ADDIS ABABA UNIVERISTY
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DEPARTMENT OF MATHEMATICS

A GRADUATE SEMINAR REPORT
ON
SOLUTION OF MIXED STRATEGIES PROBLEMS IN GAME THEORY

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ADDIS ABABA
JUNE, 2005

Contents

Preface

This paper is a report of two seminars: namely seminar I (Math 701) and seminar II (Math 702), which are conducted in the first and second semesters respectively. It comprises five chapters that organized in order of dependence of one on the other. The chapters in this seminar report are divided into sections.

This seminar report is mainly focus on how can determine solution of pure strategies and mixed strategies.
The first chapter provides basic definition of competitive strategies and explains some important terms.

In general speaking chapter 2, 3 and 5 introduce the method of solving games with pure strategies by Maximin – Minimax principle. Further we have discussed the method of solving 2 x 2 games (having no saddle point). In all such cases, to solve games both the players must determine an optimal mixture of strategies to find a saddle point and also discussed different solution methods that can be solved a mixed strategy game. On the other hand, we have discussed a particular method in chapter 4. In this method, the payoff matrix can be reduced by mere observation and in many cases it may be solved only be adopting this method. The method of reduction of the payoff matrix by this process is called the dominance property of the rows and columns of the payoff matrix.

First of all I thank the almightily God because with out his help this seminar report may not take its present form.
I express my sincere gratitude to my advisor Dr. Mobin Ahmad his genuine advice, constructive comments, invaluable suggestions and provision of materials (reference books) in preparing this seminar report.

I express my sincere gratitude to my advisor Dr. Mobin Ahmad his genuine advice, constructive comments, invaluable suggestions and provision of materials (reference books) in preparing this seminar report. I also extend my deep appreciation to the department of mathematics for its material support and for giving me the chance to type the manuscript of this seminar by myself.

23/10/97

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1. Competitive strategies

1.1 Introduction

Game theory is a mathematical theory that deals with the general features of competitive situations. In many practical problems, it is required to take decision in situation where there are two or more opposite parties with conflicting interests and the action of one depends upon the action, which the opponent takes. Such a situation is termed as a "competitive situation".
A great variety of competitive situation is commonly seen in every day life.
For example in military battles, political campaigns, elections advertising and marketing campaigns.
The competitors are referred as players. A player may be an individual, a group of individual or an organization. Theory of games started in the 20th century. The mathematical treatment of games was developed when john Van Neumann and Morgenstern published with their work" Theory of Games and Economic Behavior". The approach to competitive problems developed by J. Van Neumann (Known as the father of game theory) utilities the minimax principal, which involve the fundamental ideal of minimization of the maximum loss or the maximization of the minimum gain.

1.2 Definitions and Explanations of some important Terms.

A competitive situation is called ***a competitive game*** if it has the following properties.

I. There is finite number of participants called ***players.*** If the number of competitors is 2 then it is called a two persons game and if the number of competitors is n> 2, then it is called n-persons game.
II. Each of the n-players has available to him a list of finite number of possible courses of action. This list need not be same for each player.
III. A play is said to be played when each of the players (competitors) chooses a single course of action from the list of courses of action available to him. Here it is assumed that the choices are made simultaneously so that no player knows his opponents choice until he has decided his own course of action.
IV. Every combination of courses of action determines an outcome (which may be points, money or anything else what ever) which results in gain of payments

(Positive or negative, or zero) to each player, provided each player is playing uncompromisingly to as much as possible. Negative gain implies the losses of same amount.

V. After all participants have chosen a course of action, their respective gains are finite.

VI. The gains of the participants depend on their own action and those of other.

Definition 1.2.1 A game is said to be ***a finite game*** if it has a finite number of moves each of which involves only finite number of alternatives. A game, which is not a finite game, is called ***an infinite game***.

Move to mean a point in a game at which one of the players picks outs an alternative from some set of alternatives.

Definition 1.2.2 In a two person zero sum game, the resulting gain, can easily be represented in the form of a matrix, called **the pay - off matrix or gain matrix**. Thus a pay of matrix is a table, which shows how payments should be made at the end of a play or game. Since the game is zero-sum game the gain of one player is equal to the loss of other and vice versa. In other words one players pay off table would contain the same amounts in payoff table of other player with the sign changed.

Definition 1.2.3 Optimal strategy is the course of action or plan which puts the player in the most preferred position. Any deviation from this strategy results in a decreasing payoff for the player.

1.3 The person zero sum (or Rectangular) Games

Definition 1.3.1 Consider a game where there are n competitors, ad competitor I has N_I course of action available to him Then the total number of possible out comes to a play of the game will be $N_1, N_2, \ldots ,N_n$. Let a particular out come 0 result in a payment P (I, θ) to competitor I. The game is called Zero sum game, of for every possible out come, we have

$$\sum_{I=1}^{n} P(I,\theta) = 0$$

In other words, a game is said to be zero sum game if the sum of payments to all competitors after a play of the game is restricted to zero.

Definition 1.3.2 A game with only two players in which the gains of one player are the losses of another player, is called ***a two - person zero sum game***. In other words the game in which the algebraic sum of gains and losses of all the players is zero called zero sum games. Two person, zero sum games are also called rectangular game, because these are usually represented by a pay of matrix is rectangular form.

If player A has m strategies (or Courses of action) represented by A_1 , A_2 , $\cdots$, A_m and player B has n strategies (or Courses of action) represented by B_1 , B_2 ,$\cdots$,B_n.

The numbers m and n need not be equal. The total number of possible out comes is therefore m x n. For convenience, it is assumed that player A is always a gainer whereas player B a loser. Let a_{ij} be the payoff matrix which player A gains from player B, if player A chooses strategy "i" and player B chooses strategy "j". Then the two payoff matrixes are as follows.

Remark: In a zero sum two person game the cell entry in B's payoff matrix will be the negative of the corresponding cell entry in A's payoff matrix.

Table 1.3.1 ***(Payoff matrix)***

Player B's Strategies

Player A's Strategies		B_1	B_2	$\cdots$	B_j	$\cdots$	B_n
	A_1	a_{11}	a_{12}	$\cdots$	a_{1j}	$\cdots$	a_{1n}
	A_2	a_{21}	a_{22}	$\cdots$	a_{2j}	$\cdots$	a_{2n}
	$\vdots$	$\vdots$	$\vdots$	$\cdots$	$\vdots$	$\cdots$	$\vdots$
	A_i	a_{i1}	a_{i2}	$\cdots$	a_{ij}	$\cdots$	a_{in}
	$\vdots$	$\vdots$	$\vdots$	$\cdots$	$\vdots$	$\cdots$	$\vdots$
	A_m	a_{m1}	a_{m2}	$\cdots$	a_{mj}	$\cdots$	a_{mn}

Remark: In a Zero-sum two person game, the cell entry in B's payoff matrix will be the negative of the corresponding cell entry in A's payoff matrix.

Assumptions of the Game

1. Each player has available to him a finite number of possible strategies (courses of action). The list my not be the same for each player.
2. Player A attempts to Maximize gains and player B minimize losses.
3. The decisions of both players are made individual prior to the play with no communication between then.
4. The decisions are made simultaneously and also announced simultaneously so that neither player has an advantage resulting from direct knowledge of the other player's decision.
5. Each the players know not only possible payoffs to themselves but also of each other.

Example 1.3.1: Suppose A and B play a game, called two fingers Morra: Both simultaneously show either one or two fingers. It the number of fingers of one coincide with the number f fingers of the other then the player A wins and get Birr 1.00 from A. Thus game is a two person zero sum game, since the winning of one player are taken as losses of the other. For this game the payoff matrix to A is a follows.

Table 1.3.2 ***A's payoff matrix***

		B	
		1 finger	2 fingers
A	1 finger	+ 1	-1
	2 fingers	-1	+ 1

Note: In the two person zero sum game we shall not write the B's payoff matrix, since in B's payoff matrix the cell entries are just the negative of the corresponding cell entries in A's payoff matrix.

1.4 Strategy

The strategy of a player is the predetermined rule by which a player decides his course of action from his own list of course of action during the game. Generally, players in a game employ two types of strategies.

A. ***Pure strategy***: It is a decision, rule that is always used by the player to select the particular strategy (Course of action). Thus, each player knows in advance of all strategies out of which he always selects only one particular strategy regardless of the other player's strategy, and the objective of the players is to maximize gains or minimize losses.

B. ***Mixed strategy:*** It is courses of action that are to be selected on a particular occasion with some fixed probability. Thus, there is a probabilistic situation and objective of the players is to maximize expected gains or to minimize expected losses by making choice among pure strategies with fixed probabilities.

Mathematically, a mixed strategy to any player is a set X of m non-negative real numbers whose sum is unity. These m non-negative real numbers represent the probabilities with which each course of action (or pure strategy) should be selected, being the number pure strategies of the player.

Thus, if X, is the probability of choosing course "i", then

$$X = (x_1 , x_2 , \cdots, x_m)$$

$$where\ x_i \geq 0 \quad \forall i = 1, 2, \cdots, m. \quad and \quad \sum_{i=1}^{m} x_i = 1$$

If $x_r = 1$ and $x_i = 0$ $i \neq r$, then the mixed strategy indicate the i^{th} pure strategy.

Thus a pure strategy is a special case of a mixed strategy.

2. Pure strategies

2.1 Solution of a game.

By solving a game we mean to find the best strategies for both players and the value of the game. The value of the game is the maximum guaranteed gain to player A (Maximizing player) if both players use their best strategies. It is generally denoted by v, and is unique. If the value of a game is zero it is called ***a fair game.*** Other wise the game is ***strictly determinable***. In a game theory the best strategies for each player are determined on the basis of maximum and minimum criterion of optimality.

2.2 Maximin and minimax criterion of optimality.

It is states that if a player lists his worst possible out come of all his potential strategies, and then he will choose that strategy which corresponds to the best of these worst outcomes.

Let A's payoff matrix is given by:

Table 2.2.1 ***Payoff matrix***

		Player B's Strategies					
		B_1	B_2	$\cdots$	B_j	$\cdots$	B_n
	A_1	a_{11}	a_{12}	$\cdots$	a_{1j}	$\cdots$	a_{1n}
	A_2	a_{21}	a_{22}	$\cdots$	a_{2j}	$\cdots$	a_{2n}
Player A's Strategies	$\vdots$	$\vdots$	$\vdots$	$\cdots$	$\vdots$	$\cdots$	$\vdots$
	A_i	a_{i1}	a_{i2}	$\cdots$	a_{ij}	$\cdots$	a_{in}
	$\vdots$	$\vdots$	$\vdots$	$\cdots$	$\vdots$	$\cdots$	$\vdots$
	A_m	a_{m1}	a_{m2}	$\cdots$	a_{mj}	$\cdots$	a_{mn}

If A plays strategy 1, then he is sure of getting at least $\underset{j}{Min}\ a_{1j}$

If A plays strategy 2, then he is sure of getting at lest $\underset{j}{Min}\ a_{2j}.$

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If A plays strategy i, then he is sure of getting at lest $\underset{j}{Min}\ a_{ij}$

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… … … … … … … … … … … … … …

If A plays strategy m, then he is sure of getting at lest $\underset{j}{Min}\ a_{mn}$.

Thus, by the Maximin, minimax criterion on optimality A will choose the strategy which corresponds to the best of these worst outcomes $Min\, a_{1j}\ ,\ Min\, a_{2j}\ ,\cdots, Min\, a_{mj}$.

Thus, the maximum for A is given by $\underset{i}{Max}\left[\underset{j}{Min}\, a_{ij}\right]$.

Similarly B will choose that strategy which corresponds to the best (Minimum) of the worst out comes (Maximum losses). $\underset{i}{Max}\, a_{i1}\ ,\ \underset{i}{Max}\, a_{i2}\ ,\cdots, \underset{i}{Max}\, a_{in}$.

Thu, the minimax for B is given by $\underset{j}{Mini}\left[\underset{i}{Max}\, a_{ij}\right]$.

Theorem 2.2.1 (cf. [2]) let (a_{ij}) be the payoff matrix for two person zero-sum game (Say Player A and player B).If $\underline{v}$ the Maximin value and $\bar{v}$ minimax value of the game, then show that $\underline{v} \le \bar{v}$ *i.e.* $Maximin\ for\ A\ \le\ Minimax\ for\ B.$

Proof:

$$\text{Let}\quad \underset{i}{Max}\left[\underset{j}{Min}\ a_{ij}\right] = a_{pq} \qquad \cdots\cdots\cdots \qquad (1)$$

and

$$\underset{j}{Min}\left[\underset{i}{Max}\ a_{ij}\right] = a_{rs} \qquad \cdots\cdots\cdots \qquad (2)$$

From (1) *it follows that* a_{pq} *is the minimum element in the* p^{th} *row*

$$a_{pq} \le a_{ps} \qquad \cdots\cdots\cdots \qquad (3)$$

Since a_{ps} *is another element in* p^{th} *row* .

Also from (2) *it follows that* a_{rs} *the maximum element in the* s^{th} *column.*

$$a_{ps} \le a_{rs} \qquad \cdots\cdots\cdots \qquad (4)$$

Since *Since* a_{ps} *is another element in* s^{th} *column*.

From (3) *and* (4) *we get*

$$a_{pq} \leq a_{rs}$$

$$\underset{i}{Max}\left[\underset{j}{Min}\, a_{ij}\right] \leq \underset{j}{Min}\left[\underset{i}{Max}\, a_{ij}\right]$$

$$\text{i.e. } Maxi\min \ for\ A \quad \leq \quad Mini\max \ for\ B \qquad \cdots\cdots\cdots \quad (5)$$

Maximin a_{ij} (i.e. Maximin for A) is called the lower value of the game and is denoted by $\underline{V}$ and

Minimax a_{ij} (Minimax for B) is called the upper value of the game and is

denoted by $\bar{V}$.

If V is the value of the game then it will always satisfy the inequality.

$$Maxi\min \ for\ A \leq v \leq Mini\max\ B$$

$$\underline{v} \leq v \leq \bar{v}$$

If for a game $\underline{v} = v = \bar{v} = a_{lk}$, then the game posses a solution given by:

Best strategy for A is i^{th} pure strategy.

Best strategy for B is k^{th} pure strategy and

the value of the game $v = a_{lk}$. Such a game is called ***a game with saddle point***.□

2.3 Solution of a Rectangular game with saddle point

A saddle point of a payoff matrix is that position in the matrix where the maximum of row minima coincides with the minimum of the column maxima. The cell entry (Payoff) at the saddle point is called ***the value of the game***.

A game for which Maximin for A= Minimax for B is called ***a game with saddle point***. Thus in a game with saddle point the players use pure strategies. (i.e. they choose the same course of action throughout the game.)

For a rectangular game with saddle point the best strategies for players A and B will be those which correspond to the row and column respectively through the saddle point. The value of the game to A is the element at the saddle point and its negative is the value of game to B.

2.4 Rules to determine saddle point

Thus in order to solve a rectangular game with saddle point one has to find the saddle point. To determine the saddle point in the pay of matrix follows the following three steps.

1. Select the minimum (lowest) element in each row of the payoff matrix and write them under row minima' heading. Then select the largest element among these elements and enclose it in a circle.
2. Select the maximum (largest) element in each column of the payoff matrix and write them under column maxima heading. Then select the lowest element among these elements and enclose it in a rectangle.
3. Find out the element(s), which is same in the circle as well as rectangle and mark the position of such elements in the matrix. This element represents the value of the game and is called the saddle (or equilibrium).

Remark: If a matrix involves more than one saddle point, then there exists more then one solution of the game.

Example 2.4.1: For the game with payoff matrix:

Table 2.4.1

		Player B	
Player A	B_1	B_2	B_3
A_1	6	8	6
A_2	4	12	2

Determine the best's strategies for players A and B. Also determine the value of game. Is this game I) fair? ii) Strictly determinable?

Solution:

To find the saddle point, rectangle around the row minimums and putting circle the column maximums, we get the following tables

Table 2.4 2

Player A	B_1	B_2	B_3	Row minimums
A_1	6	8	6	6
A_2	4	12	2	2
Column Max.	6	12	6	

Obviously, the matrix has two saddle points at (1, 1) and (1, 3).

Thus, the solution to the game is given by

I) the best strategy for player A is A_1.

II) The best strategy for player B is B_1 or B_3.

III) The value of he game is 6 for A and -6 for B.

IV. The game is strictly determinable since the value of the game is not zero.

Example 2.4.2: The payoff matrix of a game is given below find the solution of the game to A and B.

Table 2.4.3

Player A	B_1	B_2	B_3	B_4	B_5
A_1	-2	0	0	5	3
A_2	3	2	1	2	2
A_3	-4	-3	0	-2	6
A_4	5	3	-4	2	-6

Solution: To find the saddle point, first construct the table for row minimums and column maximums, as shown below.

Table 2.4.4 **Player B**

Player A	B_1	B_2	B_3	B_4	B_5	*Row min.*
A_1	-2	0	0	5	3	-2
A_2	3	2	1	2	2	1
A_3	-4	-3	0	-2	6	-4
A_4	5	3	-4	2	-6	-6
Column max.	5	3	1	5	6	

From the above table we have seen that Maximin = 1 and Minimax = 1

Obviously the matrix has a saddle point at the position (2, 3).

Thus, the solution of the game is give by:

I) The best strategy for player A is A_2.

II) The best strategy for player B is B_3.

III) The value of the game is 1 for player A and -1 for player B.

Example 2.4.3: A company management and the labor union are negotiating a new three-year settlement. Each of these has 4 strategies.

I: Hard and aggressive bargaining

II: Reasoning and logical approach

III: Legalistic strategy

IV: Conciliatory approach

The costs to the company are given for every pair if strategy choice.

Table 2.4.5 ***Company strategies***

Union strategies	*I*	*II*	*III*	*IV*
I	20	15	12	35
II	25	14	8	10
III	40	2	10	5
IV	-5	4	11	0

What strategy will the two sides adopt? Also determine the value of the game.

Solution:

Applying the rule of finding out the saddle point, we obtain the saddle point as shown below.

Table 2.4.6 ***Company strategies***

Union strategies	*I*	*II*	*III*	*IV*	*Row Min.*
I	20	15	12	35	12
II	25	14	8	10	8
III	40	2	10	5	2
IV	-5	4	11	0	-5
Column Max.	40	15	12	35	

Obviously, the matrix has a saddle point at the position (1, 3).

Thus the solution of the game is given by:

I) The company will always adopt strategy III legalistic strategy.

II) The union will always adopt strategy I Hard and aggressive bargaining.

III) The value of the game is 12 for union and -12 for company.

3. Mixed Strategies

In certain cases, there is no pure strategy solution for a game, i.e no saddle point exists. In all such cases to solve games both players must determine an optimal mixture of strategies to find a saddle (equilibrium) point.

The optimal strategy mixture for each player may be determined by assigning to each strategy its probability of being chosen. The strategies so determined are called mixed strategies because they are probabilistic combination of available choices of strategy.

3.1 Solution of a rectangular game in terms of mixed strategies.

It a game does not have a saddle point; the two players cannot use Maximin, minimax (pure) strategies as their optimal strategies, then the best strategies are mixed strategies. The two players, instead of selecting pure strategies only, may play their plays according to predetermined set that consists of probabilities corresponding to each of their pure strategies.

Table 3.1.1 *A's pay off matrix*

Player B

Probabilities		y_1	y_2	$\cdots$	y_j	---	y_n
	Pure strategies	B_1	B_2	---	B_j	---	B_n
x_1	A_1	a_{11}	a_{12}	---	a_{1j}	---	a_{1n}
x_2	A_2	a_{21}	a_{22}	---	a_{2j}	---	a_{2n}
--	--	---	---	---	---	---	---
x_i	A_i	a_{i1}	a_{i2}	---	a_{ij}	---	a_{in}
--	--	---	---	---	---	---	---
x_m	A_m	a_{m1}	a_{m2}	---	a_{mj}	---	a_{mn}

Consider a rectangular game played by two players A (maximizing player) and B, with payoff matrix $(a_{ij})_{mxn}$. Here the players A and B have m and n pure strategies respectively.

Let $X=(x_1, x_2, \cdots, x_m)$ *and* $Y=(y_1, y_2, \cdots, y_n)$ be the mixed strategies of the two players A and B respectively, where $x_1, x_2, \cdots, x_m$ *and* $y_1, y_2, \cdots, y_n$ the probabilities by which A and B are, respectively, select their pure strategies

$$\sum_{i=1}^{m} x_i = 1 \quad and \quad \sum_{j=1}^{n} y_j = 1 \quad ; x_i \geq 0 \, , \; y_j \geq 0$$

$$For\ all\ i = 1 , 2 , \cdots , m. \quad and\ j = 1 , 2 , \cdots , n$$

Now expected gain to player A is:

$$a_{11}x_1 \; + \; a_{21}x_2 \; + \; \cdots\cdots \; + \; a_{i1}x_i \; + \; \cdots\cdots \; + \; a_{m1} = \sum_{i=1}^{m} a_{i1}x_i$$

$$\left(if\ B\ uses\ strategy\ B_1\ with\ probability\ Y_1\right)$$

$$a_{12}x_1 \; + \; a_{22}x_2 \; + \; \cdots\cdots \; + \; a_{i2}x_i \; + \; \cdots\cdots \; + \; a_{m2} = \sum_{i=1}^{m} a_{i2}x_i$$

$$\left(if\ B\ uses\ strategy\ B_{21}\ with\ probability\ Y_2\right)$$

$$\cdots\cdots \quad \cdots\cdots \quad \cdots\cdots \quad \cdots\cdots \quad \cdots\cdots$$

$$\cdots\cdots \quad \cdots\cdots \quad \cdots\cdots \quad \cdots\cdots \quad \cdots\cdots$$

$$a_{i1}x_1 \; + \; a_{i2}x_2 \; + \; \cdots\cdots \; + \; a_{ij}x_i \; + \; \cdots\cdots \; + \; a_{im} = \sum_{i=1}^{m} a_{ij}x_i$$

$$\left(if\ B\ uses\ strategy\ B_j\ with\ probability\ Y_j\right)$$

$$\cdots\cdots \quad \cdots\cdots \quad \cdots\cdots \quad \cdots\cdots \quad \cdots\cdots$$

$$\cdots\cdots \quad \cdots\cdots \quad \cdots\cdots \quad \cdots\cdots \quad \cdots\cdots$$

$$a_{m1}x_1 \; + \; a_{m2}x_2 \; + \; \cdots\cdots \; + \; a_{mj}x_i \; + \; \cdots\cdots \; + \; a_{mn} = \sum_{i=1}^{m} a_{mn}x_i$$

$$\left(if\ B\ uses\ strategy\ B_n\ with\ probability\ Y_n\right)$$

The expected gain to A (payoff function to A) is given by

$$E\left(x, y\right) = \sum_{i=1}^{m} \sum_{j=1}^{n} a_{ij}x_i y_j$$

By Maximin - minimax criterion,

A selects $x_i \left(x_i \geq 0\right) \sum_{i=1}^{n} x_i = 1$). This will maximize his minimum expected gain.

i.e. A selects x_i *which will*

$$\underset{x_i}{Max}[Min\left\{\sum_{i=1}^{m} a_{i1}x_i \, , \, \sum_{i=1}^{m} a_{i2}x_i \, , \, \cdots , \sum_{i=1}^{m} a_{in}x_i\right\}]$$

This value is referred to the maximum ($\underset{-}{v}$) expected value for player A.

Similarly B selects y_j ($y_j \geq 0$, $\sum_{j=1}^{n} y_j = 1$), which will minimize his maximum expected loss.

i.e. if B selects y_j which will

$$\underset{y_j}{Min}[Max\left\{\sum_{j=1}^{n} a_{1j}y_j, \sum_{j=1}^{n} a_{2j}y_j, \cdots, \sum_{j=1}^{n} a_{mj}y_j\right\}]$$

This value is referred to as the minimax ($\bar{v}$) expected value for player B.

Hence, for A the best strategy is that which maximizes the $\underset{j}{Min} \sum_{i=1}^{m} a_{ij}x_i$ and

for B best strategy is that which minimize the $\underset{i}{Max} \sum_{j=1}^{n} a_{ij}y_j$.

3.2 Important properties of optimal mixed strategies

Propositions 3.2.1

1. If one of the players' adheres to his optimal mixed strategy and other deviates from his optimal strategy, then the deviating players can only decrease his yield and cannot increase in any case (at most may be equal).
2. One of the players adheres to his optimal strategy, the value of the game does not alter if the opponent uses his supporting strategists only either singly or mixture.
3. If a fixed number c is added to each element of the payoff matrix, the optimal strategies remain unchanged while the value of the game increases by C.
4. If every element of the payoff matrix is multiplied by a constant C, then the optimal strategies remain unchanged while the value the game becomes C times the value of the original game.

Proof:

Let $v_1 , v_2 , \cdots , v_n$ be the gains to player A when A uses his optimal mixed strategy and player B uses his pure supporting strategies 1,2,..., n. respectively.

As B uses his pure strategies with probabilities $y_1 , y_2 , \cdots , y_n$ (B's optimal strategies), therefore we have

$$V = v_1 y_1 + v_2 y_2 + \cdots + v_n y_n . \qquad \cdots\cdots\cdots \qquad (1)$$

Such that all $y_j \geq 0 \ and \sum_{j=1}^{n} y_j = 0$

Now if any of $v_i's$ is greater that V, then the value of the expression on R.H.S of (1) is also greater than V, which is a contradiction.

Therefore, more of $v_i's$ is greater than V.

Hence each $v_i's$ is equal to V.

Also if B uses his supporting strategies in accordance with some other probability distribution $y_1^{'} , y_2^{'} , \cdots , y_n^{'}$. Where all $y_j^{'} \geq 0 \ and \sum_{j=1}^{n} y_j^{'}$ and

The expected yield = $vy_1^{'} + vy_2^{'} + \cdots + vy_n^{'}$

$$= v(y_1^{'} + y_2^{'} + \cdots + y_n^{'}) = v.1 = v$$

Which proves that even B uses his supporting strategies in any mixture, the value of the game remains unaltered. Similarly the result holds if B adheres to his optimal strategy.

Pr*oof of props.*3 By adding a fixed number C to every element of the payoff matrix

$(a_{ij})_{mxn}$ *the element* a_{ij} *becomes* $a_{ij} + C$

If E_1 (x, y) is the original payoff function, then $E_1(x, y) = \sum_{i=1}^{m} \sum_{j=1}^{n} x_i a_{ij} y_j$

If $E_2(x, y)$ *is the new payoff function ,then* $E_2(x, y) = \sum_{i=1}^{m} \sum_{j=1}^{n} x_i (a_{ij} + C) y_j$

$$E_2(x, y) = \sum_{i=1}^{m} \sum_{j=1}^{n} x_i a_{ij} + C \sum_{i=1}^{m} \sum_{j=1}^{n} x_i y_j$$

$$= E_1 (x, y) + C \qquad ---- \qquad (2)$$

Thus addition of C does not change the nature of E_1 (x, y). Therefore the optimal strategies, for the two games are the same.

If X^*, Y^* are the optimal strategies, for the two games, then from (2).

$E_2(X^*, Y^*) = E_1(X^*, Y^*) + C$

Therefore value of the new game = V + C

Hence the value of the game is increased by C, by the addition of a fixed constant c to every element of the payoff matrix.□

3.3 Solution of 2x2 games without saddle points

Here we consider the 2x2 game, which do not have a saddle point. So in this case the best strategies are the mixed strategies. i.e here we shall determine the probability with which each action should be selected. For this we prove the following important theorem.

Theorem 3.3.1: (cf. [2]). For any zero two persons game where the optimal strategies are not equal. Pure strategies and for which A's payoff matrix is.

The optimal strategies are (x_1, x_2) *and* (y_1, y_2) given by

Table 3.3.1

Player A	***Player B***	
	I (y_1)	***II (y_2)***
(x_1) I	a_{11}	a_{12}
(x_2) II	a_{21}	a_{22}

The optimal strategies are (x_1, x_2) *and* (y_1, y_2) given by

$$\frac{x_1}{x_x} = \frac{a_{22} - a_{21}}{a_{11} - a_{12}} \quad and \quad \frac{y_1}{y_2} = \frac{a_{22} - a_{12}}{a_{11} - a_{21}} \quad \text{and the value of the game to A}$$

$$V = \frac{a_{11}a_{22} - a_{12}a_{21}}{(a_{11} + a_{22}) - (a_{12} + a_{21})}$$

Proof:

If $(x_1 , x_2)\, and\, (y_1 , y_2)$ are the mixed strategies for players A and B respectively,

Then
$$x_1 + x_2 = 1 \qquad \ldots\ldots \qquad (1)$$
$$y_1 + y_2 = 1 \qquad \ldots\ldots \qquad (2)$$

$$x_1 \geq 0 , x_2 \geq 0 , y_1 \geq 0 , y_2 \geq 0$$

Now the expected gain to A is $a_{11}x_1 + a_{21}x_2$ when B uses strategy I and

the expected gain to A is $a_{12}x_1 + a_{22}x_2$ when B uses strategy II.

Similarly the expected loss to B is $a_{11}y_1 + a_{21}y_2$ when A uses strategy I.

Similarly the expected loss to B is $a_{11}y_1 + a_{12}y_2$ when A uses strategy Ii.

If v is the value of the game (since every rectangular game has a solution), then since A expects to get at least V.

$$\left.\begin{aligned} a_{11}x_1 + a_{21}x_2 &\geq v \\ a_{12}x_1 + a_{22}x_2 &\geq v \end{aligned}\right\} \qquad \ldots\ldots \qquad (3)$$

Also v expects to loses at most V (value of the game)

$$\left.\begin{aligned} a_{11}y_1 + a_{12}y_2 &\leq v \\ a_{21}y_1 + a_{22}y_2 &\leq v \end{aligned}\right\} \qquad \ldots\ldots \qquad (4)$$

Now the problem is to find the values of x_1 , x_2 , y_1 , y_2 satisfying (1), (2), (3) and (4)

Regarding (3) and (4) strictly equations, we have

$$a_{11}x_1 + a_{21}x_2 = v \qquad \ldots\ldots \qquad (5)$$

$$a_{12}x_1 + a_{22}x_2 = v \qquad \ldots\ldots \qquad (6)$$

$$a_{11}y_1 + a_{21}y_2 = v \qquad \ldots\ldots \qquad (7)$$

$$a_{21}y_1 + a_{22}y_2 = v \qquad \ldots\ldots \qquad (8)$$

Subtracting (6) from (5), we have

$$(a_{11} - a_{12})x_1 = (a_{22} - a_{21})x_2$$

$$Therfore\ \frac{x_1}{x_2} = \frac{a_{22} - a_{21}}{a_{11} - a_{12}} \qquad \ldots\ldots \qquad (9)$$

Similarly subtracting (8) from (7) we have

$$\frac{y_1}{y_2} = \frac{a_{22} - a_{12}}{a_{11} - a_{21}} \qquad \ldots\ldots \qquad (10)$$

From (9) we have

$$x_1\left[1+\frac{a_{11}-a_{12}}{a_{22}-a_{21}}\right]=1$$

$$x_1=\frac{a_{aa}-a_{21}}{(a_{aa}-a_{22})-(a_{12}+a_{21})} \qquad \ldots\ldots \qquad (11)$$

$$x_2=\frac{a_{11}-a_{12}}{(a_{11}+a_{22})-(a_{12}+a_{21})} \qquad \ldots\ldots \qquad (12)$$

Similarly from (2) and (10), we have

$$y_1=\frac{a_{22}-a_{12}}{(a_{11}+a_{22})-(a_{12}+a_{21})} \qquad \ldots\ldots \qquad (13)$$

$$y_2=\frac{a_{11}-a_{21}}{(a_{11}+a_{22})-(a_{12}+a_{21})} \qquad \ldots\ldots \qquad (14)$$

Substituting these value in any one of the equations (5), (6), (7), or (8), we have

$$v=\frac{a_{11}a_{22}-a_{12}a_{21}}{(a_{11}+a_{aa})-(a_{12}+a_{21})} \qquad \ldots\ldots \qquad (15)$$

The value of x_1 , x_2 , y_1 , y_2 and v satisfy (1), (2), (3) and (4). Since the game has no saddle point, therefore the largest and the second largest elements must lie on one of the diagonals. Therefore there is only the following possible ordering of the elements of the matrix.

$$\begin{array}{l} a_{11} \geq a_{22} \geq a_{12} \geq a_{21} \\ a_{11} \geq a_{22} \geq a_{21} \geq a_{12} \\ a_{22} \geq a_{11} \geq a_{12} \geq a_{21} \\ a_{22} \geq a_{11} \geq a_{21} \geq a_{12} \\ a_{12} \geq a_{21} \geq a_{11} \geq a_{22} \\ a_{12} \geq a_{21} \geq a_{22} \geq a_{11} \\ a_{21} \geq a_{12} \geq a_{11} \geq a_{22} \\ a_{21} \geq a_{12} \geq a_{22} \geq a_{11} \end{array} \qquad \text{---------} \qquad (16)$$

If can be verified that for all the ordering (16) of the elements of the pay of matrix, x_1 , x_2 , y_1 , y_2 given by (11), (12), (13), (14) are all non-negative. Hence these values of x_1 , x_2 , y_1 , y_2 given by (11), (12), (13), and (14) form the solution of the problem and the value of the game is given by (15).□

4. Dominance property

The rules of dominance are used to reduce the size of the payoff matrix. Sometimes in a rectangular game we come across a fact that one or more of the pure strategies of a player are inferior (less attractive) to at least one of the remaining strategies. In such a case this inferior strategy is never used. *In other words we can say that this inferior pure strategy is dominated by a superior pure strategy.* In such cases of dominance, we can reduce the size of the pay off matrix by removing (deleting) the pure strategies, which are dominated by the other strategies.

4.1 Rules (or principles) of Dominance

Rule 1: For player B, who is assumed to be the loser, if each element in a column, say Cr is greater than or equal to the corresponding element in another column, say Cs in the payoff matrix. Then the column Cr is dominated by column Cs, and therefore, column Cr can be deleted from the payoff matrix. In other words, player B will loss more by choosing strategy for Cr column than by choosing strategy for column Cs, therefore, he will never use strategy corresponding to column Cr.

Rule 2: For player A who is assumed to be the gainer, of each element in arrow, say R_r is less than or equal to the corresponding element in another row, say R_s, in the pay of matrix, then the row R_r is dominated by now R_s and therefore, row R_r can be deleted from the payoff matrix. In other words, player A will never use the strategy corresponding to row R_r, because he will gain less for choosing such strategy.

Rule 3: A strategy say K, can also be dominated if it is inferior (less attractive) to an average of two or more other pure strategies. In this case, if the domination is strict, then strategy k can be deleted. If strategy K dominates the convex linear combination some other pure strategies, then one of the pure strategies involved in the combination may be deleted. The domination will be decided as per days 1 and 2 above.

Remark: 1.Rules (principles) of dominance discussed are used when the payoff matrix is a profit matrix for the player A and a less matrix for player B. Other wise the principle gets reversed.

2. By using the dominance properties we always try to reduce the size of the payoff matrix to 2 x2.

Example 4.1.1: Solve the game whose pay off matrix is

Table 4.1.1

		Player B	
Player A	B_1	B_2	B_3
A_1	-1	-2	8
A_2	7	5	-1
A_3	6	0	12

Solution:

Step 1: First we try to find saddle point of the game by the rules described earliest and get the following table.

Table 4.1.2

		Player B		
Player A	B_1	B_2	B_3	*Row minimum*
A_1	-1	-2	8	-2
A_2	7	5	-1	-1
A_3	6	0	12	0
Column max.	7	5	12	

From the above table, we see that the game does not have a saddle point. Therefore, we try to reduce the size of the given payoff matrix by the use of dominance rules.

Step 2: Since every elements of column B_1 is greater than the corresponding elements of column Therefore by dominance rue 1; from player B's point of view the pure strategy B_1 is dominated by the strategy B_2. Hence delete the B_1 strategy so we get the following reduced matrix.

Table 4.1.3

	Player B	
Player A	***B_2***	***B_3***
A_1	-2	8
A_2	5	-1
A_3	0	12

Step 3: Since every element of row A_1 is less than the corresponding elements of row A_3. Therefore by dominance rule 2 from player A's point of view, the pure strategy A_1 is dominated by the strategy A_3. Hence delete the strategy A_1. So we get the following reduced 2x2 matrix.

Table 4.1.4

	Player B		
Player A	**B_2**	**B_3**	*Probability*
A_2	5	-1	x_1
A_2	0	12	x_2
Probability	y_1	y_2	

Now this reduced 2x2 matrix has no saddle point and cannot be reduced further therefore the optimal strategies will be mixed strategies.

If A chooses his A_2 and A_3 strategies with probabilities x_1 *and* x_2 are respectively and B chooses his B_2 and B_3 strategies with probabilities y_1 *and* y_2 and respectively, then by the formula for 2x2 game given in (3:3) we get

For player A $\quad x_1 = \frac{2}{3}, \quad x_2 = \frac{1}{3}$

For player B $\quad y_1 = \frac{13}{18}, \quad y_2 = \frac{5}{18}$

Hence an optimal solution of the given game is:-

Optimal strategy for player A is (0, 2/3, 1/3).

Optimal strategy for player B is (0, 13/18, 5/8).

The expected value of the game is 10/3.

5. SOLUTION METHOD FOR GAMES WITHOUT SADDLE POINT.

A mixed strategy game can be solved by different solution methods such as:

i) Algebraic Method ii) Analytic or Calculus method iii) Matrix Method
iv) Graphical Method v) Linear programming Method

5.1 Algebraic Method for the solution of a general game

This Method known as algebraic method is a direct approach to solve any game. Here we first convert the game into a system of inequalities and then solve them. The method is quite lengthily when there are more strategies. Therefore in case of large games the problem is solved by transforming it into a Linear programming problem.

Let (a_{ij}) be the payoff matrix of a rectangular game between two persons. Suppose $X = (x_1 , x_2 ,\cdots ,x_m) \; and \; Y = (y_1 , y_2 ,\cdots , y_n)$ be the mixed optimal strategies of player A and B respectively. Then A's expected gains when B used his pure strategies $1 , 2 , \cdots, n$. respectively are:

$$\sum_{i=1}^{m} a_{i1}x_i , \sum_{i=1}^{m} a_{i2}x_i ,\ldots, \sum_{i=1}^{m} a_{in}x_i .$$

If V is the value of the game, then the minimum expected gain to A is V. Therefore,

$$\sum_{i=1}^{m} a_{i1}x_i \geq V, \quad \sum_{i=1}^{m} a_{i2}x_i \geq V, \quad \cdots, \sum_{i=1}^{m} a_{in}x_i \geq V \qquad \cdots\cdots \quad (1)$$

Similarly considering B's expected losses and considering the fact that the maximum loss of B is V, we get the following system of inequalities.

$$\sum_{j=1}^{n} a_{1j}y_j \leq V, \quad \sum_{j=1}^{n} a_{2j}y_j \leq V, \cdots, \sum_{j=1}^{n} a_{mj}y_j \leq V \qquad \cdots\cdots \quad (2)$$

$$\left.\begin{array}{l} x_1 + x_2 \; + \; \cdots \; + x_m = 1 \\ y_1 + y_2 \; + \; \cdots \; + y_n = 1 \end{array}\right\} \qquad \cdots\cdots \quad (3)$$

and $y_j \geq 0 , x_i \geq 0 , \quad \forall i = 1, 2, 3,\cdots,m. \; and \; \forall j = 1, 2, 3, \cdots , n.$

Now, the problem is to find the value of $x_1 , x_2 ,\cdots,x_m, y_1, y_2 ,\cdots, y_n$ such that (1), (2) and (3) are satisfied. For this the procedure is as follows

1. Consider all the inequalities (1) and (2) as equalities and, then try to solve them. If we get a solution satisfying (1), (2) and (3), then the given problem is solved completely.

2. In case the system of equations obtained as above are in consistent (i.e does not give a solution), then we conclude that at least one of the inequalities is strict inequality. In that case we try to solve by taking one or more inequalities as strict inequalities and the other as equalities until we get a solution of the problem. Thus we have to really on trial and error method for the solution of such games by algebraic method.

Example 5.1.1. The payoff matrix for A in a two person zero sum game is given below. Determine the value of the game and the optimum strategies for both players.

Table 5.1.1

Player B

Player A	B_1	B_2	B_3	*Probability*
A_1	-1	2	1	x_1
A_2	1	-2	2	x_2
A_3	3	0	-3	x_3
Probability	y_1	y_2	y_3	

SOLUTION:

The game does not have a saddle point and cannot be reduced to 2 x 2 games by the dominance rules. Therefore, we produced to solve this game by algebraic method.

Let (x_1, x_2, x_3) and (y_1, y_2, y_3) be the optimal mixed strategies of the two players A and B respectively and V the value of the game proceeding a usual we set the following relations:

For player A,

$$-1x_1 + .x_2 + 3x_3 \geq v \quad \cdots\cdots \quad (1)$$

$$2.x_1 + -2x_2 + 4x_3 \geq v \quad \cdots\cdots \quad (2)$$

$$1.x_1 + 2x_2 + -3x_3 \geq v \quad \cdots\cdots \quad (3)$$

For player B,

$$-1y_1 + 2y_2 + 1.y_3 \leq v \quad \cdots\cdots \quad (4)$$

$$1y_1 + -2y_2 + 2y_3 \leq v \quad \cdots\cdots \quad (5)$$

$$3y_1 + 4y_2 + -3y_3 \leq v \quad \cdots\cdots \quad (6)$$

$$x_1 + x_2 + x_3 = 1 \quad \cdots\cdots \quad (7)$$
$$y_1 + y_2 + y_3 = 1 \quad \cdots\cdots \quad (8)$$

and
$$\left.\begin{aligned} x_1 \ , x_2 \ , x_3 \ \geq 0 \\ y_1 \ , y_2 \ , y_3 \ \geq 0 \end{aligned}\right\} \quad \cdots\cdots \quad (9)$$

Now the problem is to find the values $x_1, x_2, x_3, y_1, y_2, y_3$ such that all above relations are satisfied. For this first we consider the inequality (1) to (6) as strict equations (10) to (15) as follows:

$$-x_1 + x_2 + 3x_3 = v \quad \cdots\cdots \quad (10)$$
$$2x_1 - 2x_2 + 4x_3 = v \quad \cdots\cdots \quad (11)$$
$$x_1 + 2x_2 + -3x_3 = v \quad \cdots\cdots \quad (12)$$
$$-y_1 + 2y_2 + y_3 = v \quad \cdots\cdots \quad (13)$$
$$y_1 - 2y_2 + 2y_3 = v \quad \cdots\cdots \quad (14)$$
$$3y_1 + 4y_2 - 3y_3 = v \quad \cdots\cdots \quad (15)$$

Adding (10) and (12), we have $3x_2 = 2V \Rightarrow x_2 = \frac{2}{3}V$

Adding two times of (10) and (11) we have $10x_3 = 3V \Rightarrow x_3 = \frac{3}{10}V$

Putting this value in (12), we get $x_1 = \frac{17}{30}V$, substituting these values of x_1, x_2, x_3 in (7)

We have $V = \frac{15}{23}$. *Therefore* $x_1 = \frac{17}{46}$, $x_2 = \frac{10}{23}$, $x_3 = \frac{9}{46}$ which are non negative.

Again adding (13) and (14), we have $3y_3 = 2V \ \therefore \ y_3 = \frac{10}{23}$

Adding three times (13) in (15), we have $10y_2 = 2V \ \therefore \ y_2 = \frac{6}{23}$.

Putting the values of y_2, y_3, *and* V *in* (14) , we have $y_1 = \frac{7}{23}$. These values of y_1, y_2, y_3 also satisfy (8) and also the non negative restriction. Thus the constitute the solution of the given problem. Hence the solution of the game is: Optimal mixed strategy for player A $\left(\frac{17}{46}, \frac{10}{23}, \frac{9}{46}\right)$

Optimal mixed strategy for player B $\left(\frac{7}{23}, \frac{6}{23}, \frac{10}{23}\right)$ and the value of the game $v = \frac{15}{23}$

5.2 Arithmetic Method

The arithmetic method (also is know as short cut method) provides an easy method for finding optimal strategies for each player in a payoff matrix of size 2x2 without saddle point. The steps of this method are as follows:

1. Find the difference between the two values in the first row and put it against second row of the matrix, neglecting negative sign (if any)
2. Find the difference between the two values in the second row and put it against first row of the matrix, neglecting negative sign (if any)
3. Repeat steps 1 and 2 for the columns also.

The values so obtained by "swapping the differences "represent the optimal relative frequencies of play both players' strategies. These may be converted to probabilities by dividing each of them by their sum. The values of the game can be obtained by applying any of the methods discussed earlier.

Remark: The arithmetic method should not be used to solve a 2x 2 game having saddle point because the method yields an incorrect answer.

Example 5.2.1 Two competitors are competing for the market share of the similar product. The payoff matrix in terms of their advertising plan is shown below.

Table 5.2.1

	Competitor B		
Competitor A	*No Advertising*	*Medium Advertising*	*Heavy Advertising*
No. Advertising	10	5	-2
Medium Advertising	13	12	15
Heavy Advertising	16	14	10

Suggest optimal strategies for the two firms and the net outcome therefore.

Solution: -

Applying rules of dominance to deleted first column (dominated by second column) and then first row (dominated by second as well as third row) from the payoff matrix, the second payoff matrix so obtained is shown below:

Table 5.2.2

	Firm B	
Firm A	*Medium Advertising (B_2)*	*Heavy Advertising (B_3)*
Medium Advertising (A_2)	12	15
Heavy Advertising (A_3)	14	10

At the payoff matrix doesn't have saddle point, firms will use mixed strategies. Applying arithmetic method as explained earlier to get optimal mixed strategies for both the firms, the results are:

Table 5.2.3

	Firm B			
Firm A	**B_2**	**B_3**	*Probability*	
A_2	12	15	**P (A_2)**	14-10 =4
A_3	14	10	**P (A_3)**	15 -12 =3
Probability	**P (B_2)**	**P (B_3)**		
	15 -10 =5	14 -12 =2		

$$P(A_2)=\frac{4}{4+3}=\frac{4}{7} \quad ,P(A_3)=\frac{3}{4+3}=\frac{3}{7}$$

$$P(B_2)=\frac{5}{5+2}=\frac{5}{7}, \quad P(B_3)=\frac{2}{5+2}=\frac{2}{7}$$

Hence, Firm A should adopt strategy A_2 and A_3, 57% of the time and 43% of time, respectively (or with 57% and 43% probability on any play of the game respectively).Similarly, Firm B should adopt strategy B2 and B3, 71% of time and 29% of time, respectively (or with 71% and 29% of probability on any one play of the game, respectively).

<u>*Expected Gain to Firm A*</u>

I) 12 x (4/7) + 14x (3/7) = 90/7, Firm B adopt B_2.

II) 15 x 4/7) + 10 x (3/7) = 90/7, Firm B adopt B_3.

<u>*Expected Loss to Firm B*</u>.

I) 12 x (5/7) + 15 x (2/7) = 90/7, Firm A adopt A_2.

II) 14 x (5/7) + 10 x (2/7) = 90/7, Firm A adopt A_3.

5.3 Matrix Method

If the game matrix is in the form of a square matrix, then optimal strategy mix as well as value of the game may be obtained by the matrix method. The solution of a two person zero – sum game with mixed strategies with a square payoff matrix may be obtained by using the following formula.

$$\text{Player } A\text{'s optimal strategy} = \frac{[1 \quad 1]P_{adj}}{[1 \quad 1]P_{adj}\begin{bmatrix}1\\1\end{bmatrix}}$$

$$\text{Player } B\text{'s optimal strategy} = \frac{[1 \quad 1]P_{cof}}{[1 \quad 1]P_{adj}\begin{bmatrix}1\\1\end{bmatrix}}$$

$$\text{Value of the game} = \begin{pmatrix}\text{Player } A\text{'s}\\ \text{optimal strategies}\end{pmatrix}\begin{pmatrix}\text{Payoff}\\ \text{matrix } P_{ij}\end{pmatrix}\begin{pmatrix}\text{player } B\text{'s}\\ \text{optimal strategies}\end{pmatrix}$$

Where $P_{adj} = adjoint\ matrix$

$P_{cof} = cofactor\ matrix$

Player A's optimal strategies are in the form a row vector and B's optimal strategies are in the form of a column vector.

This method can be used for finding a solution of a game with size more than 2 x 2. However, rare cases, the solution violets the non-negative condition of probabilities,

$X_i \geq 0$, $Y_j \geq 0$, $\forall i = 1,2,\cdots,m$ $\forall j = 1,2,\cdots,n$ *although the requirement* $X_1 + X_2 + \cdots + X_m = 1$ *or* $Y_1 + Y_2 + \cdots + Y_n = 1$ *is met.*

Example 5.3.1

Solve the following game after reducing it to a 2x 2 game

Table 5.3.1 ***Player B***

Player A	***B_1***	***B_2***	***B_3***
A_1	1	7	2
A_2	6	2	7
A_3	5	1	6

Solution:

In the given game matrix, third row is dominated by second row and in the reduced matrix third column is dominated by the first column. So after elimination of the third row and the third column the game matrix becomes.

Table 5.3.2 ***Player B***

Player A	***B_1***	***B_2***
A_1	1	7
A_2	6	2

For this reduced matrix, let us calculate P_{adj} *and* P_{Cof} as given below

$$P_{adj} = \begin{bmatrix} 2 & -7 \\ -6 & 1 \end{bmatrix}, \quad P_{cof} = \begin{bmatrix} 2 & -6 \\ -7 & 1 \end{bmatrix}$$

$$\text{Player A's optimal strategies} = \frac{\begin{bmatrix} 1 & 1 \end{bmatrix}\begin{bmatrix} 2 & -7 \\ -6 & 1 \end{bmatrix}}{\begin{bmatrix} 1 & 1 \end{bmatrix}\begin{bmatrix} 2 & -7 \\ -6 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \end{bmatrix}} = \frac{\begin{bmatrix} -4 & -6 \end{bmatrix}}{-10} = \frac{\begin{bmatrix} 4 & 6 \end{bmatrix}}{10}$$

This solution can be broken down into the opt9imal strategy mix for player A as

$X_1 = \frac{4}{10} = \frac{2}{5}$ *and* $X_2 = \frac{6}{10} = \frac{3}{5}$, Where X_1 *and* X_2 represent the probabilities of player A's using his strategies A_1 *and* A_2 *resp.*

Similarly, the optimal strategy mixture for player B is obtained as:

$$\text{Player B's optimal strategies} = \frac{[1 \quad 1]\begin{bmatrix} 2 & -6 \\ -7 & 1 \end{bmatrix}}{[1 \quad 1]\begin{bmatrix} 2 & -7 \\ -6 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \end{bmatrix}} = \frac{[-5 \quad -5]}{-10} = \frac{[5 \quad 5]}{10}$$

This solution can also be broken down into the optimal strategy mixture for player B as

$Y_1 = \frac{5}{10} = \frac{1}{2}, \quad Y_2 = \frac{5}{10} = \frac{1}{2}$ Where Y_1 *and* Y_2 represent the probabilities of player B's using his strategies B_1 *and* B_2 *resp.*

Hence, value of the game $V = \begin{bmatrix} 2 & 3 \\ 5 & 5 \end{bmatrix}\begin{bmatrix} 1 & 7 \\ 6 & 2 \end{bmatrix}\begin{bmatrix} 1/2 \\ 1/2 \end{bmatrix} = 4$

5.4 Graphical Method for the solution of (2xn) and (mx2) games

Graphical method is used to solve 2 x n or m x 2 games. i.e. a game with mixed strategies which has only two pure strategies (un dominated) for one of the players. Since the optimal strategies for both players assigns non-zero probabilities to the same number of pure strategies, therefore if one player has only two pure strategies, the other player will also use to pure strategies only. Graphical method helps us to find which two strategies should be used. Thus the game reduces to 2 x 2 which can be solved by usual method given earlier.

Consider a 2 x n game which has no saddle point whose payoff matrix is as follows:

Table 5.4.1

		B					
		1	2	3	---	n	
A	1	a_{11}	a_{12}	a_{13}	---	a_{1n}	X_1
	2	a_{21}	a_{22}	a_{23}	---	a_{2n}	X_2

If X_1 *and* X_2 are the probabilities with which the player A uses his pure strategies, then

$$X_1 + X_2 = 1, \quad X_1 \geq 0, X_2 \geq 0$$

$$\therefore \quad X_2 = 1 - X_1.$$

The expected payoff to player A for different pure strategies used by player B is tabulated as follows:

Table 5.4.2

Pure strategies used By player B	***E(v) , Expected payoff to player A***
1	$a_{11}X_1 + a_{21}X_2 = a_{11}X_1 + a_{21}(1 - X_1) = (a_{11} - a_{21})X_1 + a_{21}$
2	$a_{12}X_1 + a_{22}X_2 = a_{12}X_1 + a_{22}(1 - X_1) = (a_{12} - a_{22})X_1 + a_{22}$
-	--- --- ---
-	--- --- ---
n	$a_{1n}X_1 + a_{2n}X_2 = a_{1n}X_1 + a_{2n}(1 - X_1) = (a_{1n} - a_{2n})X_1 + a_{2n}$

Thus, it is clear that A's expected payoff varies linearly with X_1. From the minimax criterion for mixed strategies game the player a will select that value of X_1 which will maximize his minimum expected payoff. To find this value we plot the following straight lines as function of X_1.

$$E(v) = (a_{11} - a_{21})X_1 + a_{21}.$$
$$E(v) = (a_{12} - a_{22})X_1 + a_{22}.$$
$$\ldots \quad \ldots \quad \ldots$$
$$\ldots \quad \ldots \quad \ldots$$
$$E(v) = (a_{1n} - a_{2n})X_1 + a_{2n}.$$

5.4.1 Method for plot the above lines

To plot the expected payoff lines, we draw two parallel lines one unit apart and mark a scale on each line.

These two lines represent the two strategies available to A. then we draw lines to present each of B's strategies. To represent B's 1st strategy (see payoff matrix) we join all on scale 1 to a_{21} on scale 2.This line will represent the expected payoff line will represent the expected payoff line $E(v) = (a_{11} - a_{21})x_1 + a_{21}$ with x_1 as x-axis and $E(v)$ as y-axis.

Similarly we may draw other payoff lines. The lower boundary to these lines will give the minimum expected payoff as function x_1. The highest point 'P' on this lower boundary will give the maximum expected payoff to A and hence optimum value of x_1. Then we determine only two strategies for player B corresponding to those two lines which pass through this *maximum point p*. If more than two lines pass through this point, then any two of them having opposite signs for this slops will be alternative optimum solutions. In this way the game is reduced to 2 x 2 games which can be solved easily by using formula of 3.3. We solve m x 2 games in the same manner expect that minimax point p is the lowest point on the upper most boundary.

Example 5.4.1: Solve the game whose payoff matrix is

Table 5.4.1

		Player B			
Player A	**I**	**II**	**III**	**IV**	*Probability*
I	1	3	-3	7	x_1
II	2	5	4	-6	x_2

Solution:

This 2 x 4 game has no saddle point. Therefore, we shall use graphical method to reduce this game to 2 x 2 games.

If x_1 *and* x_2 are the probabilities with which the player A uses his pure strategies, then

$x_1 + x_2 = 1, \quad x_1 \geq 0, x_2 \geq 0$

$x_2 = 1 - x_1$

The expected payoff to player A for different pure strategies used by player B may tabulate as follows:

Table 5.4.2

Pure strategies used by player B	$E(v)$, *A's expected payoff*
I	$E(v) = x_1 + 2x_2 = x_1 + 2(1 - x_1) = -x_1 + 2$
II	$E(v) = 3x_1 + 5x_2 = 3x_1 + 5(1 - x_1) = -2x_1 + 5$
III	$E(v) = -3x_1 + 4x_2 = -3x_1 + 4(1 - x_1) = -7x_1 + 4$
IV	$E(v) = 7x_1 - 6x_2 = 7x_1 - 6(1 - x_1) = 13x_1 - 6$

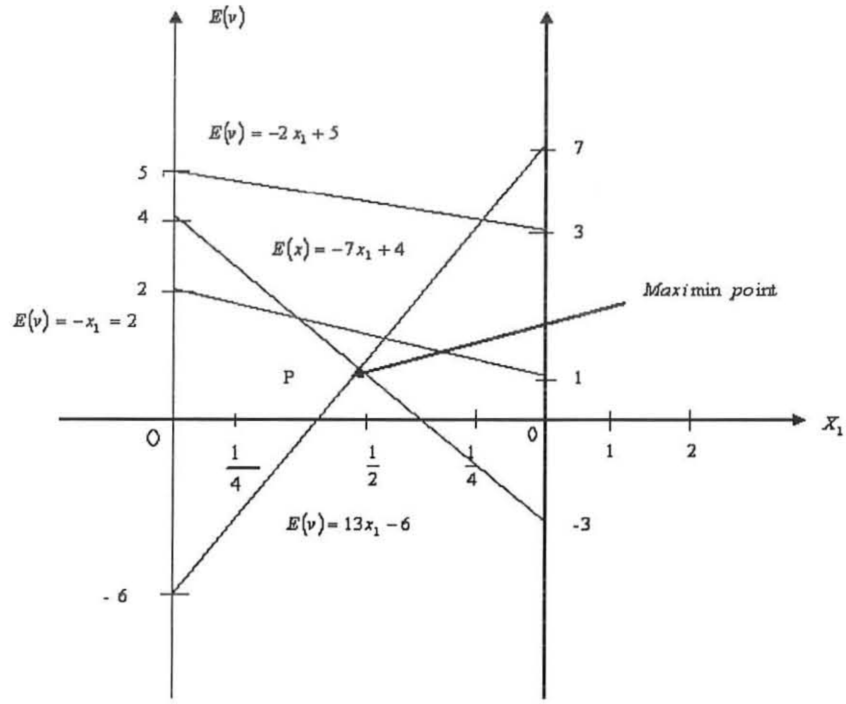
Now we draw the four payoff lines

$E(v) = -x_1 + 2$, $E(v) = -2x_1 + 5$, $E(v) = -7x_1 + 4$, $E(v) = 13x_1 - 6$ *in the graph* To

draw these lines first we draw two parallel lines one unit apart and mark a scale on each as shown in graph 5.4.1 below. These two lines represent two strategies available to A. Now join 1 on scale 1 to 2 on scale 2. This line represents B's 1st strategy.

i.e. the payoff line $E(v) = -x_1 + 2.$

Similarly joining 3 on scale 1 to 5 on scales 2, -3 on scale 1 to on scale 2 and 7 on scale 1 to -6 on scale 2 we get the other payoff lines respectively. Now lowest boundary APB to these lines give the minimum expected payoff A. The highest point p on this lowest boundary will give the maximum expected payoff and hence the expected value of x_1. Thus the best strategies for player B are III and IV pure strategies passing through point p.



Graph5.4.1

Therefore, the game is reduced to 2 x 2 game give by the following payoff matrix.

Table 5.4.4

	Player B		
Player A	***III***	***IV***	**Probability**
I	-3	7	x_1
II	4	-6	x_2
Probability	y_1	y_2	

Let y_1 *and* y_2 are the probabilities with which player B chooses his best strategies III and IV respectively. By using the formulae 3.3, we have:

$$a_{11} = -3 \, , a_{12} = 7 \, , a_{21} = 4 \, , a_{22} = -6$$

$$x_1 = \frac{1}{2}, \quad x_2 = \frac{1}{2}, \quad y_1 = \frac{13}{20}, \quad y_2 = \frac{7}{20} \quad and \quad v = \frac{1}{2}$$

Solution of the game is:

(i) *For player A optimal mixed strategies* $\left(\frac{1}{2}, \frac{1}{2}\right)$.

(ii) *For player B optimal mixed strategies* $\left(0 \, , 0 \, , \frac{13}{20}, \frac{7}{20}\right)$ *and*

(iii) *the value of the game is* $\frac{1}{2}$ *to A and* $-\frac{1}{2}$ *to B.*

5.5 General Method for solution of mixed strategies by Linear Programming Method

Every finite two person zero sum game can be expressed as a linear program on the other hand every linear program can be represented as a game. If there is no saddle point, method of dominance does not work in reducing the game and the method of matrices also fails, then linear programming offers the best method of solution. To illustrate the transformation of a game problem to a linear programming problem, consider a payoff matrix of size m x n. Let a_{ij} be the element in i^{th} *row and* j^{th} *column* of game payoff matrix and let $X = [x_1, x_2, \cdots, x_m]$ *and* $Y = [y_1, y_2, \cdots, y_n]$ be the mixed strategies of the two players A and B respectively by which A and B select their pure strategies.

A selects his optimal mixed strategies which will

$$\underset{x_i}{Max}\left[Mini\left\{ \sum_{i=1}^{m} a_{i1}x_i, \sum_{i=1}^{m} a_{i2}x_i, \cdots, \sum_{i=1}^{m} a_{in}x_i \right\} \right] \quad \text{such that}$$

$x_1 + x_2 + \cdots + x_m = 1$ and $x_i \geq 0 \quad \forall i = 1,2,\cdots,m$

Also B selects his optimal mixed strategies which will

$$\underset{y_j}{Mini}\left[Max\left\{ \sum_{j=1}^{n} a_{1j}y_j, \sum_{j=1}^{n} a_{2j}y_j, ---, \sum_{j=1}^{n} a_{mj}y_j \right\} \right] \quad \text{such that}$$

$y_1 + y_2 + \cdots + y_n = 1$ and $y_j \geq 0 \quad \forall j = 1,2,\cdots,n$

If $Mini\left[\sum_{i=1}^{m} a_{i1}x_i, \sum_{i=1}^{m} a_{i2}x_i, \cdots, \sum_{i=1}^{m} a_{in}x_i \right] = v$, then A expects to gain at least V. The criterion for A to choose his strategy is to maximize his least gain V.

Thus, v is minimum of all expected gains therefore, we have,

$$\sum_{i=1}^{m} a_{i1}x_i \geq v, \quad \sum_{i=1}^{m} a_{i2}x_i \geq v, \cdots, \sum_{i=1}^{m} a_{in}x_i \geq v$$

Thus, A's problem is to determine $x_1 + x_2 + \cdots + x_m$ to minimize Z = v

Subject to the constraints

$$\left.\begin{array}{l} a_{11}x_1 + a_{21}x_2 + \cdots + a_{m1}x_m \geq \quad v \\ a_{12}x_1 + a_{22}x_2 + \cdots + a_{m2}x_m \geq \quad v \\ \cdots \quad \cdots \quad \cdots \quad \cdots \quad \cdots \\ \cdots \quad \cdots \quad \cdots \quad \cdots \quad \cdots \\ a_{1n}\, x_1 + a_{2n}x_2 + --- + a_{mn}x_m \geq v \end{array}\right\} \quad \ldots\ldots \quad (1)$$

and $\qquad x_1 + x_2 + \cdots + x_m = 1 \qquad and \quad x_i \geq 0 \quad \forall i = 1,2,\cdots,m.$

We assume that v is positive. For $v \geq 0$, it is enough that all the elements of the payoff matrix are positive. If all the elements are not positive, then we can add a sufficient large quantity (say k) to every element of the payoff matrix so that they all become positive. By doing so the value of the game is also increased by k, but the solution remains the same. Thus we can take v to be positive. Dividing equations of (1) by v and taking

$$\frac{x_1}{v} = X_1 \quad , \frac{x_2}{v} = X_2 \quad , \cdots, \quad \frac{x_m}{v} = X_m \quad we \quad have$$

$$\begin{array}{l} a_{11}X_1 \quad + \; a_{21}X_2 \quad + \; \cdots \quad + a_{m1}X_m \geq 1 \\ a_{12}X_1 \quad + \; a_{22}X_2 \quad + \; \cdots \quad + a_{m2}X_m \geq 1 \\ \cdots \qquad \cdots \qquad \cdots \qquad \cdots \\ \cdots \qquad \cdots \qquad \cdots \qquad \cdots \\ a_{1n}X_1 \quad + \; a_{2n}X_2 \quad + \; \cdots \quad + a_{mn}X_m \geq 1 \\ and \quad X_1 + \quad X_2 \; + \quad \cdots \quad + X_m = \; \dfrac{1}{v} \\ X_1, X_2, \cdots, X_m \geq \quad 0 \end{array}$$

$$Max\ v = Mini\left(\frac{1}{v}\right)$$

Since, $\qquad = Mini\left\{\dfrac{x_1 + x_2 + \cdots \quad x_m}{v}\right\}$

$$= Mini\ \{X_1 + X_2 + \cdots + X_m\}.$$

Thus, the given rectangular game reduces to the following L.P problem.

$$MiniX^* \ \frac{1}{v} = X_1 + X_2 + \cdots + X_m.$$

Subject to the constraints,

$$\left.\begin{array}{lllllllll} a_{11}X_1 & + & a_{21}X_2 & + & \cdots & + & a_{m1}X_m & \geq & 1 \\ a_{12}X_1 & + & a_{22}X_2 & + & \cdots & + & a_{m2}X_m & \geq & 1 \\ \cdots & + & \cdots & + & \cdots & + & \cdots & & \cdots \\ \cdots & + & \cdots & + & \cdots & + & \cdots & & \cdots \\ a_{1n}X_1 & + & a_{2n}X_2 & + & \cdots & + & a_{mn}X_m & \geq & 1 \\ X_1 \geq 0 & , & X_2 \geq 0, & & \cdots, & & X_m \geq 0 & & \end{array}\right\} \quad \cdots\cdots \quad (2)$$

Now consider the problem from B's point of view who want to minimize V (because the gain of A is the loss of B) . Proceeding in the similar manner we get the following L.P problem

$$Maximize Y^* = \frac{1}{v} = Y_1 + Y_2 + \cdots + Y_n.$$

Subject to the constraints

$$\left.\begin{array}{l} a_{11}Y_1 + a_{12}Y_2 + \cdots + a_{1n}Y_n \leq 1 \\ a_{21}Y_1 + a_{22}Y_2 + \cdots + a_{2n}Y_n \leq 1 \\ \cdots + \cdots + \cdots + \cdots \quad \cdots \\ \cdots + \cdots + \cdots + \cdots \quad \cdots \\ a_{m1}Y_1 + a_{m2}Y_2 + \cdots + a_{mn}Y_n \leq 1 \\ and \quad Y_1 \geq 0, \; Y_2 \geq 0, \; \cdots, \; Y_n \geq 0 \\ Where \quad Y^* = \frac{1}{v}, \quad Y_1 = \frac{y_1}{v_1}, \quad Y_2 = \frac{y_2}{v_2}, \quad \cdots, \quad Y_n = \frac{y_n}{v_n}. \end{array}\right\} \quad \cdots\cdots \quad (3)$$

It can be seen that the L.P problems given in (2) and (3), are the duals of each other. (i.e. B's problem is the dual of the A's problem and vice versa). Therefore if one problem is solved then the other is solved automatically. After getting the values of X_i , Y_j and Mini of $\sum X_i$ which is equal to Max of $\sum Y_j$, we can find the values of x_i and y_j from $x_i = vX_i$ *and* $y_j = vY_j$.

Theorem 5.5.1: (cf. [2]). Every game can be solved interns of mixed strategies.

i.e. if mixed strategies are adopted there always exists a value of the game. i.e. $\underline{v} = v = \bar{v}$ *Where* $\underline{v}$ *and* $\bar{v}$ are Maximin and minimax values of V.

Proof:

It has been shown in 5.5 that if $X = [x_1, x_2, \cdots, x_m]$ *and* $Y = [y_1, y_2, \cdots, y_n]$ are the mixed strategies of the two players where $x_1 + x_2 + \cdots + x_m$; $y_1 + y_2 + \cdots + y_n$ are probabilities with which they choose r pure strategies, then player A's problem is

$$To \quad Minimize \quad X^* = X_1 + X_2 + \cdots + X_m.$$

$s.t$

$$\begin{array}{ccccccccc} a_{11}x_1 & + & a_{21}x_2 & + & \cdots & + & a_{m1}x_m & \geq 1 \\ a_{12}x_1 & + & a_{22}x_2 & + & \cdots & + & a_{m2}x_m & \geq 1 \\ \cdots & + & \cdots & + & \cdots & + & \cdots & \\ \cdots & + & \cdots & + & \cdots & + & \cdots & \\ a_{1n}x_1 & + & a_{2n}x_2 & + & \cdots & + & a_{mn}x_m & \geq 1 \end{array}$$

and $X_1, X_2, \cdots, X_m \geq 0$

Where $X_i = \dfrac{x_i}{v}, \quad \forall i = 1, 2, \cdots m.$

And player B's problem is:

$$To\ Maximize \quad Y^* = Y_1 + Y_2 + \cdots + Y_n \ .$$

$s.t$

$$\begin{array}{cccccccccc} a_{11}Y_1 & + & a_{12}Y_2 & + & \cdots & + & a_{1n}Y_n & \leq & 1 \\ a_{21}Y_1 & + & a_{22}Y_2 & + & \cdots & + & a_{2n}Y_n & \leq & 1 \\ \cdots & + & \cdots & + & \cdots & + & \cdots & & \cdots \\ \cdots & + & \cdots & + & \cdots & + & \cdots & & \cdots \\ a_{m1}Y_1 & + & a_{m2}Y_2 & + & \cdots & + & a_{mn}Y_n & \leq & 1 \end{array}$$

and $Y_1 + Y_2 + \cdots + Y_n \geq 0$

Where $Y_j = \dfrac{y_j}{v}, \quad \forall_j = 1, 2, \cdots, n.$

It is clear that B's problem is the dual of A's problem. But from duality thermo, we know that "If either the primal or the dual problem has a finite optimal solution, then the other problem has a finite optimal solution and the optimal values of the two objective functions are equal.

i.e. Max Y* = Mini X* or $\underline{v} = \bar{v} = v$ where v is the value of the game.□

Remark: Linear programming technique requires all variables to be non negative and therefore to obtain a non-negative value V of the game, the data to the problem. i.e. a_{ij} in the payoff table should all be non negative. If there are some negative elements in the payoff table, a constant to every element in the payoff table must be added so as to make the smallest element zero, the solution to this new game will give an optimal mixed strategy for the original game. The value of the original game then equals the value of the new game minus the constant.

5.5.1 Solving Linear programming problem by simplex method.

Computational procedure of the simplex method for the solution of a maximizations linear programming problem.

Step 1. If the problem is of minimization, convert it into the maximization problem.

Step 2. Make all bi's positive.

Step 3. convert the constraints into equations by introducing the non negative slack or surplus variables.

Also introduce artificial variables in the constraints where surplus variables are Inserted and which do not form the column of unit matrix.

Step 4. Find initial B.F.S.

Step 5. Construct starting simplex table.

Now we construct simplex table as on table 5.5.1 on the next page

Here the columns corresponding to the coefficients of $x_1, x_2, \cdots, x_n$ are shown by $y_1, y_2, \cdots, y_n$ and c_j is the row of coefficients of the variables in the objective function.

Where z denote the profit, then z = 0 corresponding to this basic feasible solution, C_B denote the coefficient of the basic variables in the objective function and X_B the numerical values of the basic variables.

Step 6. Test of starting B.F.S for optimality. This is done by computing an evaluation Δ_j

For each non – basic variable (zero variable) x_j by the formula $\Delta_j = C_j - C_B Y_j$.

Note that Δ_j for basic variables are zero.

Insert the value of Δ_j's for zero variables as a row in the starting simplex table as

Show in table 5.5.1.

i) if $\Delta_j \leq 0$, for each j, the solution under test is optimal.

a. If none of Δ_j is positive but any are zero, then other optimal solution exist with the same value of z.

b. If all Δ_j are negative, the solution under test is unique optimal solution.

ii) If $\Delta_j > 0$ for any j, i.e if any or more of Δ_j are positive, the solution under is not optimal.

c. If corresponding to maximum positive Δ_j, all the elements in the column y_j are negative or zero, then the solution under test will be unbounded.

d. If the value at least one artificial variable appearing in the basis is non zero and the optimality condition is satisfied, then we shall say that the problem has no feasible solution.

Step 7. Find in coming (or entering) and out going vectors. To improve the above Solution (which is not optimal) we find the vector entering the basis matrix (Called in coming vector) and the vector to be removed from the basis Matrix (called outgoing vector) by the following rules.

i) To find in coming vector. The incoming vector will be taken as α_k if $\Delta_k = \max \Delta_j$. If maximum value of Δ_j at more than once α_j, then any one of these may be taken as an incoming vector.

ii) To find out going vector. The out going vector β_r is taken corresponding to that value of v for which

$\frac{x_{Br}}{y_{rk}} = \overset{Mini}{i} \left\{ \frac{x_{Bi}}{y_{ik}}, y_{ik} > 0 \right\}$, when α_k is the in-coming vector. If minimum is not unique, i.e. the minimum occurs at more than one value of i, then more than one variable will vanish in the next solution. Therefore, the next solution will become a degenerate B.F.S. for which the out going vector is called in a different way.

Step 8. When α_k is the incoming vector and β_r the outgoing vector then the element $y_{rk} (= a_{rk})$ is called *the key element or Pivot element* which is at the intersection of minimum ratio arrow $(\leftarrow)$ and incoming vector arrow $(\uparrow)$. We mark this element by□.

In order to bring β_r in place of $\alpha_k(y_k)$ there should be equal to unity at the position □. (i.e. the key element $y_k(=\alpha_k)$ should be equal to unity(=1). If it is not 1 , then divide all the element of this row by this key element a_{rk}. Then subtracts appropriate multipliers of this row (containing key element) from all the other rows and obtain zero(=0) at all other positions of this column $\alpha_k(=y_k)$. Now bring β_r in place of $\alpha_k(=y_k)$ and construct new (revised) simplex table.

Step 9. Now test the above improved B.F.S for optimality as in step 6. I In this solution is not optimal, then repeat step (7) and (8) until an optimum solution is finally obtained.

		C_j	c_1	c_2	----	c_r	----	c_n	c_{n+1}	c_{n+2}	----	c_{n+m}
B	C_B	X_B	$y_1(=\alpha_1)$	$y_2(=\alpha_2)$	----	$y_k(=\alpha_k)$	----	$y_n(=\alpha_n)$	$y_{n+1}(=\beta_1)$	$y_n(=\beta_1)$	----	$y_{n+m}(=\beta_m$
y_{n+1}	$C_{B1}=0$	$x_{B1}=b_1$	$y_{11}=a_{11}$	$y_{12}=a_{12}$	----	$y_{1k}=a_{1k}$	----	$y_{1n}=a_{1n}$	1	0	----	0
y_{n+2}	$C_{B2}=0$	$x_{B2}=b_2$	$y_{21}=a_{21}$	$y_{22}=a_{22}$	----	$y_{2k}=a_{2k}$	----	$y_{2n}=a_{2n}$	0	1	----	0
----	----	----	----	----	----	----	----	----				
y_{n+r}	$C_{Br}=0$	$x_{Br}=b_r$	$y_{r1}=a_{r1}$	$y_{r2}=a_{r2}$	----	$y_{rk}=a_{rk}$	----	$y_{rm}=a_{rm}$	-	0	----	-
----	----	----	----	----	----	----	----	----	-	-	----	0
y_{n+m}	$C_{Bm}=0$	$x_{Bm}=b_m$	$y_{m1}=a_{m1}$	$y_{m2}=a_{m2}$	----	$y_{mk}=a_{mk}$	----	$y_{mn}=a_{mn}$	0	0	----	1
$Z=C_B X_B=0$		Δ_j	Δ_1	Δ_2	---	Δ_k	---	Δ_n	Δ_{n+1}	Δ_{n+2}	----	Δ_{n+m}

Starting Simplex table 5.5.1

Example 5.5.1: Find the best strategies and the value of the following game.

Table 5.5.1 — ***Player B***

		B_1	B_2	B_3
Player A	A_1	1	-1	3
	A_2	3	5	-3
	A_3	6	2	-2

Solution:

First we construct the table for row minimums and column maximums as follows:

Table 5.5.2

Player B

		B_1	B_2	B_3	*Row min*
	A_1	1	-1	3	-1
Player A	A_2	3	5	-3	-3
	A_3	6	2	-2	-2
Column Max.		6	5	3	

From the above table $Maxi\min Value\ (\underline{v}) = -1$ *and* $Mini\max Value\ (\overline{v}) = 3$ so it is clear that the value of the game (v) lies between -1 and 3. i.e. $-1 \le v \le 3$.

$\therefore$ It is possible that the value of the game may be negative or zero. Thus we add a constant k=4 to all the elements of the matrix to all the elements of the matrix so that all these elements of the matrix become positive assuming that the value of the game represented by this new matrix is non-negative and non-zero. The transformed (reduced) matrix is as follows:

Table 5.5.3

		Player B			
		B_1	**B_2**	**B_3**	**Probability**
	A_1	5	3	7	x_1
Player A	**A_2**	7	9	1	x_2
	A_3	10	6	2	x_3
Probability		y_1	y_2	y_3	

Let $X = (x_1, x_2, x_2)$ *and* $Y = (y_1, y_2, y_3)$ be the mixed strategies of the two players respectively, *where* $x_1, x_2, x_3, y_1, y_2, y_3$ are the probabilities with which they choose their pure strategies.

Therefore B's problem :

To find (y_1, y_2, y_3) *which* min*imize* v'

Subject to:

$$5y_1 + 3y_2 + 7y_3 \le v' \quad (\textit{if A uses strategy } A_1)$$
$$7y_1 + 9y_2 + y_3 \le v' \quad (\textit{if A uses strategy } A_2)$$
$$10y_1 + 6y_2 + 2y_2 \le v' \quad (\textit{if A uses strategy } A_3)$$
$$y_1 + y_2 + y_3 = 1$$

and $\quad y_1, y_2, y_3 \ge 0$

Here $v' > 0$, *dividing above equations by* v' *and putting* $Y_1 = \frac{y_1}{v'}, Y_2 = \frac{y_2}{v'}, Y_3 = \frac{y_3}{v'}$ *the B's problem reduces to the following linear program* min *g problem*

$$Max.\,Z = \frac{1}{v'} = \frac{y_1 + y_2 + y_3}{v'} = Y_1 + Y_2 + Y_3.$$

s.t.

$$5Y_1 + 3Y_2 + 7Y_3 \le 1$$
$$7Y_1 + 9Y_2 + Y_3 \le 1$$
$$10Y_1 + 6Y_2 + 2Y_3 \le 1$$

and $\quad Y_1, Y_2, Y_3 \ge 0.$

It maybe noted that problem of player A is the dual of the problem of player B. Therefore, solution of the dual problem can be obtained from the optimal simplex table of primal. To solve the problem of player B, introduce slack variables to convert the three inequalities to equalities.

The problems becomes

Proceeding with usual simplex method, the optimal solution is shown in the table below.
Simplex Table
Table 5.5.4

		C_j	1	1	1	0	0	0	Mini Ratio
B	C_B	X_B	Y_1	Y_2	Y_3	S_1	S_2	S_3	
S_1	0	1	5	3	7	1	0	0	1/3
S_2	0	1	7	9	1	0	1	0	1/9
S_3	0	1	10	6	2	0	0	0	1/6
$Z=\sum CB_i XB_i$		z_j	0	0	0	0	0	0	
$=0$		Δ_j	1	1 ↑ incoming vector	1	0	0 ↓ out going vector	0	
S_1	0	2/3	8/3	0	20/3	1	-1/3	0	1/10
Y_2	1	1/9	7/9	1	1/9	0	1/9	0	1/9
S_3	0	1/3	16/3	0	4/3	0	-2/3	1	1/4
$Z=\sum CB_i XB_i$		Z_j	7/9	1	1/9	0	1/9	0	
$=1/9$		Δ_j	2/9	0	8/9 ↑	0 ↓	-1/9	0	
Y_3	1	1/10	2/5	0	1	3/20	-1/20	0	
Y_2	1	1/10	11/15	1	0	-1/60	7/60	0	
S_3	0	1/5	24/5	0	0	-1/5	-3/5	1	
$Z=\sum CB_i XB_i$		Z_j	17/15	1	1	2/15	1/15	0	
$=1/5$		Δ_j	-2/15	0	0	-2/15	-1/15	0	

Since all the Δ_j's for zero variables are negative so this solution is optimal.

$\therefore$ The optimal solution (Mixed strategies) for B is

$Y_1 = 0 \quad , Y_2 = \frac{1}{10} \quad , Y_3 = \frac{1}{10}$ *and* $Max\ Z = \frac{1}{5}$ *, From* $Z = \frac{1}{v'}$ *we have Min* $v' = 5$.

Therfore $y_1 = Y_1 v' = 0\,(5) = 0$

$$y_2 = Y_2 v' = \frac{1}{10}(5) = \frac{1}{2}$$

$y_3 = Y_3 v' = \frac{1}{10}(5) = \frac{1}{2}$ *and the value of the original game* v *is*

$$v = v' - 4 = 1$$

Since A's strategies are the dual of the above problem. Therefore $X_1 = \frac{x_1}{v'} = \frac{2}{15}$,

$X_2 = \frac{x_2}{v'} = \frac{1}{15}$, $X_3 = \frac{x_3}{v'} = 0$ (*since values of* Δ_4 , Δ_5 , Δ_6 *with sign changed.*)

$x_1 = \frac{2}{3}$, $x_2 = \frac{1}{3}$, $x_3 = 0.$

Hence the optimal solution is :

(*i*) *Best strategy for* $A\left(\frac{2}{3}, \frac{1}{3}, 0\right)$.

(*ii*) *Best strategy for* $B\left(0, \frac{1}{2}, \frac{1}{2}\right)$

(*iii*) *Value of the game* $v = 1$

Example 5.5.2: Solve 3 x 3 game by the simplex method of linear programming whose payoff matrix is given below

Table 5.5.5 **Player B**

Player A	B_1	B_2	B_3
A_1	3	-1	-3
A_2	-3	3	-1
A_3	-4	-3	3

Solution: First apply minimax (Maximin) criteri8on to find minimax $(\bar{v})$ *and* $(\underline{v})$ value of the game. Thus, the following matrix is obtained

Table 5.5.6 ***Player B***

Player A	B_1	B_2	B_3	*Row Min.*
A_1	3	-1	-3	-3
A_2	-3	3	-1	-3
A_3	-4	-3	3	-4
Column Max.	3	3	3	

From the Table 5.5.6 Maximin value = -3, minimax value = 3

Since, Maximin value is -3, it is possible that the value of the game (v) may be negative or zero because $-3 < v < 3$. thus, a constant k is added to all elements of the matrix which is at least equal to the negative of the Maximin value, i.e. $k \geq 3$, let $k = 5$. The matrix shown in table 5.5.6

Table 5.5.7 **_Player B_**

Player A	B_1	B_2	B_3	*Probability*
A_1	8	4	2	x_1
A_2	2	8	4	x_2
A_3	1	2	8	x_3
Probability	y_1	y_2	y_3	

Let $X = (x_1, x_2, x_2)$ *and* $Y = (y_1, y_2, y_3)$ be the mixed strategies of the two players respectively, *where* $x_1, x_2, x_3, y_1, y_2, y_3$ are the probabilities with which they choose their pure strategies.

Therefore B's problem :

To find (y_1, y_2, y_3) *which* min *imize* V'

$$
\begin{aligned}
\textit{Subject to}: \quad & 8y_1 + 4y_2 + 2y_3 \leq V' \quad (\textit{if A uses strategy } A_1) \\
& 2y_1 + 8y_2 + 4y_3 \leq V' \quad (\textit{if A uses strategy } A_2) \\
& y_1 + 2y_2 + 8y_2 \leq V' \quad (\textit{if A uses strategy } A_3) \\
& y_1 + y_2 + y_3 = 1 \\
\textit{and} \quad & y_1, y_2, y_3 \geq 0
\end{aligned}
$$

Here $V' > 0$, *dividing above equations by* V' *and putting* $Y_1 = \frac{y_1}{v'}, Y_2 = \frac{y_2}{v'}, Y_3 = \frac{y_3}{v'}$ *the B's problem reduces to the following linear program* min g *problem*

$$
\begin{aligned}
Max.Z &= \frac{1}{v'} = \frac{y_1 + y_2 + y_3}{v'} = Y_1 + Y_2 + Y_3. \\
s.t. \quad & 8Y_1 + 4Y_2 + 2Y_3 \leq 1 \\
& 2Y_1 + 8Y_2 + 4Y_3 \leq 1 \\
& Y_1 + 2Y_2 + 8Y_3 \leq 1 \\
\textit{and} \quad & Y_1, Y_2, Y_3 \geq 0.
\end{aligned}
$$

It may be noted that problem of player A is the dual of the problem of player B. Therefore, solution of the dual problem can be obtained from the optimal simplex table of primal. To solve the problem of player B, introduce slack variables to convert the three inequalities to equalities. The problems becomes

Maximize $Z_p = Y_1 + Y_2 + Y_3 + 0S_1 + 0S_2 + S_3$

Subject to the constrains :

$$8Y_1 + 4Y_2 + 2Y_3 + S_1 = 1$$
$$2Y_1 + 8Y_2 + 4Y_3 + S_2 = 1$$
$$Y_1 + 2Y_2 + 8Y_3 + S_3 = 1$$

and $Y_1, Y_2, Y_3, S_1, S_2, S_3 \geq 0$

Taking $Y_1 = 0, Y_2 = 0, Y_3 = 0$, *and We have* $S_1 = 1,\ S_2 = 1, S_3 = 1$ *which is the starting Basic Feasible Solution.*

Proceeding with usual simplex method, the optimal solution is shown in the table 5.5.7 below.

Table 5.5.8 **Simplex table**

		C_j	1	1	1	0	0	0	Mini Ratio
B	C_B	X_B	Y_1	Y_2	Y_3	S_1	S_2	S_3	
S_1	0	1	8	4	2	1	0	0	1/8
S_2	0	1	2	8	4	0	1	0	1/2
S_3	0	1	1	2	8	0	0	0	1
$Z=\sum CB_i XB_i$		z_j	0	0	0	0	0	0	
$=0$		Δ_j	1 ↑ incoming vector	1	1	0 ↓ out going vector	0	0	
Y_1	1	1/8	1	1/2	1/4	1/8	0	0	1/2
S_2	0	3/4	0	7	7/2	-1/4	1	0	1/9
S_3	0	7/8	0	3/2	31/4	-1/8	0	0	1/4
$Z=\sum CB_i XB_i$		Z_j	1	1/2	1/4	1/8	0	0	
$=1/8$		Δ_j	0	1/2	3/4 ↑	-1/8 ↓	0	0	
Y_1	1	3/31	1	14/31	0	4/31	0	-1/31	3/14
S_2	0	11/31	0	-196/31	0	-6/31	1	-14/31	11/196
Y_3	1	7/62	0	6/31	1	-1/62	0	4/31	7/12
$Z=\sum CB_i XB_i$ $=13/62$		Δ_j	0	11/31 ↑	0	-7/62	0 ↓	-3/31	
Y_1	1	1/14	1	0	0	1/7	1/14	0	
Y_2	1	11/196	0	1	0	-3/98	31/196	-1/14	
Y_3	1	5/49	0	0	1	-1/98	-3/98	1/7	
$Z=\sum CB_i XB_j$ $=45/196$		Δ_j	0	0	0	-5/49	-11/96	-1/14	

Since all the Δ_j's for zero variables are negative so this solution is optimal.

$\therefore$ The optimal solution (Mixed strategies) for B is

$$Y_1=\frac{1}{4}\ ,Y_2=\frac{11}{196}\ ,Y_3=\frac{5}{49}\quad and\quad Max\ Z=\frac{45}{196}\ ,From\ Z=\frac{1}{v'}\ we\ have\ Min\ v'=\frac{196}{45}.$$

Therfore $y_1 = Y_1 v' = \frac{1}{14}\left(\frac{196}{45}\right) = \frac{14}{45}$

$y_2 = Y_2 v' = \frac{11}{196}\left(\frac{196}{45}\right) = \frac{11}{45}$

$y_3 = Y_3 v' = \frac{5}{49}\left(\frac{196}{45}\right) = \frac{20}{45}$ *and the value of the original game v is*

$v = v' - 5 = \frac{196}{45} - 5 = -\frac{29}{45}$

Since A's strategies are the dual of the above problem. Therefore $X_1 = \frac{x_1}{v'} = \frac{5}{49}$,

$X_2 = \frac{x_2}{v'} = \frac{11}{196}$, $X_3 = \frac{x_3}{v'} = \frac{1}{14}$ (since *values of* $\Delta_4, \Delta_5, \Delta_6$ *with sign changed.*)

$x_1 = \frac{5}{49}\left(\frac{196}{45}\right) = \frac{20}{45}$, $x_2 = \frac{11}{196}\left(\frac{196}{45}\right) = \frac{11}{45}$, $x_3 = \frac{1}{14}\left(\frac{196}{45}\right). = \frac{14}{45}$

Hence the optimal solution is :

(i) *Best strategy for A is* $\left(\frac{20}{45}, \frac{11}{45}, \frac{14}{45}\right)$.

(ii) *Best strategy for B* $\left(\frac{14}{45}, \frac{11}{45}, \frac{20}{45}\right)$

(iii) *Value of the game* $v = \frac{29}{45}$

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