ADDIS ABABA UNIVERSITY

Topological and Dynamical Structures of Composition Operators on Generalized Fock Spaces

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Declaration

I, Werkaferahu Seyoum Teklewold, with student ID number GSR 4111/09 hereby declare that this thesis is my own work and that it has not been previously submitted for assessment or completion of any post graduate qualification to another university or for another qualification.

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This is to certify that the thesis prepared by Werkaferahu Seyoum Teklewold, entitled **“Topological and Dynamical Structures of Composition Operators on Generalized Fock Spaces “** submitted to the Department of Mathematics of Addis Ababa University in the partial fulfilment of the requirements of the degree of Doctor of Philosophy in Mathematics complies with the regulation of the University and meets the accepted standards with originality and quality.

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Abstract

In this thesis we study various properties of composition operators acting between generalized Fock spaces $\mathcal{F}_\varphi^p$ and $\mathcal{F}_\varphi^q$ with weight functions φ grow faster than the classical Gaussian weight function $\frac{1}{2}|z|^2$ and satisfy some mild smoothness conditions. Let ψ be an analytic map on the complex plane. Then for $p \neq q$, we have shown that the composition operator $C_\psi : \mathcal{F}_\varphi^p \to \mathcal{F}_\varphi^q$ is bounded if and only if it is compact. This result shows a significance difference with the analogous result for the case when C_ψ acts between the classical Fock spaces or generalized Fock spaces where the weight functions grow slower than the Gaussian function.

We further study some topological structure of composition operators on the spaces. It is shown that the difference of two composition operators is compact if and only if both are compact, and hence cancellation phenomenon fails to exist. While each non-compact bounded composition operator is an isolated point, the set of all compact composition operators forms a connected components of the space of the operators under the operator norm topology. Moreover, Schatten $\mathcal{S}_p(\mathcal{F}_\varphi^2)$ class membership, spectra, hyponormality of the composition operators are characterized.

We also study various dynamical structure of composition operators C_ψ on the generalized Fock spaces $\mathcal{F}_\varphi^p$ and the weighted composition operators $W_{(u,\psi)}$ defined on the classical Fock spaces $\mathcal{F}_p$. It is shown that all composition operators on the spaces are power bounded. Several conditions characterizing uniformly mean ergodic composition operators are provided. We have identified operators $W_{(u,\psi)}$ that are power bounded and uniformly mean ergodic on the spaces, and these properties are described in terms of easy to apply conditions which are based merely on the values $|u(0)|$ and $|u(\frac{b}{1-a})|$, where the numbers a and b are from linear expansion of the symbol $\psi(z) = az + b$. We have proved that composition operators C_ψ and weighted composition operators $W_{(u,\psi)}$ can not be supercyclic on their respective Fock spaces. Furthermore, the operator $C_\psi,\ \psi(z) = az+b,\ |a| \leq 1$ and $b = 0$ when $|a| = 1$, is cyclic if and only if $a^n \neq a$, while $W_{(u,\psi)}$ is cyclic if and only if the corresponding composition operator is cyclic and u fails to vanish on $\mathbb{C}$. The set of periodic points of C_ψ is also determined. Conditions under which the operators satisfy the Ritt's resolvent growth conditions are also identified. In particular, we show that a non-trivial composition operator on the Fock spaces satisfies such growth condition if and only if it is compact.

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Contents

Chapter 1

Introduction

For holomorphic function ψ on a given domain G, the composition operator C_ψ on space of holomorphic functions on G is defined by $C_\psi f = f \circ \psi$. A systematic study of the composition operators was begun back in 1968 by Nordgren [48] where he proved that all the composition operator on L^2 of the unit circle induced by inner functions are bounded. Since then, the theory of the operator has experienced fast growth with investigations mainly focused on describing its operator theoretic properties in terms of function theoretic properties of the inducing functions. In particular, the theory began attracting several researchers following the work of J. Shapiro [63] on Hardy space over the unit disc $\mathbb{D}$. As a consequence of the well known Littlewood subordination principle, it is shown that every analytic self map on $\mathbb{D}$ induces a bounded composition operator on Hardy spaces and Bergman spaces. However, a self map ψ on $\mathbb{D}$ doesn't necessarily induce a bounded composition operator on the Dirichlet space on $\mathbb{D}$. A necessary and sufficient condition for ψ to induce a bounded composition operator on the Dirichlet space of open disc is given in terms of the counting function and Carleson measure; see [26, 79].

On the other hand, although the corresponding Hardy spaces of the unit disc and the upper half-plane are isomorphic, composition operators act differently in the two cases. There exist analytic self-maps of the upper half-plane which do not induce bounded composition operators on Hardy and weighted Bergman spaces over the upper half-plane. S. Elliott and M. T. Jury in [19], and S. Elliott and A. Wynn in [20] proved that C_ψ is bounded if and only if ψ has a finite angular derivative at infinity on Hardy and Bergman spaces of half-planes respectively. Moreover, both the Hardy and Bergman spaces over the upper half-plane fail to support compact composition operators unlike the unit disc case: see [49, 41]. In contrary, the Dirichlet spaces of upper half-plane support compact composition operator. We refer [50] for more details.

Later, the study of composition operators advanced on spaces of analytic functions defined on the whole complex plane $\mathbb{C}$. For instance, in the frame of Fock spaces, in 2003, Carswell, MacCluer and Schuster [11] characterized bounded and compact composition operators on the classical Fock spaces $\mathcal{F}_p, 0 < p < \infty$. They showed that only the class of linear mappings $\psi(z) = az + b,\ |a| \leq 1$ and $b = 0$ whenever $|a| = 1$ induces bounded composition operators C_ψ on $\mathcal{F}_p$. Compactness of C_ψ was described by the strict requirement $|a| < 1$. Recall that the classical Fock spaces $\mathcal{F}_p$ are spaces consisting of all entire functions f for which

$$\|f\|_p = \begin{cases} \left(\frac{p}{2\pi}\int_{\mathbb{C}} |f(z)|^p e^{-\frac{p|z|^2}{2}} dA(z)\right)^{\frac{1}{p}} < \infty, & 0 < p < \infty \\ \sup_{z\in\mathbb{C}} |f(z)| e^{-\frac{|z|^2}{2}} < \infty, & p = \infty, \end{cases}$$

where dA denotes the usual Lebesgue area measure on the complex plane $\mathbb{C}$. Here the function $\Omega(z) = \frac{1}{2}|z|^2$ is Gaussian weight function. For functions f in such spaces, the subharmonicity of $|f|^p$ implies that the local point estimate

$$|f(z)|^p e^{-\frac{p|z|^2}{2}} \leq \int_{D(z,1)} |f(w)|^p e^{-\frac{p|w|^2}{2}} dA(w)$$

holds where $D(z, 1)$ is a disc of radius 1 and center z. This further results

$$|f(z)| \leq e^{\frac{|z|^2}{2}} \|f\|_p. \tag{1.0.1}$$

By definition of the norm, the estimate in (1.0.1) is valid for $p = \infty$ as well.

The space $\mathcal{F}_2$ is a reproducing kernel Hilbert space with kernel function $K_w(z) = e^{\overline{w}z}$ and normalized kernel $k_w = \frac{K_w}{\|K_w\|_2}$. For all complex numbers w, a straightforward calculation shows that k_w belongs to the Fock spaces $\mathcal{F}_p$ and $\|k_w\|_p = 1$ for all p. Another interesting and useful property of the spaces is inclusion or nestedness. That is if $p < q$, then

$$\mathcal{F}_p \subset \mathcal{F}_q. \tag{1.0.2}$$

We refer to [77] for more background information about the spaces.

Let φ be a function defined on $[0, \infty)$ and $\varphi \geq 0$. We define the generalized Fock spaces $\mathcal{F}_\varphi^p, 0 < p \leq \infty$ associated with φ as spaces consisting of all entire functions f for which

$$\|f\|_{(\varphi,p)} = \begin{cases} \left(\int_{\mathbb{C}} |f(z)|^p e^{-p\varphi(|z|)} dA(z) \right)^{1/p}, & 0 < p < \infty \\ \sup_{z \in \mathbb{C}} |f(z)| e^{-\varphi(|z|)}, & p = \infty \end{cases}$$

is finite. In 2008, Guo and Izuchi [24] studied various aspects of the operators when $p = 2$. They showed that if $\lim_{|z| \to \infty} \varphi'(z) < \infty$, then C_ψ is bounded on $\mathcal{F}_\varphi^2$ if and only if $\psi(z) = az + b$, $|a| \leq 1$ and b is any complex number. On the other hand, if $\lim_{|z| \to \infty} \varphi'(z) = \infty$, then their result shows that C_ψ is bounded on $\mathcal{F}_\varphi^2$ if and only if $\psi(z) = az + b$, $|a| \leq 1$ and $b = 0$ when $|a| = 1$. In both cases C_ψ is compact if and only if $|a| < 1$. In 2014, T. Mengestie [51] showed that the same symbol forms $\psi(z) = az + b, |a| \leq 1$ and $b = 0$ whenever $|a| = 1$ induce bounded composition operator $C_\psi : \mathcal{F}_m^p \to \mathcal{F}_m^q$; $0 < p \leq q < \infty$, where $\mathcal{F}_m^p$ are Fock–Sobolev spaces, which are typical example of generalized Fock spaces with weight function $-m \log(1 + |z|) + \frac{1}{2}|z|^2$, which grows slower than Gaussian weight function. We may note that, if φ grows slower than the Gaussian weight function, (1.0.1) and (1.0.2) results in the inclusion property $\mathcal{F}_\varphi^p \subset \mathcal{F}_\varphi^q$ whenever $p < q$. Further topological and dynamical properties of C_ψ have also been studied: see

[24, 31].

A natural question is what happens to the symbol ψ when the weight function grows faster than the Gaussian function. The first main aim of this work is taking further the study of the operators on such spaces, and answer these and other related topological and dynamical questions.

The rest of the thesis is organized as follows. In the second chapter we set the definition of the generalized Fock spaces and establish some interesting preliminary results that will be used in our subsequent chapters apart from being interest of their own. The new results are collected from the papers [59, 60] where we characterized boundedness and compactness of embedding mapping $I_d : \mathcal{F}_\varphi^\infty \to L^p(d\mu)$. It is also shown that the set of complex polynomials is dense in $\mathcal{F}_\varphi^p$ for all $0 < p < \infty$ or $p = 0$. Density of the polynomials plays a paramount roll to study dynamical and spectral properties of the composition operators.

The third chapter is devoted to our studies on topological properties of composition operators. Bounded and compact composition operators on the spaces $\mathcal{F}_\varphi^p$, $0 < p \leq \infty$ or $p = 0$ are completely identified. We, in particular, have shown that if $p \neq q$, then the operator $C_\psi : \mathcal{F}_\varphi^p \to \mathcal{F}_\varphi^q$ is bounded if and only if it is compact. This result shows a significance difference with the analogous result for the case when C_ψ acts between the classical Fock spaces or generalized Fock spaces where the weight functions grow slower than the Gaussian function. As a consequence, it is concluded that while the set of all compact composition operators $C_\psi : \mathcal{F}_\varphi^p \to \mathcal{F}_\varphi^q$, $p \neq q$ forms a connected component of the space of all bounded composition operators in norm topology, each non-compact bounded composition operator is an isolated point. We further described the compact difference, the Schatten class $\mathcal{S}_p(\mathcal{F}_\varphi^2)$, essential norm, hyponormal, and unitary composition operators. All of these results are published in [59, 60].

In the fourth chapter we focus on the dynamics of composition operator on the generalized Fock spaces. We first determined the spectrum of the operators and applied the result to study their supercyclicity and mean ergodicity properties. The chapter presents a detailed study of our results on power bounded, mean ergodic, and uniformly mean ergodic composition operators on $\mathcal{F}_\varphi^p$, $1 \leq p \leq \infty$. Cyclicity and supercyclicity of C_ψ on $\mathcal{F}_\varphi^p$ are also characterized. The results in this chapter are from our published papers [59, 60, 62].

In the last chapter we extended our works on the dynamical properties of the composition operators to weighted composition operators. There is an extensive literature regarding boundedness, compactness, connected component and isolated points, and Schatten class membership of weighted composition operators on various spaces of analytic functional spaces including Hardy spaces, Bergman spaces, Dirichelet spaces, and Fock–Sobolov spaces. See [13, 15, 16, 17, 45, 51, 54, 55, 68] and references theerein. Not much is known about the dynamical properties of these operators even on the classical Fock spaces setting. In this chapter, we determine spectrum and some dynamical properties of the operator on the classical Fock spaces. The results are collected from our preprint works [58, 61].

We close this introduction with a few words on notations. We denote the complex plane by $\mathbb{C}$, the Euclidean unit disc $\{z \in \mathbb{C} : |z| < 1\}$ by $\mathbb{D}$, and its boundary $\{z \in \mathbb{C} : |z| = 1\}$ by $\mathbb{T}$. The Euclidean disc with center at a, and radius r is denoted by $D(a, r)$.

The notation $u(z) \lesssim v(z)$ means that there is a constant C such that $u(z) \leq Cv(z)$ holds for all z in the set of a question, and we define $u(z) \gtrsim v(z)$ in analogous manner. We write $u(z) \simeq v(z)$ if both $u(z) \lesssim v(z)$ and $u(z) \gtrsim v(z)$. We use the notation $f(r) \asymp g(r), r \geq 0$ means that as $r \to \infty$, $f(r)$ is very close to constant multiple of $g(r)$.

We also denote the space of holomorphic functions on $\mathbb{C}$ by $\mathcal{H}(\mathbb{C})$ and for a bounded operator T on $\mathcal{H}(\mathbb{C})$, we denote its resolvent set by $\varrho(T) := \mathbb{C} - \sigma(T)$.

Chapter 2

Generalized Fock Spaces $\mathcal{F}^p_\varphi$

In this chapter we present some definitions and basic results on generalized Fock spaces induced by rapidly increasing weight functions. We refer to [12, 53, 59, 60] for more details.

We begin by setting as in [12, 53] the growth and smoothness conditions for the weight function. Let $\varphi : [0, \infty) \to [0, \infty)$ be a twice continuously differentiable function. We extend φ to the whole complex plane by setting $\varphi(z) = \varphi(|z|)$. We further assume that its Laplacian, $\Delta\varphi$, is positive and set a radial differentiable function

$$\tau(z) := \begin{cases} 1 & 0 \leq |z| < 1 \\ (\Delta\varphi(z))^{-1/2} & |z| \geq 1 \end{cases}$$

satisfying the admissibility conditions

$$\lim_{r\to\infty} \tau(r) = 0 \quad \text{and} \quad \lim_{r\to\infty} \tau'(r) = 0, \tag{2.0.1}$$

and there exists a constant $C > 0$ such that $\tau(r)r^C$ increases for large r or

$$\lim_{r\to\infty} \tau'(r) \log \frac{1}{\tau(r)} = 0.$$

There are many concrete examples of weight functions φ that satisfy the above smoothness and admissibility conditions. Some of them are presented below.

Example 2.0.1. The power functions $\varphi_\alpha(r) = r^\alpha,\ \alpha > 2$ has Laplacian $\Delta\varphi_\alpha(r) = (\alpha^2 - \alpha)r^{\alpha-2}$ and hence satisfies the smoothness and admissibility conditions.

Example 2.0.2. The exponential type functions such as $\varphi_\beta(r) = e^{\beta r},\quad \beta > 0$, has Laplacian $\Delta\varphi_\beta(r) = e^{\beta r}(\beta + \frac{\beta}{r})$, which also satisfies the condition.

Example 2.0.3. Supper exponential functions $\varphi(r) = e^{e^r}$ has Laplacian $\Delta\varphi(r) \asymp e^{2r+e^r}$ which also satisfies the smoothness and admissibility conditions.

Having set forth the conditions on φ, we may now define the associated generalized Fock spaces $\mathcal{F}^p_\varphi$ as spaces consisting of all entire functions f for which

$$\|f\|_{(\varphi,p)} = \begin{cases} \left(\int_{\mathbb{C}} |f(z)|^p e^{-p\varphi(z)} dA(z) \right)^{\frac{1}{p}} < \infty, & 0 < p < \infty \\ \sup_{z\in\mathbb{C}^n} |f(z)| e^{-p\varphi(z)} < \infty, & p = \infty \end{cases}$$

where dA denotes the usual Lebesgue area measure on $\mathbb{C}$. The growth of the weight function φ has resulted various structural differences between the spaces $\mathcal{F}^p_\varphi$ and the classical spaces $\mathcal{F}_p$ and generalized Fock spaces with weight grows slower than Gaussian weight function. As an instance, while the inclusion property holds on the classical Fock spaces $\mathcal{F}_p$ and generalized Fock spaces with weight grows slower than Gaussian weight function, it fails to hold in $\mathcal{F}^p_\varphi$. Moreover, an explicit expression for the reproducing kernel in the weighted space $\mathcal{F}^2_\varphi$ is still unknown.

We note that if φ satisfies the above smoothness and admissibility condition, there is a constant $k > 0$, such that for the associated τ we have

$$|\tau(z) - \tau(w)| \le k|z - w|, \text{ for } z, w \in \mathbb{C}. \tag{2.0.2}$$

Let

$$m_\tau = \frac{\min\{1, 1/k\}}{4} \tag{2.0.3}$$

where k is the constant in (2.0.2).

From Lemma 5 of [12], if $0 < \delta < m_\tau$ and $a \in \mathbb{C}$, there is $c > 0$ such that

$$\frac{1}{c}\tau(a) \leq \tau(z) \leq c\tau(a) \tag{2.0.4}$$

if $z \in D(a, \delta\tau(a))$.

For $0 < p < \infty$, we can rewrite the above definition in polar coordinates as follows.

$$\|f\|^p_{(\varphi,p)} = \int_{\mathbb{C}} |f(z)|^p e^{-p\varphi(z)} dA(z) = 2\pi \int_0^\infty \left[\int_0^{2\pi} |f(re^{it})|^p \frac{dt}{2\pi}\right] re^{-p\varphi(r)} dr.$$

For $0 < r < \infty$, if we write $M_p(f,r) = \left[\int_0^{2\pi} |f(re^{it})|^p \frac{dt}{2\pi}\right]^{\frac{1}{p}}$, then

$$\|f\|^p_{(\varphi,p)} = 2\pi \int_0^\infty M_p^p(f,r) re^{-p\varphi(r)} dr.$$

We define $M_\infty(f,r) := \sup_{|z|=r} |f(z)|$. For each $1 \leq p \leq \infty$, the function $M_p(f,r)$ is increasing for $r \in [0,\infty)$.

We also consider the subspace $\mathcal{F}^0_\varphi$ of $\mathcal{F}^\infty_\varphi$ defined by

$$\mathcal{F}^0_\varphi = \{f \in \mathcal{F}^\infty_\varphi : \lim_{|z|\to\infty} |f(z)|e^{-\varphi(z)} = 0\}.$$

This subspace is closed in $\mathcal{F}^\infty_\varphi$ and it contains the polynomials. By [6, Theorem 1.4], the space $\mathcal{F}^\infty_\varphi$ is canonically isomorphic to the bidual of $\mathcal{F}^0_\varphi$.

For $p = 2$, the space $\mathcal{F}^2_\varphi$ is a Hilbert space endowed with the inner product

$$\langle f, g\rangle = \int_{\mathbb{C}} f(z)\overline{g(z)} e^{-2\varphi(z)} dA(z).$$

The sequence $\{e_n\}$, where $e_n = \frac{z^n}{\|z\|_{(\varphi,2)}}$ is an orthonormal basis for $\mathcal{F}^2_\varphi$. As a result, the set of polynomials is dense in $\mathcal{F}^2_\varphi$.

By Lemma 7 of [12], for subharmonic functions φ and holomorphic function f, it holds a local pointwise estimate

$$|f(z)|^p e^{-\beta\varphi(z)} \lesssim \frac{1}{\sigma^2\tau(z)^2}\int_{D(z,\sigma\tau(z))}|f(w)|^p e^{-\beta\varphi(w)}dA(w) \tag{2.0.5}$$

for all finite exponent p, any real number β, and a small positive number σ. Thus by (2.0.5) the point evaluations are bounded linear functionals on $\mathcal{F}^p_\varphi$. Therefore, there exists a reproducing kernel function K_w such that

$$f(w) = \langle f, K_w\rangle = \int_{\mathbb{C}} f(z)\overline{K_w(z)}e^{-2\varphi(z)}dA(z).$$

Hence, the space $\mathcal{F}^2_\varphi$ is a reproducing kernel Hilbert space. An explicit expression for the kernel function is still an interesting open problem. However, by Corollary 8 of [12] the reproducing kernel K_w of $\mathcal{F}^2_\varphi$ satisfies the asymptotic estimation of the norm

$$\|K_w\|^2_{(\varphi,2)} \simeq \tau(w)^{-2}e^{2\varphi(w)}, \tag{2.0.6}$$

holds for all $w \in \mathbb{C}$.

The normalized reproducing kernels have been used as a sequence of test functions to study many operator theoretic properties on the classical Fock spaces setting. On few cases, the kernel functions will be also used on generalized Fock spaces. The next lemma provides an important property of the sequence of the kernel functions in $\mathcal{F}^2_\varphi$.

Lemma 2.0.4. *The normalized reproducing kernel $K_z/\|K_z\|_{(\varphi,2)}$ converges weakly to 0 in $\mathcal{F}^2_\varphi$ when $|z| \to \infty$.*

Proof. Since holomorphic polynomials are dense in $\mathcal{F}^2_\varphi$, it suffices to show that for any non-negative integer m

$$\left|\left\langle w^m, \frac{K_z}{\|K_z\|_{(\varphi,2)}}\right\rangle\right| = \frac{|z|^m}{\|K_z\|_{(\varphi,2)}} \to 0, \quad |z| \to \infty.$$

But this holds trivially as

$$\|K_z\|_{(\varphi,2)}^2 = \sum_{n=0} |e_n(z)|^2 = \sum_{n=0}^{\infty} \frac{|z|^{2n}}{\|z^n\|_{(\varphi,2)}^2},$$

which is a power series on $|z|^2$ with positive coefficients. □

When $p \neq 2$, the lack of an explicit expression for the normalized reproducing kernel function makes it difficult to use it as main tool in the study of dynamical and topological properties of composition operators. Thus, we use another family of test functions described below.

By Proposition A and Corollary 8 of [12] where the original idea comes from [8], for a sufficiently large positive number R, there exists a number $\eta(R)$ such that for any $w \in \mathbb{C}$ with $|w| > \eta(R)$, there exists an entire function $f_{(w,R)}$ such that

$$|f_{(w,R)}(z)|e^{-\varphi(z)} \leq C \min\left\{1, \left(\frac{\min\{\tau(w), \tau(z)\}}{|z-w|}\right)^{\frac{R^2}{2}}\right\} \tag{2.0.7}$$

for all z in $\mathbb{C}$, and for some constant C that depends on φ and R. In particular, when z belongs to $D(w, R\tau(w))$, the estimate becomes

$$|f_{(w,R)}(z)|e^{-\varphi(z)} \simeq 1. \tag{2.0.8}$$

Furthermore, the functions $f_{(w,R)}$ belong to $\mathcal{F}_\varphi^p$ for all p with norms estimated by

$$\|f_{(w,R)}\|_{(\varphi,p)}^p \simeq \tau(w)^2, \qquad \eta(R) \leq |w|. \tag{2.0.9}$$

When $p = \infty$, from (2.0.7) and (2.0.8) we deduce that $f_{(w,R)}$ belong to $\mathcal{F}_\varphi^\infty$ and

$$\|f_{(w,R)}\|_{(\varphi,\infty)} \simeq 1. \tag{2.0.10}$$

2.1 The (p,q) Fock–Carleson measures

Identifying some basic properties of embedding maps or Carleson measures has been a useful tool in studying bounded and compact properties of operators on functional spaces.

A finite measure μ on $\mathbb{C}$ is said to be a *(p,q) Fock–Carleson measure* if the space $\mathcal{F}_\varphi^p$ is a subset of $L^q(\mu)$. By the closed graph theorem, this is equivalent to the existence of a positive constant C such that

$$\|f\|_{L^q(\mu)} \leq C\|f\|_{(\varphi,p)} \tag{2.1.1}$$

for all $f \in \mathcal{F}_\varphi^p$. The inequality in (2.1.1) is equivalent to the boundedness of the embedding map $I_d : \mathcal{F}_\varphi^p \to L^q(\mu)$. Observe that, if a weight φ grows slower than Gaussian weight, we have $\mathcal{F}_\varphi^p \subseteq \mathcal{F}_p$. From this and the definition, if a weight φ grows slower than Gaussian weight and μ is a (p,q) Fock–Carleson measure for $\mathcal{F}_p$ then it is Carleson measure for $\mathcal{F}_\varphi^p$ also. The (p,q) Fock–Carleson measure on $\mathcal{F}_p$ is characterized in [11] by making slight improvement of results contained in [70]. For fast growing weight φ in our setting and finite exponent, the bounded and compact properties of such maps $I_d : \mathcal{F}_\varphi^q \to L^p(\mu)$ were studied in [12]. In this section we prove the analogous results for $q = \infty$, which is one of the main tool to prove Theorem 3.1.1 latter.

Theorem 2.1.1. *Let $0 < p < \infty$ and μ be a finite Borel measure on $\mathbb{C}$, and $I_d : \mathcal{F}_\varphi^\infty \to L^p(d\mu)$ be the embedding map. Then the following statements are equivalent.*

(i) $I_d : \mathcal{F}_\varphi^\infty \to L^p(d\mu)$ is bounded;

(ii) $I_d : \mathcal{F}_\varphi^\infty \to L^p(d\mu)$ is compact;

(iii) For some $\sigma > 0$, the function $\mathcal{G}_{(\mu,p,\varphi)}$ belongs to $L^1(dA)$ where

$$\mathcal{G}_{(\mu,p,\varphi)}(z) := \frac{1}{\tau(z)^2}\int_{D(z,\sigma\tau(z))} e^{p\varphi(w)}d\mu(w). \tag{2.1.2}$$

For the proof of the theorem, we may first collect some background materials. We consider the Rademacher sequence of functions r_n as defined on [40] as

$$r_0(t) = \begin{cases} 1, & 0 \leq t < \frac{1}{2} \\ -1, & \frac{1}{2} \leq t < 1 \end{cases}$$

and

$$r_n(t) = r_0(2^n t) \text{ if } n = 1, 2, 3, \ldots.$$

Given a sequence $\{a_n\} \in \ell^2$, we define the function $\varphi(t) = \sum_n a_n r_n(t), 0 \leq t \leq 1$ which is well defined almost everywhere. These functions satisfy the following estimate for all $0 < p < \infty$.

$$\left(\int_0^1 |\varphi(t)|^p \right)^{1/p} \simeq \left(\sum_{n=1}^{\infty} |a_n|^2 \right)^{1/2}. \tag{2.1.3}$$

The estimate in (2.1.3) is known as *Khinchine's inequality.*
The following lemmas collected from [12, 53], are fundamental in the proof of the result.

Lemma 2.1.2. *[[12], Lemma 6] Let $t : \mathbb{C} \to (0, \infty)$ be a continuous function which satisfies $|t(z) - t(w)| \leq \frac{1}{4}|z - w|$ for all z and w in $\mathbb{C}$. We also assume that $t(z) \to 0$ when $|z| \to \infty$. Then there exists a sequence of points z_j in $\mathbb{C}$ satisfying the following conditions.*

(i) $z_j \notin D(z_k, t(z_k)), \quad j \neq k;$ *(ii)* $\mathbb{C} = \bigcup_j D(z_j, t(z_j));$

(iii) $\bigcup_{z \in D(z_j, t(z_j))} D(z, t(z)) \subset D(z_j, 3t(z_j));$

(iv) The sequence $D(z_j, 3t(z_j))$ is a covering of $\mathbb{C}$ with finite multiplicity $N_{\max}$.

Note that, from (2.0.2) and $0 < \delta < m_\tau$, where m_τ is defined in (2.0.3) we have

$$|\delta\tau(z) - \delta\tau(w)| \leq \frac{1}{4}|z - w|, \text{ for } z, w \in \mathbb{C}.$$

From the Lemma 2.1.2, we can formulate the following lemma.

Lemma 2.1.3. *There exists a sequence of points z_j in $\mathbb{C}$ satisfying the following conditions.*

(i) $z_j \notin D(z_k, \delta\tau(z_k)),\quad j \neq k;$ *(ii)* $\mathbb{C} = \bigcup_j D(z_j, \delta\tau(z_j));$

(iii) $\bigcup_{z\in D(z_j,\delta\tau(z_j))} D(z, t(z)) \subset D(z_j, 3\delta\tau(z_j));$

(iv) The sequence $D(z_j, 3\delta\tau(z_j))$ is a covering of $\mathbb{C}$ with finite multiplicity $N_{\max}$.

Another ingredient for the proof of the theorem is stated as the following lemma.

Lemma 2.1.4. *[[53], Lemma 2.4] Let R be sufficiently large number and $\eta(R)$ be as before. If $\{z_k\}$ is the covering sequence from Lemma 2.1.2, then*

$$F = \sum_{z_k:|z_k|>\eta(R)} a_k f_{(z_k,R)}$$

belongs to $\mathcal{F}^\infty_\varphi$ for every ℓ^∞ sequence $\{a_k\}$, and also $\|F\|_{(\varphi,\infty)} \lesssim \|a_k\|_{\ell^\infty}$.

Proof of Theorem 2.1.1. Since (ii) obviously implies (i), we plan to show $(i) \Rightarrow (iii)$ and $(iii) \Rightarrow (ii)$. We begin with $(iii) \Rightarrow (ii)$. It suffices to show $\|f_n\|_{L^p(d\mu)} \to 0$ as $n \to \infty$ for each uniformly bounded sequence f_n in $\mathcal{F}^\infty_\varphi$ that converges to zero uniformly on compact subsets of $\mathbb{C}$. Applying (2.0.5) and Fubini's theorem, for some $r > 0$:

$$\begin{aligned}\int_{\{z\in\mathbb{C}:|z|>r\}} |f_n(z)|^p d\mu(z) &\lesssim \int_{\{z\in\mathbb{C}:|z|>r/2\}} \int_{D(z,(\sigma/2)\tau(z))} \frac{e^{p\varphi(z)}|f_n(w)|^p}{\tau(z)^2 e^{p\varphi(w)}} dA(w)d\mu(z)\\ &= \int_{D(z,(\sigma/2)\tau(z))} \int_{\{z\in\mathbb{C}:|z|>r/2\}} \frac{e^{p\varphi(z)}}{\tau(z)^2}|f_n(w)|^p e^{-p\varphi(w)} d\mu(z)dA(w)\end{aligned}$$

Since $\tau(z)$ is a radially decreasing function and hence $D(z,(\sigma/2)\tau(z)) \subset \{|w| > r/2\}$ for $|z| > r$, the last double integral above is bounded by

$$\begin{aligned}&\int_{\{w\in\mathbb{C}:|w|>r/2\}} |f_n(w)|^p e^{-p\varphi(w)} \frac{1}{\tau(w)^2} \int_{D(w,\delta\tau(w)} e^{p\varphi(z)} d\mu(z)dA(w)\\ &\qquad \leq \|f_n\|^p_{(\varphi,\infty)} \int_{\{z\in\mathbb{C}:|w|>r\}} \left(\frac{1}{\tau(w)^2} \int_{D(w,\sigma\tau(w))} e^{p\varphi(z)} d\mu(z)\right) dA(w)\end{aligned}$$

from which and the assumption in (iii) we can make the last right-hand side double integral above as small as we wish. The remaining integral over the compact set satisfies

$$\lim_{n\to\infty}\int_{\{z\in\mathbb{C}:|z|\leq r\}}|f_n(z)|^p d\mu(z)=0$$

and completes the proof of the first assertion.

To prove $(i)\Rightarrow(iii)$, for each ℓ^∞ sequence (a_k) we consider the function

$$F_t=\sum_{z_k:|z_k|\geq\eta(R)}a_k r_k(t)f_{(z_k,R)}$$

where $r_k(t),\ 0<t<1$ is a sequence of Rademacher functions.

Then by Lemma 2.1.4 and our boundedness assumption

$$\int_{\mathbb{C}}|F_t(z)|^p d\mu(z)=\|I_dF_t\|^p_{L^p(\mu)}\leq\|I_d\|^p\|F_t\|^p_{(\varphi,\infty)}\lesssim\|I_d\|^p\|(a_k)\|^p_{\ell^\infty}.$$

Integrating with respect to t,

$$\int_0^1\int_{\mathbb{C}}|F_t(z)|^p d\mu(z)\lesssim\|I_d\|^p\|(a_k)\|^p_{\ell^\infty}.$$

Applying Fubini's theorem

$$\int_{\mathbb{C}}\int_0^1|F_t(z)|^p d\mu(z)\lesssim\|I_d\|^p\|(a_k)\|^p_{\ell^\infty},$$

and by using Khinchine's inequality (2.1.3) we also obtain

$$\int_{\mathbb{C}}\Bigg(\sum_{z_k:|z_k|\geq\eta(R)}|a_k|^2|f_{(z_k,R)}|^2\Bigg)^{p/2}d\mu(z)\lesssim\|I_d\|^p\|(a_k)\|^p_{\ell^\infty}.$$

This together with the finite multiplicity N_{max} of the covering sequence $D(z_k,3\delta\tau(z_k))$, where $\delta \in (0, m_\tau)$ for m_τ defined in (2.0.3), and estimate (2.0.8) imply

$$\begin{aligned}\sum_{z_k:|z_k|\geq\eta(R)} |a_k|^p \int_{D(z_k,3\delta\tau(z_k))} e^{p\varphi(z)}d\mu(z) &\lesssim \sum_{z_k:|z_k|\geq\eta(R)} |a_k|^p \int_{D(z_k,3\delta\tau(z_k))} |f_{(z_k,R)}|^p d\mu(z)\\ &= \int_{\mathbb{C}} \sum_{z_k:|z_k|\geq\eta(R)} |a_k|^p |f_{(z_k,R)}|^p \chi_{(D(z_k,3\tau(z_k)))}(z)d\mu(z)\\ &\lesssim \max\{1, N_{\max}^{1-\frac{p}{2}}\} \int_{\mathbb{C}} \Bigg(\sum_{z_k:|z_k|\geq\eta(R)} |a_k|^2 |f_{(z_k,R)}|^2 \Bigg)^{p/2} d\mu(z) \lesssim \|(a_k)\|_{\ell^\infty}^p.\end{aligned}$$

Setting in particular $a_k = 1$ for all k in the above series of estimates we obtain

$$\sum_{z_k:|z_k|\geq\eta(R)} \int_{D(z_k,3\tau(z_k))} e^{p\varphi(z)}d\mu(z) \lesssim 1. \tag{2.1.4}$$

Now we choose a positive number $\rho \geq \eta(R)$ such that whenever z_k of the covering sequence belongs to $D(0, \eta(R))$, then $D(z_k, \sigma\tau(z_k))$ is contained in $D(0, \rho)$. On the other hand, by (2.0.4) there exists a positive constant c with

$$\frac{1}{c}\tau(w) \leq \tau(z) \leq c\tau(w). \tag{2.1.5}$$

Then, applying this and (2.1.4)

$$\begin{aligned}&\int_{\{|z|\geq\rho\}} \frac{1}{\tau(z)^2} \int_{D(z,\tau(z))} e^{p\varphi(w)}d\mu(w)dA(z)\\ &\qquad\leq \sum_{z_k:|z_k|\geq\eta(R)} \int_{D(z_k,\tau(z_k))} \frac{1}{\tau(z)^2} \int_{D(z,\tau(z))} e^{p\varphi(w)}d\mu(w)dA(z)\\ &\qquad\qquad\lesssim \sum_{z_k:|z_k|\geq\eta(R)} \int_{D(z_k,3\tau(z_k))} e^{p\varphi(w)}d\mu(w) \lesssim 1.\end{aligned}$$

The remaining piece of integral over the set $\{z \in \mathbb{C} : |z| < \rho\}$ is finite since by

(2.0.8) and (2.0.9)

$$\begin{aligned}\int_{\{|z|<\rho\}} \frac{1}{\tau(z)^2} \int_{D(z,\tau(z))} & e^{p\varphi(w)} d\mu(w) dA(z) \\ & \simeq \int_{\{|z|<\rho\}} \frac{1}{\tau(z)^2} \int_{D(z,\tau(z))} |f_{(z,R)}(w)|^p d\mu(w) dA(z)\end{aligned}$$

$$\lesssim \int_{\{|z|<\rho\}} \frac{1}{\tau(z)^2} \|I_d\|^p \|f_{(z,R)}\|^p_{(\varphi,\infty)} dA(z) \simeq \int_{\{|z|<\rho\}} \frac{1}{\tau(z)^2} dA(z) \lesssim \frac{\rho^2}{\tau(\rho)^2} < \infty,$$

and completes the proof of the theorem.
As a consequence of Theorem 2.1.1 and Theorem 1 in [12], unlike the classical setting the inclusion property fails to hold. That is $\mathcal{F}^p_\varphi \nsubseteq \mathcal{F}^q_\varphi$ and $\mathcal{F}^q_\varphi \nsubseteq \mathcal{F}^p_\varphi$ for all $0 < p, q \leq \infty$.

2.2 Density of complex polynomials

The next result about the density of polynomials in $\mathcal{F}^p_\varphi$ is an important tool in proving various results in the sequel.

Theorem 2.2.1. *Suppose* $0 < p < \infty$ *or* $p = 0$*, and* $f \in \mathcal{F}^p_\varphi$*. Then there is a sequence of polynomials* $\{P_n\}$ *such that* $\|P_n - f\|_{(\varphi,p)} \to 0$ *as* $n \to \infty$*. This assertion fails to hold if* $p = \infty$*.*

Proof. If $p = 0$, the result follows from Theorem 1.4 in [6]. Assume now that $0 < p < \infty$. For $f \in \mathcal{F}^p_\varphi$ and $0 < r < 1$, define a sequence of dilation functions f_r by $f_r(z) = f(rz)$. Then it suffices to show that $\|f_r - f\|_{(\varphi,p)} \to 0$ as $r \to 1^-$ and $\|f_r - P_n\|_{(\varphi,p)} \to 0$ as $n \to \infty$ where P_n is a sequence of complex polynomials. To show the first, we may compute

$$\begin{aligned}\|f_r\|^p_{(\varphi,p)} = \int_{\mathbb{C}} |f(rz)|^p e^{-p\varphi(z)} dA(z) &= \frac{1}{r^2} \int_{\mathbb{C}} |f(w)|^p e^{-p\varphi(r^{-1}w)} dA(w) \\ &= \frac{1}{r^2} \int_{\mathbb{C}} |f(w)|^p e^{-p\varphi(w)} e^{-p(\varphi(r^{-1}w)+\varphi(w))} dA(w).\end{aligned}$$

Since φ is an increasing radial function and $0 < r < 1$ we have $e^{-p(\varphi(r^{-1}w)-\varphi(w))} \leq$

1 for all $w \in \mathbb{C}$. Applying Lebesgue dominated convergence theorem,

$$\lim_{r\to 1^-} \|f_r\|^p_{(\varphi,p)} = \lim_{r\to 1^-} \frac{1}{r^2} \int_{\mathbb{C}} |f(w)|^p e^{-p\varphi(w)} \left(e^{-p(\varphi(r^{-1}w)-\varphi(w))} \right) dA(w) = \|f\|^p_{(\varphi,p)},$$

showing that $\|f_r\|^p_{(\varphi,p)} \to \|f\|^p_{(\varphi,p)}$ and $f_r(z) \to f(z)$ as $r \to 1^-$.
Therefore,

$$\lim_{r\to 1^-} \|f_r - f\|_{(\varphi,p)} = 0. \tag{2.2.1}$$

Next, we fix some $r \in (0,1)$, $\alpha \in (r^2, \frac{1}{2})$ and proceed to show that $f_r \in \mathcal{F}^2_{(\varphi,\alpha)}$ and $\mathcal{F}^2_{(\varphi,\alpha)} \subset \mathcal{F}^p_\varphi$ where

$$\mathcal{F}^2_{(\varphi,\alpha)} := \left\{ f \text{ entire : } \|f\|^2_{(2,\alpha)} = \int_{\mathbb{C}} |f(z)|^2 e^{-2\alpha\varphi(z)} dA(z) < \infty \right\}.$$

To prove the first, we may apply (2.0.5) and estimate

$$\|f_r\|^2_{(2,\alpha)} = \int_{\mathbb{C}} |f(rw)|^2 e^{-2\alpha\varphi(w)} dA(w) \lesssim \|f\|^2_{(\varphi,p)} \int_{\mathbb{C}} \frac{1}{\tau(rw)^{\frac{4}{p}}} e^{2\varphi(wr)-2\alpha\varphi(w)} dA(w).$$

By definition of τ and φ, we also observe that

$$\frac{1}{\tau(rw)^{\frac{4}{p}}} \lesssim e^{\varphi(wr)} \text{ as } |w| \to \infty.$$

Taking this into account and the fact that $\alpha > r^2$ we further estimate

$$\begin{aligned}\int_{\mathbb{C}} \frac{1}{\tau(rw)^{\frac{4}{p}}} e^{2\varphi(wr)-2\alpha\varphi(w)} dA(w) &\lesssim \int_{\mathbb{C}} e^{4\varphi(wr)-2\alpha\varphi(w)} dA(w) \\ &\lesssim \int_{\mathbb{C}} e^{2r^2\varphi(w)-2\alpha\varphi(w)} dA(w) < \infty,\end{aligned}$$

here we used the fact that φ grows faster than the classical function $|z|^2/2$ and hence $\varphi(rw) \lesssim \frac{r^2}{2}\varphi(w)$ whenever $|w| \to \infty$.
For the inclusion property, we consider $h \in \mathcal{F}^2_{(\varphi,\alpha)}$ and applying (2.0.5) again and

proceed to estimate

$$\begin{aligned}\int_{\mathbb{C}} |h(z)|^p e^{-p\varphi(z)} dA(z) &\lesssim \|h\|_{(2,\alpha)}^p \int_{\mathbb{C}} \frac{e^{p\alpha\varphi(z)-p\varphi(z)}}{\tau(z)^p} dA(z) \\ &\leq \|h\|_{(2,\alpha)}^p \int_{\mathbb{C}} e^{2p\alpha\varphi(z)-p\varphi(z)} dA(z) \lesssim \|h\|_{(2,\alpha)}^p.\end{aligned}$$

Now, since the set of all holomorphic complex polynomials is dense in the Hilbert space $\mathcal{F}^2_{(\varphi,\alpha)}$, taking P_n be the n^{th} Taylor polynomial of f_r, we deduce from the inclusion property that

$$\|f_r - P_n\|_{(\varphi,p)} \leq C\|f_r - P_n\|_{(2,\alpha)} \to 0$$

as $n \to \infty$. From this and (5.1.2), the result follows.
For $p = \infty$, since the set of polynomials is dense in $\mathcal{F}^0_\varphi$, and $\mathcal{F}^0_\varphi$ is a closed proper subset of $\mathcal{F}^\infty_\varphi$, the set can not be dense in $\mathcal{F}^\infty_\varphi$.

Chapter 3

Topological Properties of Composition Operators on $\mathcal{F}_\varphi^p$

In the first part of this chapter, we present results that characterize bounded and compact composition operators on $\mathcal{F}_\varphi^p,\ 0 < p \leq \infty$. Our result shows that if $p \neq q$, then the operator $C_\psi : \mathcal{F}_\varphi^p \to \mathcal{F}_\varphi^q$ is compact if and only if it is bounded, which is the main novelty here. When $p = q$, the result simply generalizes those results on classical Fock space setting.

In the remaining sections of the chapter we study various topological properties which include the compact differences, isolated and essentially isolated points, and connected components of the space of bounded composition operators under the operator norm topology. Unitary composition operator on $\mathcal{F}_\varphi^2$ is characterized. Furthermore, we prove that only normal composition operators are hyponormal on the space $\mathcal{F}_\varphi^2$.

3.1 Bounded and compact composition operators on $\mathcal{F}_\varphi^p$

In this section we present our result on bounded and compact composition operators acting between the generalized Fock spaces whose inducing weight functions grow faster than Gaussian weight function.

Theorem 3.1.1. *Let* $0 < p, q \leq \infty$ *and* ψ *be a non-constant holomorphic map on the complex plane* $\mathbb{C}$. *If* $p \neq q$, *then the following statements are equivalent.*

(a) $C_\psi : \mathcal{F}_\varphi^p \to \mathcal{F}_\varphi^q$ *is bounded;*

(b) $C_\psi : \mathcal{F}_\varphi^p \to \mathcal{F}_\varphi^q$ *is compact;*

(c) $\psi(z) = az + b$ *for some complex numbers* a *and* b *such that* $|a| < 1$.

As compared to other generalized Fock spaces, this equivalence works only when the Laplacian of the weight function is unbounded over the complex plane. It is rather interesting to notice that the classical Gaussian weight function (up to scalar multiple) is a cut-off weight for the equivalency. Moreover, when the weight function φ grows at most the classical weight case and $\limsup_{|z|\to\infty} \varphi^{'}(z) = \infty$, result in [24] shows that C_ψ is bounded on $\mathcal{F}_\varphi^2$ if and only if $\psi(z) = az + b$ with $|a| \leq 1$, and $b = 0$ whenever $|a| = 1$. Applying (p,q) Fock-Carleson measure on $\mathcal{F}_p$, which is (p,q) Fock-Carleson measure on $\mathcal{F}_\varphi^p$ also, the same conclusion holds when $C_\psi : \mathcal{F}_\varphi^p \to \mathcal{F}_\varphi^q$, $0 < p \leq q \leq \infty$. Compactness is obtained if and only if $|a| < 1$. When the weight function φ satisfies the growth condition $\limsup_{|z|\to\infty} \varphi^{'}(z) < \infty$ another result in [24] shows that C_ψ is bounded on $\mathcal{F}_\varphi^p$ if and only if $\psi(z) = az + b$ with $|a| \leq 1$, and compactness is achieved if and only if $|a| < 1$.

Proof. We may first reformulate the boundedness and compactness problems of C_ψ in terms of embedding maps between $\mathcal{F}_\varphi^p$ and $\mathcal{F}_\varphi^q$. We set a pullback measure $\mu_{(\psi,q)}$ on $\mathbb{C}$ as

$$\mu_{(\psi,q)}(E) = \int_{\psi^{-1}(E)} e^{-q\varphi(w)} dA(w) \tag{3.1.1}$$

for every Borel subset E of $\mathbb{C}$. Then we observe

$$\|C_\psi f\|_{\varphi,q}^q = \int_{\mathbb{C}} |f(\psi(z))|^q e^{-q\varphi(z)} dA(z) = \int_{\mathbb{C}} |f(z)|^q d\mu_{(\psi,q)}(z). \tag{3.1.2}$$

From this, it follows that $C_\psi : \mathcal{F}^p_\varphi \to \mathcal{F}^q_\varphi$ is bounded if and only if the embedding map $I_d : \mathcal{F}^p_\varphi \to L^q(\mu_{(\psi,q)})$ is bounded. To study this reformulation further, we may consider following two cases:

Case 1: $0 < q < p \leq \infty$. By [12, Theorem 1] and Theorem 2.1.1, boundedness or compactness of I_d holds if and only if for some $\delta > 0$ and $0 < q < \infty$ the function

$$\mathcal{T}(z) := \frac{1}{\tau(z)^2} \int_{D(z,\delta\tau(z))} e^{q\varphi(w)} d\mu_{(\psi,q)}(w) = \frac{1}{\tau(z)^2} \int_{D(z,\delta\tau(z))} \frac{e^{q\varphi(w)}}{e^{q\varphi(\psi^{-1}(w))}} dA(\psi^{-1}(w))$$

belongs to $L^{\frac{p}{p-q}}(\mathbb{C}, dA)$ if $0 < p < \infty$, and belongs to $L(\mathbb{C}, dA)$ if $p = \infty$. We plan to show that this holds if and only if ψ has the form $\psi(z) = az + b$ with $|a| < 1$. If $0 < q < p < \infty$, assuming the latter and applying Hölder's inequality

$$\begin{aligned} \int_{\mathbb{C}} |\mathcal{T}(z)|^{\frac{p}{p-q}} dA(z) &= \int_{\mathbb{C}} \left(\frac{1}{\tau(z)^2} \int_{D(z,\delta\tau(z))} \frac{e^{q\varphi(w)}}{e^{q\varphi(\psi^{-1}(w))}} dA(\psi^{-1}(w)) \right)^{\frac{p}{p-q}} dA(z) \\ &\lesssim \int_{\mathbb{C}} \tau(z)^{-2} \int_{D(z,\delta\tau(z))} \frac{e^{\frac{qp}{p-q}\varphi(w)}}{e^{\frac{qp}{p-q}\varphi(\psi^{-1}(w))}} dA(\psi^{-1}(w)) dA(z) =: \mathcal{T}_1 \end{aligned}$$

Since $w \in D(z, \delta\tau(z))$, by inequality (2.0.4) there exists a positive constant c with

$$\frac{1}{c}\tau(w) \leq \tau(z) \leq c\tau(w).$$

Then, for any $\zeta \in D(z, \delta\tau(z))$

$$|\zeta - w| \leq |\zeta - z| + |z - w| \leq 2\delta\tau(z) \leq 2\delta c\tau(w) = \beta\tau(w), \quad \beta := 2\delta c.$$

This shows that $D(z, \delta\tau(z)) \subset D(w, \beta\tau(w))$ which together with Fubini's theorem and inequality (2.0.4) again imply

$$\begin{aligned}
\mathcal{T}_1 &= \int_{\mathbb{C}} \tau(z)^{-2} \int_{\mathbb{C}} \chi_{D(z,\delta\tau(z))}(w) \frac{e^{\frac{qp}{p-q}\varphi(w)}}{e^{\frac{qp}{p-q}\varphi(\psi^{-1}(w))}} dA(\psi^{-1}(w))dA(z)\\
&\leq \int_{\mathbb{C}} \frac{e^{\frac{qp}{p-q}\varphi(w)}}{e^{\frac{qp}{p-q}\varphi(\psi^{-1}(w))}} \left(\int_{\mathbb{C}} \chi_{D(w,\beta\tau(w))}(z)\tau(z)^{-2}dA(z)\right) dA(\psi^{-1}(w))\\
&= \int_{\mathbb{C}} \frac{e^{\frac{qp}{p-q}\varphi(w)}}{e^{\frac{qp}{p-q}\varphi(\psi^{-1}(w))}} \left(\int_{D(w,\beta\tau(w))} \tau(z)^{-2}dA(z)\right) dA(\psi^{-1}(w))\\
&\simeq \int_{\mathbb{C}} \frac{e^{\frac{qp}{p-q}\varphi(w)}}{e^{\frac{qp}{p-q}\varphi(\psi^{-1}(w))}} dA(\psi^{-1}(w)) < \infty.
\end{aligned}$$

For $p = \infty$ and $0 < q < \infty$, applying the same procedure as in the above proof, we get that

$$\begin{aligned}
\int_{\mathbb{C}} |\mathcal{T}(z)|dA(z) &= \int_{\mathbb{C}} \tau(z)^{-2} \int_{\mathbb{C}} \chi_{D(z,\delta\tau(z))}(w) \frac{e^{q\varphi(w)}}{e^{q\varphi(\psi^{-1}(w))}} dA(\psi^{-1}(w)dA(z)\\
&\leq \int_{\mathbb{C}} \frac{e^{q\varphi(w)}}{e^{q\varphi(\psi^{-1}(w))}} dA(\psi^{-1}(w)) < \infty.
\end{aligned}$$

On the other hand, if $\mathcal{T}$ is $L^{\frac{p}{p-q}}$ integrable over $\mathbb{C}$, then $C_\psi : \mathcal{F}^p_\varphi \to \mathcal{F}^q_\varphi$ is bounded, and applying C_ψ to the sequence of test functions $f_{(w,R)}$ and using a weaker version of the point estimate in (2.0.5)

$$\|f_{(w,R)}\|_{(\varphi,p)} \gtrsim \|C_\psi f_{(w,R)}\|_{(\varphi,q)} \gtrsim |f_{(w,R)}(\psi(z))|\tau(z)^{\frac{2}{q}}e^{-\varphi(z)}$$

for all points $w, z \in \mathbb{C}$. Setting, in particular, $w = \psi(z)$ and invoking the estimates in (2.0.8) and (2.0.9) gives

$$\tau(\psi(z))^{\frac{2}{p}} \gtrsim \tau(z)^{\frac{2}{q}} e^{\varphi(\psi(z))-\varphi(z)}. \tag{3.1.3}$$

Since ψ is a non-constant entire function, the left-hand side of (3.1.3) tends to zero as $|z| \to \infty$. So does the right-hand side and that happens only if

$$\sup_{z\in\mathbb{C}} e^{\varphi(\psi(z))-\varphi(z)} < \infty. \tag{3.1.4}$$

Similarly if $\mathcal{T}$ is in $L(\mathbb{C}, dA)$ then $C_\psi : \mathcal{F}_\varphi^\infty \to \mathcal{F}_\varphi^q$ is bounded, and applying C_ψ to the sequence of test functions $f_{(w,R)}$ and using the point estimate in (2.0.5)

$$1 \simeq \|f_{(w,R)}\|_{(\varphi,p)} \gtrsim \|C_\psi f_{(w,R)}\|_{(\varphi,q)} \gtrsim |f_{(w,R)}(\psi(z))|\tau(z)^{\frac{2}{q}} e^{-\varphi(z)}$$

Setting, in particular, $w = \psi(z)$ and invoking the estimates in (2.0.8) and (2.0.9) gives

$$1 \gtrsim \tau(z)^{\frac{2}{q}} e^{\varphi(\psi(z))-\varphi(z)}. \tag{3.1.5}$$

for all $z \in \mathbb{C}$. Since ψ is a non-constant entire function, the right-hand side of (3.1.5) should be bounded, and that happens only if

$$\sup_{z\in\mathbb{C}} e^{\varphi(\psi(z))-\varphi(z)} < \infty. \tag{3.1.6}$$

(3.1.4) and (3.1.6) hold only when $\varphi(\psi(z)) - \varphi(z)$ is uniformly bounded over $\mathbb{C}$ which further implies

$$\limsup_{|z|\to\infty} \frac{\varphi(\psi(z))}{\varphi(z)} \leq 1. \tag{3.1.7}$$

Since ψ is entire function it has power series expansion $\psi(z) = \sum_{n=0}^\infty a_n z^n$. Now, for each $\xi \in \mathbb{T}$ we define a function ψ_ξ by $\psi_\xi(\lambda) = \psi(\xi\lambda)$ for $\lambda > 0$. By (3.1.7) we have

$$\limsup_{\lambda\to\infty} \frac{\varphi(\psi_\xi(\lambda))}{\varphi(\lambda)} \leq 1.$$

Since $\psi_\xi(\lambda) = \sum_{n=0}^\infty (a_n\xi^n)\lambda^n$, and φ is a non constant radially increasing function,

$$\limsup_{\lambda\to\infty} \frac{\varphi(\psi_\xi(\lambda))}{\varphi(\lambda)} = \limsup_{\lambda\to\infty} \frac{\varphi(a_0 + a_1\xi\lambda + a_2\xi^2\lambda^2 + ...)}{\varphi(\lambda)} \leq 1$$

holds only if $a_n\xi^n = 0$ for all $n \geq 2$ and $\xi \in \mathbb{T}$, and hence $a_n = 0$ for all $n \geq 2$. Thus ψ has the form of $\psi(z) = az + b, |a| \leq 1$. We further claim that $|a| < 1$. If not, setting $\zeta = \psi^{-1}(w)$ and hence $w = \psi(\zeta) = a\zeta$, and $dA(\psi^{-1}(w)) = \frac{1}{|a|^2}dA(\zeta) = dA(\zeta)$, and using again the $L^{\frac{p}{p-q}}$ or L^1 integrability of $\mathcal{T}$

$$\begin{aligned}\int_{\mathbb{C}} |\mathcal{T}(z)|^{\frac{p}{p-q}} dA(z) &= \int_{\mathbb{C}} \tau(z)^{-\frac{2p}{p-q}} \bigg(\int_{D(z,\delta\tau(z))} \frac{e^{q\varphi(w)}}{e^{q\varphi(\psi^{-1}(w))}} dA(\psi^{-1}(w)) \bigg)^{\frac{p}{p-q}} dA(z) \\ &\gtrsim \int_{\mathbb{C}} \tau(z)^{-\frac{2p}{p-q}} \bigg(\int_{D(z/a,\delta\tau(z))} \frac{e^{q\varphi(aw)}}{e^{q\varphi(w)}} dA(w) \bigg)^{\frac{p}{p-q}} dA(z) \\ &= \int_{\mathbb{C}} \tau(z)^{-\frac{2p}{p-q}} \tau(z/a)^{\frac{2p}{p-q}} dA(z) = \infty.\end{aligned}$$

Similarly

$$\begin{aligned}\int_{\mathbb{C}} |\tau(z)| dA(z) &= \int_{\mathbb{C}} \tau(z)^{-2} \bigg(\int_{D(z,\delta\tau(z))} \frac{e^{q\varphi(w)}}{e^{q\varphi(\psi^{-1}(w))}} dA(\psi^{-1}(w)) \bigg) dA(z) \\ &= \int_{\mathbb{C}} \tau(z)^{-2} \bigg(\int_{D(\frac{z}{a},\delta\tau(z))} \frac{e^{q\varphi(a\zeta)}}{e^{q\varphi(\zeta)}} dA(\zeta) \bigg) dA(z) \\ &= \int_{\mathbb{C}} \tau(z)^{-2} \tau(z)^2 dA(\zeta) dA(z) = \infty.\end{aligned}$$

which is a contradiction whenever $|a| = 1$.

Case 2: Assume $0 < p < q \leq \infty$. Here it suffices to show $(a) \Rightarrow (c)$ and $(c) \Rightarrow (b)$, since compactness obviously implies boundedness.

Assume first $0 < p < q < \infty$. Invoking the reformulation in (3.1.2) again, $C_\psi : \mathcal{F}_\varphi^p \to \mathcal{F}_\varphi^q$ is bounded if and only if the embedding map $I_d : \mathcal{F}_\varphi^p \to L^q(\mu_{(\psi,q)})$ is bounded. By Theorem 1 in [12], the map $I_d : \mathcal{F}_\varphi^p \to L^q(\mu_{(\psi,q)})$ is bounded if and only if for some $\delta > 0$,

$$\sup_{w \in \mathbb{C}} \frac{1}{\tau(w)^{2q/p}} \int_{D(w,\delta\tau(w))} e^{q\varphi(z)} d\mu_{(\psi,q)}(z) < \infty.$$

Using (3.1.1), we may rewrite this condition again as

$$I := \sup_{w\in\mathbb{C}} \frac{1}{\tau(w)^{2q/p}} \int_{D(w,\delta\tau(w))} e^{q\varphi(z)} d\mu_{(\psi,q)}(z)$$
$$= \sup_{w\in\mathbb{C}} \frac{1}{\tau(w)^{2q/p}} \int_{D(w,\delta\tau(w))} e^{q(\varphi(z)-\varphi(\psi^{-1}(z))} dA(\psi^{-1}(z)) < \infty. \tag{3.1.8}$$

Assume that (a) holds. Then (3.1.8) holds and show that $\psi(z) = az + b$ for some $|a| < 1$. Applying (2.0.5) and estimating further on the right-hand side of (3.1.8) gives

$$I \gtrsim \tau(\psi(w))^{\frac{2p-2q}{p}} e^{q\left(\varphi((\psi(w)))-\varphi(w)\right)}$$

for all w in $\mathbb{C}$ which implies

$$\tau(\psi(w))^{2\frac{(q-p)}{p}} \gtrsim e^{q\left(\varphi((\psi(w)))-\varphi(w)\right)}. \tag{3.1.9}$$

We claim that

$$\limsup_{|w|\to\infty}(\varphi(\psi(w)) - \varphi(w)) < 0.$$

If not, then there exists a sequence $w_j \in \mathbb{C}$ such that $|w_j| \to \infty$ as $j \to \infty$ and

$$\limsup_{j\to\infty} \varphi(\psi(w_j)) - \varphi(w_j)) \geq 0.$$

This along with (3.1.9) and applying the admissibility assumptions on (2.0.1), and the fact that ψ is a non-constant entire function, we get

$$0 = \limsup_{j\to\infty} \tau(\psi(w_j))^{2\frac{(q-p)}{p}} \gtrsim \limsup_{j\to\infty} e^{q\left(\varphi((\psi(w_j)))-\varphi(w_j)\right)}.$$
$$= e^{\limsup_{j\to\infty} q\left(\varphi((\psi(w_j)))-\varphi(w_j))\right)} \geq 1,$$

which is a contradiction. By the growth assumption on φ we see that $\psi(z) = az+b$

for some a, b in $\mathbb{C}$ and $|a| < 1$.

Next, we assume that ψ has the above linear form with $|a| < 1$, and proceed to show that C_ψ is a compact map. Using the preceding embedding formulation and Theorem 1 in [12], $C_\psi : \mathcal{F}_\varphi^p \to \mathcal{F}_\varphi^q$ is compact if and only if

$$\lim_{|w|\to\infty} \frac{1}{\tau(w)^{2q/p}} \int_{D(w,\delta\tau(w))} e^{q\varphi(z)-q\varphi(\psi^{-1}(z))} dA(\psi^{-1}(z)) = 0. \tag{3.1.10}$$

Since $|a| < 1$, the integrand above is a decaying function. Thus,

$$\begin{aligned} &\frac{1}{\tau(w)^{2q/p}} \int_{D(w,\delta\tau(w))} e^{q\varphi(z)-q\varphi(\psi^{-1}(z))} dA(\psi^{-1}(z)) \\ \lesssim\ & \frac{\tau(w)^2}{\tau(w)^{2q/p}} e^{q\varphi(w)-q\varphi(\psi^{-1}(w))} = \tau(az+b)^{\frac{2p-2q}{p}} e^{q\varphi(az+b)-q\varphi(z))}. \end{aligned} \tag{3.1.11}$$

By definition of φ and τ, we notice that the last quantity in (3.1.11) tends to zero as $|w| \to \infty$ and hence (3.1.10) holds. Thus we proved $(a) \Rightarrow (c)$ and $(c) \Rightarrow (b)$ for the case $0 < p < q < \infty$.

Next we assume that $q = \infty$ and $0 < p < \infty$. If (a) holds, then using the sequence of test functions $f_{(w,R)}$ described in (2.0.7),

$$\begin{aligned} \|C_\psi\| &\geq \sup_{\{w:|w|>\eta(R)\}} \frac{\|C_\psi f_{(w,R)}\|_{(\varphi,\infty)}}{\|f_{(w,R)}\|_{(\varphi,p)}} \\ &\simeq \sup_{\{w:|w|>\eta(R)\}} \frac{1}{\tau(w)^{2/p}} \sup_{z\in\mathbb{C}} |f_{(w,R)}(\psi(z))| e^{-\varphi(z)} \\ &\geq \sup_{\{w:|w|>\eta(R)\}} \frac{1}{\tau(w)^{2/p}} \sup_{z\in D(w,R\tau(w))} |f_{(w,R)}(\psi(z))| e^{-\varphi(z)} \end{aligned}$$

for all $z \in \mathbb{C}$. Taking in particular $w = \psi(z)$ and eventually applying (2.0.8)

$$\begin{aligned} \|C_\psi\| &\geq \sup_{\{\psi(z):|\psi(z)|>\eta(R)\}} \frac{1}{\tau(\psi(z))^{2/p}} |f_{(\psi(z),R)}(\psi(z))| e^{-\varphi(\psi(z)} e^{\varphi(\psi(z))-\varphi(z)} \\ &\simeq \sup_{\{\psi(z):|\psi(z)|>\eta(R)\}} \frac{e^{\varphi(\psi(z))-\varphi(z)}}{\tau(\psi(z))^{2/p}}. \end{aligned} \tag{3.1.12}$$

Since by our admissibility assumption $\frac{1}{(\tau(\psi(z)))^2}$ increases rapidly, the condition in

(3.1.12) holds only if

$$\limsup_{|\psi(z)|\to\infty} \left(\varphi(\psi(z)) - \varphi(z)\right) < 0$$

from which and repeating the arguments used in the proof of the first case, we must have $\psi(z) = az + b$ with $|a| < 1$.

To prove $(c) \Rightarrow (b)$ consider a uniformly bounded sequence $\{f_n\}_n$ in $\mathcal{F}_\varphi^p$ that converges to zero uniformly on compact subsets of $\mathbb{C}$.

Then for a positive number R

$$\|C_\psi f_n\|_{(\varphi,\infty)} \leq \sup_{\{z\in\mathbb{C}:|az+b|\leq R\}} |f_n(\psi(z))|e^{-\varphi(z)} + \sup_{\{z\in\mathbb{C}:|az+b|>R\}} |f_n(\psi(z))|e^{-\varphi(z)}$$

The first summand on the right-hand side above obviously converges to zero as $n \to \infty$. The second does the same, since applying (2.0.5) and the assumption $|a| < 1$,

$$\sup_{\{z\in\mathbb{C}:|az+b|>R\}} |f_n(\psi(z))|e^{-\varphi(z)} \lesssim \|f_n\|_{(\varphi,p)} \sup_{\{z\in\mathbb{C}:|az+b|>R\}} \frac{e^{\varphi(\psi(z))-\varphi(z)}}{\tau(az+b)^{2/p}} \to 0$$

as $R \to \infty$ which completes the required assertion that $C_\psi f_n$ converges to zero in $\mathcal{F}_\varphi^\infty$. Hence C_ψ is compact. □

The next result simply extends the classical results with the same form.

Theorem 3.1.2. *Let* $0 < p \leq \infty$ *or* $p = 0$*, and* ψ *be a nonconstant holomorphic map on the complex plane* $\mathbb{C}$*. Then* $C_\psi : \mathcal{F}_\varphi^p \to \mathcal{F}_\varphi^p$ *is*

(a) bounded if and only if $\psi(z) = az + b$ *for some complex numbers* a *and* b *such that* $|a| \leq 1$*, and* $b = 0$ *whenever* $|a| = 1$*.*

(b) compact if and only if $\psi(z) = az + b$ *for some complex numbers* a *and* b *such that* $|a| < 1$*.*

Analogous results were proved on the classical Fock spaces and generalized Flock spaces with weight function growing slower than Gaussian weight: See for instance

[11, 24, 51]. Our theorem shows that the result remains the same when the weight function grows faster than the Gaussian weight function.

Proof. We prove the theorem by considering three separate cases.
Case 1: $p = \infty$. *(a)* Assume that C_ψ is bounded on $\mathcal{F}_\varphi^\infty$. Using the estimate in (2.0.10) and applying C_ψ to the sequence of the test functions $f_{(w,R)}$

$$1 \gtrsim \|C_\psi f_{(w,R)}\|_{(\varphi,\infty)} \geq \Big(|f_{w,R)}(\psi(z))|e^{-\varphi(\psi(z))}\Big)e^{\varphi(\psi(z))-\varphi(z)} \tag{3.1.13}$$

for all $z \in \mathbb{C}$. Replacing $\psi(z)$ in place of w in (3.1.13) and applying (2.0.8), we get that

$$1 \gtrsim e^{\varphi(\psi(z))-\varphi(z)}. \tag{3.1.14}$$

Such estimate holds only when $\varphi(\psi(z))-\varphi(z)$ is uniformly bounded over $\mathbb{C}$ which further implies

$$\limsup_{|z|\to\infty} \frac{\varphi(\psi(z))}{\varphi(z)} \leq 1.$$

Applying the same argument in the proof of Theorem 3.1.1, we get that $\psi(z) = az + b$, $|a| \leq 1$. Next, we show that $b = 0$ whenever $|a| = 1$. Assume on the contrary that $|a| = 1$ and $b \neq 0$. Choose a complex number ω with $|\omega| = 1$ and $Re(\omega b) > 0$. Setting $z = t\omega$ for a positive number t, we compute

$$\begin{aligned} e^{\varphi(\psi(z)-\varphi(z)} &= e^{\varphi(az+b)-\varphi(z)} = e^{\varphi(at\omega+b)-\varphi(t\omega)} \\ &= e^{\varphi\left(\sqrt{t^2+2tRe(\omega b)+|b|^2}\right)-\varphi(t)} \to \infty \end{aligned}$$

as $t \to \infty$ since φ is a strictly increasing function, which contradicts (3.1.14). Conversely, if ψ has the linear form, then for each f in $\mathcal{F}_\varphi^\infty$

$$\|C_\psi f\|_{(\varphi,\infty)} \leq \sup_{z\in\mathbb{C}} e^{\varphi(az+b)-\varphi(z)} \sup_{z\in\mathbb{C}} |f(az+b)|e^{-\varphi(az+b)} \leq \|f\|_{(\varphi,\infty)}.$$

(b) To prove the sufficiency, we consider a uniformly bounded sequence $\{f_n\}_n$ in $\mathcal{F}_\varphi^\infty$ that converges to zero uniformly on compact subsets of $\mathbb{C}$. Then for a positive number R

$$\|C_\psi f_n\|_{(\varphi,\infty)} \leq \sup_{\{z\in\mathbb{C}:|z|\leq R\}} |f_n(\psi(z))|e^{-\varphi(z)} + \sup_{\{z\in\mathbb{C}:|z|>R\}} |f_n(\psi(z))|e^{-\varphi(z)}.$$

The first summand on the right-hand side above obviously converges to zero as $n \to \infty$. The second does the same since

$$\begin{aligned}\sup_{\{z\in\mathbb{C}:|z|>R\}} |f_n(\psi(z))|e^{-\varphi(z)} &\lesssim \|f_n\|_{(\varphi,\infty)} \sup_{\{z\in\mathbb{C}:|z|>R\}} e^{\varphi(\psi(z))-\varphi(z)}\\ &\lesssim \sup_{\{z\in\mathbb{C}:|z|>R\}} e^{\varphi(az+b)-\varphi(z)} \to 0, \quad R\to\infty\end{aligned}$$

which completes the required assertion that $C_\psi f_n \to 0$ in $\mathcal{F}_\varphi^\infty$. Next, we prove the necessity of the condition. If C_ψ is compact on $\mathcal{F}_\varphi^\infty$, then from the above boundedness argument we already have $\psi(z) = az + b, |a| \leq 1$ and $b = 0$ when $|a| = 1$. Thus, we proceed to show that $|a| < 1$. Since the sequence of test functions $f_{(w,R)}$ is uniformly bounded and converges to zero on compact subset of $\mathbb{C}$, applying C_ψ to this sequence again and eventually setting $w = \psi(z)$, we obtain

$$0 = \lim_{|z|\to\infty} \|C_\psi f_{(\psi(z),R)}\|_{(\varphi,\infty)} \geq \lim_{|z|\to\infty} e^{\varphi(az+b)-\varphi(z)} \tag{3.1.15}$$

from which we arrive at the claim. If not, $|a| = 1$ implies that $\varphi(az+b) - \varphi(z) = \varphi(z) - \varphi(z) = 1$ which leads to contradiction.

Case 2: $0 < p < \infty$. *(a)*. From (3.1.2), $C_\psi : \mathcal{F}_\varphi^p \to \mathcal{F}_\varphi^p$ is bounded if and only if the embedding map $I_d : \mathcal{F}_\varphi^p \to L^p(\mu_{(\psi,q)})$ is bounded . By the first part of Theorem 1 in [12], boundedness of i_d holds if and only if for some $\delta > 0$,

$$\sup_{z\in\mathbb{C}} \frac{1}{\tau(z)^2} \int_{D(z,\delta\tau(z))} e^{p\varphi(w)} d\mu_{(\psi,p)}(w) < \infty.$$

Applying (2.0.5),

$$\sup_{z\in\mathbb{C}} \frac{1}{\tau(z)^2}\int_{D(z,\delta\tau(z))} e^{p\varphi(w)}d\mu_{(\psi,p)}(w)$$

$$= \sup_{z\in\mathbb{C}} \frac{1}{\tau(z)^2}\int_{D(z,\delta\tau(z))} e^{p\varphi(w)-p\varphi(\psi^{-1}(w))}d\mu(w)$$

$$\gtrsim \frac{1}{\tau(z)^2}\int_{D(z,\delta\tau(z))} e^{p\varphi(w)-p\varphi(\psi^{-1}(w))}d\mu(w) \gtrsim e^{p\varphi(z)-p\varphi(\psi^{-1}(z))}. \tag{3.1.16}$$

Replacing $\psi(z)$ in place of z in (3.1.16), we get that for all $z \in \mathbb{C}$,

$$\sup_{z\in\mathbb{C}} \frac{1}{\tau(z)^2}\int_{D(z,\delta\tau(z))} e^{p\varphi(w)}d\mu_{(\psi,p)}(w) \gtrsim e^{p\varphi(\psi(z))-p\varphi(z)},$$

which is the same as inequality (3.1.14).

Thus by an argument following (3.1.14), we conclude that $\psi(z) = az + b, |a| \leq 1$ and $b = 0$ whenever $|a| = 1$. Conversely, if ψ has the linear form, for each $f \in \mathcal{F}^p_\varphi$,

$$\begin{aligned}\|C_\psi f\|^p_{(\varphi,p)} &= \int_{\mathbb{C}} |f(\psi(z))|^p e^{-p\varphi(z)}dA(z)\\ &= \int_{\mathbb{C}} |f(\psi(z))|^p e^{-p\varphi(\psi(z))}e^{p\varphi(\psi(z))-p\varphi(z)}dA(z)\\ &\leq \sup_{z\in\mathbb{C}} e^{p\varphi(\psi(z))-p\varphi(z)}\int_{\mathbb{C}} |f(\psi(z))|e^{-p\varphi(\psi(z))}dA(z) \leq \|f\|^p_{(\varphi,\infty)}.\end{aligned}$$

(b) Using the embedding reformulation in (3.1.2), $C_\psi : \mathcal{F}^p_\varphi \to \mathcal{F}^p_\varphi$ is compact if and only if $I_d : \mathcal{F}^p_\varphi \to L^p(\mu)$ is compact. By Theorem 1 in [12], this is equivalent to

$$\lim_{|z|\to\infty} \frac{1}{\tau(z)^2}\int_{D(w,\sigma\tau(w))} e^{p\varphi(z)}d\mu_{(\psi,p)} = 0. \tag{3.1.17}$$

We plan to show (3.1.17) holds if and only if $\psi(z) = az + b, |a| < 1$. Observe that

$$\begin{aligned}\lim_{|z|\to\infty} \frac{1}{\tau(z)^2} \int_{D(z,\sigma\tau(z))} & e^{p\varphi(z)} d\mu_{(\psi,p)}(w) \\ &= \lim_{|z|\to\infty} \frac{1}{\tau(z)^2} \int_{D(z,\sigma\tau(z))} e^{p\varphi(w)-p\varphi(\psi^{-1}(w))} d\mu(z) \\ &\qquad \simeq \lim_{|z|\to\infty} e^{p\varphi(z)-p\varphi(\psi^{-1}(z))}.\end{aligned}$$

Now, if $\psi(z) = az + b, |a| < 1$, then $\lim_{|z|\to\infty} e^{p\varphi(z)-p\varphi(\psi^{-1}(z))} = 0$, and hence C_ψ is compact.

Conversely, if C_ψ is compact, then

$$\begin{aligned}\lim_{|z|\to\infty} \frac{1}{\tau(z)^2} \int_{D(z,\sigma\tau(z))} e^{p\varphi(z)} d\mu_{(\psi,p)}(w) &\simeq \lim_{|z|\to\infty} e^{p\varphi(z)-p\varphi(\psi^{-1}(z))} \\ &= \lim_{|z|\to\infty} e^{p\varphi(\psi(z))-p\varphi((z))} = 0\end{aligned}$$

From this we have

$$\sup_{z\in\mathbb{C}} e^{p\varphi(\psi(z))-p\varphi((z))} < \infty.$$

This yields again $\psi(z) = az + b, |a| \le 1$ and $b = 0$ whenever $|a| = 1$. We claim that $|a| < 1$, otherwise

$$\lim_{|z|\to\infty} e^{p\varphi(\psi(z))-p\varphi((z))} = \lim_{|z|\to\infty} e^{p\varphi(az)-p\varphi((z))} = 1 \neq 0.$$

Case 3: $p = 0$. The composition operator C_ψ is bounded (resp. compact) on $\mathcal{F}^0_\varphi$ if and only if it is bounded (resp. compact) on $\mathcal{F}^\infty_\varphi$. In fact, every bounded composition operator C_ψ on $\mathcal{F}^\infty_\varphi$ maps polynomials into polynomials. This implies by density that such operator C_ψ maps $\mathcal{F}^0_\psi$ into itself. On the other hand, if C_ψ is bounded on $\mathcal{F}^0_\varphi$ then its bidual, which coincides with $C_\psi : \mathcal{F}^\infty_\varphi \to \mathcal{F}^\infty_\varphi$, is also bounded. Now the result follows from case 1. □

3.2 Essential norms of composition operators on $\mathcal{F}_\varphi^p$ and $\mathcal{F}_p$

For a bounded linear operator T on Banach space X, the essential norm of T denoted by $\|T\|_e$ is the distance from the operator to the space of compact operators. That is

$$\|T\|_e = \inf\{\|T-K\|;\ K \text{ is a compact operator }\}.$$

By definition it follows that $\|T\|_e \leq \|T\|$, and $\|T\|_e = 0$ if and only if T is compact. When the strict inequality $\|T\|_e < \|T\|$ holds, the operator T is norm-attaining [28, Proposition 2.2]. We say that a bounded linear operator T on a Banach space X attains its norm on X if there exists a function $f \in X$ with norm 1 such that $\|T\| = \|Tf\|$. A function f with these properties is an extremal function for the norm of T.

Computing the values of the norm and essential norm of composition operator is not an easy task. Interestingly, we have proved that norm and essential norm of non-compact composition operator C_ψ on $\mathcal{F}_\varphi^2$ or $\mathcal{F}_p, 0 < p \leq \infty$ is equal to 1. We have also estimated essential norms for C_ψ on the spaces $\mathcal{F}_\varphi^p$ for $p \geq 1$ and $p \neq 2$ as stated below.

Theorem 3.2.1. *Let $1 \leq p \leq \infty$ and ψ be a non-constant holomorphic map on $\mathbb{C}$ that induces a bounded operator C_ψ on $\mathcal{F}_\varphi^p$. Then if C_ψ is not compact on $\mathcal{F}_\varphi^p$ for $p \geq 1$, then its essential norm is comparable with its operator norm and*

$$1 \geq \|C_\psi\| \geq \|C_\psi\|_e \gtrsim 1. \tag{3.2.1}$$

On the Hilbert space $\mathcal{F}_\varphi^2$ case, we have equality and

$$\|C_\psi\|_e = \|C_\psi\| = 1. \tag{3.2.2}$$

If we replace $\mathcal{F}_\varphi^p$ by the classical Fock space $\mathcal{F}_p$, then the equality (3.2.2) *holds.*

On classical Fock space $\mathcal{F}_2$, it has been proved in [31] that the norm and essential norm of C_ψ are equal. For the proof, they used Hilbert spaces techniques based on an explicit expression of the reproducing kernels. In the Hilbert space $\mathcal{F}_\varphi^2$, an explicit expression for the kernel function is still an open problem. Yet, we managed this difficulty by using our result in Lemma 2.0.4, and we have obtained the precise values of the norms, which is the same as for the classical Fock space $\mathcal{F}_2$, namely that $\|C_\psi\|_e = \|C_\psi\| = 1$. This equality holds in all classical Fock spaces $\mathcal{F}_p,\ 1 \le p \le \infty$. For $\mathcal{F}_\varphi^p,\ p \ne 2$, we used sequence of the test functions $f_{(w,R)}$ to get the estimate in (3.2.1).

Proof. If C_ψ is bounded but not compact, then by Theorem 3.1.2, $\psi(z) = az$ where $|a| = 1$. Consequently, $\varphi(\psi(z)) = \varphi(az) = \varphi(|az|) = \varphi(z)$. With this, we find an upper bound for the norm of the operator

$$\|C_\psi f\|_{(\varphi,p)}^p = \int_{\mathbb{C}} \frac{|f(\psi(z))|^p}{e^{p\varphi(z)}} dA(z) \le \sup_{z\in\mathbb{C}} \left(e^{p\varphi(\psi(z))-p\varphi(z)}\right) \int_{\mathbb{C}} \frac{|f(\psi(z))|^p}{e^{p\varphi(\psi(z))}} dA(z)$$
$$= \sup_{z\in\mathbb{C}} e^{p\varphi(\psi(z))-p\varphi(z)} \|f\|_{(\varphi,p)}^p = \|f\|_{(\varphi,p)}^p.$$

Therefore,

$$1 \ge \|C_\psi\| \ge \|C_\psi\|_e. \tag{3.2.3}$$

A common way to prove lower bounds for essential norms is to find a suitable weakly null sequence of functions f_n and use the fact that

$$\|C_\psi\|_e \ge \limsup_{n\to\infty} \|C_\psi f_n\|_{(\varphi,p)}. \tag{3.2.4}$$

On classical Fock spaces, the sequence of the reproducing kernels does this job. Since no explicit expression is known for the kernel function in our current setting, we will instead use the sequence of functions

$$f^*_{(w,R)} = f_{(w,R)}/\|f_{(w,R)}\|_{(\varphi,p)} \tag{3.2.5}$$

as described by the properties in (2.0.7), (2.0.8), and (2.0.9). Obviously, the sequence $f^*_{(w,R)}$ is uniformly bounded, and due to the relation in (2.0.7), $f^*_{(w,R)} \to 0$ uniformly on compact subset of $\mathbb{C}$ as $|w| \to \infty$. With this, we proceed to make further estimates on the right-hand side of the norm in (3.2.4).

For any compact operator K on $\mathcal{F}^p_\varphi$,

$$\begin{aligned}\|C_\psi - K\| &\geq \limsup_{|w|\to\infty} \|(C_\psi - K)f^*_{(w,R)}\|_{(\varphi,p)} \\ &\geq \limsup_{|w|\to\infty} \left(\|C_\psi f^*_{(w,R)}\|_{(\varphi,p)} - \|K f^*_{(w,R)}\|_{(\varphi,p)}\right) = \limsup_{|w|\to\infty} \|C_\psi f^*_{(w,R)}\|_{(\varphi,p)},\end{aligned}$$

which leads to

$$\|C_\psi\|_e \geq \limsup_{|w|\to\infty} \|C_\psi f^*_{(w,R)}\|_{(\varphi,p)}.$$

Making use of (2.0.5) for some small positive number δ

$$\begin{aligned}\|C_\psi\|_e &\geq \limsup_{|w|\to\infty} \left\|C_\psi f^*_{((\psi(w),R)}\right\|_{(\varphi,p)} \\ &\simeq \limsup_{|w|\to\infty} \frac{1}{\tau(w)^{\frac{2}{p}}} \left(\int_{\mathbb{C}} |f_{((\psi(w),R)}(\psi(z))|^p e^{-p\varphi(z)} dA(z)\right)^{\frac{1}{p}} \\ &\geq \limsup_{|w|\to\infty} \frac{1}{\tau(w)^{\frac{2}{p}}} \left(\int_{D(\psi(w),\delta\tau(\psi(w)))} |f_{((\psi(w),R)}(\psi(z))|^p e^{-p\varphi(\psi(z))} dA(z)\right)^{\frac{1}{p}} \\ &\gtrsim \limsup_{|w|\to\infty} \frac{\tau(\psi(w))^{\frac{2}{p}} |f_{((\psi(w),R)}(\psi(w))|^p e^{-p\varphi(\psi(w))}}{\tau(w)^{\frac{2}{p}}} \\ &\simeq \limsup_{|w|\to\infty} \frac{\tau(\psi(w))^{\frac{2}{p}}}{\tau(w)^{\frac{2}{p}}} = \limsup_{|w|\to\infty} \frac{\tau(w)^{\frac{2}{p}}}{\tau(w)^{\frac{2}{p}}} = 1\end{aligned}$$

which completes the proof of the lower estimate.

For the Hilbert space case, applying Lemma 2.0.4, we have

$$\begin{aligned}\|C_\psi\|_e \geq \limsup_{|w|\to\infty} \big\| \|K_w\|_{(\varphi,2)}^{-1} C_\psi K_w \big\|_{(\varphi,2)} \\ = \limsup_{|w|\to\infty} \|K_w\|_{(\varphi,2)}^{-1} \bigg(\int_{\mathbb{C}} |K_w(\psi(z))|^2 e^{-2\varphi(z)} dA(z) \bigg)^{1/2} \\ = \limsup_{|w|\to\infty} \|K_w\|_{(\varphi,2)}^{-1} \bigg(\int_{\mathbb{C}} |K_w(az)|^2 e^{-2\varphi(az)} dA(z) \bigg)^{1/2} = 1,\end{aligned}$$

from which and (3.2.3) we arrive at the asserted equality.

For the proof on the classical setting, first we consider $p = \infty$.

Using the sequence of the normalized reproducing kernels $K_w/\|K_w\|_2$,

$$\begin{aligned}\|C_\psi\|_e \geq \limsup_{|w|\to\infty} \big\| \|K_w\|_2^{-1} \|C_\psi K_w\big\|_\infty \\ = \limsup_{|w|\to\infty} \|K_w\|_2^{-1} \sup_{z\in\mathbb{C}} |K_w(az)| e^{-\frac{1}{2}|z|^2} \\ = \limsup_{|w|\to\infty} \|K_w\|_2^{-1} \sup_{z\in\mathbb{C}} e^{-\frac{1}{2}|w-az|^2} e^{-\frac{1}{2}|w|^2} = \sup_{z\in\mathbb{C}} e^{-\frac{1}{2}|w-az|^2} = 1,\end{aligned}$$

from which and (3.2.3) we arrive at conclusion of equality.

If $1 \leq p < \infty$, simplifying Theorem 3.7 of [72] gives $1 \leq \|C_\psi\|_e \leq 2$. On other hand, the norm is computed by

$$\|C_\psi\| = \sup_{z\in\mathbb{C}} e^{\frac{1}{2}(|\psi(z)|^2 - |z|^2)} = \sup_{z\in\mathbb{C}} e^{\frac{1}{2}(|az|^2 - |z|^2)} = 1.$$

Combining these observations with the fact that $\|C_\psi\|_e \leq \|C_\psi\|$ we deduce

$$\|C_\psi\|_e = \|C_\psi\| = 1.$$

Thus, (3.2.2) holds in all classical Fock spaces $\mathcal{F}_p$ and $1 \leq p \leq \infty$. □

Before closing this section we ask whether C_ψ is norm attaining on $\mathcal{F}_\varphi^p$ or $\mathcal{F}_p$. The question of norm attaining composition operator was first explicitly studied by Hammond [27] in 2003 in the setting of the Hardy space H^2. Motivated by this, Martin [42] characterized norm attaining composition operator on the Block space. Martin's result states that every bounded composition operator C_ψ on Block space is norm attaining. Our next corollary shows that the result holds true if the space is $\mathcal{F}_p$ or $\mathcal{F}^p_\varphi, 1 \leq p \leq \infty$ also.

Corollary 3.2.2. *Let* $1 \leq p \leq \infty$ *and* C_ψ *be a bounded operator on* $\mathcal{F}^p_\varphi$. *Then* C_ψ *is norm-attaining and every unit norm function in* $\mathcal{F}^p_\varphi$ *is extremal. The same statement holds on the classical Fock spaces* $\mathcal{F}_p$.

Proof. If C_ψ is compact on $\mathcal{F}^p_\varphi$ or $\mathcal{F}_p$ $1 \leq p \leq \infty$, then $\|C_\psi\|_e < \|C_\psi\|$, and hence C_ψ is norm attaining by [28, Proposition 2.2].
Next, if C_ψ is not compact on $\mathcal{F}^p_\varphi$, then $\psi(z) = az$, and $\varphi(z) = \varphi(az)$. Thus for $1 \leq p < \infty$,

$$\|C_\psi f\|^p_{(\varphi,p)} = \int_{\mathbb{C}} |f(az)|^p e^{-p\varphi(z)} dA(z) = \int_{\mathbb{C}} |f(az)|^p e^{-p\varphi(az)} dA(z) = \|f\|^p_{(\varphi,p)},$$

and for $p = \infty$,

$$\|C_\psi f\|^p_{(\varphi,\infty)} = \sup_{z\in\mathbb{C}} f(az)^{e^{-\varphi(z)}} = \sup_{z\in\mathbb{C}} f(az)^{e^{-\varphi(az)}} = \|f\|_{(\varphi,\infty)}.$$

Thus, for $1 \leq p \leq \infty$,

$$\|C_\psi f\|_{(\varphi,p)} = \|f\|_{(\varphi,p)} \tag{3.2.6}$$

The same argument yields (3.2.6), if $\mathcal{F}^p_\varphi$ is replaced by $\mathcal{F}_p$.
Therefore C_ψ is norm-attaining and every unit norm function in $\mathcal{F}^p_\varphi$ or $\mathcal{F}_p$ is extremal.

3.3 Schatten class composition operators

If T is compact operator on Hilbert space $\mathcal{H}$, it admits a Schmidt decomposition and there exist orthonormal sets $\{e_n\}$ and $\{\sigma_n\}$ in $\mathcal{H}$ such that

$$Tx = \sum_{n=1}^{\infty} \lambda_n \langle x, e_n \rangle \sigma_n,$$

where λ_n is the singular value of T, i.e. it is the eigenvalue of $|T| = (T^*T)^{1/2}$. Given $0 < p < \infty$. The Schatten p-class of $\mathcal{H}$, denoted by $\mathcal{S}_p(\mathcal{H})$ consists of those compact operators $T : \mathcal{H} \to \mathcal{H}$ whose sequence of singular values $\{\lambda_n\}_{n=0}^{\infty}$ belongs to the space ℓ^p.

For $1 \leq p < \infty$, $\mathcal{S}_p(\mathcal{H})$ is a Banach space with respect to the norm

$$\|T\|_{\mathcal{S}_p} = \left(\sum_{n=0}^{\infty} |\lambda_n|^p \right)^p.$$

The Schatten classes are nested, with $\mathcal{S}_p \subset \mathcal{S}_q$ if $0 < p < q < \infty$.

Note that, if

$$Tx = \sum_{n=1}^{\infty} \lambda_n \langle x, e_n \rangle \sigma_n$$

is the canonical decomposition of compact operator T, then

$$T^*x = \sum_{n=1}^{\infty} \overline{\lambda_n} \langle x, e_n \rangle \sigma_n$$

is the canonical decomposition of compact operator T^*. This lead to the fact that, T is in $\mathcal{S}_p(\mathcal{H})$ if and only if T^* is in $\mathcal{S}_p(\mathcal{H})$ and $\|T\|_{\mathcal{S}_p(\mathcal{H})} = \|T^*\|_{\mathcal{S}_p(\mathcal{H})}$. We refer to [78] for more information about the Schatten class. Our next theorem describes that all compact composition operators on $\mathcal{F}_\varphi^2$ are in the Schatten $\mathcal{S}_p$ class for all $0 < p < \infty$.

Theorem 3.3.1. *Suppose that C_ψ is compact on $\mathcal{F}_\varphi^2$. Then it belongs to the*

Schatten $\mathcal{S}_p(\mathcal{F}^2_\varphi)$ *class for all* $0 < p < \infty$.

As seen in [34, 76], there are compact composition operators that are not in Schatten p-classes on the Hardy and Bergman spaces. On the other hand, on classical Fock spaces, all compact composition operators are in Schatten p-class for all $0 < p < \infty$ [31]. Our theorem shows that the result remains valid on generalized Fock spaces generated by fast growing weight functions.
For the proof of the theorem we require the following lemma.

Lemma 3.3.2. *Let* C_ψ *be compact operator on* $\mathcal{F}^2_\varphi$ *and* $k_z = \frac{K_z}{\|K_z\|_{(\varphi,2)}}$ *be the normalized kernel function. Then*

$$tr(C^*_\psi C_\psi)^{p/2} = \int_{\mathbb{C}} \langle (C^*_\psi C_\psi)^{p/2} k_z, k_z\rangle e^{-2\varphi(z)} dA(z)$$

Proof. Let $\{e_n\}$ be an orthonormal basis for $\mathcal{F}^2_\varphi$. Since

$$k_z = \sum_n \langle k_z, e_n\rangle e_n$$

we have

$$\langle (C^*_\psi C_\psi)^{p/2} k_z, k_z\rangle = \sum_n \langle (C^*_\psi C_\psi)^{p/2} e_n, e_n\rangle |e_n(z)|^2.$$

Therefore

$$\begin{aligned}\int_{\mathbb{C}} \langle (C^*_\psi C_\psi)^{p/2} k_z, k_z\rangle e^{-2\varphi(z)} dA(z) &= \int_{\mathbb{C}} \sum_n \langle (C^*_\psi C_\psi)^{p/2} e_n, e_n\rangle |e_n(z)|^2 e^{-2\varphi(z)} dA(z)\\ &= \sum_n \langle (C^*_\psi C_\psi)^{1/2} e_n, e_n\rangle \int_{\mathbb{C}} |e_n(z)|^2 e^{-2\varphi(z)} dA(z)\\ &= \sum_n \langle (C^*_\psi C_\psi)^{1/2} e_n, e_n\rangle = tr(C^*_\psi C_\psi)^{p/2}.\end{aligned}$$

Proof of Theorem 3.3.1. Since Schatten class membership has the nested property in the sense that $\mathcal{S}_p \subseteq \mathcal{S}_q$ for $p \leq q$, it suffices to verify the theorem only for the case when p is in the range $0 < p \leq 2$.

By Theorem 1.26 in [78], a compact operator C_ψ belongs to the Schatten $\mathcal{S}_p$ class if and only if the positive operator $(C_\psi^* C_\psi)^{p/2}$ belongs to the trace class $\mathcal{S}_1$. Furthermore, $C_\psi \in \mathcal{S}_p$ if and only if $C_\psi^* \in \mathcal{S}_p$, and $\|C_\psi\|_{\mathcal{S}_p} = \|C_\psi^*\|_{\mathcal{S}_p}$. Thus, we may estimate the trace of $\big(C_\psi C_\psi^*\big)^{p/2}$. Applying Lemma 3.3.2,

$$\begin{aligned} \operatorname{tr}\big((C_\psi C_\psi^*)^{\frac{p}{2}}\big) &= \int_{\mathbb{C}} \Big\langle \big(C_\psi C_\psi^* k_z\big)^{\frac{p}{2}}, k_z \Big\rangle e^{-2\varphi(z)} dA(z) \\ \leq \int_{\mathbb{C}} \Big\langle C_\psi C_\psi^* k_z, k_z \Big\rangle^{\frac{p}{2}} e^{-2\varphi(z)} dA(z) &= \int_{\mathbb{C}} \|C_\psi^* k_z\|^p_{(\varphi,2)} e^{-2\varphi(z)} dA(z), \end{aligned} \tag{3.3.1}$$

where the inequality holds by Proposition 1.31 in [78], since $0 < p \leq 2$, $C_\psi C_\psi^*$ is a positive operator, and $k_z = K_z/\|K_z\|_{(\varphi,2)}$ is a unit norm vector. On the other hand, by the reproducing property of the kernel function, we have the adjoint property

$$C_\psi^* K_w(z) = \big\langle C_\psi^* K_w, K_z \big\rangle = \big\langle K_w, C_\psi K_z \big\rangle = \overline{\big\langle C_\psi K_z, K_w \big\rangle} = K_{\psi(w)}(z).$$

From this estimate and (2.0.6), we have that

$$\|C_\psi^* k_w\|_{(\varphi,2)} \simeq \frac{\tau(w)}{\tau(\psi(w))} e^{\varphi(\psi(w)) - \varphi(w)}.$$

This along with (3.3.1) and compactness of C_ψ implies

$$\begin{aligned} \operatorname{tr}\big(\big(C_\psi C_\psi^*\big)^{\frac{p}{2}}\big) &\leq \int_{\mathbb{C}} \bigg(\frac{\tau(w)}{\tau(\psi(w))}\bigg)^p e^{p(\varphi(\psi(w)) - 3\varphi(w))} dA(z) \\ = \int_{\mathbb{C}} \bigg(\frac{\tau(w)}{\tau(aw+b)}\bigg)^p e^{p(\varphi(\psi(w)) - \varphi(w))} dA(z) &\lesssim \int_{\mathbb{C}} e^{p(\varphi(\psi(w)) - 3\varphi(w))} dA(z) < \infty, \end{aligned}$$

from which we conclude that $\operatorname{tr}\big(\big(C_\psi C_\psi^*\big)^{\frac{p}{2}}\big)$ is finite.

3.4 Connected components and isolated points

In the present section we consider some topological structures of bounded compositions operators $C_\psi : \mathcal{F}_\varphi^p \to \mathcal{F}_\varphi^q$ for all $0 < p, q < \infty$. We denote by $C(\mathcal{F}_\varphi^p, \mathcal{F}_\varphi^q)$ the space of such operators equipped with the operator norm topology. A natural point of interest is to identify the isolated points and connected component of the space $C(\mathcal{F}_\varphi^p, \mathcal{F}_\varphi^q)$ which we give below in Theorem 3.4.2. Before that we may prove the following Lemma.

Lemma 3.4.1. *Let $0 < p, q < \infty$, $\psi_n(z) = a_n z + b_n$, and $\psi(z) = az + b$ where (a_n) and (b_n) are sequences of complex numbers such that $0 < |a_n| \leq 1$ for all n, $a_n \to a \neq 0$ and $b_n \to b$ as $n \to \infty$. Then for any $f \in \mathcal{F}_\varphi^p$ and $C_\psi, C_{\psi_n} \in C(\mathcal{F}_\varphi^p, \mathcal{F}_\varphi^q)$*

$$\lim_{n\to\infty} \|C_{\psi_n} f - C_\psi f\|_{(\varphi,q)} = 0. \tag{3.4.1}$$

Proof. Assuming $0 < |a| \leq 1$, and $0 < |a_n| \leq 1$ we compute

$$\begin{aligned} \|C_{\psi_n} f\|_{(\varphi,q)}^q &= \int_{\mathbb{C}} |f(a_n z + b_n)|^q e^{-q\varphi(z)} dA(z) \\ &= \int_{\mathbb{C}} |f(a_n z + b)|^q e^{-q\varphi(a_n z + b_n)} \Big(e^{q\varphi(a_n z + b_n) - q\varphi(z)} \Big) dA(z) \\ &= \int_{\mathbb{C}} |f(w)|^q e^{-q\varphi(w)} \Big(|a_n|^{-2} e^{q\varphi(w) - q\varphi((w - b_n)/a_n)} \Big) dA(w). \end{aligned}$$

Since $|a_n| \leq 1$, the quantity $e^{q\varphi(w) - q\varphi((w-b_n)/a_n)}$ is uniformly bounded on $\mathbb{C}$. Moreover $|a_n| \to |a| \neq 0$ and $a_n \neq 0$ for all n. From this we have $|a_n|^{-2} \to |a|^{-2}$, and hence the sequence $\{1/|a_n|^2\}$ is also bounded.

Applying Lebesgue dominated convergence theorem and smoothness of the weight function φ, we obtain

$$\begin{aligned} \lim_{n\to\infty} \|C_{\psi_n} f\|_{(\varphi,q)}^q &= \lim_{n\to\infty} \int_{\mathbb{C}} |f(w)|^q e^{-q\varphi(w)} \Big(|a_n|^{-2} e^{q\varphi(w) - q\varphi((w-b_n)/a_n)} \Big) dA(w) \\ &= \int_{\mathbb{C}} |f(w)|^q e^{-q\varphi(w)} \Big(|a|^{-2} e^{q\varphi(w) - q\varphi((w-b)/a)} \Big) dA(w) \\ &= \int_{\mathbb{C}} |f(az+b)|^q e^{-q\varphi(z)} dA(z) = \|C_\psi f\|_{(\varphi,q)}^q \end{aligned}$$

from which (3.4.1) follows.

Now we state our main theorem on this section.

Theorem 3.4.2. *Let* $0 < p, q < \infty$ *and* C_ψ *be in* $C(\mathcal{F}^p_\varphi, \mathcal{F}^q_\varphi)$. *If*

(a) $p \neq q$, *then the space* $C(\mathcal{F}^p_\varphi, \mathcal{F}^q_\varphi)$ *is connected.*

(b) $p = q$, *then* C_ψ *is an isolated point if and only if it is not compact. In this case, the set of all compact composition operators on* $\mathcal{F}^p_\varphi$ *is a connected component of* $C(\mathcal{F}^p_\varphi, \mathcal{F}^p_\varphi)$.

In [25] it was shown that there exists non compact composition operators in the connected component of the compact ones on Hardy space $\mathcal{H}^2$. On the other hand, it was verified in [72] that all non compact composition operator on the space of all bounded $C_\psi : \mathcal{F}_p \to \mathcal{F}_q,\ 0 < p \leq q < \infty$ are isolated. Our result also shows that each non compact composition operator in $C(\mathcal{F}^p_\varphi, \mathcal{F}^p_\varphi)$ is an isolated point. By Theorem 3.1.1, if $p \neq q$, all bounded composition operator are compact, and hence no point is isolated in $C(\mathcal{F}^p_\varphi, \mathcal{F}^q_\varphi)$.

Proof. (a) Assume that $p \neq q$. We plan to show that $C(\mathcal{F}^p_\varphi, \mathcal{F}^q_\varphi)$ is connected. Aiming to argue in the direction of contradiction, suppose there exists an isolated point $C_\psi \in C(\mathcal{F}^p_\varphi, \mathcal{F}^q_\varphi)$. Since $p \neq q$, by Theorem 3.1.1, C_ψ is a compact operator and hence $\psi(z) = az+b,\ |a| < 1$. Then, if $a \neq 0$ choose two sequences of numbers (a_n) with $|a_n| < 1$ and $(a_n) \neq 0$ for all n and (b_n) such that $a_n \to a$ and $b_n \to b$ as $n \to \infty$. It follows that $\psi_n(z) = a_n z + b_n \to\ az + b = \psi(z)$. Then for any $f \in \mathcal{F}^p_\varphi$, by Lemma 3.4.1

$$\|C_{\psi_n} f - C_\psi f\|_{(\varphi,q)} \to 0 \ \text{ as } \ n \to \infty. \tag{3.4.2}$$

If $a = 0$, make $a_n = 0$ for all n. Here also it follows that $\psi_n(z) = a_n z + b_n \to az + b = \psi(z)$. In this case direct calculation yields (3.4.2). Using this we find

$$\lim_{n\to\infty} \|C_{\psi_n} - C_\psi\|_{(\varphi,q)} \leq \lim_{n\to\infty} \sup_{\|f\|_{(\varphi,q)}\leq 1} \|C_{\psi_n} f - C_\psi f\|_{(\varphi,q)} = 0$$

contradicting our assumption.

(b) Let $p = q$ and assume that $C_\psi \in C(\mathcal{F}^p_\varphi, \mathcal{F}^p_\varphi)$ is not compact. Then by

Theorem 3.1.1, $\psi(z) = az, |a| = 1$. We proceed to show that C_ψ is isolated. That is there exists a positive number c such that

$$\|C_\psi - C_{\psi_1}\| \geq c \tag{3.4.3}$$

for all $C_{\psi_1} \in C(\mathcal{F}_\varphi^p, \mathcal{F}_\varphi^p)$ for which $\psi_1 \neq \psi$. We may first consider the forms $\psi_1(z) = a_1 z, \quad |a_1| = 1$ and $a_1 \neq a$. Since the polynomials are contained in $\mathcal{F}_\varphi^p$,

$$\begin{aligned} \|C_\psi - C_{\psi_1}\| &\geq \sup_{n\geq 0} \|z^n\|_{(\varphi,p)}^{-1} \big\|(C_\psi - C_{\psi_1})z^n\big\|_{(\varphi,p)} \\ &= \sup_{n\geq 0} \|z^n\|_{(\varphi,p)}^{-1} |a^n - a_1^n| \|z^n\|_{(\varphi,p)} = \sup_{n\geq 0} |a^n - a_1^n| \geq 2. \end{aligned} \tag{3.4.4}$$

On the other hand, if C_{ψ_1} is compact, then $\psi_1 = a_1 z + b, \quad |a_1| < 1$ and using the unit norm sequence of functions $f_{(w,R)}^*$ in (3.2.5)

$$\begin{aligned} \|C_\psi - C_{\psi_1}\| &= \sup_{w\in\mathbb{C}} \big\|(C_\psi - C_{\psi_1}) f_{(w,R)}^*\big\|_{(\varphi,p)} \\ &\geq \sup_{w\in\mathbb{C}} \left(\|C_\psi f_{(w,R)}^*\|_{(\varphi,p)} - \left\|C_{\psi_1} f_{(w,R)}^*\right\|_{(\varphi,p)} \right) \\ &\gtrsim \sup_{w\in\mathbb{C}} \left(1 - \left\|C_{\psi_1} f_{(w,R)}^*\right\|_{(\varphi,p)} \right). \end{aligned} \tag{3.4.5}$$

Now, $f_{(w,R)}^* \to 0$ weakly as $|w| \to \infty$, and as C_{ψ_1} is compact, we have

$$\left\|C_{\psi_1} f_{(w,R)}^*\right\|_{(\varphi,p)} \to 0$$

as $|w| \to \infty$. This together with (3.4.5) for sufficiently big $|w|$ gives

$$\|C_\psi - C_{\psi_1}\| \gtrsim 1. \tag{3.4.6}$$

From (3.4.6) and (3.4.4), the claim in (3.4.3) follows. $\square$

A natural question following Theorem 3.4.2 is whether every isolated composition operator in $C(\mathcal{F}_\varphi^p, \mathcal{F}_\varphi^p)$ is still isolated under the essential norm topology which is weaker than the topology induced by the operator norm. Our next main result shows that this is in deed the case.

Theorem 3.4.3. *Let* $1 \leq p < \infty$. *Then a composition operator* C_ψ *in* $C(\mathcal{F}_\varphi^p, \mathcal{F}_\varphi^p)$ *is essentially isolated if and only if it is isolated.*

The isolated and essentially isolated points of the space of the operators on the classical Fock spaces have not been also identified as far as we know. The method we use to prove the result can be easily adopted to the classical setting by using the sequence of the normalized reproducing kernels in stead of using the sequence of the functions $f^*_{w,R}$ to conclude the analogous results.

Proof. Since the essential norm topology is weaker than the operator norm topology, each essentially isolated point is isolated. Thus, we consider an operator $C_{\psi_1} \in C(\mathcal{F}_\varphi^p, \mathcal{F}_\varphi^p)$, and assume that it is isolated in the operator norm topology. Then we plan to show that it is also essentially isolated. We may let $\psi_1(z) = a_1 z$ with $|a_1| = 1$. It suffices to show that for all bounded composition operators $C_{\psi_2} \in C(\mathcal{F}_\varphi^p, \mathcal{F}_\varphi^p)$, the estimate

$$\|C_{\psi_1} - C_{\psi_2}\|_e \gtrsim 1$$

holds. If ψ_2 is not compact either, then we may set $\psi_2(z) = a_2 z$ where $a_1 \neq a_2$ and $|a_2| = 1$. Then for any compact operator Q on $\mathcal{F}_\varphi^p$ we have

$$\begin{aligned}\|(C_{\psi_1} - C_{\psi_2}) - Q\| &\geq \limsup_{|w|\to\infty} \|((C_{\psi_1} - C_{\psi_2}) - Q) f^*_{(w,R)}\|_{(\varphi,p)} \\ &\geq \limsup_{|w|\to\infty} \|(C_{\psi_1} - C_{\psi_2}) f^*_{(w,R)}\|_{(\varphi,p)} - \|Q f^*_{(w,R)}\|_{(\varphi,p)} \\ &= \limsup_{|w|\to\infty} \|(C_{\psi_1} - C_{\psi_2}) f^*_{(w,R)}\|_{(\varphi,p)}.\end{aligned}$$

Arguing as in the preceding proof and setting $w = \psi_1(z_0)$ we find ,

$$\begin{aligned}\|C_{\psi_1} - C_{\psi_2}\|_e &\gtrsim \limsup_{|z_0|\to\infty} \Big(|f_{(\psi_1(z_0),R)}(\psi_1(z_0))| - |f_{(\psi_1(z_0),R)}(\psi_2(z_0))|\Big)e^{-\varphi(z_0)}\\ &\gtrsim \limsup_{|z_0|\to\infty}\left(1-\left(\frac{\tau(z_0)}{|z_0||a_1-a_2|}\right)^{\frac{R^2}{2}}\right) = 1.\end{aligned}$$

On the other hand, if C_{ψ_2} is compact, we set $\psi_2(z) = a_2 z + b$ with $|a_2| < 1$, and repeating the preceding arguments

$$\begin{aligned}\|C_{\psi_1} - C_{\psi_2}\|_e &\gtrsim \limsup_{|z_0|\to\infty} \Big(|f_{(\psi_1(z_0),R)}(\psi_1(z_0))| - |f_{(\psi_1(z_0),R)}(\psi_2(z_0))|\Big)e^{-\varphi(z_0)}\\ &\gtrsim \limsup_{|z_0|\to\infty}\left(1-\left(\frac{\min\{\tau(z_0),\tau(a_2z_0+b_2)\}}{|z_0(a_1-a_2)+b_2|}\right)^{\frac{R^2}{2}}\right) = 1,\end{aligned}$$

and completes the proof.

3.5 Difference of compact composition operators

A natural question to pose now is when the difference of two operators from $C(\mathcal{F}_\varphi^p, \mathcal{F}_\varphi^q)$ becomes compact. It turns out that the difference is compact if and only if both of the operators are compact.

Theorem 3.5.1. *Let $0 < p < \infty$ and $C_{\psi_1}, C_{\psi_2} \in C(\mathcal{F}_\varphi^p, \mathcal{F}_\varphi^p)$ where $\psi_1 \neq \psi_2$. Then $C_{\psi_1} - C_{\psi_2}$ is compact on $\mathcal{F}_\varphi^p$ if and only if both C_{ψ_1} and C_{ψ_2} are compact.*

The theorem shows that cancellation property of the inducing maps plays no roll for compactness of the difference. On the contrary, it is worth mentioning that compactness of the differences of two composition operators on the weighted Bergman spaces over the unit disc has been characterized by some suitable cancellation property of the inducing symbols at each boundary points [43]. Such property makes it possible for each composition operator in the difference not necessarily to be compact.

Proof. If both operators are compact, obviously the difference is also compact. Thus, we shall prove the other implication, i.e. assuming the difference is compact, we need to verify that both composition operators are compact. We plan to argue in the direction of contradiction again, and assume that one of them C_{ψ_1} is not compact. It follows that C_{ψ_2} is not compact either since for a weakly convergent sequence (f_n) in $\mathcal{F}_\varphi^p$ with $\lim_{n\to\infty} \|C_{\psi_1} f_n\|_{(\varphi,p)} \neq 0$, we have

$$0 < \lim_{n\to\infty} \|C_{\psi_1} f_n\|_{(\varphi,p)} \leq \lim_{n\to\infty} \|(C_{\psi_1} - C_{\psi_2}) f_n\|_{(\varphi,p)} + \lim_{n\to\infty} \|C_{\psi_2} f_n\|_{(\varphi,p)}.$$

Thus, we may set $\psi_1(z) = a_1 z$ and $\psi_2(z) = a_2 z$ where $a_1 \neq a_2$ and $|a_j| = 1, j = 1, 2$. Since the unit norm sequence $f^*_{(w,R)}$ is weakly convergent, compactness of the difference operator implies

$$\|(C_{\psi_1} - C_{\psi_2}) f^*_{(w,R)}\|_{(\varphi,p)} \to 0 \quad \text{as} \quad |w| \to \infty. \tag{3.5.1}$$

On the other hand, we have a lower estimate

$$\begin{aligned}
\|(C_{\psi_1} - C_{\psi_2}) f^*_{(w,R)}\|^p_{(\varphi,p)} &= \int_{\mathbb{C}} |C_{\psi_1} f^*_{(w,R)}(z) - C_{\psi_2} f^*_{(w,R)}(z)|^p e^{-p\varphi(z)} dA(z) \\
&\geq \int_{D(z_0,\tau(z_0))} |C_{\psi_1} f^*_{(w,R)}(z) - C_{\psi_2} f^*_{(w,R)}(z)|^p e^{-p\varphi(z)} dA(z).
\end{aligned}$$

From this and applying (2.0.5) and (2.0.9) we estimate

$$\begin{aligned}
\|(C_{\psi_1} - C_{\psi_2}) f^*_{(w,R)}\|_{(\varphi,p)} &\gtrsim \tau(z_0)^{\frac{2}{p}} |C_{\psi_1} f^*_{(w,R)}(z_0) - C_{\psi_2} f^*_{(w,R)}(z_0)| e^{-\varphi(z_0)} \\
&\simeq \frac{\tau(z_0)^{\frac{2}{p}}}{\tau(w)^{\frac{2}{p}}} |f_{(w,R)}(\psi_1(z_0)) - C_{\psi_2} f_{(w,R)}(\psi_2(z_0))| e^{-\varphi(z_0)}.
\end{aligned}$$

Setting $w = \psi_1(z_0)$ on the right-hand side above, applying (2.0.7) and (2.0.8) and observing that $\tau(\psi_1(z_0)) = \tau(\psi_2(z_0)) = \tau(z_0)$ leads to

$$\begin{aligned}\|(C_{\psi_1} - C_{\psi_2})f^*_{(w,R)}\|_{(\varphi,p)} &\gtrsim \frac{\tau(z_0)^{\frac{p}{2}}}{\tau(\psi_1(z_0))^{\frac{p}{2}}}|f_{(\psi_1(z_0),R)}(\psi_1(z_0)) - f_{(\psi_1(z_0),R)}(\psi_2(z_0))|e^{-\varphi(z_0)}\\ &\geq \Big(|f_{(\psi_1(z_0),R)}(\psi_1(z_0))| - |f_{(\psi_1(z_0),R)}(\psi_2(z_0))|\Big)e^{-\varphi(z_0)}\\ \gtrsim \left(e^{\varphi(z_0)} - e^{\varphi(z_0)}\left(\frac{\tau(z_0)}{|z_0||a_1 - a_2|}\right)^{\frac{R^2}{2}}\right)e^{-\varphi(z_0)} &= 1 - \left(\frac{\tau(z_0)}{|z_0||a_1 - a_2|}\right)^{\frac{R^2}{2}} = 1\end{aligned}$$

when $|z_0| \to \infty$ which contradicts the fact in (3.5.1).

3.6 Unitary and hyponormal composition operators

In this section we characterize mappings ψ which induce unitary composition operators C_ψ, and describe the relationship between hyponormal and normal composition operators C_ψ on the spaces $\mathcal{F}^2_\varphi$. Recall that a bounded linear operator T on a complex Hilbert space $\mathcal{H}$ is said to be hyponormal if $T^*T \geq TT^*$ where T^* is the adjoint of T. The operator is normal if $TT^* = T^*T$, and unitary whenever $TT^* = T^*T = I$, where I is the identity operator on $\mathcal{H}$. Our next main result shows that only non compact composition operators are unitary.

Theorem 3.6.1. *Let $\psi(z) = az + b$ induces a bounded composition operator C_ψ on $\mathcal{F}^2_\varphi$. Then C_ψ is unitary if and only if $|a| = 1$.*

Proof. Assume that $|a| = 1$. Then by Theorem 3.1.2, $b = 0$ and $C_\psi(z) = az$, with $|a| = 1$. We need to show that C_ψ is surjective and preserves the inner product on $\mathcal{F}^2_\varphi$. Thus, for each f, g in $\mathcal{F}^2_\varphi$:

$$\begin{aligned}\langle C_\psi f, C_\psi g\rangle &= \int_{\mathbb{C}} f(az)\overline{g(az)}e^{-2\varphi(z)}dA(z)\\ &= \frac{1}{|a|^2}\int_{\mathbb{C}} f(w)\overline{g(w)}e^{-2\varphi(w)}dA(w) = \langle f, g\rangle.\end{aligned}$$

which shows that the operator preserves the inner product. It remains to show that the operator is also surjective. But this follows easily since $C_\psi^{-1} = C_{\psi^{-1}}$ exists in this case. Conversely, if C_ψ is unitary, then

$$\|K_z\|^2_{(\varphi,2)} = \|C_\psi K_z\|^2_{(\varphi,2)} = \|C^*_\psi K_z\|^2_{(\varphi,p)} = \|K_{\psi(z)}\|^2_{(\varphi,p)}$$

Considering the asymptotic relation in (2.0.6) we further have

$$\frac{e^{2\varphi(z)}}{\tau(z)^2} \simeq \frac{e^{2\varphi(az+b)}}{\tau(az+b)^2}.$$

By definition of τ and the admissibility condition on the weight function φ, the above estimate holds for $|z| \to \infty$ only if $b = 0$ and $|a| = 1$. which holds only if $b = 0$ and $|a| = 1$. ☐

Our next result shows that hyponormal composition operators must be normal in $\mathcal{F}^2_\varphi$.

Theorem 3.6.2. *Let $\psi(z) = az + b$ induce a bounded composition operator C_ψ on $\mathcal{F}^2_\varphi$. Then C_ψ is normal if and only if it is hyponormal.*

On the classical Fock space, this result was proved in [31]. Our result shows that this property is independent of the fast growth of the inducing weight function.

Proof. Obviously normal operators are hyponormal. Conversely assume that C_ψ is hyponormal. If C_ψ is compact hyponormal, then by [[5],Corollary 2] it is normal. If C_ψ is bounded but not compact,then by Theorem 3.6.1 it is unitary and hence normal. ☐

An interesting related property is the notion of essentially normal. Recall that a bounded composition operator C_ψ is essentially normal if the commutator $[C^*_\psi, C_\psi] = C^*_\psi C_\psi - C_\psi C^*_\psi$ is compact. Then, the following is an immediate consequence of Theorem 3.6.1 and Theorem 3.1.1.

Corollary 3.6.3. *Let $\psi(z) = az + b$ induce a bounded composition operator C_ψ on $\mathcal{F}^2_\varphi$. Then C_ψ is essentially normal.*

Proof. By Theorem 3.1.1 either $|a| = 1$ in which case by Theorem 3.6.1, it is unitary and hence the operator is normal or $|a| < 1$ and the operator becomes compact. Since normal and compact operators are essentially normal, the corollary trivially holds.

Chapter 4

Spectral and Dynamical Properties of Composition Operators on $\mathcal{F}_\varphi^p$

We recall that the spectrum $\sigma(T)$ of a bounded operator T on a Banach space is the set $\sigma(T) = \{\lambda \in \mathbb{C} : T - \lambda I \text{ is not invertible}\}$. The spectrum of an operator on a finite-dimensional vector space is precisely the set of its eigenvalues. However, an operator on an infinite-dimensional space may have additional elements in its spectrum, and may have no eigenvalues. In fact every bounded linear operator on a complex Banach space must have a non-empty spectrum.

In Section 4.1, we describe the spectrum of the composition operators on the spaces $\mathcal{F}_\varphi^p$ and $\mathcal{F}_p$, $0 < p \leq \infty$. The result obtained will be applied later while studying its dynamical structures. We may remind that the study of the dynamics of an operator is mainly concerned with the behaviour of its iterates. If $\mathcal{H}$ is a Banach space and T is a bounded linear operator on $\mathcal{H}$, then for $x \in \mathcal{H}$, $T^n x,\ n = 0, 1, 2, ...$ is a sequence of its iterates or orbits corresponding to vector x. For the composition operator C_ψ, a simple computation shows that the iterate

C_ψ^n is itself a composition operator induced by the n^{th} iterate of ψ. That is

$$C_\psi^n = C_{\psi^n}, \quad \psi^n = \underbrace{\psi \circ \psi \circ \psi \circ \ldots \circ \psi}_{n \text{ times}}, \tag{4.0.1}$$

which obviously makes the study of the dynamical properties a natural subject. Furthermore, the relation in (4.0.1) indicates that the dynamical behaviour of a composition operator is heavily dependent on the dynamical properties of its inducing map ψ.

We denote the n-th ergodic mean of T by T_n and

$$T_n := \frac{1}{n} \sum_{m=1}^{n} T^m.$$

T is said to be *power bounded* if $\sup_{n\in\mathbb{N}} \|T^n\| < \infty$ and T is said to be *polynomially bounded* if there exist a constant $M > 1$ such that

$$\|p(T)\| \leq M\|p\|_\infty$$

for every polynomial p, where $\|p\|_\infty = \sup\{|p(z)| : \ z \in \mathbb{C}, \ |z| \leq 1\}$. We say that T is *mean ergodic* if there exists a bounded operator P on a Banach space $\mathcal{H}$ such that for each f in $\mathcal{H}$

$$\lim_{n\to\infty} \|T_n f - Pf\| = 0$$

and *uniformly mean ergodic* if the pointwise convergence above is uniform;

$$\lim_{n\to\infty} \|T_n - P\| = 0.$$

Power bounded, mean ergodic and uniformly mean ergodic composition operator have been studied over various spaces of analytic functions on the unit disc $\mathbb{D}$. For example; Bonet and Domański [7] on spaces of analytic functions over some domain in a Stein manifold, E. Wolf [74] on $H_v^\infty(\mathbb{D})$, Beltrán-Meneu, Gómez-

Collado, Jordá and Jornet [3] on a disc algebra $A(\mathbb{D})$ and the space $H^\infty(\mathbb{D})$ of bounded analytic function.

In Section 4.2, we prove that all bounded composition operators on $\mathcal{F}_\varphi^p$; $1 \leq p \leq \infty$ are power bounded. In Section 4.3, we present our results on mean ergodic and uniform mean ergodic composition operator.

Another basic dynamical properties of bounded operators the notions of cyclic and supercyclic. A bounded linear operator T on a Banach space $\mathcal{H}$ is said to be cyclic if there exists a vector x in $\mathcal{H}$ such that the linear span of its orbit under T,

$$\text{Orb}(T,x) = \{T^n x : n = 0, 1, 2, ..\},$$

is dense in $\mathcal{H}$. Such a vector x is called cyclic for the operator T. The operator is hypercyclic if the orbit itself is dense in $\mathcal{H}$, and supercyclic if there exists a vector x in $\mathcal{H}$ such that the projective orbit,

$$\text{Projorb}(T,x) = \{\lambda T^n x : \quad \lambda \in \mathbb{C}, n = 0, 1, 2, ..\},$$

is dense in $\mathcal{H}$. Clearly any hypercyclic operator is cyclic, but the cyclic operators form a much larger class while supercyclicity is an intermediate property between the two. It is worth mentioning that if an operator T has a hypercyclic vector, then each element in the orbit of such vector is also hypercyclic which implies that a hypercyclic operator has a dense set of hypercyclic vectors.

We may mention that the cyclicity and supercyclicity dynamical structures have not been studied on the classical Fock spaces settings either except for the Hilbert space case which was considered in [24, 31]. As can be seen in Section 4.4, our approach, which neither uses Hilbert spaces techniques nor the fast growth property of the weight function φ, shows that the same result holds on $\mathcal{F}_\varphi^p$ as well as on the classical Fock space $\mathcal{F}^p$ for all $1 \leq p < \infty$.

In the last section of the chapter we study periodic points in $\mathcal{F}_\varphi^p$ under the composition operators.

4.1 The spectrum of composition operators

We now state our result on the spectrum $\sigma(C_\psi)$ of the composition operators acting on $\mathcal{F}_\varphi^p$ or $\mathcal{F}_\varphi^p$.

Theorem 4.1.1. *Let* $1 \leq p \leq \infty$ *and* $\psi(z) = az+b$ *induces a bounded composition operator* C_ψ *on* $\mathcal{F}_\varphi^p$ *or* $\mathcal{F}_p$*. Then*

$$\sigma(C_\psi) = \overline{\{a^n, n = 0, 1, 2, 3, ...\}}.$$

The result clearly shows that the spectral property of C_ψ depends on the derivative of the symbol ψ^n. In addition, the operator admits only point spectrum except zero. The spectrum $\sigma(C_\psi)$ could be also finite if there exists a positive integer k such that $a^k = a$.

K. Guo and K. Izuchi [24] described spectrum of bounded composition operator C_ψ on Fock type spaces $\mathcal{F}_\varphi^2$. Our result shows that $\sigma(C_\psi)$ is independent of the exponent p in the range mentioned above and on the condition whether the Laplacian of the weight function over the whole complex plane is bounded.

Proof. Since the complex polynomials are contained in all the spaces $\mathcal{F}_\varphi^p$, for $a \neq 1$ setting

$$u_n(z) = \Big(z - \frac{b}{1-a}\Big)^n, \ n = 0, 1, 2, 3, ...$$

we obtain

$$C_\psi u_n(z) = \left(az + b - \frac{b}{1-a}\right)^n = a^n u_n(z)$$

which shows once side of the inclusion

$$\sigma(C_\psi) \supseteq \overline{\{a^n, \ n = 0, 1, 2, 3, ...\}}. \tag{4.1.1}$$

To prove the reverse inclusion, we consider two cases. First, if $|a| < 1$, then the operator is compact and its spectrum contains only zero and eigenvalues. Thus, we may consider a non-zero eigenvalue λ in $\sigma(C_\psi)$ with corresponding non-zero eigenvector f and show that λ must be of the form a^n for some positive integer n. Thus,

$$C_\psi f(z) = f(az + b) = \lambda f(z) \tag{4.1.2}$$

for all z in $\mathbb{C}$. Now if f has no zero at $z_0 := \frac{b}{1-a}$, then from (4.1.2) we observe that $f(z_0) = \lambda f(z_0)$ and hence $\lambda = 1 = a^0$. On the other hand, if f has zero at z_0 of order m, we may write

$$f(z) = (z - z_0)^m g(z)$$

where $g(z_0) \neq 0$. Then substituting f by this factorization in (4.1.2) and differentiating both sides of the equation m times and eventually setting $z = z_0$, we only get

$$a^m m! g(z_0) = \lambda m! g(z_0) \tag{4.1.3}$$

as all the other terms have factor $z - z_0$ and vanish. Now, $g(z_0)$ is non-zero and (4.1.3) holds only if $\lambda = a^m$ as asserted.

Second, if $|a| = 1$, then $\|C_\psi\| = 1$, and hence the spectral radius of C_ψ is 1. Therefore, $\sigma(C_\psi)$ is contained in the closed unit disc $\overline{\mathbb{D}}$. Assume $0 \neq \lambda \in \overline{\mathbb{D}}$ is not in $\overline{\{a^n,\ n = 0, 1, 2, 3, ...\}}$. Then we plan to show that $C_\psi - \lambda I$ is invertible by explicitly computing its inverse operator. To this end, we define a linear map T_λ on the polynomials by $T_\lambda : z^n \longmapsto \frac{1}{a^n - \lambda} z^n$ for all $n = 0, 1, 2, 3...$ Since $\lambda \notin \overline{\{a^n,\ n = 0, 1, 2, 3, ...\}}$, we have $a^n - \lambda \neq 0$. Therefore, T_λ is well-defined on the polynomials and

$$T_\lambda(C_\psi z^n - \lambda z^n) = z^n = (C_\psi z^n - \lambda I z^n)(T_\lambda(z^n)). \tag{4.1.4}$$

Now, if $p < \infty$, then by Theorem 2.2.1, the set of polynomials is dense in $\mathcal{F}_\varphi^p$. This along with (4.1.4) shows that T_λ is the inverse of $C_\psi - \lambda I$ and hence no such λ can be in $\sigma(C_\psi)$.
For $p = \infty$, the polynomials are dense only in the closed subspace given by

$$\mathcal{F}_\varphi^0 = \big\{ f \in \mathcal{F}_\varphi^\infty : \lim_{|z|\to\infty} |f(z)|e^{-\varphi(z)} = 0 \big\}.$$

Thus, we argue differently and consider the Taylor series expansion of each function $f \in \mathcal{F}_\varphi^\infty$ at $z = 0$; $f(z) = \sum_{n=0}^\infty a_n z^n$ and define

$$T_\lambda f(z) = \sum_{n=0}^\infty \frac{a_n}{a^n - \lambda} z^n.$$

Observe that T_λ is bounded and is the inverse of $C_\psi - \lambda I$.
The proof for the classical Fock space setting follows exactly in the same way, and completes the proof.

4.2 Power bounded composition operators

The main result of this section states that all bounded composition operator on $\mathcal{F}_\varphi^p$ are power bounded.

Theorem 4.2.1. *Let $1 \le p \le \infty$ and C_ψ be bounded on $\mathcal{F}_\varphi^p$. Then C_ψ is power bounded.*

Power bounded composition operators are not identified on the classical Fock spaces either. As easily seen from the proof below, Theorem 4.2.1 is also valid on such spaces.

Proof. We split the proof into two cases.

Case 1: $1 \le p < \infty$. We first consider the case when $|a| < 1$. In polar coordinates $a = |a|e^{i\phi}$ for some ϕ. For each $n \in \mathbb{N}$ consider a translation $\tau_n(z) = z + \frac{b(1-a^n)}{1-a}$. Then for any $f \in \mathcal{F}^p_\varphi$ and $z = re^{it} \in \mathbb{C}$,

$$\begin{aligned}\int_0^{2\pi} \left| f\left(a^n re^{it} + \frac{b(1-a^n)}{1-a}\right)\right|^p \frac{dt}{2\pi} &= \int_0^{2\pi} \left|f \circ \tau_n(a^n re^{it})\right|^p \frac{dt}{2\pi} \\ &= \int_0^{2\pi} \left|f \circ \tau_n((|a|^n r)e^{i(t+n\phi)})\right|^p \frac{dt}{2\pi} = M_p^p(f \circ \tau_n, (|a|^n r)).\end{aligned}$$

Since M_p^p is increasing and $|a|^n \le |a|$, we have

$$\begin{aligned}M_p^p(f \circ \tau_n, (|a|^n r)) \le M_p^p(f \circ \tau_n, (|a| r)) &= \int_0^{2\pi} \left|f \circ \tau_n(|a|re^{i(t+n\phi)})\right|^p \frac{dt}{2\pi} \\ = \int_0^{2\pi} |f \circ \tau_n((|a|e^{i\phi})re^{i(t+(n-1)\phi)})|^p \frac{dt}{2\pi} &= \int_{(n-1)\phi}^{2\pi+(n-1)\phi} |f \circ \tau_n((|a|e^{i\phi})re^{it})|^p \frac{dt}{2\pi} \\ = \int_{(n-1)\phi}^{2\pi+(n-1)\phi} \left|f\left(are^{it} + \frac{b(1-a^n)}{1-a}\right)\right|^p \frac{dt}{2\pi} &= \int_0^{2\pi} \left|f\left(are^{it} + \frac{b(1-a^n)}{1-a}\right)\right|^p \frac{dt}{2\pi}.\end{aligned}$$

Thus

$$\int_0^{2\pi} \left|f\left(a^n re^{it} + \frac{b(1-a^n)}{1-a}\right)\right|^p \frac{dt}{2\pi} \le \int_0^{2\pi} \left|f\left(are^{it} + \frac{b(1-a^n)}{1-a}\right)\right|^p \frac{dt}{2\pi}.$$

Now, for any $n \in \mathbb{N}$ and $f \in \mathcal{F}^p_\varphi$,

$$\begin{aligned}\|C_\psi^n f\|^p_{(\varphi,p)} &= \int_{\mathbb{C}} \left|f\left(a^n z + \frac{b(1-a^n)}{1-a}\right)\right|^p e^{-p\varphi(z)} dA(z) \\ &= 2\pi \int_0^\infty \int_0^{2\pi} \left|f\left(a^n re^{it} + \frac{b(1-a^n)}{1-a}\right)\right|^p \frac{dt}{2\pi} re^{-p\varphi(r)} dr \\ &\le 2\pi \int_0^\infty \int_0^{2\pi} \left|f\left(are^{it} + \frac{b(1-a^n)}{1-a}\right)\right|^p \frac{dt}{2\pi} re^{-p\varphi(r)} dr \\ &= \int_{\mathbb{C}} \left|f\left(az + \frac{b(1-a^n)}{1-a}\right)\right|^p e^{-p\varphi(z)} dA(z).\end{aligned} \tag{4.2.1}$$

From (4.2.1) and the fact that $|a| < 1$,

$$\begin{aligned}
\|C_\psi^n f\|_{(\varphi,p)}^p &\leq \int_{\mathbb{C}} \left| f\left(az + \frac{b(1-a^n)}{1-a}\right)\right|^p e^{-p\varphi(z)} dA(z) \\
&= \int_{\mathbb{C}} \left| f\left(az + \frac{b(1-a^n)}{1-a}\right)\right|^p e^{-p\varphi\left(az+\frac{b(1-a^n)}{1-a}\right)} e^{p\varphi\left(az+\frac{b(1-a^n)}{1-a}\right)-p\varphi(z)} dA(z) \\
&\leq \sup_{z\in\mathbb{C}} e^{p\varphi\left(az+\frac{b(1-a^n)}{1-a}\right)-p\varphi(z)} \int_{\mathbb{C}} \left| f\left(az + \frac{b(1-a^n)}{1-a}\right)\right|^p e^{-p\varphi\left(az+\frac{b(1-a^n)}{1-a}\right)} dA(z) \\
&= \frac{1}{|a|^2} \sup_{z\in\mathbb{C}} e^{p\varphi\left(az+\frac{b(1-a^n)}{1-a}\right)-p\varphi(z)} \int_{\mathbb{C}} |f(w)|^p e^{-p\varphi(w)} dA(w) \\
&\leq \frac{1}{|a|^2} \sup_{z\in\mathbb{C}} e^{p\varphi\left(|a||z|+\frac{2|b|}{|1-a|}\right)-p\varphi(|z|)} \|f\|_{(\varphi,p)}^p.
\end{aligned}$$

Hence

$$\|C_\psi^n\| \leq \frac{1}{|a|^2} \sup_{z\in\mathbb{C}} e^{\varphi\left(|a||z|+\frac{2|b|}{|1-a|}\right)-\varphi(|z|)}.$$

For the case $|a| = 1$, if $f \in \mathcal{F}_\varphi^p$, then

$$\begin{aligned}
\|C_\psi^n f\|_{(\varphi,p)}^p &= \int_{\mathbb{C}} |f(\psi^n(z))|^p e^{-p\varphi(z)} dA(z) \\
&= \int_{\mathbb{C}} |f\big(a^n z\big)|^p e^{-p\varphi(a^n z)} dA(z) = \|f\|_{(\varphi,p)}^p.
\end{aligned}$$

From which we deduce that

$$\sup_{n\in\mathbb{N}} \|C_\psi^n\| = 1.$$

Case 2 : $p = \infty$. For $|a| < 1$, set $R_0 = \frac{4|b|}{(1-|a|)|1-a|}$. Then for each $|z| \geq R_0$ and $n \in \mathbb{N}$,

$$|\psi^n(z)| = \left|a^n z + \frac{b(1-a^n)}{(1-a)}\right| \leq |a||z| + \frac{2|b|}{|1-a|} \leq \left(|a| + \frac{1-|a|}{2}\right)|z| < |z|.$$

Taking $f \in \mathcal{F}_\varphi^\infty$ with $\|f\|_{(\varphi,\infty)} = 1$, we have $|f(z)| \leq e^{\varphi(z)}$ for any $z \in \mathbb{C}$.

Thus

$$\sup_{|z|>R_0} |C_\psi^n f(z)| e^{-\varphi(z)} = \sup_{|z|>R_0} \left| f\left(a^n z + \frac{b(1-a^n)}{1-a}\right)\right| e^{-\varphi(z)}$$
$$\leq \sup_{|z|>R_0} e^{\left(\varphi\left(a^n z + \frac{b(1-a^n)}{1-a}\right) - \varphi(z)\right)} \leq 1,$$

and

$$\sup_{|z|\leq R_0} |C_\psi^n f(z)| e^{-\varphi(z)} \leq \sup_{|z|\leq R_0} \left| f\left(a^n z + \frac{b(1-a^n)}{1-a}\right)\right| \sup_{|z|\leq R_0} e^{-\varphi(z)}$$
$$= \max_{|z|=R_0} \left| f\left(a^n z + \frac{b(1-a^n)}{1-a}\right)\right| \leq \max_{|z|=R_0} e^{\varphi\left(a^n z + \frac{b(1-a^n)}{1-a}\right)} \leq e^{\varphi(R_0)}.$$

From this we get that, for each $n \in \mathbb{N}$,

$$\|C_\psi^n\| = \sup_{\|f\|_{(\varphi,\infty)}=1} \|C_\psi^n f\|_{(\varphi,\infty)} \leq 1 + e^{\varphi(R_0)}.$$

If $|a| = 1$, then

$$\|C_\psi^n f\|_{(\varphi,\infty)} = \sup_{z\in\mathbb{C}} |f(a^n z)| e^{-\varphi(z)} = \sup_{z\in\mathbb{C}} |f(a^n z)| e^{-\varphi(a^n z)} = \|f\|_{(\varphi,\infty)},$$

which shows

$$\sup_{n\in\mathbb{N}} \|C_\psi^n\| = 1.$$

□

Corollary 4.2.2. *Every bounded composition operator C_ψ on $\mathcal{F}_\varphi^0$ is power bounded.*

Proof. The conclusion follows from Theorem 3.1.2 and Theorem 4.2.1, since the restriction of a power bounded operator is clearly power bounded.

Nagy [69] showed that every power bounded operator T such that T^{-1} exists and is power bounded is polynomially bounded . This leads to the following corollary.

Corollary 4.2.3. *Let $1 \leq p \leq \infty$ and C_ψ be a non-compact bounded operator on $\mathcal{F}_\varphi^p$. Then it is polynomially bounded.*

4.3 Mean ergodic and uniformly mean ergodic composition operators

In this section we identify mean ergodic and uniformly mean ergodic composition operators on $\mathcal{F}^p_\varphi$. Our first result in this perspective states

Theorem 4.3.1. *The composition operator* $C_\psi : \mathcal{F}^p_\varphi \to \mathcal{F}^p_\varphi$, $\psi(z) = az + b,\ |a| < 1$, *for* $1 \leq p < \infty$ *or* $p = 0$, *satisfies, for each* $f \in \mathcal{F}^p_\varphi$,

$$\lim_{n\to\infty} \|C^n_\psi f - C_{\frac{b}{1-a}} f\|_{(\varphi,p)} = 0.$$

Proof. *Case 1:* $1 \leq p < \infty$. Let $f \in \mathcal{F}^p_\varphi$, $1 \leq p < \infty$. Since C_ψ is power bounded, there is a constant $M > 0$ such that

$$\|C^n_\psi f\|_{(\varphi,p)} \leq M\|f\|_{(\varphi,p)}$$

for every $n \in \mathbb{N}$. Equivalently

$$\int_{\mathbb{C}} \left|C^n_\psi f(z)\right|^p e^{-p\varphi(z)} dA(z) \leq \int_{\mathbb{C}} M^p |f(z)|^p e^{-p\varphi(z)} dA(z). \tag{4.3.1}$$

Moreover, by continuity of f,

$$\lim_{n\to\infty} \left|C^n_\psi f(z) - f\left(\frac{b}{1-a}\right)\right| = 0,$$

and using (4.3.1),

$$\begin{aligned}
&\int_{\mathbb{C}} \left|C^n_\psi f(z) - f\left(\frac{b}{1-a}\right)\right|^p e^{-p\varphi(z)} dA(z) \\
&\qquad \leq \int_{\mathbb{C}} 2^p \left[\left|C^n_\psi f(z)\right|^p + \left|f\left(\frac{b}{1-a}\right)\right|^p\right] e^{-p\varphi(z)} dA(z) \\
&\qquad = \int_{\mathbb{C}} 2^p \left|C^n_\psi f(z)\right|^p e^{-p\varphi(z)} dA(z) + \int_{\mathbb{C}} 2^p \left|f\left(\frac{b}{1-a}\right)\right|^p e^{-p\varphi(z)} dA(z) \\
&\qquad \leq \int_{\mathbb{C}} 2^p M^p |f(z)|^p e^{-p\varphi(z)} dA(z) + \int_{\mathbb{C}} 2^p \left|f\left(\frac{b}{1-a}\right)\right|^p e^{-p\varphi(z)} dA(z) < \infty.
\end{aligned}$$

Applying Lebesgue dominated convergent theorem on the sequence

$$g_n(z) := \left| f\left(a^n z + \frac{b(1-a^n)}{1-a}\right) - f\left(\frac{b}{1-a}\right)\right|^p e^{-p\varphi(z)},$$

we get that

$$\lim_{n\to\infty} \|C_\psi^n f - C_{\frac{b}{1-a}} f\|_{(\varphi,p)}^p = \lim_{n\to\infty} \int_{\mathbb{C}} g_n(z)dA(z) = \int_{\mathbb{C}} \lim_{n\to\infty} g_n(z)dA(z) = 0.$$

Case 2: $p = 0$. Let $f \in \mathcal{F}_\varphi^0$. Since $|a| < 1$, $C_\varphi^n f$ is also in $\mathcal{F}_\varphi^0$. To show this

$$\begin{aligned}
\lim_{|z|\to\infty} \left|C_\varphi^n f(z)\right| e^{-\varphi(z)} &= \lim_{|z|\to\infty} \left| f\left(a^n z + \frac{b(1-a^n)}{1-a}\right)\right| e^{-\varphi(z)} \\
&= \lim_{|z|\to\infty} \left| f\left(a^n z + \frac{b(1-a^n)}{1-a}\right)\right| e^{-\varphi\left(a^n z+\frac{b(1-a^n)}{1-a}\right)} e^{\varphi\left(a^n z+\frac{b(1-a^n)}{1-a}\right)} e^{-\varphi(z)} \\
&\le \sup_{z\in\mathbb{C}} e^{\varphi\left(a^n z+\frac{b(1-a^n)}{1-a}\right)-\varphi(z)} \lim_{|z|\to\infty} |f(z)| e^{-\varphi(z)} = 0.
\end{aligned}$$

Moreover, as $n \to \infty$

$$\psi^n(z) = a^n z + \frac{b(1-a^n)}{1-a} \to \frac{b}{1-a}$$

uniformly on compact subsets K of $\mathbb{C}$, since

$$\begin{aligned}
\left|\psi^n(z) - \frac{b}{1-a}\right| &= \left|a^n z + \frac{b(1-a^n)}{1-a} - \frac{b}{1-a}\right| \\
&= |a^n| \left|z - \frac{b}{1-a}\right| \le |a^n| \left(\max_{w\in K} |z| + \left|\frac{b}{1-a}\right|\right) \to 0.
\end{aligned}$$

From this we have

$$f(\psi^n(z)) \to f\left(\frac{b}{1-a}\right)$$

uniformly on the compact subsets of $\mathbb{C}$. That is, for each compact set K in $\mathbb{C}$,

$$\sup_{z\in K} \left| f(\psi^n(z)) - f\left(\frac{b}{1-a}\right)\right| \to 0 \tag{4.3.2}$$

as $n \to \infty$.

Given $\varepsilon > 0$. Since $f \in \mathcal{F}^0_\varphi$, and hence $C^n_\psi f$ and constant functions are in $\mathcal{F}^0_\varphi$, we find $r_0 > 0$ such that

$$\left|C^n_\varphi f(z)\right| e^{-\varphi(z)} < \frac{\varepsilon}{4} \text{ and } |f(\frac{b}{1-a})|e^{-\varphi(z)} < \frac{\varepsilon}{4}$$

if $|z| > r_0$. Then, for each $|z| > r_0$ and $n \in \mathbb{N}$, we have

$$\left|C^n_\psi f(z) - f\left(\frac{b}{1-a}\right)\right| e^{-\varphi(z)} \leq \left|C^n_\varphi f(z)\right| e^{-\varphi(z)} + \left|f\left(\frac{b}{1-a}\right)\right| e^{-\varphi(z)} < \frac{\varepsilon}{2}.$$

We apply (4.3.2) to the compact set $K_0 = \{z \in \mathbb{C} : |z| \leq r_0\}$ to find n_0 such that if $z \in K_0$ and $n \geq n_0$ we have

$$\left|f(\psi^n(z)) - f\left(\frac{b}{1-a}\right)\right| < \frac{\varepsilon}{2S},$$

with $S := \max_{z \in K_0} e^{-\varphi(z)}$. If $n \geq n_0$ and $z \in \mathbb{C}$, we have

$$\left|C^n_\psi f(z) - f\left(\frac{b}{1-a}\right)\right| e^{-\varphi(z)} < \varepsilon.$$

Thus

$$\lim_{n\to\infty} \|C^n_\psi f - C_{\frac{b}{1-a}} f\|_{(\varphi,\infty)} = 0.$$

The next theorem ensures that all compact composition operators on $\mathcal{F}^p_\varphi$, $1 \leq p < \infty$ or $p = 0$ are mean ergodic.

Theorem 4.3.2. *The composition operator* $C_\psi : \mathcal{F}^p_\varphi \to \mathcal{F}^p_\varphi$, $\psi(z) = az + b$, $|a| < 1$, *for* $1 \leq p < \infty$ *or* $p = 0$, *is mean ergodic.*

Proof. We show that

$$\lim_{n\to\infty} \left\| \frac{1}{n} \sum_{k=1}^{n} C^k_\psi f - C_{\frac{b}{1-a}} f \right\|_{(\varphi,p)} = 0.$$

By Theorem 4.3.1, for $\epsilon > 0$ there is a positive integer N such that

$$\|C_\psi^n f - C_{\frac{b}{1-a}} f\|_{(\varphi,p)} < \epsilon$$

whenever $n > N$. Thus

$$\begin{aligned}
&\lim_{n\to\infty}\left\|\frac{1}{n}\sum_{k=1}^{n} C_\psi^k f - C_{\frac{b}{1-a}} f\right\|_{(\varphi,p)} \\
&\quad \le \lim_{n\to\infty}\frac{1}{n}\left[\sum_{k=1}^{N}\left\|C_\psi^k f - C_{\frac{b}{1-a}} f\right\|_{(\varphi,p)} + (n-N)\epsilon\right] < \epsilon.
\end{aligned}$$

Since ϵ is arbitrary, the result follows.

We recall that a bounded operator is called quasicompact if there exist a positive integer m and a compact operator K such that

$$\|T^m - K\| < 1.$$

From the definition we observe that every compact operator is quasicompact. We state the following proposition which is taken from *[75] Theorem 4 and Corollary on page 204-205* .

Proposition 4.3.3. *If T is power bounded and quasicompact on a Banach space, then T is uniformly mean ergodic.*

Theorem 4.3.4. *Let $\psi(z) = az + b, |a| < 1$. Then the operator $C_\psi : \mathcal{F}_\varphi^p \to \mathcal{F}_\varphi^p$, $1 \le p \le \infty$ or $p = 0$, is uniformly mean ergodic and*

$$\lim_{n\to\infty}\left\|\frac{1}{n}\sum_{k=1}^{n} C_\psi^k - C_{\frac{b}{1-a}}\right\| = 0. \tag{4.3.3}$$

Proof. The operator $C_\psi : \mathcal{F}_\varphi^p \to \mathcal{F}_\varphi^p$, $1 \le p \le \infty$ or $p = 0$, defined by $\psi(z) = az + b, |a| < 1$ is compact by Theorem 3.1.2, and it is power bounded by Theorem 4.2.1 and Corollary 4.2.2. Therefore, C_ψ is uniformly mean ergodic for $1 \le p \le \infty$ by the Proposition 4.3.3. Combining this with Theorem 4.3.2 we get, for $1 \le p < \infty$

or $p = 0$,

$$\lim_{n\to\infty} \left\| \frac{1}{n}\sum_{k=1}^{n} C_\psi^k - C_{\frac{b}{1-a}} \right\| = 0. \tag{4.3.4}$$

Next, we show (4.3.3) for $p = \infty$. From the well-known fact $\|T\| = \|T^{'}\| = \|T^{''}\|$ for any bounded operator T on a Banach space, and from (4.3.4) we have

$$\lim_{n\to\infty} \left\| \frac{1}{n}\sum_{k=1}^{n} C_\psi^k - C_{\frac{b}{1-a}} \right\|_{(\varphi,0)} = 0$$

implies

$$\lim_{n\to\infty} \left\| \left(\frac{1}{n}\sum_{k=1}^{n} C_\psi^k\right)^{''} - \left(C_{\frac{b}{1-a}}\right)^{''} \right\|_{(\varphi,0)} = 0.$$

Since $\mathcal{F}_\varphi^\infty$ is canonically isomorphic to the bidual of $\mathcal{F}_\varphi^0$, and the bi-transpose operator $C_\psi^{''}$ of $C_\psi : \mathcal{F}_\varphi^0 \to \mathcal{F}_\varphi^0$ coincides with composition operator $C_\psi : \mathcal{F}_\varphi^\infty \to \mathcal{F}_\varphi^\infty$, we get that

$$\lim_{n\to\infty} \left\| \frac{1}{n}\sum_{k=1}^{n} C_\psi^k - C_{\frac{b}{1-a}} \right\|_{(\varphi,\infty)} = 0.$$

It is well-known that every periodic operator is uniformly mean ergodic. We include the proof of the next result for the sake of completeness.

Proposition 4.3.5. *The composition operator C_ψ, where $\psi(z) = e^{i\theta}z$, $0 < \theta \leq 2\pi$ and $\frac{2k\pi}{\theta} = m$ for some positive integers m, k, is uniformly mean ergodic on $\mathcal{F}_\varphi^p$, $1 \leq p \leq \infty$ or $p = 0$.*

Proof. Consider the smallest positive integer m such that $\frac{2k\pi}{\theta} = m$ for some positive integers k. In this case the sequence C_ψ^n is periodic with period m. Any $n \in \mathbb{N}$ can be written in the form of $n = ml + j$ for some $l \in \mathbb{N}$ and

$j = 0, 1, 2, ..., m-1$. Thus

$$\begin{aligned}\lim_{n\to\infty} \left\| \frac{1}{n}\sum_{k=1}^{n} C_\psi^k - \frac{1}{m}\sum_{k=1}^{m} C_\psi^k \right\|_{(\varphi,p)} &= \lim_{l\to\infty} \frac{1}{(ml+j)} \left\| \sum_{k=1}^{j} C_\psi^k - \frac{j}{m}\sum_{k=1}^{m} C_\psi^k \right\|_{(\varphi,p)} \\ &\leq \lim_{l\to\infty} \frac{1}{(ml+j)} \left(\sum_{k=1}^{j} \|C_\psi^k\|_{(\varphi,p)} + \frac{j}{m}\sum_{k=1}^{m} \|C_\psi^k\|_{(\varphi,p)} \right) = 0.\end{aligned}$$

The following lemma is a main tool for the proof of Theorem 4.3.7.

Lemma 4.3.6. *[[3], Lemma 2.1] Let $\{T_n\}$ be a sequence of equicontinuous operator on locally convex space E. If $\{T_n\}$ is pointwise convergent to a continuous operator T on some dense set $D \subseteq E$, then $\{T_n\}$ is pointwise convergent to T in E.*

From the definition of power bounded, if T on some Banach space X is power bounded, there is $M > 0$ such that

$$\|T^n f\| \leq M\|f\| \text{ for all } f \in X.$$

This implies that $\{T^n\}_n$ is equicontinous. Thus, the assertion in Lemma 4.3.6 holds true if equicontinuous is replaced by power bounded.

An operator T is uniformly mean ergodic if and only if it is power bounded and either 1 is in $\mathbb{C}\backslash\sigma(T)$ or 1 is a pole of order 1 of the resolvent $R_T(\lambda) = (T - \lambda I)^{-1}$ [Theorem 3.16 in [18]]. In particular, if 1 is in $\sigma(T)$ and an accumulation point of $\sigma(T)$, then T is not uniformly mean ergodic. Our next two result shows non compact composition operator C_ψ, $\psi(z) = az, |a| = 1$ and a is not root of unity, is not uniformly mean ergodic.

Theorem 4.3.7. *If $\psi(z) = az$, $|a| = 1$, and a is not root of unity, then the composition operator $C_\psi : \mathcal{F}_\varphi^p \to \mathcal{F}_\varphi^p$, $1 \leq p < \infty$ or $p = 0$ is mean ergodic but not uniformly mean ergodic.*

Proof. First we show that

$$\lim_{n\to\infty}\left\|\frac{1}{n}\sum_{k=1}^{n}C_{\psi}^{k}f-C_0f\right\|_{(\varphi,p)}=0 \tag{4.3.5}$$

when f is a polynomial. It is enough to check it when f belongs to the sequence of monomials $\{1, z, z^2, ...\}$. If $f(z) = 1$, then $C_\psi f$ is the constant function 1, and hence (4.3.5) holds. If $f(z) = z^m$ for some $m \in \mathbb{N}$,

$$\begin{aligned}\left\|\frac{1}{n}\sum_{k=1}^{n}C_{\psi}^{k}f\right\|_{(\varphi,p)} = \left\|\frac{1}{n}\sum_{k=1}^{n}a^{mk}z^{m}\right\|_{(\varphi,p)} &= \frac{|a^m||1-a^{mn}|}{n|1-a^m|}\left\|z^m\right\|_{(\varphi,p)}\\ &\le \frac{2}{n|1-a^m|}\left\|z^m\right\|_{(\varphi,p)} \to 0\end{aligned}$$

as $n \to \infty$. Since the set of polynomials is dense in $\mathcal{F}_{\varphi}^{p}$ and C_φ is power bounded and hence $\{C_\varphi^n\}_n$ is equicontinuous on $\mathcal{F}_{\varphi}^{p}$ by Lemma 4.3.6 we have ,

$$\lim_{n\to\infty}\left\|\frac{1}{n}\sum_{k=1}^{n}C_{\psi}^{k}f-C_0f\right\|_{(\varphi,p)}=0$$

for every $f \in \mathcal{F}_{\varphi}^{p}$. This implies that $C_\psi : \mathcal{F}_{\varphi}^{p} \to \mathcal{F}_{\varphi}^{p}$ is mean ergodic.

Next, we prove that C_ψ is not uniformly mean ergodic. By Theorem 4.1.1, $\sigma(C_\psi) = \overline{\{a^n : n = 0, 1, 2, ...\}}$. This shows $1 \in \sigma(C_\psi)$ and 1 is an accumulation point of $\sigma(C_\psi)$. Therefore by Theorem 3.16 in [18] C_ψ is not uniformly mean ergodic.

Recall that a Banach space X is a *Grothendieck space* if every sequence (x_n) in X' which is convergent to 0 for the weak topology $\sigma(X', X)$ is also convergent to 0 for the weak topology $\sigma(X', X'')$. The space X has the *Dunford-Pettis property* if for any sequence (x_n) in X which is convergent to 0 for the weak topology $\sigma(X, X')$ and any sequence (x_n') in X' which is convergent to 0 for the weak topology $\sigma(X', X'')$ one gets $\lim_{n\to\infty}\langle x_n, x_n'\rangle$. The spaces ℓ^∞ or $H^\infty(\mathbb{D})$ are examples of Grothendieck spaces with Dunford-Pettis property [37]. We record the following proposition from Lotz, [38]

Proposition 4.3.8. *Let X be a Banach space, which is a Grothendieck space with the Dunford-Pettis property. Let $T \in L(X)$ be a power bounded operator. Then T is mean ergodic if and only if T is uniformly mean ergodic.*

Our next Theorem characterizes ergodicity of non-compact composition operator C_{az} on $\mathcal{F}_{\varphi}^{\infty}$ when a is not root of unity.

Theorem 4.3.9. *If $\psi(z) = az$, $|a| = 1$, and a is not root of unity, the composition operator $C_{\psi} : \mathcal{F}_{\varphi}^{\infty} \to \mathcal{F}_{\varphi}^{\infty}$, is not mean ergodic, hence not uniformly mean ergodic.*

Proof. Since $1 \in \sigma(C_{\psi}) = \overline{\{a^n : n = 0, 1, 2, ...\}}$ and 1 is an accumulation point of $\sigma(C_{\psi})$, we apply Theorem 3.16 in [18] to conclude that C_{ψ} is not uniformly mean ergodic on $\mathcal{F}_{\varphi}^{\infty}$. On the other hand, Theorem 1.1 in [39] implies that $\mathcal{F}_{\varphi}^{\infty}$ is isomorphic to ℓ^{∞} or $H^{\infty}(\mathbb{D})$. Hence $\mathcal{F}_{\varphi}^{\infty}$ is Grothendieck spaces with Dunford-Pettis property. By a result of Lotz [38], Proposition 4.3.8 every power bounded mean ergodic operator on a Grothendieck Banach space with the Dunford-Pettis property is uniformly mean ergodic. Therefore, C_{ψ} is not mean ergodic in $\mathcal{F}_{\varphi}^{\infty}$.

4.4 Cyclic composition operators

In [31], Cyclic and supercyclic properties of composition operators were studied on classical Fock spaces $\mathcal{F}_2$. It is a natural problem to ask what happen to these phenomena for other exponents p, and on the generalized Fock spaces $\mathcal{F}_{\varphi}^{p}$. We start with the stronger hypercyclic property. Our next theorem describes that no bounded composition operator on $\mathcal{F}_{\varphi}^{p}$ can be hypercyclic.

Theorem 4.4.1. *Let $1 \leq p \leq \infty$ and $\psi(z) = az + b$ induces bounded composition operator C_{ψ} on $\mathcal{F}_{\varphi}^{p}$. Then the operator C_{ψ} can not be hypercyclic.*

When the weight function becomes $\varphi_m(z) = |z|^m, 0 < m \leq 1$, then as shown in [24], each nontrivial translation operator acting on the corresponding Fock-type spaces is hypercyclic. Since a bounded C_{ψ} on such spaces happens if and only if $\psi(z) = az + b$ with $|a| \leq 1$, by setting $a = 1$ and $b \neq 0$, we observe that C_{ψ}

reduces to the translation operator T_b which is hypercyclic.

Proof. If $|a| < 1$, then the operator C_ψ is compact and hence by Corollary 1.22 in [2], it can not be hypercyclic. On the other hand, if $|a| = 1$, we may deny the assertion and assume that the operator is hypercyclic with hypercyclic vector f. By extracting a subsequence ψ^{n_k} such that $\psi^{n_k}z \to az$ as $k \to \infty$, we observe that for any univalent function g in the orbit of f,

$$g(z) = \lim_{k\to\infty} C_{\psi^{n_k}} f(z) = \lim_{k\to\infty} C_{\psi^{n_k}} f(z) = f(az).$$

It follows that f itself is a univalent function and hence its orbit contains only univalent functions which is a contradiction.

We may now ask for the weaker supercyclicity property. Our next theorem shows C_ψ is not supercyclic either.

Theorem 4.4.2. *Let $1 \le p \le \infty$ and $\psi(z) = az + b$ be a non-constant map on $\mathbb{C}$ that induces a bounded composition operator C_ψ on $\mathcal{F}^p_\varphi$. Then C_ψ can not be supercyclic on $\mathcal{F}^p_\varphi$.*

The supercyclicity problem has not been solved in the classical Fock spaces settings either except for the Hilbert space case which was studied in [31]. Our approach, which neither uses Hilbert spaces techniques nor the fast growth property of the weight function, shows that the same result holds for all $1 \le p \le \infty$ on the classical spaces as well.

Proof. First we prove that C_ψ can not be supercyclic on $\mathcal{F}^p_\varphi$, $1 \le p < \infty$. We set $\psi(z) = az+b$ and argue in the direction of contradiction, and assume that C_ψ has a supercyclic vector $f \in \mathcal{F}^p_\varphi$. If $0 < |a| < 1$, then ψ fixes the point $b/(1-a)$. It follows that $f(b/(1-a)) \neq 0$. If not, the projective orbit contains only functions which vanishes at $b/(1-a)$. Now for each function g in the projective orbit of f, there exists a sequence (λ_{n_k}) such that

$$\lim_{k\to\infty} \|\lambda_{n_k} C_\psi^{n_k} f - g\|_{(\varphi,p)} = 0.$$

Then we compute

$$g\Big(\frac{b}{1-a}\Big) = \lim_{k\to\infty} \lambda_{n_k} C_{\psi}^{n_k} f\Big(\frac{b}{1-a}\Big) = \lim_{k\to\infty} \lambda_k C_{\psi^{n_k}} f\Big(\frac{b}{1-a}\Big) = f\Big(\frac{b}{1-a}\Big) \lim_{k\to\infty} \lambda_{n_k},$$

where we used here the fact that norm convergence implies point-wise convergence. Thus, for all $z \in \mathbb{C}$, applying the fact that $a^{n_k} \to 0$ as $k \to \infty$

$$\begin{aligned} g(z) = \lim_{k\to\infty} \lambda_{n_k} C_{\psi}^{n_k} f(z) &= \lim_{k\to\infty} \lambda_{n_k} f\Big(a^{n_k} z + \frac{b(1-a^{n_k})}{1-a}\Big) \\ &= \left[f\left(\frac{b}{1-a}\right)\right]^{-1} g\Big(\frac{b}{1-a}\Big) \lim_{k\to\infty} f\Big(a^{n_k} z + \frac{b(1-a^{n})}{1-a}\Big) \\ &= \left[f\left(\frac{b}{1-a}\right)\right]^{-1} g\Big(\frac{b}{1-a}\Big) f\Big(\frac{b}{1-a}\Big) = g\Big(\frac{b}{1-a}\Big), \end{aligned}$$

showing that only constant functions are in the projective orbit of f resulting a contradiction. If $|a| = 1$, then $\psi(z) = az$ and it fixes the origin. We may choose a univalent function $g \in \mathcal{F}_{\varphi}^{p}$ such that $g(0) \neq 0$, and pick a subsequence ψ^{n_k} such that $\psi^{n_k}(z) \to az$ as $k \to \infty$. Then

$$g(z) = \lim_{k\to\infty} \lambda_{n_k} C_{\psi}^{n_k} f(z) = \lim_{k\to\infty} \lambda_{n_k} f(a^{n_k} z) = g(0) f(az).$$

It follows that

$$f(z) = \frac{1}{g(0)} g\Big(\frac{z}{a}\Big) \tag{4.4.1}$$

is univalent. Consequently, the projective orbits of f contains only univalent functions which is again a contradiction. Next, we prove the result on $\mathcal{F}_{\varphi}^{\infty}$. Since C_{ψ} is bounded, by Theorem 3.1.2, we can set $\psi(z) = az + b, \ \ |a| \leq 1$. If $|a| < 1$, then C_{ψ} is compact. The spectrum of a compact supercyclic operator on an infinite dimensional complex Banach space contains only the zero element: see [2, p. 29]. On the other hand, by Theorem 4.1.1, the spectrum $\sigma(C_{\psi})$ contains infinitely many elements showing that C_{ψ} can not be supercyclic in this case. When $|a| = 1$, then the same argument as above gives the same conclusion.

From Theorem 4.4.1 and Theorem 4.4.2, we observe that no orbit or projective orbit is dense in the space $\mathcal{F}_\varphi^p$. Now we turn our attention to the density of span of an orbit in $\mathcal{F}_\varphi^p$.

Theorem 4.4.3. *Let $\psi(z) = az + b$ be a non-constant map on $\mathbb{C}$ that induces a bounded composition operator C_ψ on $\mathcal{F}_\varphi^p$. Then*

(i) C_ψ is cyclic on $\mathcal{F}_\varphi^p$, $1 \leq p < \infty$ or $p = 0$ if and only if $a^n \neq a$ for all $n > 1$. Furthermore, a function $h \in \mathcal{F}_\varphi^p$ with Taylor series expansion

$$h(z) = \sum_{n=0}^{\infty} a_n \left(z - \frac{b}{1-a} \right)^n$$

is a cyclic vector for C_ψ if and only if $a_n \neq 0$ for all $n \in \mathbb{Z}_+ := \{0, 1, 2, 3, ...\}$.

(ii) C_ψ can not be cyclic on $\mathcal{F}_\varphi^\infty$.

The theorem was proved in Hilbert Fock spaces for the case where φ is Gaussian weight $\varphi(z) = \frac{1}{2}|z|^2$ and for the case $\varphi(z) = |z|^s$, $s < 1$ in [31] and [24] respectively. Their proof were based on Hilbert space properties. In our proof we will follow the same approach but replacing all the Hilbert space arguments by other general arguments. Our argument shows that the same result holds for all $1 \leq p < \infty$ or $p = 0$ on the classical spaces as well.

Proof. *(i)* Let us first assume that C_ψ is cyclic and prove the necessity of the condition. Arguing on the contrary, if $a^k = a$ for some $k \geq 2$, then $|a| = 1$ and hence $\psi(z) = az$. For any cyclic vector f_0 in $\mathcal{F}_\varphi^p$, it follows that $C_\psi^k f_0(z) = f_0(a^k z) = f_0(az) = C_\psi f_0(z)$ which implies

$$\{C_\psi^n f_0, n \in \mathbb{Z}_+\} = \{C_\psi^n f_0 : \; n = 0, 1, 2, 3, ...k\}.$$

This shows that the closed linear span of the orbit is finite dimensional, and hence C_ψ can not be cyclic. Conversely, suppose $\psi(z) = az + b$ and $a^n \neq a$ for every

$n \geq 2$ which obviously implies that $a \neq 1$. Then we proceed to show that there exists a cyclic vector $h \in \mathcal{F}_\varphi^p$ with Taylor series expansion at $z = \frac{b}{1-a}$

$$h(z) = \sum_{n=0}^{\infty} a_n \Big(z - \frac{b}{1-a} \Big)^n.$$

Let us first make a short argument verifying the necessity that for h to be a cyclic vector, $a_n \neq 0$ for all $n \in \mathbb{Z}_+$. If $a_n = 0$ for some $n = m$, it follows from the fact that

$$C_\psi^k h(z) = \sum_{n=0}^{\infty} a_n a^{kn} \Big(z - \frac{b}{1-a} \Big)^n,$$

all functions f in the closed linear span of $\big\{ C_\psi^k h : k \in \mathbb{Z}_+ \big\}$ satisfy $\frac{d^m}{dz^m} f \big|_{z=\frac{b}{1-a}} = 0$ which contradicts the cyclic behaviour of h. We may now consider the case when $|a| = 1$ and hence $b = 0$. This together with the assumption $a^n \neq a$ for every $n \geq 2$ imply

$$\overline{\{a^k, k \in \mathbb{Z}_+\}} = \mathbb{T} = \{z \in \mathbb{C} : |z| = 1\}.$$

Thus, for each $w \in \mathbb{T}$ there exists a sequence $\{k_j\}_j$ in $\mathbb{Z}_+$ such that $a^{k_j} \to w$ as $j \to \infty$. Let $\psi_w(z) = wz$. Then we claim that

$$\lim_{j \to \infty} \|C_\psi^{k_j} h - C_{\psi_w} h\|_{(\varphi,p)} = 0 \tag{4.4.2}$$

for $1 \leq p < \infty$ or $p = 0$.

If $1 \leq p < \infty$, using the radial property $\varphi(a^{k_j} z) = \varphi(z)$ and change of variables, we compute

$$\begin{aligned}\lim_{j\to\infty}\|C_\psi^{k_j}h\|^p_{(\varphi,p)} &= \lim_{j\to\infty}\int_{\mathbb{C}}|h(a^{k_j}z)|^p e^{-p\varphi(a^{k_j}z)}dA(z)\\ =\lim_{j\to\infty}\frac{1}{|a^{k_j}|^2}\int_{\mathbb{C}}|h(z)|^p e^{-p\varphi(z)}dA(z) &= \frac{1}{|w|^2}\int_{\mathbb{C}}|h(z)|^p e^{-p\varphi(z)}dA(z)\\ &= \int_{\mathbb{C}}|h(wz)|^p e^{-p\varphi(z)}dA(z) = \|C_{\psi_w}h\|^p_{(\varphi,p)}\end{aligned}$$

from which (4.4.2) follows.

Next, we show that (4.4.2) holds if $p = 0$ also. To this end, first we show that $C_\psi^{k_j}h - C_{\psi_w}h$ belongs to $\mathcal{F}^0_\varphi$ for any $j \in \mathbb{Z}$.

$$\begin{aligned}&\lim_{|z|\to\infty}|C_\psi^{k_j}h(z) - C_{\psi_w}h(z)|e^{-\varphi(z)}\\ &\qquad\le \lim_{|z|\to\infty}|C_\psi^{k_j}h(z)|e^{-\varphi(z)} + \lim_{|z|\to\infty}|C_{\psi_w}h(z)|e^{-\varphi(z)}\\ &\qquad= \lim_{|z|\to\infty}h(a^{k_j}z)|e^{-\varphi(z)} + \lim_{|z|\to\infty}|h(wz)|e^{-\varphi(z)}\\ &\qquad= \lim_{|z|\to\infty}h(a^{k_j}z)|e^{-\varphi(a_{k_j}z)} + \lim_{|z|\to\infty}|h(wz)|e^{-\varphi(wz)} = 0. \qquad (4.4.3)\end{aligned}$$

For any $R > 0$.

$$\begin{aligned}\lim_{j\to\infty}\|C_\psi^{k_j}h - C_{\psi_w}h\|_{(\varphi,0)} &= \lim_{j\to\infty}\sup_{z\in\mathbb{C}}|C_\psi^{k_j}h(z) - C_{\psi_w}h(z)|e^{-\varphi(z)}\\ \le \lim_{j\to\infty}\left[\sup_{|z|\le R}|h(a^{k_j}z) - h(wz)|e^{-\varphi(z)}\right] &+ \lim_{j\to\infty}\left[\sup_{|z|>R}|h(a^{k_j}z) - h(wz)|e^{-\varphi(z)}\right]\\ = \sup_{|z|\le R}\left[\lim_{j\to\infty}|h(a^{k_j}z) - h(wz)|\right]e^{-\varphi(z)} &+ \lim_{j\to\infty}\left[\sup_{|z|>R}|h(a^{k_j}z) - h(wz)|e^{-\varphi(z)}\right]\\ &= 0 + \lim_{j\to\infty}\left[\sup_{|z|>R}|h(a^{k_j}z) - h(wz)|e^{-\varphi(z)}\right].\end{aligned}$$

Applying (4.4.3) and letting $R \to \infty$ the last term is also 0. Therefore

$$\lim_{j\to\infty}\|C_\psi^{k_j}h - C_{\psi_w}h\|_{(\varphi,0)} = 0.$$

This verifies that $C_{\psi_w}h$ belongs to the closed linear span of $\left\{C_{\psi}^{k}h : k \in \mathbb{Z}_+\right\} \subseteq \mathcal{F}_{\varphi}^{p},\ 1 \leq p < \infty$ or $p = 0$.

The mapping $G : \mathbb{T} \to \mathcal{F}_{\varphi}^{p}$ defined by $G(w) = C_{\psi_w}h$ is continuous, and can be extended to analytic function $\tilde{G}$ in $\mathbb{D}$ with $\tilde{G}(w) = G(w)$ on the boundary of $\mathbb{D}$. Then, by Cauchy Integral Formula, using $C_{\psi_w}(z) = G(w)(z) = \tilde{G}(w)(z))$

$$a_n z^n = \frac{1}{2\pi i}\int_{|w|=1} \frac{C_{\psi_w}h(z)}{w^{n+1}}dw.$$

Hence the set of polynomials $a_n z^n, \quad n \in \mathbb{Z}_+$ belongs to the closed linear span of $\left\{C_{\psi}^{k}h : k \in \mathbb{Z}_+\right\}$. From this, the fact that $a_n \neq 0$ for all $n \in \mathbb{Z}_+$, and Theorem 2.2.1, the conclusion of the theorem follows for this case.

It remains to show the case when $0 < |a| < 1$. For each $m \in \mathbb{Z}_+$, we decompose the function h as $h = h_m + g_m$ where

$$h_m(z) = \sum_{n=0}^{m} a_n\left(z - \frac{b}{1-a}\right)^n \quad \text{and} \quad g_m(z) = \sum_{n=m+1}^{\infty} a_n\left(z - \frac{b}{1-a}\right)^n. \qquad (4.4.4)$$

Using induction we plan to prove that for every $m \in \mathbb{Z}_+$

$$h_m \in \overline{\operatorname{span}\left\{C_{\psi}^{k}h : k \in \mathbb{Z}_+\right\}}.$$

To this end, consider a function g in $\mathcal{F}_{\varphi}^{p}$ and observe that

$$C_{\psi}^{k}g(z) = g\left(a^k z + \frac{b(1-a^k)}{1-a}\right).$$

Since $|a| < 1$, we also have $a^k z + \frac{b(1-a^k)}{1-a} \to \frac{b}{1-a}$ and by Theorem 4.3.1,

$$\lim_{k\to\infty} \|C_{\psi}^{k}g - C_{\frac{b}{1-a}}g\|_{(\varphi,p)} = 0.$$

It follows form this and (4.4.4) that

$$\lim_{k\to\infty} \|C_{\psi}^{k}g_0 - C_{\frac{b}{1-a}}g_0\|_{(\varphi,p)} = \lim_{k\to\infty} \|C_{\psi}^{k}g_0\|_{(\varphi,p)} = 0$$

from which we further deduce

$$\|C_\psi^k h - a_0\|_{(\varphi,p)} = \|C_\psi^k(a_0 + g_0) - a_0\|_{(\varphi,p)} \leq \|C_\psi^k(a_0) - a_0\|_{(\varphi,p)} + \|C_\psi^k(g_0)\|_{(\varphi,p)} \to 0$$

as $k \to \infty$. Therefore,

$$h_0 \in \overline{\operatorname{span}\left\{C_\psi^k h : k \in \mathbb{Z}_+\right\}}.$$

Suppose now that $h_0, h_1, ...h_{N-1} \in \overline{\operatorname{span}\left\{C_\psi^k h : k \in \mathbb{Z}_+\right\}}$. Then by the decomposition in (4.4.4) it holds that $g_{N-1} \in \overline{\operatorname{span}\left\{C_\psi^k h : k \in \mathbb{Z}_+\right\}}$, and hence

$$C_\psi^j g_{N-1} \in \overline{\operatorname{span}\left\{C_\psi^k h : k \in \mathbb{Z}_+\right\}} \tag{4.4.5}$$

for every $j \in \mathbb{Z}_+$. We next compute

$$\begin{aligned} C_\psi^j g_{N-1}(z) = C_\psi^j \sum_{n=N}^{\infty} a_n \left(z - \frac{b}{1-a}\right)^n &= \sum_{n=N}^{\infty} a_n a^{jn} \left(z - \frac{b}{1-a}\right)^n \\ &= a^{jN} \left(z - \frac{b}{1-a}\right)^N \sum_{n=N}^{\infty} a_n a^{j(n-N)} \left(z - \frac{b}{1-a}\right)^{n-N} \\ &= a^{jN} \left(z - \frac{b}{1-a}\right)^N C_\psi^j \sum_{n=N}^{\infty} a_n \left(z - \frac{b}{1-a}\right)^{n-N} \\ &= a^{jN} \left(z - \frac{b}{1-a}\right)^N C_\psi^j f_{N-1}(z) \end{aligned} \tag{4.4.6}$$

where $C_\psi^j = C_{\psi^j}$, $\psi^j(z) = a^j z + \frac{b(1-a^j)}{1-a}$ and

$$f_{N-1}(z) = a_N + \sum_{n=N+1}^{\infty} a_n \left(z - \frac{b}{1-a}\right)^{n-N}.$$

From (4.4.5) and (4.4.6) we also obtain

$$\left(z - \frac{b}{1-a}\right)^N C_\psi^j f_{N-1} \in \overline{\operatorname{span}\left\{C_\psi^k h : k \in \mathbb{Z}_+\right\}}. \tag{4.4.7}$$

By Theorem 4.3.1 we have that

$$\lim_{j\to\infty}\|C_{\psi^j}f_{N-1}-a_N\|_{(\varphi,p)}=\lim_{j\to\infty}\|C_{\psi^j}f_{N-1}-C_{\frac{b}{1-a}}f_{N-1}\|_{(\varphi,p)}=0. \tag{4.4.8}$$

We further claim that,

$$\Gamma_j(z):=\left(z-\frac{b}{1-a}\right)^N C_{\psi^j}f_{N-1}\to a_N\left(z-\frac{b}{1-a}\right)^N=:\Gamma(z) \tag{4.4.9}$$

in $\mathcal{F}^p_\varphi$ as $j\to\infty$ as well. We may compute

$$\begin{aligned}\|\Gamma_j\|_p^p&=\int_{\mathbb{C}}\left|\left(z-\frac{b}{1-a}\right)^N C^j_\psi f_{N-1}(z)\right|^p e^{-p\varphi(z)}dA(z)\\&=\int_{\mathbb{C}}\left|f_{N-1}\Big(a^jz+\frac{b(1-a^j)}{1-a}\Big)\right|^p e^{-p\varphi\left(a^jz+\frac{b(1-a^j)}{1-a}\right)}U_j(z)dA(z)\end{aligned}$$

where

$$U_j(z)=\left|z-\frac{b}{1-a}\right|^{pN}e^{p\varphi\left(a^jz+\frac{b(1-a^j)}{1-a}\right)-p\varphi(z)}$$

We also observe that since φ is an increasing weight function, and $|a^j|<1$, the sequence of functions U_j are uniformly bounded over $\mathbb{C}$. Furthermore, since norm convergence in $\mathcal{F}^p_\varphi$ implies pointwise convergence, by (4.4.8) for each $z\in\mathbb{C}$

$$C_{\psi^j}f_{N-1}(z)\to C_{\frac{b}{1-a}}f_{N-1}(z)$$

as $j\to\infty$. With this, an application of Lebesques convergence theorem implies

$$\begin{aligned}\lim_{j\to\infty}\|\Gamma_j\|_p^p&=\lim_{j\to\infty}\int_{\mathbb{C}}\left|f_{N-1}\Big(a^jz+\frac{b(1-a^j)}{1-a}\Big)\right|^p e^{-p\varphi\left(a^jz+\frac{b(1-a^j)}{1-a}\right)}U_j(z)dA(z)\\&=\int_{\mathbb{C}}\left|C_{\frac{b}{1-a}}f_{N-1}(z)\right|^p\left|z-\frac{b}{1-a}\right|^{pN}e^{-p\varphi(z)}dA(z)=\|\Gamma\|_p^p.\end{aligned}$$

Thus the claim in (4.4.9) follows for $1\le p<\infty$.

If $p = 0$, when $f(z) = \sum_{n=0}^{\infty} a_n z^n \in \mathcal{F}_{\varphi}^{0}$

$$\lim_{j\to\infty} \left\| \left[\left(z - \frac{b}{1-a} \right)^N C_{\psi^j} f_{N-1} \right] - \left[a_N \left(z - \frac{b}{1-a} \right)^N \right] \right\|_{(\varphi,0)}$$
$$= \lim_{j\to\infty} \left\| C_{\varphi}^{j} F \right\|_{(\varphi,0)} = \lim_{j\to\infty} \left\| C_{\varphi}^{j} F - F\left(\frac{b}{1-a} \right) \right\|_{(\varphi,0)}$$

where

$$F(z) = \sum_{n=N+1}^{\infty} a_n \left(z - \frac{b}{1-a} \right)^n$$

Since $f \in \mathcal{F}_{\varphi}^{0}$, we have $F \in \mathcal{F}_{\varphi}^{0}$ and hence $C_{\psi}^{j} F$ is also in $\mathcal{F}_{\varphi}^{0}$.
Thus by Theorem 4.3.1,

$$\lim_{j\to\infty} \left\| C_{\psi}^{j} F - F\left(\frac{b}{1-a} \right) \right\|_{(\varphi,0)} = 0.$$

which shows

$$\left(z - \frac{b}{1-a} \right)^N C_{\psi^j} f_{N-1} \to a_N \left(z - \frac{b}{1-a} \right)^N$$

in norm in $\mathcal{F}_{\varphi}^{0}$. Thus, the claim in (4.4.9) follows for $p = 0$ also, which along with (4.4.7) gives

$$a_N \left(z - \frac{b}{1-a} \right)^N \in \overline{\operatorname{span}\left\{ C_{\psi}^{k} h : k \in \mathbb{Z}_+ \right\}}, \text{ and } \ h_N \in \overline{\operatorname{span}\left\{ C_{\psi}^{k} h : k \in \mathbb{Z}_+ \right\}}.$$

Therefore,

$$h_m \in \overline{\operatorname{span}\left\{ C_{\psi}^{k} h : k \in \mathbb{Z}_+ \right\}},$$

for every $m \in \mathbb{Z}_+$ which in turn results in

$$a_n \left(z - \frac{b}{1-a} \right)^n \in \overline{\operatorname{span}\left\{ C_{\psi}^{k} h : k \in \mathbb{Z}_+ \right\}}$$

for every $n \in \mathbb{Z}_+$. Then, since $a_n \neq 0$ for all $n \in \mathbb{Z}_+$, by Theorem 2.2.1 the assertion of the theorem follows.

(ii) By [[39],Theorem 1.1], $\mathcal{F}_\varphi^\infty$ is isomorphic to ℓ^∞, and ℓ^∞ is not separable. Hence $\mathcal{F}_\varphi^\infty$ is not separable. But $\overline{span\{C_\psi^k h : k = 0, 1, 2, ...\}}$ is separable for any $h \in \mathcal{F}_\varphi^\infty$. To show this, let S be a subspace of $span\{C_\psi^k h : k = 0, 1, 2, ...\}$ consisting all linear combination of elements of $span\{C_\psi^k h : k = 0, 1, 2, ...\}$ formed using only scalar coefficients, whose imaginary and real part is from $\mathbb{Q}$. We observe that S is countable. Suppose that $f_1, f_2, ..., f_m \in \{C_\psi^k h : k = 0, 1, 2, ...\}$ and $\alpha_1, \alpha_2, ..., \alpha_m \in \mathbb{C}$. For each j such that $j = 1, 2, ..., m$ there is a sequence $\{\alpha_{j,n}\}_{n=1}^\infty$, whose imaginary and real part are in $\mathbb{Q}$ converging to α_j, and it follows from the continuity of the vector space operations that the sequence $\{\alpha_{1,n} f_1 + \alpha_{2,n} f_2 + ... \alpha_{m,n} f_m\}_{n=1}^\infty$ in S converging to $\alpha_1 f_1 + \alpha_2 f_2 + ... \alpha_m f_m$. Thus the countable set S is dense in $span\{C_\psi^k h : k = 0, 1, 2, ...\}$, and hence $\overline{span\{C_\psi^k h : k = 0, 1, 2, ...\}}$ is separable. Therefore, C_ψ can not be cyclic on $\mathcal{F}_\varphi^\infty$.

4.5 Periodic points under the composition operators

We say that a point x in X is periodic under an operator T on X if there is some $n \in \mathbb{N}$ such that $T^n x = x$. In this section we determined periodic points in $\mathcal{F}_\varphi^p$ under the composition operator.

Lemma 4.5.1. *Let $\alpha, \beta \in \mathbb{C}$, $\alpha \neq 0$, $\alpha^n \neq 1$ for every $n \in \mathbb{N}$ and f is analytic function on $\mathbb{C}$. If $f(\alpha z + \beta) = f(z)$ for each $z \in \mathbb{C}$, then f is constant.*

Proof. The mapping $\phi(z) = \alpha z + \beta$ fixes the point $z_0 := \frac{\beta}{1-\alpha}$. Define $g(z) = f(z) - f(z_0)$. Then $g(z_0) = 0$ and $g^{(n)}(z) = f^{(n)}(z)$ for every $n \in \mathbb{N}$.

Since $f(\alpha z + \beta) = f(z)$ for each $z \in \mathbb{C}$, we have $\alpha^n f^{(n)}(\alpha z + \beta) = f^{(n)}(z)$, and hence $\alpha^n f^{(n)}(z_0) = f^{(n)}(z_0)$. This yields $f^{(n)}(z_0) = 0$ for every $n \in \mathbb{N}$ since $\alpha^n \neq 1$. Thus $g^{(n)}(z_0) = 0$ for every $n = 0, 1, 2, ...$ Hence $g(z) = 0$ for every $z \in \mathbb{C}$. Therefore, $f(z) = f(z_0)$ for every $z \in \mathbb{C}$.

Observe that the assumption that $\alpha^n \neq 1$ for every $n \in \mathbb{N}$ is necessary in Lemma 4.5.1. For example if we take $\alpha = 1$ and $\beta = i2\pi$, then the function $f(z) = e^z$ satisfies $f(\alpha z + \beta) = f(z + i2\pi) = e^{z+i2\pi} = e^z = f(z)$ for each $z \in \mathbb{C}$. Our next theorem describes periodic elements of C_ψ in $\mathcal{F}^p_\varphi$, $1 \leq p \leq \infty$ or $p = 0$.

Theorem 4.5.2. *Let* C_ψ *be a bounded composition operator on* $\mathcal{F}^p_\varphi$, $1 \leq p \leq \infty$ *or* $p = 0$.

(a) Every $f \in \mathcal{F}^p_\varphi$ *is periodic point for* C_ψ *if* $\psi(z) = az, |a| = 1$ *and* $a^m = 1$ *for some* $m \in \mathbb{N}$.

(b) Only constant functions in $\mathcal{F}^p_\varphi$ *are periodic points for* C_ψ *if* $\psi(z) = az + b, |a| < 1$ *or* $\psi(z) = az, |a| = 1, a^n \neq 1$ *for all* $n \in \mathbb{N}$.

Proof. (a) If $\psi(z) = az, |a| = 1$ and $a^m = 1$ for some $m \in \mathbb{N}$, then for each $f \in \mathcal{F}^p_\varphi$, $C^m_\psi f(z) = f(a^m z) = f(z)$ for every $z \in \mathbb{C}$.
(b) Assume that $\psi(z) = az+b, |a| < 1$ and $b \in \mathbb{C}$ or $\psi(z) = az, |a| = 1, a^n \neq 1$ for all $n \in \mathbb{N}$. If $f \in \mathcal{F}^p_\varphi$ is a periodic point of C_ψ, there is $s \in \mathbb{N}$ such that $C^s_\psi f = f$. Then

$$C^s_\psi f(z) = f(\psi^s(z)) = f\left(a^s z + \frac{b(1-a^s)}{1-a}\right) = f(z).$$

Our assumptions on the symbol ψ imply that $(a^s)^n \neq 1$ for each $n \in \mathbb{N}$. We can apply Lemma 4.5.1 to conclude that f must be constant.

Chapter 5

Dynamics of Weighted Composition Operators on The Fock Spaces

The theory of weighted composition operators lies at the interface of analytic function theory and operator theory, and its study traces back to the sixties in the work of Forelli [22] where it was shown that the isometries in the Hardy spaces H^p whenever $1 < p < \infty, p \neq 2$ are weighted composition operators. De Leeuw [30] later showed the same holds true on the space H^1 as well. Since then the operator has become a natural object of study and its investigations has rapidly evolved in function related operator theory. A number of researchers have studied the operator over various settings mainly with the aim to express its spectral, topological and dynamical properties in terms of the function theoretic properties of the inducing pairs of symbols (u, ψ): see for example [15, 21, 30, 45, 51, 73]. Recall that for holomorphic functions u and ψ on a given domain $\mathbb{C}$, the weighted composition operator $W_{(u,\psi)}$ on spaces of holomorphic functions $\mathcal{F}$ on $\mathbb{C}$ is defined by $W_{(u,\psi)}f = u \cdot f \circ \psi$. The operator generalizes both the composition C_ψ and multiplication M_u operators since it can be represented as $W_{(u,\psi)} = M_u C_\psi$, where $M_u f = u \cdot f$ and $C_\psi f = f \circ \psi$. This representation has partly contributed for the rapid development of the theory. For further detailed studies of the operator on spaces of functions defined over the unit disc, we may refer for example on Hardy space [13, 14], on Bergman space [16], on disc algebra [66] and the references therein.

In 2007, Ueki [73] considered the operators on the classical Fock spaces $\mathcal{F}_2$ and characterized the bounded and compact $W_{u,\psi}$ in terms of Berezin-type integral transform. In [51], T. Mengestie considered a more general setting namely, Fock-Sobolev spaces, which include all the classical Fock spaces and characterized various properties of the operators including boundedness, compactness, essential norm and Schatten class membership in terms of generalized Berezin type integral transforms.

Later, Le [45] considered the Hilbert space $\mathcal{F}_2$ setting and obtained a simpler condition namely that $W_{(u,\psi)}$ is bounded on $\mathcal{F}_2$ if and only if u belongs to $\mathcal{F}_2$ and

$$\sup_{z\in\mathbb{C}} |u(z)|e^{\frac{1}{2}\left(|\psi(z)|^2-|z|^2\right)} < \infty. \tag{5.0.1}$$

and compact if and only if

$$\lim_{|z|\to\infty} |u(z)|e^{\frac{1}{2}\left(|\psi(z)|^2-|z|^2\right)} = 0. \tag{5.0.2}$$

He further proved that (5.0.1) implies $\psi(z) = az + b$ with $|a| \leq 1$. In [57], T. Mengestie and M. Worku proved that the Berezin-type integral condition used to describe the boundedness of generalized Volterra-type integral operators $V_{(g,\psi)}$ on the Fock spaces $\mathcal{F}_p$ is equivalent to a simple condition as in (5.0.1). Because of the Littlewood-Paley type description of the Fock spaces, by simply replacing $|g'(z)|/(1 + |z|)$ by $|u(z)|$ in the results there, it has been known that (5.0.1) in fact describes the bounded weighted composition operators on all the spaces $\mathcal{F}_p,\ 1 \leq p < \infty$, with norm bounds

$$\sup_{z\in\mathbb{C}} |u(z)|e^{\frac{1}{2}\left(|\psi(z)|^2-|z|^2\right)} \leq \|W_{(u,\psi)}\|_p \leq \frac{1}{|a|^2} \sup_{z\in\mathbb{C}} |u(z)|e^{\frac{1}{2}\left(|\psi(z)|^2-|z|^2\right)}. \tag{5.0.3}$$

The same conclusion as in (5.0.3) was also reported later in [72] for $p < \infty$. For $p = \infty$, the corresponding relation holds in fact with equality

$$W_{(u,\psi)}\|_\infty = \sup_{z\in\mathbb{C}} |u(z)|e^{\frac{1}{2}\left(|\psi(z)|^2-|z|^2\right)}. \tag{5.0.4}$$

As indicated in the proof of [[45], Proposition 2.1], an interesting consequence of (5.0.1) is that if $|a| = 1$, then a simple argument with Liouville's theorem gives that the weight function u has the form $u(z) = u(0)K_{-\overline{a}b}(z)$. This representation of u will play an important roll in the rest of the paper. Thus, we may formulate it as a lemma for the purpose of easy further referencing.

Lemma 5.0.3. *Let* $1 \leq p \leq \infty,\ u,\ \psi \in \mathcal{H}(\mathbb{C})$ *and* $W_{(u,\psi)}$ *be bounded on* $\mathcal{F}_p$, *and hence* $\psi(z) = az + b,\ |a| \leq 1$. *If* $|a| = 1$, *then*

$$u(z) = u(0)K_{-\overline{a}b}(z).$$

In (5.0.2), compactness of $W_{(u,\psi)}$ has been described by the fact that $\psi(z) = az + b, |a| \leq 1$ and $|u(z)|e^{\frac{1}{2}(|\psi(z)|^2-|z|^2)} \to 0$ as $|z| \to \infty$. The later condition implies that $|a| < 1$ but not conversely. Very recently, Carroll and Gilmore [10], used the idea of order of analytic function and proved the following analogues result.

Lemma 5.0.4. *Let* $1 \leq p \leq \infty,\ u,\ \psi \in \mathcal{H}(\mathbb{C})$ *and* $\psi(z) = az + b,\ |a| < 1$, *and assume that* u *is non-vanishing. Then* $W_{(u,\psi)}$ *is compact on* $\mathcal{F}_p$ *if and only if* u *has the form*

$$u(z) = e^{a_0+a_1z+a_2z^2}$$

for some constants a_0, a_1, a_2 *such that* $|a_2| < \frac{1-|a|^2}{2}$.

In this chapter we treat dynamics of weighted composition on Fock spaces. It has been well-known that the spectrum of an operator plays a vital roll in the study of its dynamical properties; see for example [23]. Thus, in the section to follow, we first determine the spectrum of weighted composition operators on the Fock spaces $\mathcal{F}_p,\ 1 \leq p \leq \infty$.

5.1 Spectrum of weighted composition operators

We may begin by stating our result.

Theorem 5.1.1. *Let* $1 \leq p \leq \infty,\ u,\ \psi \in \mathcal{H}(\mathbb{C})$ *and* $W_{(u,\psi)}$ *be bounded on* $\mathcal{F}_p$ *and hence* $\psi(z) = az + b$ *with* $|a| \leq 1$. *Then if*

(i) $W_{(u,\psi)}$ *is compact and hence* $|a| < 1$, *then*

$$\sigma(W_{(u,\psi)}) = \Big\{0,\ u\Big(\frac{b}{1-a}\Big)a^m, \quad m \in \mathbb{N}_0\Big\}. \tag{5.1.1}$$

(ii) $|a| = 1$, *then*

$$\sigma(W_{(u,\psi)}) = \begin{cases} \overline{\Big\{u(0)e^{\frac{a|b|^2}{a-1}}a^m : m \in \mathbb{N}_0\Big\}}, \ a \neq 1 \\ \Big\{z : |z| = |u(0)|e^{\frac{|b|^2}{2}}\Big\}, \quad a = 1,\ b \neq 0 \\ \{u(0)\}, \quad a = 1, \quad b = 0. \end{cases}$$

.

We now remark a few points. First, observe that the number $b/(1-a)$ in (5.1.1) is the fixed point of the symbol ψ. We also note that when $a \neq 1$ and $|a| = 1$ the expression in the spectrum can be expressed in terms of this fixed point. That is from Lemma 5.0.3, it follows that $u(0)e^{\frac{a|b|^2}{a-1}}a^m = u(b/(1-a))a^m$. In this case, the the spectrum contains finite number of points only when a is a root of unity.

Proof. (i). Let $W_{(u,\psi)}$ be compact and hence $|a| < 1$. Here our proof is based on an argument that goes back to [33]. We set $z_0 = b/(1-a)$ and plan to show that the range of $W_{(u,\psi)} - a^n u(z_0)I$ fails to contain the complex polynomial z^n. Setting $n = 1$ and arguing in the direction of contradiction, assume that there exists an $f \in \mathcal{F}_p$ such that

$$u(z)f(\psi(z)) - au(z_0)f(z) = z. \tag{5.1.2}$$

If $u(z_0) = 0$ or a= 0, then $a^n u(z_0) = 0$ and belongs to the spectrum. Thus, we may assume that z_0 is not in the zero set of u and $a \neq 0$. First assume that $z_0 = 0$. Then taking $z = 0$ in (5.1.2), we obtain that $f(0) = 0$. On the other hand, differentiating both sides of equation (5.1.2) and setting again $z = 0$

$$u'(0)f(\psi(0)) + u(0)\psi'(0)f'(\psi(0)) - au(0)f'(0) = 1$$

which results the contradiction $0 = 1$. Similarly for $n > 1$, differentiating both side of the equation

$$u(z)f(\psi(z)) - a^n u(z_0)f(z) = z^n$$

repeatedly and eventually setting $z = 0$, we obtain $f^{(m)}(0) = 0$ for all $m < n$ while for $m = n$ we get again the contradiction $0 = n!$.
If $z_0 \neq 0$, then we may set $\psi_1(z) = az$,

$$u_1(z) = \frac{u(z + z_0)}{\|K_{-z_0}\|_2^2} e^{-\overline{z_0}z + \overline{z_0}(az + az_0 + b)}$$

and observe that $\psi_1(0) = 0$ and $u_1(0) = u(z_0)$. A straightforward calculation shows that

$$W_{(u_2,\psi_2)} W_{(u,\psi)} W^{-1}_{(u_2,\psi_2)} = W_{(u_1,\psi_1)}$$

and $W^{-1}_{(u_2,\psi_2)} = W_{(u_3,\psi_3)}$ where $u_2(z) = k_{-z_0}(z)$, $\psi_2(z) = z + z_0$, $u_3 = k_{z_0}$, and $\psi_3(z) = z - z_0$. It follows that the weighted composition operators $W_{(u_1,\psi_1)}$ and $W_{(u,\psi)}$ are similar and have the same spectrum, and our conclusion follows from case one. Therefore, the set in the right-hand side of (5.1.1) in this case is contained in the spectrum.

Conversely, if $|a| < 1$, then $W_{(u,\psi)}$ is compact and its spectrum contains only zero and eigenvalues. Thus, we consider a nonzero eigenvalue $\lambda \in \sigma(C_\psi)$ and show that it is of the form $u(z_0)a^n$ for some positive integer n. If f is a corresponding nonzero eigenvector, then

$$W_{(u,\psi)}f(z) = u(z)f(az+b) = \lambda f(z) \tag{5.1.3}$$

for all z in $\mathbb{C}$. If f is not zero at z_0, then (5.1.3) implies $u(z_0) = \lambda$ and hence $\lambda = a^0 u(z_0)$. On the other hand, if f has zero at z_0 of order m, we may write $f(z) = (z - z_0)^m g(z)$ where $g(z_0) \neq 0$. Then substituting f by this in (5.1.3) and differentiating both sides of the equation m times and eventually setting $z = z_0$, we only get

$$a^m u(z_0) m! g(z_0) = \lambda m! g(z_0) \tag{5.1.4}$$

as all the other terms have factor $z - z_0$ and vanish. Now, $g(z_0)$ is non-zero and (5.1.4) holds only if $\lambda = a^m u(z_0)$ as asserted.

The argument in the proof of part (ii) is divided into three cases depending on the values of a and b.

Case 1. Let $|a| = 1$ and $a \neq 1$. For simplicity we first set $\psi_{z_0}(z) = z - z_0$ and claim that the weighted composition operator induced by (k_{z_0}, ψ_{z_0}) is an isometric bijective map on $\mathcal{F}_p$ with inverse $W_{(k_{-z_0}, \psi_{z_0}^{-1})}$. To this claim, for every $f \in \mathcal{F}_p$

$$\begin{aligned}
\|W_{(k_{z_0},\psi_{z_0})}f\|_p^p &= \frac{p}{2\pi}\int_{\mathbb{C}} |k_{z_0}(z)|^p |f(z-z_0)|^p e^{-\frac{p}{2}|z|^2} dA(z)\\
&= \frac{p}{2\pi}\int_{\mathbb{C}} |f(z-z_0)|^p e^{-\frac{p}{2}|z-z_0|^2}\left(|k_{z_0}(z)|^p e^{\frac{p}{2}|z-z_0|^2-\frac{p}{2}|z|^2}\right) dA(z)\\
&= \frac{p}{2\pi}\int_{\mathbb{C}} |f(z-z_0)|^p e^{-\frac{p}{2}|z-z_0|^2} dA(z) = \|f\|_p^p
\end{aligned}$$

for all $1 \leq p < \infty$ which also holds true for $p = \infty$. This shows that the operator is a linear isometry and hence satisfies the injectivity condition $W^{-1}_{(k_{z_0},\psi_{z_0})}W_{(k_{z_0},\psi_{z_0})} = I$. On the other hand, for each $f \in \mathcal{F}_p$

$$W_{(k_{z_0},\psi_{z_0})}W_{(k_{-z_0},\psi_{z_0}^{-1})}f(z) = k_{z_0}(z)k_{-z_0}(z-z_0)f(z) = f(z)$$

which also shows that $W_{(k_{z_0},\psi_{z_0})}W^{-1}_{(k_{z_0},\psi_{z_0})} = I$, and hence the claim.
Next, using $z_0 = b/1-a$ and Lemma 5.0.3 for every $f \in \mathcal{F}_p$ we compute

$$\begin{aligned} &W_{(k_{-z_0},\psi_{z_0}^{-1})}W_{(u,\psi)}W_{(k_{z_0},\psi_{z_0})}f(z) \\ &\quad = k_{-z_0}(z)u(\psi_{z_0}^{-1}(z)))k_{z_0}(\psi(\psi_{z_0}^{-1}(z))f(\psi_{z_0}(\psi(\psi_{z_0}^{-1}(z)))) \\ &= k_{-z_0}(z)u(0)K_{-\overline{a}b}(z+z_0)k_{z_0}(az+b+az_0)f(az) = u(0)e^{\frac{|b|^2}{1-\overline{a}}}C_{\Psi_0}f(z) \end{aligned}$$

where C_{ψ_0} the composition operator induced by the symbol $\Psi_0(z) = az$. This shows that $W_{(u,\psi)}$ is similar to the composition operator, up to a multiple, C_{Ψ_0}. Thus, $\sigma(W_{(u,\psi)}) = u(0)e^{\frac{|b|^2}{1-\overline{a}}}\sigma(C_{\Psi_0})$. Using the spectrum of C_{Ψ_0} from Theorem 4.1.1 and observing that $(1-\overline{a})^{-1} = a/(a-1)$ when $|a| = 1$ and $a \neq 1$, we arrive at the desired conclusion.
Case 2. Let $a = 1$ and $b \neq 0$. Applying Lemma 5.0.3,

$$W_{u,\psi} = u.C_\psi = u(0)K_{-b}C_\psi = u(0)e^{\frac{|b|^2}{2}}k_{-b}C_\psi = u(0)e^{\frac{|b|^2}{2}}W_{(k_{-b},\psi)} \tag{5.1.5}$$

The weighted composition operator $W_{(k_{-b},\psi)}$ is unitary. Recall that the spectrum of a unitary operator lies on the unit circle $\mathbb{T}$. We claim that the spectrum of $W_{(k_{-b},\psi)}$ is $\mathbb{T}$. To prove the claim, for any nonzero $w \in \mathbb{C}$ and $f \in \mathcal{F}_p$ we have

$$\begin{aligned} &W_{(k_{-w},\psi_w^{-1})}W_{(k_{-b},\psi)}W_{(k_w,\psi_w)}f(z) \\ &\quad = k_{-w}(z)k_{-b}(z+w)k_w(z+b+w)f(z+b) \\ &\quad\quad = k_{-b}(z)e^{2i\Im(\overline{w}b)}f(z+b) = e^{2i\Im(\overline{w}b)}W_{(k_{-b},\psi)}f(z) \end{aligned}$$

which shows that $W_{(k_{-b},\psi)}$ is similar to $e^{2i\Im(\overline{w}b)}W_{(k_{-b},\psi)}$ for any $w \in \mathbb{C}$. Since

$b \neq 0$, and $e^{2i\Im(\overline{w}b)}$ is unimodular, and the spectrum of a unitary operator lies on the unit circle $\mathbb{T}$, it follows that the whole unit circle constitutes the spectrum. Therefore, combining this with (5.1.5)

$$\sigma(W_{(u,\psi)}) = u(0)e^{\frac{|b|^2}{2}}\sigma(W_{(k_{-b},\psi)}) = u(0)e^{\frac{|b|^2}{2}}\mathbb{T} = \left\{z : |z| = |u(0)|e^{\frac{|b|^2}{2}}\right\}.$$

Case 3. Let $a = 1$ and $b = 0$. In this case, the operator $W_{(u,\psi)}$ reduces to the multiplication operator M_u where its spectrum has been already identified in Lemma 2.3 of [56], and completes the proof the theorem.

5.2 Power bounded weighted composition operators

For the operator $W_{(u,\psi)}$ and $f \in \mathcal{H}(\mathbb{C})$, each element of the orbit has the form

$$W^n_{(u,\psi)}f = f \circ \psi^n \cdot u_n, \quad u_n(z) := \prod_{j=0}^{n-1} u(\psi^j(z)). \tag{5.2.1}$$

for each non negative integer n and $\psi^0 = I$ the identity map on $\mathbb{C}$.

First, we consider the following key lemma which provides necessary conditions for power bounded $W_{(u,\psi)}$. The lemma further gives a good restriction on the growth of the sequence $(\|u_n\|_p)_n$ and the value $|u(z_0)|$ where z_0 is a fixed point of ψ.

Lemma 5.2.1. *Let $1 \leq p \leq \infty$ and $u, \psi \in \mathcal{H}(\mathbb{C})$. If $W_{(u,\psi)}$ is power bounded on $\mathcal{F}_p$, then*

(i) $|u(z_0)| \leq 1$ where z_0 is a fixed point of ψ.

(ii) $(\|u_n\|_p)_n$ is a bounded sequence.

Proof. (i). Since the constant function $\mathbf{1}$ belongs to the spaces $\mathcal{F}_p$ with $\|\mathbf{1}\|_p = 1$, using the point-wise estimate in (1.0.1)

$$\|W^n_{(u,\psi)}\| \geq \|W^n_{(u,\psi)}\mathbf{1}\|_p \geq |W^n_{(u,\psi)}\ \mathbf{1}(z_0)|e^{-\frac{|z_0|^2}{2}} = |u_n(z_0)|e^{-\frac{|z_0|^2}{2}} = |u(z_0)|^n e^{-\frac{|z_0|^2}{2}}$$

from which the inequalities

$$\infty > \sup_{n\in\mathbb{N}} \|W^n_{(u,\psi)}\| \geq e^{-\frac{|z_0|^2}{2}} \sup_{n\in\mathbb{N}} |u(z_0)|^n$$

hold only if $|u(z_0)| \leq 1$.

To prove (ii), for $p = \infty$ arguing as above we have

$$\|W^n_{(u,\psi)}\| \geq \|W^n_{(u,\psi)}\mathbf{1}\|_\infty \geq |W^n_{(u,\psi)}\ \mathbf{1}(z)|e^{-\frac{|z|^2}{2}} = |u_n(z)|e^{-\frac{|z|^2}{2}}.$$

Taking the supremum with respect to first with z and then with n give the required assertion. On the other hand, if $p < \infty$, then

$$\|W^n_{(u,\psi)}\|^p \geq \|W^n_{(u,\psi)}\mathbf{1}\|_p^p = \frac{p}{2\pi}\int_{\mathbb{C}} |u_n(z)|^p e^{-\frac{p|z|^2}{2}} dA(z) = \|u_n\|_p^p$$

from which the conclusion follows again.

The next simple lemma will be crucial in the proof of Theorem 5.2.5.

Lemma 5.2.2. *Let $a \in \mathbb{C}$ and $|a| < 1$. Then for all $n \in \mathbb{N}$*

$$\frac{|1-a^2|}{1-|a|^2} \geq \frac{|1-a^{2n}|}{1-|a|^{2n}}. \tag{5.2.2}$$

Proof. Applying triangular inequality,

$$\begin{aligned}\frac{|1-a^{2n}|}{|1-a^2|} &= \left|1 + a^2 + (a^2)^2 + ... + (a^2)^{n-1}\right| \leq 1 + |a^2| + |(a^2)^2| + ... + |(a^2)^{(n-1)}| \\ &= 1 + |a|^2 + (|a|^2)^2 + ... + (|a|^2)^{n-1} = \frac{1-|a|^{2n}}{1-|a|^2}\end{aligned}$$

from which (5.2.2) follows.

We now state the main results on power boundedness. Depending on whether

$|a| = 1$ or $|a| < 1$, we give two main results as in Theorem 5.2.3 and Theorem 5.2.5.

Theorem 5.2.3. *Let* $1 \leq p \leq \infty$, $u, \psi \in \mathcal{H}(\mathbb{C})$ *and* $W_{(u,\psi)}$ *be bounded on* $\mathcal{F}_p$, *and hence* $\psi(z) = az + b$, $|a| \leq 1$. *If* $|a| = 1$, *then the following statements are equivalent.*

(i) $W_{(u,\psi)}$ *is power bounded on* $\mathcal{F}_p$*;*

(ii) $(\|u_n\|_p)_n$ *is a bounded sequence;*

(iii) $|u(0)| \leq e^{-\frac{|b|^2}{2}}$.

It is interesting that we have an easy to apply equivalent conditions for the power boundedness of the weighted composition operators. Part (iii) of the condition is also independent of underlying space or the exponents p. We recall that a bounded linear operator is a contraction when its norm is bounded by 1. In view of this, we may add one more equivalent condition to the above list in the theorem, namely that $W_{(u,\psi)}$ is power bounded on $\mathcal{F}_p$ if and only if it is a contraction.

Proof. The statement (i) implies (ii) is proved in Lemma 5.2.1. On the other hand, if (ii) holds, then using (1.0.1)

$$\infty > \sup_{n\in\mathbb{N}_0} \|u_n\|_p \geq \sup_{n\in\mathbb{N}_0} |u_n(z)|e^{-\frac{|z|^2}{2}} \tag{5.2.3}$$

for each $z \in \mathbb{C}$. If $a = 1$, then $\psi^j(z) = z + jb$ and using Lemma 5.0.3,

$$u_n(z) = u(0)^n \prod_{j=0}^{n-1} K_{-b}(z + jb) = u(0)^n e^{l_n(z)}$$

where

$$l_n(z) := -\bar{b}\sum_{j=0}^{n-1}(z + jb) = -\bar{b}nz - \frac{|b|^2}{2}n(n-1).$$

It follows that

$$u_n(z) = u(0)^n e^{-\frac{|b|^2}{2}n(n-1)} K_{-nb}(z) \tag{5.2.4}$$

for all $z \in \mathbb{C}$. Considering (5.2.4) and applying the estimate in (5.2.3) at $z = -nb$

$$\begin{aligned}\sup_{n\in\mathbb{N}_0} \|u_n\|_p &\gtrsim \sup_{n\in\mathbb{N}_0} |u_n(-nb)| e^{-\frac{|nb|^2}{2}}\\ &= \sup_{n\in\mathbb{N}_0} \left| u(0) e^{\frac{|b|^2}{2}} \right|^n e^{-|b|^2 n^2} K_{-nb}(-nb) = \sup_{n\in\mathbb{N}_0} \left| u(0) e^{\frac{|b|^2}{2}} \right|^n\end{aligned}$$

and hence the statement in (iii) follows. On the other hand, if $a \neq 1$ and $|a| = 1$, then set z_0 be the fixed point of ψ and eventually applying Lemma 5.0.3

$$\begin{aligned}|u_n(z_0)| = \left| u(0) K_{-\bar{a}b}\left(\frac{b}{1-a}\right) \right|^n &= \left| u(0) e^{-a\bar{b}\left(\frac{b}{1-a}\right)} \right|^n\\ &= |u(0)|^n \, e^{-n\Re\left(\frac{a|b|^2}{1-a}\right)} = |u(0)|^n e^{\frac{n|b|^2}{2}}\end{aligned}$$

and the conclusion follows after taking this in (5.2.3) again.

It remains to prove (iii) implies (i). First observe that for each $n \in \mathbb{N}$, the operator $W^n_{(u,\psi)}$ itself is a weighted composition operator and $W^n_{(u,\psi)} = W_{(u_n,\psi^n)}$. Then applying (5.0.1) together with the analysis after it, we find that $W_{(u,\psi)}$ is power bounded if and only if

$$\sup_{n\in\mathbb{N}} \sup_{z\in\mathbb{C}} |u_n(z)| e^{\frac{1}{2}\left(\left|a^n z + \frac{b(1-a^n)}{1-a}\right|^2 - |z|^2\right)} < \infty.$$

Thus, for $|a| = 1$, we apply (5.0.3) and obtain the norm

$$\|W^n_{(u,\psi)}\| = \sup_{z\in\mathbb{C}} |u_n(z)| e^{\frac{1}{2}(|\psi^n(z)|^2 - |z|^2)}. \tag{5.2.5}$$

Our next task is to simplify (5.2.5). If $a = 1$, then the representation in (5.2.4) implies

$$\begin{aligned}\|W^n_{(u,\psi)}\| &= \sup_{z\in\mathbb{C}} e^{\frac{1}{2}(|z+nb|^2-|z|^2)}|u(0)|^n e^{-\frac{|b|^2}{2}n(n-1)}|K_{-nb}(z)| \\ &= \sup_{z\in\mathbb{C}} \Big(|u(0)|e^{\frac{|b|^2}{2}}\Big)^n e^{\Re(nb\overline{z})}|K_{-nb}(z)| = \Big(|u(0)|e^{\frac{|b|^2}{2}}\Big)^n \qquad (5.2.6)\end{aligned}$$

from which the statement follows.

Next, assume $a \neq 1$ and $|a| = 1$. Then $\psi^j(z) = a^j z + b\frac{1-a^j}{1-a}$. By using Lemma 5.0.3 again $u_n(z) = u(0)^n e^{h_n(z)}$ where

$$h_n(z) := -a\bar{b}\sum_{j=0}^{n-1}\Big(a^j z + b\frac{1-a^j}{1-a}\Big) = -a\bar{b}z\frac{1-a^n}{1-a} - \frac{a|b|^2 n}{1-a} + \frac{a|b|^2(1-a^n)}{(1-a)^2}.$$

Thus, we have

$$u_n(z) = u(0)^n e^{-\frac{a|b|^2 n}{1-a} + \frac{a|b|^2(1-a^n)}{(1-a)^2}} K_{-\overline{a}b\frac{1-\overline{a}^n}{1-\overline{a}}}(z)$$

from which and (5.2.5)

$$\|W^n_{(u,\psi)}\| = \sup_{z\in\mathbb{C}} e^{\frac{1}{2}\left|a^n z + b\frac{1-a^n}{1-a}\right|^2 - \frac{1}{2}|z|^2}|u(0)|^n\left|e^{-a\bar{b}z\frac{1-a^n}{1-a} - \frac{a|b|^2 n}{1-a} + \frac{a|b|^2(1-a^n)}{(1-a)^2}}\right|$$

where

$$\left|a^n z + b\frac{1-a^n}{1-a}\right|^2 - |z|^2 = |b|^2\left|\frac{1-a^n}{1-a}\right|^2 + 2\Re\Big(a^n z\bar{b}\frac{1-\overline{a}^n}{1-\overline{a}}\Big)$$

and

$$\left|e^{-a\bar{b}z\frac{1-a^n}{1-a} - \frac{a|b|^2 n}{1-a} + \frac{a|b|^2(1-a^n)}{(1-a)^2}}\right| = e^{\Re\left(-a\bar{b}z\frac{1-a^n}{1-a} - \frac{a|b|^2 n}{1-a} + \frac{a|b|^2(1-a^n)}{(1-a)^2}\right)}.$$

On the other hand,

$$a^n z\bar{b}\frac{1-\overline{a}^n}{1-\overline{a}} - a\bar{b}z\frac{1-a^n}{1-a} = z\bar{b}(a^n-1)\Big(\frac{1}{1-\overline{a}} + \frac{a}{1-a}\Big) = 0$$

and combining all the above

$$\begin{aligned}\|W^n_{(u,\psi)}\| &= |u(0)|^n e^{\frac{|b|^2}{2}\left|\frac{1-a^n}{1-a}\right|^2 + \Re\left(\frac{a|b|^2(1-a^n)}{(1-a)^2} - \frac{a|b|^2 n}{1-a}\right)} \\ &\leq |u(0)|^n e^{\frac{|b|^2}{2}\left|\frac{2}{1-a}\right|^2 + \Re\left(\frac{a|b|^2(1-a^n)}{(1-a)^2}\right) - \Re\left(\frac{a|b|^2 n}{1-a}\right)} \\ &\leq |u(0)|^n e^{\frac{|b|^2}{2}\left|\frac{2}{1-a}\right|^2 + \left|\frac{a|b|^2(1-a^n)}{(1-a)^2}\right|} e^{-n|b|^2\Re\left(\frac{a}{1-a}\right)} \\ &\leq e^{\frac{|b|^2}{2}\left|\frac{2}{1-a}\right|^2 + \frac{2|b|^2}{|1-a|^2}} \Big(|u(0)| e^{-|b|^2\Re\left(\frac{a}{1-a}\right)}\Big)^n \\ &= e^{\frac{|b|^2}{2}\left|\frac{2}{1-a}\right|^2 + \frac{2|b|^2}{|1-a|^2}} \Big(|u(0)| e^{\frac{|b|^2}{2}}\Big)^n. \qquad (5.2.7)\end{aligned}$$

Thus, power boundedness follows whenever $|u(0)| \leq e^{-\frac{|b|^2}{2}}$. □

It should be also noted that for the case $a \neq 1$ and $|a| = 1$, the above conditions are also equivalent to $\left|u\left(\frac{b}{1-a}\right)\right| \leq 1$ since an application of Lemma 5.0.3 implies

$$\begin{aligned}\left|u\Big(\frac{b}{1-a}\Big)\right| = |u(0)|\left|K_{-\overline{a}b}\Big(\frac{b}{1-a}\Big)\right| = |u(0)|\left|e^{-\frac{a|b|^2}{1-a}}\right| \\ = |u(0)| e^{-|b|^2\Re\left(\frac{a}{1-a}\right)} = |u(0)| e^{\frac{|b|^2}{2}},\end{aligned}$$

where $\Re$ denotes the real part of the given complex number. This inspires us to ask whether a similar condition works for the remaining case namely that when $|a| < 1$. In this case, as will be explained later, the powers of weighted composition operators are again weighted composition operators. This together with the relations in (5.0.3) and (5.0.4) ensure that the following necessary and sufficient conditions hold whenever $|a| < 1$.

Proposition 5.2.4. *Let $1 \leq p \leq \infty$, $u, \psi \in \mathcal{H}(\mathbb{C})$ and $W_{(u,\psi)}$ be bounded on $\mathcal{F}_p$. Let $\psi(z) = az + b$ and $|a| < 1$.*

1. *If $W_{(u,\psi)}$ is power bounded on $\mathcal{F}_p$, then*

$$\left|u\Big(\frac{b}{1-a}\Big)\right| \leq 1 \qquad (5.2.8)$$

2. $W_{(u,\psi)}$ *is power bounded on* $\mathcal{F}_p$*,* $p < \infty$ *if*

$$\Big|u\Big(\frac{b}{1-a}\Big)\Big| \leq |a|^{\frac{2}{p}} \tag{5.2.9}$$

When $p \to \infty$, the right-hand side in (5.2.9) tends to 1. Thus, the condition in (5.2.8) is both necessary and sufficient for $W_{(u,\psi)}$ to be power bounded on the space $\mathcal{F}_\infty$. In particular when $W_{(u,\psi)}$ is compact, we record our next main result which holds true on all the spaces $\mathcal{F}_p$.

Theorem 5.2.5. *Let* $1 \leq p \leq \infty$*,* $u, \psi \in \mathcal{H}(\mathbb{C})$ *and* $W_{(u,\psi)}$ *be bounded on* $\mathcal{F}_p$*, and* $\psi(z) = az + b$*, with* $|a| < 1$*. Let* u *be non-vanishing and* $W_{(u,\psi)}$ *be compact. Then the following statements are equivalent.*

(i) $W_{(u,\psi)}$ *is power bounded on* $\mathcal{F}_p$*;*

(ii) $(\|u_n\|_p)_n$ *is a bounded sequence;*

(iii) $|u\big(\frac{b}{1-a}\big)| \leq 1$.

As in Theorem 5.2.3, condition (iii) is simple to apply and independent of the exponents p. Observe that from the two theorems above, it is easy to see that a bounded composition operator C_ψ is always power bounded while the multiplication operator M_u is not in general; see Corollary 5.3.5.

Proof. The statement (i) implies (ii) follows from Lemma 5.2.1 again. Assuming (ii), we proceed to show that (iii) holds. Using (1.0.1) we estimate

$$\infty > \sup_{n\in\mathbb{N}_0} \|u_n\|_p \geq \sup_{n\in\mathbb{N}_0} |u_n(z_0)| e^{-\frac{|z_0|^2}{2}}. \tag{5.2.10}$$

where $z_0 = b/1 - a$ is the fixed point of ψ. Moreover, observe that

$$|u_n(z_0)| = \prod_{j=0}^{n-1} u\big(\psi^j(z_0)\big) = |u(z_0)|^n$$

which together with (5.2.10) gives statement (iii).

Next, we prove (iii) implies (i). If $p = \infty$, applying the relation in (5.0.4) for the weighted composition operator $W_{(u_n,\psi^n)}$ we get

$$\|W_{(u_n,\psi^n)}\| = \|W^n_{(u,\psi)}\| = \sup_{z\in\mathbb{C}} |u_n(z)| e^{\frac{1}{2}\left(\left|a^n z + \frac{b(1-a^n)}{1-a}\right|^2 - |z|^2\right)}.$$

Thus, $W_{(u_n,\psi^n)}$ is power bounded on $\mathcal{F}_\infty$ if and only if

$$\sup_{n\in\mathbb{N}} \sup_{z\in\mathbb{C}} |u_n(z)| e^{\frac{1}{2}\left(\left|a^n z + \frac{b(1-a^n)}{1-a}\right|^2 - |z|^2\right)} < \infty. \tag{5.2.11}$$

Therefore, by using the assumption $\left|u\left(\frac{b}{1-a}\right)\right| \leq 1$, we plan to show that (5.2.11) holds. First we consider Lemma 5.0.4 and compute

$$u_n(z) = \prod_{j=0}^{n-1} u(\psi^j(z)) = e^{S_n(z)}$$

where

$$\begin{aligned} S_n(z) &= \sum_{j=0}^{n-1} \left(a_0 + a_1\Big(a^j z + \frac{(1-a^j)b}{1-a}\Big) + a_2\Big(a^j z + \frac{(1-a^j)b}{1-a}\Big)^2 \right) \\ &= na_0 + \frac{a_1 bn}{1-a} - \frac{a_1 b(1-a^n)}{(1-a)^2} + \frac{a_1(1-a^n)}{1-a} z + \frac{a_2(1-a^{2n})}{1-a^2} z^2 \\ &\quad + \frac{a_2 b^2}{(1-a)^2}\left(n - \frac{2(1-a^n)}{1-a} + \frac{1-a^{2n}}{1-a^2}\right) + \frac{2a_2 zb}{1-a}\left(\frac{1-a^n}{1-a} - \frac{1-a^{2n}}{1-a^2}\right) \\ &= na_0 + \frac{a_1 bn}{1-a} - \frac{a_1 b(1-a^n)}{1-a} + \frac{a_2 b^2}{(1-a)^2}\left(n - \frac{2(1-a^n)}{1-a} + \frac{1-a^{2n}}{1-a^2}\right) \\ &\quad + \left(\frac{a_1(1-a^n)}{1-a} + \frac{2a_2 b}{1-a}\left(\frac{1-a^n}{1-a} - \frac{1-a^{2n}}{1-a^2}\right)\right) z + \frac{a_2(1-a^{2n})}{1-a^2} z^2. \end{aligned} \tag{5.2.12}$$

Now taking this into account and the fact that $W^n_{(u,\psi)} = W_{(u_n,\psi^n)}$, the corresponding notation in (5.0.1) becomes

$$\begin{aligned} M(u_n,\psi^n) &= \sup_{z\in\mathbb{C}} |u_n(z)| e^{\frac{1}{2}\left(\left|a^n z+\frac{b(1-a^n)}{1-a}\right|^2-|z|^2\right)} \\ &= e^{c_n}\sup_{z\in\mathbb{C}} e^{\Re(t_n z)+\Re(p_n z^2)-q_n|z|^2} \\ &\leq e^{c_n}\sup_{z\in\mathbb{C}} e^{\Re(t_n z)+(|p_n|-q_n)|z|^2} \end{aligned} \tag{5.2.13}$$

where c_n is the real part of the expression

$$na_0+\frac{a_1bn}{1-a}-\frac{a_1b(1-a^n)}{(1-a)^2}+\frac{a_2b^2}{(1-a)^2}\left(n-\frac{2(1-a^n)}{1-a}+\frac{1-a^{2n}}{1-a^2}\right)+\left|\frac{b(1-a^n)}{1-a}\right|^2,$$

$$t_n=\frac{a_1(1-a^n)}{1-a}+\frac{2a_2b}{1-a}\left(\frac{1-a^n}{1-a}-\frac{1-a^{2n}}{1-a^2}\right)+\overline{\frac{(1-a^n)b}{1-\overline{a}}}a^n,$$

$$p_n=\frac{a_2(1-a^{2n})}{1-a^2}, \quad \text{and} \quad q_n=\frac{1-|a|^{2n}}{2}.$$

Now to estimate the supremum in (5.2.13), we claim that

$$|p_n|-q_n=|a_2|\left|\frac{1-a^{2n}}{1-a^2}\right|-\frac{1-|a|^{2n}}{2}<0.$$

Observe that the inequality holds if and only if

$$|a_2|<\frac{(1-|a|^{2n})}{\left|1-a^{2n}\right|}\frac{\left|1-a^2\right|}{2}. \tag{5.2.14}$$

This follows immediately from Lemma 5.2.2 as $|a_2|<\frac{1-|a|^2}{2}$.
It follows from this and (5.2.13) that

$$M(u_n,\psi^n)\lesssim e^{c_n}.$$

On the other hand, since $|a| < 1$

$$c_n \leq n\Re\Big(a_0 + \frac{a_1 b}{1-a} + \frac{a_2 b^2}{(1-a)^2}\Big) + \frac{|2a_1 b|}{|1-a|^2} \\ + \frac{|a_2 b^2|}{|1-a|^2}\Big(\frac{2}{|1-a|} + \frac{2}{|1-a^2|}\Big) + \Big|\frac{2b}{1-a}\Big|^2$$

and hence

$$e^{c_n} \lesssim e^{n\Re\left(a_0 + \frac{a_1 b}{1-a} + \frac{a_2 b^2}{(1-a)^2}\right)} = \Big|u\Big(\frac{b}{1-a}\Big)\Big|^n.$$

from which and the assumption that $\Big|u\big(\frac{b}{1-a}\big)\Big| \leq 1$, the condition in (5.2.11) follows.

Next, we consider the case when $p < \infty$ and consider first the case $a = 0$. Then $\psi^n(z) = b$, $u_n(z) = u(z)(u(b))^{n-1}$ and applying (1.0.1),

$$\begin{aligned}\|W^n_{(u,\psi)} f\|_p^p &= \frac{p}{2\pi}\int_{\mathbb{C}} \Big|f\Big(b\Big)\Big|^p |u_n(z)|^p e^{-\frac{p}{2}|z|^2} dA(z) \\ &= \frac{p}{2\pi}\int_{\mathbb{C}} \Big|f\Big(b\Big)\Big|^p |u(z)|^p |u(b)|^{(n-1)p} e^{-\frac{p}{2}|z|^2} dA(z) \\ &= |f(b)|^p |(u(b))|^{(n-1)p} \|u\|_p^p \leq |(u(b))|^{(n-1)p} \|u\|_p^p e^{|b|^2} \|f\|_p^p.\end{aligned}$$

from which we arrive at the claim. Here note that since $W_{u,\psi}$ is bounded, the multiplier u belongs to $\mathcal{F}_p$ for all p. If $a \neq 0$, then applying the local point estimate in (2.0.5),

$$\begin{aligned}\|W^n_{(u,\psi)} f\|_p^p &= \frac{p}{2\pi}\int_{\mathbb{C}} \Big|f\Big(a^n z + \frac{b(1-a^n)}{1-a}\Big)\Big|^p |u_n(z)|^p e^{-\frac{p}{2}|z|^2} dA(z) \\ &\leq \frac{p}{2\pi}\int_{\mathbb{C}} e^{\frac{p}{2}\left(\left|a^n z + \frac{b(1-a^n)}{1-a}\right|^2 - |z|^2\right)} |u_n(z)|^p \\ &\quad \times \int_{D\left(a^n z + \frac{b(1-a^n)}{1-a}, 1\right)} |f(w)|^p e^{-\frac{p}{2}|w|^2} dA(w) dA(z) \\ &= \frac{p}{2\pi}\int_{\mathbb{C}}\int_{\mathbb{C}} e^{\frac{p}{2}\left(\left|a^n z + \frac{b(1-a^n)}{1-a}\right|^2 - |z|^2\right)} |u_n(z)|^p \\ &\quad \chi_{D\left(a^n z + \frac{b(1-a^n)}{1-a}, 1\right)}(w) |f(w)|^p e^{-\frac{p}{2}|w|^2} dA(w) dA(z). \end{aligned} \tag{5.2.15}$$

Observe that if $w \in D\big(a^n z + \frac{b(1-a^n)}{1-a}, 1\big)$, then

$$\begin{aligned} 1 \geq \left| w - a^n z - \frac{b(1-a^n)}{1-a} \right| &= |a|^n \left| \frac{w}{a^n} - z - \frac{b(1-a^n)}{a^n(1-a)} \right| \\ &= |a|^n \left| z - \Big(\frac{w}{a^n} - \frac{b(1-a^n)}{a^n(1-a)} \Big) \right|, \end{aligned}$$

which holds true if and only if

$$\frac{1}{|a|^n} \geq \left| z - \Big(\frac{w}{a^n} - \frac{b(1-a^n)}{a^n(1-a)} \Big) \right|.$$

Thus, w belongs to the disk

$$D\left(a^n z + \frac{b(1-a^n)}{1-a}, 1 \right)$$

if and only if z belongs to

$$D\left(\frac{w}{a^n} - \frac{b(1-a^n)}{1-a}, \frac{1}{|a|^n} \right).$$

Making use of this and Fubini's theorem in (5.2.15)

$$\begin{aligned} \|W^n_{(u,\psi)} f\|_p^p = \frac{p}{2\pi} \int_{\mathbb{C}} |f(w)|^p e^{-\frac{p}{2}|w|^2} \\ \times \left(\int_{\mathbb{C}} e^{\frac{p}{2}\left(\left| a^n z + \frac{b(1-a^n)}{1-a} \right|^2 - |z|^2 \right)} |u_n(z)|^p \chi_{D\left(\frac{w}{a^n} - \frac{b(1-a^n)}{1-a}, \frac{1}{|a|^n} \right)}(z) dA(z) \right) dA(w). \end{aligned}$$

Using Lemma 5.0.4 and simplifying like the case for $p = \infty$, we get

$$e^{\frac{p}{2}\left(\left| a^n z + \frac{b(1-a^n)}{1-a} \right|^2 - |z|^2 \right)} |u_n(z)|^p \leq e^{pc_n} e^{p|t_n||z| - p\left(\frac{1-|a|^{2n}}{2} - \left| \frac{a_2(1-a^{2n})}{1-a^2} \right| \right)|z|^2}$$

for all $z \in \mathbb{C}$. Since $W_{(u,\psi)}$ is compact and $|a_2| < \frac{1-|a|^2}{2}$, it follows that

$$q_n - |p_n| = \frac{1-|a|^{2n}}{2} - \left| \frac{a_2(1-a^{2n})}{1-a^2} \right| > 0$$

and

$$\int_{\mathbb{C}} e^{\frac{p}{2}\left(\left|a^n z+\frac{b(1-a^n)}{1-a}\right|^2-|z|^2\right)}|u_n(z)|^p\chi_{D\left(\frac{w}{a^n}-\frac{b(1-a^n)}{1-a},\frac{1}{|a|^n}\right)}(z)dA(z)$$
$$\leq \int_{\mathbb{C}} e^{pc_n}e^{p|t_n||z|-p\left(\frac{1-|a|^{2n}}{2}-\left|\frac{a_2(1-a^{2n})}{1-a^2}\right|\right)|z|^2}dA(z) \lesssim e^{pc_n}.$$

Hence,

$$\|W^n_{(u,\psi)}f\|_p \lesssim e^{c_n}\|f\|_p$$

and the conclusion follows as in the last part of $p=\infty$, and completes the proof of the theorem.

5.3 Uniformly mean ergodic weighted composition operators

Having identified conditions under which $W_{(u,\psi)}$ is power bounded, we next turn our attention to the mean and uniformly mean ergodic properties of $W_{(u,\psi)}$ on $\mathcal{F}_p$. The first result in this arena reads as follows.

Theorem 5.3.1. *Let $1 \leq p \leq \infty$ and $W_{(u,\psi)}$ be a compact power bounded operator on $\mathcal{F}_p$, and hence $\psi(z) = az+b$ such that $|a|<1$. Let u is non-vanishing on $\mathbb{C}$. Then $W_{(u,\psi)}$ is uniformly mean ergodic, and*

$$\lim_{n\to\infty}\left\|\frac{1}{n}\sum_{k=1}^{n}W^k_{(u,\psi)}-W_{(u_\infty,\frac{b}{1-a})}\right\| = 0, \quad u_\infty(z)=\prod_{j=0}^{\infty}u(\psi^j(z)). \tag{5.3.1}$$

Proof. By Proposition 4.3.3, $W_{(u,\psi)}$ is uniformly mean ergodic. To prove the limit in (5.3.1), we argue as follows. First observe that u_∞ is a well-defined product as $W_{(u,\psi)}$ is power bounded, Lemma 5.2.1 and (1.0.1) imply

$$|u_\infty(z)| = \lim_{n\to\infty}|u_n(z)| \leq \lim_{n\to\infty}e^{\frac{1}{2}|z|^2}\|u_n\|_p \lesssim e^{\frac{1}{2}|z|^2}.$$

Case 1: Let $1 \leq p < \infty$. Then for each $f \in \mathcal{F}_p$ we claim that

$$\lim_{n\to\infty} \|W^n_{(u,\psi)} f - W_{(u_\infty, \frac{b}{1-a})} f\|_p = 0. \tag{5.3.2}$$

Observe that (5.3.2) implies (5.3.1). Since $W_{(u,\psi)}$ is power bounded, by Lemma 5.2.1, u_n belongs to $\mathcal{F}_p$ for all $n \in \mathbb{N}$. On the other hand, using the representation of u_n in (5.2.12) and applying Theorem 5.2.5 and Lemma 5.2.2

$$|u_n(z)| \lesssim e^{|t_n z| + \left|\frac{a_2(1-a^{2n})}{1-a^2}\right| |z|^2} \leq e^{|t_n z| + |a_2| \frac{1+|a|^{2n}}{|1-a^2|} |z|^2}$$

where

$$t_n = \frac{a_1(1-a^n)}{1-a} + \frac{2a_2 b}{1-a}\left(\frac{1-a^n}{1-a} - \frac{1-a^{2n}}{1-a^2}\right) + \overline{\frac{(1-a^n)b}{1-\overline{a}}} a^n,$$

and letting $n \to \infty$

$$|u_\infty(z)| \lesssim e^{\left(\frac{2|a_1|}{|1-a|} + \frac{2|a_2 b|}{|1-a|}\left(\frac{2}{|1-a|} + \frac{2}{|1-a^2|}\right)\right)|z| + \frac{|a_2|}{|1-a^2|}|z|^2}.$$

The compactness condition $|a_2| < \frac{1-|a|^2}{2}$ implies $\frac{|a_2|}{|1-a^2|} < \frac{1}{2}$, which shows that $u_\infty \in \mathcal{F}_p$. Moreover, by continuity,

$$\lim_{n\to\infty} \left| W^n_{(u,\psi)} f(z) - W_{(u_\infty, \frac{b}{1-a})} f(z) \right| = 0,$$

and since $W_{(u,\psi)}$ is power bounded, there is a constant $\alpha > 0$ such that for every $n \in \mathbb{N}$

$$\int_{\mathbb{C}} \left| W^n_{(u,\psi)} f(z) \right|^p e^{-\frac{p}{2}|z|^2} dA(z) \leq \int_{\mathbb{C}} \alpha^p |f(z)|^p e^{-\frac{p}{2}|z|^2} dA(z).$$

Thus, by Lebesgue dominated convergence theorem

$$\lim_{n\to\infty} \int_{\mathbb{C}} |W^n_{(u,\psi)} f(z)|^p e^{-\frac{1}{2}|z|^2} dA(z) = \int_{\mathbb{C}} |W_{(u_\infty, \frac{b}{1-a})} f(z)|^p e^{-\frac{1}{2}|z|^2} dA(z).$$

Consequently, we have

$$\begin{aligned}&\int_{\mathbb{C}}\left|W^n_{(u,\psi)}f(z)-W_{(u_\infty,\frac{b}{1-a})}f(z)\right|^p e^{-\frac{p}{2}|z|^2}dA(z)\\&\qquad\leq\int_{\mathbb{C}}2^p\left(\alpha^p\left|f(z)\right|^p+\left|W_{(u_\infty,\frac{b}{1-a})}f(z)\right|^p\right)e^{-\frac{p}{2}|z|^2}dA(z)<\infty.\end{aligned}$$

Applying Lebesgue dominated convergent theorem again on the sequence

$$g_n(z):=\left|W^n_{(u,\psi)}f(z)-W_{(u_\infty,\frac{b}{1-a})}f(z)\right|^p e^{-\frac{p}{2}|z|^2},$$

we get that

$$\lim_{n\to\infty}\|W^n_{(u,\psi)}f-W_{(u_\infty,\frac{b}{1-a})}f\|_p^p=\frac{p}{2\pi}\lim_{n\to\infty}\int_{\mathbb{C}}g_n(z)dA(z)=\int_{\mathbb{C}}\lim_{n\to\infty}g_n(z)dA(z)=0$$

as claimed.

Case 2: $p=\infty$: The subspace $\mathcal{F}_0$ defined by

$$\mathcal{F}_0=\{f\in\mathcal{F}_\infty:\lim_{|z|\to\infty}|f(z)|e^{-\frac{1}{2}|z|^2}=0\}.$$

is closed subspace in $\mathcal{F}_\infty$ and it contains polynomials. Moreover, the polynomials are dense in $\mathcal{F}_0$, and $\mathcal{F}_\infty$ is canonically isomorphic to the bidual of $\mathcal{F}_0$. We proceed to show first that (5.3.2) holds for each $f\in\mathcal{F}_0$. By (4.3.2),

$$f(\psi^n(z))\to f\left(\frac{b}{1-a}\right)$$

uniformly on compact subset K of $\mathbb{C}$. Next, we show that $u_n\to u_\infty$ uniformly on compact subset of $\mathbb{C}$ also. Since $u_\infty\in\mathcal{F}_p$

$$\int_{\mathbb{C}}|u_n(z)-u_\infty(z)|^p e^{-\frac{p}{2}|z|^2}dA(z)\leq\int_{\mathbb{C}}2^p\left(\alpha^p+|u_\infty(z)|^p\right)e^{-\frac{p}{2}|z|^2}dA(z)<\infty.\quad(5.3.3)$$

Applying Lebesgue dominated convergence theorem, for $z\in K$, where K is compact subset of $\mathbb{C}$,

$$\begin{aligned}\lim_{n\to\infty}|u_n(z)-u_\infty(z)| &\le \lim_{n\to\infty} e^{\frac{1}{2}|z|^2}\|u_n-u_\infty\|_1 \le \left(\max_{z\in K} e^{\frac{1}{2}|z|^2}\right)\lim_{n\to\infty}\|u_n-u_\infty\|_1\\ &= \left(\max_{z\in K} e^{\frac{1}{2}|z|^2}\right)\lim_{n\to\infty}\int_{\mathbb{C}}|u_n(w)-u_\infty(w)|e^{-\frac{1}{2}|w|^2}dA(w)=0.\end{aligned}$$

From this we have

$$u_n(z)f(\psi^n(z)) \to u_\infty(z)f\left(\frac{b}{1-a}\right)$$

uniformly on the compact subsets of $\mathbb{C}$. That is, for each compact set K in $\mathbb{C}$,

$$\sup_{z\in K}\left|u_n(z)f(\psi^n(z))-u_\infty(z)f\left(\frac{b}{1-a}\right)\right|\to 0 \tag{5.3.4}$$

as $n\to\infty$. Next, with $f\in\mathcal{F}_0$ and each n,

$$\begin{aligned}\lim_{|z|\to\infty}|W^n_{(u,\psi)}f(z)|e^{-\frac{1}{2}|z|^2} &= \lim_{|z|\to\infty}|u_n(z)|\left|f\Big(a^nz+\frac{b(1-a^n)}{(1-a)}\Big)\right|^2 e^{-\frac{1}{2}|z|^2}\\ &\le \sup_{z\in\mathbb{C}}\left(|u_n(z)|e^{\frac{1}{2}\left(|a^nz+\frac{b(1-a^n)}{(1-a)}|^2-|z|^2\right)}\right)\\ &\times\lim_{|z|\to\infty}\left|f\Big(a^nz+\frac{b(1-a^n)}{1-a}\Big)\right|^2 e^{-\frac{1}{2}|a^nz+\frac{b(1-a^n)}{(1-a)}|^2}\\ &\lesssim \lim_{|z|\to\infty}\left|f\Big(a^nz+\frac{b(1-a^n)}{1-a}\Big)\right|^2 e^{-\frac{1}{2}|a^nz+\frac{b(1-a^n)}{(1-a)}|^2}=0.\end{aligned}$$

Note that the last inequality above holds since $W_{(u,\psi)}$ is power bounded the supremum above is uniformly bounded. That is

$$\sup_n\sup_{z\in\mathbb{C}}\left(|u_n(z)|e^{\frac{1}{2}\left(|a^nz+\frac{b(1-a^n)}{(1-a)}|^2-|z|^2\right)}\right)\lesssim|u(b/(1-a))|^n\le 1.$$

Furthermore, $u_\infty\in\mathcal{F}_0$ since it belongs to $\mathcal{F}_p$ for all $p\le\infty$.

Now given $\varepsilon>0, f\in\mathcal{F}_0$ and since $u_\infty\in\mathcal{F}_0$, we can find $r_0>0$ such that $|W^n_{(u,\psi)}f(z)|e^{-\frac{1}{2}|z|^2}<\varepsilon/2$ and $|u_\infty(z)||f(\frac{b}{1-a})|e^{-\frac{1}{2}|z|^2}<\varepsilon/2$ whenever $|z|>r_0$.

Then, for each $|z| > r_0$ and $n \in \mathbb{N}$, we have

$$\left|W^n_{(u,\psi)}f(z) - u_\infty(z)f\left(\frac{b}{1-a}\right)\right| e^{-\frac{1}{2}|z|^2} \leq \left|W^n_{(u,\psi)}f(z)\right| e^{-\frac{1}{2}|z|^2} + \left|u_\infty(z)f\left(\frac{b}{1-a}\right)\right| e^{-\frac{1}{2}|z|^2} < \varepsilon.$$

We apply (5.3.4) to the compact set $K_0 = \{z \in \mathbb{C} : \ |z| \leq r_0\}$ to find n_0 such that if $z \in K_0$ and $n \geq n_0$ we have

$$\left|u_n(z)f(\psi^n(z)) - u_\infty(z)f\left(\frac{b}{1-a}\right)\right| < \frac{\varepsilon}{2S},$$

with $S := \max_{z \in K_0} e^{-\frac{1}{2}|z|^2}$. If $n \geq n_0$ and $z \in \mathbb{C}$, we have

$$\left|W^n_{(u,\psi)}f(z) - u_\infty(z)f\left(\frac{b}{1-a}\right)\right| e^{-\frac{1}{2}|z|^2} < \varepsilon.$$

Thus

$$\lim_{n\to\infty} \|W^n_{(u,\psi)}f - u_\infty f\left(\frac{b}{1-a}\right)\|_\infty = 0.$$

Next, we show (5.3.1) for $p = \infty$. Since $\mathcal{F}_\infty$ is canonically isomorphic to the bidual of $\mathcal{F}_0$, and the bi-transpose operator $W''_{(u.\psi)}$ of $W_{(u,\psi)} : \mathcal{F}_0 \to \mathcal{F}_0$ coincides with composition operator $W_{(u,\psi)} : \mathcal{F}_\infty \to \mathcal{F}_\infty$, the conclusion follows from the well-known fact that $\|T\| = \|T'\| = \|T''\|$ for any bounded operator T on a Banach space. $\square$

The preceding result assures that $W_{(u,\psi)}$ with, $\psi(z) = az + b$ is always uniformly mean ergodic whenever it is compact and power bounded. Now, we consider the case when $\psi(z) = az + b$ and $|a| = 1$. Note that power boundedness in this case implies that either $|u(0)| = e^{-\frac{|b|^2}{2}}$ or $|u(0)| < e^{-\frac{|b|^2}{2}}$. In 1939, Lorch [36] proved that every power bounded operator on a reflexive Banach space is mean ergodic. The same result was latter obtained in reflexive Frechet spaces [1]. Accordingly, as the spaces $\mathcal{F}_p$ are reflexive for all $1 < p < \infty$, every power bounded $W_{(u,\psi)}$ is mean ergodic. Thus, for such spaces we will consider conditions under which the ergodicity becomes uniform.

Theorem 5.3.2. *(i) Let $1 \leq p \leq \infty$, $\psi(z) = az + b$ with $|a| = 1$ and $|u(0)| < e^{-\frac{|b|^2}{2}}$. Then $W_{(u,\psi)}$ is uniformly mean ergodic on $\mathcal{F}_p$, and*

$$\lim_{n\to\infty} \left\| \frac{1}{n} \sum_{k=0}^{n} W_{(u,\psi)}^k \right\| = 0.$$

(ii) Let $1 \leq p \leq \infty$,and $\psi(z) = az$ with $|a| = 1$. If both $u(0)$ and a are roots of unity, then $W_{(u,\psi)}$ is uniformly mean ergodic on $\mathcal{F}_p$.

By Theorem 4.3.7 and Theorem 4.3.9, the composition operator C_ψ is not uniformly mean ergodic on $\mathcal{F}_p$, $1 \leq p \leq \infty$ whenever $|a| = 1$ and a is not root of unity. Now the weight function u makes it possible to enrich uniformity by taking the value $|u(0)|$ smaller.

Proof. (i) Applying the assumption along with (5.2.6) and (5.2.7)

$$\begin{aligned} \lim_{n\to\infty} \left\| \frac{1}{n} \sum_{k=0}^{n} W_{(u,\psi)}^k \right\| \leq \lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n} \left\| W_{(u,\psi)}^k \right\| &\lesssim \lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n} \left(|u(0)| e^{\frac{|b|^2}{2}} \right)^k \\ &\leq \lim_{n\to\infty} \frac{2n^{-1}}{1 - |u(0)| e^{\frac{|b|^2}{2}}} = 0 \end{aligned}$$

as claimed.

(ii) By assumption there exist numbers $m, N \in \mathbb{N}$ such that $a^N = 1 = u(0)^m$. Consider the smallest positive integer $N_0 \leq mN$ such that $a^{N_0} = u(0)^{N_0} = 1$. In this case the sequence $W_{u,\psi}^n$ is periodic with period N_0. Any $n \in \mathbb{N}$ can be written in the form of $n = N_0 l + j$ for some $l \in \mathbb{N}$ and $j = 0, 1, 2, ..., N_0 - 1$. Thus

$$\begin{aligned} \lim_{n\to\infty} \left\| \frac{1}{n} \sum_{k=1}^{n} W_{(u,\psi)}^k - \frac{1}{N_0} \sum_{k=1}^{N_0} W_{(u,\psi)}^k \right\| &= \lim_{l\to\infty} \frac{1}{(N_0 l + j)} \left\| \sum_{k=1}^{j} W_{(u,\psi)}^k - \frac{j}{N_0} \sum_{k=1}^{N_0} W_{(u,\psi)}^k \right\| \\ &\leq \lim_{l\to\infty} \frac{1}{(N_0 l + j)} \left(\sum_{k=1}^{j} \|W_{(u,\psi)}^k\| + \frac{j}{N_0} \sum_{k=1}^{N_0} \|W_{(u,\psi)}^k\| \right) = 0 \end{aligned}$$

and completes the proof.

Our next result consider the cases when the uniform ergodicity fails.

Theorem 5.3.3. *Let* $1 \leq p \leq \infty$ *and* $W_{(u,\psi)}$ *is bounded on* $\mathcal{F}_p$ *with* $\psi(z) = az$, $|a| = 1$. *Let* $1 \neq u(0)a^m$ *for all* $m \in \mathbb{N}$ *or* $u(0) = 1$ *and* a *is not root of unity. Then* $W_{(u,\psi)}$ *is*

(i) mean ergodic on $\mathcal{F}_p$ *for all* $p < \infty$, *and for each* $f \in \mathcal{F}_p$

$$\lim_{n\to\infty} \left\| \frac{1}{n} \sum_{k=0}^{n} W^k_{(u,\psi)} f - W_{(u(0),0)} f \right\|_p = 0,$$

(ii) not uniformly mean ergodic on $\mathcal{F}_p$ *for all* $1 \leq p \leq \infty$, *and not mean ergodic on* $\mathcal{F}_\infty$ *either.*

Proof. (i) Assume that $1 \neq u(0)a^m$ for all $m \in \mathbb{N}$. We first check when f belongs to the set of monomials. If $f = 1$, then the result holds trivially. Thus, for $z^m, m \geq 1$

$$\left\| \frac{1}{n} \sum_{k=1}^{n} W^k_{(u,\psi)} z^m \right\|_p = \left\| \frac{1}{n} \sum_{k=1}^{n} a^{mk} u_k z^m \right\|_p .$$

Since $a^k u(0) \neq 1$ for each $k \in \mathbb{N}$ and $u_k(z) = u(0)^k$,

$$\begin{aligned} \left\| \frac{1}{n} \sum_{k=1}^{n} a^{mk} u_k z^m \right\|_p = \left\| \frac{1}{n} \sum_{k=1}^{n} u(0)^k a^{mk} z^m \right\|_p &= \left\| \frac{z^m}{n} \frac{a^m(1 - u(0)^n a^{mn})}{1 - a^m u(0)} \right\|_p \\ &\leq \frac{2\|z^m\|_p}{n|1 - a^m u(0)|} \to 0 \end{aligned}$$

as $n \to \infty$. Since the set of polynomials is dense in $\mathcal{F}_p$ and $W_{(u,\psi)}$ is power bounded on $\mathcal{F}_p$ by Theorem 5.2.1, we have (see e.g. Lemma 2.1 in [3]),

$$\lim_{n\to\infty} \left\| \frac{1}{n} \sum_{k=1}^{n} W^k_{(u,\psi)} f - f(0) \right\|_p = \lim_{n\to\infty} \left\| \frac{1}{n} \sum_{k=0}^{n} W^k_{(u,\psi)} f - W_{(u(0),0)} f \right\|_p = 0$$

for every $f \in \mathcal{F}_p$.

(ii) Assume now on the contrary that $W_{u,\psi}$ is uniformly mean ergodic. By the relations in (5.2.6) and (5.2.7), it holds that $\frac{1}{n}\|W^n_{(u,\psi)}\| \to 0$ as $n \to \infty$. Then, by the classical result of Lin [35]), $\mathrm{Im}(I - W_{(u,\psi)})$ is closed where $\mathrm{Im}(I - W_{(u,\psi)})$ denotes the range of $I - W_{(u,\psi)}$, and hence

$$\begin{aligned}\mathrm{Im}(I - W_{(u,\psi)}) = \overline{\mathrm{Im}(I - W_{(u,\psi)})} &= \left\{ f \in \mathcal{F}_p : \lim_{n\to\infty} \frac{1}{n}\sum_{k=1}^{n} W^k_{(u,\psi))} f = 0 \right\} \\ &= \left\{ f \in \mathcal{F}_p : \lim_{n\to\infty} \frac{1}{n}\sum_{k=1}^{n} u(0)^k f(\psi^k) = 0 \right\}. \quad (5.3.5)\end{aligned}$$

where the last equality follows after an application of Lemma 5.0.3. Furthermore,

$$\mathcal{F}_p = \mathrm{Im}(I - W_{(u,\psi)}) \bigoplus \mathrm{Ker}(I - W_{(u,\psi)}). \quad (5.3.6)$$

We claim that $\mathrm{Ker}(I - W_{(u,\psi)})$ contains only the constant functions. In deed, if f belongs to it, then since $u(0)f(az) = f(z)$ for each $z \in \mathbb{C}$, we have $u(0)a^n f^{(n)}(az) = f^{(n)}(z)$, and hence $u(0)a^n f^{(n)}(0) = f^{(n)}(0)$. This yields $f^{(n)}(0) = 0$ for every $n \in \mathbb{N}$ since $u(0)a^n \neq 1$. If we define $g(z) = f(z) - f(0)$, it follows that $g^{(n)}(0) = 0$ for every $n = 0, 1, 2, ...$ Hence $g(z) = 0$ for every $z \in \mathbb{C}$. Therefore, $f(z) = f(0)$ for every $z \in \mathbb{C}$.

Next, we show that the constant functions belong to the set in (5.3.5) and contradicts (5.3.6). Thus, if $h = \alpha$ is a non-zero constant function, then $W^k_{(u,\psi))} h(z) = u(0)^k h(\psi^k(z)) = u(0)^k \alpha$ for every $z \in \mathbb{C}$. Thus

$$\begin{aligned}\left\| \frac{1}{n}\sum_{k=1}^{n} h(\psi^k) \right\|_p \le \frac{\|\alpha\|_p}{n} \left| \sum_{k=1}^{n} u(0)^k \right| &= \frac{\|\alpha\|_p}{n} \left| \frac{u(0) - u(0)^{n+1}}{1 - u(0)} \right| \\ &\le \frac{2\|\alpha\|_p}{n|1 - u(0)|} \to 0 \quad \text{as} \quad n \to \infty.\end{aligned}$$

Now we consider the case when $1 = u(0)$ and a is not a root of unity. The proof of part (i) and the fact that $\mathrm{Ker}(I - W_{(u,\psi)})$ contains only constant functions follow exactly in the same way as above. But now the non-zero constant functions do not belong to (5.3.5). Thus, we modify the argument as follows. Consider the function $h(z) = z + 1$. It follows that h belongs to neither $\mathrm{Ker}(I - W_{(u,\psi)})$ nor to the set in (5.3.5). For the latter case, observe that $\frac{1}{n}\sum_{k=1}^{n} h(\psi^k) \to 1$ as $n \to \infty$. In deed

$$\left\| \frac{1}{n} \sum_{k=1}^{n} h(\psi^k) - 1 \right\|_p \leq \frac{\|z\|_p}{n} \left| \sum_{k=1}^{n} a^k \right| \leq \frac{2\|z\|_p}{n|1-a|} \to 0 \ \text{ as } \ n \to \infty.$$

and contradicts (5.3.6) again. Hence $W_{(u,\psi)}$ is not uniformly mean ergodic on $\mathcal{F}_\varphi^p$, $1 \leq p \leq \infty$. It remains to show that $W_{(u,\psi)}$ is not mean ergodic on $\mathcal{F}_\infty$ either. But this follows from Theorem 4.3.8. □

We remark that when $|a| = 1$ and $|u(0)| = e^{-|b|^2/2}$, $b \neq 0$, the operators are isometric bijective with $W_{(u,\psi)}^{-1} = W_{(v,\psi^{-1})}$ where $v(z) := \overline{u(0)} K_{\overline{a}b}(z)$. This can be seen as for every $f \in \mathcal{F}_p$

$$\begin{aligned} \|W_{(u,\psi)} f\|_p^p &= \frac{p}{2\pi} |u(0)|^p \int_{\mathbb{C}} |K_{-\overline{a}b}(z)|^p |f(az+b)|^p e^{-\frac{p}{2}|z|^2} dA(z) \\ = \frac{p}{2\pi} |u(0)|^p \int_{\mathbb{C}} &|f(z+b)|^p e^{-\frac{p}{2}|az+b|^2} \left(|K_{-\overline{a}b}(z)|^p e^{\frac{p}{2}|az+b|^2 - \frac{p}{2}|z|^2} \right) dA(z) \\ &= \frac{1}{|a|^2} |u(0)|^p e^{\frac{p|b|^2}{2}} \|f\|_p^p = \|f\|_p^p \end{aligned}$$

for all $1 \leq p < \infty$ which also holds true for $p = \infty$. This shows that the operator is a linear isometry and hence satisfies the injectivity condition $W_{(u,\psi)}^{-1} W_{(u,\psi)} = I$. On the other hand, for each $f \in \mathcal{F}_p$

$$\begin{aligned} &W_{(u,\psi)} W_{(v,\psi^{-1})} f(z) = u(z) \overline{u(0)} K_{\overline{a}b}(\psi(z)).f(\psi^{-1}(\psi(z))) \\ &= u(z) \overline{u(0)} K_b(az+b) f(z) = |u(0)|^2 K_{-\overline{a}b}(z) K_{\overline{a}b}(az+b) f(z) = f(z) \end{aligned}$$

which also shows that $W_{(u,\psi)} W_{(u,\psi)}^{-1} = I$. As shown below and Theorem 5.1.1, the spectrum of some of these class of operators are contained in the unit circle.
The uniformly mean ergodic results in part (ii) of Theorem 5.3.2 and Theorem 5.3.3 deal with when $\psi(z) = az + b$ form with $|a| = 1$ and $b = 0$. The case for $b \neq 0$ is our next point of interest.

Corollary 5.3.4. *Let* $1 \le p \le \infty$, $\psi(z) = az + b$, $|a| = 1$ *and* a *is not a root of unity,* $u\left(\frac{b}{1-a}\right) = 1$ *and hence* $|u(0)| = e^{-\frac{|b|^2}{2}}$. *Then* $W_{(u,\psi)}$ *can not be uniformly mean ergodic on* $\mathcal{F}_p$.

Proof. First observe that in this case Theorem 5.1.1 and since for $|a| = 1$ and $a \neq 1$,

$$\left|u(0)e^{\frac{a|b^2|}{a-1}}a^m\right| = |u(0)|e^{|b|^2\Re\left(\frac{a}{a-1}\right)} = |u(0)|e^{|b|^2\Re\left(\frac{a(\overline{a}-1)}{(a-1)(\overline{a}-1)}\right)} = |u(0)|e^{\frac{|b^2|}{2}} = 1,$$

the spectrum $\sigma(W_{(u,\psi)})$ is contained in the unit circle $\mathbb{T}$. Furthermore as a is not a root of unity in the case when $a \neq 1$, it follows that 1 is an accumulation point of the spectrum of $W_{(u,\psi)}$. Moreover 1 is in the spectrum of $W_{(u,\psi)}$ since $u(0)e^{\frac{a|b^2|}{a-1}}a^0 = u\left(\frac{b}{1-a}\right)a^0 = 1$. Then an application of Theorem 3.16 of [18] gives the conclusion.

5.3.1 The multiplication operator

We now conclude the section by specializing the main results made in the above section to the multiplication operator M_u acting on Fock spaces. Note that from Lemma 2.3 in [52], it is known that the operator M_u is bounded on $\mathcal{F}_p$ if and only if u is a constant function. The same conclusion can be also easily drawn by applying the condition in (5.0.1) along with Liouville's Theorem.

Corollary 5.3.5. *Let* $1 \le p \le \infty$ *and* $u \in \mathcal{H}(\mathbb{C})$ *such that* M_u *is bounded on* $\mathcal{F}_p$. *Then the following statements are equivalent.*

(i) M_u *is power bounded on* $\mathcal{F}_p$*;*

(ii) $|u(0)| \le 1$*;*

(iii) M_u *is mean ergodic on* $\mathcal{F}_p$*;*

(iv) M_u *is uniformly mean ergodic on* $\mathcal{F}_p$.

Proof. The equivalency of (i) and (ii) is an immediate deduction from Theorem 5.2.3. Thus, we shall show (ii) ⇒ (iii), (iii)⇒ (iv), and (iv) ⇒ (i). For the first, simplifying the proof of Theorem 5.3.2 part (i) for the case $b = 0$ and $a = 1$, we get $W^k_{(u,\psi)} = M^k_u$ and

$$\Big\|\frac{1}{n}\sum_{k=1}^{n} M_u^k f\Big\|_p = \Big\|\frac{f}{n}\sum_{k=1}^{n} u_k(0)\Big\|_p = \Big\|\frac{f}{n}\sum_{k=1}^{n} u(0)^k\Big\|_p = \|f\|_p\Big|\frac{1}{n}\sum_{k=1}^{n} u(0)^k\Big|. \quad (5.3.7)$$

Consider first the case when $u(0) \neq 1$ and $|u(0)| \leq 1$. Then

$$\frac{\|f\|_p}{n}\Big|\sum_{k=1}^{n} u(0)^k\Big| = \frac{\|f\|_p}{n}\frac{|u(0)||1-u(0)^n|}{|1-u(0)|} \leq \frac{2\|f\|_p}{n|1-u(0)|} \to 0$$

as $n \to \infty$. Thus, $\frac{1}{n}\sum_{k=1}^{n} M_u^k$ converges pointwise to zero. If $u(0) = 1$, then M_u reduces to the identity map and the assertion follows trivially.

Next, we show that (iii) implies (iv). That is the above convergence is uniform on the operator norm. Now the assumption implies that $\frac{M_u^n f}{n} \to 0$ as $n \to \infty$ for all $f \in \mathcal{F}_p$. In particular for $f = \mathbf{1}$, the statement $\frac{M_u^n \mathbf{1}}{n} = \frac{u(0)^n}{n} \to 0$ holds only if $|u(0)| \leq 1$. Now, for $u(0) = 1$, the operator reduces again to the identity map. Thus, we consider the case when $u(0) \neq 1$, and argue

$$\begin{aligned}\Big\|\frac{1}{n}\sum_{k=1}^{n} M_u^k\Big\| &= \sup_{\|f\|_p=1}\Big\|\frac{1}{n}\sum_{k=1}^{n} M_u^k f\Big\|_p = \sup_{\|f\|_p=1}\Big\|\frac{1}{n}\sum_{k=1}^{n} u(0)^k f\Big\|_p \\ &\leq \frac{1}{n}\sum_{k=1}^{n}|u(0)|^k = \frac{|u(0)||1-|u(0)|^n|}{n(1-|u(0)|)} \leq \frac{2}{n|1-u(0)|} \to 0,\ n \to \infty.\end{aligned}$$

Now assume that (iv) holds. Then $\|M_u^n\|/n \to 0$ as $n \to \infty$. On the other hand, from (5.2.6) we get $\|M_u^n\| = |u(0)|^n$ which implies that $\|M_u^n\|/n \to 0$ only when $|u(0)| \leq 1$. Therefore, by Theorem 5.2.3, the operator is power bounded. $\square$

5.4 Cyclicity of weighted composition operators

Cyclicity and supercyclicity of weighted composition operator on $\mathcal{F}_2$ are studied by T. Mengestie [64]. Using Hilbert space techniques, he characterized cyclicity of $W_{(u,\psi)}$ on $\mathcal{F}_2$ and showed that $W_{(u,\psi)}$ is not supercyclic on $\mathcal{F}_2$. It is a natural

problem to study what happens to this cyclicity and supercyclicity phenomena on other Fock spaces $\mathcal{F}_p, 1 \leq p \leq \infty$. In this section we present our result of cyclicity, supercyclicity and hypercyclicity of $W_{(u,\psi)}$ on $\mathcal{F}_p, 1 \leq p \leq \infty$.
For the sake of expository reasons for the structures being considered, we start with the stronger hypercyclicity property and prove the following.

Proposition 5.4.1. *Let $1 \leq p < \infty$ and (u, ψ) be a pair of entire functions which induces a bounded weighted composition operator $W_{(u,\psi)}$ on $\mathcal{F}_p$. Then, the operator $W_{(u,\psi)}$ can not be hypercyclic on $\mathcal{F}_p$.*

Proof. To verify this, suppose, for the sake of contradiction, that $W_{(u,\psi)}$ is hypercyclic on $\mathcal{F}_p$ with hypercyclic vector f. Since $W_{(u,\psi)}$ is bounded, we have $\psi(z) = az + b,\ |a| \leq 1$ and ψ fixes the point $z_0 := b/(1-a)$ when $a \neq 1$. Then, for each $g \in \mathcal{F}_p$, there exists a sequence (n_k) such that

$$\|g - W_{(u,\psi)}^{n_k} f\|_p \to 0 \text{ as } k \to \infty.$$

By (1.0.1), norm convergence implies pointwise convergence on Fock spaces, from which and relation (5.2.1)

$$g(z_0) = \lim_{k\to\infty} W_{(u,\psi)}^{n_k} f(z_0) = f(z_0) \lim_{k\to\infty} (u(z_o))^{n_k}.$$

This implies that $f(z_0) \neq 0$. If not, the orbit will contain only functions that vanish at z_0 which does not represent the whole space $\mathcal{F}_p$. Hence we may write

$$\frac{g(z_0)}{f(z_0)} = \lim_{k\to\infty} (u(z_o))^{n_k}. \tag{5.4.1}$$

Since the constant functions are contained in $\mathcal{F}_p$ for all ranges of p, the relation in (5.4.1) shows that each such function is a limit point of a subsequence of $\{(u(z_o))^n : n \in \mathbb{N}\}$. Said differently, the set $\{(u(z_o))^n : n \in \mathbb{N}_0\}$ is dense in the complex plane $\mathbb{C}$ which is not always the case either.
When $a = 1$ and $b = 0$, then $W_{(u,\psi)}$ reduces to the multiplication operator which

obviously is not hypercyclic. On the other hand, if $a = 1$ and $b \neq 0$, then the argument becomes more interesting. First, observe that $\psi^j(z) = z + jb$ and using (5.0.3),

$$u_n(z) = u(0)^n \prod_{j=0}^{n-1} K_{-b}(z + jb) = u(0)^n e^{l_n(z)}$$

where $l_n(z) := -\bar{b}\sum_{j=0}^{n-1}(z + jb) = -\bar{b}nz - \frac{|b|^2}{2}n(n-1)$. It follows that

$$u_n(z) = u(0)^n e^{-\frac{|b|^2}{2}n(n-1)} K_{-nb}(z).$$

Now for each $n \in \mathbb{N}$ and any possible hypercyclic vector f, we compute

$$\begin{aligned} \|W^n_{(u,\psi)}f\|_p^p &= \frac{p}{2\pi}\int_{\mathbb{C}} |u_n(z)|^p |f(\psi^n(z))^p e^{-\frac{p}{2}|z|^2} dA(z) \\ = \frac{p}{2\pi}|u(0)|^{np} e^{-\frac{p|b|^2}{2}n(n-1)} &\int_{\mathbb{C}} |K_{-nb}(z)|^p |f(\psi^n(z))^p e^{-\frac{p}{2}|z|^2} dA(z) \\ &= |u(0)|^{np} e^{-\frac{p|b|^2}{2}n(n-1)} \|f\|_p^p < \infty \end{aligned} \tag{5.4.2}$$

as $n \to \infty$ since $b \neq 0$. On the other hand, for fixed m, setting $g_m(z) = mz$, there exists a sequence $W^{n_k}_{(u,\psi)}f$ such that

$$\|W^{n_k}_{(u,\psi)}f - g_m\|_p \to 0$$

whenever $k \to \infty$. But a computation using Gamma function shows that

$$\|g_m\|_p^p \simeq m^p \to \infty$$

as $m \to \infty$ which contradicts (5.4.2). □

Having observed the absence of hypercyclicity structure, we ask for the weaker supercyclicity dynamical property of the operator. One of our main results to follow affirms that the Fock spaces support no supercyclic weighted composition operator either.

Theorem 5.4.2. *Let $1 \leq p \leq \infty$ and (u, ψ) be a pair of functions in $\mathcal{H}(\mathbb{C})$ which induces a bounded weighted composition operator $W_{(u,\psi)}$ on $\mathcal{F}_p$. Then $W_{(u,\psi)}$ can not be supercyclic.*

The result shows that the projective orbit of any given vector under $W_{(u,\psi)}$ is not dense enough to ensure supercyclicity on the spaces $\mathcal{F}_p$, and the weighted composition operators exhibit the same supercyclic phenomena as the unweighed ones.

Proof. Now we give the proof of the theorem, and set $\psi(z) = az + b$, with $|a| \leq 1$ as before. For the case $|a| < 1$ or $|a| = 1$ and $a \neq 1$, we argue as follows. The map ψ fixes the point $z_0 = \frac{b}{1-a}$. Assume on the contrary that there exists a supercyclic vector f in $\mathcal{F}_p$. First we claim that u is zero free on $\mathbb{C}$. If u vanishes at point w, then (5.2.1) implies that every element in the orbit of f vanishes at w which extends to the projective orbit which obviously is not the case. Observe that f can not have zero in $\mathbb{C}$ either. This is because all the elements in the projective orbit will also vanish at a possible zero which extends to the closure. Thus, by Proposition 4 of [4], for any two different numbers $z, w \in \mathbb{C}$,

$$\overline{\left\{ \frac{u_n(z) f(\psi^n(z))}{u_n(w) f(\psi^n(w))} \right\}} = \mathbb{C}. \tag{5.4.3}$$

Let $r > 0$ be given. Then $K = \{z \in \mathbb{C} : |z - z_0| \leq r\}$ is a compact neighbourhood of z_0 which also contains $\psi(K)$ since for each $z \in K$

$$|\psi(z) - z_0| = |az + b - z_0| \leq |az - az_0| + |az_0 - z_0 + b| \leq |a|r \leq r.$$

Now, if we set $w = \psi(z), z \in K, z \neq z_0$ and consider the expression in (5.4.3)

$$\left| \frac{u_n(z) f(\psi^n(z))}{u_n(w) f(\psi^n(w))} \right| = \left| \frac{u_n(z) f(\psi^n(z))}{u(\psi^n(z)) f(\psi^{n+1}(z))} \right| \leq C$$

for all $n \in \mathbb{N}$ where

$$C = \frac{\max_{z\in K}|u(z)| \cdot \max_{z\in K}|f(z)|}{\min_{z\in K}|u(z)| \cdot \min_{z\in K}|f(z)|}.$$

This obviously contradicts the relation in (5.4.3).

It remains to show the case for $a = 1$. This is rather immediate as by Lemma 5.0.3, we have $u(z) = u(0)K_{-b}(z)$ and

$$\begin{aligned} W_{(u,\psi)}f(z) = u(0)K_{-b}(z)f(\psi(z)) &= u(0)\|K_{-b}\|_2 k_{-b}(z)f(\psi(z)) \\ &= u(0)e^{\frac{|b|^2}{2}}W_{(k_{-b},\psi)}f(z). \end{aligned} \tag{5.4.4}$$

Then we show that $W_{(k_{-b},\psi)}$ is an isometric bijective operator with inverse map $W^{-1}_{(k_{-b},\psi)} = W_{(k_b,\psi^{-1})}$. Thus, for $f \in \mathcal{F}_p$

$$\begin{aligned} \|W_{(k_{-b},\psi)}f\|_p^p &= \frac{p}{2\pi}\|K_{-b}\|_2^{-p}\int_{\mathbb{C}}|K_{-b}(z)|^p|f(z+b)|^p e^{-\frac{p}{2}|z|^2}dA(z) \\ &= \frac{p}{2\pi}\|K_{-b}\|_2^{-p}\int_{\mathbb{C}}|f(z+b)|^p e^{-\frac{p}{2}|z+b|^2}\Big(|K_{-b}(z)|^p e^{\frac{p}{2}|z+b|^2-\frac{p}{2}|z|^2}\Big)dA(z) \\ &= \|K_{-b}\|_2^{-p}e^{\frac{p|b|^2}{2}}\|f\|_p^p = \|f\|_p^p \end{aligned}$$

for all $1 \leq p < \infty$. This shows that the operator is a linear isometry and hence satisfies the injectivity condition $W^{-1}_{(k_{-b},\psi)}W_{(k_{-b},\psi)} = I$. On the other hand, for each $f \in \mathcal{F}_p$

$$W_{(k_{-b},\psi)}W^{-1}_{(k_{-b},\psi)}f(z) = k_{-b}(z)k_b(\psi(z)).f(\psi^{-1}(\psi(z))) = f(z)$$

which also shows that $W_{(k_{-b},\psi)}W^{-1}_{(k_{-b},\psi)} = I$.

Thus, by (5.4.4) the operator $W_{(u,\psi)}$ is normal in this case. Consequently, from a result of Hilden and Wallen [29], it follows that $W_{(u,\psi)}$ can not be supercyclic and completes the proof. □

If $u = 1$, then $W_{(u,\psi)}$ is just the composition operator C_ψ. On the other hand, if ψ is the identity map, then $W_{(u,\psi)}$ reduces to the multiplication operator M_u. With this, we obtain the following immediate consequence of Theorem 5.4.2.

Corollary 5.4.3. *Let $1 \leq p < \infty$ and (u, ψ) be a pair of entire functions, and let the operators C_ψ and M_u be bounded on $\mathcal{F}_p$. Then, both C_ψ and M_u are not supercyclic on $\mathcal{F}_p$.*

Having observed that no projective orbit of a vector in $\mathcal{F}_p$ is big enough to span the whole space, in this section, we consider again another yet weaker dynamical structure namely cyclicity. Suppose that there exist numbers $m, k \in \mathbb{N}$ such that $a^k = 1 = u(0)^m$. Then set $q \leq mN$ be the smallest positive integer such that $a^q = u(0)^q = 1$. It follows that the sequence $W_{(u,\psi)}$ is periodic with period q. Hence the operator fails to be cyclic. This is because the closed linear span of the orbit of any vector under $W^n_{(u,\psi)}$ becomes finite dimensional. Our next main result affirms that periodicity of a fails in general to produce cyclic weighted composition operators.

Theorem 5.4.4. *Let $1 \leq p < \infty$ and (u, ψ) be a pair of entire functions which induces a bounded weighted composition operator $W_{(u,\psi)}$ on $\mathcal{F}_p$ and hence $\psi(z) = az + b$ with $|a| \leq 1$ and u is a non-zero function. Then the following statements are equivalent.*

(i) $W_{(u,\psi)}$ is cyclic on $\mathcal{F}_p$;

(ii) $a^k \neq a$ for all positive integer $k \geq 2$ and u fails to vanish on $\mathbb{C}$.

The novelty of Theorem 5.4.4, besides it provides a simple condition to apply, is the absence of any non-trivial condition on the weight function u. So long as the weight function is non-zero, the weighted operator inherits the cyclicity property from the unweighted case. We may also remark that both Theorem 5.4.2 and Theorem 5.4.4 are independent of the underlying classical Fock spaces.

Proof. We now give a short proof to the theorem. As noted before for $p = 2$, this was proved in [64]. Now let us indicate how to extend that to an arbitrary Banach space. In fact the argument goes through by applying the notion of duality. By Theorem 2.23 and Theorem 2.24 in [78], the dual space of $\mathcal{F}_p$ can be identified as $\mathcal{F}_q$ where $1 = \frac{1}{p} + \frac{1}{q}$ under the integral pairing

$$\langle f, g\rangle = \lim_{R\to\infty} \frac{1}{\pi} \int_{|z|<R} f(z)\overline{g(z)} e^{-|z|^2} dA(z).$$

Using this, in the proof of Theorem 2.1 in [64] exhibit that the reproducing kernels K_z are cyclic vectors for $W_{(u,\psi)}$ for every $z \in \mathbb{C}$ and completes the proof.
Since all kernel functions are cyclic, we remark that both the operators C_ψ and $W_{(u,\psi)}$ belong to the class of operators which have dense set of cyclic vectors on the classical Fock spaces. □

5.5 The Ritt's resolvent condition for weighted composition operators

The other dynamical structure treated in this chapter is Resolvent condition. We recall that the resolvent function $R(\lambda, T)$ defined on the resolvent set $\varrho(T)$ is an operator valued function given by $R(\lambda, T) = (\lambda I - T)^{-1}$. If the operator T is power bounded and hence $\sigma(T)$ contained in the closed unit disc, then using the series expansion of the resolvent function

$$\|R(\lambda, T)\| = \left\| \sum_{n=0}^{\infty} \frac{T^n}{\lambda^{n+1}} \right\| \leq \sum_{n=0}^{\infty} \frac{\|T^n\|}{|\lambda|^{n+1}} = \frac{\sup_{n\in\mathbb{N}_0} \|T^n\|}{|\lambda| - 1}. \qquad (5.5.1)$$

Thus, every power bounded operator satisfies the so called Kreiss's resolvent condition. For infinite dimensional spaces,condition (5.5.1) does not imply power boundedness as it only gives $\|T^n\| = O(n)$ as $n \to \infty$; see for example [67].
Now we are interested in a more stronger resolvent growth condition that implies power boundedness, namely, the Ritt's resolvent condition [71]. An operator T

satisfies such a condition if there exists a positive constant M such that

$$\|R(\lambda, T)\| \leq \frac{M}{|\lambda - 1|} \tag{5.5.2}$$

for all $\lambda \in \mathbb{C}$ with $|\lambda| > 1$. Both the Kreiss's and Ritt's resolvent conditions play important rolls in numerical analysis; see [9] and references therein for more details.

In Theorem 5.1.1, we described the spectrum of the weighted composition operators on Fock spaces and used the result to prove some results on power boundedness and ergodicity. Now we use it again to show the following interesting result on the Ritt's resolvent growth condition.

Theorem 5.5.1. *Let $1 \leq p < \infty$ and (u, ψ) be a pair of entire functions which induces a bounded weighted composition operator $W_{(u,\psi)}$ on $\mathcal{F}_p$ and hence $\psi(z) = az + b$, $|a| \leq 1$. Then if*

(i) $|a| = 1$, then $W_{(u,\psi)}$ satisfies the Ritt's resolvent condition if and only if it satisfies any one of the following

  - *$a = 1$, $b = 0$ and $|u(0)| < 1$ or $u(0) = 1$.*
  - *$a \neq 1$, $b = 0$ and $|u(0)| < e^{-\frac{|b|^2}{2}}$*
  - *$a = 1$, $b \neq 0$ and $|u(0)| < e^{-\frac{|b|^2}{2}}$*

(ii) $W_{(u,\psi)}$ is compact and satisfies the Ritt's resolvent growth condition, then either $|u\big(\frac{b}{1-a}\big)| < 1$ or $u\big(\frac{b}{1-a}\big) = 1$. Conversely, if $|u\big(\frac{b}{1-a}\big)| < 1$, then $W_{(u,\psi)}$ satisfies the Ritt's resolvent growth condition.

Proof. In 1999, Nagy and Zemanek [69] proved that a power bounded operator T in complex Banach space satisfies the Ritt's resolvent condition if and only if

$$\sup_{n\in\mathbb{N}} n\|T^{n+1} - T^n\| < \infty. \tag{5.5.3}$$

As before let $\psi(z) = az + b$, $|a| \leq 1$, and set $z_0 = \frac{b}{1-a}$ when $a \neq 1$. By Theorem 5.1.1, the spectrum of the operators have also the following description.

If $W_{(u,\psi)}$ is compact and hence $|a| < 1$, then

$$\sigma(W_{(u,\psi)}) = \Big\{0,\ u\big(z_0\big)a^m,\quad m \in \mathbb{N}_0\Big\}.$$

On the other hand, if $|a| = 1$, then

$$\sigma(W_{(u,\psi)}) = \begin{cases} \overline{\Big\{u\big(z_0\big)a^m : m \in \mathbb{N}_0\Big\}},\ a \neq 1 \\ \Big\{z : |z| = |u(0)|e^{\frac{|b|^2}{2}}\Big\}, \quad a = 1,\ b \neq 0 \\ \{u(0)\}, \quad a = 1, \quad b = 0 \end{cases} .$$

Now from [46] and [47, Theorem 4.5.4] if $W_{(u,\psi)}$ satisfies the Ritt's condition, then

$$\sigma(W_{(u,\psi)}) \cap \mathbb{T} \subset \{1\}. \tag{5.5.4}$$

Hence, for $|a| = 1$, it follows that either $a = 1$, $b \neq 0$ and $|u(0)| < e^{-\frac{|b|^2}{2}}$, or $a \neq 1$, $b = 0$ and $|u(0)| < e^{-\frac{|b|^2}{2}}$, or $a = 1$, $b = 0$ and $|u(0)| < 1$ or $u(0) = 1$.
Conversely, if $a = 1$, $b = 0$ and $|u(0)| \leq 1$ or $u(0) = 1$, then $W_{(u,\psi)}$ is trivially power bounded. Furthermore, $W_{(u,\psi)}$ reduces to the multiplication operator M_u. By [52, Lemma 2.3], it is known that M_u is bounded on $\mathcal{F}_p$ if and only if u is a constant function $u = u(0)$. Then $\|T^{n+1} - T^n\| = |u(0)|^n|1 - u(0)|$ which implies that condition (5.5.3) holds.
On the other hand, if $a \neq 1$ and $|u(z_0)| < 1$, then applying (5.2.6) and (5.2.7), we have

$$\|W^n_{(u,\psi)}\| \lesssim \big(|u(0)|e^{\frac{|b|^2}{2}}\big)^n = \big|u(0)e^{\frac{a|b|^2}{a-1}}\big|^n$$

since

$$\big|u(0)e^{\frac{a|b|^2}{a-1}}\big| = |u(0)|e^{|b|^2\Re\left(\frac{a}{a-1}\right)} = |u(0)|e^{|b|^2\Re\left(\frac{a(\overline{a}-1)}{(a-1)(\overline{a}-1)}\right)} = |u(0)|e^{\frac{|b|^2}{2}}.$$

From which it follows that

$$\sup_{n\in\mathbb{N}} n\|W^{n+1}_{(u,\psi)} - W^{n}_{(u,\psi)}\| \lesssim \sup_{n\in\mathbb{N}} n\Big(|u(0)|e^{\frac{|b|^2}{2}}\Big)^{n} + \sup_{n\in\mathbb{N}} n\Big(|u(0)|e^{\frac{|b|^2}{2}}\Big)^{n+1} < \infty. \tag{5.5.5}$$

For $|a| < 1$, first observe that for each $n \in \mathbb{N}$, the operator $W^n_{(u,\psi)}$ itself is a weighted composition operator induced by the symbol (u_n, ψ^n) and $W^n_{(u,\psi)} = W_{(u_n,\psi^n)}$. Aiming to check condition (5.5.3), we first find an estimate to the norm of the difference of the two weighted composition operators,

$$\big\|W^{n+1}_{(u,\psi)} - W^{n}_{(u,\psi)}\big\| = \big\|W_{(u_{n+1},\psi^{n+1})} - W_{(u_n,\psi^n)}\big\|.$$

Applying the difference to the normalized reproducing kernels k_w and relation (1.0.1)

$$\begin{aligned}\big\|W_{(u_{n+1},\psi^{n+1})} - W_{(u_n,\psi^n)}\big\| &\geq \big\|W_{(u_{n+1},\psi^{n+1})}k_w - W_{(u_n,\psi^n)}k_w\big\|_p\\ &\geq \Big|u_{n+1}(z)e^{\overline{w}\psi^{n+1}(z)} - u_n(z)e^{\overline{w}\psi^{n}(z)}\Big|e^{-\frac{|z|^2+|w|^2}{2}}\end{aligned} \tag{5.5.6}$$

for all $w, z \in \mathbb{C}$. In particular, setting $z = w = z_0$ it readily follows from (5.5.6) and (5.2.1) that

$$\big\|W_{(u_{n+1},\psi^{n+1})} - W_{(u_n,\psi^n)}\big\| \geq |u_n(z_0)||u(z_0) - 1| = |u(z_0)|^n|u(z_0) - 1|, \tag{5.5.7}$$

from which the relation in (5.5.3) for $\|W_{(u,\psi)}\|$ holds only if $|u(z_0)| < 1$ or $u(z_0) = 1$.

Conversely, by Theorem 5.2.5, $W_{(u,\psi)}$ is power bounded if and only if $|u(z_0)| \leq 1$, and $\|W^n_{(u,\psi)}\| \simeq |u(z_0)|^n$. Then we repeat the argument in (5.5.5) to show that (5.5.3) holds whenever $|u(z_0)| < 1$, and completes the proof.

It would be desirable to know whether the necessity condition $u(z_0) = 1$ could be sufficient as well. We in fact conjecture that it should be. In the following we provide several results in favor of our conjecture.

Corollary 5.5.2. *Let $1 \leq p < \infty$ and (u, ψ) be a pair of entire functions which induces a bounded weighted composition operator $W_{(u,\psi)}$ on $\mathcal{F}_p$ and hence $\psi(z) = az + b,\ |a| \leq 1$. If $a = 0$, then $W_{(u,\psi)}$ satisfies the Ritt's resolvent condition on $\mathcal{F}_p$ if and only if either $u(b) = 1$. or $|u(b)| < 1$*

Proof. The necessity of the condition follows from (5.5.7) as a particular case. Thus, we shall verify the sufficiency. A simple computation shows that for each $f \in \mathcal{F}_p$

$$\big\|W_{(u_{n+1},\psi^{n+1})}f - W_{(u_n,\psi^n)}f\big\|_p^p \leq |u(b)|^{pn}\|f\|_p^p|u(b) - 1|^p.$$

It follows that

$$\big\|W_{(u_{n+1},\psi^{n+1})} - W_{(u_n,\psi^n)}\big\| \leq |u(b)|^n|u(b) - 1|$$

from which it is easy to see that (5.5.3) holds.
Observe that the multiplication operator M_u is power bounded if and only if $|u(0)| \leq 1$. This gives the following consequence.

Corollary 5.5.3. *Let $1 \leq p < \infty$ and u be an entire functions. Then M_u satisfies the Ritt's resolvent condition on $\mathcal{F}_p$ if and only if it is power bounded.*

The composition operator is one of the other cases where we have $u(z_0) = 1$. For this we show that the Ritt's resolvent condition is in fact equivalent to the stronger unconditional Ritt's condition on $\mathcal{F}_p$. We recall that an operator T on a Banach space $\mathcal{X}$ satisfies the unconditional Ritt's condition if there exists a non negative constant K such that

$$\Big\|\sum_{k=1} a_k\big(T^k - T^{k-1}\big)\Big\| \leq K \sup_k\{|a_k|\} \tag{5.5.8}$$

for any finite sequence (a_k) of complex numbers. We note in passing that the notion of the unconditional Ritt's condition is the discrete analogue of the H^∞

calculus for sectorial operators [44]. N. Kalton and P. Portal [32] proved that the unconditional Ritt's condition implies the Ritt's resolvent condition in general, but not conversely. But for the composition operators, it turns out that the two are equivalent.

Theorem 5.5.4. *Let $1 \leq p < \infty$ and ψ be an entire function which induces a bounded composition operator C_ψ on $\mathcal{F}_p$. Then the following are equivalent*

(i) C_ψ satisfies the Ritt's resolvent condition on $\mathcal{F}_p$;

(ii) C_ψ is compact or C_ψ is the identity map on $\mathcal{F}_p$;

(iii) C_ψ satisfies the unconditional Ritt's condition on $\mathcal{F}_p$.

Proof. We first show that the statements in (i) and (ii) are equivalent. Recall that C_ψ is bounded if and only if $\psi(z) = az + b,\ |a| \leq 1$ and $b = 0$ whenever $|a| = 1$. Compactness is described by the strict inequality $|a| < 1$. If $|a| = 1$, then by Theorem 5.5.2 the composition operator C_ψ satisfies the Ritt's resolvent condition on $\mathcal{F}_p$ if and only if $a = 1$, that means ψ reduces to the identity operator. Thus, we shall proceed to the case when $|a| < 1$. We need to prove the sufficiency. Applying the same approach in the proof of [[65], Theorem 2.1], for any $f \in \mathcal{F}_p$, we have

$$\left\|C_{\psi^{n+1}} - C_{\psi^n}\right\|^p \leq C|a|^{pn} \tag{5.5.9}$$

for some constant C. From which and since $|a| < 1$, the relation in (5.5.3) holds for C_ψ. Next we show that the statements in (i) and (iii) are also equivalent. From the discussion above, we have already mentioned that (iii) implies (i). Thus, it remains to show (i) implies (iii). But this is rather immediate since by (5.5.3), we observe that $\|C_{\psi^n} - C_{\psi^{n+1}}\|$ is bounded by an exponentially decreasing sequence with n. Thus

$$\left\|\sum_{k=1} a_k\big(C_{\psi^k} - C_{\psi^{k-1}}\big)\right\| \leq \sup_k\{|a_k|\} \sum_{k=1} \|C_{\psi^k} - C_{\psi^{k-1}}\| \lesssim \sup_k\{|a_k|\}.$$

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