COLLEGE OF NATURAL SCIENCE
DEPARTMENT OF MATHEMATICS

GRADUATE PROJECT REPORT ON ASYMPTOTICS FOR BESSEL FUNCTIONS

Submited in partial fulfilment of requirements for the Degree of Master of Science in Mathematics

Prepared by: TANGUT ESUBALEW
Advisor: TADESSE ABDI(Ph.D)

AUGUST, 2014
Addis Ababa, Ethiopia

Contents

Addis Ababa University
Department of Mathematics

The undersigned here by certify that they have read and recommend to the school of graduate studies for acceptance of a project entitled **Asymptotics For Bessel Functions** by Tangut Esubalew in partial fulfillment of the requirements for the degree of master of Science.

Date:August,2014

Advisor: ____________________

Dr.Tadesse Abdi

Examining committee:____________________

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August,2014

Acknowledgement

First and foremost I would like to give my thanks to the **God** who gave me the chance to live and have all such experiences in learning up to this level of which I did n't think of. Secondly, I deeply thank my advisor, Dr.Taddese Abdi, whose help, advice and material support was invaluable. Finally, I would also like to express my deepest thanks to Dr.Mengistu, my friends and my family, for there helps directly or indirectly and to all my instructors for the long lasting they gave me during my course work.

Abstract

The main objective of the present paper is to address how to solve Bessel's equation and describe its solution, Bessel functions. Additionally we discuss the approximate solution of Bessel equations as x tends to infinity(for larger interval of x).

Introduction

Bessel function is defined for a first time by the mathematician Daniel Bernoulli and generalized by Friedrich Bessel. A differential equation of the form

$$x^2y^{''} + xy^{'} + (x^2 - p^2)y = 0, x > 0 \ \text{ and } \ p \geq 0$$

Where p is arbitrary real or complex number is called a Bessel equation and its solution is known as Bessel function. This equation arises in problem involving vibrations or heat conduction in regions possessing circular symmetry; therefore Bessel function have many application in physics and engineering in connection with the propagation of waves, elasticity ,fluid motion. This seminar report consists three chapters. The first chapter remained about the power series, second order linear differential equation, singularity point and then gamma function which help to express factorial. In the second chapter I will discuss the Bessel equation and its solution which is Bessel functions. I will also discuss about properties of Bessel function and some recurrence formula. The third chapter discuss about asymptotics for Bessel functions.

Chapter 1

Preliminaries

1.1 Power series

An infinite series of the form

$$\sum_{n=0}^{\infty} a_n(x-x_0)^n = a_0 + a_1(x-x_0) + a_2(x-x_0)^2 + \dots \tag{1.1}$$

Where $a's$ are independent of x is called a **power series** in x about the point x_0. A common special case is the series about $x_0 = 0$

$$\sum_{n=0}^{\infty} a_n x^n = a_0 + a_1 x + a_2 x^2 + \dots \tag{1.2}$$

We now establish that the power series representation is unique.

$$If \quad f(x) = \sum_{n=0}^{\infty} a_n x^n, -R_a < x < R_a \;\; and \;\; g(x) = \sum_{n=0}^{\infty} b_n x^n, -R_b < x < R_b \tag{1.3}$$

With over lapping intervals , including the origin, then $a_n = b_n$ for all n ; that is, we assume two (different) power series representations and then proceed to show that the two are actually identical. From equation (1.3)

$$\sum_{n=0}^{\infty} a_n x^n = \sum_{n=0}^{\infty} b_n x^n, -R < x < R \tag{1.4}$$

Where R is the smaller of R_a and R_b.Setting $x = 0$ to eliminate all but the constant terms, we obtain $a_0 = b_0$ now exploiting the differentiability of our power series, we differentiate equation (1.4) with respect to x, we get

$\sum_{n=0}^{\infty} na_n x^{n-1} = \sum_{n=0}^{\infty} nb_n x^{n-1}$

We again set $x = 0$ to isolate the new constant terms and find $a_1 = b_1$.By repeating this process n times we get $a_n = b_n$.Which shows that the two series coincide.
Therefore our power series representation is unique.

1.2 Second Order Linear Equation

The general second order linear differential equation is

$$A(x)y'' + P(x)y' + Q(x)y = R(x) \tag{1.5}$$

Where $P(x), A(x), Q(x)$ and $R(x)$ are functions of x alone or constants. If $R(x) = 0$, then equation (1.5) reduces to

$$A(x)y^{''} + P(x)y^{'} + Q(x)y = 0 \tag{1.6}$$

and it is called homogeneous linear equation. If $R(x)$ is non zero, then equation (1.5) is called non-homogeneous.

Definition 1.2.1. *The n-functions $f_1, f_2, f_3, \cdots, f_n$ are called linearly dependent on $a \leq x \leq b$ if and only if there exist constants $a_1, a_2, a_3, \cdots, a_n$ not all zero such that $a_1f_1+a_2f_2+a_3f_3+\ldots+a_nf_n = 0$ for all $x \in [a, b]$; and they are linearly independent on $a \leq x \leq b$ if and only if $a_1f_1+a_2f_2+a_3f_3+\cdots+a_nf_n = 0$ implies $a_1 = a_2 = \ldots = a_n = 0$.*

Definition 1.2.2. *Let $f_1, f_2, f_3, \cdots, f_n$ be a real functions in which each has (n-1) derivatives on [a,b].The determinant*

$$\begin{bmatrix} f_1 & \cdots & f_n \\ \vdots & \ddots & \vdots \\ f_1^{n-1} & \cdots & f_n^{n-1} \end{bmatrix}$$

is called the Wronskian of the n-functions $f_1, f_2, f_3, \ldots, f_n$ and is denoted by $W(f_1, f_2, f_3, \ldots, f_n)(x)$.

Now let us consider the homogeneous linear equation

$$a_0(x)y^n + a_1(x)y^{n-1} + \ldots + a_{n-1}(x)y^1 + a_n(x)y = 0 \tag{1.7}$$

Where $a_0, a_1, a_2, \ldots, a_{n-1}, a_n$ are continuous on [a,b]

Theorem 1.2.1. *A necessary and sufficient condition those n-solutions* $f_1, f_2, f_3, \cdots, f_n$ *of the* n^{th} *order homogeneous linear equation (1.7) be linearly independent solutions on [a,b] if the value of the Wronskian is non-zero for all* $x \in [a, b]$ *.*

Theorem 1.2.2. *If* $Y_1(x)$ *and* $Y_2(x)$ *are linearly independent solution of the homogeneous equation (1.6), then* $c_1Y_1(x) + c_2Y_2(x)$ *is the general solution of equation (1.6).*

1.3 Ordinary Points and Singular points of Differential Equations

Consider the linear differential equation

$$A(x)y^{''}(x) + P(x)y^{'}(x) + Q(x)y(x) = 0 \tag{1.8}$$

Where A,P and Q are polynomials containing no common factors. Suppose we want to solve (1.8) in some interval containing the point x_0. If $A(x_0) \neq 0$, then the point x_0 is called an ordinary point. A point that is not an ordinary point of a differential equation is called a singular point.
Suppose x_0 is a singular point.Multiplying through by $\frac{(x-x_0)^2}{A(x)}$,we may rewrite (1.8) as
$(x - x_0)^2y^{''} + (x - x_0)\ p(x)\ y^{'} + q(x)y = 0$

where $p(x) = \frac{(x-x_0)P(x)}{A(x)}$, $q(x) = \frac{(x-x_0)^2Q(x)}{A(x)}$

We say that x_0 is a regular singular point if the rational functions $p(x)$ and $q(x)$ have no singularity at x_0.Otherwise it is an irregular singular point.

Example 1.3.1. *Consider the equation*

$$x^2(x - 2)^2y^{''} + (x - 2)y^{'} + 3x^2y = 0$$

then,since at x=0 and 2, $A(x) = x^2(x - 2)^2 = 0$,

so 0 and 2 are its singular points.

And $p(x) = \frac{(x-x_0)P(x)}{A(x)}$ *and* $q(x) = \frac{(x-x_0)^2Q(x)}{A(x)}$

$\Rightarrow p(x) = \frac{(x-x_0)(x-2)}{x^2(x-2)^2}$ *and* $q(x) = \frac{(x-x_0)^2.3x^2}{x^2(x-2)^2}$

Now, At $x_0 = 0$ *,*$p(0) = \frac{1}{x(x-2)} \Rightarrow p(0) \neq 0$

and $q(0) = \frac{3x^2}{(x-2)^2} \Rightarrow q(0) = 0.$

Therefore 0 is an irregular singular point.

And at $x_0 = 2$*,* $p(2) = \frac{1}{x^2}, \quad \Rightarrow p(2) = \frac{1}{4} \neq 0$

and $q(2) = 3, \quad \Rightarrow q(2) \neq 0$

Therefore $p(x)$ *and* $q(x)$ *are non zero at* $x = 2$*,*

so, 2 is a regular singular point.

Remark 1.3.1. *An Indicial equation is the key to solving a second-order linear differential equation with non-constant coefficients.*

Definition 1.3.1. *An expression like*
$\sum_{m=0}^{\infty} a_m(x-a)^m = a_0 + a_1(x-a) + a_2(x-a)^2 + \cdots$
that gives each coefficient in terms of the preceding ones is called a ***recurrence relation****.*

1.4 Solution near Regular Singular Points

A power series method may not work for solution about regular singular point. In this case we shall assume an expansion of the form

$$y(x) = \sum_{n=0}^{\infty} a_n x^{n+r} \tag{1.9}$$

If r happens to be an integer, then equation (1.9) is just a power series, but often r will not be an integer.
The general series given by equation (1.9) is called a **Frobenius series** and the use of such a series to find solutions about singular points is called the **method of Frobenius** .To determine the coefficients in the Frobenius series ,we substitute equation (1.9) in to the differential equation and equate the coefficients of the various power of x to zero.

1.5 Gamma Functions

In 1700_s ,Euler introduced a function that yields n! when n is a positive integer ,the function is called the **gamma function** and is defined by the integral expression, $\Gamma(n) = \int_0^\infty e^{-x}x^{n-1}dx$, When n is positive.

1.5.1 Property of Gamma Functions

$i, \Gamma(n+1) = n\Gamma(n)$

$ii, \Gamma(1) = 1$

$iii, \Gamma(n+1) = n!$

$iv, \frac{p+k}{\Gamma(k+p+1)} = \frac{1}{\Gamma(p-1+k+1)}$

$v, \Gamma(n+\frac{1}{2}+1) = \frac{(2n+1)!}{(2^{2n+1}\ \ n!)}\sqrt{\pi}$

$vi, \Gamma(n+\frac{1}{2}) = \frac{(2n)!}{(2^{2n}\ \ n!)}\sqrt{\pi}$

proof:

$$\begin{aligned}\Gamma(n+1) &= \int_{n=0}^{\infty} e^{-x}x^n dx \\ &= -[e^{-x}x^n]_0^\infty + n\int_0^\infty e^{-x}x^{n-1}dx(Integrating\ \ by\ \ part) \\ &= n\int_0^\infty e^{-x}x^{n-1}dx = n\Gamma(n)\end{aligned}$$

$$\therefore \Gamma(n+1) = n\int_0^\infty e^{-x}x^{n-1}dx = n\Gamma(n) \tag{1.10}$$

From (1.10) it is evident that if the value of $\Gamma(n)$ is known for n between two successive positive integers the value $\Gamma(n)$ for any positive value of n can be determined by the successive application of (1.10).

Now replacing n by (n-1) in (1.10) we get

$\Gamma(n) = (n-1)\Gamma(n-1)$
Similarly $\Gamma(n-1) = (n-2)\Gamma(n-2)$ e.t.c.Hence (1.10) yields
$\Gamma(n+1) = n(n-1)(n-2)\cdots 3.2.1.\Gamma(1)$

Since by definition $\Gamma(1) = \int_0^\infty e^{-x}dx = 1$
Therefore $\quad \Gamma(n+1) = n(n-1)(n-2)(n-3)\cdots 3.2.1. = n!$ Provided that n is a positive integer.
Note: $\Gamma(-n) = \infty$ when n =0 or a positive integer.

Chapter 2

Bessel Equation and Bessel Functions

2.1 Introduction

Definition 2.1.1. *A differential equation of the form*

$$x^2y^{''} + xy^{'} + (x^2 - p^2)y = 0 \tag{2.1}$$

is called ***Bessels equation*** *of order p , where p is non-negative constant. The solution of this equation is called* ***Bessel function***.

2.2 Solution of Bessels Equation

Equation (2.1) is an ordinary differential equation of the second order. Since $x = 0$ is a regular singular point of the differential equation (2.1); and all other values of x are ordinary points; so we can apply Frobenius method.
Let the series solution of (2.1) be of the form

$$y = \sum_{m=0}^{\infty} a_m x^{r+m}, a_0 \neq 0 \tag{2.2}$$

Where the first term is non zero and r is some arbitrary constant .Then differentiating it we get $y' = \sum_{m=0}^{\infty}(r+m)a_m x^{r+m-1}$

$y'' = \sum_{m=0}^{\infty}(r+m)(r+m-1)a_m x^{r+m-2}$

Then substituting in (2.1) ,we get

$$x^2\sum_{m=0}^{\infty}a_m(r+m)(r+m-1)x^{r+m-2)}+x\sum_{m=0}^{\infty}a_m(r+m)x^{r+m-1}+(x^2-p^2)\sum_{m=0}^{\infty}a_m x^{r+m}=0$$

$$\sum_{m=0}^{\infty}a_m(r+m)(r+m-1)x^{r+m}+\sum_{m=0}^{\infty}a_m(r+m)x^{r+m}+\sum_{m=0}^{\infty}a_m x^{r+m+2}-p^2\sum_{m=0}^{\infty}a_m x^{r+m}=0$$

$\sum_{m=0}^{\infty}a_m[(r+m)^2-p^2]x^{r+m}+\sum_{m=0}^{\infty}a_m x^{r+m+2}=0$

$\sum_{m=0}^{\infty}a_m[(r+m)^2-p^2]x^{r+m}+\sum_{m=2}^{\infty}a_{m-2}x^{r+m}=0$

By uniqueness of power series the coefficients of each power of x on the left hand side of the last equation must vanish individually. Equating coefficients of the series to zero gives

$$a_0(r^2-p^2)=0 \qquad (if \quad m=0) \tag{2.3}$$

$$a_1[(r+1)^2-p^2]=0 \qquad (if \quad m=1) \tag{2.4}$$

$$a_m[(r+m)^2-p^2]+a_{m-2}=0 \qquad (if \quad m\geq 2) \tag{2.5}$$

Since $a_0\neq 0$,then from (2.3) we get the indicial equation $(r+p)(r-p)=0$
With indicial roots $r=p$ and $r=-p$.

Case i. First solution of Bessel Equation

Setting $r=p$, in (2.5) gives the recurrence relation
$a_m[(p+m)^2-p^2]+a_{m-2}=0, \qquad m\geq 2$
$a_m[p^2+2mp+m^2-p^2]+a_{m-2}=0, \qquad m\geq 2$

$$So, \quad a_m=\frac{-1}{m(2p+m)}a_{m-2}, \quad m\geq 2 \tag{2.6}$$

This is two term recurrence relation .
Putting m=1 in (2.6) gives that $a_1=0$(since $a_{-1}=0$)
Thus (2.6) also shows that $a_3=a_5=\cdots=0$
i.e all $a's$ with odd subscripts are zero.
To obtain the remaining coefficients, let us put $m=2,4,6\cdots$ in (2.6), then
$a_2=\frac{-a_0}{2(2p+2)}$

$a_4=\frac{-a_2}{4(2p+4)}=\frac{a_0}{2.4(2p+2)(2p+4)}$

$a_6 = \frac{-a_4}{6(2p+6)} = \frac{-a_0}{2.4.6.(2p+2)(2p+4)(2p+6)}$

$\vdots$

Thus putting this value in

$$y = \sum_{m=0}^{\infty} a_m x^{r+m} \quad andweget$$
$$= a_0x^r + a_1x^{1+r} + a_2x^{2+r} + a_3x^{3+r} + a_4x^{4+r} + \cdots$$
$$= a_0x^r + a_2x^{2+r} + a_4x^{4+r} + a_6x^{6+r} + \cdots$$
$$= a_0x^r - \frac{a_0}{2(2p+2)}x^{2+r} + \frac{a_0}{2.4(2p+2)(2p+4)}x^{4+r} - \frac{a_0}{2.4.6(2p+2)(2p+4)(2p+6)}x^{6+r} + \cdots$$
$$= a_0x^r[1 - \frac{x^2}{2^2(p+1)} + \frac{x^4}{2^4.1.2(p+1)(p+2)} - \frac{x^6}{2^6.1.2.3(p+1)(p+2)(p+3)} + \cdots]$$
$$= a_0x^r[1 - \frac{(\frac{x}{2})^2}{1!(p+1)} + \frac{(\frac{x}{2})^4}{2!(p+1)(p+2)} - \frac{(\frac{x}{2})^6}{3!(p+1)(p+2)(p+3)} + \cdots] \quad (2.7)$$

Now choose $a_0 = \frac{1}{2^p p!}$ and since $r = p$ then (2.7)becomes

$$y = (\tfrac{x}{2})^p[\tfrac{1}{p!} - \tfrac{(\frac{x}{2})^2}{1!(p+1)!} + \tfrac{(\frac{x}{2})^4}{2!(p+2)!} - \tfrac{(\frac{x}{2})^6}{3!(p+3)!} + \cdots]$$

Now using properties of gamma function to generalize factorials we have

$$y = (\frac{x}{2})^p[\frac{1}{\Gamma(p+1)} - \frac{(\frac{x}{2})^2}{1!\Gamma(p+2)} + \frac{(\frac{x}{2})^4}{2!\Gamma(p+3)} - \frac{(\frac{x}{2})^6}{3!\Gamma(p+4)} + \cdots] \quad (2.8)$$

Which is solution of Bessel equation for all $p \geq 0$ and it is called the Bessel function of the first kind of order p and let us denote it by $J_p(x)$. Thus

$$J_p(x) = \sum_{k=0}^{\infty} \frac{(-1)^k}{k!\Gamma(p+k+1)}(\frac{x}{2})^{p+2k} \quad (2.9)$$

Case ii. Second solution of Bessels equation If $r = -p, p > 0$
It is not necessary to repeat all the above steps. Let us replace p by $-p$ in (2.8), and then we have

$$J_{-p}(x) = (\frac{x}{2})^{-p}[\frac{1}{\Gamma(-p+1)} - \frac{(\frac{x}{2})^2}{1!\Gamma(-p+2)} + \frac{(\frac{x}{2})^4}{2!\Gamma(-p+3)} - \frac{(\frac{x}{2})^6}{3!\Gamma(-p+4)} + \cdots] \quad (2.10)$$

$$or \quad J_{-p}(x) = \sum_{k=0}^{\infty} \frac{(-1)^k}{k!\Gamma(-p+k+1)}(\frac{x}{2})^{-p+2k} \quad (2.11)$$

Now if p is positive but is not an integer, the solution (2.11) is not bounded (i.e $J_{-p}(x)$ will be infinite), while $J_p(x)$ is finite, this is because of $J_p(x)$ contains a positive power of x only, on the other hand $J_{-p}(x)$ contains a negative power of x .This implies that the two solutions are linearly independent. Therefore $J_p(x)$ and $J_{-p}(x)$ are two independent solution of (2.1) when p is not integer.

Note :

1. The general solution of Bessel equation when p is not integer is given by $y = c_1 J_p(x) + c_2 J_{-p}(x)$ Where c_1 and c_2 are arbitrary constant.

2.3 Relation Between $J_p(x)$and $J_{-p}(x)$

Theorem 2.3.1. *If p is a non-negative integer,then* $J_{-p}(x) = (-1)^p J_p(x)$

Proof 2.3.1. *By definition we have*

$J_{-p}(x) = \sum_{k=0}^{\infty} \frac{(-1)^k}{k!\Gamma(-p+k+1)} (\frac{x}{2})^{-p+2k}$

Since $p > 0, \Gamma(-p + k + 1)$ *is infinite for* $k = 0, 1, 2, 3, \cdots , p - 1$ *.Thus k must be taken from p to infinite because the gamma function is not defined at 0 and negative integers. i.e*

$$\begin{aligned} J_{-p}(x) &= \sum_{k=p}^{\infty} \frac{(-1)^k}{k!\Gamma(-p+k+1)} (\frac{x}{2})^{-p+2k} \\ &= \sum_{m=0}^{\infty} \frac{(-1)^{m+p}}{(m+p)!\Gamma(m+1)} (\frac{x}{2})^{-p+2(m+p)} \\ &= \sum_{m=0}^{\infty} \frac{(-1)^m(-1)^p}{m!\Gamma(m+p+1)} (\frac{x}{2})^{2m+p} \\ &= (-1)^p \sum_{m=0)}^{\infty} \frac{(-1)^m}{m!\Gamma(m+p+1)} (\frac{x}{2})^{2m+p} \\ &= (-1)^p J_p(x) \end{aligned}$$

Note:

1.The second solution simply reproduced the first; we have failed second independent solution for Bessels equation by this series technique when p is an integer.

2. $J_p(-x) = (-1)^p J_p(x)$ For $p = 0, 1, 2, \cdots$

Proof:

$$\begin{aligned} J_p(-x) &= \sum_{k=0}^{\infty} \frac{(-1)^k}{k!\Gamma(p+k+1)} (\frac{-x}{2})^{p+2k} \\ &= \sum_{k=0}^{\infty} \frac{(-1)^k(-1)^{p+2k}}{k!\Gamma(p+k+1)} (\frac{x}{2})^{p+2k} \\ &= (-1)^p \sum_{k=0}^{\infty} \frac{(-1)^{3k}}{k!\Gamma(p+k+1)} (\frac{x}{2})^{p+2k} \\ &= (-1)^p J_p(x). \end{aligned}$$

Thus, $J_p(-x) = (-1)^p J_p(x)$

3. J_p is even if p is even and odd if p is odd.

2.4 Properties of Bessel Function

2.4.1 Recurrence Relation (formula) for J_p

1.$\frac{d}{dx}[x^p J_p(x)] = x^p J_{p-1}(x)$

2.$\frac{d}{dx}[x^{-p} J_p(x)] = -x^{-p} J_{p+1}(x)$

3.$J_p'(x) = J_{p-1}(x) - \frac{p}{x} J_p(x)$ or $xJ_p'(x) = xJ_{p-1}(x) - pJ_p(x)$

4.$J_p'(x) = \frac{p}{x} J_p(x) - J_{p+1}(x)$ or $xJ_p'(x) = pJ_p(x) - xJ_{p+1}(x)$

5.$J_p'(x) = \frac{1}{2}[J_{p-1}(x) - J_{p+1}(x)]$ or $2J_p'(x) = J_{p-1}(x) - J_{p+1}(x)$

6.$J_{p-1}(x) + J_{p+1}(x) = \frac{2p}{x} J_p(x)$ or $2pJ_p(x) = x[J_{p-1}(x) + J_{p+1}(x)]$

Proof:

1. By definition of $J_p(x)$ we have

$$\begin{aligned} x^p J_p(x) &= x^p \sum_{k=0}^{\infty} \frac{(-1)^k}{k!\Gamma(k+p+1)} (\frac{x}{2})^{p+2k} \\ &= \sum_{k=0}^{\infty} \frac{(-1)^k x^p}{k!\Gamma(k+p+1)} (\frac{x}{2})^{p+2k} \end{aligned}$$

Then,

$$\begin{aligned} \frac{d}{dx}[x^p J_p(x)] &= \frac{d}{dx} \sum_{k=o}^{\infty} \frac{(-1)^k x^p}{k!\Gamma(k+p+1)} (\frac{x}{2})^{p+2k} \\ &= \sum_{k=0}^{\infty} \frac{(-1)^k (\frac{1}{2})^{p+2k}}{k!\Gamma(k+p+1)} \frac{d}{dx} (x^{2p+2k}) \\ &= \sum_{k=0}^{\infty} \frac{(-1)^k (\frac{1}{2})^{p+2k}}{k!\Gamma(k+p+1)} (2p+2k) x^{2p+2k-1} \\ &= x^p \sum_{k=0}^{\infty} \frac{(-1)^k}{k!\Gamma(p-1+k+1)} (\frac{x}{2})^{p+2k-1} \\ &= x^p J_{p-1}(x) \end{aligned}$$

2.

$$\begin{aligned} \frac{d}{dx}[x^{-p} J_p(x)] &= \frac{d}{dx}[x^{-p} \sum_{k=0}^{\infty} \frac{(-1)^k}{k!\Gamma(k+p+1)} (\frac{x}{2})^{p+2k}] \\ &= \sum_{k=0}^{\infty} \frac{(-1)^k (\frac{1}{2})^{p+2k}}{k!\Gamma(k+p+1)} \frac{d}{dx} (x^{2k}) \\ &= \sum_{k=0}^{\infty} \frac{(-1)^k (\frac{1}{2})^{p+2k} 2k \ x^{2k-1}}{k(k-1)!\Gamma(k+p+1)} \\ &= \sum_{k=1}^{\infty} \frac{(-1)(-1)^{k-1} (\frac{1}{2})^{p+2k-1}}{(k-1)!\Gamma(p+1+k)} x^{2k-1} \\ &= -\sum_{k+1=1}^{\infty} \frac{(-1)^k (\frac{1}{2})^{p+2k+1}}{k!\Gamma(k+p+2)} \frac{x^{p+2k+1}}{x^p} (changing\ k\ to\ k+1\ to\ start\ the\ sum\ at\ 0) \\ &= -x^{-p} \sum_{k=0}^{\infty} \frac{(-1)^k}{k!\Gamma(p+1+k+1)} (\frac{x}{2})^{p+1+2k} \\ &= -x^{-p} J_{p+1}(x) \end{aligned}$$

3. From recurrence formula (1),

$$\begin{aligned}\frac{d}{dx}[x^p J_p(x)] &= x^p J_{p-1}(x)\\ px^{p-1}J_p(x) + x^p J_p'(x) &= x^p J_{p-1}(x)\\ J_p'(x) &= J_{p-1}(x) - \frac{p}{x}J_p(x)\end{aligned}$$

4. From recurrence formula (2) ,

$$\begin{aligned}\frac{d}{dx}[x^{-p} J_p(x)] &= -x^{-p} J_{p+1}(x)\\ -px^{-p-1}J_p(x) + x^{-p} J_p'(x) &= -x^{-p} J_{p+1}(x)\\ J_p'(x) &= -J_{p+1}(x) + \frac{p}{x}J_p(x)\end{aligned}$$

5.From recurrence formula (3) and (4) i.e.

$$+\begin{cases} J_p'(x) = J_{p-1}(x) - \frac{p}{x}J_p(x)\\ \underline{J_p'(x) = \frac{p}{x}J_p(x) - J_{p+1}(x)}\end{cases}$$

$$2J_p'(x) = J_{p-1}(x) - J_{p+1}(x)$$

$$J_p'(x) = \frac{1}{2}[J_{p-1}(x) - J_{p+1}(x)]$$

6. From recurrence formula (3) and (4)

$$-\begin{cases} J_p'(x) = J_{p-1}(x) - \frac{p}{x}J_p(x)\\ \underline{J_p' = \frac{p}{x}J_p(x) - J_{p+1}(x)}\end{cases}$$

$$0 = -\tfrac{2p}{x}J_p(x) + J_{p-1}x + J_{p+1}x$$

$$Hence \quad J_{p-1}(x) + J_{p+1}(x) = \frac{2p}{x}J_p(x)$$

Based on this recursion formula let us do some example:

$$i. \quad \frac{1}{2}xJ_p(x) = (p+1)J_{p+1}(x) - (p+3)J_{p+3}(x) + (p+5)J_{p+5}(x) + \cdots$$

Solution:
Recurrence relation (6) is $2pJ_p(x) = x[J_{p-1}(x) + J_{p+1}(x)]$ replacing p by $p+1$ in this relation we have

$$2(p+1)J_{p+1}(x) = x[J_{p+1-1}(x) + J_{p+1+1}(x)]$$

$$i.e. \quad \frac{1}{2}xJ_p(x) = (p+1)J_{p+1}(x) - \frac{1}{2}xJ_{p+2}(x) \tag{2.12}$$

again replacing p by $p+2$ in (2.12) we have

$$\frac{1}{2}xJ_{p+2}(x) = (p+3)J_{p+3}(x) - \frac{1}{2}xJ_{p+4}(x) \tag{2.13}$$

Putting the value of $\frac{1}{2}xJ_{P+2}(x)$ from (2.13) in (2.12) we get

$$\frac{1}{2}xJ_p(x) = (p+1)J_{p+1}(x) - (p+3)J_{p+3}(x) + \frac{1}{2}xJ_{p+4}(x) \tag{2.14}$$

Replacing p by $p+4$ in (2.12) we have

$$\frac{1}{2}xJ_{p+4}(x) = (p+5)J_{p+5}(x) - \frac{1}{2}xJ_{p+6}(x)$$

Thus (2.14) becomes

$$\frac{1}{2}xJ_p(x) = (p+1)J_{p+1}(x) - (p+3)J_{p+3}(x) + (p+5)J_{p+5}(x) - \cdots$$

2.5 Bessels function of the second kind of order p

When p is an integer J_p and J_{-p} are not linearly independent. To determine the second linearly independent solution of the Bessel equation, let us define

$$Y_p(x) = \begin{cases} \frac{J_p(x)\cos(p\pi) - J_{-p}(x)}{\sin(p\pi)} & \text{if } p \text{ is not an integer.} \\ \lim_{\mu\to p} \frac{J_\mu(x)\cos(\mu\pi) - J_{-\mu}(x)}{\sin(\mu\pi)} & \text{if } p \text{ is an integer.} \end{cases} \tag{2.15}$$

The function Y_p is called ***the Bessel function of the second kind of order*** p. It is also known as the ***Neumann function***.
Now when p is a non-integer Y_p is clearly a solution of the Bessel equation, because it is a linear combination of two linearly independent solutions $J_p(x)$ and $J_{-p}(x)$.
For integer,$\mu = p$ and $p = 0, 1, 2, \cdots$ equation (2.15) becomes

$$Y_p(x) = \lim_{\mu\to P} \frac{J_\mu(x)\cos(\mu\pi) - J_{-\mu}(x)}{sin(\mu\pi)}$$

Since $\cos(p\pi) = (-1)^p, \sin(p\pi) = 0$ and $J_{-p}(x) = (-1)^p J_p(x)$, so the limit is indeterminate form of $\frac{0}{0}$.
Then by L'Hospital rule

$$\begin{aligned} Y_p(x) &= \lim_{\mu \to p} \frac{\frac{\partial}{\partial \mu}[J_\mu(x)\cos(\mu\pi) - J_{-\mu}(x)]}{\frac{\partial}{\partial \mu}[\sin(\mu\pi)]} \\ &= [\frac{-\pi \sin(\mu\pi) J_\mu(x) + \cos(\mu\pi)\frac{\partial}{\partial \mu}J_\mu(x) - \frac{\partial}{\partial \mu}J_{-\mu}(x)}{\pi \cos(\mu\pi)}]_{\mu=p} \\ &= \frac{1}{\pi}[\frac{\partial}{\partial \mu}J_\mu(x) - (-1)^p \frac{\partial}{\partial \mu}J_{-\mu}(x)]_{\mu=p} \end{aligned}$$

Let us now show that $Y_p(x)$ so defined is a solution of the Bessels equation. By definition J_μ and $J_{-\mu}$ respectively, satisfies the following differential:

$$x^2 J''_\mu(x) + xJ'_\mu(x) + (x^2 - \mu^2)J_\mu(x) = 0$$

$$x^2 J''_{-\mu}(x) + xJ'_{-\mu}(x) + (x^2 - \mu^2)J_{-\mu}(x) = 0$$

Differentiating with respect to μ we have

$$x^2 \frac{d^2}{dx^2}(\frac{\partial}{\partial \mu}J_\mu) + x\frac{d}{dx}(\frac{\partial}{\partial \mu}J_\mu) + (x^2 - \mu^2)\frac{\partial}{\partial \mu}J_\mu - 2\mu J_\mu = 0$$

$$x^2 \frac{d^2}{dx^2}(\frac{\partial}{\partial \mu}J_{-\mu}) + x\frac{d}{dx}(\frac{\partial}{\partial \mu}J_{-\mu}) + (x^2 - \mu^2)\frac{\partial}{\partial \mu}J_{-\mu} - 2\mu J_{-\mu} = 0$$

Multiplying the second equation by $(-1)^p$ and subtracting it from the first equation, we have
$x^2 \frac{d^2}{dx^2}(\frac{\partial}{\partial \mu}J_\mu - (-1)^p \frac{\partial}{\partial \mu}J_{-\mu}) + x(\frac{\partial}{\partial \mu}J_\mu - (-1)^p \frac{\partial}{\partial \mu}J_{-\mu}) + (x^2 - \mu^2)[\frac{\partial}{\partial \mu}J_\mu - (-1)^p \frac{\partial}{\partial \mu}J_{-\mu}] - 2\mu(J_\mu - (-1)^p J_{-\mu}) = 0$

Taking the *limit* $\to p$, the last term drops out because $J_p - (-1)^p J_{-p} = 0$. Clearly the Neumann function expressed in (2.15) satisfies the Bessels equation.
Problem 4: *Show that $J_p(x)$ and $Y_p(x)$ are linearly independent for all p.*
Proof: Let us see the Wronskian

$$\begin{aligned}
W[J_p, Y_p] &= \begin{vmatrix} J_p & J_p(x)\cot(p\pi) - J_{-p}(x)\csc(p\pi) \\ J_p' & J_p'(x)\cot(p\pi) - J_{-p}'(x)\csc(p\pi) \end{vmatrix} \\
&= \cot(p\pi)\begin{vmatrix} J_p & J_p \\ J_p' & J_p' \end{vmatrix} - \csc(p\pi)\begin{vmatrix} J_p & J_{-p} \\ J_p' & J_{-p}' \end{vmatrix} \\
&= -\csc(p\pi) - \frac{2}{\pi x}\sin(p\pi) \\
&= -\frac{2}{\pi x} \neq 0
\end{aligned}$$

Hence, the function $J_p(x)$ and $Y_p(x)$ are linearly independent.
Note: The general solution of the Bessel equation can be written as

$$y_p(x) = c_1 J_p(x) + c_2 Y_P(x)$$

Chapter 3

Asymptotics for Bessel function

Bessel functions are widely used in mathematics,engineering and physics because they are at the heart of so many important applications.To gain insight in to the theory of Bessel functions we will represent them by formulas and integrals involving more familiar functions,such as the cosine and sine functions.
Let us start by driving an important integral representation for Bessel functions and then use it to show a surprising connection with Fourier series.

3.1 Integral Representation of Bessel Functions

Theorem 3.1.1. *Let $p = 0, \pm 1, \pm 2, \cdots$,then for all x,we have*

$$J_p(x) = \frac{1}{\pi}\int_0^{\pi} \cos(p\theta - x\sin\theta)d\theta$$

Proof 3.1.1. *case 1:-If $p \geq 0$.*
Let x and θ be real numbers and set $z = e^{i\theta}$ then we can expand $e^{\frac{x}{2}(z-\frac{1}{z})}$ in Laurent series.

$$\begin{aligned} e^{\frac{x}{2}(z-\frac{1}{z})} &= e^{\frac{x}{2}z}.e^{-\frac{x}{2z}} \\ &= e^{\frac{x}{2}z}\sum_{k=0}^{\infty}\frac{1}{k!}(-\frac{x}{2z})^k \\ &= \sum_{k=0}^{\infty}\frac{(-1)^k}{k!2^k}x^k\frac{e^{\frac{x}{2}z}}{z^k} \end{aligned}$$

For fixed x,the series is absolutely convergent for all θ,so we can multiply both sides by z^{-p},

$$\begin{aligned} e^{\frac{x}{2}(z-\frac{1}{z})}.z^{-p} &= \sum_{k=0}^{\infty}\frac{(-1)^k}{k!2^k}x^k.\frac{e^{\frac{x}{2}z}}{z^k}.z^{-p} \\ &= \sum_{k=0}^{\infty}\frac{(-1)^k}{k!2^k}x^k\frac{e^{\frac{x}{2}z}}{z^{k+p}} \end{aligned}$$

then integrate term by term and get

$$\frac{1}{2\pi}\int_0^{2\pi} e^{\frac{x}{2}(z-\frac{1}{z})}.z^{-p}d\theta = \sum_{k=0}^{\infty}\frac{(-1)^k}{k!2^k}x^k\frac{1}{2\pi}\int_0^{2\pi}\frac{e^{\frac{x}{2}z}}{z^{k+p}}d\theta$$

claim 1:- $\frac{1}{2\pi}\int_0^{2\pi} e^{\frac{x}{2}(z-\frac{1}{z})}.z^{-p}d\theta = \frac{1}{\pi}\int_0^{\pi}\cos(\theta p - x\sin\theta)d\theta$

claim 2:- $J_p(x) = \sum_{k=0}^{\infty}\frac{(-1)^k}{k!2^k}x^k\frac{1}{2\pi}\int_0^{2\pi}\frac{e^{\frac{x}{2}z}}{z^{k+p}}d\theta$

We need the identities

$$\begin{aligned} &z - \tfrac{1}{z} = e^{i\theta} - e^{-i\theta} = 2i\sin\theta \quad \textit{and} \\ &z^{-p} = e^{-ip\theta}(\textit{since} z = e^{i\theta}) \end{aligned}$$

proof of first claim:-*For any integer p,write*

$$\begin{aligned} e^{\frac{x}{2}(z-\frac{1}{z})}.z^{-p} &= e^{ix\sin\theta}.e^{-ip\theta} = e^{i(x\sin\theta - p\theta)} \\ &= \cos(x\sin\theta - p\theta) + i\sin(x\sin\theta - p\theta) \end{aligned}$$

Since,the terms on the right side are 2π-periodic functions of θ,then we have

$$\begin{aligned} \frac{1}{2\pi}\int_0^{2\pi} e^{\frac{x}{2}(z-\frac{1}{z})}.z^{-p}d\theta &= \frac{1}{2\pi}\int_{-\pi}^{\pi}\cos(x\sin\theta - p\theta)d\theta + i\frac{1}{2\pi}\int_{-\pi}^{\pi}\sin(x\sin\theta - p\theta)d\theta \\ &= \frac{1}{2\pi}[\int_{-\pi}^{0}\cos(x\sin\theta - p\theta)d\theta + \int_0^{\pi}\cos(x\sin\theta - p\theta)d\theta] \\ &(\textit{since} \quad i\frac{1}{2\pi}\int_{-\pi}^{\pi}\sin(x\sin\theta - p\theta)d\theta = 0) \\ &= \frac{1}{2\pi}[2\int_0^{\pi}\cos(x\sin\theta - p\theta)d\theta] \\ &= \frac{1}{\pi}\int_0^{\pi}\cos(x\sin\theta - p\theta)d\theta \end{aligned}$$

proof of second claim:- *It is enough to show that*

$$\frac{1}{2\pi}\int_0^{2\pi}\frac{e^{\frac{x}{2}z}}{z^{k+p}}d\theta = (\frac{x}{2})^{k+p}\frac{1}{(k+p)!} = \sum_{j=0}^{\infty}\frac{1}{j!}(\frac{x}{2})^j z^{j-k-p}$$

$$= \sum_{j=0}^{\infty}\frac{1}{j!}(\frac{x}{2})^j e^{i(j-k-p)\theta}$$

Thus

$$\frac{1}{2\pi}\int_0^{\infty}\frac{e^{\frac{x}{2}z}}{z^{k+p}}d\theta = \sum_{j=0}^{\infty}\frac{1}{j!}(\frac{x}{2})^j\frac{1}{2\pi}\int_0^{2\pi}e^{i(j-k-p)\theta}d\theta$$

But By the orthogonality of the complex exponential system

$$\frac{1}{2\pi}\int_0^{2\pi}e^{i(j-k-p)\theta}d\theta = \begin{cases} 0, if & j \neq k+p \\ 1, if & j = k+p \end{cases}$$

Thus,

$$\frac{1}{2\pi}\int_0^{2\pi}\frac{e^{\frac{x}{2}z}}{z^{k+p}}d\theta = \sum_{j=k+p}\frac{1}{j!}(\frac{x}{2})^j$$

$$= \frac{1}{(k+p)!}(\frac{x}{2})^{k+p}$$

Therefore

$$\sum_{k=0}^{\infty}\frac{(-1)^k}{k!}(\frac{x}{2})^k\frac{1}{2\pi}\int_0^{2\pi}\frac{e^{\frac{x}{2}z}}{z^{k+p}}d\theta = \sum_{k=0}^{\infty}\frac{(-1)^k}{k!}(\frac{x}{2})^k.\frac{1}{(k+p)!}(\frac{x}{2})^{k+p}$$

$$= \sum_{k=0}^{\infty}\frac{(-1)^k}{k!(k+p)!}(\frac{x}{2})^{2k+p}$$

$$= \sum_{k=0}^{\infty}\frac{(-1)^k}{k!\Gamma(k+p+1)}(\frac{x}{2})^{2k+p}$$

$$= J_p(x)$$

Since

$$\sum_{k=0}^{\infty}\frac{(-1)^k}{k!2^k}x^k\frac{1}{2\pi}\int_0^{2\pi}\frac{e^{\frac{x}{2}z}}{z^{k+p}}d\theta = \frac{1}{2\pi}\int_0^{2\pi}e^{\frac{x}{2}(z-\frac{1}{z})}.z^{-p}d\theta$$

Thus,

$$J_p(x) = \frac{1}{\pi}\int_0^{\pi}\cos(x\sin\theta - p\theta)d\theta$$

case 2:-*If* $p < 0$

proof:-*Let* $I_p(x) = \frac{1}{\pi}\cos(p\theta - x\sin\theta)d\theta$

We prove in case 1 for $p \geq 0, I_p(x) = J_p(x)$.

We have defined

$$J_{-p}(x) = (-1)^p J_p(x)$$

Thus in order to show it holds for $p < 0$*,it is enough to show that*

$$I_{-p}(x) = (-1)^p I_p(x).$$

i.e.
a.show that $I_{-p}(x) = \frac{1}{\pi}\int_0^\pi \cos(p\theta + x\sin\theta)d\theta$ *and*
b.Using the addition formula for the cosine to expand the integrands in I_p *and* I_{-p}*, then we conclude that* $I_{-p} = (-1)^p I_p$

corollary:-J_0 *is bounded by 1.*
i.e. Show that for all $x, |J_0(x)| \leq 1$.

proof:-Since from the above theorem we have

$J_p(x) = \frac{1}{\pi}\int_0^\pi \cos(p\theta - x\sin\theta)d\theta$.

Thus,for $p = 0$,we have

$$|J_0(x)| = \frac{1}{\pi}|\int_0^\pi \cos(x\sin\theta)d\theta| \leq \frac{1}{\pi}\int_0^\pi |\cos(x\sin\theta)d\theta|$$

$$\leq \frac{1}{\pi}\int_0^\pi d\theta \quad (\because |\cos(x\sin\theta)| \leq 1)$$

$$= 1$$

Thus,$|J_0(x)| \leq 1$

3.2 Generating Function for Bessel Function($J_p(x)$)

A generating function is a formal power series in one indeterminate,whose coefficients encode information about a sequence of number a_n that is indexed by the natural numbers.

Definition 3.2.1. *A generating function is a clothes line on which we hang up a sequence of numbers for display.*

Theorem 3.2.1. *For all x, all θ and $z = e^{i\theta}$, we have*
$e^{\frac{x}{2}(z-\frac{1}{z})} = e^{ix\sin\theta} = \sum_{p=-\infty}^{\infty} J_p(x)e^{ip\theta}$

Proof.

$$
\begin{aligned}
e^{\frac{xz}{2}}.e^{-\frac{x}{2z}} &= \sum_{r=0}^{\infty} \frac{(\frac{1}{2}xz)^r}{r!}.\sum_{k=0}^{\infty} \frac{(\frac{\frac{-1}{2}x}{z})^k}{k!} (using Laurent Series) \\
&= \sum_{r=0}^{\infty}\sum_{k=0}^{\infty} \frac{(-1)^k(\frac{1}{2}x)^{k+r}z^{r-k}}{r!k!} \\
&= \sum_{p=-\infty}^{\infty}\Big(\sum_{r-k=p \quad and \quad r,k\geq 0} \frac{(-1)^k(\frac{x}{2})^{r+k}}{r!k!}\Big)z^p \\
&= \sum_{p=-\infty}^{\infty}\sum_{k=0}^{\infty} \frac{(-1)^k(\frac{x}{2})^{p+2k}}{(p+k)!k!}z^p \\
&= \sum_{p=-\infty}^{\infty} J_p(x)z^p
\end{aligned}
$$

□

Thus, $e^{ix\sin\theta} = \sum_{p=-\infty}^{\infty} J_p(x)e^{ip\theta}$

Note:-The function $e^{\frac{x}{z}(z-\frac{1}{z})}$is the generating function for Bessel function of the first kind.

Proposition 3.2.1. *For all x*

$\cos x = J_0(x) + 2\sum_{p=1}^{\infty}(-1)^p J_{2p}(x)$ *and*

$\sin x = 2\sum_{p=0}^{\infty}(-1)^p J_{2p+1}(x)$

Proof. Since $e^{ix\sin\theta} = \sum_{p=-\infty}^{\infty} J_p(x)e^{ip\theta}$
Now,put $\theta = \frac{\pi}{2}$,then for all x

$e^{ix\sin\frac{\pi}{2}} = \sum_{p=-\infty}^{\infty} J_p(x)e^{ip\frac{\pi}{2}}$

$\cos x + i\sin x = \sum_{p=-\infty}^{\infty} J_p(x)[\cos(p\frac{\pi}{2}) + i\sin(p\frac{\pi}{2})]$

$\cos x = \sum_{p=-\infty}^{\infty} J_p(x)\cos(p\frac{\pi}{2})$ and

$\sin x = \sum_{p=-\infty}^{\infty} \sin(p\frac{\pi}{2})J_p(x)$

Using $\cos(p\frac{\pi}{2}) = 0$ if p is odd and $(-1)^m$ if $p = 2m$ (even)

and $J_{-2p}(x) = (-1)^{2p} J_{2p}(x) = J_{2p}(x)$,we obtain

$$\begin{aligned}
\cos x &= \sum_{p=-\infty}^{\infty} J_p(x) \cos(p\frac{\pi}{2}) \\
&= J_0(x) + \sum_{p=1}^{\infty} (-1)^p J_{2p}(x) + \sum_{p=-1}^{-\infty} (-1)^p J_{2p}(x) \\
&= J_0(x) + \sum_{p=1}^{\infty} (-1)^p J_{2p}(x) + \sum_{p=1}^{\infty} (-1)^p J_{-2p}(x) \\
&= J_0(x) + \sum_{p=1}^{\infty} (-1)^p (J_{2p}(x) + J_{-2p}(x)) \\
&= J_0(x) + 2\sum_{p=1}^{\infty} (-1)^p J_{2p}(x)
\end{aligned}$$

$$\therefore \cos(x) = J_0(x) + 2\sum_{p=1}^{\infty} (-1)^p J_{2p}(x)$$

similarly $\sin x = 2\sum_{n=0}^{\infty} (-1)^p J_{2p+1}(x)$ □

3.3 Methods of Stationary phase

Lemma 3.3.1. *Suppose that f(t) is a real-value function with a Taylor Series centered at t_0 in the interval [a,b],such that $f^{'}(t_0) = 0, f^{'}(t) \neq 0$ for all $t \neq t_0$, and $f^{''}(t) \neq 0$.Let g(t) be an arbitrary smooth complex-valued function on [a,b],then for large x*

$$\int_a^b e^{ixf(t)} g(t) dt \sim \sqrt{\frac{2\pi}{x}} g(t_0) \frac{e^{i(xf(t_0) \pm \frac{\pi}{4})}}{\sqrt{|f''(t_0)|}}$$

where we use the plus sign if $f^{''}(t_0) > 0$ and the minus sign if $f^{''}(t_0) < 0$.The function f(t) is called a phase function and the point $t_0$is called a stationary point.

Proof. Let $I = \int_a^b e^{ixf(t)} g(t) dt$
case 1:-Suppose that $f^{''}(t_0) > 0$

Expand the function f(t) in a Taylor series about t_o:

$f(t) = f(t_0) + f^{'}(t_0)(t - t_0) + \frac{f^{''}(t_0)}{2}(t - t_0)^2 + \cdots$

since $f^{'}(t_0) = 0$ and $f^{''}(t_0) > 0$ then approximately

$f(t) = f(t_0) + \frac{f^{''}(t_0)}{2}(t - t_0)^2$

and $g(t) = g(t_0)$

Thus,

$$
\begin{aligned}
I &\approx \int_a^b e^{ix(f(t_0)+\frac{f^{''}(t_0)}{2}(t-t_0)^2)} g(t_0)dt \\
&= g(t_0)e^{ixf(t_0)} \int_a^b e^{ix(\frac{f^{''}(t_0)}{2}(t-t_0)^2)} dt \\
&= g(t_0)e^{ixf(t_0)} \int_a^b e^{i([\frac{xf^{''}(t_0)}{2}]^{\frac{1}{2}}(t-t_0))^2} dt \\
&= \sqrt{\frac{2}{x}} g(t_0) \frac{e^{ixf(t_0)}}{\sqrt{|f''(t_0)|}} \int_A^B e^{iu^2} du
\end{aligned}
$$

where $u = [\frac{xf^{''}(t_0)}{2}]^{\frac{1}{2}}(t - t_0), du = [\frac{xf^{''}(t_0)}{2}]^{\frac{1}{2}}dt, A = [\frac{xf^{''}(t_0)}{2}]^{\frac{1}{2}}(a - t_0)$

and $B = [\frac{xf^{''}(t_0)}{2}]^{\frac{1}{2}}(b - t_0)$

As $x \to \infty, A \to -\infty$ and $B \to \infty$ and the integral converges to

$$
\begin{aligned}
\int_{-\infty}^{\infty} e^{iu^2} &= \int_{-\infty}^{\infty} \cos(u^2)du + i \int_{-\infty}^{\infty} \sin(u^2)du \\
&= \sqrt{\pi}[\cos\frac{\pi}{4} + i \sin\frac{\pi}{4}] \\
&= \sqrt{\pi} e^{i\frac{\pi}{4}}
\end{aligned}
$$

The last displayed integrals are known as the Fresnel integrals.

Thus

$$
\begin{aligned}
I = \int_a^b e^{ixf(t)} g(t)dt &\sim \sqrt{\frac{2}{x}} g(t_0) \frac{e^{ixf(t_0)}}{\sqrt{|f''(t_0)|}} \sqrt{\pi} e^{i\frac{\pi}{4}} \\
\therefore \int_a^b e^{ixf(t)} g(t)dt &\sim \sqrt{\frac{2\pi}{x}} g(t_0) \frac{e^{i(xf(t_0)+\frac{\pi}{4})}}{\sqrt{|f''(t_0)|}}
\end{aligned}
$$

case 2:-Suppose $f^{''}t_0 < 0$.

proof:-Expand the function f(t) in Taylor series about t_0:

$f(t) = f(t_0) + f^{'}(t_0)(t - t_0) - \frac{f^{''}(t_0)}{2}(t - t_0)^2 + \cdots$

Since $f^{'}(t_0) = 0$ and $f^{''}(t_0) < 0$,then approximately

$f(t) = f(t_0) - \frac{f^{''}}{2}(t - t_0)^2$

and $g(t) = g(t_0)$

Thus,

$$\begin{aligned}
\int_a^b e^{ixf(t)}g(t)dt &\approx \int_a^b e^{ix(f(t_0) - \frac{f^{''}(t_0)}{2}(t-t_0)^2)}.g(t_0)dt \\
&= g(t_0)e^{ixf(t_0)} \int_a^b e^{-ix\frac{f^{''}(t_0)}{2}(t-t_0)^2)}dt \\
&= g(t_0)e^{ixf(t_0)} \int_a^b e^{-i([\frac{xf^{''}(t_0)}{2}]^{\frac{1}{2}}(t-t_0))^2}dt \\
&= \sqrt{\frac{2}{x}}g(t_0)\frac{e^{ixf(t_0)}}{\sqrt{|f''(t_0)|}} \int_A^B e^{-iu^2}du
\end{aligned}$$

where $u = [\frac{xf^{''}(t_0)}{2}]^{\frac{1}{2}}(t - t_0), du = [\frac{xf^{''}(t_0)}{2}]^{\frac{1}{2}}dt, A = [\frac{xf^{''}(t_0)}{2}]^{\frac{1}{2}}(a - t_0)$

and $B = [\frac{xf^{''}(t_0)}{2}]^{\frac{1}{2}}(b - t_0)$

As $x \to \infty, A \to -\infty$ and $B \to \infty$ and the integral converges to

$$\begin{aligned}
\int_{-\infty}^{\infty} e^{-iu^2} &= \int_{-\infty}^{\infty} \cos(u^2)du - i\int_{-\infty}^{\infty} \sin(u^2)du \\
&= \sqrt{\pi}[\cos\frac{\pi}{4} - i\sin\frac{\pi}{4}] \\
&= \sqrt{\pi}e^{-i\frac{\pi}{4}}
\end{aligned}$$

The last displayed integrals are known as the Fresnel integrals.

Thus

$$I = \int_a^b e^{ixf(t)}g(t)dt \sim \sqrt{\frac{2}{x}}g(t_0)\frac{e^{ixf(t_0)}}{\sqrt{|f''(t_0)|}}\sqrt{\pi}e^{-i\frac{\pi}{4}}$$

$$\therefore \int_a^b e^{ixf(t)} g(t)dt \sim \sqrt{\frac{2\pi}{x}} g(t_0) \frac{e^{i(xf(t_0)-\frac{\pi}{4})}}{\sqrt{|f''(t_0)|}}$$

□

3.4 Asymptotics Formula

As x tends to ∞,the graph oscillates like a cosine or sine wave and its amplitude decays like a negative power of x.
In general we have the following asymptotics formula

Theorem 3.4.1. *Let $p \geq 0$ be an integer ,then for all large x , we have $J_p(x) \sim \sqrt{\frac{2}{\pi x}} \cos(x - \frac{\pi}{4} - \frac{p\pi}{2}) + O(\frac{1}{x^{\frac{3}{2}}})$*
Where the big oh notation means that the approximation is of the order $\frac{1}{x^{\frac{3}{2}}}$.
(OR The expression "big oh of $\frac{1}{x^{\frac{3}{2}}}$" means that there is a constant $c > 0$ such that $|J_p(x) - \sqrt{\frac{2}{\pi x}} \cos(x - \frac{\pi}{4})| \leq \frac{c}{x^{\frac{3}{2}}}$ for all x.Thus the error in the approximation tends to zero like $\frac{1}{x^{\frac{3}{2}}}$.)

Proof. The formula can be derived using the method of stationary phase.
Recall the integral representation that we derived in the proof of the first claim in Theorem 3.1.1:

$$\begin{aligned}
J_p(x) &= \frac{1}{2\pi} \int_0^{2\pi} e^{\frac{x}{2}(z-\frac{1}{z})} z^{-p} d\theta \\
&= \frac{1}{2\pi} \int_0^{2\pi} e^{ix \sin\theta} e^{-ip\theta} d\theta \\
&= \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{ix \sin\theta} e^{-ip\theta} d\theta \\
&= \frac{1}{2\pi} \int_0^{\pi} e^{ix \sin\theta} e^{-ip\theta} d\theta + \frac{1}{2\pi} \int_0^{\pi} e^{-ix \sin\theta} e^{ip\theta} d\theta
\end{aligned}$$

(changing the variable θ to $-\theta$ on $[-\pi, 0]$)

Now consider the first integral

$\frac{1}{2\pi} \int_0^{\pi} e^{ix \sin\theta} e^{-ip\theta} d\theta$

In Lemma 3.3.1, take $f(\theta) = \sin\theta$ and $g(\theta) = e^{-ip\theta}$
$f^{'}(\theta) = 0$ only at $\theta = \frac{\pi}{2}$ in the interval $[0, \pi]$ and $f''(\frac{\pi}{2}) = -\sin\frac{\pi}{2} = -1 \neq 0$.

From Lemma 3.3.1,

$$\frac{1}{2\pi}\int_0^\pi e^{ix\sin\theta}.e^{-ip\theta}d\theta \sim \frac{1}{2\pi}\sqrt{\frac{2\pi}{x}}e^{-ip\frac{\pi}{2}}e^{i(x-\frac{\pi}{4})}$$
$$= \frac{1}{2\pi}\sqrt{\frac{2\pi}{x}}e^{i(x-\frac{\pi}{4}-p\frac{\pi}{2})}$$

Thus,

$$\frac{1}{2\pi}\int_0^\pi e^{ix\sin\theta}.e^{-ip\theta}d\theta \sim \frac{1}{2}\sqrt{\frac{2}{\pi x}}e^{i(x-\frac{\pi}{4}-p\frac{\pi}{2})} \tag{3.1}$$

Similarly, the asymptotics formula for the second integral $\frac{1}{2\pi}\int_0^\pi e^{-ix\sin\theta}.e^{ip\theta}d\theta$ is

In Lemma 3.3.1,take $f(\theta) = -\sin\theta$ and $g(\theta) = e^{ip\theta}$
$f'(\theta) = 0$ only at $\theta = \frac{\pi}{2}$ in the interval $[0,\pi]$
and$f''(\frac{\pi}{2}) = \sin\frac{\pi}{2} = 1 \neq 0$
From Lemma 3.3.1,

$$\frac{1}{2\pi}\int_o^\pi e^{-ix\sin\theta}e^{ip\theta}d\theta \sim \frac{1}{2\pi}\sqrt{\frac{2\pi}{x}}e^{ip\frac{\pi}{2}}\frac{e^{i(-x+\frac{\pi}{4})}}{\sqrt{|1|}}$$
$$= \frac{1}{2\pi}\sqrt{\frac{2\pi}{x}}e^{-i(x-\frac{\pi}{4}-p\frac{\pi}{2})}$$

$$Thus, \quad \frac{1}{2\pi}\int_0^\pi e^{-ix\sin\theta}.e^{ip\theta}d\theta \sim \frac{1}{2}\sqrt{\frac{2}{\pi x}}e^{-i(x-\frac{\pi}{4}-p\frac{\pi}{2})} \tag{3.2}$$

Now add the two asymptotics and simplifying it

$$\begin{aligned}
J_p(x) &= \frac{1}{2\pi}\int_0^\pi e^{ix\sin\theta}.e^{-ip\theta}d\theta + \frac{1}{2\pi}\int_0^\pi e^{-ix\sin\theta}.e^{ip\theta}d\theta \\
&\sim \frac{1}{2}\sqrt{\frac{2}{\pi x}}e^{i(x-\frac{\pi}{4}-p\frac{\pi}{2})} + \frac{1}{2}\sqrt{\frac{2}{\pi x}}e^{-i(x-\frac{\pi}{4}-p\frac{\pi}{2})} \\
&= \frac{1}{2}\sqrt{\frac{2}{\pi x}}[e^{i(x-\frac{\pi}{4}-p\frac{\pi}{2})} + e^{-i(x-\frac{\pi}{4}-p\frac{\pi}{2})}] \\
&= \frac{1}{2}\sqrt{\frac{2}{\pi x}}2\cos(x-\frac{\pi}{4}-p\frac{\pi}{2}) \\
&= \sqrt{\frac{2}{\pi x}}\cos(x-\frac{\pi}{4}-p\frac{\pi}{2})
\end{aligned}$$

$$\therefore J_p(x) \sim \sqrt{\frac{2}{\pi x}}\cos(x-\frac{\pi}{4}-p\frac{\pi}{2})$$

□

Summary

In this work, we have discussed the first and the second solutions of Bessel equations, which are called Bessel functions, properties of Bessel functions,some recurrence formulas, Generating function for Bessel Function($J_p(x)$)and finally we have discussed the approximate solution of Bessel equation as x tends to infinity(for larger intervals of x.)

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