THE AGING SPRING PROBLEM AND BESSEL'S DIFFERENTIAL EQUATION

ADDIS ABABA UNIVERSITY

COLLEGE OF NATURAL SCIENCES

DEPARTMENT OF MATHEMATICS.

SUBMITTED IN PARTIAL FULFILMENT OF THE REQUIRMENT FOR THE DEGREE OF MASTER OF SCIENCE IN MATHEMATICS.

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August, 2014

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The undersigned hereby certify that they have read and recommend to the department of Mathematics for acceptance of this project entitled **"BESSEL'S DIFFERENTIAL EQUATION AND THE AGING SPRING PROBLEM"** by **Tilahun Belayneh** in partial fulfillment of the requirements for the degree of Master of Science in Mathematics.

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Acknowledgements

First and foremost I would like to give my thanks to the Holly Savior who gave me the chance to alive and have all such learning experiences. Secondly, my heart felt gratitude goes to my advisor Dr. Tadesse Abdi for his constructive comments and friendly approaches all through the work. Next I would like to thank my classmates and colleague for their constant support in sharing ideas and materials. Finally my thanks go to my family for their financial and moral support.

Abstract

In this paper we study the aging spring equation without damping

$$\ddot{x}(t) + e^{-at+b} x(t) = 0 \, ,$$

or the most general one with damping

$$\ddot{x}(t) + d\dot{x}(t) + e^{-at} x(t) = 0$$

which is a special cases of homogeneous linear second order ODE with variable coefficients. We will show that, with a suitable change of variables, this equation of aging spring problem can be transformed into Bessel differential equation and hence the solution can be expressed in terms of Bessel functions.

Notations

$J_p(x)$: is the solution to Bessel's equation is referred to as a Bessel function of the first kind.

$Y_p(x)$: is the solution to Bessel's equation is referred to as a Bessel function of the second kind or sometimes the Weber function or the Neumann function.

$I_p(x)$: is the modified Bessel's equation is referred to as a modified Bessel function of the first kind.

$K_p(x)$: is the modified Bessel's equation is referred to as a modified Bessel function of the second kind or sometimes the Weber function or the Neumann function.

$\gamma \approx 0.5772$: Euler's constant

Introduction

The vibration of a spring which is governed by hooks law assumes that the stiffness of the spring is constant over time. As a result the governing equation is a second order linear ODE with constant coefficients. However, in reality this is true only for values of *t* in some range. Thus, as the value of *t* grows large, the restoring force of the spring is weakening with time and the governing equation of the aging spring problem without damping is given by

$$\ddot{x}(t) + e^{-at+b} x(t) = 0 . \qquad \ldots\ldots\ldots\ldots \qquad (*)$$

Here, in this equation the stiffness of the spring, i.e. the restoring force is a decaying exponential of time thereby showing weakening of the restoring force with time. The more realistic situation is reflected in the one with damping, that is expressed as

$$\ddot{x}(t) + d\dot{x}(t) + e^{-at} x(t) = 0 \qquad \ldots\ldots\ldots\ldots \qquad (2*)$$

These aging spring equations (*) and (**) are nothing but special cases of homogeneous linear second order ordinary differential equations with variable coefficients

$$\ddot{x}(t) + p(t)\dot{x}(t) + q(t)x(t) = 0 . \qquad \ldots\ldots\ldots\ldots \qquad (3*)$$

At this point we recall that, if t^* is an ordinary point of the above ODE (3*), i.e. if *p*(t) and *q*(t) are analytic at t^* then the ODE admits a power series solution in a neighborhood of t^*.

Bessel functions are defined as particular solutions of a linear differential equation of the second order

$$t^2\ddot{x}(t) + t\,\dot{x}(t) + (t^2 - n^2)x(t) = 0$$

known as Bessel's equation. And it is often used as model of real physical problems. Bessel functions are named for Friedrich Wilhelm Bessel (1784 – 1846), however, Daniel Bernoulli is generally credited with being the first to introduce the concept of Bessel functions in 1732. He used the function of zero order as a solution to the problem of an oscillating chain suspended at one end. In 1764, Leonhard Euler employed Bessel functions of both zero and integral orders in an analysis of vibrating of a stretched membrane. Bessel, while receiving named credit for these functions, did not incorporate them into his work as an astronomer until 1817. The Bessel function was the result of Bessel study of determining the motion of the three bodies moving under mutual gravitation. In 1824, he incorporated Bessel functions in a study of planetary perturbations where the Bessel functions appear as coefficients in a series expansion of the indirect perturbation of a planet, which is the motion of the sun caused by the perturbing body. Some classes of differential equations are important because they arise in many different applications. It may be necessary to have a way to compute these solutions quickly, even though these solutions are required in the neighborhood of a singular point. Such a solution, if it can only be defined as a series, rather than in terms of elementary functions, is called *a special function.* Some of the special functions we may run into are Bessel functions, Legendre functions, etc.

This paper presents solution of the aging spring equation in terms of the Bessel functions of the first and second kind arising from the solution of transformed second order differential equation and. The content of this paper is organized as follows. *In the first chapter,* we will see basic concepts and definitions. *In the second chapter*, we introduce the (modified) Bessel differential equation and deduce from it the (modified) Bessel functions of first and second kind. This will be done via a power series approach. *In the third chapter,* we will prove some properties of Bessel functions. There are of course more interesting facts; in particular, there are connections between Bessel functions and the usual trigonometry functions and much more. Finally, *in the fourth chapter*, we will see how other differential equation, the aging spring equation can be converted in to the general solution of Bessel equation of order 0.

Chapter 1

Basic Concepts and Definitions

1.1 Initial Definitions and Basic Theory

- ***Second Order ODE***

Definition 1.1.1 [1] *The general linear homogeneous second order ODE is given by*

$$y'' + p(x)y' + q(x)y = 0 \tag{1.1.1}$$

has coefficients p and q that are not both constants.

However, sometimes we can write a solution y(x) as a power series:

$$y(x) = \sum_{n=0}^{\infty} a_n(x - x_0)^n$$

where we use ODE (1.1.1) *to where we determine the coefficients* a_n.

- ***Infinite Series***

Definition 1.1.2 [5] *An infinite series of the form*

$$\sum_{n=0}^{\infty} a_n x^n = a_0 + a_1 x + a_2 x^2 + \dots \tag{1.1.2}$$

is called a power series in x.

Definition 1.1.3 [6] *The series of the form*

$$\sum_{n=0}^{\infty} a_n(x - x_0)^n = a_0 + a_1(x - x_0) + a_2(x - x_0)^2 + \dots \tag{1.1.3}$$

is a power series in $x - x_0$.

The seri es (1.1.2) *is s aid t o be convergent at a point x if the l imit,* $\lim_{m\to\infty} \sum_{n=0}^{m} a_n x^n$ *exists, and in this case the sum of the series is the value of this limit. A power series may or may not converge at either endpoint of its interval of convergence.*

Definition 1.1.4 [6] *If a function y(x) has a convergent Taylor series*

$$y(x) = \sum_{n=0}^{\infty} a_n(x - x_0)^n$$

in some interval about $x = x_0$, *then* $y(x)$ *is said to be analytic at* x_0. *In this case the* a_n*'s are necessarily given by* $a_n = \frac{y^{[n]}(x_0)}{n!}$.

Since all derivatives of analytic functions exist, the derivatives y' and y'' *of y can be obtained by differentiating that series term by term, producing series with the same radius of convergence as the series for y. If we substitute these series in to ODE* (1.1.1), *we can determine the coefficients* a_n. *To begin with,* a_0 and a_1 *are equal to the initial values* $y(x_0)$ *and* $y'(x)$ *respectively.*

❖ ***Recurrence Formulas***

Definition 1.1.5 [5] *A recurrence formula for the coefficients* a_n *is a formula that defines each* a_n *in terms of the coefficients* $a_0, a_1, \ldots, a_{n-1}$.

To find such a formula, we have to express each of the terms in ODE (1.1.1) *as power series about* $x = x_0$. *This is the point at which the initial conditions are given. Then we combine these series to obtain a single power series which according to ODE* (1.1.1), *must be zero for all x near* x_0. *This implies that the coefficient of each power of* $x - x_0$ *must be equal to zero, which yields an equation for each* a_n *in terms of the preceding coefficients* $a_0, a_1, \ldots, a_{n-1}$.

❖ ***Ordinary Points***

Definition 1.1.6 [1] *If* $p(x)$ *and* $q(x)$ *are both analytic at* x_0, *then* x_0 *is called an ordinary point for the differential equation* $y'' + p(x)y' + q(x)y = 0$.

Remark 1.1.1 *Any point that is not an ordinary point of ODE* (1.1.1) *is called a singular point.*

Theorem 1.1.1 [1] ***Ordinary Points Theorem***

If x_0 *is an ordinary point of ODE* (1.1.1) *that is, if* $p(x)$ *and* $q(x)$ *are both analytic at* x_0, *then the general solution of ODE* (1.1.1) *is given by the series*

$$y(x) = \sum_{n=0}^{\infty} a_n(x - x_0)^n = a_0 y_1(x) + a_1 y_2(x) \qquad (1.1.4)$$

Where a_0 *and* a_1 *are arbitrary and for each* $n \geq 2$, a_n *can be written in terms of* a_0 *and* a_1. *When this is done, we get the right- hand term in formula* (1.1.4), *where* $y_1(x)$ and $y_2(x)$ *are linearly independent solutions of ODE* (1.1.1) *that are analytic at* x_0.

Proof : see [1].

- ***Regular Singular Points***

Definition 1.1.7 [1] *A singular point of the ODE* (1.1.1) *is a regular singular point if both* $(x-x_0)\,p(x)$ *and* $(x-x_0)^2 q(x)$ *are analytic at* x_0.

In this case we'll have to modify the method to find a series solution to the ODE (1.1.1). *Since* $(x-x_0)\ p(x)$ *and* $(x-x_0)^2 q(x)$ *are analytic at* x_0, *they have a power series expansions centered at* x_0*:*

$$(x-x_0)p(x) = p_0 + p_1(x-x_0) + p_2(x-x_0)^2 + \dots$$

$$(x-x_0)^2 q(x) = q_0 + q_1(x-x_0) + q_2(x-x_0)^2 + \dots$$

As we shall soon see, the constant coefficients p_0 *and* q_0, *in these two series are particularly important. The roots of the quadratic equation called the indicial equation*

$$r(r-1) + p_0 r + q_0 = 0 \tag{1.1.5}$$

are used in solution formula (1.1.6) below.

Theorem 1.1.2 [1] ***Frobenius' Theorem***

If x_0 *is a regular singular point of ODE* (1.1.1), *then there is at least one series solution at* x_0 *of the form*

$$y_1(x) = (x-x_0)^{r_1} \sum_{n=0}^{\infty} a_n(x-x_0)^n = \sum_{n=0}^{\infty} a_n(x-x_0)^{n+r_1} \tag{1.1.6}$$

where r_1 *is the larger of the two roots* r_1 *and* r_2 *of the indicial equation.*

Proof : see [2].

Here are a few things to keep in mind when finding a Frobenius series.

1. *The roots of the indicial equation may not be integers, in which case the series representation of the solution would not be a power series, but is still a valid series.*
2. *If* $r_1 - r_2$ *in not an integer, then the smaller root* r_2 *of the indicial equation generates a second solution of the form*

   $$y_2(x) = (x-x_0)^{r_2} \sum_{n=0}^{\infty} b_n(x-x_0)^n$$

   which is linearly independent of the first solution $y_1(x)$.
3. *When* $r_1 - r_2$ *is an integer, a second solution of the form*

   $$y_2(x) = C y_1(x) ln(x-x_0) \sum_{n=0}^{\infty} b_n(x-x_0)^{n+r_2}$$

   exists, where the values of the coefficients b_n *are determined by finding a recurrence formula, and C is a constant. The solution* $y_2(x)$ *is linearly independent of* $y_1(x)$.

1.2 Notions of Bessel Differential Equation

❖ ***Bessel Differential Equation***

Definition 1.2.1 [1] *For each non- negative constant p, the associated Bessel equation of order p is*

$$x^2 y'' + x y' + (x^2 - p^2) y = 0 \tag{1.2.1}$$

This can also be written in the form

$$y'' + p(x) y' + q(x)y = 0$$

With $P(x) = \frac{1}{x}$ *and* $q(x) = 1 - \frac{p^2}{x^2}$.

Solutions of equation (1.2.1) *are Bessel functions. These functions appear frequently in applications involving cylindrical geometry and have been extensively studied. Bessel functions are the most widely used functions in science and engineering.*

Definition 1.2.1 [16] *The modified Bessel equation of order p is given by*

$$x^2y'' + xy' - (x^2 + p^2) y = 0 \tag{1.2.2}$$

Since Bessel' differential equation is a second-order differential equation, there must be two linearly independent solutions.

❖ ***Gamma Function***

We recall that

$$k! = k \times (k-1). . . 3 \times 2 \times 1.$$

Euler was able to give a correct definition to k! when k is not a positive integer. He invented the Euler-gamma function in the year 1729. That is,

Definition 1.2.2 [2] *The gamma function,* $\Gamma(x)$, *is defined for* $x > 0$ *as follows:*

$$\Gamma(x) = \int_0^\infty t^{x-1} e^{-t} dt \tag{1.2.3}$$

This integral is improper and converges for all $x > 0$. *The basic property of the gamma function is*

$$\Gamma(x+1) = x\,\Gamma(x)$$

To prove this we use integration by parts as follows

$$\Gamma(x+1) = \lim_{b\to\infty} \int_0^b t^{x-1}e^{-t}dt$$

$$= \lim_{b\to\infty} \left(-t^x e^{-t} \Big|_0^b + x\int_0^b t^{x-1}e^{-t}dt\right)$$

$$= x\left(\lim_{b\to\infty} \int_0^b t^{x-1}e^{-t}dt\right)$$

$$= x\,\Gamma(x) \quad , \quad \text{since } \frac{b^x}{e^b} \to 0 \text{ as } b \to \infty$$

$$\Rightarrow \quad \Gamma(x+1) = x\,\Gamma(x)$$

We can easily find the values of the gamma function at the positive integers

For example, $\Gamma(1) = \int_0^\infty e^{-t}dt = 1$

The basic property now gives $\Gamma(2) = 1\,\Gamma(1) = 1!$

$\Gamma(3) = 2\,\Gamma(2) = 2.1 = 2!$

$\Gamma(4) = 3\,\Gamma(3) = 3.2.1 = 3!$

and in general $\Gamma(n+1) = n!$

For any integer $n \geq 0$, *where we have set* $0! = 1$.

For this reason the gamma function is sometimes called the generalized factorial function.

Chapter 2

The Bessel functions

2.1 Bessel functions of first and second kind

❖ ***Solving Bessel Equation***

For any non-negative constant p, the differential equation

$$x^2 y'' + x y' + (x^2 - p^2)\, y = 0$$

is known as Bessel's equation of order *p,* and its solutions are the Bessel functions of order *p.*

Second order linear differential equations with the form

$$y''(x) + p(x)\, y'(x) + q(x)\, y(x) = 0 \qquad (2.1.1)$$

With neither $p(x)$ nor $q(x)$ are analytic at $x = x_0$, but with both $(x\text{-}x_0)\, p(x)$ and $(x\text{-}x_0)^2 q(x)$ are analytic, are said to be equations with regular singular points. Writing equation (1.2.1) as

$$y''(x) + \frac{1}{x} y'(x) + \left(1 - \frac{p^2}{x^2}\right) y(x) = 0 \qquad (2.1.2)$$

We see that Bessel's equation is such a regular singular point equation, with the singular point $x_0 = 0$. Solutions to such equations can be found using the technique of Frobenius series. A Frobenius series associated with the singular point $x_0 = 0$ has the form

$$y(x) = x^r (a_0 + a_1 x + a_2 x^2 + \text{---})$$

$$y(x) = \sum_{n=0}^{\infty} a_n x^{n+r}$$

where r is to be determined, and $a_0 \neq 0$.

So, as suggested by the method of Frobenius, we try for a solution

$$y(x) = \sum_{n=0}^{\infty} a_n x^{n+r} \qquad (2.1.3)$$

$$y'(x) = \sum_{n=0}^{\infty} (n+r) a_n x^{n+r-1} \qquad (2.1.4)$$

$$y''(x) = \sum_{n=0}^{\infty} (n+r)(n+r-1) a_n x^{n+r-2} \qquad (2.1.5)$$

Substituting equations (2.1.3), (2.1.4) and (2.1.5) in to equation (1.2.1) yields,

$$x^2\sum_{n=0}^{\infty} (n+r)(n+r-1)a_n x^{n+r-2} + x\sum_{n=0}^{\infty} (n+r)a_n x^{n+r-1} + (x^2-p^2)\sum_{n=0}^{\infty} a_n x^{n+r} = 0$$

$$\sum_{n=0}^{\infty} (n+r)(n+r-1)a_n x^{n+r} + \sum_{n=0}^{\infty} (n+r)a_n x^{n+r} + \sum_{n=0}^{\infty} a_n x^{n+r+2} - \sum_{n=0}^{\infty} p^2 a_n x^{n+r} = 0$$

Making the change of index in the third term, gives

$$\sum_{n=0}^{\infty} (n+r)(n+r-1)a_n x^{n+r} + \sum_{n=0}^{\infty} (n+r)a_n x^{n+r} + \sum_{n=2}^{\infty} a_{n-2} x^{n+r} - \sum_{n=0}^{\infty} p^2 a_n x^{n+r} = 0$$

Writing the terms corresponding to $n = 0$ and $n = 1$ separately gives

$$a_0 (r^2 - p^2)\, x^r + a_1 [(r+1)^2 - p^2]\, x^{r+1} + \sum_{n=2}^{\infty} (a_n [(r+n)^2 - p^2] + a_{n-2})\, x^{r+n} = 0$$

Equating coefficients of the series to zero gives

$$a_0 (r^2 - p^2) = 0 \qquad (n = 0) \qquad (2.1.6)$$

$$a_1 [(r+1)^2 - p^2] = 0 \qquad (n = 1) \qquad (2.1.7)$$

$$a_n [(r+n)^2 - p^2] + a_{n-2} = 0 \qquad (n \geq 2) \qquad (2.1.8)$$

From equation (2.1.6), since $a_0 \neq 0$, we get the indicial equation

$(r + p)(r - p) = 0$, with indicial roots $r_1 = p$ and $r_2 = -p$. we assume here that the indicial roots are real and that $r_2 \leq r_1$. Thus, the Bessel differential equation has a regular point at $x = 0$ for every p the equation has at least one solution of $y(x)$ of the form

$$y(x) = \sum_{n=0}^{\infty} a_n x^{n+r}$$

where $a_0 \neq 0$.

❖ ***First Solution Of Bessel's Equation***

Setting $r = p$, equation (2.1.8) gives the recurrence relation

$$a_n = \frac{-1}{n(n+2p)}\, a_{n-2} \qquad n \geq 2$$

This is a two-step recurrence relation, so the even and odd-indexed terms are determined separately.

We deal with the odd-indexed terms first with $r = p$ equation (2.1.7) becomes

$$a_1 [(p + 1)^2 - p^2] = 0$$

Which implies that $a_1 = 0$, recall that $p \geq 0$ in the equation (1.2.1) and so $a_3 = a_5 = - - - = 0$.

To make it easier to find a pattern for the even-indexed terms we rewrite the recurrence relation with $n = 2k$ and get

$$a_{2k} = \frac{-1}{2k(2k+2p)} a_{2k-2} = \frac{-1}{2^2 k(k+p)} a_{2(k-1)} \qquad k \geq 1$$

This gives

$$a_2 = \frac{-1}{2^2(1+p)} a_0;$$

$$a_4 = \frac{-1}{2^2 2(2+p)} a_2; = \frac{1}{2^4 2!(1+p)(2+p)} a_0;$$

$$a_6 = \frac{-1}{2^2 3(3+p)} a_4; = \frac{-1}{2^6 3!(1+p)(2+p)(3+p)} a_0;$$

$$\cdots$$

$$a_{2k} = \frac{(-1)^k}{2^{2k} k!(1+p)(2+p)(3+p)\ \ldots\ (k+p)} a_0$$

Substituting these coefficients in to equation (2.1.3) gives one solution to Bessel's equation

$$y = a_0 \sum_{k=0}^{\infty} \frac{(-1)^k}{2^{2k} k!(1+p)(2+p)(3+p)\ \ldots\ (k+p)} x^{2k+p} \qquad (2.1.9)$$

Where $a_0 \neq 0$ is arbitrary and it is convenient to choose $a_0 = \frac{1}{2^p p!}$, so together we get the p^{th} Bessel functions.

$$y(x) = \sum_{k=0}^{\infty} \frac{(-1)^k}{k!(k+p)!} \left(\frac{x}{2}\right)^{2k+p} \qquad (2.1.10)$$

We have a problem! So far, we have allowed p to be any real number and then we need to apply the basic property of the gamma function and we also want to construct a solution for complex order p, thus equation (2.1.10) yields the first solution denoted by J_p and called *Bessel functions of order p.*

$$J_p(x) = \sum_{k=0}^{\infty} \frac{(-1)^k}{\Gamma(k+1)\ \Gamma(p+k+1)} \left(\frac{x}{2}\right)^{2k+p} \qquad (2.1.11)$$

To get an idea of the behavior of the Bessel functions we sketch the graphs of the first six Bessel functions of this kind are shown in the figure 1.

Le $p = n, n = 0, 1, 2, \ldots$

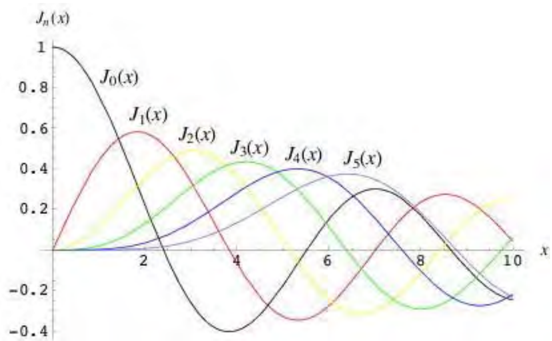

Figure 1 - $J_n(x)$: Bessel function of the first kind of order n.

Note that J_p is bounded at 0. As we will see shortly, this property is not shared by the second linearly independent solution.

❖ ***Second Solution Of Bessel's Equation***

If in equation (2.1.3) we replace r by the second indicial root $-p$, we arrives at the solution

$$J_{-p}(x) = \sum_{k=0}^{\infty} \frac{(-1)^k}{\Gamma(k+1)\ \Gamma(-p+k+1)} \left(\frac{x}{2}\right)^{2k+p} \tag{2.1.12}$$

Equation (2.1.11) and (2.1.12) both are called *the Bessel functions of order p of the first kind.*

It turns out that if p is not an integer, then J_p *and* J_{-p} are two linearly independent solutions.

It can be shown that the wronskian of J_p *and* J_{-p} is given by

$$W(J_p, J_{-p}) = \frac{-2\sin p\pi}{\pi x}$$

Then they also form a fundamental of solutions when p is not an integer.

When $p = n$, we have $\Gamma(p+k+1) = \Gamma(n+k+1) = (n+k)!$ and so the Bessel functions of order n is

$$J_n(x) = \sum_{k=0}^{\infty} \frac{(-1)^k}{k!(n+k)!} \left(\frac{x}{2}\right)^{2k+n} \tag{2.1.13}$$

Equation (2.1.11) is valid unless $-n \in \mathbb{N}$. But in this case we can just write

$$J_{-n}(x) := \sum_{k=n}^{\infty} \frac{(-1)^k}{\Gamma(k+1)\ \Gamma(-n+k+1)} \left(\frac{x}{2}\right)^{2k-n}$$

$$= \sum_{k=0}^{\infty} \frac{(-1)^{k+n}}{\Gamma(k+1)\ \Gamma(n+k+1)} \left(\frac{x}{2}\right)^{2k+2n-n}$$

$$= (-1)^n J_n(x)$$

$$\Rightarrow \quad J_{-n}(x) = (-1)^n J_n(x) \qquad (2.1.14)$$

This shows that equation (2.1.11) and (2.1.12) are linearly dependent when p is an integer. Thus it requires an extra effort to find another linearly independent solution. Since $(-1)^n = \cos n\pi$ the function $J_p(x)\cos p\pi - J_{-p}(x)$ is a solution of the Bessel equation, which vanishes if $n \in \mathbb{N}_0$. Therefore, we define (p is not an integer)

$$Y_p(x) = \frac{\cos(p\pi)J_p(\mathrm{x}) - J_{-p}(\mathrm{x})}{\sin(p\pi)} \qquad (2.1.15)$$

The function Y_p is called a Bessel function of the second kind of order p and also known as *Neumann functions* that are developed by a linear combination of Bessel function of the first order. The case when p is an integer n is defined by

$$Y_n(x) := \lim_{p\to n} \frac{\cos(p\pi)J_p(\mathrm{x}) - J_{-p}(\mathrm{x})}{\sin(p\pi)} \qquad (2.1.16)$$

$$= \lim_{p\to n} Y_p(x),$$

exists**,** as can be shown by L'Hospital rule. Y_p and J_p are two linearly independent solutions of the Bessel equation for all p is an element of complex number. This can be shown by computing the wronskian determinant.

Let $p = n,\ n = 0, 1, 2, ..$

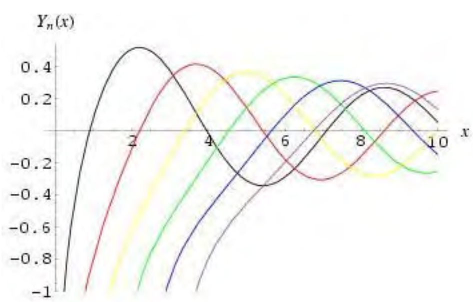

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Figure 2 - $Y_n(x)$: Bessel functions of the second kind (Neumann functions) of order n.

As illustrated in figure 2, we have

$$\lim_{x \to 0^+} Y_p(x) = -\infty$$

In particular, the Bessel functions of the second kind are not bounded near 0.

❖ ***General Solutions of Bessel's Equation Of order p***

We summarize our analysis of Bessel's differential equation of (1.2.1) as follows.

The general solution of Bessel's differential equation of (1.2.1) of order p is given by

$$y(x) = c_1 J_p(x) + c_2 Y_p(x) \tag{2.1.17}$$

Where J_p is given by equation (1.1.11) and Y_p is also given by equation (1.1.15), when p is not an integer, a general solution is also given by

$$y(x) = c_1 J_p(x) + c_2 J_{-p}(x) \tag{2.1.18}$$

Where J_p is given by equation (1.1.11) and J_{-p} is given by equation (1.1.12), and c_1 & c_2 are arbitrary constants from the boundary conditions.

2.2 Modified Bessel Functions Of First and Second Kind

The modified Bessel equation is given by

$$x^2 y'' + x y' - (x^2 + p^2) y = 0 \tag{2.2.1}$$

which transform from equation (1.2.1) when x is replaced with ix.

Modified Bessel functions are found as solutions to the modified Bessel equation. The solutions to equation (2.2.1) are analog to the previous section we can compute a solution using the technique of Frobenius series or power series approach. This gives

$$I_p(x) = \sum_{k=0}^{\infty} \frac{1}{\Gamma(k+1)\ \Gamma(p+k+1)} \left(\frac{x}{2}\right)^{2k+p} \tag{2.2.2}$$

$$I_{-p}(x) = \sum_{k=0}^{\infty} \frac{1}{\Gamma(k+1)\ \Gamma(-p+k+1)} \left(\frac{x}{2}\right)^{2k-p}$$

Both $I_p(x)$ and $I_{-p}(x)$ are a set of functions known as the modified Bessel functions of the first kind. The general solution of the modified Bessel functions is expressed as a combination of $I_p(x)$ and $I_{-p}(x)$ which is given by

$$y(x) = A I_{-p}(x) - B I_p(x) \tag{2.2.3}$$

Where $I_p(x)$ and $I_{-p}(x)$ are given by equation (2.2.2) and A & B are arbitrary constants determined from the boundary conditions.

As before, we search for a second linearly independent solution for p is not an integer be given by

$$K_p(x) = \frac{\pi}{2} \frac{I-p(x) - Ip(x)}{sin(p\pi)} \tag{2.2.4}$$

and its limit for $p \to n \in \mathbb{Z}$ exists.

The function $K_p(x)$ are known as modified Bessel functions of the second kind. A plot of the Neumann functions ($Y_p(x)$) and modified Bessel functions ($I_p(x)$) is shown in figure- 3.

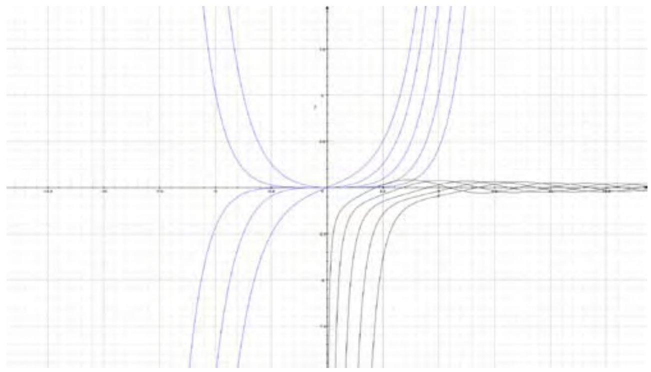

Figure 3- The Neumann functions(black) and The modified Bessel functions(blue) for integer orders p = 0 to p = 5.

A plot of the modified second kind Bessel functions ($K_p(x)$) is shown in figure-4.

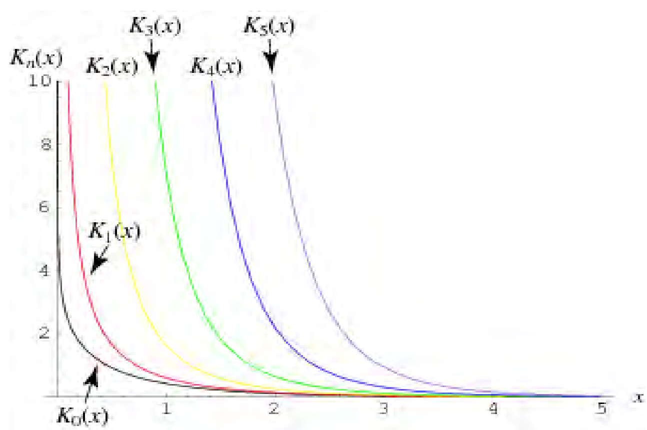

Figure 4- The Modified Bessel Functions of the Second Kind for orders p = 0 to p = 5.

When we see the graphs of Bessel functions, we observed that the following points.
Bessel functions of the first and second kind have an infinite number of zeros at the value of x goes to ∞, the zeros of the functions can be seen in the crossing points of the graphs in figure-1 and figure-2.

The modified Bessel functions of the first kind ($I_p(x)$) have only one zero at the point $x = 0$, and the modified Bessel functions of the second kind ($K_p(x)$) do not have zeros as shown in the figure-3 & figure-4 respectively.

The zeros of Bessel functions are of great importance in applications.

Chapter 3
Properties of the Bessel functions

3.1 Generating function

Many facts about Bessel functions can be proved by using its generating function. Here we want to determine the generating function. The Bessel functions of integral order are linked together by the fact that

$$e^{\frac{x}{2}(z-\frac{1}{z})} = J_0(x) + \sum_{n=1}^{\infty} J_n(x)\,[Z^n + (-1)^n z^{-n}].$$

Since $J_{-n}(x) = (-1)^n J_n(x)$, this is often written in the following theorem.

Theorem 3.1.1 **[17]** *We have*

$$e^{\frac{x}{2}(z-\frac{1}{z})} = \sum_{n=-\infty}^{\infty} J_n(x)\, z^n \qquad (3.1.1)$$

i.e. $e^{\frac{x}{2}(z-\frac{1}{z})}$ *is the generating function of* $J_n(x)$.

Proof: we have

$$e^{\frac{x}{2}z}\, e^{-\frac{x1}{2z}} = \sum_{m=0}^{\infty} \frac{(\frac{x}{2})^m}{m!}\, z^m \sum_{k=0}^{\infty} \frac{(-1)^k(\frac{x}{2})^k}{k!}\, z^{-k}$$

$$= \sum_{n=-\infty}^{\infty} \left(\sum_{m-k=n} \frac{(-1)^k\left(\frac{x}{2}\right)^{m+k}}{k!}\, z^{-k} \right) z^n \qquad m\,,\,k \geq 0$$

$$= \sum_{n=-\infty}^{\infty} \left(\sum_{k=0}^{\infty} \frac{(-1)^k}{(n+k)!k!}\left(\frac{x}{2}\right)^{2k} \left(\frac{x}{2}\right)^n\right) z^n$$

$$= \sum_{n=-\infty}^{\infty} J_n(x)\, z^n$$

A simple consequence of theorem (3.1.1) is *the addition formula*. The result is given in the following lemma.

***Lemma 3.1.1* [2]** *We have*

$$J_n(x+y) = \sum_{k=-\infty}^{\infty} J_{n-k}(x)J_k(y). \tag{3.1.2}$$

To prove this, we notice first that

$$e^{\frac{x}{2}(z-\frac{1}{z})}\ e^{\frac{y}{2}(z-\frac{1}{z})} = e^{\frac{[(x+y)]}{2}(z-\frac{1}{z})} = \sum_{n=-\infty}^{\infty} J_n(x+y)z^n .$$

However, the product of the two exponentials on the left is also

$$\left[\sum_{j=-\infty}^{\infty} J_j(x)z^j\right]\left[\sum_{k=-\infty}^{\infty} J_k(y)z^k\right] = \sum_{n=-\infty}^{\infty}\left[\sum_{k=-\infty}^{\infty} J_{n-k}(x)J_k(y)\right]z^n,$$

And equation (3.1.2) follows at once on equating the coefficients of z^n in these expressions.

When $n = 0$, equation (3.1.2) can be written as

$$\begin{aligned} J_0(x+y) &= \sum_{k=-\infty}^{\infty} J_{-k}(x)J_k(y) \\ &= J_0(x)J_0(y) + \sum_{k=1}^{\infty} J_{-k}(x)J_k(y) + \sum_{k=1}^{\infty} J_k(x)J_{-k}(y) \\ &= J_0(x)J_0(y) + \sum_{k=1}^{\infty} (-1)^k[J_k(x)J_k(y) + J_k(x)J_k(y)] \\ &= J_0(x)J_0(y) + \sum_{k=1}^{\infty} (-1)^k 2J_k(x)J_k(y) \end{aligned}$$

or

$$J_0(x+y) = J_0(x)J_0(y) - 2J_1(x)J_1(y) + 2J_2(x)J_2(y) - \ldots \tag{3.1.3}$$

If we replace y by $-x$ and use the fact that $J_n(x)$ is even or odd according as n is even or odd, then equation (3.1.3) yields the remarkable identity

$$1 = J_0(x)^2 + 2J_1(x)^2 + 2J_2(x)^2 + \ldots, \tag{3.1.4}$$

which shows that $|J_0(x)| \le 1$ and $|J_n(x)| \le \frac{1}{\sqrt{2}}$ for $n = 1, 2, \ldots$

We can also use this generating function to prove some standard results

***Lemma 3.1.2* [17]** *we have*

$$\cos x = J_0(x) + 2\sum_{n=1}^{\infty} (-1)^n J_{2n}(x) \tag{3.1.5}$$

$$\sin x = 2\sum_{n=0}^{\infty} (-1)^n J_{2n+1}(x), \tag{3.1.6}$$

$$1 = J_0(x) + 2\sum_{n=1}^{\infty} J_{2n}(x) \tag{3.1.7}$$

Proof: Directly from equation (3.1.1) with $z = e^{i\emptyset}$, $i \sin \emptyset = \frac{1}{2}(z - \frac{1}{z})$ we get

$$\cos(x\sin\emptyset)+i\sin(x\sin\emptyset)=e^{ix\sin\emptyset}=\sum_{n=-\infty}^{\infty} J_n(x)\, e^{in\emptyset} = \sum_{n=-\infty}^{\infty} J_n(x)\ [\cos(n\emptyset)+i\sin(n\emptyset)]$$

so

$$\cos(x\sin\emptyset) = J_0(x) + 2\sum_{n=1}^{\infty} J_{2n}(x)\cos(2n\emptyset)$$

and

$$\sin(x\sin\emptyset) = 2\sum_{n=0}^{\infty} J_{2n+1}(x)\sin((2n+1)\emptyset).$$

This gives the results with $\emptyset = \frac{\pi}{2}$ and $\emptyset = 0$, respectively.

***Lemma 3.1.3* [17]** *we have*

$$J_n(-x) = J_{-n}(x) = (-1)^n J_n(x) \qquad \textit{for all } n \in \mathbb{Z} \tag{3.1.8}$$

Proof: To show the first equality, we make the change of variables $x \rightarrow -x$, $z \rightarrow z^{-1}$ in the generating function. Then the function does not change and we get

$$\sum_{n=-\infty}^{\infty} J_n(-x)\, z^{-n} = \sum_{n=-\infty}^{\infty} J_n(x)\, z^{n} = \sum_{n=-\infty}^{\infty} J_{-n}(x)\, z^{-n}$$

and comparing coefficients gives the results. The second equality follows analog with the change of variable $z \rightarrow -z^{-1}$.

***Lemma 3.1.4* [17]** *we have for any $n \in \mathbb{Z}$*

$$2J'_n(x) = J_{n-1}(x) - J_{n+1}(x), \tag{3.1.9}$$

$$\frac{2n}{x}J_n(x) = J_{n+1}(x) + J_{n-1}(x), \tag{3.1.10}$$

$$\frac{d}{dx}(x^n Jn(x)) = x^n J_{n-1}(x), \tag{3.1.11}$$

Proof: The first statement follows, when we differentiate equation (3.1.1) with respect to *x:*

$$\sum_{n=-\infty}^{\infty} J_n(x)\, z^n = \frac{1}{2}\left(z - \frac{1}{z}\right) e^{\frac{x}{2}\left(z-\frac{1}{z}\right)} = \frac{1}{2}z \sum_{n=-\infty}^{\infty} J_n(x)\, z^n - \frac{1}{2z} \sum_{n=-\infty}^{\infty} J_n(x)\, z^n$$

$$= \sum_{n=-\infty}^{\infty} \frac{1}{2}(J_{n-1}(x) - J_{n+1}(x))\, z^n$$

Comparing coefficients yields the result.

The second statement follows completely analog when considering the derivative with respect to *z*. Adding the two equations and multiplying with $\frac{x^n}{2}$ gives the third result.

3.2 Special values

In this section we want to examine how the Bessel functions look like when we plug in some special values.

***Lemma 3.2.1* [16]** *If $p \in \frac{1}{2} + \mathbb{Z}$, the Bessel functions are elementary functions. We have*

$$J_{\frac{1}{2}}(x) = Y_{-\frac{1}{2}}(x) = \sqrt{\frac{2}{\pi x}}\, sin(x) ,$$

$$J_{-\frac{1}{2}}(x) = -Y_{\frac{1}{2}}(x) = \sqrt{\frac{2}{\pi x}}\, cos\,(x),$$

$$I_{\frac{1}{2}}(x) = \sqrt{2/\pi x}\, sin\, h(x),$$

$$I_{-\frac{1}{2}}(x) = \sqrt{\frac{2}{\pi x}}\, cos\, h\,(x),$$

$$K_{\frac{1}{2}}(x) = K_{-\frac{1}{2}}(x) = \sqrt{\frac{\pi}{2x}}\, e^{-x}.$$

Proof: These formulas follow directly from the series representations of J_p and I_p, the definitions of Y_p and K_p and the properties of the gamma function.
For example, we have

$$J_{\frac{1}{2}(x)} = \sqrt{\frac{x}{2}} \sum_{k=0}^{\infty} \frac{(-1)^k}{\Gamma(k+1)\Gamma\left(k+\frac{3}{2}\right)} (\frac{x}{2})^{2k}$$

$$= \sqrt{\frac{x}{2}} \sum_{k=0}^{\infty} \frac{(-1)^k}{k!\, \frac{(2k+1)!\sqrt{\pi}}{k!\, 2^{2k+1}}} (\frac{x}{2})^{2k}$$

$$= \sqrt{\frac{2}{\pi x}} \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)!} x^{2k+1}$$

$$= \sqrt{\frac{2}{\pi x}} sinx$$

and

$$Y_{-\frac{1}{2}}(x) = \frac{cos\left(-\frac{\pi}{2}\right)\ J_{-\frac{1}{2}}(x) - J_{\frac{1}{2}}(x)}{sin\,(-\frac{\pi}{2})} = J_{\frac{1}{2}(x)}$$

The rest follows analog with the recurrence formulas.

3.3 Integral representations

When $t = e^{i\theta}$, the exponent on the left side of equation (3.1.1) becomes

$$x\ \frac{e^{i\theta}-e^{-i\theta}}{2} = ix\ sin\theta,$$

and equation(3.1.1) itself assumes the form

$$e^{ix\ sin\theta} = \sum_{n=-\infty}^{\infty} J_n(x)e^{in\theta} \qquad (3.3.1)$$

Since $e^{ix\ sin\theta} = \cos(x\ \sin\theta) + i\sin(x\sin\theta)$ and $e^{in\theta} = \cos n\theta + i\sin n\theta,$

equating real and imaginary parts in (3.3.1) yields

$$\cos(x\sin\theta) = \sum_{n=-\infty}^{\infty} J_n(x)\cos n\theta \qquad (3.3.2)$$

and $$\sin(x \sin\theta) = \sum_{n=-\infty}^{\infty} J_n(x) \sin n\theta \tag{3.3.3}$$

If we now use the relations

$$J_{-n}(x) = (-1)^n J_n(x), \quad \cos(-n\theta) = \cos n\theta, \text{ and } \sin(-n\theta) = -\sin n\theta,$$

then equations (3.3.2) and (3.3.3) become

$$cos(x\, sin\emptyset) = J_0(x) + 2\sum_{n=1}^{\infty} J_{2n}(x)\, cos(2n\emptyset) \tag{3.3.4}$$

and

$$sin(xsin\emptyset) = 2\sum_{n=1}^{\infty} J_{2n-1}(x)\, sin((2n-1)\emptyset) \tag{3.3.5}$$

As a special case of equation (3.3.4), we note that $\theta = 0$ yields the interesting series

$$1 = J_0(x) + 2J_2(x) + 2J_4(x) + \ldots$$

Also, on putting $\theta = \frac{\pi}{2}$ in equations (3.3.4) and (3.3.5), we obtain the formulas

$$\cos x = J_0(x) - 2J_2(x) + 2J_4(x) - \ldots$$

and

$$\sin x = 2J_1(x) - 2J_3(x) + 2J_5(x) - \ldots$$

which demonstrate once again the close ties between the Bessel functions and the trigonometric functions. The most important application of equations (3.3.2) and (3.3.3) is to the proof of *Bessel's integral formula*

$$J_n(x) = \frac{1}{\pi}\int_0^{\pi} \cos(n\theta - x\sin\theta)d\theta. \tag{3.3.6}$$

To establish this, we multiply equation (3.3.2) by $\cos m\theta$,equation (3.3.3) by $\sin m\theta$, and add:

$$\cos(m\theta - x\sin\theta) = \sum_{n=-\infty}^{\infty} J_n(x)\cos(m-n)\theta.$$

When both sides of this are integrated from $\theta = 0\ to\ \theta = \pi$, the right side reduces to

$\pi J_m(x)$, and replacing *m* by *n* yields formula (3.3.6).

3.4 zeros and Bessel series

i. From our previous section, we know that every nontrivial solution of Bessel's equation has infinitely many zeros. That is for every value of p, the function $J_p(x)$ has an infinite number of positive zeros. This is true in particular of $J_0(x)$, the zeros of this function are known to a high degree of accuracy, and their values are given in many volumes of mathematical tables. The first five are approximately 2.4048, 5.5201, 8.6537, 11.7915, and 14.9309; their successive differences are 3.1153, 3.1336, 3.1378, and 3.1394. The corresponding positive zeros and differences for $J_1(x)$ are 3.8317, 7.0156, 10.1735, 13.3237, and 16.4706; and 3.1839, 3.1579, 3.1502, and 3.1469. Notice how these differences confirm the guarantees given by the following two cases: Let x_1 and x_2 be successive positive zeros of a nontrivial solution $J_p(x)$ of Bessel's equation, then we have If $0 \leq p < \frac{1}{2}$, then $x_2 - x_1$ is less than π and approaches π as $x_1 \to \infty$.

ii. If $p > \frac{1}{2}$, then $x_2 - x_1$ is greater than π and approaches π as $x_1 \to \infty$.

What is the purpose of this concern with the zeros of $J_p(x)$? It is often necessary in mathematical physics to expand a given function in terms of Bessel functions, where the particular type of expansion depends on the problem at hand. The simplest and most useful expansions of this kind are series of the form

$$f(x) = \sum_{n=1}^{\infty} a_n J_p(\lambda_n x) = a_1 J_p(\lambda_1 x) + a_2 J_p(\lambda_2 x) + \ . \ . \ . \ , \qquad (3.4.1)$$

Where $f(x)$ is defined on interval $0 \leq x \leq 1$ and the λ_n's are the positive zeros of some fixed Bessel function $J_p(x)$ with $p \geq 0$. We have chosen the interval $0 \leq x \leq 1$ only for the sake of simplicity, and all the formulas given below can be adapted by a simple of variable to the case of a function defined on an interval of the form $0 \leq x \leq a$. The role of such expansions in physical problems in similar to that of Fourier and Legendre series.

In the light of our previous experience with Fourier and Legendre series, we expect the determination of the coefficients in equation (3.4.1) to depend on certain integral properties of the functions $J_p(\lambda_n x)$. What we need here is the fact that

$$\int_0^1 x J_p(\lambda_m x) J_p(\lambda_n x) dx = \begin{cases} 0, & if\ m \neq n\ , \\ \frac{1}{2} J_{p+1}(\lambda_n)^2, & if\ m = n\ . \end{cases} \qquad (3.4.2)$$

these formulas say that the functions $\sqrt{x}J_p(\lambda_n x)$ are orthogonal on the interval $0 \le x \le 1$. If an expansion of the form (3.4.1) is assumed to be possible, then multiplying through by $xJ_p(\lambda_m x)$, formally integrating term by term from 0 to 1, and using (3.4.2) yields

$$\int_0^1 x\, f(x) J_p(\lambda_m x)\, dx = \frac{a_m}{2} J_{p+1}(\lambda_m)^2;$$

and on replacing *m* by *n* we obtain the following formula for a_n:

$$a_n = \frac{2}{J_{p+1}(\lambda_n)^2} \int_0^1 x\, f(x) J_p(\lambda_n x)\, dx \qquad (3.4.3)$$

The series (3.4.1), with its coefficients calculated by (3.4.3), is called ***the Bessel series or the Fourier-Bessel series*** of the function ***f(x)***. We state without proof a rather deep theorem that gives conditions under which this series actually converges and the sum *f(x).*

Theorem 3.4.1* [2]** ***Bessel Expansion Theorem

Assume that f(x) and f'(x) have at most a finite number of jump discontinuities on the interval $0 \le x \le 1.$ *if* $0 < x < 1$, *then the Bessel series* (3.4.1) *converges to f*(*x*) *when x is a point of continuity of this function, and converges to* $\frac{1}{2}\,[\,f(x-) + f(x+)]$ *when x is a point of discontinuity.*

It is natural to wonder what happens at the end of the interval. At x = 1, the series converges to zero regardless of the nature of the function because every $J_p(\lambda_n)$ is zero. The series also converges at x = 0, to zero if $p > 0$ and to $f(0+)$ if $p = 0$.

As an illustration, we compute the Bessel series of the function $f(x) = 1$ for interval $0 \le x \le 1$ in terms of the function $J_0(\lambda_n x)$, where it is understood that the λ_n's are the positive zeros of $J_0(x)$.

In this case, equation (3.4.3) is

$$a_n = \frac{2}{J_1(\lambda_n)^2} \int_0^1 x\, J_0(\lambda_n x)\, dx.$$

since

$$\int x\, J_0(x) dx = x\, J_1(x) + c,$$

we see that

$$\int_0^1 x\, J_0(\lambda_n x)\, dx = [\frac{1}{\lambda_n} x J_1(\lambda_n x)]_0^1 = \frac{J_1(\lambda_n)}{\lambda_n},$$

so

$$a_n = \frac{2}{\lambda_n J_1(\lambda_n)}$$

It follows that

$$1 = \sum_{n=1}^{\infty} \frac{2}{\lambda_n J_1(\lambda_n)} J_0(\lambda_n) \qquad (0 \le x < 1)$$

is the desired Bessel series.

3.5 Bessel inequality

Another widely used inequality for vectors in inner product space is Bessel inequality. The Cauchy-Schwarz inequality follows from the Bessel inequality. In this section we prove the Bessel inequality.

Recall that: *let* $(H, \langle ., . \rangle)$ *be an i nner pr oduct ov er real or c omplex num ber f ield k and* $\{e_i\}_{i \in I}$ *a countable family of orthonormal vectors in H, i.e.*

$$\langle e_i, e_j \rangle = \begin{cases} 0, & if \;\; i \neq j \\ 1, & if \;\; i = j \end{cases} \qquad , \qquad i, j \in I$$

where I is the set of indices.

Lemma 3.5.1* [3]** ***Bessel's inequality

Let $\{e_j\}_{j \ge 1}$ *be an or thonormal s ystem i n a H ilbert s pace H , t hen f or al l* $x \in H$

$$\sum_{j \ge 1} \left| \langle x, e_j \rangle \right|^2 \le \|x\|^2.$$

(Note that the case n = 1 is the Cauchy-Schwartz inequality)

Proof: let $\alpha_k = \langle x, e_k \rangle$, then we have

$$\left\| x - \sum_{k=1}^{n} \alpha_k e_k \right\|^2 = \langle \, x - \sum_{k=1}^{n} \alpha_k e_k \;\; , \;\; x - \sum_{k=1}^{n} \alpha_k e_k \; \rangle$$

$$= \|x\|^2 - \langle \sum_{k=1}^{n} \alpha_k e_k \;, x \rangle - \langle x \,, \sum_{k=1}^{n} \alpha_k x_k \;\; \rangle + \sum_{k=1}^{n} |\alpha_k|^2$$

$$= \|x\|^2 - \sum_{k=1}^{n} \alpha_k \overrightarrow{\langle x, e_k \rangle} - \sum_{k=1}^{n} \overrightarrow{\alpha_k} \langle x, e_k \rangle + \sum_{k=1}^{n} |\alpha_k|^2$$

$$= \|x\|^2 - \sum_{k=1}^{n} |\langle x, e_k \rangle|^2 + \sum_{k=1}^{n} |\langle x, e_k \rangle - \alpha_k|^2$$

$$= \|x\|^2 - \sum_{k=1}^{n} |\langle x, e_k \rangle|^2$$

We have

$$\sum_{k=1}^{n} |\langle x, e_k \rangle|^2 = \|x\|^2 - \left\| x - \sum_{k=1}^{n} \alpha_k e_k \right\|^2 \leq \|x\|^2$$

Let $n \to \infty$ in the last in equality. We have a sequence of non-negative numbers, where the sum of the numbers is bounded from above. Hence lemma 2.5 follows.

The inner products $\langle x, e_j \rangle$ in lemma 3.5.1 are called *the Fourier coefficients* of x with respect to the orthonormal system $\{e_j\}_{j \geq 1}$.

Chapter 4

Aging Spring Problem

Bessel functions of the first and the second kind are the most commonly found forms of the Bessel functions in applications. The next step is to show that other differential equations can be converted into Bessel equation of order 0 (J_0). An interesting example of this type is the aging spring problem.

4.1 Aging Spring problem

Aging spring problem with damping is an equation with time-varying coefficients which is given by

$$mx'' + bx' + ke^{-vt}x = 0 \tag{4.1.1}$$

This models a mass-spring system where the restoring force of the spring is weakening over time. The new problem here is that the spring constant is replaced by decaying exponential function, and series multiplication must be used to obtain the recurrence relation.

Using the exponential series

$$e^{-vt} = 1 - vt + \frac{(-vt)^2}{2!} + \frac{(-vt)^3}{3!} + \cdots = \sum_{n=0}^{\infty} \frac{(-vt)^n}{n!} \ ,$$

& letting

$$x(t) = \sum_{n=0}^{\infty} a_n t^n$$

We can write the term $ke^{-vt}x$ in the series form as follows:

$$ke^{-vt}x = k\left(\sum_{n=0}^{\infty} \frac{(-vt)^n}{n!}\right)\left(\sum_{n=0}^{\infty} a_n t^n\right) = k\sum_{n=0}^{\infty}\left(\sum_{j=0}^{n} a_{n-j}\frac{(-v)^j}{j!}\right)t^n$$

The differential equation (4.1.1) becomes

$$m\sum_{n=2}^{\infty} n(n-1)a_n t^{n-2} + b\sum_{n=1}^{\infty} na_n t^{n-1} + \ k\sum_{n=0}^{\infty}\left(\sum_{j=0}^{n} a_{n-j}\frac{(-v)^j}{j!}\right)t^n \equiv 0.$$

Changing the index to $s = n-2$ in the first series and $s = n-1$ in the second,

$$\sum_{s=0}^{\infty} m(s+2)(s+1)a_{s+2}t^s + \sum_{s=o}^{\infty} b(s+1)a_{s+1}t^s + \sum_{s=0}^{\infty} k\left(\sum_{j=0}^{s} a_{s-j}\frac{(-v)^j}{j!}\right)t^s \equiv 0.$$

Then for $s = 0,\ 1,\ 2, \ldots$, each coefficient of t^smust be zero; that is,

$$m(s+2)(s+1)a_{s+2} + b(s+1)a_{s+1} + k\left(\sum_{j=0}^{s} a_{s-j}\frac{(-v)^j}{j!}\right) = 0\,,$$

And the recurrence relation is

$$a_{s+2} \ = \ \frac{-b(s+1)a_{s+1} - k\left(\sum_{j=0}^{s} a_{s-j}\frac{(-v)^j}{j!}\right)}{m(s+1)(s+2)}\,, \qquad s = 0, 1, 2, \ldots$$

This recurrence relation will be used to solve an initial-value problem for a particular aging spring problem.

From our previous section, we defined the function $J_p(x)$ which is called the Bessel function of order p of the first kind. The series converges and bounded for all x. If p is not an integer, it can be shown that a second solution of Bessel's equation is $J_{-p}(x)$ that the general solution of Bessel's equation is a linear combination of $J_p(x)$ and $J_{-p}(x)$.

For the special case $p = 0$, we get the function $J_0(x)$ used in the Aging spring model:

i.e.

$$J_0(x) \ = \sum_{n=0}^{\infty} \frac{(-1)^n}{(n!)^2}\left(\frac{x}{2}\right)^{2n} = \ 1 - \frac{x^2}{4} + \frac{x^4}{64} - \frac{x^6}{2304} + \ \ldots$$

Note that even though $x = 0$ is a singular point of the Bessel equation of order zero, the value of $J_0(x)$ is finite [$J_0(0) = 1$].

When p is an integer we have to work much harder to get a second solution that is linearly independent of $J_p(x)$. The result is a function $Y_p(x)$ called the Bessel function of order p of the second kind. The general formula for $Y_p(x)$ is extremely complicated.

We show only the special case $Y_0(x)$, used in the Aging spring model.

$$Y_0(x) = \frac{2}{\pi}\left[\left(\gamma + ln\frac{x}{2}\right)J_0(x) + \sum_{n=0}^{\infty}\frac{(-1)^{n+1}H_n}{(n!)^2}\left(\frac{x}{2}\right)^{2n}\right]$$

Where $H_n = 1 + \frac{1}{2} + \frac{1}{3} + . . . + \frac{1}{n}$ and γ is an ending decimal which is known as Euler's constant:

$$\gamma = \lim_{n\to\infty}(H_n - \ln n) \approx 0.5772.$$

The general solution to Bessel's equation of integer order p is given by formula (2.1.17). An important thing to note here the value of $Y_p(x)$ at $x = 0$ does reflect the singularity at $x = 0$; in fact, $Y_p(x) \to -\infty \; as \; x \to 0^+$, so that a solution having the form given in formula (2.1.17) is bounded if $C_2 = 0$.

4.2 Conversion of the Aging Spring Problem without damping to Bessel's equation of order 0 (J_0).

It will only be possible to do the conversion for the *un-damped* aging spring problem:

$$x''(t) + e^{-at+b}x(t) = 0 \qquad (a > 0), \quad t > 0, \qquad (4.2.1)$$

The equation (4.2.1) models the vibrations of a spring whose spring constant is tending to zero with time.

We will make a change of independent variable of the form

$$s = \frac{2}{a}\, e^{-\frac{1}{2}(at-b)} \qquad (4.2.2)$$

This will convert equation (4.2.1) in to the form of Bessel's equation of order 0:

$$s^2X''(s) + sX'(s) + s^2X(s) = 0 \qquad (4.2.3)$$

To show this, first note that by differentiating (4.2.2)

$$\frac{ds}{dt} = -e^{-\frac{1}{2}(at-b)}$$

If we let X(s)≡ $x(t)$ and $e^{-at+b} = \frac{a^2}{4}s^2$, then the derivatives $x' and\ x''$ can be found in terms of the new variable X using the chain rule, as follows:

$$x' = \frac{dx}{dt} = \frac{dX}{ds}\frac{ds}{dt} = X'\left(-e^{-\frac{1}{2}(at-b)}\right)$$

and

$$x'' = \frac{d^2x}{dt^2} = \frac{d}{ds}\left[X'\left(-e^{-\frac{1}{2}(at-b)}\right)\right] = X''e^{-at+b} + X'\frac{a}{2}e^{-\frac{1}{2}(at-b)}$$

Now equation (4.2.1) becomes

$$X''e^{-at+b} + X'^{\frac{a}{2}e^{-\frac{1}{2}(at-b)}} + Xe^{-at+b} = 0,$$

then multiplying by e^{at-b} and using equation (4.2.2), we get

$$X'' + \frac{1}{s}X' + X = 0,$$

finally multiplying by s^2 , then the equation becomes

$$s^2X'' + sX' + s^2X = 0,$$

which is Bessel's equation of order 0.

The general solution of equation (4.2.3) is given by

$$X(s) = C_1 J_0(s) + C_2 Y_0(s)$$

but $X(s) = x(t),$ and $s = \frac{2}{a}e^{-\frac{1}{2}(at-b)}$

so,

$$x(t) = c_1J_0\left(\frac{2}{a}e^{-\frac{1}{2}(at-b)}\right) + c_2Y_0\left(\frac{2}{a}e^{-\frac{1}{2}(at-b)}\right) \qquad (4.2.4)$$

where $c_1\ and\ c_2$ are arbitrary constants, J_0 is the Bessel function of order 0, and Y_0 is the Bessel function of order 0 of the second kind.

4.3 The Aging Spring Stretches

❖ ***The behavior of equation (4.2.4) as*** $\boldsymbol{t} \to \infty$

Case –i **:** If $c_1 = 0$ and $c_2 \neq 0$, then

$$x(t) = c_2 Y_0\left(\frac{2}{a} e^{-\frac{1}{2}(at-b)}\right),$$

As $t \to \infty$, $s(t) \to 0$ and $Y_0(s) \to -\infty$. In this case, $x(t)$ could approach either $+\infty$ or $-\infty$ depending on the sign of c_2. $x(t)$ would approach infinity linearly as near 0, $Y_0(x) \approx \ln x$

$$\text{so,} \quad x(t) \approx \ln\left(\frac{2}{a} e^{-\frac{1}{2}(at-b)}\right),$$

Case –ii **:** If $c_1 \neq 0$ and $c_2 = 0$, then

$$x(t) = c_1 J_0\left(\frac{2}{a} e^{-\frac{1}{2}(at-b)}\right)$$

As $t \to \infty$, $s(t) \to 0$, $J_0(s) \to 1$ and $x(t) \to c_1$. In this case the solution is bounded.

Case –iii **:** If $c_1 \neq 0$ and $c_2 \neq 0$, then

$$x(t) = c_1 J_0\left(\frac{2}{a} e^{-\frac{1}{2}(at-b)}\right) + c_2 Y_0\left(\frac{2}{a} e^{-\frac{1}{2}(at-b)}\right)$$

As $t \to \infty$, $s(t) \to 0$, $J_0(s) \to 1$ and $Y_0(s) \to -\infty$. Since Y_0 will dominate, the solution will behave like ***case i***. It makes sense to have unbounded solutions because eventually the spring wears out and does not affect the motion. Newton's laws tell us the mass will continue with unperturbed momentum, i.e. as $t \to \infty$, $x'' = 0$ and so $x(t) = c_1 t + c_2$, a linear function, which is unbounded if $c_1 \neq 0$.

Summary

The Bessel functions appear in many diverse scenarios, particularly situations involving aging spring model. The most difficult aspect of working with the Bessel function is first determine that they can be applied through reduction of the system equation to Bessel differential or modified equation, and the coefficients values on the argument of the Bessel function, and also the general formula for the Bessel function of order p of the second kind ($Y_p(x)$) is extremely complicated. This topic can be greatly expanded upon, and the reader is highly encouraged to review the applications and development presented in [4.1].

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