HARMONIC FUNCTION

COLLEGE OF COMPUTATIONAL AND NATURAL SCIENCES
DEPARTMENT OF MATHEMATICS

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The undersigned hereby certify that they have read and recommend to the department of mathematics for acceptance of this project entitled "**Harmonic Function**" by **Abebe Erjabo** in partial fulfillment of the requirements for the degree of Master of Science in mathematics.

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Abstract

Harmonic functions are closely connected to analytic functions. Since the real and imaginary parts of analytic functions are harmonic functions.
The theories of harmonic functions have many interesting features. Among these interesting features are mean value property; maximum principle and these properties with Poisson integral enables to solve Dirichlet problem.
The Poisson integral formula shows that if $u(z)$ is harmonic in a disk and continuous on the closed disk, then its value at any interior point is completely determined by its value on the boundary circle.
These facts suggest the following two questions:
(a) Given a real valued bounded piecewise continuous function $u(e^{i\theta})$ on the unit circle, do we obtain a harmonic function $v(z)$ through the Poisson integral;

$$v(z) = \frac{1}{2\pi}\int_0^{2\pi} u(e^{i\theta})P_r(\theta - t)d\theta, z = re^{it}?$$

(b) If so, do the boundary values of $v(z)$ agree with $u(e^{i\theta})$?
The answer is affirmative and the unique solution is given by;

$$v(z) = \begin{cases} pu, if z \in D(0;1) \\ u(z), if z \in \partial D(0;1) \end{cases}$$

The above two questions leads to the problem of finding a function that is harmonic in a region and has pre assigned values on the boundary which is known as the Dirichlet problem.

Introduction

This project is devoted to study of harmonic functions on a plane and on the disk. These functions are closely connected to analytic functions, since the real and imaginary parts of analytic functions are harmonic functions. The study of harmonic functions is important in mathematics, physics and Engineering, and there are many results in the theory of harmonic functions that are not connected directly with complex analysis. However, in this project we consider that, part of the theory of harmonic functions that grows out of the Cauchy theory and Mobius transformations. These functions have many interesting property, like mean value property (MVP) and maximum principle. One of the most important aspects of harmonic functions is that they arise as functions that solve boundary value problems for analytic functions, known as the Dirichlet problem. This project organized in to three chapters. In the first chapter, we consider basic properties of real and complex functions theory with some topological concept. It includes basic definition and notations, class of analytic function and continuous functions. The second chapter deals with harmonic functions as a real part and imaginary part of analytic function and interesting property of harmonic functions such that mean value property and maximum principle and integral representation, that develops from the theory of conformal mapping. The third chapter deals with the solution of Dirichlet problem on the disk.

Chapter 1

Preliminaries

In this chapter we consider basic properties of real and complex function theory with some topological concepts. Several results and techniques of this chapter are frequently used in later chapters.

1.1 Definitions and Notations

Let $\mathbb{C}$ denotes the set of complex numbers and $\mathbb{R}^n$ denotes the Euclidean space of n dimensional for $n > 1$. However, in this project we use for $n = 2$. Throughout this project the letters G always will denote plane open set. Let $z \in \mathbb{C}$, we identify a point (x, y) with $z = x + iy$.

Recall that the following notations: If a is a point in space $\mathbb{C}$ and $r > 0$, then we write:
(i) A set $D(a, r) = \{z \in \mathbb{C} : |z - a| < r\}$ is called an open ball (*disk*) center at a and radius r.
(ii) A set $D[a, r] = \{z \in \mathbb{C} : |z - a| \leq r\}$ is called a closed ball (*disk*) center at a and radius r.
(iii) A set $S = \{z \in \mathbb{C} : |z - a| = r\}$ is called a sphere (*circle*) center at a and radius r.
(iv) The open unit ball is the set $D(a, 1) = \{z \in \mathbb{C} : |z - a| < 1\}$.
(v) The closed unit ball is given by $D[a; 1] = \{z \in \mathbb{C} : |z - a| \leq 1\} = S \cup D(a; 1)$.

Definition 1.1.1. An arbitrary set of points N contained in a ball $D(a; r)$ for some $r > 0$ is called *neighborhood of a*.

Definition 1.1.2. A set which is a neighborhood of all of its points is called an *open set.* That is, if O is an open set and $a \in O$, then there exists $r > 0$ such that $D(a; r) \subseteq O$ or for each $a \in O, \exists r > 0$ such that $x \in O$ whenever $|x - a| < r$.

Definition 1.1.3. A set F is said to be *closed* if and only if it contains all of its limit points. or its complement is open.

Definition 1.1.4. *A closure of a set E,* denoted by $\overline{E}$ or $cl(E)$ consists of E together with a limit points of all convergent sequences of point E.

Definition 1.1.5. A set which is contained in some ball is called a *bounded set.*

Definition 1.1.6. *A region G is connected* if the only subsets of G which are both open and closed are empty and G itself. Or if E a set, a pair of sets E_1 and E_2 are called a partition of E if $E_1 \bigcup E_2 = E$ and $E_1 \bigcap E_2 = \emptyset$, for $E_1, E_2 \neq \emptyset$. if a set does not permit partition then it is called connected set.

Definition 1.1.7. By a region we shall mean a non-empty connected open sub set of the complex plane. The union of disjoint region will form plane open set G.

1.2 The Class of Continuous and Analytic functions

Definition 1.2.1. A function f is continuous at a point z_o in region G if all three of the following conditions are satisfies.
1. $\lim_{z \to z_o} f(z)$ exists.
2. $f(z_o)$ exists.
3. $\lim_{z \to z_o} f(z) = f(z_o)$
Recall that a function of a complex variable is said to be continuous in a region G if it is continuous at each point in G.

Definition 1.2.2. If G is an open set in $\mathbb{C}$ and $f : G \longrightarrow \mathbb{C}$ then f is differentiable at a point a in G, if

$$\lim_{x \to a} \frac{f(x) - f(a)}{x - a}$$

exists; the value of the limit is denoted by $f'(a)$ and is called the derivative of f at a. If f is differentiable at each point of G we say that f is differentiable on G.
Note that; if f is differentiable on G then $f^{'}(a)$ defines a function $f^{'} : G \longrightarrow \mathbb{C}$. if $f^{'}$ is continuous then we say that f is continuously differentiable. If $f^{'}$ is differentiable then f is twice differentiable; continuing, a differentiable function such that each successive derivative is again differentiable is called infinitely differentiable.

Definition 1.2.3. *A function $f : G \to \mathbb{C}$ is analytic if f is continuously differentiable on G.*

1.3 The Role of Conformal Mapping and the Mőbius Transformation

Definition 1.3.1. A bijective analytic function $f : u \longrightarrow v$, with u and v is open in $\mathbb{C}$ is called a conformal map. Given such a mapping f, we say that u and v are conformal equivalent. A conformal map $f : u \longrightarrow v$ from one open set to another can be used to transfer analytic function on u to v and vice versa, that is $h : v \longrightarrow \mathbb{C}$ is analytic if and only if hof is analytic on u and $g : u \longrightarrow \mathbb{C}$ is analytic if and only if gof^{-1} is analytic on v.

Definition 1.3.2. A Mőbius transformation (sometimes known as fractional linear transformation or bilinear transformation) is any function of the form

$$w = f(z) = \frac{az + b}{cz + d}$$

with restriction that $ad \neq bc$ (so that w is not a constant function).
Notice that; since $f'(z) = \frac{ad-bc}{(cz+d)^2}$ does not vanish, the Mőbius transformation $f(z)$ is conformal at ever point except at its pole $z = \frac{-d}{c}$. Here we consider the special maps of unit disk $D(0,1)$ into itself.
The following lemma supplies us the important class of examples,

Lemma 1.3.1. For $a \in \mathbb{C}, \mid a \mid < 1$ we define $\phi_a(z) = \frac{z-a}{1-z\bar{a}}$. Then each ϕ_a is a conformal self map of the unit disk.
Note; In general for $a \in \mathbb{C}, \mid a \mid < 1$, then the analytic function $\phi_a = \frac{z-a}{1-z\bar{a}}$ has the following properties.
(1) ϕ_a is analytic and invertible on a neighborhood of $\bar{D}(0,1)$;
(2) $\phi_a : D(0,1) \longrightarrow D(0,1)$ is one to one and onto;
(3) $\phi_a^{-1} = \phi_{-a}$.
(4) $\phi_a = 0$.

1.4 Cauchy's Integral Formula

Theorem 1.4.1. *(Cauchy's Intigral Formula) Let γ be a simple closed positively oriented contour, if f is analytic in a simple connected domain G containing γ and z_0 is any point inside, γ then*

$$f(z_0) = \frac{1}{2\pi i} \int_\gamma \frac{f(z)}{(z - z_0)} dz.$$

Chapter 2

Harmonic Functions

2.1 Introduction

The theories of harmonic functions have many interesting features. Among these interesting features are mean value property; maximum principle and these properties with Poisson integral enables to solve Dirichlet problem. Harmonic functions are the solutions of Laplaces equation and hence play a crucial role in many areas of mathematics, physics and engineering.

Definition 2.1.1. If G is an open subset of $\mathbb{C}$, then a function $u : G \longrightarrow \mathbb{R}$ is *harmonic* if u has continuous second partial derivatives and

$$\frac{d^2u}{dx^2} + \frac{d^2u}{dy^2} = 0$$

in region G. This equation is called Laplace equation.

Theorem 2.1.1. *Let u and v be real valued functions defined on a region G and suppose that u and v have continuous partial derivatives.*
Then $f : G \longrightarrow \mathbb{C}$ defined by

$$f(z) = u(z) + iv(z) \tag{2.1}$$

is analytic if and only if $Ref = u$ and $Imf = v$ are harmonic functions which satisfy the Cauchy-Riemann equations.

Proof. Suppose that u and v are continuous partial derivatives and f is analytic. We want to show u and v satisfy a Cauchy Riemann equation. Let $Ref(x + iy) = u(x, y)$ and $Imf(x + iy) = v(x, y)$, where $z = x + iy$ in G

then $f(x+iy) = u(x,y) + iv(x,y)$. Suppose that at a point $z_0 = (x_0, y_0)$ the derivative of f exists there. We also show how to write

$$f'(z_0) = \lim_{h \to 0} \frac{f(z_0+h) - f(z_0)}{h} \tag{2.2}$$

exist. Writing $z_0 = x_0 + iy_0$ and $h = \Delta x + i\Delta y$, we then have

$$Re[f'(z_0)] = \lim_{(\Delta x, \Delta y) \to (0,0)} Re[\frac{f(z_0+h) - f(z_0)}{h}] \tag{2.3}$$

$$Im[f'(z_0)] = \lim_{(\Delta x, \Delta y) \to (0,0)} Im[\frac{f(z_0+h) - f(z_0)}{h}] \tag{2.4}$$

where

$$\frac{f(z_0+h) - f(z_0)}{h} = \frac{u(x_0+\Delta x, y_0+\Delta y) - u(x_0, y_0) + i[v(x_0+\Delta x, y_0+\Delta y) - v(x_0, y_0)]}{\Delta x + i\Delta y}. \tag{2.5}$$

It is important now to keep in mind that expression (2.3) and (2.4) are valid as $(\Delta x, \Delta y)$ tends to $(0,0)$ in any manner that we may choose.In particular, let $(\Delta x, \Delta y)$ tends to $(0,0)$ horizontal through the points $(\Delta x, 0)$. This means that $\Delta y = 0$ in equation (2.5) and we find that

$$Re[f'(z_0)] = \lim_{\Delta x \to 0} \frac{u(x_0+\Delta x, y_0) - u(x_0, y_0)}{\Delta x}.$$

$$Im[f'(z_0)] = \lim_{\Delta x \to 0} \frac{i[v(x_0+\Delta x, y_0) - v(x_0, y_0)]}{\Delta x}.$$

That is

$$f'(z_0) = u_x(x_o, y_0) + iv_x(x_o, y_0) \tag{2.6}$$

where $u_x(x_o, y_0)$ and $v_x(x_0, y_0)$ denoted the first order partial derivatives with respect to x of the functions u and v at (x_o, y_o). We might have let (x_o, y_o) tends to zero vertically through the points $(0, \Delta y)$. in that case,$\Delta x = 0$ in equation (2.5)and we obtain the expression

$$f^{'}(z_o) = v_y(x_o, y_o) - iu_y(x_o, y_o) \ \ for \ \ f^{'}(z_o). \tag{2.7}$$

this time in terms of the first order partial derivatives u and v with respect to y, evaluated at (x_o, y_o). Equations (2.6)and (2.7) not only gives $f^{'}(z_o)$ in terms of partial derivatives of the component u and v. but they also proved necessary conditions for the existence of $f^{'}(z_o)$. For, on equating the real and imaginary parts of right hand sides of these equations,we see that the existence of $f^{'}(z_o)$ requires that

$$u_x(x_o, y_o) = v_y(x_o, y_o) \ and \ u_y(x_o, y_o) = -v_x(x_o, y_o) \tag{2.8}$$

Equation (2.8)are the cauchy-Remain equations. Since f is analytic in G we start with the observation that the first order partiale derivatives of its component functions must satisfy the cauchy-Riemann equations throuhg G. Differentiating both sides of these equations with respect to x, we have equation

$$u_{xx} = v_{yy} \ \ and \ \ u_{yy} = -v_{xx}$$

say that (3). Likewise, differentiations with respect to y yields

$$u_{xy} = v_{yy} \ \ and \ \ u_{yy} = -v_{xy}$$

Say equation (4). Now by theorem in advanced calculus, the continuity of partial derivatives of u and v ensures that

$$u_{yx} = u_{xy} \ and \ v_{yx} = v_{xy}$$

It then follows from equation (3) and (4) that

$$u_{xx} + u_{yy} = 0 \ \ and \ \ v_{xx} + v_{yy} = 0$$

That is, u and v are harmonic in G. □

Corollary 2.1.1. *Let $f(z) = u(x, y)+iv(x, y)$ is analytic in a region G, then both the real and imaginary parts of f are harmonic in G.*

Proof. Since $f(z)$ is analytic, u and v satisfying Cauchy-Riemann equations. That is: $u_x = v_y$ and $u_y = -v_x$.
Now the Laplace equation is given for the real part by

$$\begin{aligned} \Delta u &= \frac{d^2u(x, y)}{dx^2} + \frac{d^2u(x, y)}{dy^2} \\ &= \frac{dv_x(x, y)}{dx} + \frac{du_y(x, y)}{dy} \\ &= \frac{dv_y(x, y)}{dx} - \frac{dv_x(x, y)}{dy} \\ &= v_{yx} - v_{xy} = 0. \end{aligned}$$

From the fact v has continuous a second order continuous partial derivative,

$$v_{yx} = v_{xy}.$$

Similarly the imaginary parts of analytic functions are harmonic. □

Definition 2.1.2. If $f : G \longrightarrow \mathbb{C}$ is an analytic function then $u = Ref$ and $v = Imf$ is called harmonic conjugates. Every harmonic function on a simply connected region has a harmonic conjugate. If u is a harmonic function on G and D is a disk that is contained in G then there is a harmonic function v on D such that $u + iv$ is analytic on D. In other words each harmonic function has a harmonic conjugate locally.

Example 2.1.1. $u(x, y) = x^3 - 3x^2y - 3xy^2 + y^3$ is harmonic. Find the harmonic conjugate on u.

To find a harmonic conjugate, first we solve the cauchy-Riemann equation. that is

$$\frac{du}{dx} = \frac{dv}{dy}, \ and \ , \frac{-du}{dy} = \frac{dv}{dx}$$

$$\frac{-du}{dx} = \frac{dv}{dx} = 3x^2 + 6xy - 3y^2$$

$$\int dv = \int (3x^2 + 6xy - 3y^2)dx$$

$$v = x^3 + 3x^2y - 3xy^2 + h(x)$$

$$v(x, y) = x^3 + 3x^2y - 3xy^2 + h(x)$$

where $h(y)$is the constant of the integration is a function of y alone. To determine h we substitute our formula in the second Cauchy-Riemann equation.

$$3x^2 - 6xy + h^{'}(y) = \frac{dv}{dy} = \frac{du}{dx} = 3x^2 - 6xy - 3y^2.$$

Therefore $h^{'}(y) = -y^2$ and so $h(y) = -y^3 + c$. where c is a real constant. we conclude that every harmonic conjugate $u(x, y)$has the form

$$v(x, y) = x^3 + 3xy^2 - 3xy^3 - y^3 + c.$$

Proposition 2.1.1. *If $u : G \longrightarrow \mathbb{R}$ is harmonic then u infinitely differentiable.*

Proof. Let $z_0 = x_0 + iy_0 \in G$ and let chose $\delta > 0$ such that $D(z_0, \delta) \subset G$. Then from definition above u has a harmonic conjugate v on $D(z_0, \delta)$ that is, $f = u + iv$ is analytic on D and hence infinitely differentiable on $D(z_0, \delta)$ Therefore u is infinitely differentiable. □

2.2 Properties of Harmonic Functions

Mean value Property (MVP)

Mean value properties is one of the properties of the harmonic function which is analogous to the Cauchy integral formula. It is used to find the value of a harmonic function at the center of the disk from its value at the boundary as well as on the surface on the disk.

Theorem 2.2.1. ***(MVT)*** *If $u : G \longrightarrow \mathbb{R}$ is a harmonic function and $\bar{B}(a, r)$ is a closed disk contained in G, then*

$$u(a) = \frac{1}{2\pi}\int_0^{2\pi} u(a + re^{i\theta})d\theta.$$

Proof. Let D is a disk such that $\bar{B}(a; r) \subset D \subset G$ and let f is analytic on D and $u = Ref$,then by Cauchy integral formula we have that:

$$f(a) = \frac{1}{2\pi i}\int_{|z-a|=r} \frac{f(z)}{z-a}dz,$$

where $z = a + re^{i\theta}$,then

$$\begin{aligned} f(a) = \frac{1}{2\pi i}\int_0^{2\pi} \frac{f(a + re^{i\theta})}{re^{i\theta}}dz &= \frac{1}{2\pi}\int_0^{2\pi} f(a + re^{i\theta})ire^{i\theta}ire^{i\theta}d\theta \\ &= \frac{1}{2\pi}\int_0^{2\pi} u(a + re^{i\theta})d\theta. \end{aligned}$$

Then by taking real parts,

$$u(a) = \frac{1}{2\pi}\int_0^{2\pi} u(a + re^{i\theta})d\theta$$

□

Definition 2.2.1. A continuous function $u : G \longrightarrow \mathbb{R}$ has the mean value property (MVP) if whenever $\bar{B}(a; r) \subset G$, then

$$u(a) = \frac{1}{2\pi}\int_0^{2\pi} u(a + re^{i\theta})d\theta$$

2.3 Maximum principle for Harmonic Function

Theorem 2.3.1. *(Maximum principle)(First version) Let G be a region and suppose that u is a continuous real value function on G with the MVP. If there is a point a in G such that $u(a) \geq u(z)$ for all z in G then u is a constant function.*

Proof. Assume that $u(a) = M$ for some $a \in G$ and $u(z) \leq M$ for all $z \in G$. Since G is open it contains an open disk $D(a, R)$. Since u is continuous on G with the MVP $r < R$, we have

$$M = U(a) = \frac{1}{2\pi} \int_0^{2\pi} u(a + re^{i\theta}) d\theta \leq \frac{1}{2\pi} \int_0^{2\pi} M d\theta = M.$$

If $u(a + re^{i\theta})$ is strictly smaller then M at some point $\theta_0 \in [0, 2\pi]$, then by continuity the same is true for all θ sufficiently close to θ_o. In this case

$$\frac{1}{2\pi} \int_0^{2\pi} u(a + re^{i\theta}) d\theta$$

is strictly smaller than

$$\frac{1}{2\pi} \int_0^{2\pi} M d\theta$$

which contradicts to the above formula. Thus $u(a + re^{i\theta}) = M$ for all $r < R$ and $\theta \in [0, 2\pi]$. Or in other words

$$u(z) = M \quad \textit{for all} \quad z \in D(a, R)$$

□

Theorem 2.3.2. *(Maximum Principle) (Second Version) Let G be a region and let u and v be two bounded continuous real valued functions on G that have a MVP. If for each point a in the extended boundary $\partial_\infty G$,*

$$\limsup_{z \to a} u(z) \leq \liminf_{z \to a} v(z).$$

Then either $u(z) \leq v(z)$ for all z in G or $u = v$

Proof. Fix a in $\partial_\infty G$ and for each $\delta > 0$, let $G_\sigma = G \bigcap B(a, \sigma)$ Then according to the hypothesis.

$$\begin{aligned} 0 &\geq \lim_{\sigma \to 0} [\sup\{u(z) : z \in G_\sigma\} - \inf\{V(z) : z \in G_\sigma\}] \\ &= \lim_{\sigma \to 0} \{\sup\{u(z) : z \in G_\sigma\} + \sup\{-v(z) : z \in G_\sigma\} \\ &\geq \lim_{\sigma \to 0} \sup\{u(z) - v(z) : z \in G_\sigma\} \end{aligned}$$

so $\lim_{z\to 0} sup[u(z) - v(z)] \leq 0$. for each a in $\partial_\infty G$. So it is sufficient theorem under the assuption that $v(z) = 0$ for all z in G. That is assume

$$\limsup_{z\to a} u(z) \leq 0$$

for all a in $\partial_\infty G$ and show that either $u(z) < 0$ for all z in G or $u \equiv 0$ By first version of the maximum principle, it suffices to show that $u(z) \leq 0$ for all $w(z)$in G. □

Corollary 2.3.1. *Let G be a bounded region and suppose that* $w : G \longrightarrow \mathbb{R}$ *is a continuous function that satisfies the* (MVP) *on G. If* $w(z) = 0, \forall z \in \partial G$, *then* $w(z) = 0, \forall z \in G$.

Proof. first take $w = u$ and $v = 0$ in the theorem of maximum principle of second version; so $w(z) < 0$ for all z or $w(z) \equiv 0$ Now take $w = v$ and $u = 0$ in theorem of maximum principle of second version; so either $w(z) > 0$ for all z or $w(z) \equiv 0$. Hence both of these hold $w(z) \equiv 0$ □

Theorem 2.3.3. *(Minimum principle) Let G be a region and suppose that u is continuous real valued function on G with the MVP. If there is a point* $a \in G$ *such that* $u(a) \leq u(z) \forall z \in G$, *then u is a constant function.*

2.4 Integral Representation Of Harmonic Functions On Disk

In this section we introduce the Poisson kernel function and we develop the Poisson integral formula which is integral representation of harmonic function on the unit disk. Then we extend the result for arbitrary disk. The Poisson integral formula we show, among other things, that every real harmonic functions is locally the real part of analytic function and it will yield information the boundary behaviour of certain classes of analytic functions in an open disk.

Definition 2.4.1. Poisson kernel is the function defined;

$$P_r(\theta) = \sum_{n=-\infty}^{\infty} r^{|n|} e^{in\theta}$$

$for\ 0 \leq r < 1, and - \infty < \theta < \infty.$

Proposition 2.4.1. *The Poisson kernel satisfies the following properties:*
(i) $P_r(\theta) = Re[\frac{1+z}{1-z}]$*, where* $z = re^{i\theta}$
(ii) $P_r(\theta) = \frac{1-r^2}{1+r^2-2r\cos\theta}$*, where* $z = re^{i\theta} \in D$ *and* D *is unit disk*
(iii) $\frac{1}{2\pi}\int_{-\pi}^{\pi} P_r(\theta)d\theta = 1$
(iv) $P_r(\theta) > 0$ *for all* θ
(v) $P_r(\theta) = P_r(-\theta)$ *and* p_r *is periodic in* θ *with period* 2π
(vi) $P_r(\theta) < P_r(\delta)$ *if* $0 < \delta < |\theta| \leq \pi$;
(vii) For each $\delta > 0$, $\lim_{r\to 1} P_r(\theta)$ *=0, uniformly in* θ *for* $\pi \geq |\theta| \geq \delta$.

Proof. (i) Let $z = re^{i\theta}, 0 \leq r < 1$ then,

$$\begin{aligned}\frac{1+re^{i\theta}}{1-re^{i\theta}} &= (1+z)(1+z+z^2+z^3+...)\\ &= 1+2\sum_{n=1}^{\infty} z^n\\ &= 1+2\sum_{n=1}^{\infty} r^n e^{in\theta}\end{aligned}$$

$$\begin{aligned}Re[\frac{1+re^{i\theta}}{1-re^{i\theta}}] &= Re(1+2\sum_{n=1}^{\infty} r^n e^{in\theta})\\ &= 1+\sum_{n=1}^{\infty} r^n (e^{in\theta+e^{-in\theta}})\\ &= p_r(\theta)\end{aligned}$$

□

Proof. (ii) Let

$$\begin{aligned}
\frac{1+re^{i\theta}}{1-re^{i\theta}} &= \frac{1+r\cos\theta+ir\sin\theta}{1-r\cos\theta-ir\sin\theta} \\
&= \frac{1+r\cos\theta+ir\sin\theta}{1-r\cos\theta-ir\sin\theta} \times \frac{1-r\cos\theta+ir\sin\theta}{1-r\cos\theta+ir\sin\theta} \\
&= \frac{1+2ir\sin\theta-r^2\sin^2\theta}{-} r^2\cos^2\theta 1-2r\cos\theta+r^2\cos^2\theta+r^2\sin^2\theta \\
&= \frac{1+2ir\sin\theta-r^2}{1-2r\cos\theta+r^2} \\
&= Re[\frac{1+re^{i\theta}}{1-re^{i\theta}}] \\
&= \frac{1-r^2}{1-2r\cos\theta+r^2} = P_r(\theta)
\end{aligned}$$

□

Proof. (iii)

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} P_r(\theta)d\theta = 1$$

$$\begin{aligned}
\frac{1}{2\pi}\int_{-\pi}^{\pi} P_r(\theta)d\theta &= \frac{1}{2\pi}\int_{-\pi}^{\pi} \Sigma_{n=-\infty}^{\infty} r^{|n|}e^{in\theta}d\theta \\
&= \frac{1}{2\pi}\int_{-\pi}^{\pi} d\theta + \frac{1}{2\pi}\int_{-\pi}^{\pi} \Sigma_{n=1}^{\infty} r^n e^{in\theta}d\theta + \frac{1}{2\theta}\int_{-\pi}^{\pi} \Sigma_{n=1}^{\infty} r^n e^{-in\theta}d\theta \\
&= \frac{1}{2\pi}[(\pi-(-\pi)] + \frac{1}{2\pi}\Sigma_{n=1}^{\infty} r^n \int_{-\pi}^{\pi}[e^{in\theta}+e^{-in\theta}] \\
&= \frac{1}{2\pi}(2\pi) + \frac{1}{2\pi}\Sigma_{n=1}^{\infty} r^n \int_{-\pi}^{\pi}[\cos n\theta + i\sin n\theta + \cos n\theta - i\sin n\theta]d\theta \\
&= 1 + \frac{1}{2\pi}2r^n\Sigma_{n=1}^{\infty}\int_{-\pi}^{\pi}[\cos n\theta]d\theta \\
&= 1 + 0 = 1
\end{aligned}$$

□

Proof. (iv)
From (ii) we have that,

$$P_r(\theta) = \frac{1-r^2}{1+r^2-2r\cos\theta}$$

Since $1 + r^2 - 2r\cos\theta \geqslant 1 + r^2 - 2r = (1-r)^2 > 0$, for $0 \leq r < 1$, So

$$\frac{1-r^2}{1+r^2-2r\cos\theta} > 0 \quad as \quad 0 \leq r < 1.$$

Therefore $P_r(\theta) > 0$ for all θ. □

Proof. (v)
$P_r(\theta)$ is given by

$$P_r(\theta) = \frac{1-r^2}{1+r^2-2r\cos\theta}.$$

Since $\cos\theta$ is even,

$$P_r(-\theta) = \frac{1-r^2}{1+r^2-2r\cos\theta} = P_r(\theta)$$

and since $\cos\theta$ is periodic in θ with $2\pi, P_r(\theta)$ is periodic in θ with 2π. □

Proof. (vi)
For $\delta > 0, \theta \in (0, \pi]$ such that $\delta < \theta$; we have that $\cos\delta > \cos\theta$. Hence

$$1 + r^2 - 2r\cos\delta < 1 + r^2 - 2r\cos\theta.$$

From these we get that;

$$\frac{1-r^2}{1+r^2-2r\cos delta} > \frac{1-r^2}{1+r^2-2r\cos\theta}.$$

Then $P_r(\delta) > P_r(\theta)$ □

The Mean value property for harmonic function shows that the value of the function at the center of the circle is the average of the values on the boundary of the circle. That is;

$$u(0) = \frac{1}{2\pi}\int_0^{2\pi} u(e^{i\theta})d\theta$$

This formula says that under the assumption that $u(z)$ is harmonic in D and continuous on $\bar{D}$; the value of the function at the center of the circle $D(0,1)$ is the average of its value on the circle. However the Poisson integral formula gives the average value for any arbitrary points inside the circle. Here we use Mőbius Transformation; Suppose a is an arbitrary point in D; from section of the role of conformal mapping we have that;

$$\phi_a(z) = W = TZ = \frac{z-a}{1-\bar{a}z} \tag{2.9}$$

, where T is a Mőbius transformation.
Then Mőbius transformation T maps the unit disk conformally on itself and the boundary on to the boundary, mapping point $'a$ to the origin (center). Therefore, $(uoT^{-1})(w) = u(T^{-1}(w) = u(z)$ is harmonic on $D(0,1)$ and continuous on $\bar{D}$. So we apply the mean value property, we obtain

$$(uoT^{-1})(0) = u(a) = \frac{1}{2\pi}\int_0^{2\pi}(uoT^{-1})(e^{i\psi})d\psi = \frac{1}{2\pi}\int_0^{2\pi}(uoT^{-1}e^{i\psi})d\psi \tag{2.10}$$

Since T is a bijection of the boundary circle to itself and $(uoT^{-1}e^{i\psi})$ is 2π periodic with respect to ψ
we may change the variable of integration by setting;

$$e^{i\theta} = T^{-1}e^{i\psi}$$

that is,

$$e^{i\psi} = Te^{i\theta} = \frac{e^{i\theta} - a}{1 - \bar{a}e^{i\theta}} \tag{2.11}$$

Then by differentiating both sides of (2.10),we get

$$\frac{d}{d\theta}(e^{i\psi} = \frac{e^{i\theta} - a}{1 - \bar{a}e^{i\theta}})$$

$$\Longrightarrow d\psi = \frac{(1 - |a|^2)}{1 - |ae^{-i\theta}|^2}d\theta$$

Taking $a = re^{it}$ for $0 \le r < 1$, then we obtain,

$$\Longrightarrow d\psi = \frac{1 - r^2}{1 + r^2 - 2r\cos(\theta - t)}d\theta \tag{2.12}$$

Hence from equation (2.10), (2.11) and (2.12), we get that

$$u(a) = \frac{1}{2\pi}\int_0^{2\pi} u(e^{i\theta})\frac{1 - r^2}{1 + r^2 - 2r\cos(\theta - t)}d\theta \tag{2.13}$$

If we make change of variable $\theta \to \theta - t$ and use the 2π-periodicity of $P_r(\theta)$, we obtain an alternative form of equation (2.16)

$$u(a) = \frac{1}{2\pi}\int_0^{2\pi} u(e^{i(\theta - t)}p_r(\theta)d\theta \tag{2.14}$$

The integral given in equation (2.14) is called Poisson integral formula and denoted by Pu

2.5 Harnack's Inequality

If $u : \bar{B}(a;R) \longrightarrow \mathbb{R}$ is continuous, harmonic in $B(a;R)$, and $u \geq 0$ then for $o \leq r < R$ and all θ; $\frac{R-r}{R+r}u(a) \leq u(a+re^{i\theta}) \leq \frac{R+r}{R-r}u(a)$.

Proof. From poisson integral formula

$$u(z) = \frac{1}{2\pi}\int_0^{2\pi} u(a+Re^{i\psi})\frac{R^2-r^2}{R^2+r^2-2Rr\cos(\theta-\psi)}d\psi.$$

where $z = a+re^{i\theta}$. Consider

$$|R-r| = ||Re^{i\theta}| - |re^{i\psi}|| \leq |Re^{i\theta} - re^{i\psi}| \leq |Re^{i\theta}| + |re^{i\psi}| = R+r.$$

this impels that $R-r \leq R+r$. Since $0 \leq r < 1, R-r > 0$
we have

$$(R-r)^2 \leq |Re^{i\theta} - re^{i\psi}|^2 \leq (R+r)^2.$$

Then

$$\frac{1}{(R+r)^2} \leq \frac{1}{|Re^{i\theta} - re^{i\psi}|} \leq \frac{1}{(R-r)^2}.$$

Now multiply by $R^2 - r^2$ we get that

$$\frac{R-r}{R+r} \leq \frac{R^2-r^2}{|Re^{i\theta} - re^{i\psi}|^2} \leq \frac{R+r}{R-r}.$$

Since $u(a+Re^{i\psi}) > 0$, then multiplying the above equation by these we get that

$$\frac{R-r}{R+r}\frac{u(a+Re^{i\psi})}{2\pi} \leq \frac{R^2-r^2}{|Re^{i\theta} - re^{i\psi}|^2}\frac{u(a+Re^{i\psi})}{2\pi} \leq \frac{R+r}{R-r}\frac{u(a+Re^{i\psi})}{2\pi}.$$

Then we integrate with respect to ψ from 0 to 2π we get that

$$\begin{aligned}\frac{1}{2\pi}\int_0^{2\pi}\frac{R-r}{R+r}u(a+Re^{i\psi})d\psi \leq & \frac{1}{2\pi}\int_0^{2\pi}\frac{R^2-r^2}{|Re^{i\theta} - re^{i\psi}|^2}u(a+Re^{i\psi})d\psi \\ \leq & \frac{1}{2\pi}\int_0^{2\pi}\frac{R+r}{R-r}u(a+Re^{i\psi})d\psi.\end{aligned}$$

By mean value property and poisson integral formula we have

$$\frac{R-r}{R+r}u(a) \leq u(z) \leq \frac{R+r}{R-r}u(a)$$

□

2.6 Sub Harmonic and Super Harmonic Functions

Definition 2.6.1. Let G be a region and let $\psi : G \to \mathbb{R}$ be continuous function. ψ is a sub harmonic function if whenever $\bar{B}(a;r) \subset G$, then

$$\psi(a) \leq \frac{1}{2\pi} \int_0^{2\pi} \psi(a + re^{i\theta}) d\theta$$

and ψ is super harmonic function if whenever $\bar{B}(a;r) \subset G$, then

$$\psi(a) \geq \frac{1}{2\pi} \int_0^{2\pi} \psi(a + re^{i\theta}) d\theta$$

Theorem 2.6.1. *(Maximum principle) (Third version)Let G be a region and let $\psi : G \longrightarrow \mathbb{R}$ be a sub harmonic function. If there is a point a in G with $\psi(a) \geq \psi(z)$ for all z in G, then ψ is a constant function.*

Proof. Assume that $\psi(a) = M$ for some point $a \in G$ and $\psi(z) \leq M$ for all point $z \in G$ since G is open it contains an open disk $D(a; R)$.
Then by MVP we have;

$$M = \psi(a) \geq \frac{1}{2\pi} \int_0^{2\pi} \psi(a + re^{i\theta}) d\theta \leq \frac{1}{2\pi} \int_0^{2\pi} \psi M d\theta = M$$

If $\psi(a + re^{i\theta})$ is strictly smaller than M for some point θ_o in $[0, 2\pi]$ by continuity of ψ , it is true for θ which is strictly smaller than

$$\frac{1}{2\pi} \int_0^{2\pi} \psi M d\theta$$

contradicts our above formula
So that $\psi(a + re^{i\theta}) = M$ for all $\theta \in [0, 2\pi]$ and $r < R$,or in other word $\psi(z) = M$ for all $z \in D(a; R)$ □

Theorem 2.6.2. *(Maximum principle) (fourth version) Let G be a region and let ψ and φ be bounded real valued functions defined on G such that φ is sub harmonic and ψ super harmonic .*
If for each point a in $\partial_\infty G$

$$\lim_{z \to a} \sup \varphi(z) \leq \lim_{z \to a} \inf \psi(z)$$

Then either $\varphi(z) < \psi(z)$ for all z in G or $\varphi = \psi$ and φ is harmonic.

Chapter 3

Dirichlet Problem

3.1 Introduction

In this chapter we construct harmonic functions on D that behave in a prescribed manner near ∂D. The Poisson integral formula shows that if $u(z)$ is harmonic in a disk and continuous on the closed disk, then its value at any interior point is completely determined by its value on the boundary circle. on the other hand, the Poisson integral is meaning full for every **(bounded piecewise)** continuous function $u(e^{i\theta})$ on the circle (or even for Lebesgue integration functions. Here we consider the Dirichlet problem for simple but important case where the domain is a disk, $D = \{z \in \mathbb{C} : |z - z_o| \leq \rho\}$. These facts suggest the following two questions:
(a) Given a real valued bounded piecewise continuous function $u(e^{i\theta})$ on the unit circle, do we obtain a harmonic function $v(z)$ through the Poisson integral;

$$v(z) = \frac{1}{2\pi}\int_0^{2\pi} u(e^{i\theta})P_r(\theta - t)d\theta, z = re^{it}?$$

(b) If so, do the boundary values of $v(z)$ agree with $u(e^{i\theta})$?
The answer is affirmative and the unique solution is given by;

$$v(z) = \begin{cases} pu, if z \in D(0;1) \\ u(z), if z \in \partial D(0;1) \end{cases}$$

The above two questions leads to the problem of finding a function that is harmonic in a region and has pre assigned values on the boundary which is known as the Dirichlet problem.

3.2 Dirichlet problem on Disk and Solutions

Theorem 3.2.1. *Let $D = z : |z| < 1$ and suppose that $f : \partial D \longrightarrow \mathbb{R}$ is a continuous function. Then there is a continuous function $u : \bar{D} \longrightarrow \mathbb{R}$ such that;*
(i) $u(z) = f(z)$ for z in ∂D.
(ii) u is harmonic in D, moreover u is unique and is defined by the formula;

$$u(re^{i\theta}) = \frac{1}{2\pi}\int_{-\pi}^{\pi} P_r(\theta - t)f(e^{it})dt;$$

for $0 \leq r < 1$, $0 \leq \theta \leq 2\pi$

Proof. From proposition 2.15 (iii) we have

$$u(re^{i\theta}) = \frac{1}{2\pi}\int_{-\pi}^{\pi} P_r(t)dt = 1$$

and we have that

$$u(e^{i\theta}) = f(e^{i\theta}) = \frac{1}{2\pi}\int_{-\pi^{\pi}} f(e^{i\pi})P_r(t)dt.$$

for $0 \leq r < 1$ and from poisson integral formula we have

$$u(re^{i\theta}) = f(re^{i\theta}) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(e^{i(\theta-t)})P_r(t)dt.$$

for $0 \leq r < 1$ and $0 \leq \theta \leq 2\pi$.
Again from $P_r(t) > 0$, for all t and $0 \leq r < 1$.

$$|u(re^{i\theta}) - u(e^{i\theta})| = |\frac{1}{2\pi}\int_{-\pi}^{\pi} f(e^{i(\theta-t)}) - f(e^{i\theta})P_r(t)dt| \leq \frac{1}{2\pi}\int_{-\pi^{\pi}} |f(e^{i(\theta-t)}) - f(e^{i\theta})|P_r(t)dt.$$

From hypothesis we have that f is uniformly continuous on ∂D,we may select for any $\in > 0, \exists\delta > 0$ such that

$$|f(e^{i(\theta-t)}) - f(e^{i\theta})| \leq \frac{\epsilon}{2}$$

provide that $|t| < \delta$. Now we break the integral into two pieces and make the obvious estimate on each piece.

$$I_1; = \frac{1}{2\pi}\int_{|t|<\delta} |f(e^{i(\theta-t)}) - f(e^{i\theta})|P_r(t)dt.$$

$$I_2 = \frac{1}{2\pi}\int_{\delta\leq|t|} |f(e^{i(\theta-t)}) - f(e^{i\theta})|P_r(t)dt.$$

Then

$$I_1 \leq \frac{\epsilon}{2}\frac{1}{2\pi}\int_{-\pi}^{\pi} P_r(t)dt.$$

therefore $I_1 \leq \frac{\epsilon}{2}$. and

$$I_2 = \frac{1}{2\pi}\int_{\delta\leq|t|} |f(e^{i(\theta-t)})f(e^{i\theta})|P_r(t)dt \leq \frac{1}{2\pi}\int_{\delta\leq|t|} |f(e^{i(\theta-t)})| + |f(e^{i\theta})|P_r(t)dt.$$

From property of poisson kernel, proposition 2.15(vII) for each

$$\delta > 0, \lim_{r\to 1} P_r(\theta) = 0$$

as uniformly in θ for $\delta \leq |\theta| \leq \pi$ and $\max \delta \leq |\theta| \leq \pi$, $P_r(\theta) \to 0$ as $r \to 1$ and letting $M = \max f(e^{it}) : \delta \leq |t| \leq \pi$ we have that.

$$0 < P_r(t) < \frac{\epsilon}{4M}.$$

now

$$\begin{aligned}
I_2 \leq& \frac{1}{2\pi}\int_{\delta\leq|t|} |f(e^{i(\theta-t)})|P_r(t)dt + \frac{1}{2\pi}\int_{\delta\leq|t|} |f(e^{it})|P_r(t)dt \\
\leq& \frac{1}{2\pi} \times 2M \times \max_{\delta\leq|t|\leq\pi} P_r(t) \int dt_{\delta\leq|t|\leq\pi} < \frac{1}{2\pi} 2M \max P_r(t) \int_{-\pi}^{\pi} dt \\
=& 2M \times \max_{\delta\leq|t|\leq\pi} P_r(t) = 2M \times \frac{\epsilon}{4M} = \frac{\epsilon}{2}
\end{aligned}$$

Therefore $I_2 \leq \frac{\epsilon}{2}$ for r sufficiently close to 1. Thus we have shown that

$$|u(re^{i\theta}) - u(e^{i\theta})| \leq I_1 + I_2 \leq \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$$

For r sufficiently close to 1. Therefore u is continuous on $\bar{D}$ □

Corollary 3.2.1. *If $u : \bar{D} \longrightarrow \mathbb{R}$ is a continuous function that is harmonic in D, then*

$$u(re^{i\theta}) = \frac{1}{2\pi}\int_{-\pi}^{\pi} P_r(\theta - t)f(e^{it})dt;$$

for $0 \leq r < 1$ *and all* θ. *Moreover,u is the real part of the analytic function.*

$$f(z) = \frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{e^{it}+z}{e^{it}-z} u(e^{it})dt$$

Proof. The first part of the corollary is a direct consequence of the theorem. The second part of the corollary will be;

$$f(re^{i\theta}) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(re^{i\theta})P_r(\theta - t)dt, wherez = re^{i\theta}$$

We have that

$$P_r(\theta) = Re(\frac{1+z}{1-z}), wherez = re^{i\theta}$$

$$P_r(\theta - t) = Re(\frac{e^{it}+re^{i\theta}}{e^{it}-re^{i\theta}})$$

We have

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} P_r(\theta - t)dt = 1, then$$

$$\begin{aligned} f(z) &= \frac{1}{2\pi}\int_{-\pi}^{\pi} f(z)P_r(\theta - t)dt \\ &= \frac{1}{2\pi}\int_{-\pi}^{\pi} f(z)Re(\frac{e^{it}+re^{i\theta}}{e^{it}-re^{i\theta}})dt \\ &= \frac{1}{2\pi}\int_{-\pi}^{\pi} u(e^{it})(\frac{e^{it}+re^{i\theta}}{e^{it}-re^{i\theta}})dt \end{aligned}$$

□

Corollary 3.2.2. *Let* $a \in \mathbb{C}, \rho > 0$ *and suppose* h *is a continuous real valued function on* $\{z : |z - a| = \rho\}$*; then there is a unique continuous function* $W : \bar{B}(a;\rho) \longrightarrow \mathbb{R}$ *such that* W *is harmonic on* $B(a;\rho)$ *and* $W(z) = h(z)$ *for* $|z - a| = \rho$

Proof. Consider $f(e^{i\theta}) = h(a + \rho e^{i\theta})$; then f is continuous on ∂D. Then by above theorem there exists a continuous function $u : \bar{D} \longrightarrow R$ such that u is harmonic on D and;

$$u(e^{i\theta}) = f\partial(e^{i\theta}) = h(a + \rho e^{i\theta})$$

Then we set

$$u(z) = u(\frac{z-a}{\rho}) = W(z) = h(z)$$

Therefore $u(\frac{z-a}{\rho})$ is the desired function on $\bar{B}(a;\rho)$ □

Theorem 3.2.2. *If $u : G \longrightarrow \mathbb{R}$ is a continuous function which has a MVP then u is harmonic.*

Proof. Let $a \in G$ and chose ρ such that $\bar{B}(a;\rho) \subset G$; it is sufficient to show that u is harmonic on $B(a;\rho)$. But according to the above corollary there is a continuous function

$$w : \bar{B}(a;\rho) \longrightarrow \mathbb{R}$$

which is harmonic in

$$B(a;\rho) \quad \textit{and;}\ W(a+\rho e^{i\theta}) = u(a+\rho e^{i\theta}) \quad \textit{for all} \quad \theta$$

Since $u - w$ satisfies the MVP and

$$(u-w)(z) = 0 \quad \textit{for} \quad |z-a| = \rho,$$

it follows from corollary above that $u \equiv w \in B(a;\rho)$ in particular, u must be harmonic.

As was mentioned at the beginning of this section, one of the purposes in studying sub harmonic function is that they enter in to the solution of the Dirichlet problem. Indeed, the fourth version of the maximum principle gives an insight into how this occurs. If G is a region and $u : \bar{G} \longrightarrow \mathbb{R}$ is a continuous function ($\bar{G}$= the closure in $\mathbb{C}_\infty$) which is harmonic in G. Then

$$\psi(z) \leq u(z \quad \textit{for all} \quad z \in G$$

and for all sub harmonic functions ψ which satisfy

$$\lim_{z\to a} \sup \psi(z) \leq u(a) \quad \textit{for all} \quad a \in \partial_\infty G.$$

Since u is itself such a sub harmonic function we arrive at the result that $u(z) = \sup \psi(z) : \psi$ is a sub harmonic and

$$\lim_{z\to a} \sup \psi(z) \leq u(a) \quad \textit{for all} \quad a \in \partial_\infty G. \tag{3.1}$$

Equation (3.1) says that if $f : \partial_\infty G \longrightarrow \mathbb{R}$ is a continuous function and if f can be extended to a function u that is harmonic on G, then u can be obtained from a set of sub harmonic function which are defined solely in terms of the boundary values f. This leads to the following definition □

Definition 3.2.1. If G is a region and $f : \partial_\infty G \longrightarrow \mathbb{R}$ is a continuous function then the Perron Family, $P(f, G)$, consists of all sub harmonic function $\psi : G \longrightarrow \mathbb{R}$ such that $\limsup_{z\to a} \psi(z) \leq f(a)$ for all a in $\partial_\infty G$.

Since f is continuous, there is a constant M such that $|f(a)| \leq M$,for all a in $\partial_\infty G$. So the constant function M is in $P(f, G)$ and the Perron family is never empty If $u : \bar{G} \longrightarrow \mathbb{R}$ is a continuous function which is harmonic in G and $f = U \setminus \partial_\infty G$ then (1) becomes;

$$u(z) = sup\psi(z) : \psi \in P(f, G), for\ eachz \in G \tag{3.2}$$

Conversely, if f is given and u is defined by (3.1) then u must be the solution of the Dirichlet problem with the boundary values f; that is provided the Dirichlet problem can be solved. In order to show that (3.2) is a solution of the Dirichlet problem, the following two questions must be answered affirmatively.
(a) Is u harmonic in G ?
(b) Does $\lim_{z\to a} u(z) = f(a)$? for each $a \in \partial_\infty G$
The first question can always be answered yes and this is shown in the next theorem. The second question sometimes has a negative answer. However, it is possible to impose geometrical restrictions on G which guarantee that the answer to (b) is always yes for any continuous function f.

Definition 3.2.2. Let G be a region and let $f : \partial_\infty G \longrightarrow \mathbb{R}$ be a continuous function. The harmonic function u obtained in the preceding theorem is called the perron function associated with f.

Definition 3.2.3. A region G is called a Dirichlet Region if the Dirichlet problem can be solved for G. That is G is a Dirichlet Region if for each continuous function $f : \partial_\infty G \longrightarrow \mathbb{R}$ there is a continuous function $u : \bar{G} \longrightarrow \mathbb{R}$ such that u is harmonic in G and $u(z) = f(z)$ for all z in $\partial_\infty G$.

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