


# ELECTRON-POSITRON PAIR PRODUCTION IN THE FIELD OF SUPERSTRONG OPPOSITELY DIRECTED LASER BEAMS

By

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# Table of Contents

**Table of Contents** **iv**

**List of Figures** **v**

**Abstract** **vi**

**Acknowledgments** **vii**

**1 Introduction** **1**

**2 Time Dependent Perturbation Theory With Application to Pair Production Process** **5**
2.1 Time Dependent Perturbation Theory . . . . . 5
2.2 Electron-Positron Pair Creation . . . . . 8

**3 Nonlinear Solution of Dirac Equation for Vacuum Electrons in the Strong Electromagnetic Radiation Field** **19**

**4 Multiphoton Probabilities of Nonlinear Production of Electron-Positron Pair** **32**
4.1 The Probability of the n-Photon $e^-$,$e^+$ Pair Production . . . . . 32
4.2 The Angular Distribution of a n-Photon Differential Probability of the $e^-$,$e^+$ Pair Production . . . . . 35
4.3 The Total Probability Density of the $e^-, e^+$ Pair Production in the Field of Strong Opposite E.M Waves in Vacuum . . . . . 38
4.4 Numerical Data Analysis Approximation for Number of Pair Production in Vacuum . . . . . 39

**5 Conclusion** **42**

# List of Figures

2.1 Feynman diagrams for pair creation: (A) Direct and (B) Exchange process. 8

2.2 The graph of differential cross-section as a function of the scattering angle $\theta$ for photon energies $\omega$. . . . . 18

4.1 [9] The angular dependence of the partial differential probability of n-photon electron-positron production per second, per $cm^3$ space volume with laser frequency 1.17eV and intensity $1.35 \times 10^{22} w/m^2$, at which the most probable energy of created particles is $2^{1/2}m$. . . . . 36

# Abstract

This thesis presents the multiphoton electron-positron pair production through nonlinear channels by superintense oppositely directed laser pulses of the same frequencies in vacuum. On the basis of the Dirac model the resonance approximation for vacuum induced transitions is developed. The analytic formulas for energetic and angular distributions and total number of electron-positron pair in the limit of a short interaction time are obtained. The obtained approximate nonlinear solution of the Dirac equation is applicable in the field when the energy of interaction with the Dirac vacuum is comparable to the electron rest mass. Using intense electromagnetic field of a laser beam we can obtain one $e^-, e^+$ pair. Passing to higher harmonics the number of photons decreases while it produced one $e^-, e^+$ of pair. However, for the number of $e^-, e^+$ pair to be create the applicability value of incident radiation (q) will be in the range of 1.4-1.43 and laser frequency $(\omega, 3\omega, 5\omega)$ for this process.

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# Chapter 1

# Introduction

The production of electron-positron pair from light fields is one of the most intriguing phenomena that is predicted by quantum electrodynamics. This is a principally a non-linear process that can be observable in superintense laser fields through multiphoton channels. The appearance of super power ultrashort laser pulses exceeding the intensities $10^{18}\omega/cm^2$ in optical region or $10^{16}\omega/cm^2$ in near-infrared region, when the energy of the interaction of electron with the field over a wavelength exceeds the electron rest energy, opens actual possibilities to observe the $e^-, e^+$ pair production from these fields. Generally, the interaction of such fields with the vacuum makes available the revelation of many nonlinear quantum-electrodynamic phenomena. For, a description of those phenomena the Feynman diagrammatic techniques of QED ($eA/\hbar\omega << 1$ ; e is the electric charge, $\hbar$ is the plank constant, A and $\omega$ are the amplitude of the vector potential and the wave frequency, respectively) is certainly not applicable. Besides, as it is known, the conservation laws for the $e^-, e^+$ pair production process in the field of a plane monochromatic wave (which laser radiation resembles, very closely) can not be satisfied in vacuum without a third body. A third body can serve a $\gamma$ quanta and the first experimental observation of electron-positron pair from intense light field and $\gamma$ quanta [1] took place at the Standford Linear Accelerator Center (**SLAC**)[2]. As a third body may also serve as an ion/nucleus the papers [3,4] are devoted to the experimental observation of electron-positron pair production from intense light fields in the plasma. On the other hands the conservation

laws for the pair production in the field of a plane monochromatic wave can be satisfied in a plasmalike medium where an electromagnetic (e.m) wave propagates with a phase velocity larger than the speed of light in vacuum. In this case

$$\omega^2/c^2 - k^2 > 0 \tag{1.0.1}$$

(c is the light speed in vacuum, $k$ is the wave vector), which means that we have a "photon with nonzero rest mass" providing the creation of the particles with the rest masses. The process of the electron-positron pair production by the monochromatic photon fields in the plasmalike media with the macroscopic dispersing properties, in the wide region of fully degenerated plasma, as well as in plasmas with the sharp nonstationary properties (particularly, in a laser produced plasma) by various approximations have been investigated [5]. Specially, for the solution of non-linear laser-vacuum interaction problem an approach based on the Dirac model of vacuum in the given strong e.m classical field has been evolved [5]. To avoid the negative impact of a plasma's particles on the created electron and positrons the vacuum versions of the pair production by superintense laser pulses in the presence of a third body are of great interest. The satisfaction of conservation laws for the pair $e^-, e^+$ production process in the e.m field is equivalent to the satisfaction of the condition.

$$E^2 - H^2 > 0 \tag{1.0.2}$$

where **E**, **H** are the electric and magnetic strengths of the field. The latter is obvious in the frame of reference where there is only electric field that provides the creation of pair (in the opposite case we would have only a magnetic field that cannot produce a pair). The condition (1.0.2) can be satisfied in the stationary maxima of a standing wave being formed by two opposite waves (laser beams). In the present work the multiphoton process of pair production in vacuum in the field of opposite laser beams of the same frequencies is investigated. The nonlinear solution of Dirac equation has been found for such laser intensities ($\xi = eA/mc^2 \sim 1$) when the pairs are essentially produced at the lengths

$l << \lambda$, where $\lambda$ is the wavelength of laser radiation. On the other hand, these lengths are much more than the characteristic quantum size of a e: electromagnetic vacuum: $l >> \lambda_c($ $\lambda_c = \hbar/mc$ is the Compton wavelength of an electron; for the production of $e^-, e^+$ pair at the Compton wavelengths $\lambda_c$ e.m fields with electric strength $E_c \sim 10^{16}$ v/cm are required).

Note that the $e^-, e^+$ pair production process in a homogeneous periodic electric field has been studied by various methods. However, the multiphoton probabilities were obtained by using the perturbation theory for the very weak fields, when $eA/\hbar\omega << 1$ (for the special case $\theta = 90$, as well [6]). The contribution of the wave intensity in the law of dispersion becomes essential when $\xi \geqslant 1$ since even for the most powerful laser fields existing, $\xi \leqslant 1$, the dispersion law $n(\omega, \xi^2) < 1$ of a plasma can be regarded as linear, but the pair-production process in such fields will have an essential nonlinear behavior. The multiphoton degree n is defined by the condition (reaction threshold).

$$N\hbar\omega \geq 2mc^2/[1 - n^2(\omega)]^{1/2} \tag{1.0.3}$$

This thesis presents the electron-positron pair-production process through nonlinear channels by a strong laser field in a plasma. The problem is solved with the help of the Dirac model, according to which all negative-energy states are filled with electrons and the interaction of the wave occurs with that vacuum.

The main contribution of the author in the thesis is a numerical data analysis for the $e^-, e^+$ pair production to be created. Simulations have been used for the pairs to be created where we take the values of incident energy (E), laser frequency ($\omega$) and invariant parameter ($\xi$) for this process.

Relativistic units with ($\hbar = c = 1$) are used through the thesis. The electron mass is m=0.511 Mev. For production of an electron-positron pair, the center-of-mass energy of

the scattering photons must be at least $2mc^2 = 1.02$ MeV. The metric tensor used is:

$$g^{\mu\nu} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

where the Dirac $\gamma$ matrices satisfy the anticommutation relations$\{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu}$ .

The thesis is organized as follows: Chapter 2 deals with the time-dependent perturbation theory with application to electron-positron pair creation process. The cross section is calculated for this process. Chapter 3 deals with a nonlinear solution of the Dirac equation for vacuum electrons in the field of strong opposite e.m waves of the same frequencies (standing wave) in the resonance approximation, chapter 4 deals with the calculation of the multiphoton probabilities of $e^-, e^+$ pair production in the above mentioned approximation and present the energetic and angular distributions and total number of pair created by the fundamental and higher harmonics of a conventional laser with respect to changes in the electric field strength and the energy spectrum of the created electrons (positrons) in the limit of short interaction time. All numerical results have been produced using Mathematica for different values of photon energy (1.17eV, $3\omega$, $5\omega$...) at the intensity of $1.35\times10^{22}W/m^2$. Finally, chapter 5 is devoted to conclusion.

# Chapter 2

# Time Dependent Perturbation Theory With Application to Pair Production Process

Time dependent perturbation technique for solving the time-dependent *Schrödinger* equation is a powerful general approximation technique and used to find an approximate solution of *Schrödinger* time dependent equation. We review first the time ordered perturbation expansion of the S-matrix from which we obtain covariant expressions for transition amplitudes of various QED processes.

## 2.1 Time Dependent Perturbation Theory

The time-independent perturbation theory is very successful when the strength of the perturbation (as in case of QED where $\alpha = \frac{e^2}{4\pi\hbar c} \simeq \frac{1}{137} << 1$) is small. Even when the perturbation is time-independent, it is useful to study the time-dependence of the system, for example in scattering.

We recall that the state ket in interaction picture can be expressed in terms of the state ket in *Schrödinger* picture as,

$$\mid \alpha, t_0; t >_I = e^{iH_0 t/\hbar} \mid \alpha, t_0; t >_s \tag{2.1.1}$$

where the operators $\theta_I$ can be expressed in terms of corresponds operators in *Schrödinger* picture $\theta_s$ as,

$$\theta_I = e^{iH_0t}\theta_s e^{-iH_0t} \tag{2.1.2}$$

Thus,

$$H_I = e^{iH_0t}H_s e^{-iH_0t} \tag{2.1.3}$$

where $H_S$ is time dependent Hamiltonian in *Schrödinger* picture which can be expressed as,

$$H_S = H_I(s) + H_0(s) \tag{2.1.4}$$

where $H_0$ is unperturbed Hamiltonian,

$$H_I = \int d^3(x)H_I^{'} = \int d^3(x)(-ie\overline{\Psi}\gamma_\mu\Psi A_\mu(x)); H_I = -ie\overline{\Psi}\gamma_\mu\Psi A_\mu. \tag{2.1.5}$$

The fundamental differential equation that characterizes the time evaluation of a state ket in the interaction picture is:

$$i\hbar\frac{\partial}{\partial t} \mid \alpha, t_0; t >_I = H_I \mid \alpha, t_0; t >_I \tag{2.1.6}$$

If $U_I(t, t_0)$ is the time evolution operator in interaction picture then,

$$\mid \alpha, t_0; t >_I = U_I(t, t_0) \mid \alpha, t_0; t_0 >_I \tag{2.1.7}$$

Substituting equation (2.1.7) in to equation (2.1.6), we get the equation of motion of $U_I(t, t_0)$:

$$i\frac{\partial}{\partial t}U_I(t, t_0) = H_I U_I(t, t_0) \tag{2.1.8}$$

We must solve this operator differential equation subject to the initial condition imposing the initial condition $U_I(t, t_0) \mid_{t=t_0}$=1. First, let us note that the differential equation together with the initial condition is equivalent to the following integral equation: the solution of (2.1.8) becomes,

$$U_I(t, t_0) = 1 - i\int_{t_0}^{t} H_I(t_1)U_I(t_1, t_0)dt_1 \tag{2.1.9}$$

We can obtain an approximation solution to this equation by iteration [7] as:

$$U(t,t_0) = 1 - i\int_{t_0}^{t} H_I(t_1)dt_1 + (-i)^2\int_{t_0}^{t} dt_1\int_{t_0}^{t_1} dt_2 H_I(t_1)H_I(t_2)...$$
$$+(-i)^n\int_{t_0}^{t} dt_1\int_{t_0}^{t_1} dt_2...\int_{t_0}^{t_{n-1}} dt_n H_I(t_1)H_I(t_2)...H_I(t_n)... \qquad (2.1.10)$$

If $\mid i >$ is the initial state of the system at time $t_0$ and $\mid f >$ is the final state of the same system at time t, then $|< f \mid U_I(t,t_0) \mid i >|^2 = \mid U_{fi}(t,t_0) \mid^2$ is the transition probability of the system in state $\mid f >$ in time t if it was in state $\mid i >$ at initial time $t_0$ it approaches a definite limit as, $t_0 \to$ -$\infty$ and t = $+\infty$ . The S-matrix,

S=U($+\infty, -\infty$) it connects state vector from $t_0$=-$\infty$, to t=$+\infty$ according to $\mid \alpha,\infty >$=S($\mid \alpha$,-$\infty >$). The explicit form of S-matrix can be easily read from equation (2.1.10) as,

$$S = S^{(0)} + S^{(1)} + S^{(2)} + S^{(3)} + ... \qquad (2.1.11)$$

Thus the matrix element of S between initial $|i >$ and final $|f >$ states is,

$$S_{fi} =< f|[1 - i\int_{-\infty}^{\infty} dt_1 H_I(t_1) + (-i)^2\int_{-\infty}^{\infty} dt_1\int_{-\infty}^{t_1} dt_2 H_I(t_1)H_I(t_2)$$
$$+...(-i)^n\int_{-\infty}^{\infty} dt_1\int_{-\infty}^{t_1} dt_2...\int_{-\infty}^{t^{n-1}} dt_n H_I(t_1)H_I(t_2)...H_I(t_n)]|i > . \qquad (2.1.12)$$

In QED, the S-matrix expansion turns out to results that agree extremely with observations just by considering the first few terms; this, of course, is because the dimensionless coupling constant $\alpha = \frac{e^2}{4\pi\hbar c} = 1/137$ is small compared to unity. Thus we obtain the corrections for various orders of S-matrix as:

$$S_{fi}^{(0)} = \delta_{fi}. \qquad (2.1.13)$$

$$S_{fi}^{(1)}(t) = \frac{-i}{\hbar}\int_{t_0}^{t} < f|H_I(t')|i > dt' = \frac{-i}{\hbar}\int_{t_0}^{t} e^{i\omega_{fi}t'} H_{fi}(t')dt'. \qquad (2.1.14)$$

$$S_{fi}^{(2)}(t) = \left(\frac{-i}{\hbar}\right)^2 \sum_m \int_{t_0}^{t} dt' \int_{t_0}^{t'} dt'' e^{i\omega_{fm}t'} H_{fm}(t') e^{i\omega_{mi}t''} H_{mi}(t''). \qquad (2.1.15)$$

where we have used, $e^{i(E_f - E_i)t/\hbar} = e^{i\omega_{fi}t}$.

The transition probability for $|i > \to |f >$ with $f \neq i$ is obtained by

$$P(i \to f) = |S_{fi}^{(1)}(t) + S_{fi}^{(2)}(t) + ...|^2. \qquad (2.1.16)$$

## 2.2 Electron-Positron Pair Creation

The Breit-Wheeler process $\gamma_1 + \gamma_2 \rightarrow e^- + e^+$ still has not been directly observed in the laboratory because of the difficulty in preparing colliding beams of gamma rays. However, a related and even more interesting process has recently been observed by using the intense electromagnetic field of a laser beam. Here a High-energy gamma ray collides with a number n of optical photons of energy $\hbar\omega$ to form an electron-positron pair:

$$\gamma_1 + n\omega \rightarrow e^- + e^+ \tag{2.2.1}$$

We have studied the process $\gamma_1 + n\omega \rightarrow e^- + e^+$ as a mathematical preliminary in this thesis. There are two diagrams contributing to the above equation (2.2.1) .

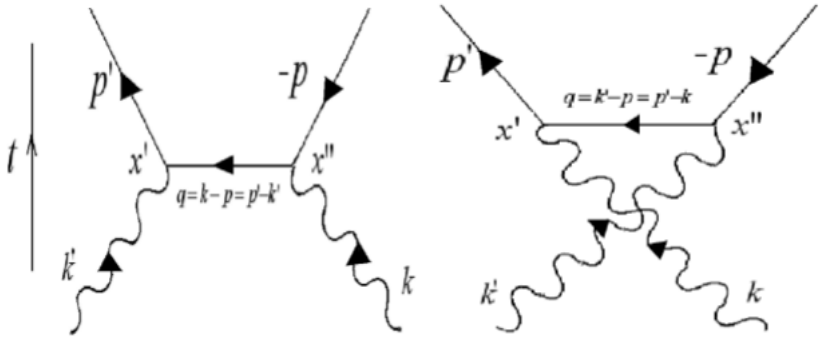



Figure 2.1: Feynman diagrams for pair creation: (A) Direct and (B) Exchange process.

In the first diagram (A), the incident photon (k, $\varepsilon$) is absorbed by the incident negative energy electron of (-p, s) and then the electron absorbs a photon ($k^{'}$, $\varepsilon^{'}$) and goes into final state ($p^{'}$, $s^{'}$), we call this the direct process. In the second diagram (B) the incident negative energy electron of (-p, s) absorbs a photon ($k^{'}$, $\varepsilon^{'}$) before it absorbs a photon of (k, $\varepsilon$) and goes to the final state ($p^{'}$, $s^{'}$), we call this the exchange process. Note that the

two diagrams differ only by the type of the absorbed photons.

For the process (2.2.1) we solve the Dirac equation:

$$(i\gamma_\mu\partial_\mu - m)\Psi(x) = e\gamma_\mu A_\mu(x)\Psi(x). \tag{2.2.2}$$

By first solving the unit source problem,

$$(i\gamma_\mu\partial_\mu - m)k_F(x,x^{'}) = \delta^{(4)}(x-x^{'}). \tag{2.2.3}$$

We obtained approximate solution to the equation (2.2.2) accurate to any desired power of e (depending on the type of interaction considered) by solving the unit source problem (2.2.3) iteratively as [7]:

$$\begin{aligned}\Psi(x) = \Psi_0(x) + \int d^4x^{'}k_F(x,x^{'})[-e\gamma_\mu A_\gamma(x^{'})]\Psi_0(x^{'}) \\ + \int d^4x^{'}\int d^4x^{''}k_F(x,x^{'})[-e\gamma_\mu A_\mu(x^{'})]k_F(x^{'},x^{''})[-e\gamma_\nu A_\nu(x^{''})]\Psi_0(x^{''}) \\ + \int d^4x^{'}\int d^4x^{''}\int d^4x^{'''}k_F(x,x^{'})[-e\gamma_\mu A_\mu(x^{'})]k_F(x^{'},x^{''}) \\ [-e\gamma_\nu A_\nu(x^{''})]k_F(x^{''},x^{'''})[-e\gamma_\lambda A_\lambda(x^{'''})]\Psi_0(x^{'''}) + ...\end{aligned} \tag{2.2.4}$$

Since the electron-positron pair creation (2.2.1) is second order in e (see figure 2.1), we consider second order perturbation expansion of $\Psi(x)$.

$$\begin{aligned}\Psi(x) &\simeq \Psi_0(x) + \int d^4x^{'}\int d^4x^{''}k_F(x,x^{'})[-e\gamma_\mu A_\mu(x^{'})]k_F(x^{'},x^{''})[-e\gamma_\nu A_\nu(x^{''})]\Psi_0(x^{''}) \\ &\simeq \Psi_0(x) + \int d^3x^{'}\left[\int_{-\infty}^{t}dt^{'} + \int_{t}^{+\infty}dt^{'}\right]\int d^4(x^{''})k_F(x,x^{'}) \\ &\quad [-e\gamma_\mu A_\mu(x^{'})]k_F(x^{'},x^{''})[-e\gamma_\nu A_\nu(x^{''})]\Psi_0(x^{''}).\end{aligned} \tag{2.2.5}$$

and thus neglect first and higher order terms. Using the spinor form of the electron propagator:

$$k_F(x,x^{'}) = \lim_{V\to\infty}\sum_{\overrightarrow{p},s}\frac{m}{EV}u^{(s)}(p)\overline{u}^{(s^{'})}(\overrightarrow{p}^{'})e^{ip.(x-x^{'})}; t > t^{'}$$

and

$$k_F(x,x^{'}) = -\lim_{V\to\infty}\sum_{\overrightarrow{p},s}\frac{m}{EV}v^{(s)}(p)\overline{v}^{(s^{'})}(\overrightarrow{p}^{'})e^{-ip.(x-x^{'})};t^{'}>t \tag{2.2.6}$$

where u and v are electron and positron spinors respectively. Let's consider a potentials problem assuming the validity of the second order perturbation theory.

Substituting equation (2.2.6) in equation (2.2.5), we get;

$$\begin{aligned}\Psi(x) = \Psi_0(x) + \int d^3x^{'}\int_{-\infty}^{t}dt^{'}\int d^4x^{''}\sqrt{\frac{M}{E^{'}V}}\overline{u}^{s}(\overrightarrow{p}^{'})e^{-ip^{'}.x^{''}}[-e\gamma_\mu A_\mu(x^{'})]\\ k_F(x^{'},x^{''})[-e\gamma_\nu A_\nu(x^{''})]\Psi_0(x^{''})\left[\sqrt{\frac{M}{E^{'}V}}u^{s}(\overrightarrow{p})e^{ip.x}\right]\\ +\int d^3x^{'}\int_{t}^{+\infty}dt^{'}\int d^4x^{''}\sqrt{\frac{M}{E^{'}V}}\overline{v}^{s}(\overrightarrow{p}^{'})e^{ip^{'}.x^{''}}[-e\gamma_\mu A_\mu(x^{'})]\\ k_F(x^{'},x^{''})[-e\gamma_\nu A_\nu(x^{''})]\Psi_0(x^{''})\left[\sqrt{\frac{M}{E^{'}V}}v^{s}(\overrightarrow{p})e^{-ip.x}\right].\end{aligned} \tag{2.2.7}$$

Equation (2.2.5) then becomes,

$$\Psi(x) = \Psi_0(x) + \sum_{\overrightarrow{p}^{'},s^{'}}C^{(+)}_{\overrightarrow{p}^{'},s^{'}}(t)\sqrt{\frac{m}{E^{'}V}}u^{(s)}(\overrightarrow{p})e^{ip.x} + \sum_{\overrightarrow{p}^{'},s^{'}}C^{(-)}_{\overrightarrow{p}^{'},s^{'}}(t)\sqrt{\frac{M}{E^{'}V}}v^{(s)}(\overrightarrow{p})e^{-ip.x}. \tag{2.2.8}$$

Therefore the second-order pair production amplitude is:

$$\begin{aligned}S_{fi} = C^{(+)}_{\overrightarrow{p}^{'},s^{'}}(+\infty)\\ = -ie^2\int d^4x^{'}\int d^4x^{''}\overline{\Psi}_f(x^{'})\Big[\not{A}(x^{'};k^{'})k_F(x^{'},x^{''})\not{A}(x^{''};k)\\ +\not{A}(x^{'};k)k_F(x^{'},x^{''})\not{A}(x^{''};k^{'})\Big]\Psi_i(x^{''}).\end{aligned} \tag{2.2.9}$$

Due to the amplitudes of the two indistinguishable processes, which differ in the association of momentum and spin variables (p, s) and $(s^{'},p^{'})$ with the outgoing photon lines, have to be added coherently.

$$S^2_{fi} = S^2_{fi}(dir.) + S^2_{fi}(exch.) = S^2_{fi}|_D + S^2_{fi}|_X \tag{2.2.10}$$

$$S^2_{fi}(dir.) = (-ie)^2\int d^4x^{'}\int d^4x^{''}\overline{\Psi}_f(x^{'})\left[\not{A}(x^{'};k_2)k_F(x^{'},x^{''})\not{A}(x^{''};k_1)\right]\Psi_i(x^{''}),$$

$$S_{fi}^2(exch.) = (-ie)^2 \int d^4x^{'} \int d^4x^{''} \overline{\Psi}_f(x^{'}) \left[ \not{A}(x^{'};k_1) k_F(x^{'},x^{''}) \not{A}((x^{''};k_2) \right] \Psi_i(x^{''}). \tag{2.2.11}$$

where $\not{A}$=$\gamma_\mu A_\mu$ and we have summed both the amplitudes for the Direct (figure 2.1 (A)) and Exchange (figure 2.1 (B)) processes since they only differ by the exchange of absorbed photons in the final states so that Bose-Einstein statistics is obeyed.

were we have the wave functions

$$\overline{\Psi}_{p_-}(x^{'}) = \overline{\Psi}_f(x^{'}) = \sqrt{\frac{m}{E_2 V}} \overline{u}^{(s_-)}(\overrightarrow{p}_-) e^{ip_-.x^{'}}$$

$$\Psi_{p_+}(x^{''}) = \Psi_i(x^{''}) = \sqrt{\frac{m}{E_1 V}} v^{(s_+)}(\overrightarrow{p}_+) e^{ip_+.x^{''}},$$

Using the Feynman electron propagator:

$$k_F(x^{'},x^{''}) = \frac{-i}{(2\pi)^4} \int d^4q \left( \frac{-i\gamma.q + m}{q^2 - m^2} \right) e^{iq.(x^{'}-x^{''})}, \tag{2.2.12}$$

and the free photon wave function,

$$A_\mu(x;k) = \sqrt{4\pi} \frac{\varepsilon^\mu}{\sqrt{2\omega V}} (e^{-ik.x} + e^{ik.x}), \tag{2.2.13}$$

where the first and second terms, $e^{-ik.x}$ and $e^{ik.x}$ of the photon wave function in above equation corresponding respectively to absorption and emission of a photon of four-momentum k.

The S-matrix element (2.2.9) becomes,

$$S_{fi}^2 = \frac{-ie^2}{(2\pi)^4} \sqrt{\frac{m}{E_1 V}} \sqrt{\frac{m}{E_2 V}} \frac{4\pi}{\sqrt{(2\omega_1 V)(2\omega_2 V)}} \int d^4x^{'} \int d^4x^{''} \int d^4q \overline{u}^{(s_-^{'})}(\overrightarrow{p}_-)$$

$$\left[ \not{\varepsilon}^{'} \left( \frac{\not{q} + m}{q^2 - m^2} \right) \not{\varepsilon} e^{ix^{'}.(p^{'}-k^{'}-q)} e^{ix^{''}.(q-k+p)} + \not{\varepsilon} \left( \frac{\not{q} + m}{q^2 - m^2} \right) \not{\varepsilon}^{'} e^{ix^{'}.(p^{'}-k-q)} e^{ix^{''}.(q-k^{'}+p)} \right]$$

$$v^{(s_+)}(\overrightarrow{p}_+). \tag{2.2.14}$$

where we substitute $p \to p_+$, $p^{'} \to p_-$ , $k \to k_1$ , $k^{'} \to k_2$, $\varepsilon \to \varepsilon_1$, $\varepsilon' \to \varepsilon_2$, with the energy-momentum conservation written as $p_- + p_+$=$k_1 + k_2$, where $p_+$ and $p_-$ are 4-momenta of the position and the electron, $k_1$ and $k_2$ are 4-momenta of two photons and

$\varepsilon_1$ and $\varepsilon_2$ are the polarization four-vector of the photon. Where momenta and energies are related by $(p_1+p_2)^2$=$(k_1+k_2)^2$=$2(k_1k_2)$=$2\varepsilon^2$.

Let's us expand the integrations inside, from the above equation:

$$S_{fi}^2 = -ie^2\sqrt{\frac{m}{E_1V}}\sqrt{\frac{m}{E_2V}}\frac{4\pi}{\sqrt{(2\omega_1V)(2\omega_2V)}}\int\frac{d^4q}{(2\pi)^4}\overline{u}^{(s_-)}(\overrightarrow{p}_-)$$
$$\left[\not{\varepsilon}_2\left(\frac{1}{-\not{p}_+ + \not{k}_1 - m_0}\right)\not{\varepsilon}_1\int d^4x^{'}e^{ix^{'}.(p_- - k_2 - q)}\right.$$
$$\int d^4x^{''}e^{ix^{''}.(q-k_1+p_+)} + \not{\varepsilon}_1\left(\frac{1}{-\not{p}_+ + \not{k}_2 - m_0}\right)\not{\varepsilon}_2\int d^4x^{'}e^{ix^{'}.(p_- - k_1 - q)}$$
$$\left.\int d^4x^{''}e^{ix^{''}.(q-k_2+p_+)}\right]v^{(s_+)}(\overrightarrow{p}_+). \tag{2.2.15}$$

Performing the space-time integrations $\int d^4x^{'}\ \int d^4x^{''}\ \int d^4q$ and using the properties of $\delta$-function.

Therefore we get,

$$S_{fi}^2 = -ie^2\sqrt{\frac{m}{E_1V}}\sqrt{\frac{m}{E_2V}}\frac{4\pi}{\sqrt{(2\omega_1V)(2\omega_2V)}}(2\pi)^4\delta^4(p_- - k_1 - k_2 + p_+)$$
$$\overline{u}^{(s_-)}(\overrightarrow{p}_-)\left[\not{\varepsilon}_2\left(\frac{-\not{p}_+ + \not{k}_1 - m_0}{-2p_+.k_1}\right)\not{\varepsilon}_1 + \not{\varepsilon}_1\left(\frac{-\not{p}_+ + \not{k}_2 - m_0}{-2p_+.k_2}\right)\not{\varepsilon}_2\right]v^{(s_+)}(\overrightarrow{p}_+). \tag{2.2.16}$$

Thus,

$$S_{fi}^2 = -ie^2\sqrt{\frac{m}{E_1V}}\sqrt{\frac{m}{E_2V}}\frac{4\pi}{\sqrt{(2\omega_1V)(2\omega_2V)}}\overline{u}^{(s_-)}(\overrightarrow{p}_-)(2\pi)^4\delta^4(p_- - k_1 - k_2 + p_+)$$
$$|M_{fi}|v^{(s_+)}(\overrightarrow{p}_+). \tag{2.2.17}$$

where the invariant amplitude is given by

$$|\overline{M}_{fi}^{pair}| = \overline{u}^{(s_-)}(\overrightarrow{p}_-)\left[\not{\varepsilon}_2\left(\frac{-\not{p}_+ + \not{k}_1 - m_0}{-2p_+.k_1}\right)\not{\varepsilon}_1 + \not{\varepsilon}_1\left(\frac{-\not{p}_+ + \not{k}_2 - m_0}{-2p_+.k_2}\right)\not{\varepsilon}_2\right]v^{(s_+)}(\overrightarrow{p}_+). \tag{2.2.18}$$

S-matrix reduces to:

$$S_{fi}^2 = -ie^2\sqrt{\frac{m}{E_1V}}\sqrt{\frac{m}{E_2V}}\frac{4\pi}{\sqrt{(2\omega_1V)}(\sqrt{2\omega_2V})}(2\pi)^4\delta^4(p_+ - k_1 + p_- - k_2)$$
$$\overline{u}^{(s_-)}(\overrightarrow{p}_-)\Gamma v^{(s_+)}(\overrightarrow{p}_+). \tag{2.2.19}$$

where,

$$\Gamma = \not{\epsilon}_2 \left( \frac{\not{k}_1 - \not{p}_+ - m_0}{-2k_1.p_+} \right) \not{\epsilon}_1 + \not{\epsilon}_1 \left( \frac{\not{k}_2 - \not{p}_+ - m_0}{-2k_2.p_+} \right) \not{\epsilon}_2, \tag{2.2.20}$$

Note that from the $\delta$-function we have the energy-momentum conservation:

$$k_1 + k_2 = p_- + p_+. \tag{2.2.21}$$

Spins $s_-$ and $s_+$ as follows

$$|\overline{M}_{fi}|^2 = \frac{1}{2} \sum_{s_-=1}^{2} \sum_{s_+=1}^{2} |\overline{u}^{(s_-)}(\overrightarrow{p}_-)\Gamma v^{(s_+)}(\overrightarrow{p}_+)|^2$$

$$= \frac{1}{2} Tr \left( \frac{\not{p}_- + m_o}{2m_o} \Gamma \frac{\not{p}_+ - m_o}{2m_o} \overline{\Gamma} \right) \tag{2.2.22}$$

with $\overline{\Gamma}$=$\gamma^0\Gamma^\dagger\gamma^0$ and we have used the spin sum relations

$$\sum_{s_-=1}^{2} u_\alpha^{(s_-)}(\overrightarrow{p}_-)\overline{u}_\beta^{(s_-)}(\overrightarrow{p}_-) = \left( \frac{\not{p}_- + m_o}{2m_o} \right)_{\alpha\beta} and$$

$$\sum_{s_+=1}^{2} v_\alpha^{(s_+)}(\overrightarrow{p}_+)\overline{v}_\beta^{(s_+)}(\overrightarrow{p}_+) = \left( \frac{\not{p}_+ - m_o}{2m_o} \right)_{\alpha\beta} \tag{2.2.23}$$

We simplify $\Gamma$ (2.2.22) into the form,

$$\Gamma = \frac{\not{\epsilon}_2\not{\epsilon}_1\not{k}_1}{2k_1.p_+} + \frac{\not{\epsilon}_1\not{\epsilon}_2\not{k}_2}{2k_2.p_+}, \tag{2.2.24}$$

where we have used the free Dirac equation for the antiparticle unit spinor $v^{(s_+)}(\overrightarrow{p}_+)$

$$(\not{p}_+ + m)v^{(s_+)}(\overrightarrow{p}_+) = 0. \tag{2.2.25}$$

and the anticommutation relations,

$$\not{p}_+\not{\epsilon}_1 = -\not{\epsilon}_1 p_+, \not{p}_-\not{\epsilon}_2 = -\not{\epsilon}_2\not{p}_-$$

$$\not{k}_1\not{\epsilon}_1 = -\not{\epsilon}_1\not{k}_1, \not{k}_2\not{\epsilon}_2 = -\not{\epsilon}_2\not{k}_2 \tag{2.2.26}$$

This leads to,

$$\overline{\Gamma} = \gamma^0\Gamma^\dagger\gamma^0 = \frac{\not{k}_1\not{\epsilon}_1\not{\epsilon}_2}{2k_1.p_+} + \frac{\not{k}_2\not{\epsilon}_2\not{\epsilon}_1}{2k_2.p_+} \tag{2.2.27}$$

where we have used the relations,

$$\overline{\not{a}} = \gamma^0 \not{a}^\dagger \gamma^0 = \not{a}$$

$$\overline{\not{a}\not{b}...\not{p}} = \gamma^0 \not{p}^\dagger \not{b}^\dagger ... \not{a}^\dagger \gamma^0 = \not{p}...\not{b}\not{a}. \tag{2.2.28}$$

Substituting equations (2.2.24) and (2.2.27) into equation (2.2.22), we get

$$|\overline{M}_{fi}|^2 = Tr\left[\frac{\not{p}_- + m_o}{2m_o}\left(\frac{\not{\varepsilon}_2\not{\varepsilon}_1\not{k}_1}{2k_1.p_+} + \frac{\not{\varepsilon}_1\not{\varepsilon}_2\not{k}_2}{2k_2.p_+}\right)\frac{\not{p}_+ - m_0}{2m_o}\left(\frac{\not{k}_1\not{\varepsilon}_1\not{\varepsilon}_2}{2k_1.p_+} + \frac{\not{k}_2\not{\varepsilon}_2\not{\varepsilon}_1}{2k_2.p_+}\right)\right]. \tag{2.2.29}$$

The calculation of the trace in (2.2.29) is not easily done since products of up to 8$\gamma$ matrices are involved. We simplify the trace calculation, where a trace involves at least two factors which are dot product of $\gamma$-matrices with same four-vectors, we anticommute the $\gamma$-matrices such that the same four-factors come alongside each other so that we can make use of the identity $\not{a}\not{a}=a^2$ to get rid of two $\gamma$-matrices from the product and thus considerably simplifying the calculation. We also make use of $k^2 = 0$, $\varepsilon_1^2$=$\varepsilon_2^2$=-1, $k_1.p_-$=$k_2.p_+$ and $k_1.\varepsilon_2 = p_-.\varepsilon_2$.

Using the property of Traces [8]:

$$|\overline{M}_{fi}|^2 = \frac{1}{4m_o^2}2\left[\frac{k_2.p_+}{k_1.p_+} + \frac{k_1.p_+}{k_2.p_+} + 4(\varepsilon_2.\varepsilon_1)^2 - 2\right] \tag{2.2.30}$$

We now insert the kinematics of the Laboratory frame (L) i.e, $k_1.p_+$=$\omega_1 m_o$ , $k_2.p_+$=$\omega_2 m_o$. This leads to,

$$|\overline{M}_{fi}^{pair}|^2 = \frac{1}{2m_o^2}\left[\frac{\omega_2}{\omega_1} + \frac{\omega_1}{\omega_2} + 4(\varepsilon_2.\varepsilon_1)^2 - 2\right]. \tag{2.2.31}$$

Finally we arrive at the following expression:

$$S_{fi}^2 = -ie^2\sqrt{\frac{m}{E_1 V}}\sqrt{\frac{m}{E_2 V}}\frac{4\pi}{\sqrt{(2\omega_1 V)}(\sqrt{2\omega_2 V})}(2\pi)^4\delta^4(p_+ + p_- - k_1 - k_2)$$

$$\frac{1}{2m_o^2}\left[\frac{\omega_2}{\omega_1} + \frac{\omega_1}{\omega_2} + 4(\varepsilon_2.\varepsilon_1)^2 - 2\right]. \tag{2.2.32}$$

The transition rate is given by

$$W_{fi} = \frac{|S_{fi}|^2}{VT}$$

$$= e^4 \frac{m^2}{E_1 E_2 V^2} \frac{(4\pi)^2}{4\omega_1 \omega_2 V^2} (2\pi)^4 \delta^4(p_+ - k_1 + p_- - k_2) |M_{fi}|^2 \tag{2.2.33}$$

We have used the square of the $\delta$-function.

From this relation:

$$|(2\pi)^4 \delta^4(p_+ - k_1 + p_- - k_2)|^2 = (2\pi)^4 V T \delta^4(p_+ - k_1 + p_- - k_2). \tag{2.2.34}$$

Transition rate becomes;

$$W_{fi} = e^4 \frac{m^2}{E_1 E_2 V^2} \frac{(4\pi)^2}{4\omega_1 \omega_2 V^2} (2\pi)^4 \delta^4(p_+ + p_- - k_1 - k_2) \frac{1}{2m_o^2} \left[ \frac{\omega_2}{\omega_1} + \frac{\omega_1}{\omega_2} + 4(\varepsilon_2.\varepsilon_1)^2 - 2 \right]. \tag{2.2.35}$$

The general expression for the differential cross-section of pair creation is by dividing the rate by the incident flux, $|\overrightarrow{J}_{inc}| = \frac{|\overrightarrow{v}_{rel}|}{v}$ divided by the number of target particles per unit volume 1/V and finally, multiplying by the density of final particle states $\left[ V \frac{d^3 \overrightarrow{p}_-}{(2\pi)^3} \right]$ $\left[ V \frac{d^3 \overrightarrow{p}_+}{(2\pi)^3} \right]$.

Then $d\sigma$ becomes,

$$d\sigma = W_{fi} \frac{1}{|\overrightarrow{J}_{inc}| \frac{1}{V}} \left[ V \frac{d^3 \overrightarrow{p}_-}{(2\pi)^3} \right] \left[ V \frac{d^3 \overrightarrow{p}_+}{(2\pi)^3} \right] \tag{2.2.36}$$

where we can use the phase space volume $d^3 \overrightarrow{p} = \overrightarrow{p}^2 d\overrightarrow{p}\, d\Omega$,

$$d\sigma_{pair} = \frac{\alpha^2}{2} \frac{4\pi}{2\omega_1} \frac{4\pi}{2\omega_2} |M_{fi}^{pair}|^2 (2\pi)^4 \delta^4(p_+ + p_- - k_1 - k_2) \frac{m_0 d^3 \overrightarrow{p}_+}{(2\pi)^3 E_+} \frac{m_0 d^3 \overrightarrow{p}_-}{(2\pi)^3 E_-}. \tag{2.2.37}$$

where the factor 1/2 results from the photon flux factor $1/|v_1 - v_2|$, where $|v_1 - v_2|$ is the relative velocity of the photons. The relative velocity of two collinearly colliding photons is 2 in any frame of reference.

The differential cross-section now becomes:

$$\frac{d\sigma}{d\Omega} = \frac{2\alpha^2 m_o}{\omega_1 E |\overrightarrow{V}_{rel}|} \int d\omega_2 \omega_2 \frac{d^3 p_-}{2E_-} |M_{fi}|^2 \delta^4(p_- + p_+ - k_1 - k_2). \tag{2.2.38}$$

Use the next identity to solve equation (2.2.38)

$$\frac{d^3 p_-}{2E_-} = \int d^4 p_- \delta(p_-^2 - m_o^2) \Theta(p_{-o}).$$

$$\frac{d\sigma}{d\Omega} = \frac{2\alpha^2 m_o}{\omega_1 E|\overrightarrow{V}_{rel}|}\int d\omega_2\omega_2 \int d^4p_-|M_{fi}|^2\delta^4(p_- + p_+ - k_1 - k_2)\delta(p_-^2 - m_o^2)\Theta(p_{-o}).$$

$$\frac{d\sigma}{d\Omega} = \frac{2\alpha^2 m_o}{\omega_1 E|\overrightarrow{V}_{rel}|}\int d\omega_2\omega_2|M_{fi}|^2\delta\left[(k_1 + k_2 - p_+)^2 - m_o^2\right]\Theta(\omega_1 - E + \omega_2).$$

$$\frac{d\sigma}{d\Omega} = \frac{2\alpha^2 m_o}{\omega_1 E|\overrightarrow{V}_{rel}|}\int_0^{\omega_1 - E} d\omega_2\omega_2|M_{fi}|^2\delta\left[2k_1.k_2 - 2p_+.(k_1 + k_2)\right],$$

where the kinematic variables in the invariant matrix element $|M_{fi}|^2$ must now obey the condition $p_- = k_1 + k_2 - p_+$, and the constant $\alpha = \frac{e^2}{4\pi}$ is used. In this frame $p_+ = (m_o, 0)$, the incident beam consists of photons with unit velocity, such that $|\overrightarrow{V}_{rel}| = 2$.

The differential cross section now becomes

$$\frac{d\sigma}{d\Omega} = \frac{\alpha^2 m_o}{\omega_1}\int_0^{\omega_1 - E} d\omega_2\omega_2|M_{fi}|^2\delta\left[2\omega_1\omega_2(1 - \cos\theta) - 2m_o(\omega_1 + \omega_2)\right],$$

$$\frac{d\sigma}{d\Omega} = \frac{\alpha^2 m_o}{\omega_1}\int_0^{\omega_1 - E} d\omega_2\omega_2|M_{fi}|^2\delta\left[(2\omega_1(1 - \cos\theta) - 2m_o)\left(\omega_2 - \frac{2m_o\omega_1}{2\omega_1(1 - \cos\theta) - 2m_o}\right)\right],$$

$$\frac{d\sigma}{d\Omega} = \frac{\alpha^2}{2\omega_1}\frac{1}{(\omega_1/m_o)(1 - \cos\theta) - 1}\int_0^{\omega_1 - E} d\omega_2\omega_2|M_{fi}|^2\delta\left(\omega_2 - \frac{\omega_1}{(\omega_1/m_o)(1 - \cos\theta) - 1}\right),$$

$$\frac{d\sigma}{d\Omega} = \frac{\alpha^2}{2}\left(\frac{1}{(\omega_1/m_o)(1 - \cos\theta) - 1}\right)^2\left||M_{fi}|^2\right|_{\omega_2 = \frac{\omega_1}{(\omega_1/m_o)(1-\cos\theta)-1}}. \tag{2.2.39}$$

where $\theta$ is the angle between the direction of the momentum of produced $e^{-s}$ and amplitude of the electric field strength of one incident wave (between the incident photons). By substituting the value of $|M_{fi}|^2$ from equation (2.2.31) to equation (2.2.39), then the differential cross-section becomes,

$$\frac{d\sigma}{d\Omega}(\lambda_2, \lambda_1) = \alpha^2\frac{1}{2}\frac{1}{2m_o^2}\left|\frac{\omega_2^2}{\omega_1^2}\left[\frac{\omega_2}{\omega_1} + \frac{\omega_1}{\omega_2} + 4(\varepsilon_2.\varepsilon_1)^2 - 2\right]\right|_{\omega_2 = \frac{\omega_1}{(\omega_1/m_o)(1-\cos\theta)-1}}. \tag{2.2.40}$$

The energy of the incident and scattered photons are connected by

$$\omega_2 = \frac{\omega_1}{(\omega_1/m_o)(1 - \cos\theta) - 1}. \tag{2.2.41}$$

We evaluate the unpolarized differential cross-section for scattering of photons, by averaging and summing over the photon polarizations $\lambda_1$ and $\lambda_2$ electron and positron spins

in the rest frame of the positron.

Let's now calculate the unpolarized cross-section by,

$$\frac{d\sigma}{d\Omega} = \frac{1}{2}\sum_{s_-,s_+}\frac{d\sigma}{d\Omega}(s_-, s_+; \lambda_1, \lambda_2). \tag{2.2.42}$$

$$= \frac{\alpha^2}{4m_o^2}\left(\frac{1}{(\omega_1/m_o)(1-\cos\theta)-1}\right)^2\left[\frac{1}{(\omega_1/m_o)(1-\cos\theta)-1}+\right.$$

$$\left.(\omega_1/m_o)(1-\cos\theta)-1+(\varepsilon_2.\varepsilon_1)^2-2\right]. \tag{2.2.43}$$

where,

$$k_1 = (0,0,1), \qquad k_2 = (\sin\theta\cos\varphi, \sin\theta\sin\varphi, \cos\theta)$$

$$\varepsilon_{(1)}^{(1)} = (0,1,0,0), \qquad \varepsilon_{(1)}^{(2)} = (0, \sin\varphi, -\cos\varphi, 0)$$

$$\varepsilon_{(2)}^{(1)} = (0,0,1,0), \qquad \varepsilon_{(2)}^{(2)} = (0, \cos\theta\cos\varphi, \cos\theta\sin\varphi, -\sin\theta). \tag{2.2.44}$$

with orthogonality relations satisfies,

$$\varepsilon_{(1)}^{(1)}.k_1 = \varepsilon_{(1)}^{(2)}.k_2 = 0,$$

$$\varepsilon_{(1)}^{(2)} = \varepsilon_{(2)}^{(2)} = -1.$$

Using $(\varepsilon_{(2)}, \varepsilon_{(1)})$ polarizations we obtain,

$$\sum_{\lambda_1,\lambda_2=1}^{2}(\varepsilon_{(\lambda_1)}.\varepsilon_{(\lambda_2)})^2 = 1+\cos^2\theta. \tag{2.2.45}$$

The differential cross-section (2.2.44), we get,

$$\frac{d\sigma}{d\Omega} = \frac{\alpha^2}{4m_o^2}\left(\frac{1}{(\omega_1/m_o)(1-\cos\theta)-1}\right)^2\left\{\frac{1}{(\omega_1/m_o)(1-\cos\theta)-1}+\right.$$

$$\left.(\omega_1/m_o)(1-\cos\theta)-1-\sin^2\theta\right\}. \tag{2.2.46}$$

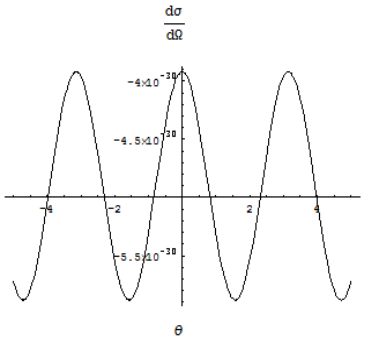



Figure 2.2: The graph of differential cross-section as a function of the scattering angle $\theta$ for photon energies $\omega$.

Lastly we integrate over the photon solid angle $d\Omega$, in order to drive the total cross-section. Where $d\Omega = \sin\theta d\theta d\varphi$, so we can substitute to the above equation and integrate both side, we will get,

$$\sigma_{pair} = \alpha^2 \frac{1}{4m_o^2} \int \sin\theta d\theta d\varphi \left( \frac{1}{(\omega_1/m_o)(1-\cos\theta)-1} \right)^2$$

$$\left\{ \frac{1}{(\omega_1/m_o)(1-\cos\theta)-1} + (\omega_1/m_o)(1-\cos\theta) - 1 - \sin^2\theta \right\}. \tag{2.2.47}$$

$$\sigma_{pair} = \alpha^2 \frac{\pi}{2m_o^2} \int_{-1}^{1} dz \left\{ \frac{1}{[(\omega_1/m_o)(1-z)-1]^3} + \frac{1}{[(\omega_1/m_o)(1-z)-1]} \right.$$

$$\left. - \frac{(1-z^2)}{[(\omega_1/m_o)(1-z)-1]^2} \right\}. \tag{2.2.48}$$

where we have set $\mathbf{z} = \cos\theta$, this integral can be calculated a closed form simple expressions can be derived for high and low photon energies.

**(a)** For small photon energies ($\omega_1 << m_o$):

$$\sigma_{pair} \simeq \alpha^2 \frac{\pi}{2m_o^2} \left[ -z - z - (z - \frac{z^3}{3}) \right]_{-1}^{1} = -\alpha^2 \frac{\pi}{2m_o^2} \frac{-16}{3} = -\frac{8\pi}{3} r_o^2. \tag{2.2.49}$$

where $r_o = \frac{\alpha}{m_o} = \frac{e^2}{m_o c^2} \simeq 2.8fm$ is the classical radius of the electron.

**(b)** For High photon energies ($\omega_1 >> m_o$):

$$\sigma_{pair} = \frac{-2[8 + (-1+z)\omega(8 + 3(-1+z)\omega]}{3[1 + (-1+z)\omega]^3} \tag{2.2.50}$$

# Chapter 3

# Nonlinear Solution of Dirac Equation for Vacuum Electrons in the Strong Electromagnetic Radiation Field

The Nonlinear interaction of charged particles with strong laser fields will be considered in general aspects from the point of view of both Nonlinear quantum FEL (free electron laser) schemes, and classical laser accelerator problems at the some time, the subject of which is Nonlinear electromagnetic processes with consideration of linear schemes of FELs taking in to account. As an intense laser pulse propagates through an underdense plasma, $\omega_p^2/\omega_0^2 << 1$, the effect excites a large amplitude plasma wave (wake-field) with phase velocity ($V_p$) approximately equal to the group velocity of the laser pulse [9].

$$V_p = c(1 - \frac{\omega_p^2}{\omega_0^2})^{1/2} \tag{3.0.1}$$

Let a plane transverse linearly polarized electromagnetic waves with frequency $\omega$ and amplitude of vector potential $A_0$.

$$\overrightarrow{A}_1 = \overrightarrow{A}_0 \cos(\omega t - \overrightarrow{k}.\overrightarrow{r}), \overrightarrow{A}_2 = \overrightarrow{A}_0 \cos(\omega t + \overrightarrow{k}.\overrightarrow{r}). \tag{3.0.2}$$

propagate in the opposite direction in vacuum.

To solve the problem of n-photon production of an $e^-, e^+$ pair in the given radiations fields (3.0.2) we shall make use of the Dirac model-all vacuum negative-energy states are filled with electrons. The Dirac equation in the field (3.0.2) has the form (here we set

$\hbar = c = 1$)

$$\overrightarrow{A}(\overrightarrow{r}, t) = \overrightarrow{A}_0 \cos(\omega t - \overrightarrow{k}.\overrightarrow{r}) + \overrightarrow{A}_0 \cos(\omega t + \overrightarrow{k}.\overrightarrow{r})$$
$$= 2\overrightarrow{A}_0 (\cos \overrightarrow{k}.\overrightarrow{r}) \cos \omega t. \tag{3.0.3}$$

is formed by two oppositely propagating laser beams of frequency $\omega$ and wavelength $\lambda$. It is assumed that pairs will be created at in the areas of antinodes and in spatial dimensions $l << \lambda$. Thus, $\overrightarrow{k}.\overrightarrow{r} = 2\pi l/\lambda$ can be ignored leading to,

$$\overrightarrow{A}(\overrightarrow{r}, t) = 2\overrightarrow{A}_0 \cos \omega t. \tag{3.0.4}$$

(as in the case for optical as well as X-ray Free Electron Laser systems). With out loss of generality $A = (0, 2A \cos \omega t, 0)$, p=$(p_x = p \sin\theta, p_y = p \cos\theta, 0)$, where, $(p_z = 0)$.

$$\overrightarrow{A}'(t') = \frac{\overrightarrow{A}_0}{2}[exp(i\omega' t) + exp(-i\omega' t)], \tag{3.0.5}$$

where, $\omega'$=$\omega\sqrt{1-n^2}$ ,

It is easily to noted that with Equation (3.0.5) taken in to account the reaction threshold condition (1.0.3), is obtained from the laws of the conservation of energy $E_- + E_+ = n\hbar\omega$ and momentum $p_- + p_+ = n\hbar k' = 0$ in the C.M frame.

We can write Dirac equation for electron in presence of e.m field $A_\mu = (\phi, \overrightarrow{A})$ using principle of minimal coupling as,

$$i\frac{\partial \Psi}{\partial t} = \left\{ c\overrightarrow{\alpha}.[\overrightarrow{p} - \frac{e\overrightarrow{A}(t)}{c}] + \hat{\beta}mc^2 \right\} \Psi. \tag{3.0.6}$$

Thus can in turn be expressed as,

$$i\frac{\partial \Psi}{\partial t} = \left\{ \overrightarrow{\alpha}.[\overrightarrow{p} - 2e\overrightarrow{A}_0 \cos(k.\overrightarrow{r}) \cos \omega t] + \hat{\beta}m \right\} \Psi, \tag{3.0.7}$$

$$\overrightarrow{\alpha} = \begin{pmatrix} 1 & \overrightarrow{\sigma} \\ \overrightarrow{\sigma} & 0 \end{pmatrix}$$

$$\hat{\beta} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

where the Dirac matrices $\overrightarrow{\alpha}$, $\hat{\beta}$ will be chosen in the standard representation, with $\overrightarrow{\sigma}$ the Pauli matrices. According to the Dirac model the electron-positron pair production by e.m wave field occurs when the vacuum electrons with initial negative energies $E_0 < 0$ due to n-photon absorption pass to the final state with positive energies E=$E_0$+n$\omega$ > 0. Since we study the case of superstrong laser fields in which the pairs are essentially produced at the length $l << \lambda$ and on the other hand the Hamiltonian of the interaction $H_{int} \sim \overrightarrow{p}.(\overrightarrow{A}_1+\overrightarrow{A}_2)$, then the significant contribution in the process of $e^-$, $e^+$ pair creation will be conditioned by the areas of stationary maxima in the direction along the electric field strength of the standing wave. Consequently, we can neglect the inhomogeneity of the field in considering problem, i.e., the Equation (3.0.7) will reduce to the following equation [9]:

$$i\frac{\partial \Psi}{\partial t} = \left\{ \overrightarrow{\alpha}.(\overrightarrow{p} - 2e\overrightarrow{A}_0 \cos\omega t) + \hat{\beta}m \right\} \Psi. \tag{3.0.8}$$

In this case we can introduce the time-dependent wave function $\Psi = \Psi(\overrightarrow{r}, t)$.

$$i\hbar\frac{\partial \Psi}{\partial t} = H\Psi. \tag{3.0.9}$$

$$H = H^o + \lambda H^{'}(t). \tag{3.0.10}$$

where, $H^o$ is the time-independent part, $H^{'}$ is the time-dependent part. The time-dependent *Schrödinger* equation before the perturbation (before interaction) is given by;

$$H^o\Psi_n^{(0)} = E_n^{(0)}\Psi_n^{(0)}.$$

Using the time-dependent wave function $\Psi_n^{(0)} = \Psi_n^{(0)}(\overrightarrow{r}, t)$. Equation (3.0.9) can be written as,

$$H^o\Psi_n^{(0)}(\overrightarrow{r}, t) = i\hbar\frac{\partial}{\partial t}\Psi_n^{(0)}(\overrightarrow{r}, t). \tag{3.0.11}$$

In the general solution of the unperturbed system is given by the complete set of orthonormal solutions

$$\Psi_n^{(0)} = \sum_{n=1}^{4} a_n\Psi_n^{(0)}. \tag{3.0.12}$$

The general solution is normalized, if $< \Psi^{(0)}|\Psi^{(0)} >= 1$ (or) $\sum_n a_n^\dagger a_n = 1$. To find the time-dependent wave function *Schrödinger* equation is;

$$[H^o + \lambda H^{'}(t)]\Psi = i\hbar\frac{\partial\Psi}{\partial t}. \tag{3.0.13}$$

Since in the C.M frame the interaction Hamiltonian does not depend on the space coordinates, the solution of (3.0.8) can be represented in the form of linear combination of free solutions of the Dirac equation with amplitudes $a_n(t)$ depending only on time:

$$\Psi_p(\overrightarrow{r}, t) = \sum_{n=1}^{4} a_n(t)\Psi_n^{(0)}(\overrightarrow{r}, t) \tag{3.0.14}$$

Here,

$$\Psi_{1,2}^{(0)}(\overrightarrow{r}, t) = \left(\frac{E+m}{2E}\right)^{1/2} \begin{pmatrix} \varphi_{1,2} \\ \frac{\overrightarrow{\sigma}\,\overrightarrow{p}}{E+m}\varphi_{1,2} \end{pmatrix} e^{i(\overrightarrow{p}.\overrightarrow{r}-Et)}$$

$$\Psi_{3,4}^{(0)}(\overrightarrow{r}, t) = \left(\frac{E+m}{2E}\right)^{1/2} \begin{pmatrix} \frac{-\overrightarrow{\sigma}\,\overrightarrow{p}}{E+m}X_{3,4} \\ X_{3,4} \end{pmatrix} e^{i(\overrightarrow{p}.\overrightarrow{r}+Et)} \tag{3.0.15}$$

where,

$$E = \sqrt{p^2+m^2}, \varphi_1 = X_3 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \varphi_2 = X_4 = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \tag{3.0.16}$$

The solution of equation (3.0.8) in the form of equation (3.0.14) corresponds to an expansion of the wave function in a complete set of orthonormal functions of the electrons (positrons) with specified momentum [with energies E=$\pm(p^2+m^2)^{1/2}$ and spin projections $S_z = \pm 1/2$]. The latter are normalized to one particle per unit volume. Since in the C.M frame there is symmetry with respect to the direction $\mathbf{A}_o$ (the OY axis) we can take, without loss of generality the vector $\mathbf{P}$ to lie in the X-Y plane ($p_z = 0$).

Substituting equation (3.0.14) into equation (3.0.13), we get,

$$i\hbar\sum_n[\frac{\partial}{\partial t}a_n(t)]\Psi_n^{(0)} + i\hbar\sum_n a_n(t)\frac{\partial}{\partial t}\Psi_n^{(0)} = \sum_n a_n(t)H^o\Psi_n^{(0)} + \sum_n a_n(t)\lambda H^{'}\Psi_n^{(0)}. \tag{3.0.17}$$

The first term of the left-hand side and the second term of the right-hand side of this equation are equal, these two terms represent the unperturbed *Schrödinger* equation and

they can be removed from the equation. Now we consider one specific wave function $\Psi_m^{(o)}$ of the orthonormal set $\Psi_n^{(0)}$.

$$< \Psi_m^{(0)} | \Psi_n^{(0)} >= \delta_{mn}$$

Now multiplying equation (3.0.17) by the Hermitian conjugate functions $\Psi_m^{(0)^*}$ and taking into account orthogonality of the eigenfunctions (3.0.15).

Thus,

$$\frac{da_m(t)}{dt} = \frac{-i}{\hbar}\lambda \sum_{n=1}^{\infty} a_n(t) < \Psi_m^{(0)^*} | H^{'} | \Psi_n^{(0)} >,$$

yields,

$$\frac{da_m(t)}{dt} = \frac{-i}{\hbar} \sum_{n=1}^{\infty} a_n(t) \int \Psi_m^{(0)^*} H^{'} \Psi_n^{(0)} d\tau. \tag{3.0.18}$$

where m=1,2,3,4... this is the fundamental result of time-dependent perturbation theory. It gives the rate of change of $m^{th}$ component of the expansion n which describes the true time-varying wave function of the system. The rate of change of the amplitude $a_m$ depends upon the magnitude of the other amplitudes and also upon a set of matrix elements, $\int \Psi_m^{(0)^*} H^{'} \Psi_n^{(0)} d\tau$, which connects by means of $H^{'}$, the pure state $\Psi_m^{(0)}$ with each of the other pure state $\Psi_n^{(0)}$. The time-dependent operator $H^{'}$ causes the amplitude of each of the pure vibrations to change in some definite manner. Since the sums adds over all states with subscript n a set of matrix elements connects the state $\Psi_m^{(0)}$, via the perturbation Hamiltonian $H^{'}$, with all other wave functions $\Psi_n^{(0)}$:

$$i\hbar\frac{da_0}{dt} = a_0 \left\langle \Psi_0^{(0)} | H^{'} | \Psi_0^{(0)} \right\rangle + a_1 \left\langle \Psi_0^{(0)} | H^{'} | \Psi_1^{(0)} \right\rangle + ... + a_j \left\langle \Psi_0^{(0)} | H^{'} | \Psi_j^{(0)} \right\rangle + ...$$

$$i\hbar\frac{da_1}{dt} = a_0 \left\langle \Psi_1^{(0)} | H^{'} | \Psi_0^{(0)} \right\rangle + a_1 \left\langle \Psi_1^{(0)} | H^{'} | \Psi_1^{(0)} \right\rangle + ... + a_j \left\langle \Psi_1^{(0)} | H^{'} | \Psi_j^{(0)} \right\rangle + ...$$

$$\begin{array}{cccc} . & . & . & . \\ . & . & . & . \\ . & . & . & . \end{array}$$

$$i\hbar\frac{da_j}{dt} = a_0 \left\langle \Psi_j^{(0)} | H^{'} | \Psi_0^{(0)} \right\rangle + a_1 \left\langle \Psi_j^{(0)} | H^{'} | \Psi_1^{(0)} \right\rangle + ... + a_j \left\langle \Psi_j^{(0)} | H^{'} | \Psi_j^{(0)} \right\rangle + ... \tag{3.0.19}$$

$$a_j = a_j^0 + \lambda a_j^{'} + \lambda^2 a_j^{''} + .... \tag{3.0.20}$$

where $\frac{da_j}{d\lambda}$=$a_j$ , $\frac{d^2a_j}{d\lambda^2} = a_{j''}$. Substituting equation (3.0.20) into equation (3.0.19) yields for $a_j$:

$$i\hbar\frac{d}{dt}\left(a_j^0 + \lambda a_j^{'} + \lambda^2 a_j^{''}\right) = \left(a_0^0 + \lambda a_0^{'} + \lambda^2 a_0^{''}\right)\left\langle \Psi_j^{(0)}|\lambda H^{'}|\Psi_0^{(0)}\right\rangle + ... + \left(a_j^0 + \lambda a_j^{'} + \lambda^2 a_j^{''}\right)\left\langle \Psi_j^{(0)}|\lambda H^{'}|\Psi_j^{(0)}\right\rangle + .. \quad (3.0.21)$$

If $\lambda = 0$( no perturbation);

$$\frac{da_0^0}{dt} = 0, \frac{da_1^0}{dt} = 0, \frac{da_2^0}{dt} = 0; ...$$

We next consider only the first term correction terms and will neglect the second order correction terms. Equation (3.0.21) yields,

$$i\hbar\frac{da_0^{'}}{dt} = a_0^0\left\langle \Psi_0^{(0)}|H^{'}|\Psi_0^{(0)}\right\rangle + a_1^0\left\langle \Psi_0^{(0)}|H^{'}|\Psi_1^{(0)}\right\rangle + ... + a_j^0\left\langle \Psi_0^{(0)}|H^{'}|\Psi_j^{(0)}\right\rangle + ..$$

$$i\hbar\frac{da_1^{'}}{dt} = a_0^0\left\langle \Psi_1^{(0)}|H^{'}|\Psi_0^{(0)}\right\rangle + a_1^0\left\langle \Psi_1^{(0)}|H^{'}|\Psi_1^{(0)}\right\rangle + ... + a_j^0\left\langle \Psi_1^{(0)}|H^{'}|\Psi_j^{(0)}\right\rangle + ...$$

. . . .
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$$i\hbar\frac{da_j^{'}}{dt} = a_0^0\left\langle \Psi_j^{(0)}|H^{'}|\Psi_0^{(0)}\right\rangle + a_1^0\left\langle \Psi_j^{(0)}|H^{'}|\Psi_1^{(0)}\right\rangle + ... + a_j^0\left\langle \Psi_j^{(0)}|H^{'}|\Psi_j^{(0)}\right\rangle + ... \quad (3.0.22)$$

This set of equations represents an approximate first-order solution of the perturbed problem. The great advantage of this set of equations is that the perturbed wave function can be calculated from the stationary unperturbed parameters $a_j^0$.

For the new amplitude solution $a_n(t)$ equation (3.0.18) becomes,

$$\frac{da_1(t)}{dt} = \frac{-i}{\hbar}a_1(t)\int \Psi_1^{(0)^*}H^{'}\Psi_1^{(0)}d\tau + a_2(t)\int \Psi_1^{(0)^*}H^{'}\Psi_2^{(0)}d\tau + ... + a_k(t)\int \Psi_1^{(0)^*}H^{'}\Psi_k^{(0)}d\tau$$

$$\frac{da_2(t)}{dt} = \frac{-i}{\hbar}a_1(t)\int \Psi_2^{(0)^*}H^{'}\Psi_1^{(0)}d\tau + a_2(t)\int \Psi_2^{(0)^*}H^{'}\Psi_2^{(0)}d\tau + ... + a_k(t)\int \Psi_2^{(0)^*}H^{'}\Psi_k^{(0)}d\tau$$

. . . .
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$$\frac{da_k(t)}{dt} = \frac{-i}{\hbar}a_1(t)\int \Psi_k^{(0)^*} H' \Psi_1^{(0)} d\tau + a_2(t)\int \Psi_k^{(0)^*} H' \Psi_2^{(0)} d\tau + ... + a_k(t)\int \Psi_k^{(0)^*} H' \Psi_k^{(0)} d\tau \tag{3.0.23}$$

From equation (3.0.15) we have,

$$\Psi_1^{(0)}(\overrightarrow{r},t) = \left(\frac{E+m}{2E}\right)^{1/2} \begin{pmatrix} 1 \\ 0 \\ 0 \\ \frac{p_+}{E+m} \end{pmatrix} e^{-iEt}$$

$$\Psi_2^{(0)}(\overrightarrow{r},t) = \left(\frac{E+m}{2E}\right)^{1/2} \begin{pmatrix} 0 \\ 1 \\ \frac{p_-}{E+m} \\ 0 \end{pmatrix} e^{-iEt}$$

$$\Psi_3^{(0)}(\overrightarrow{r},t) = \left(\frac{E+m}{2E}\right)^{1/2} \begin{pmatrix} -1 \\ 0 \\ 0 \\ \frac{p_+}{E+m} \end{pmatrix} e^{iEt}$$

$$\Psi_4^{(0)}(\overrightarrow{r},t) = \left(\frac{E+m}{2E}\right)^{1/2} \begin{pmatrix} 0 \\ 1 \\ \frac{p_-}{E+m} \\ 0 \end{pmatrix} e^{iEt} \tag{3.0.24}$$

Using equation (3.0.24) we obtain a set of differential equations for the unknown functions $a_n(t)$:

For <u>m=1</u>, we have,

$$\frac{da_1(t)}{dt} = i\alpha e\left(a_1(t)\int \Psi_1^{(0)^\dagger} A(t')\Psi_1^{(0)} d\tau \right.$$
$$\left. + a_2(t)\int \Psi_1^{(0)^\dagger} A(t')\Psi_2^{(0)} d\tau + a_3(t)\int \Psi_1^{(0)^*} A(t')\Psi_3^{(0)} d\tau + a_4(t)\int \Psi_1^{(0)^*} A(t')\Psi_4^{(0)} d\tau\right). \tag{3.0.25}$$

Let's evaluate the above integrals:

$$\int \Psi_1^{(0)^\dagger} A(t')\Psi_2^{(0)} d\tau = \int \Psi_1^{(0)^\dagger} A(t')\Psi_4^{(0)} d\tau = 0$$

where as,

$$\int \Psi_1^{(0)^\dagger} \Psi_1^{(0)} d\tau = 1.$$

But,

$$\Psi_1^{(0)^*} \Psi_3^{(0)} = \left[ \left( \frac{E+m}{2E} \right)^{1/2} \begin{pmatrix} 1 \\ 0 \\ 0 \\ \frac{p_+}{E+m} \end{pmatrix} e^{-iEt} \right]^\dagger \left[ \left( \frac{E+m}{2E} \right)^{1/2} \begin{pmatrix} -1 \\ 0 \\ 0 \\ \frac{p_+}{E+m} \end{pmatrix} e^{iEt} \right]$$

Thus,

$$\Psi_1^{(0)^*} \Psi_3^{(0)} = (\frac{E+m}{2E}) \left( \begin{pmatrix} 1 & 0 & 0 & \frac{p_-}{E+m} \end{pmatrix} \begin{pmatrix} -1 \\ 0 \\ 0 \\ \frac{p_+}{E+m} \end{pmatrix} \right) e^{2iEt}$$

where, $p^2 = p_z^2 + p_- p_+$ .

$$\Psi_1^{(0)^*} \Psi_3^{(0)} = (\frac{E+m}{2E}) \left( -1 + \frac{p_- p_+}{(E+m)^2} \right) e^{2iEt}.$$

$$\frac{da_1(t)}{dt} = i\alpha e \left( a_1(t) \int_\infty^{t'} 2A_0 \cos\omega t dt + a_3(t) \frac{E+m}{2E} \left( -1 + \frac{p_- p_+}{(E+m)^2} \right) \right.$$
$$\left. \int_\infty^{t'} 2A_0 \cos\omega t e^{2iEt} dt \right). \tag{3.0.26}$$

where,

$$A(t) = 2A_0 \cos\omega t.$$

is the vector potential of total field corresponding to stationary maxima of the interferential field in equation (3.0.4) (in the area $l << \lambda$).Thus,

$$\frac{da_1(t)}{dt} = i\alpha e \left( a_1(t) 2\frac{A_0}{\omega} \sin\omega t + a_3(t) \frac{E+m}{2E} \left( -1 + \frac{p_- p_+}{(E+m)^2} \right) A_0 \right.$$
$$\left. \int_\infty^{t'} (e^{i\omega t} + e^{-i\omega t}) e^{2iEt} dt \right). \tag{3.0.27}$$

$$\frac{da_1(t)}{dt} = i\alpha e a_1(t) 2\frac{A_0}{\omega} \sin\omega t + i\alpha e a_3(t) \frac{E+m}{2E} \left( -1 + \frac{p_- p_+}{(E+m)^2} \right) A_0$$

$$\int_{\infty}^{t'} e^{i(\omega+2E)t} d\tau + e^{i(2E-\omega)t} dt. \tag{3.0.28}$$

Let's evaluate the integral in equation (3.0.28):

$$\int_{\infty}^{t'} (e^{i(\omega+2E)t} + e^{i(2E-\omega)t}) dt = \frac{1}{i(2E+\omega)} |e^{i(2E+\omega)t}|_{-\infty}^{t'} + \frac{1}{i(2E-\omega)} |e^{i(2E-\omega)t}|_{-\infty}^{t'}.$$

$$= \left( \frac{e^{i\omega t'}}{i(2E+\omega)} + \frac{e^{-i\omega t'}}{i(2E-\omega)} \right) e^{i2Et'}.$$

$$= \frac{2Ee^{i2Et'}}{4E^2-\omega^2} \left( e^{i\omega t'} + e^{-i\omega t'} \right) - \frac{\omega e^{i2Et'}}{4E^2-\omega^2} \left( e^{i\omega t'} - e^{-i\omega t'} \right).$$

$$\int_{\infty}^{t'} (e^{i(\omega+2E)t} + e^{i(2E-\omega)t}) dt = \frac{e^{i2Et'}}{4E^2-\omega^2} \left[ 4E\cos\omega t' - 2i\omega \sin\omega t' \right]. \tag{3.0.29}$$

Therefore we can write the final expression by substituting equation (3.0.29) into equation (3.0.28):

$$\frac{da_1(t)}{dt} = i\alpha e \left( a_1(t) 2\frac{A_0}{\omega} \sin\omega t + a_3(t) \frac{E+m}{2E} \left( -1 + \frac{p_- p_+}{(E+m)^2} \right) \frac{A_0 e^{i2Et'}}{4E^2-\omega^2} \right.$$

$$\left. [4E\cos\omega t' - 2i\omega \sin\omega t'] \right). \tag{3.0.30}$$

Similarly, for m=4, we get,

$$\frac{da_4(t)}{dt} = i\alpha e \left( a_1(t) \int \Psi_4^{(0)^\dagger} A(t') \Psi_1^{(0)} d\tau + a_2(t) \int \Psi_4^{(0)^\dagger} A(t') \Psi_2^{(0)} d\tau + a_3(t) \right.$$

$$\left. \int \Psi_4^{(0)^*} A(t') \Psi_3^{(0)} d\tau + a_4(t) \int \Psi_4^{(0)^*} A(t') \Psi_4^{(0)} d\tau \right). \tag{3.0.31}$$

We can cancel out the two integrals from the above expression,

$$\int \Psi_4^{(0)^\dagger} A(t') \Psi_1^{(0)} d\tau = \int \Psi_4^{(0)^\dagger} A(t') \Psi_3^{(0)} d\tau = 0.$$

Now,

$$\int \Psi_4^{(0)^\dagger} \Psi_4^{(0)} d\tau = 1.$$

So we have only,

$$\frac{da_4(t)}{dt} = i\alpha e \left( a_4(t) \int \Psi_4^{(0)^\dagger} A(t') \Psi_4^{(0)} d\tau + a_2(t) \int \Psi_4^{(0)^\dagger} A(t') \Psi_2^{(0)} d\tau \right). \tag{3.0.32}$$

$$\Psi_4^{(0)^*}\Psi_2^{(0)} = \left[\left(\frac{E+m}{2E}\right)^{1/2}\begin{pmatrix}0\\1\\\frac{p_-}{E+m}\\0\end{pmatrix}e^{iEt}\right]^{\dagger}\left[\left(\frac{E+m}{2E}\right)^{1/2}\begin{pmatrix}0\\1\\\frac{p_-}{E+m}\\0\end{pmatrix}e^{-iEt}\right].$$

Thus,

$$\Psi_4^{(0)^*}\Psi_2^{(0)} = (\frac{E+m}{2E})\left(\begin{pmatrix}0 & 1 & \frac{p_+}{E+m} & 0\end{pmatrix}\begin{pmatrix}0\\1\\\frac{p_-}{E+m}\\0\end{pmatrix}\right)e^{-2iEt}.$$

$$\Psi_4^{(0)^*}\Psi_2^{(0)} = (\frac{E+m}{2E})\left(1+\frac{p_+p_-}{(E+m)^2}\right)e^{-2iEt}.$$

$$\frac{da_4(t)}{dt} = i\alpha e\left(a_4(t)\int_{\infty}^{t'}2A_0\cos\omega t dt + a_2(t)\frac{E+m}{2E}\left(1+\frac{p_+p_-}{(E+m)^2}\right)\right.$$
$$\left.\int_{\infty}^{t'}2A_0\cos\omega t e^{-2iEt}dt\right). \tag{3.0.33}$$

Finally, we obtain a set of differential equation for the unknown function $a_4(t)$ as,

$$\frac{da_4(t)}{dt} = i\alpha e\left(a_4(t)2\frac{A_0}{\omega}\sin\omega t + a_2(t)\frac{E+m}{2E}\left(1+\frac{p_+p_-}{(E+m)^2}\right)\frac{A_0e^{-i2Et'}}{\omega^2-4E^2}\right.$$
$$\left.[4E\cos\omega t' + 2i\omega\sin\omega t']\right) \tag{3.0.34}$$

A similar system of equations is also obtained for the amplitudes $a_2(t)$ and $a_3(t)$. According to the assumed Dirac model we have the following initial conditions:

$$|a_3(-\infty)|^2 = |a_4(-\infty)|^2 = 1, |a_1(-\infty)|^2 = |a_2(-\infty)|^2 = 0. \tag{3.0.35}$$

i.e., before the interaction (the wave are turned on adiabatically at t=$-\infty$) there are only the electrons with negative energies. From the condition of conservation of the norm we have:

$$\sum_{n=1}^{4}|a_n(t)|^2 = 2 \tag{3.0.36}$$

which expresses the equality of the number of created electrons and positrons whose creation probability is, respectively ,$|a_{1,2}(t)|^2$ and $1-|a_{3,4}(t)|^2$.

If we want to allow for transitions between one energy level and another we must introduce a time-dependent potential (quantum dynamics) which simplifies the set of equations for the amplitudes. The new amplitudes $c_1(t)$, $c_4(t)$ satisfy the same initial condition (3.0.35) $|c_1(-\infty)| = 0$, $|c_4(-\infty)| = 1$. For the new amplitudes $c_1(t)$, $c_4(t)$ we have the following set of equations:

$$i\hbar\frac{\partial}{\partial t}|\alpha, t_o; t >_I = H_I|\alpha, t_o; t >_I . \tag{3.0.37}$$

where,

$$|\alpha, t_o; t >_I = \sum_n c_n(t)|n > .$$

$$i\hbar\frac{\partial}{\partial t} < n|\alpha, t_o; t >_I = \sum_m < n|H_I|m >< m|\alpha, t_o; t >_I . \tag{3.0.38}$$

where,

$$c_n(t) =< n|\alpha, t_o; t >_I$$

Thus,

$$i\hbar\frac{d}{dt}c_n(t) = \sum_m H_{nm}e^{-i\omega_{nm}t}c_m(t). \tag{3.0.39}$$

where,

$$\omega_{nm} = \frac{E_n - E_m}{\hbar} = -\omega_{mn}$$

Finally we get,

$$\frac{d}{dt}c_n(t) = \frac{-i}{\hbar}\sum_m H_{nm}e^{-i\omega_{nm}t}c_m(t). \tag{3.0.40}$$

Let's use for m=1,2,3,4...

$$\frac{dc_1(t)}{dt} = \frac{-i}{\hbar}\left[H_{11}e^{-i\omega_{11}t}c_1(t) + H_{12}e^{-i\omega_{12}t}c_2(t) + H_{13}e^{-i\omega_{13}t}c_3(t) + H_{14}e^{-i\omega_{14}t}c_4(t) + ...\right].$$

$$\frac{dc_4(t)}{dt} = \frac{-i}{\hbar}\left[H_{41}e^{i\omega_{41}t}c_1(t) + H_{42}e^{i\omega_{42}t}c_2(t) + H_{43}e^{i\omega_{43}t}c_3(t) + H_{44}e^{i\omega_{44}t}c_4(t) + ...\right]. \tag{3.0.41}$$

where, $\Delta_n = \delta_f + \delta_n$ is the resulting detuning.

For the new amplitudes $c_1(t)$ and $c_4(t)$ we have the following set of equations:

$$\frac{dc_1(t)}{dt} = \frac{-i}{\hbar}H_{14}e^{-i\omega_{14}t}c_4(t).$$

$$\frac{dc_4(t)}{dt} = \frac{-i}{\hbar} H_{41} e^{i\omega_{41}t} c_1(t). \tag{3.0.42}$$

Because of space homogeneity the generalized momentum of a particle is conserved. So that, the real transitions in the field occur from a -E negative energy level to the positive +E energy level (excitation of the Dirac vacuum) by the absorption of n-photons. Consequently, the multiphoton probabilities of $e^-$, $e^+$ pair production will have maximal values for the resonant transitions 2E=n$\omega$. The latter just is the conservation law of the pair production process at which both electrons and positrons will be created back to back according to zero total momentum $p_{e^-} + p_{e^+}$=0. So, we can utilize the resonant approximation, as in a two-level atomic system in the monochromatic wave filed. Generally, in this approximation, at detuning of resonance $\Delta_n = 2E - n\omega \ll \omega$, we have the following set of equations for the certain n-photon transition amplitudes $c_1^{(n)}(t)$ and $c_4^{(n)}(t)$:

$$c_1^{(n)}(t) = \frac{i|f_n|}{\Omega_n} e^{i\frac{\delta_n}{2}t} \sin(\Omega_n t), \tag{3.0.43}$$

$$c_4^{(n)}(t) = e^{-i\frac{\delta_n}{2}t} [\cos(\Omega_n t) + \frac{i\Delta_n}{2\Omega_n} \sin(\Omega_n t)]. \tag{3.0.44}$$

Therefore we get the set of differential equation;

$$\frac{dc_1^{(n)}(t)}{dt} = i f_n exp(i\Delta_n t) c_4^{(n)}(t),$$

$$\frac{dc_4^{(n)}(t)}{dt} = i f_n exp(-i\Delta_n t) c_1^{(n)}(t). \tag{3.0.45}$$

where,

$$f_n = \frac{E}{2p_y} \left(1 - \frac{p_y^2}{E^2}\right)^{1/2} n\omega J_n \left(4\xi \frac{m}{E} \frac{p_y}{\omega}\right). \tag{3.0.46}$$

and $J_n$ is the ordinary Bessel function. We include in equation (3.0.46) the quantity

$$\xi = \frac{e|E_o|}{mc\omega}, \tag{3.0.47}$$

which is the dimensionless relativistic invariant parameter of the wave intensity ($E_o$ is the amplitude of the electric field strength of the one incident wave).

which is valid for the slow varying function $c_1^{(n)}(t)$ and $c_4^{(n)}(t)$, i.e., at the following condition:

$$\left|\frac{c_{1,4}^{(n)}(t)}{dt}\right| << |c_{1,4}^{(n)}(t)|\omega. \tag{3.0.48}$$

The solution of equation (3.0.45) has the following form:

$$|c_1^{(n)}(t)|^2 = \frac{f_n^2}{\Omega_n^2}\sin^2(\Omega_n\tau),$$

$$|c_4^{(n)}(t)|^2 = 1 - \frac{f_n^2}{\Omega_n^2}\sin^2(\Omega_n\tau). \tag{3.0.49}$$

where $\Omega_n = \sqrt{f_n^2 + \frac{\Delta_n^2}{4}}$ is the "Rabi frequency" of Dirac vacuum at the interaction with a periodic e.m field and $\tau$ is the interaction time. As is seen from Equation (3.0.49) with this frequency the probability of pair production oscillates in the standing wave field during the whole interaction time similar to Rabi oscillations in two level-atomic systems. In this case the Rabi frequency has a nonlinear dependence on the amplitudes of the opposite e.m wave fields. Considerable number of electron-positron pairs can be produced by a proper choice of intensity and duration of laser pulses.

# Chapter 4

# Multiphoton Probabilities of Nonlinear Production of Electron-Positron Pair

According to equations (3.0.30), and (3.0.49), for the transition amplitude of the electron from the Dirac vacuum to the state with positive energy (in a definite spinor state) in the field we have [9];

$$|a_1(t)|^2 = |c_1(t)|^2. \tag{4.0.1}$$

To obtain the probability amplitude for the production of electron and positrons after the wave has been turned off we introduce a small detuning of the resonance ($2E = n\omega\hbar + \triangle_n$), corresponding to an n-photon transition: $2E = n\omega\hbar + \hbar\Gamma(\Gamma << \omega)$.

## 4.1 The Probability of the n-Photon $e^-$,$e^+$ Pair Production

The production probability of the $e^-$,$e^+$ pair, summed over the spin states, is determined by the quantity [9].

$$|a_1(t)|^2 + |a_2(t)|^2 = 2|a_1(t)|^2 \equiv 2|c_1(t)|^2. \tag{4.1.1}$$

The differential probability of a n-photon $e^-$,$e^+$ pair production process per unit time and unit space volume, $Vd^3\overrightarrow{p}/(2\pi\hbar)^3$ (the normalization volume V=1) in the center-of-mass

frame of the produced particles is given by,

$$d\omega_n^c = \frac{dw_n^c(t)}{t} = 2\lim_{t\to\infty}\frac{|c_1(t)|^2}{t}\frac{d^3\overrightarrow{p}}{(2\pi\hbar)^3}. \tag{4.1.2}$$

$$d\omega_n^c = 2|c_1^{(n)}(t)|^2 = \frac{2|f_n|^2}{\Omega^2}\sin^2(\Omega\tau). \tag{4.1.3}$$

where,

$$f_n = \frac{E}{2p_y}\left(1-\frac{p_y^2}{E^2}\right)^{1/2} n\omega J_n\left(\frac{4\xi_0 m p_y}{E\omega}\right). \tag{4.1.4}$$

Thus,

$$|f_n|^2 = \frac{E^2}{4p_y^2}\left(1-\frac{p_y^2}{E^2}\right) n^2\omega^2 J_n^2\left(\frac{4\xi_0 m p_y}{E\omega}\right). \tag{4.1.5}$$

So,

$$d\omega_n^c = \frac{E^2}{2p^2\cos^2\theta}\left(1-\frac{p^2\cos^2\theta}{E^2}\right) n^2\omega^2 J_n^2\left(\frac{4\xi_0 m p\cos\theta}{E\omega}\right)\frac{\sin^2(\Omega\tau)}{\Omega_n^2}. \tag{4.1.6}$$

where $\theta$ is the angle between the directions of the momentum of produced electrons (positrons) and the amplitude of the total field electric strength of one incident wave. The condition of applicability of this resonant approximation (3.0.48) is equivalent to the condition:

$$\Omega_n \ll \omega. \tag{4.1.7}$$

Let us consider the case of short interaction time, when

$$\Omega_n t \ll 1. \tag{4.1.8}$$

In the case we can determine a probability of multiphoton pair production per unit time according to the following definition of the Dirac $\delta$-function.

$$\frac{\sin^2(\Omega_n\tau)}{\Omega_n^2} \to 2\pi\hbar\tau\delta(2E-n\hbar\omega). \tag{4.1.9}$$

We obtain

$$d\omega_n^c = \frac{E^2}{2p^2\cos^2\theta}\left(1-\frac{p^2\cos^2\theta}{E^2}\right) n^2\omega^2 J_n^2\left(\frac{4\xi_0 m p\cos\theta}{E\omega}\right)\frac{\sin^2(\Omega\tau)}{(\Omega_n)^2}\frac{V d^3\overrightarrow{p}}{(2\pi)^3}. \tag{4.1.10}$$

Substituting equation (4.1.9) in equation (4.1.10), the differential probability becomes,

$$d\omega_n^c = \frac{1}{2\pi^2} f_n^2 \delta(2E - n\omega) V d^3\overrightarrow{p}. \tag{4.1.11}$$

Since the considered field is only time dependent [see equation (3.0.7)], thus, we can utilize the resonant approximation, as in a two-level atomic system in the monochromatic wave field. Generally, in this approximation, at detuning of resonance $\Delta_n = 2E - n\omega << \omega$. Thus,

$$d\omega_n^c = n^2\omega^2 \left(\frac{E^2 - p^2\cos^2\theta}{8\pi^2 p^2 cos^2\theta}\right) J_n^2 \left(\frac{4\xi_0 mp\cos\theta}{E\omega}\right) \delta(2E - n\omega) V d^3\overrightarrow{p}. \tag{4.1.12}$$

$\xi_0 = \frac{e|E_0|}{mc\omega}$, is the dimensionless relativistic invariant parameter of the wave intensity ($E_0$ is the amplitude of the electric field strength of one incident wave).

From the above relation we have,

$$d\omega_n^c = \frac{n^2\omega^2(E^2 - p^2\cos^2\theta)}{4\pi^2 p^2\cos^2\theta} J_n^2 \left(\frac{4e|E_0|p\cos\theta}{E\omega^2}\right) \delta(E - \frac{n\omega}{2}) V d^3\overrightarrow{p}. \tag{4.1.13}$$

where V=1. In the center of mass frame if one regards ($\overrightarrow{p}$) as the momentum of electron emitted from Dirac vacuum then ($-\overrightarrow{p}$) will be the momentum of positron.

Thus we obtain,

$$d\omega_n^c = \frac{n^2\omega^2(E^2 - p^2 + p^2\sin^2\theta)}{4\pi^2 p^2\cos^2\theta} J_n^2 \left(\frac{4e|E_0|p\cos\theta}{E\omega^2}\right) \delta(E - \frac{n\omega}{2}) d^3\overrightarrow{p}. \tag{4.1.14}$$

As is seen from equation (4.1.14), the pair production probability decreases highly in the directions perpendicular to the field ($p'_y = 0$), and the obtained approximate nonlinear solution describes the process behavior well at the angle not too close to $\pi/2$.

We know from the relation of relativistic energy, that is;

$$E^2 = (pc)^2 + (mc^2)^2;$$

where,

$$p = \left(\frac{n^2\omega^2 - 4m^2}{4}\right)^{1/2} \tag{4.1.15}$$

Thus,

$$d\omega_n^c = \frac{n^2\omega^2(m^2c^4 + p^2\sin^2\theta)}{4\pi^2 p^2\cos^2\theta} J_n^2\left(\frac{4e|E_0|p\cos\theta}{E\omega^2}\right)\delta(E - \frac{n\omega}{2})d^3\overrightarrow{p}. \tag{4.1.16}$$

We have,

$$\delta(E - \frac{n\omega}{2})d^3\overrightarrow{p} = \delta(E - E')d^3\overrightarrow{p} = |\overrightarrow{p}|Ed\Omega = |\overrightarrow{p}|\frac{n\omega}{2}d\Omega$$

The multiphoton probabilities of $e^-$,$e^+$ pair production will have maximal values for the resonant transitions:

$$\Delta_n = 2E - n\omega = 0$$

$$n = 2E/\omega \tag{4.1.17}$$

where n is the number of pairs of $e^-$,$e^+$ produced at the values of q.

Thus,

$$d\omega_n^c = \frac{n^2\omega^2(m^2c^4 + p^2\sin^2\theta)}{4\pi^2 p^2\cos^2\theta} J_n^2\left(\frac{4e|E_0|p\cos\theta}{\frac{n\omega^3}{2}}\right)|\overrightarrow{p}|\frac{n\omega}{2}d\Omega. \tag{4.1.18}$$

$$d\omega_n^c = \frac{n^3\omega^3(m^2c^4 + p^2\sin^2\theta)}{8\pi^2 p^2\cos^2\theta} J_n^2\left(\frac{4e|E_0|p\cos\theta}{\frac{n\omega^3}{2}}\right)|\overrightarrow{p}|d\Omega. \tag{4.1.19}$$

Analogously one can describe the multiphoton pair production process in a wiggler by a superstrong laser pulse of relativistic intensities. The two interference waves are formed which propagate with the phase velocities $V_{ph} > c$ and $V_{ph} < c$ . According to the conditions (1.0.1) and (1.0.2) the wave propagating with the phase velocity $V_{ph} > c$ will be responsible for the pair production process [9].

## 4.2 The Angular Distribution of a n-Photon Differential Probability of the $e^-$,$e^+$ Pair Production

The angular distribution of a n-photon partial probability of the $e^-$,$e^+$ pair production for the most probable number of photons n=$N \backsimeq 2\sqrt{2}m/\omega$ is presented in Fig.4.1.

By integrating over the electron (positron) energy, we obtain the angular distribution of the n-photon differential probability density (in the space volume V=1) of created

electrons (positrons).

Substituting equation (4.1.15) in equation (4.1.19), we get;

$$\frac{d\omega_n^c}{d\Omega} = \frac{n\omega}{8\pi^2} f_n^2 (n^2\omega^2 - 4m^2)^{1/2}. \tag{4.2.1}$$

Thus,

$$\frac{d\omega_n^c}{d\Omega} = \frac{n^3\omega^3}{32\pi^2} \left( \frac{m^2 + \sin^2\theta(n^2\omega^2 - 4m^2)/4}{(\frac{n^2\omega^2 - 4m^2}{4})^{1/2}\cos^2\theta} \right) J_n^2 \left( \frac{4e|E_0| \left( \frac{n^2\omega^2 - 4m^2}{4} \right)^{1/2} \cos\theta}{\frac{n\omega^3}{2}} \right). \tag{4.2.2}$$

which can in turn be expressed as,

$$\frac{d\omega_n^c}{d\Omega} = \frac{n^3\omega^3}{64\pi^2} \left( \frac{4m^2\cos^2\theta + n^2\omega^2\sin^2\theta}{\cos^2\theta(n^2\omega^2 - 4m^2)^{1/2}} \right) J_n^2 \left( \frac{4e|E_0| \left(n^2\omega^2 - 4m^2\right)^{1/2} \cos\theta}{n\omega^3} \right). \tag{4.2.3}$$

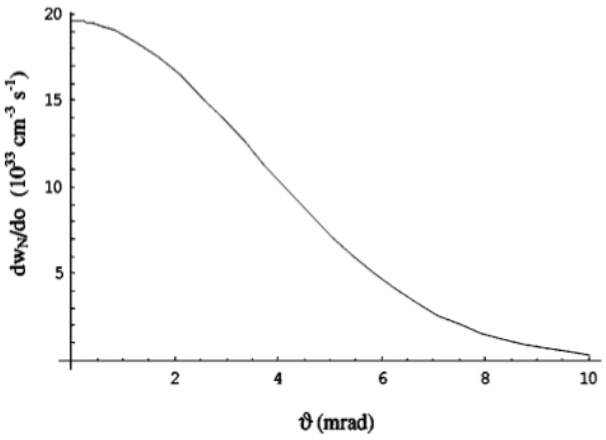



Figure 4.1: [9] The angular dependence of the partial differential probability of n-photon electron-positron production per second, per $cm^3$ space volume with laser frequency 1.17eV and intensity $1.35 \times 10^{22} w/m^2$, at which the most probable energy of created particles is $2^{1/2}m$.

.

The curve for this dependence shows that the probability of $N$-photon pair production process has a peak at the angle $\theta = 0$ in agreement with the approach [10] developed for the solution of this problem (assumed approximation $l \ll \lambda$).

By the appropriate transformations the processes of $e^-$,$e^+$ pair production in these e.m

field configurations can be reduced to the considered pair production process (as in the case of plasma) in this thesis. But, when we put the value of intensities; $|E_0|^2 = \frac{(mc\omega)^2}{e^2}\xi^2$ (since $\xi = \frac{e|E_0|}{mc\omega}$).

Finally we have from equation (4.2.3),

$$\frac{d\omega_n^c}{d\Omega} = \frac{n^3\omega^3(n^2\omega^2\sin^2\theta + 4m^2cos^2\theta)}{64\pi^2\cos^2\theta[n^2\omega^2 - 4m^2]^{1/2}} J_n^2\left(\frac{4\xi m\left(1-\frac{4m^2}{n^2\omega^2}\right)^{1/2}\cos\theta}{\omega}\right). \tag{4.2.4}$$

where $d\Omega = \sin\theta d\theta d\varphi$ is the differential solid angle in the Lab frame due to the linear polarization of the wave. In the case of circular polarization the probability distribution has azimuthal symmetry.

The total angular distribution for multiphoton pair production,

$$\frac{dw}{d\Omega} = \sum_{n=n_0}^{\infty} \frac{dw_n}{d\Omega}, \tag{4.2.5}$$

(where $n = 2mc^2/\hbar\omega$ is the threshold number of photons for the pair production process to occur). To obtain the total probability density of the $e^-$,$e^+$ pair production i.e the number of pairs created per unit time in the interaction volume (v), we shall integrate the expression (4.2.5) over solid angle taking in to account (4.2.4). For the integration over $\theta$ we turn to the new variable y=cos$\theta$.

Putting,

$$d\Omega = \sin\theta d\theta d\varphi;$$

i.e,

$$d\Omega = dyd\varphi$$

We get,

$$d\omega_n^c = \frac{n^3\omega^3(n^2\omega^2 - 4p^2\cos^2\theta)}{64\pi^2\cos^2\theta[n^2\omega^2 - 4m^2]^{1/2}} J_n^2(y)(dy)d\varphi. \tag{4.2.6}$$

Integrating over $d\Omega$, we get;

$$\int_0^{\omega} d\omega_n^c = \int \frac{n^5\omega^5 - n^3\omega^3\cos^2\theta 4p^2}{64\pi^2\cos^2\theta p} J_n^2\left(\frac{4\xi m\left(1-\frac{4m^2}{n^2\omega^2}\right)^{1/2}\cos\theta}{\omega}\right)\sin\theta d\theta d\varphi. \tag{4.2.7}$$

The angle $\varphi$ varies from 0 to $2\pi$, while $\theta$ varies from 0 to $\theta_{max}$, which is determined from the energy and momentum conservation laws.

Thus we can express $\omega_n^c$ as,

$$\omega_n^c = \sum_{n=n_0}^{\infty} \frac{n^5\omega^5 - n^3\omega^3\cos^2\theta 4p^2}{64\pi^2\cos^2\theta p} J_n^2\left(\frac{4\xi m\left(1-\frac{4m^2}{n^2\omega^2}\right)^{1/2}\cos\theta}{\omega}\right)\int_0^1 dy\int_0^{2\pi} d\varphi. \quad (4.2.8)$$

Thus,

$$\omega_n^c = \sum_{n=n_0}^{\infty} \frac{n^5\omega^5}{32\pi p}\int\left[\frac{1}{y^2} - \frac{4p^2}{n^2\omega^2}\right] J_n^2\left(\frac{4\xi m}{\omega}(1-\frac{4m^2}{n^2\omega^2})^{1/2}y\right) dy. \quad (4.2.9)$$

As for the angular distribution of the probability of n-photon pair production in the Lab frame, it can be obtained from the expression $d\omega_n^c(t)$ (see equation 4.1.2) [10].

## 4.3 The Total Probability Density of the $e^-, e^+$ Pair Production in the Field of Strong Opposite E.M Waves in Vacuum

Taking in to account the results of the integrations over the solid angle (4.2.9) we obtain the total probability density per unit time per unit volume of the $e^-, e^+$ pair production in the field of strong opposite e.m waves in vacuum $\omega = \sum_{n=n_o}\omega_n$ [10] as,

$$w = \sum_{n=n_0}^{\infty} \frac{n^5\omega^5}{32\pi p}\left[\left[\frac{2Z_0^2}{4n^2-1} - 1\right] J_n^2(Z_0) + \frac{Z_0^2 J_{n-1}^2(Z_0)}{2n(2n-1)} + \frac{Z_0^2 J_{n+1}^2(Z_0)}{2n(2n+1)} - \frac{4p^2}{n^2\omega^2}\frac{Z_0^{2n}}{(2n+1)(n!)^2 2^{2n}}\right.$$

$$\left. *F_3\left(n+\frac{1}{2}, n+\frac{1}{2}; n+1, 2n+1, n+\frac{3}{2}; -Z_0^2\right)\right]. \quad (4.3.1)$$

Here, $F_3\left(n+\frac{1}{2}, n+\frac{1}{2}; n+1, 2n+1, n+\frac{3}{2}; -Z_0^2\right)$ is the generalized hypergeometric function, and

$$Z_0 = \left(\frac{4m\xi}{\omega}\right)\left(1-\frac{4m^2}{n^2\omega^2}\right)^{1/2}.$$

The quantity $\omega$ is a relativistic invariant, and so equation (4.3.1) defines the pair production probability in the Lab frame as well, it can be obtained from the expression $\mathrm{d}\omega_c^n(t)$ (see equation (4.1.2)).

The total number of pairs $N$ created for a given laser characteristics can be estimated by [10],

$$N \sim \omega V \tau. \tag{4.3.2}$$

where $V \sim \sigma^2 l$ is the space-volume, $\sigma$ is the cross section radius, $l << \lambda$ as stated above and $\tau$ is the interaction time. For focused optical lasers in the diffraction limit $\sigma \sim 10^{-3} cm$, $\lambda = 1.074 \times 10^{-6} m$ and $\tau \sim 10^{-14} s$. The angular distribution of the number of pairs created from an n-photon process is,

$$\frac{dN_n}{d\Omega} = \frac{d\omega_n}{d\Omega} V \tau. \tag{4.3.3}$$

where $N = \omega_n V \tau$ is the number of pairs created from that process. For a rough estimation we take $\sigma \sim 10^{-5}$m, $\tau \sim 10^{-14} s$, and $l \sim 0.1\lambda/k$ (where k=1,3,5 for the the corresponding harmonics) demonstrates the fact that when going to higher harmonics the area close to the antinodes that the pair creation essentially happens decreases. Performing a numerical summation over n for above mentioned values of parameters in the formula (4.3.1), for a number of pairs created per unit time per unit volume we obtain: w$\sim 2 \times 10^{34} cm^{-3} s^{-1}$. In actual cases the total number of the created particles depends on space volume of laser beam interaction, which will be determined by a laser beam diameter $\sigma$ in the cross-section and durations of pulses $\tau$ [10].

## 4.4 Numerical Data Analysis Approximation for Number of Pair Production in Vacuum

The numerical data analysis of the above work is now presented. Thus also constitutes the contribution on the author in this thesis. The physical interpretation of the numerical computations that will follows, is played by the function $f_n$ (Rabi frequency on exact resonance). For a given value of the $\xi$ and n, as can be seen from equation (4.1.4).

$$n = 2E/\hbar\omega = 2qmc^2/\hbar\omega, q \geq 1, \tag{4.4.1}$$

where $E = q(2mc^2)$ is the total incident energy of radiation to excite $e^-$,$e^+$ pair. We take the variation of the number q (is units of rest energy ) from $1.4 \leq q \leq 1.43$. At $\theta = 0$, a suitable expression for $f_n$ can be obtained from (4.1.4) with $E = mc^2$, $p = (1/c)\sqrt{E^2 - m^2c^4}$, and using the asymptotic behavior of the Bessel function $J_n(x)$ at $x \simeq n$. In fact, as can be seen from (4.4.1) from optical lasers where $\omega$ is very small (of the order of eV), n is very large and as $\xi \lesssim 1$, the argument of the Bessel function in (4.1.4), now becomes $x = \frac{2n\xi}{q}\left(1 - \frac{1}{q^2}\right)^{\frac{1}{2}}$. Thus to obtain executable numerical computations, we shall from now on adopt this asymptotic behavior of the Bessel function by writing,

$$J_n(x) = J_n(n \sec ha) = \frac{1}{\sqrt{2\pi n \tanh a}} e^{(n \tanh a - na)}. \tag{4.4.2}$$

where $a = \sec h^{-1}(\frac{2\xi}{q}(1 - \frac{1}{q^2})^{\frac{1}{2}})$. The range change of $E_o$ (and thus of $\xi$) is very small even for higher harmonics because of the extreme sensitivity of the Bessel function and its approximation in $\xi$.

Then $f_n$ is given by [9,10],

$$f_n = \frac{1}{4}(q^2 - 1)^{-1/2} n\hbar\omega \frac{exp(n \tanh a - na)}{\sqrt{2\pi n \tanh a}}.$$

Thus we get,

$$f_n = \frac{1}{4}(q^2 - 1)^{-1/2} \left[ n\omega \frac{e^{nTanh[ArcSech[\frac{2\xi}{q}(1-\frac{1}{q^2})^{1/2}]] - nArcSech[\frac{2\xi}{q}(1-\frac{1}{q^2})^{1/2}]}}{\sqrt{2\pi n Tanh[ArcSech[\frac{2\xi}{q}(1 - \frac{1}{q^2})^{1/2}]]}} \right]. \tag{4.4.3}$$

Using equation (4.4.1) and (4.4.3), the envelop of $f_n$ as a function of q can be plotted for fixed values of $\xi$. This allows us to investigate the envelop of $f_n$, from the electric field strength or frequency of radiation or from both points of view. The most probable process corresponds to the peaks of the curves which will be labeled with the triplet $(\xi, q_p, n_p)$.

We now give the number of photons needed to produce one electron-positron pair for different values of q.

For q=1.41:

$$E_1 = (1.41)(0.511MeV) \sim 0.721MeV.$$

$$n_1 = \frac{2qmc^2}{\hbar\omega} = \frac{2(1.41)(0.511MeV)}{1.17eV} = 1.232 \times 10^6.$$

For q=1.42:

$$E_2 = (1.42)(0.511MeV) \sim 0.726MeV.$$

$$n_2 = \frac{2qmc^2}{\hbar\omega} = \frac{2(1.42)(0.511MeV)}{3.51eV} = 4.135 \times 10^5.$$

For q=1.43:

$$E_3 = (1.43)(0.511MeV) \sim 0.731MeV.$$

$$n_3 = \frac{2qmc^2}{\hbar\omega} = \frac{2(1.43)(0.511MeV)}{5.85eV} = 2.498 \times 10^5. \tag{4.4.4}$$

In equation (4.4.4), $E_1$, $E_2$, $E_3$ are the total excitation energies and $n_1$, $n_2$, $n_3$ are the number of photons needed at values $q_1$=1.41, $q_2$=1.42 and $q_3$=1.43 respectively to produce one $e^-$, $e^+$ pair.

**Table 4.1** The number of photons for the production of one $e^-$, $e^+$ pair.

| $\omega$ | q | n |
|---|---|---|
| $1.17eV$ | 1.41 | $1.232 \times 10^6$ |
| $3(1.17eV)$ | 1.42 | $4.135 \times 10^5$ |
| $5(1.17eV)$ | 1.43 | $2.498 \times 10^5$ |

In particular, emphasis has been given in the implementation of higher harmonics such as $3\omega$, $5\omega$ while the electric field strengths required, are obtained by increasing the laser energy rather than focusing to the diffraction limit ($\sigma \sim \lambda$). The need of higher harmonics is dictated by the limitation imposed by the upper value of electric filed $E_o$ of the fundamental due to the condition $\xi = \frac{eE_o}{mc\omega} \lesssim 1$. However for the developed approximation, the values of $\xi$ where these occur are not applicable as $\xi > 1$. In order to work with $\xi \lesssim 1$ but increases the $E_o$ higher $\omega$ values are necessarily.

# Chapter 5

# Conclusion

In this thesis we have included the pair production process as a mathematical preliminary. The main work in the thesis is detailed in chapter 3 and 4. In chapter 3 we have calculated nonlinear solution of Dirac equation for vacuum electrons in the strong e.m field. In chapter 4 which contains the main contribution of author in this thesis the multiphoton probabilities of nonlinear pair production which contains the probability of the n-photon, the differential probability of n-photon, the angular distribution of n-photon and the total probability density of the n-photon $e^-$,$e^+$ pair production. The probability of the n-photon $e^-$,$e^+$ pair production with a certain energy E which is a periodic function of the interaction time is similar to the Rabi oscillations in atomic systems. In this case the Rabi frequency has a nonlinear dependence on the amplitudes of the oppositely directed to the laser beams. Considerable number of electron-positron pair can be produced by a proper choice of intensity and duration of laser pulses. A quite interesting case was when dealing with higher harmonics (increase of the laser frequency) is to investigate the number of photons, as we change from $\omega$ to $2\omega$, $3\omega$ can lead $E_0$ to increase by the same amount as $\omega$. Therefore passing to higher harmonics the value of n decreases rapidly leading to produce one electron-positron pair, while, the applicability of the incident radiation (q) will be in the range of 1.4-1.43.

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**Declaration**

This Thesis is my original work, has not been presented for a degree in any other University and that all the sources of material used for the thesis have been dully acknowledged.

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**Place and time of submission: Addis Ababa University, April 2012**

This Thesis has been submitted for examination with my approval as University advisor.

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