2, \ldots, \ell_n$.

We have,

$$\ell_1 = \int \frac{g(x)w_1(x)}{W(x)}dx \ , \quad \ell_2 = \int \frac{g(x)w_2(x)}{W(x)}dx \quad , \quad \ell_3 = \int \frac{g(x)w_3(x)}{W(x)}dx ,....$$

$$\ell_n = \int \frac{g(x)w_n(x)}{W(x)}dx$$

Thus,

$$y_p(x) = y_1(x)\ell_1(x) + y_2(x)\ell_2(x) + ... + y_n(x)\ell_n(x)$$

$$\Rightarrow y_p(x) = y_1(x) \int \frac{g(x)w_1(x)}{W(x)}dx + y_2(x) \int \frac{g(x)w_2(x)}{W(x)}dx + ... + y_n(x) \int \frac{g(x)w_n(x)}{W(x)}dx$$

We should also note that in the derivation process here we assumed that the coefficient of the $y^{(n)}$ term was a one and that has been factored in to the formula above. If the coefficient of the term is not one, then we will need to make sure and divide it out before trying to use the formula.

Example 1.3.11

Solve the non homogeneous equation

$$x^3y^{III} - 3x^2y^{II} + 6xy^{I} - 6y = x^4 \ell nx \ (x > 0)$$

Solution

Step1: General solution of the homogeneous differential equation.

Substitution of y = x^m and the derivatives in to the homogeneous ordinary differential equation and deletion of the factor x^m gives,

$$m(m-1)\ (m-2)\ -3m\ (m-1) + 6m - 6 = 0.$$

Hence the roots are 1, 2, 3 and give as

$$y_1 = x,\ y_2 = x^2 \ and\ y_3 = x^3$$

Hence, the corresponding general solution of the homogeneous ordinary differential equation is

$$y_h = c_1 x + c_2 x^2 + c_3 x^3.$$

Step2: Determinants needed

$$W = \begin{vmatrix} x & x^2 & x^3 \\ 1 & 2x & 3x^2 \\ 0 & 2 & 6x \end{vmatrix} = 2x^3$$

$$w_1 = \begin{vmatrix} 0 & x^2 & x^3 \\ 0 & 2x & 3x^2 \\ 1 & 2 & 6x \end{vmatrix} = x^4$$

$$w_2 = \begin{vmatrix} x & 0 & x^3 \\ 1 & 0 & 3x^2 \\ 0 & 1 & 6x \end{vmatrix} = -2x^3$$

$$w_3 = \begin{vmatrix} x & x^2 & 0 \\ 1 & 2x & 0 \\ 0 & 2 & 1 \end{vmatrix} = x^2$$

Step3: Integration

First the standard form of the given ordinary differential equation is

$$y''' - \frac{3}{x}y'' + \frac{6}{x^2}y' - \frac{6}{x^3}y = x\ell nx.$$

$$\frac{w_1}{W} = \frac{x}{2}, \qquad \frac{w_2}{W} = -1, \qquad \frac{w_3}{W} = \frac{1}{2x}$$

Therefore,

$$y_p(x) = y_1(x)\int\frac{g(x)\,w_1(x)}{W(x)}\,dx + y_2(x)\int\frac{g(x)W_2(x)}{W(x)}\,dx + y_3(x)\int\frac{g(x)w_3(x)}{W(x)}\,dx$$

$$= x\int\frac{x}{2}\,x\ell nx dx - x^2\int x\ell nx\,dx + x^3\int\frac{1}{2x}\,x\,\ell n\,x\,dx$$

$$= \frac{x}{2}\left(\frac{x^3}{3}\ell nx - \frac{x^3}{9}\right) - x^2\left(\frac{x^2}{2}\ell nx - \frac{x^2}{4}\right) + \frac{x^3}{2}\left(x\ell nx - x\right)$$

Simplification gives

$$y_p(x) = \frac{1}{6}x^4\ (\ell nx - \frac{11}{6})$$

Thus,

$$y = y_h + y_p = c_1x + c_2x^2 + c_3x^3 + \frac{1}{6}x^4(\ell nx - \frac{11}{6})$$

1.4 The Application of Fredholm Alternative Theorem on Higher Ordinary Differential Equations

Recall the theorem (Fredholm Alternative) we proved before says that
For the equation,

$$y(x) = f(x) + \lambda \int_a^b k(x,t)\ y(t)\ dt \text{...} (3.4.1)$$

where $x, t \in [a, b]$ and f is known function and y is unknown function, for each (fixed) λ exactly one of the following statements is true

1) If the homogeneous equation possesses trivial solution $y = 0$, then (1.4.1) has unique solution.
2) Suppose the homogeneous equation possesses a finite maximal linearly independent set of, say, *r* continuous solutions $y_1, y_2, \ldots\ldots, y_r\ (r > 0)$.

   In this case (1.4.1) possesses a maximal linearly independent set of *r* continuous solutions $z_1, z_2 \ldots\ldots, z_r$ and (3.4.1) has solutions if and only if

   $\int_a^b f(x) z_k\ (x)\ dx = 0 \quad (k = 1, \ldots\ldots, r)$ are all met. When they are, the complete

   solution of (1.4.1) is given by $y(x) = g(x) + \sum_{i=1}^{r} c_i y_i(x),\ \ x \in [a, b]$. Where $c_1 \ldots\ldots, c_r$

   are constants and g : $[a\ , b] \rightarrow R$ is any continuous solution of (1.4.1).

Now applying the theorem to higher ordinary differential equation, we have the following statements.

Consider the n^th^ order non homogeneous problem $L[y] = f(x)\ on\ [a\ , b]$ subject to $B_j[y] = 0\ for\ j = 1, 2 \ldots n$ and the associated homogeneous problem.

$$L[y]=0 \text{ on } [a\,,b] \text{ subject to } B_j[y]=0 \text{ for } j=1,2,\ldots n$$

If the homogeneous problem has only the trivial solution then the inhomogeneous problem has a unique solution,

If the homogeneous problem has m independent solutions $\{y_1\,, y_2\,,\ldots\ldots, y_m\}$, then there are two possibilities:

- If *f(x)* is orthogonal to each of the homogeneous solutions then there are an infinite number of solutions of the form

$$y=y_p+\sum_{j=1}^{m} c_j y_j$$

- If f(x) is not orthogonal to each of the homogeneous solutions then there are no inhomogeneous solutions.

Now let us illustrate the above basic idea considering the following examples.

Example 1.4.1

Consider $y'' + y=\cos 2x,\ y'(0)=y(\pi)=1$

Here there are no solutions to the homogeneous equation that satisfy the homogeneous boundary conditions.

To check this, note that all solutions of the homogeneous equation have the form

$$y_h=c_1\cos x+c_2\sin x$$

$$y'_h(0)=0\Rightarrow c_2=0$$

$$y_h(\pi)=0\Rightarrow c_1=0$$

Hence only the trivial solution exists as a solution to the homogeneous equation. From the Fredholm Alternative Theorem, we see that the inhomogeneous problem has a unique solution. To find the solution, let us apply undeterminant coefficient idea.

Let $y=k\cos 2x$ (see table 1, III,1)

$$y=k\cos 2x$$

$$y'=-2k\sin 2x$$

$$y'' = -4k\cos 2x$$

$$y'' + y = -4k\cos 2x - k\cos 2x = \cos 2x$$

$$= -3k\cos 2x$$
$$= \cos 2x$$

$$\Rightarrow \text{k} = -\frac{1}{3}$$

Thus,

$$y_p = \frac{-1}{3}\cos 2x + c_1\cos x + c_2\sin x$$

$$y'(0) = 1 \Rightarrow c_2 = 1$$

$$y(\pi) = 1 \Rightarrow -\frac{1}{3} - c_1 = 1$$

$$\Rightarrow c_1 = -\frac{4}{3}$$

Thus the solution is

$$y = -\frac{1}{3}\cos 2x - \frac{4}{3}\cos x + \sin x$$

This assures the Fredholm Alternative Theorem.

Example1.4.2 Consider $y'' + y = \cos 2x + \frac{1}{3}$ with $y(0) = 0, y(\pi) = 0$. Here, from the characteristic equation $\lambda^2 + 1 = 0$ where we have $\lambda = \pm i$. We have that the homogeneous solution are *sin x* and *cos x*. Among the two, only *sin x* satisfies the boundary condition. Hence the homogeneous equation does not possess trivial solution.

Now check if *sin x* is orthogonal to $\cos 2x + \frac{1}{3}$.

$$\int_0^\pi \sin x(\cos 2x + \frac{1}{3})\,dx = \int_0^\pi (\frac{1}{2}\sin 3x - \frac{1}{2}\sin x + \frac{1}{3}\sin x)\,dx = 0$$

Since *sin x* is orthogonal to the inhomogenity, there are an infinite number of solutions to the problem for y by Fredholm Alternative theorem.

Thus, the general solution for y is

$$y = -\frac{1}{3}\cos 2x + c_1 \cos x + c_2 \sin x$$

Applying the boundary conditions

$$y(0) = \frac{2}{3} \Rightarrow c_1 = 1$$

$$y(\pi) = -\frac{4}{3} \Rightarrow -\frac{4}{3} = -\frac{4}{3}$$

Thus, we see that c_2 is arbitrary. There are an infinite number of solutions of the form

$$y = \frac{-1}{3}\cos 2x + \cos x + c\sin x$$

Example 1.4.3 Consider

$$y'' + y = \cos 2x - \frac{x}{\pi} - 1,\ \ y(0) = 0,\ y(\pi) = 0.$$

Here again *sinx* and *cosx* are two linearly independent solutions to the homogeneous equation. But *sin x* only satisfies the boundary conditions.

But $\int_0^\pi \sin x\,(\cos 2x - \frac{x}{\pi} - 1)\,dx = \int_0^\pi \sin x \cos 2x dx - \frac{1}{\pi}\int_0^\pi x \sin x dx - \int_0^\pi \sin x dx$

$$= -\frac{11}{3} \neq 0$$

Thus, $\cos 2x - \frac{x}{\pi} - 1$ is not orthogonal to *sin x*. Therefore, applying the Fredholm Alternative Theorem, there is no solution to the given inhomogeneous problem.

References

1. Franz E.Hohn; *Introduction to Linear Algebra* (Macmillan,New York,**1972**)
2. Jack Goldberg,Merle C.Potter; *Differential Equations,A System Approach*(Viacom,USA,**1998**)
3. M.I.Krasnov,A.L.Kiselyov,G.I.Makarenko;*A book of Problems in Ordinary Differential Equations* (Mir,Moscow,**1989**)
4. Peter J. Collins;*Differential and Integral Equations*,(Oxford University,New York,**2006**)
5. Robert H. Martin,Jr; *Ordinary Differential Equations*,(Mc Graw.Hill,**1983**)
6. R.Kent Nagle, Edward B.Saff; *Fundamentals of Differential Equations*,2nd ed.(Benjamin,California,**1989**)
7. Shepley L.Ross; *Introduction to Ordinary Differential Equations*,4th ed.(USA,**1989**)