SOLUTIONS OF THE DIRAC EQUATION IN THE PRESENCE OF A UNIFORM BACK-GROUND MAGNETIC FIELD

By

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Dedicated To My Family.

Table of Contents

Abstract

This thesis which is entitled as "Solutions of the Dirac Equation in the Presence of a Uniform Back-ground Magnetic Field" tried to find exact solutions of the Dirac equation under the assumption of some gauge conditions and taking the field as a uniform field. Since theoretical evidences indicates that most of the bodies in our universe posses a complicated magnetic field, we need to find the solutions of the Dirac equation in the presence of this field, by assuming the field as being uniform. To do this we began by considering the Dirac equation and we tried to find its exact solution. To find the exact solutions, we used a gauge condition,

$$\vec{A} = A_x i = -yBi$$

where B is the magnitude of the magnetic field. In addition some quantum mechanical concepts was used. Once the Dirac equation is exactly solved, we can proceed to calculate elementary particle decays and scattering cross-sections in the presence of a magnetic field. In this thesis particularly we focussed on the nature of the solutions of the Dirac equation in the presence of a uniform magnetic field and orthogonality relations.

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Introduction

A successful relativistic wave equation was first obtained by the British physicist P.A.M Dirac in 1928, who made a fundamental contribution to the early development of both quantum mechanics and quantum electrodynamics. He formulated the Dirac equation which describes the behavior of fermions and which led to the prediction of the presence of anti mater. In the same way as the schrodinger equation can not be derived from the classical mechanics because it is essentially new physics, any relativistic wave equation can only be guessed by the process of induction, and its truth or other wise must be established by testing its consequence against experiment[1].

Dirac went off to search for an equation that was consistent with special theory of relativity and avoided the non-positive definiteness of probability density, ρ. An equation which is linear in time derivative that is no more ρ less than zero, an equation that can reproduce the energy momentum relation and an equation which is relativistically covariant that is whose form is preserved under a lorentz boost are the demands on an equation describing a relativistic theory of quantum mechanics[2].

The Dirac equation is a relativistic quantum mechanical wave equation formulated by Paul Dirac, and provides a description of elementary spin -$\frac{1}{2}$ particles, such as electrons, consistence with the principle of both quantum mechanics and the theory of special relativity. The equation also demands the existence of anti-particles and actually predated their experimental discovery, making the discovery of the positron, the anti particle of the electron, which is one of the greatest triumphs of modern theoretical physics[3].

The classical relativistic relation $E^2 = p^2c^2 + m^2c^4$ expresses the square of the total energy E of a free particle interims of the square of its momentum p. It follows that there is no restriction on the sign of E and the equation has a full set of solutions for E less than $-mc^2$ as well as for E greater than mc^2 for the case of stationary state and free particle. Classically the negative energies are rejected as being unpleasant and there is no problem in particle as there is no mechanism for reaching the negative energy states from the positive ones. However, in the quantum mechanics case, negative energy states could be reached by a discontinuous quantum transition, and spontaneous transition to the states of even-lower energy might be expected[1].

A characteristic feature of the Dirac equation is that the spin of the particle is built in to the theory from the beginning, and can not be added after wards as Pauli added the electron spin to the schrodinger non relativistic equation. This feature provides a useful gauge of the applicability of a particular equation to the description of a particular kind of particle[4].

Out line of the thesis

The main purpose of this thesis is to find the solutions of the Dirac equation in the presence of a uniform back-ground magnetic field in which the solutions have a great contribution in the calculation of the scattering of particles in the presence of a very strong magnetic field.

The first chapter of this thesis is devoted to the discussion of the concepts of magnetic field. In addition magnetic vector potential, gauge conditions, spin matrices, some astronomical objects with a very strong magnetic field as well as notations and conventions are discussed. The next chapter, is devoted to the discussion of relativistic wave equations, the Klein- Gordon(relativistic form of schrodinger equation) in which its probability density is discussed and the Dirac equation and its consequences like probability density, current density, continuity equation as well as the solutions of the Dirac equation for a free particle are discussed.

The solutions of the Dirac equation for a particle in the presence of a uniform back ground magnetic field is briefly discussed in chapter three. At last discussions and conclusions are included.

Chapter 1

Introduction

1.1 Magnetic Field and its Origin in the Universe

Magnetic fields are most frequently found in various length scales and various positions in our universe. The first ever magnetic field in the universe arose with in $370,000$ years of the big bang, a new analysis suggests. The work relies on standard physics, unlike some previous theories, and may shed light on how the very first star grew[5].

Relativity confined magnetic fields like those in the earth and sun are generated by the turbulent mixing of conducting fluids in their cores. But large scale fields tangled with in galaxies and clusters of galaxies are harder to explain by fluid mixing alone. This is because most galaxies have rotated only a few dozen times since they formed. Due to this the origin of the magnetic fields in the galaxies and in the cluster of galaxies is unknown but most say in general one needs some initial, small magnetic field. Some researchers have tried to explain the origin of this so called ”seed” field by invoking new physical mechanism-such as the coupling of electromagnetic fields with exotic particles or gravity in the first instants after the Big Bang. There have been many models in this direction, most of which relay on new physics, and therefore not convincing. Now, researchers led by Kiyotomo chief of the national astronomical observatory of Japan in Tokyo Have used standard physics to explain the seed field. They say the field began before the first atom formed, when the universe was a hot soup of protons, electrons and photons-a state that lasted for the first $370,000$ after the big bang.

Photons exert a pressure on electrons than protons and also scatter off electrons more often. The researchers found that the difference in the movement of electrons and protons generated a rotating electric current, which produces a magnetic field. Magnetic fields in addition to gravity play a critical role in the formation process of various objects and their dynamical evolution in the universe. The presence of the magnetic field has an important consequence since the magnetic force can locally be much greater than the gravitational force[5].

When we come to the average magnitude of the magnetic field of some objects in our universe, the earth's magnetic field which can deflect compass needles is, 0.6 gauss. The magnetic field in a strong sun spot with dark magnetized area on the solar surface is 100 gauss. The strongest sustained that is steady magnetic field achieved so far in the laboratory which can be generated by huge electromagnets is 4.5×10^5 gauss. The strongest man-made field ever achieved which can be made by using focused explosive charges is 10^7 gauss. The strongest field ever detected on non neutron stars which can be found in a strongly magnetized compact white dwarf stars is 10^8 gauss. Typical surface polar magnetic fields of radio pulsars which are a familiar kind of spinning neutron stars is $\sim 10^{12} - 10^{13}$ gauss, Magnetars which are soft gama repeaters, and anomalous X-ray pulsars is $10^{14} - 10^{16}$ gauss[5].

Physicists have not made fields stronger than 4.5×10^5 gauss in the lab because the magnetic stress of such fields exceed the tensile strength of terrestrial materials[6].

1.2 Atoms in a Very Strong Magnetic Field

The strongest magnetic field that one can ever likely to encounter is 10^4 gauss if we have a magnetic resonance imaging (MRI) scan for medical diagnosis. Such fields pose no threat to ones health, hardly affecting the atoms in ones body. Fields in excess of 10^9 gauss however, would be instantly lethal. Such fields strongly distort atoms, compressing atomic electron clouds in to cigar shapes, thus rendering the chemistry of life impossible. In fields much stronger than 10^9 gauss, atoms are compressed in to thin needles. At 10^{14} gauss atomic needles have widths of one percent of their length and hundreds of times thinner than un magnetized atoms[5].

1.3 Magnetic Vector Potential ($\vec{A}$)

The vector potential provides a mathematical way of defining a magnetic field in classical electromagnetism. It is not directly observable, only the field it describes may be measured. If the magnetic vector potential is time-dependent, it also contributes to the electric field.

Since the magnetic field is divergence free that is $\nabla.B = 0$ (Gauss law for magnetism) this guaranties that $\vec{A}$ is fully specified by its divergence and curl, which is Helmholtz's theorem. Magnetic fields generated by steady and unsteady current satisfy
$\nabla.B = 0$. This immediately allows to write,

$$\vec{B} = \nabla \times \vec{A} \tag{1.3.1}$$

since the divergence of a curl is automatically zero, that is

$$\nabla.\vec{B} = 0 = \nabla.(\nabla \times \vec{A}) = 0$$

Eqn.(1.3.1) implies a magnetic vector potential is a three dimensional vector whose curl is the magnetic field. The vector potential $\vec{A}$ is used extensively when studying the lagrangian in classical mechanics and in quantum mechanics, such as the schrodinger equation for charged particles or the Dirac equation.

1.4 Gauge Choice and Transformation

It should be noted that the above definition eqn.(1.3.1) does not define magnetic vector potential uniquely because by definition, we can arbitrary add curl free components to the magnetic potential with out changing the observed magnetic field. Thus, their is a degree of freedom available when choosing $\vec{A}$. This condition is known as gauge invariance. According to eqn.(1.3.1) the magnetic field is invariant under the transformation,

$$\vec{A} \rightarrow \vec{A} - \nabla\Phi \tag{1.4.1}$$

In other words the vector potential is undetermined with in the gradient of a scalar field.

The above equation, eqn.(1.4.1), is known as gauge transformation. The choice of a particular function Φ is refered to us a choice of the gauge. We are free to fix the gauge to be what ever we like. The most sensible choice is the one which makes our equation as simple as possible. The usual gauge for $\vec{A}$ is such that.

$$\nabla.\vec{A} = 0 \tag{1.4.2}$$

This particular choice is known as the coulomb gauge.

1.5 Convention for Magnetic Field and Magnetic Vector Potential

1.5.1 Convention for Magnetic Field

To find the solutions of the Dirac equation in the presence of a uniform magnetic field, we consider the field to be directed along the $Z-$axis so that the field always be

$$\vec{B} = B\hat{z} \tag{1.5.1}$$

where $\hat{z}$ is the unit vector along the $Z-$ axis.

1.5.2 Convention for Magnetic Vector Potential

From the discussion of gauge condition, the vector potential can be chosen in many equivalent ways, but the most sensible is the one which makes our equation simple. So through out this thesis we take,

$$\vec{A} = A_x\hat{i} = -yB\hat{i} \tag{1.5.2}$$

where $\hat{i}$ is the unit vector along the $X-$axis.

1.6 Pauli Matrices

The pauli matrices are a set of three 2×2 complex Hermitian and unitary matrices. Usually designated by $\vec{\sigma}$.They are

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{1.6.1}$$

The names refers to Wolf gang Pauli, who suggested them.

1.6.1 Algebraic Properties

When we square the Pauli matrices we can get an identity matrix.

$$\sigma_x^2 = \sigma_y^2 = \sigma_z^2 = I \tag{1.6.2}$$

The determinants and traces of the Pauli matrix are $det(\sigma_i) = -1$ and $Tr(\sigma_i) = 0$ respectively.where $i = x, y, z$

1.6.2 Commutation Relations

The pauli matrices obey the following commutation and anti commutation relations as:

$$[\sigma_i, \sigma_j] = 2i\epsilon_{ijk}\sigma_k \tag{1.6.3}$$

and

$$\{\sigma_i, \sigma_j\} = 2\sigma_{ij} \tag{1.6.4}$$

where ϵ_{ijk} is the levi-Civita symbol, in which

$$\epsilon_{ijk} = \begin{cases} +1 & \text{if(ijk)is an even permutation} \\ -1 & \text{if(ijk) is an odd permutation} \\ 0 & \text{if any index is repeated} \end{cases} \tag{1.6.5}$$

and δ_{ij} is the kroneker delta,

$$\delta_{ij} = \begin{cases} 1 & \text{if } i = j; \\ 0 & \text{if } i \neq j. \end{cases}$$

The above two relations eqn.(1.6.3) and eqn.(1.6.4) imply that

$$\sigma_i\sigma_j = \delta_{ij} + i\sum_k \epsilon_{ijk}\sigma_k \tag{1.6.6}$$

The above equation, eqn.(1.6.6), can be used to prove the relation

$$(\vec{\sigma}.\vec{A})(\vec{\sigma}.\vec{B}) = \vec{A}.\vec{B} + i\vec{\sigma}.(\vec{A} \times \vec{B}) \tag{1.6.7}$$

Where $\vec{A}$ and $\vec{B}$ are any two vectors or operators.

Chapter 2

Relativistic Wave Equations

In this chapter we extend the non-relativistic schrodinger wave equation,

$$i\hbar\frac{\partial\Psi(r,t)}{\partial t} \quad = \quad E\Psi(r,t)$$

to the description of the motion of a particle with a speed approaching to that of light. This extension can be made in many ways, each of which is consistent with the lorentz transformation equations of the special theory of relativity. Here we consider two relativistic equations, the scalar equation due to Klein-Gordon that describes a particle with out spin, and the more complicated equation due to Dirac that describes an electron.

In discussing these equations, we devote our attention mainly to the discussions that can be made from them and do not attempt to establish their Lorentz invariance. The invariance of the Dirac equation is inferred quite convincingly from its symmetry between the space and time coordinate, that is on treating space and time on equal footing.

2.1 The Dirac Equation

It was thought historically that the non positive definiteness of ρ was a big problem. Dirac went off to search for an equation that was consistent with theory of relativity and had a positive definite probability density.

Dirac approach the problem of finding a relativistic wave equation by starting from the hamiltonian form

$$i\hbar\frac{\partial\Psi}{\partial t} = H\Psi \tag{2.1.1}$$

the classical relativistic hamiltonian for a free particle is the positive square root of the relation

$$E^2 = p^2c^2 + m^2c^4$$

However, if this is substituted in eqn.(2.2.1) and $\mathbf{P}$ is replaced by $-i\nabla$, the resulting wave equation is unsymmetrical with respect to space and time derivative, and hence not relativistic. Dirac therefore modified the hamiltonian in such a way as to make it linear in the space and time derivative.

2.1.1 Free Particle Equation

The simplest hamiltonian that is linear in the momentum and mass term can be written as:

$$H = c\vec{\alpha}.\mathbf{P} + \beta mc^2 \tag{2.1.2}$$

Where $\vec{\alpha}$ and β are quantities to be determined. Substituting in to eqn.(2.2.1) leads to the wave equation

$$\begin{aligned} i\hbar\frac{\partial\Psi}{\partial t} &= (c\vec{\alpha}.\mathbf{P} + \beta mc^2)\Psi && (2.1.3)\\ &= (-ci\hbar\vec{\alpha}.\nabla + \beta mc^2)\Psi && (2.1.4) \end{aligned}$$

We now consider the four quantities, $\alpha_x, \alpha_y, \alpha_z$ and β. If eqn.(2.2.4) is to describe a free particle wave equation, there can be no terms in the Hamiltonian that depends on the space coordinates or time. Instead the space and time derivatives appear only in $\mathbf{p}$ and E and not in α_i and β.

One can say more about α_i and β by requiring that any solution of Ψ of eqn.(2.2.4) also be a solution of Klein-Gordon equation,

$$-\hbar^2 \frac{\partial^2 \Psi}{(\partial t)^2} = -\hbar^2 c^2 \nabla^2 \Psi + m^2 c^4 \Psi \tag{2.1.5}$$

However, the converse need not be true. This is a reasonable requirement since in the absence of external field, the wave packet solutions of eqn.(2.2.4), whose motion resembles those of classical particle must have the relativistic relation,

$$E^2 = p^2 c^2 + m^2 c^4 \tag{2.1.6}$$

In terms of operators eqn.(2.2.6) becomes,

$$-\hbar^2 \frac{\partial^2}{(\partial t)^2} \Psi = -\hbar^2 c^2 \nabla^2 \Psi + m^2 c^4 \Psi \tag{2.1.7}$$

In order to get eqn.(2.2.7) from eqn.(2.2.4) we consider

$$\begin{aligned} i\hbar \frac{\partial}{\partial t} i\hbar \frac{\partial}{\partial t} \Psi &= [-ci\hbar \vec{\alpha}.\nabla + \beta mc^2][-ci\hbar \vec{\alpha}.\nabla + \beta mc^2]\Psi \\ &= [\frac{\hbar c}{i}(\alpha_x \frac{\partial}{\partial x} + \alpha_y \frac{\partial}{\partial y} + \alpha_z \frac{\partial}{\partial z}) + \beta mc^2][\frac{\hbar c}{i}(\alpha_x \frac{\partial}{\partial x} + \alpha_y \frac{\partial}{\partial y} + \alpha_z \frac{\partial}{\partial z}) + \beta mc^2]\Psi \end{aligned}$$

which implies

$$\begin{aligned} -\hbar^2 \frac{\partial^2 \Psi}{(\partial t)^2} &= \frac{\hbar^2 c^2}{i^2}[\alpha_x^2 \frac{\partial \Psi}{\partial x}\frac{\partial \Psi}{\partial x} + \alpha_x \alpha_y \frac{\partial \Psi}{\partial x}\frac{\partial \Psi}{\partial y} + \alpha_x \alpha_z \frac{\partial \Psi}{\partial x}\frac{\partial \Psi}{\partial z}] + \frac{\hbar c}{i}\alpha_x \beta mc^2 \frac{\partial \Psi}{\partial x} \\ &+ \frac{\hbar^2 C^2}{i^2}[\alpha_y \alpha_x \frac{\partial \Psi}{\partial y}\frac{\partial \Psi}{\partial x} + (\alpha_y)^2 \frac{\partial \Psi}{\partial y}\frac{\partial \Psi}{\partial y} + \alpha_y \alpha_z \frac{\partial \Psi}{\partial y}\frac{\partial \Psi}{\partial z}] + \frac{\hbar c}{i}\alpha_y \beta mc^2 \frac{\partial \Psi}{\partial y} \\ &+ \frac{\hbar^2 c^2}{i^2}[\alpha_z \alpha_x \frac{\partial \Psi}{\partial z}\frac{\partial \Psi}{\partial x} + \alpha_z \alpha_y \frac{\partial \Psi}{\partial z}\frac{\partial \Psi}{\partial y} + \alpha_z^2 \frac{\partial \Psi}{\partial z}\frac{\partial \Psi}{\partial z}] + \frac{\hbar c}{i}\alpha_z \beta mc^2 \frac{\partial \Psi}{\partial z} \\ &+ \beta mc^2 \frac{\hbar c}{i}[\alpha_x \frac{\partial \Psi}{\partial x} + \alpha_y \frac{\partial \Psi}{\partial y} + \alpha_z \frac{\partial \Psi}{\partial z}] + [\beta mc^2]^2 \end{aligned}$$

$$= \frac{\hbar^2 c^2}{i^2}[\alpha_x^2 \frac{\partial^2\Psi}{(\partial x)^2} + \alpha_y^2 \frac{\partial^2\Psi}{(\partial y)^2} + \alpha_z^2 \frac{\partial^2\Psi}{(\partial z)^2}] + \frac{\hbar^2 c^2}{i^2}[\alpha_x\alpha_y + \alpha_y\alpha_x]\frac{\partial^2\Psi}{\partial x\partial y}$$

$$+ \frac{\hbar^2 c^2}{i^2}[\alpha_x\alpha_z + \alpha_z\alpha_x]\frac{\partial^2\Psi}{\partial x\partial z} + [\alpha_y\alpha_z + \alpha_z\alpha_y]\frac{\partial^2\Psi}{\partial z\partial y} + \frac{\hbar c}{i}[\alpha_x\beta + \beta\alpha_x]\frac{\partial\Psi}{\partial x}$$
$$+ \frac{\hbar c}{i}[\alpha_y\beta + \beta\alpha_y]\frac{\partial\Psi}{\partial y} + \frac{\hbar c}{i}[\alpha_z\beta + \beta\alpha_z]\frac{\partial\Psi}{\partial z} + \beta^2 m^2 c^4\Psi$$

$$-\hbar^2\frac{\partial^2\Psi}{(\partial t)^2} = -\hbar^2 c^2[\alpha_i\frac{\partial\Psi}{\partial x^i}]^2 - \hbar^2 c^2[\alpha_i\alpha_j + \alpha_j\alpha_i]\frac{\partial^2\Psi}{\partial x^i\partial x^j} + \frac{\hbar c}{i}[\alpha_i\beta + \beta\alpha_i]\frac{\partial\Psi}{\partial x^i} + \beta^2 m^2 c^4\Psi \tag{2.1.8}$$

Comparing eqn.(2.2.8) with eqn.(2.2.7) we find

$$\alpha_i^2 = \mathbf{1} \quad , \quad \beta^2 = \mathbf{1}$$

$$\alpha_i\alpha_j + \alpha_j\alpha_i = 0$$

$$\alpha_i\beta + \beta\alpha_i = 0$$

In compact form these relations may be rewritten as

$$\{\alpha_i, \alpha_j\} = 2\delta_{ij} \tag{2.1.9}$$

$$\{\beta, \alpha_{i\}} = 0 \tag{2.1.10}$$

Since α_i and β anti-commute, rather than commute with each other, they can not be numbers, rather matrices. And since the Hamiltonian given by eqn.(2.2.2.) is hermitian each of the four matrices α_i and β must be hermitian.

2.1.2 Matrix representations of α and β

The matrix for β and α_i's can be represented schematically as

$$\beta = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix} \qquad and \qquad \alpha_i = \begin{pmatrix} 0 & \sigma_i \\ \sigma_i & 0 \end{pmatrix} \tag{2.1.11}$$

where each element is a matrix of two rows and two columns and I is an identity matrix.

2.2 Probability Density and Current Density

Consider the Dirac equation in the form,

$$i\hbar\frac{\partial\Psi}{\partial t} = -i\hbar c\alpha_k\frac{\partial\Psi}{\partial x^k} + \beta mc^2\Psi \tag{2.2.1}$$

and in terms of its hermitian conjugate,

$$-i\hbar\frac{\partial\Psi^\dagger}{\partial t} = i\hbar c\frac{\partial\Psi^\dagger}{\partial x^k}\alpha_k + mc^2\Psi^\dagger\beta \tag{2.2.2}$$

Here $(\alpha_k)^\dagger = \alpha_k$ and $(\beta)^\dagger = \beta$. Now multiplying eqn.(2.3.1) from the left by$\Psi^\dagger$ and eqn.(2.3.2) on the right by Ψ, and rearranging one can find

$$\frac{\partial}{\partial x^0}(\Psi^+\Psi) + \frac{\partial}{\partial x^k}(\Psi^+\alpha_K\Psi) \quad = \quad 0 \tag{2.2.3}$$

This looks like the familiar continuity equation. We identify $\Psi^\dagger\Psi$ as ρ, the probability density. We can also identify $\Psi^\dagger\alpha_K\Psi$ as J, the probability current and the vector nature of J is contained in α_k. Notice that the probability density in this case is positive definite.

2.3 Free particle Solutions of the Dirac Equation

As with the schrodinger equation, the simplest solution of the Dirac equation are those for a free particle. The Dirac Hamiltonian from eqn.(2.2.2) takes the form

$$H = c\vec{\alpha}.\mathbf{p} + \beta mc^2$$

where α and β are described as in eqn.(2.2.11). Using the operators $\mathbf{p} = -i\hbar\nabla$ in coordinate basis, the Dirac equation, eqn.(2.2.1) for a free particle reads,

$$[-i\hbar c\vec{\alpha}.\nabla + \beta mc^2]\Psi = i\hbar\frac{\partial}{\partial t}\Psi$$

Since the matrix operators on the left side of the Dirac equation are a 4×4, the wave function Ψ is actually a four component column vector. In order to generate an eigne value problem, we look for a solution in separated form of space and time:

$$\Psi(r,t) = \Psi(r)e^{\frac{-iEt}{\hbar}} \tag{2.3.1}$$

When substituted in Dirac equation it gives the relation which is the required eigenvalue equation,

$$[-i\hbar c\vec{\alpha}.\nabla + \beta mc^2]\Psi(r) = E\Psi(r) \tag{2.3.2}$$

Note that, since H is only a function of p, then $[\mathbf{p}, H] = 0$ so that the eigenvalue p of $\mathbf{p}$ can be used to characterize the states. In particular, we look for free-particle (plane-wave) solution in momentum (Fourier) space

$$\Psi(p) = U_p e^{\frac{i\mathbf{p}.\mathbf{r}}{\hbar}} \tag{2.3.3}$$

where U_p is a four component vector which satisfies

$$[c\vec{\alpha}.\mathbf{p} + \beta mc^2]U_p = EU_p \tag{2.3.4}$$

Since the matrices on the left hand side of this equation are expressible in terms of 2×2 blocks, we look for U_p in the form of a column vector composed of two two-component vectors:

$$U_p = \begin{pmatrix} \Phi \\ \chi \end{pmatrix} \tag{2.3.5}$$

therefor writing the equation in matrix form, we find

$$\begin{pmatrix} mc^2 & c\sigma.\mathbf{p} \\ c\sigma.\mathbf{p} & -mc^2 \end{pmatrix} \begin{pmatrix} \Phi \\ \chi \end{pmatrix} = \begin{pmatrix} E & 0 \\ 0 & E \end{pmatrix} \begin{pmatrix} \Phi \\ \chi \end{pmatrix} \tag{2.3.6}$$

Which yields two equations

$$(E - mc^2)\Phi - c\sigma.\mathbf{p}\chi = 0 \tag{2.3.7}$$

$$(E + mc^2)\chi - c\sigma.\mathbf{p}\Phi = 0 \tag{2.3.8}$$

From eqn.(2.4.8)follows

$$\chi = \frac{c\sigma.\mathbf{p}}{E + mc^2}\Phi \tag{2.3.9}$$

using this a single equation for Φ can be obtained as

$$(E - mc^2)(E + mc^2)\Phi - c^2(\sigma.\mathbf{p})^2\Phi = 0 \tag{2.3.10}$$

However, from eqn.(1.6.7) follws

$$(\sigma.\mathbf{p})^2 = (\sigma.\mathbf{p})(\sigma.\mathbf{p}) = \mathbf{p}.\mathbf{p} + i\sigma.(\mathbf{p} \times \mathbf{p}) = p^2$$

Hence, we have the condition

$$[E^2 - m^2c^4 - c^2p^2]\Phi = 0 \tag{2.3.11}$$

Since $\Phi \neq 0$ the equation is only be satisfied if the quantity in the bracket

$$E = \pm\sqrt{p^2c^2 + m^2c^4}$$

which means the eigne values are either positive or negative. An appropriate solution for Φ when $E > 0$ is to take

$$\Phi = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \qquad or \qquad \Phi = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \tag{2.3.12}$$

If this is the case, then

$$\chi = \frac{c\sigma.\mathbf{p}}{E + mc^2} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \qquad or \qquad \chi = \frac{c\sigma.\mathbf{p}}{E + mc^2} \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

But

$$\begin{aligned}
\sigma.\mathbf{p} &= \sigma_x p_x + \sigma_y p_y + \sigma_z p_z \\
&= \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} p_x & 0 \\ 0 & p_x \end{pmatrix} + \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \begin{pmatrix} p_y & 0 \\ 0 & p_y \end{pmatrix} + \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} p_z & 0 \\ 0 & p_z \end{pmatrix} \\
&= \begin{pmatrix} p_z & p_x - ip_y \\ p_x + ip_y & -p_z \end{pmatrix}
\end{aligned}$$

substituting this we get

$$\chi = \begin{pmatrix} \frac{cp_z}{E+mc^2} \\ \frac{c(p_x+ip_y)}{E+mc^2} \end{pmatrix} \qquad or \qquad \begin{pmatrix} \frac{c(p_x-ip_y)}{E+mc^2} \\ \frac{-cp_z}{E+mc^2} \end{pmatrix} \tag{2.3.13}$$

So that the full solution U_p from eqn.(2.4.5)becomes

$$U_p = \begin{pmatrix} 1 \\ 0 \\ \frac{cp_z}{E+mc^2} \\ \frac{c(p_x+ip_y)}{E+mc^2} \end{pmatrix} \qquad or \qquad U_p = \begin{pmatrix} 0 \\ 1 \\ \frac{c(p_x-ip_y)}{E+mc^2} \\ \frac{-cp_z}{E+mc^2} \end{pmatrix} \tag{2.3.14}$$

Again a similar procedure can be followed for negative energy spinors which have energy eigne values $E < 0$ or $E = -|E|$, and starting from the two lower components of the spinors, solving for Φ in terms of χ and defining,

$$\chi = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \qquad or \qquad \chi = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \tag{2.3.15}$$

We get

$$\Phi = \frac{c\sigma.\mathbf{p}}{E - mc^2}\chi \tag{2.3.16}$$

substituting $E = -|E|$

$$\Phi = \frac{-c\sigma.\mathbf{p}}{|E| + mc^2}\chi \tag{2.3.17}$$

By the same process as the positive energy, the negative energy spinors which are denoted by V_p can be done to be:

$$V_p = \begin{pmatrix} \frac{-cp_z}{|E|+mc^2} \\ \frac{-c(p_x+ip_y)}{|E|+mc^2} \\ 1 \\ 0 \end{pmatrix} \quad or \quad V_p = \begin{pmatrix} \frac{-c(p_x-ip_y)}{|E|+mc^2} \\ \\ \frac{cp_z}{|E|+mc^2} \\ 0 \\ 1 \end{pmatrix} \tag{2.3.18}$$

we call the positive and negative energy solutions, U_p and V_p as positive and negative energy spinors respectively.

Chapter 3

Solutions of the Dirac Equation in the Presence of a Uniform Back-ground Magnetic Field

3.1 Solutions of the Dirac Equation

In this chapter we try to find the exact solutions of the Dirac equation in the presence of a uniform back-ground magnetic field. In doing so, we use some assumed magnetic field and gauge condition. Since the field is uniform, we do not have a magnetic field which depend on time.

In the presence of a magnetic field, the Dirac equation for a particle of mass m, charge eQ and wave function $\Psi(r,t)$is given by

$$i\hbar\frac{\partial}{\partial t}\Psi(r,t) = H_B\Psi(r,t) \tag{3.1.1}$$

From eqn.(2.2.2) the Dirac Hamiltonian H_B in the presence of a magnetic field is given by

$$H_B = c\vec{\alpha}.\Pi + \beta mc^2 \tag{3.1.2}$$

Where Π is the kinematic momentum of the charged fermions and e is the charge of a proton. The kinematic momentum Π in terms of the canonical momentum and magnetic vector potential is given by

$$\Pi = \mathbf{p} - eQ\vec{A} \tag{3.1.3}$$

Where Q is the sign of the fermions charge and it can be ± 1. By using natural units eqn.(3.1.1) and eqn.(3.1.2) can be rewritten as

$$i\frac{\partial}{\partial t}\Psi(r,t) = H_B \Psi(r,t) \tag{3.1.4}$$

and

$$H_B = \vec{\alpha}.\Pi + \beta m \tag{3.1.5}$$

Substituting eqn.(3.1.3) in to eqn.(3.1.5) and eqn.(3.1.5) in to eqn.(3.1.4)

$$\begin{aligned} i\frac{\partial}{\partial t}\Psi(r,t) &= (\vec{\alpha}.\Pi + \beta m)\Psi(r,t) && (3.1.6) \\ &= [\vec{\alpha}.(\mathbf{p} - eQ\vec{A}) + m\beta]\Psi(r,t) && (3.1.7) \end{aligned}$$

Where $\vec{\alpha}$ and β are the Dirac matrices and $\vec{A}$ is the vector potential. Since $\mathbf{p}$ is an operator we can write as $-i\nabla$. It follows

$$\begin{aligned} i\frac{\partial}{\partial t}\Psi(r,t) &= \vec{\alpha}.[(-i\nabla - eQ\vec{A}) + m\beta]\Psi(r,t) && (3.1.8) \\ &= [-i\vec{\alpha}.\nabla - eQ\vec{\alpha}.\vec{A} + m\beta]\Psi(r,t) && (3.1.9) \end{aligned}$$

To deal with a given wave function $\Psi(r,t)$ and its time evolution one can use time evolution operator, $U(t)$ of the schrodinger picture in which:

$$U(t) \;=\; e^{-iEt}$$

and

$$\Psi(r,t) = U(t)\Psi(r)$$

with this one can write the solution for $\Psi(r,t)$ as

$$\Psi(r,t) = e^{-iEt}\Psi(r)$$

As before the matrix on the left is expressible in terms of a $2\times\mathbf{1}$ block column matrix. Hence

$$\Psi(r,t) = e^{-iEt}\begin{pmatrix}\Phi\\ \chi\end{pmatrix} \tag{3.1.10}$$

Where Φ and χ are the column matrices, and E is the energy of the particle. With this substitution we can rewrite eqn.(3.1.9) as

$$i\frac{\partial}{\partial t}\left[e^{-iEt}\begin{pmatrix}\Phi\\ \chi\end{pmatrix}\right] = [-i\vec{\alpha}.\nabla - eQ\vec{\alpha}.\vec{A} + m\beta]\left[e^{-iEt}\begin{pmatrix}\Phi\\ \chi\end{pmatrix}\right] \tag{3.1.11}$$

which gives

$$i(-i)Ee^{-iEt}\begin{pmatrix}\Phi\\ \chi\end{pmatrix} = [-i\vec{\alpha}.\nabla - eQ\vec{\alpha}.\vec{A} + m\beta]e^{-iEt}\begin{pmatrix}\Phi\\ \chi\end{pmatrix} \tag{3.1.12}$$

Eliminating the exponential term from both sides, we get

$$E\begin{pmatrix}\Phi\\ \chi\end{pmatrix} = [-i\vec{\alpha}.\nabla - eQ\vec{\alpha}.\vec{A} + m\beta]\begin{pmatrix}\Phi\\ \chi\end{pmatrix} \tag{3.1.13}$$

We use the Pauli-Dirac representation of the dirac matrix, $\vec{\alpha}$ as

$$\alpha_i = \begin{pmatrix}0 & \sigma_i\\ \sigma_i & 0\end{pmatrix} \qquad and \qquad \beta = \begin{pmatrix}I & 0\\ 0 & -I\end{pmatrix} \tag{3.1.14}$$

where each block represents a 2×2 matrix. With this representation, we can write eqn.(3.1.13) as

$$EI\begin{pmatrix}\Phi\\ \chi\end{pmatrix} = \left[-i\begin{pmatrix}0 & \vec{\sigma}\\ \vec{\sigma} & 0\end{pmatrix}.\nabla - eQ\begin{pmatrix}0 & \vec{\sigma}\\ \vec{\sigma} & 0\end{pmatrix}.\vec{A} + m\begin{pmatrix}I & 0\\ 0 & -I\end{pmatrix}\right]\begin{pmatrix}\Phi\\ \chi\end{pmatrix} \tag{3.1.15}$$

I is a 2×2 block identity matrix. So

$$\begin{aligned} E\begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix}\begin{pmatrix} \Phi \\ \chi \end{pmatrix} &= \left[\begin{pmatrix} 0 & -i\vec{\sigma}.\nabla \\ -i\vec{\sigma}.\nabla & 0 \end{pmatrix} - eQ\begin{pmatrix} 0 & \vec{\sigma}.\vec{A} \\ \vec{\sigma}.\vec{A} & 0 \end{pmatrix} + \begin{pmatrix} m & 0 \\ 0 & -m \end{pmatrix}\right]\begin{pmatrix} \Phi \\ \chi \end{pmatrix} \\ \begin{pmatrix} E\Phi \\ E\chi \end{pmatrix} &= \begin{pmatrix} m & -i\vec{\sigma}.\nabla - eQ\vec{\sigma}.\vec{A} \\ -i\vec{\sigma}.\nabla - eQ\vec{\sigma}.\vec{A} & -m \end{pmatrix}\begin{pmatrix} \Phi \\ \chi \end{pmatrix} \end{aligned}$$

$$\begin{pmatrix} E\Phi \\ E\chi \end{pmatrix} = \begin{pmatrix} m\Phi - [i\vec{\sigma}.\nabla + eQ\vec{\sigma}.\vec{A}]\chi \\ -[i\vec{\sigma}.\nabla + eQ\vec{\sigma}.\vec{A}]\Phi - m\chi \end{pmatrix}$$

Equating terms we get

$$E\Phi = m\Phi - [i\vec{\sigma}.\nabla + eQ\vec{\sigma}.\vec{A}]\chi \tag{3.1.16}$$

and

$$E\chi = -[i\vec{\sigma}.\nabla + eQ\vec{\sigma}.\vec{A}]\Phi - m\chi \tag{3.1.17}$$

This implies

$$[E - m]\Phi = [-i\vec{\sigma}.\nabla - eQ\vec{\sigma}.\vec{A}]\chi \tag{3.1.18}$$

and

$$[E + m]\chi = [-i\vec{\sigma}.\nabla - eQ\vec{\sigma}.\vec{A}]\Phi \tag{3.1.19}$$

we can rewrite these as

$$[E - m]\Phi = \vec{\sigma}.[-i\nabla - eQ\vec{A}]\chi \tag{3.1.20}$$

$$[E + m]\chi = \vec{\sigma}.[-i\nabla - eQ\vec{A}]\Phi \tag{3.1.21}$$

Solving for χ from eqn.(3.1.21), as

$$\chi = \frac{\vec{\sigma}.[-i\nabla - eQ\vec{A}]}{E + m}\Phi \tag{3.1.22}$$

and substituting it in to eqn.(3.1.20) we find

$$[E-m]\Phi = \vec{\sigma}.[-i\nabla - eQ\vec{A}]\frac{\vec{\sigma}.[-i\nabla - eQ\vec{A}]}{E+m}\Phi \tag{3.1.23}$$

From this follows

$$\begin{aligned}
[E-m][E+m]\Phi &= [\vec{\sigma}.(-i\nabla - eQ\vec{A})][\vec{\sigma}.(-i\nabla - eQ\vec{A})]\Phi \\
[E^2-m^2]\Phi &= [\vec{\sigma}.(-i\nabla - eQ\vec{A})]^2\Phi \\
&= [\vec{\sigma}.(-i\nabla - eQ\vec{A})]^2\Phi
\end{aligned} \tag{3.1.24}$$

it is possible to write

$$\begin{aligned}
[\vec{\sigma}.\Pi][\vec{\sigma}.\Pi]\Phi &= [\Pi.\Pi + i\vec{\sigma}.(\Pi\times\Pi)]\Phi \\
&= \Pi^2\Phi + i\vec{\sigma}.[\Pi\times\Pi]\Phi
\end{aligned}$$

we can express the kinematic momentum in terms of an operator ∇ and the vector potential $\vec{A}$, and we get

$$\begin{aligned}
[\vec{\sigma}.\Pi][\vec{\sigma}.\Pi]\Phi &= [(-i\nabla - eQ\vec{A}).(-i\nabla - eQ\vec{A}) + i\vec{\sigma}.(-i\nabla - eQ\vec{A})\times(-i\nabla - eQ\vec{A})]\Phi \\
&= [-\nabla.\nabla + ieQ\nabla.\vec{A} + ieQ\vec{A}.\nabla + (eQ)^2(\vec{A}.\vec{A})]\Phi \\
&+ i\vec{\sigma}.[-\nabla\times\nabla + ieQ\nabla\times\vec{A} + ieQ\vec{A}\times\nabla + (eQ)^2(\vec{A}\times\vec{A})]\Phi \\
&= [-\nabla^2 + ieQ\nabla.\vec{A} + ieQ\vec{A}.\nabla + (eQ)^2\vec{A}^2]\Phi + i\vec{\sigma}.[ieQ\nabla\times\vec{A} + ieQ\vec{A}\times\nabla]\Phi \\
&= [-\nabla^2 + (eQ)^2\vec{A}^2 + ieQ(\nabla.\vec{A} + \vec{A}.\nabla)]\Phi - eQ\vec{\sigma}.[(\nabla\times\vec{A} + \vec{A}\times\nabla)]\Phi
\end{aligned} \tag{3.1.25}$$

But

$$\begin{aligned}(\nabla.\vec{A}+\vec{A}.\nabla)\Phi &= \nabla.(\vec{A}\Phi)+\vec{A}.\nabla\Phi\\ &= (\nabla.\vec{A})\Phi+\vec{A}.\nabla\Phi+\vec{A}.\nabla\Phi\\ &= (\nabla.\vec{A})\Phi+2\vec{A}.\nabla\Phi\\ &= (\nabla.\vec{A}+2\vec{A}.\nabla)\Phi\end{aligned}$$

and

$$\begin{aligned}(\nabla\times\vec{A}+\vec{A}\times\nabla)\Phi &= (\nabla\times(\vec{A}\Phi)+\vec{A}\times\nabla\Phi\\ &= (\nabla\times\vec{A})\Phi-\vec{A}\times\nabla\Phi+\vec{A}\times\nabla\Phi\\ &= (\nabla\times\vec{A})\Phi\end{aligned}$$

Substituting these two relations in eqn.(3.1.25)

$$\begin{aligned}[\vec{\sigma}.\Pi][\vec{\sigma}.\Pi]\Phi &= [-\nabla^2+(eQ)^2\vec{A}^2]\Phi+ieQ[\nabla.\vec{A}+2\vec{A}.\nabla]\Phi+i^2eQ\vec{\sigma}.\vec{B}\Phi\\ &= [-\nabla^2+e^2Q^2(\vec{A})^2+ieQ(\frac{\partial}{\partial x}A_x+\frac{\partial}{\partial y}A_y+\frac{\partial}{\partial z}A_z)\\ &+ 2ieQ(A_x\frac{\partial}{\partial x}+A_y\frac{\partial}{\partial y}+A_z\frac{\partial}{\partial z})-eQ(\sigma_xB_x+\sigma_yB_y+\sigma_zB_z)]\Phi\end{aligned} \tag{3.1.26}$$

Here we work with a constant magnetic field $\vec{B}$. With the convention we mentioned in chapter one section five, with out loss of any generality the field can be taken along the $z-$direction and the background gauge field which gives rise to the magnetic field along the $z-$direction can be chosen in many ways but for convenience we took,

$A_0 = A_y = A_z = 0$ and $A_x = -yB$. With this choice eqn.(3.1.26) becomes

$$\begin{aligned}
[\vec{\sigma}.\Pi][\vec{\sigma}.\Pi]\Phi &= [-\nabla^2 + e^2Q^2(-yB)^2 + ieQ\frac{\partial}{\partial x}(-yB) + 2ieQ(-yB\frac{\partial}{\partial x}) - eQ\sigma_z B_z]\Phi \\
&= [-\nabla^2 + (eQB)^2y^2 + 2ieQ(-yB)\frac{\partial}{\partial x} - eQ\sigma_z B_z]\Phi \\
&= [-\nabla^2 + (eQB)^2y^2 - eQB(2iy\frac{\partial}{\partial x} + \sigma_z)]\Phi \qquad (3.1.27)
\end{aligned}$$

Again with the substitution of eqn.(3.1.27) in to eqn.(3.1.24), we get

$$[E^2 - m^2]\Phi = [-\nabla^2 + (eQB)^2y^2 - eQB(2iy\frac{\partial}{\partial x} + \sigma_z)]\Phi \qquad (3.1.28)$$

here σ_z is the diagonal Pauli matrix.

In eqn.(3.1.28) one can notice that the coordinates x and z do not appear in the equation except through the derivatives, so we can write the solution Φ for the above equation as

$$\Phi = e^{i\mathbf{p}.\mathbf{X}_{\not y}} f(y) \qquad (3.1.29)$$

where: $f(y)$ is a two component matrix which can only depend on y coordinate, x is a spatial coordinate and $\mathbf{X}_{\not y}$ is a vector X with its y component set equal to zero. This means $\mathbf{p}.\mathbf{X}_{\not y}$=$p_x x + p_z z$. Substituting eqn.(3.1.29) in to eqn.(3.1.28) we get

$$\begin{aligned}
[E^2 - m^2]e^{i\mathbf{p}.\mathbf{X}_{\not y}} f(y) &= [-\nabla^2 + (eQB)^2y^2 - eQB(2iy\frac{\partial}{\partial x} + \sigma_z)]e^{i\mathbf{p}.\mathbf{X}_{\not y}} f(y) \\
&= [-\frac{\partial^2}{(\partial x)^2} - \frac{\partial^2}{(\partial y)^2} - \frac{\partial^2}{(\partial z)^2} + (eQB)^2y^2]e^{i(p_x x + p_z z)} f(y) \\
&- [2ieQBy\frac{\partial}{\partial x} + eQB\sigma_z]e^{i(p_x x + p_z z)} f(y) \\
(E^2 - m^2)e^{i(p_x x + p_z z)} f(y) &= [p_x^2 f(y) + p_z^2 f(y) - \frac{\partial^2 f(y)}{(\partial y)^2} + (eQB)^2y^2 f(y)]e^{i(p_x x + p_z z)} \\
&- [2ieQBy(ip_x)f(y) + eQB\sigma_z f(y)]e^{i(p_x x + p_z z)}
\end{aligned}$$

Eliminating the exponential term from both side, we get

$$[E^2-m^2]f(y) = p_x^2 f(y)+p_z^2 f(y)-\frac{d^2 f(y)}{(dy)^2}+(eQB)^2 y^2 f(y)+2eQByp_x f(y)-eQB\sigma_z f(y) \tag{3.1.30}$$

Here there are two independent solutions for $f(y)$, which may be taken as solutions, to be the eigenstates of the diagonal Pauli matrix σ_z with eigenvalue S in which $S = \pm 1$ and this values of S corresponds to the spin of the fermions (spin up or spin down). We know that $f(y)$ is a two component matrix. We can set the two values of this function which correspond to the two eigenvalue S as

$$f_+(y) = \begin{pmatrix} F_+(y) \\ 0 \end{pmatrix} \quad , \quad f_-(y) = \begin{pmatrix} 0 \\ F_-(y) \end{pmatrix} \tag{3.1.31}$$

then

$$\sigma_z f_+(y) = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} F_+(y) \\ 0 \end{pmatrix} = \begin{pmatrix} F_+(y) \\ 0 \end{pmatrix} = f_+(y) \tag{3.1.32}$$

and

$$\sigma_z f_-(y) = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} 0 \\ F_-(y) \end{pmatrix} = \begin{pmatrix} 0 \\ -F_-(y) \end{pmatrix} = -f_-(y) \tag{3.1.33}$$

Therefore, in terms of S, one can write

$$\sigma_z f_S(y) = S f_S(y) \tag{3.1.34}$$

To find the differential equation satisfied by F_S, we substitute eqn.(3.1.31) in to eqn.(3.1.30) separately

$$\begin{aligned}[E^2 - m^2]f_+(y) &= p_x^2 f_+(y) + p_z^2 f_+(y) - \frac{d^2 f_+(y)}{(dy)^2} + (eQB)^2 y^2 f_+(y) \\ &+ 2eQByp_x f_+(y) - eQB\sigma_z f_+(y)\end{aligned} \tag{3.1.35}$$

$$\begin{aligned}[E^2 - m^2]\begin{pmatrix} F_+(y) \\ 0 \end{pmatrix} &= p_x^2 \begin{pmatrix} F_+(y) \\ 0 \end{pmatrix} + p_z^2 \begin{pmatrix} F_+(y) \\ 0 \end{pmatrix} - \frac{d^2}{(dy)^2}\begin{pmatrix} F_+(y) \\ 0 \end{pmatrix} \\ &+ (eQB)^2 y^2 \begin{pmatrix} F_+(y) \\ 0 \end{pmatrix} + 2eQByp_x \begin{pmatrix} F_+(y) \\ 0 \end{pmatrix} - eQB \begin{pmatrix} F_+(y) \\ 0 \end{pmatrix} \\ [E^2 - m^2]F_+(y) &= p_x^2 F_+(y) + p_z^2 F_+(y) - \frac{d^2}{(dy)^2} F_+(y) + (eQB)^2 y^2 F_+(y) \\ &+ 2eQByp_x F_+(y) - eQBF_+(y)\end{aligned} \tag{3.1.36}$$

and

$$\begin{aligned}[E^2 - m^2]f_-(y) &= p_x^2 f_-(y) + p_z^2 f_-(y) - \frac{d^2 f_-(y)}{(dy)^2} + (eQB)^2 y^2 f_-(y) \\ &+ 2eQByP_x f_-(y) + eQBf_-(y) \\ [E^2 - m^2]\begin{pmatrix} 0 \\ F_-(y) \end{pmatrix} &= p_x^2 \begin{pmatrix} 0 \\ F_-(y) \end{pmatrix} + p_z^2 \begin{pmatrix} 0 \\ F_-(y) \end{pmatrix} - \frac{d^2}{(dy)}^2 \begin{pmatrix} 0 \\ F_-(y) \end{pmatrix} \\ &+ (eQB)^2 y^2 \begin{pmatrix} 0 \\ F_-(y) \end{pmatrix} + 2eQByp_x \begin{pmatrix} 0 \\ F_-(y) \end{pmatrix} + eQB \begin{pmatrix} 0 \\ F_-(y) \end{pmatrix} \\ [E^2 - m^2]F_-(y) &= px^2 F_-(y) + p_z^2 F_-(y) - \frac{d^2 F_-(y)}{(dy)^2} + (eQB)^2 y^2 F_-(y) \\ &+ 2eQByp_x F_-(y) + eQBF_-(y)\end{aligned} \tag{3.1.37}$$

From eqn.(3.1.36) and eqn.(3.1.37) we get a generalized differential equation,

$$[E^2 - m^2]F_S(y) = p_x^2 F_S(y) + p_z^2 F_S(y) - \frac{d^2 F_S(y)}{(dy)^2} + (eQB)^2 y^2 F_S(y) + 2eQByp_x F_S(y) - eQBSF_S(y) \tag{3.1.38}$$

Rearranging terms,

$$\begin{aligned}
[\frac{d^2}{(dy)^2} + [E^2 - m^2] - p_x^2 - p_z^2 - (eQB)^2 y^2 - 2eQByp_x + eQBS]F_S(y) &= 0 \\
\frac{d^2 F_S(y)}{(dy)^2} - [p_x^2 + 2eQByp_x + (eQB)^2 y^2]F_S(y) + [E^2 - m^2 - p_z^2 + eQBS]F_S(y) &= 0 \\
\frac{d^2 F_S(y)}{(dy)^2} - [eQBy + p_x]^2 F_S(y) + [E^2 - m^2 - p_z^2 + eQBS]F_S(y) &= 0
\end{aligned}$$

this can be rewritten as

$$\frac{d^2 F_S(y)}{(dy)^2} - [eQB(y + \frac{p_x}{eQB})]^2 F_S(y) + [E^2 - m^2 - p_z^2 + eQBS]F_S(y) = 0$$

But one can write

$$eQB = \sqrt{e|Q|B}\sqrt{e|Q|B}$$

Since the square of any real number is always positive, we need to take the absolute value of Q. With this substitution we can get

$$\frac{d^2 F_S(y)}{(dy)^2} - [\sqrt{e|Q|B}\sqrt{e|Q|B}(y + \frac{p_x}{eQB})]^2 F_S(y) + [E^2 - m^2 - p_z^2 + eQBS]F_S(y) = 0$$

$$\frac{d^2 F_S(y)}{(dy)^2} - [\sqrt{e|Q|B}]^2[\sqrt{e|Q|B}(y + \frac{p_x}{eQB})]^2 F_S(y) + [E^2 - m^2 - p_z^2 + eQBS]F_S(y) = 0$$

$$\frac{d^2 F_S(y)}{(dy)^2} - e|Q|B[\sqrt{e|Q|B}(y + \frac{p_x}{eQB})]^2 F_S(y) + [E^2 - m^2 - p_z^2 + eQBS]F_S(y) = 0$$

Dividing by $e|Q|B$

$$\frac{1}{e|Q|B}\frac{d^2 F_S(y)}{(dy)^2} - [\sqrt{e|Q|B}(y + \frac{p_x}{eQB})]^2 F_S(y) + \frac{[E^2 - m^2 - p_z^2 + eQBS]}{e|Q|B}F_S(y) = 0 \tag{3.1.39}$$

Eqn.(3.1.39) is a second order differential equation, which can be simplified by defining a dimensionless variable ξ,

$$\xi = \sqrt{e|Q|B}(y + \frac{p_x}{eQB}) \tag{3.1.40}$$

Differentiating with respect to the variable y we can take the complete derivative of eqn.(3.1.40) to get

$$d\xi = \sqrt{e|Q|B}dy$$

and

$$(d\xi)^2 = (\sqrt{e|Q|B}dy)^2 = e|Q|B(dy)^2 \tag{3.1.41}$$

which implies

$$(dy)^2 = \frac{(d\xi)^2}{e|Q|B}$$

With this substitution eqn.(3.1.39) takes the form

$$(\frac{e|Q|B}{e|Q|B})\frac{d^2F_S(\xi)}{(d\xi)^2} - \xi^2F_S(\xi) + \frac{[E^2 - m^2 - p_z^2 + eQBS]}{e|Q|B}F_S(\xi) = 0$$

This further simplified by letting a_S as

$$a_S = \frac{[E^2 - m^2 - p_z^2 + eQBS]}{e|Q|B} \tag{3.1.42}$$

Consequently, we get a very simplified relation,

$$[\frac{d^2}{(d\xi)^2} - \xi^2 + a_S]F_S(\xi) = 0 \tag{3.1.43}$$

The easiest way to solve this differential equation is to find a recursion formula by starting from the assumption,

$$F_S(\xi) = e^{\frac{-\xi^2}{2}}\Phi(\xi) \tag{3.1.44}$$

Differentiating this with respect to ξ we get

$$\frac{dF_S}{d\xi} = -\xi e^{\frac{-\xi^2}{2}}\Phi(\xi) + e^{\frac{-\xi^2}{2}}\frac{d\Phi(\xi)}{d\xi} \tag{3.1.45}$$

A further differentiation gives

$$\begin{aligned}
\frac{d^2F_S}{(d\xi)^2} &= -\xi(-\xi)e^{\frac{-\xi^2}{2}\Phi(\xi)} - e^{\frac{-\xi^2}{2}}\Phi(\xi) - \xi e^{\frac{-\xi^2}{2}}\frac{d\Phi(\xi)}{d\xi} \\
&- \xi e^{\frac{-\xi^2}{2}}\frac{d\Phi(\xi)}{d\xi} + e^{\frac{-\xi^2}{2}}\frac{d^2\Phi(\xi)}{(d\xi)^2} \\
&= e^{\frac{-\xi^2}{2}}\Phi(\xi) + \xi^2 e^{\frac{-\xi^2}{2}}\Phi(\xi) - 2\xi e^{\frac{-\xi^2}{2}}\frac{d\Phi(\xi)}{d\xi} + e^{\frac{-\xi^2}{2}}\frac{d^2\Phi(\xi)}{(d\xi)^2} \\
&= e^{\frac{-\xi^2}{2}}\frac{d^2\Phi(\xi)}{(d\xi)^2} - 2\xi e^{\frac{-\xi^2}{2}}\frac{d\Phi(\xi)}{d\xi} - e^{\frac{-\xi^2}{2}}(1-\xi^2)\Phi(\xi) \qquad (3.1.46)
\end{aligned}$$

Substituting these two values in to eqn.(3.1.43) and using eqn.(3.1.44) we get

$$\begin{aligned}
e^{\frac{-\xi^2}{2}}\frac{d^2\Phi(\xi)}{(d\xi)^2} - 2\xi e^{\frac{-\xi^2}{2}}\frac{d\Phi(\xi)}{d\xi} - e^{\frac{-\xi^2}{2}}(1-\xi^2)\Phi(\xi) &\;- \\
\xi^2 e^{\frac{-\xi^2}{2}}\Phi(\xi) + a_S e^{\frac{-\xi^2}{2}} &= 0
\end{aligned}$$

Eliminating the exponential terms,

$$\frac{d^2\Phi(\xi)}{(d\xi)^2} - 2\xi\frac{d\Phi(\xi)}{d\xi} - \Phi(\xi) + a_S\Phi(\xi) = 0$$

This implies

$$\frac{d^2\Phi(\xi)}{(d\xi)^2} - 2\xi\frac{d\Phi(\xi)}{d\xi} + (a_S - 1)\Phi(\xi) = 0 \qquad (3.1.47)$$

This is also a second order differential equation. To find the solution of this differential equation and the energy eigenvalues we suggest a power series solution for $\Phi(\xi)$ as

$$\Phi(\xi) = \sum_{k=o} a_k \xi^{(k+\nu)} \qquad (3.1.48)$$

Thus

$$\frac{d}{d\xi}\Phi(\xi) = \sum_{k=1} a_k(k+\nu)\xi^{(k+\nu-1)}$$

and

$$\frac{d^2\Phi(\xi)}{(d\xi)^2} = \sum_{k=2} a_k(k+\nu)(k+\nu-1)\xi^{(k+\nu-2)}$$

Substituting these eqn.(3.1.47) becomes

$$\sum_{k=2} a_k(k+\nu)(k+\nu-1)\xi^{(k+\nu-2)} - 2\xi\sum_{k=1} a_k(k+\nu)\xi^{(k+\nu-1)} + (a_S-1)\sum_{k=o} a_k\xi^{(k+\nu)} = 0$$

Replacing $k-2$ by k, in the first series which forces the index of summation to start from zero we find

$$\begin{aligned}\sum_{k=0} a_{k+2}(k+\nu+2)(k+\nu+1)\xi^{(k+\nu)} - 2\sum_{k=0} a_k(k+\nu)\xi^{(k+\nu)} + (a_S-1)\sum_{k=o} a_k\xi^{(k+\nu)} &= 0\\ \sum_{k=0}[a_{k+2}(k+\nu+2)(k+\nu+1) - 2a_k(k+\nu) + (a_S-1)a_k]\xi^{(k+\nu)} &= 0\end{aligned}$$

Since ξ is different from zero, we can equate the term in the bracket equal to zero, so that

$$a_{k+2}(k+\nu+2)(k+\nu+1) - 2a_k(k+\nu) + (a_S-1)a_k = 0$$

which implies

$$a_{k+2}(k+\nu+2)(k+\nu+1) - [2(k+\nu) - (a_S-1)]a_k = 0$$

From this we can find a recursion formula as

$$a_{k+2} = \frac{2(k+\nu)-(a_S-1)}{(k+\nu+2)(k+\nu+1)}a_k = \frac{2k+2\nu+1-a_S}{(k+\nu+2)(k+\nu+1)}a_k \qquad (3.1.49)$$

In order to obtain physically feasible wave function, the series must break off after some terms. To get the term at which the series breaks off, we put the numerator in the recursion formula eqn.(3.1.49) equal to zero

$$2k+2\nu+1-a_S = 0$$

this implies

$$a_S = 2k+2\nu+1$$

This will turn Φ in to a polynomial rather than a power series. Then the function Φ can be set equal to the product of this polynomial and some normalization constant N_ν. To get this polynomial function consider eqn.(3.1.47),

$$\frac{d^2\Phi(\xi)}{(d\xi)^2} - 2\xi\frac{d\Phi(\xi)}{d\xi} + (a_S - 1)\Phi(\xi) = 0$$

This equation is like a Hermite equation of order ν

$$\frac{d^2H_\nu(\xi)}{(d\xi)^2} - 2\xi\frac{dH_\nu(\xi)}{d\xi} + 2\nu H_\nu(\xi) = 0 \tag{3.1.50}$$

whose solution is given by

$$H_\nu(\xi) = e^{(\xi^2)}(-1)^\nu[\frac{d^\nu \exp(-\xi^2)}{(d\xi)^\nu}] \tag{3.1.51}$$

When we compare the two equations eqn.(3.1.47) and eqn.(3.1.50), we observe similarities except in the last term. So we can compare the two equations to find the value of a_s as

$$a_s - 1 = 2\nu$$

$$a_s = 2\nu + 1 \tag{3.1.52}$$

From eqn.(3.1.42) we have

$$a_S = \frac{E^2 - m^2 - p_z^2 + eQBS}{e|Q|B}$$

When matched with the preceding equation gives

$$E^2 - m^2 - p_z^2 + eQBS = (2\nu + 1)e|Q|B$$

From this we can write the value of the energy eigenvalue as

$$E^2 = m^2 + p_z^2 - eQBS + (2\nu + 1)e|Q|B \tag{3.1.53}$$

This implies that the energy eigenvalue can have two values as

$$E = \pm\sqrt{m^2 + p_z^2 - eQBS + (2\nu + 1)e|Q|B} \tag{3.1.54}$$

We can write Φ as the product of some normalization constant N_ν and H_ν, that is

$$\Phi(\xi) = N_\nu H_\nu(\xi) \tag{3.1.55}$$

So eqn.(3.1.44) can be rewritten as

$$F_S(\xi) = e^{\frac{-\xi^2}{2}}\Phi(\xi) = N_\nu e^{\frac{-\xi^2}{2}} H_\nu(\xi) \tag{3.1.56}$$

The normalization constant N_ν can be determined by defining the normalization condition,

$$\int_{-\infty}^{\infty} |F_S(y)|^2 dy = 1 \tag{3.1.57}$$

From eqn.(3.1.40) we have

$$\xi = \sqrt{e|Q|B}(y + \frac{p_x}{eQB})$$

and hence

$$d\xi = \sqrt{e|Q|B}dy$$

which implies

$$dy = \frac{d\xi}{\sqrt{e|Q|B}}$$

Substituting this in to eqn.(3.1.57) we find

$$\int_{-\infty}^{\infty} (N_\nu)^2 (e^{\frac{-\xi^2}{2}})^2 (H_\nu(\xi))^2 \frac{d\xi}{\sqrt{e|Q|B}} = 1$$

$$\frac{(N_\nu)^2}{\sqrt{e|Q|B}} \int_{-\infty}^{\infty} e^{-\xi^2} |H_\nu(\xi)|^2 d\xi = 1$$

The Hermite polynomials are known to satisfy the relation

$$\int_{-\infty}^{\infty} e^{-\xi^2} H_n(\xi) H_m(\xi) d\xi = 2^\nu \nu! \sqrt{\Pi}\delta_{m,n}$$

Which when substituted in the preceding equation gives

$$\frac{(N_\nu)^2}{\sqrt{e|Q|B}} * 2^\nu \nu! \sqrt{\Pi} = 1$$

This implies

$$N_\nu = (\frac{\sqrt{e|Q|B}}{2^\nu \nu! \sqrt{\Pi}})^{\frac{1}{2}} \tag{3.1.58}$$

Hence, $F_s(\xi)$ takes a simplified final form

$$F_s(\xi) = (\frac{\sqrt{e|Q|B}}{2^\nu \nu! \sqrt{\Pi}})^{\frac{1}{2}} e^{\frac{-\xi^2}{2}} H_\nu(\xi) \tag{3.1.59}$$

for simplicity let us denote this solution by I_ν as

$$F_s(\xi) = (\frac{\sqrt{e|Q|B}}{2^\nu \nu! \sqrt{\Pi}})^{\frac{1}{2}} e^{\frac{-\xi^2}{2}} H_\nu(\xi) = I_\nu(\xi) \tag{3.1.60}$$

3.1.1 Electron Energy Eigne Value(Q=-1)

For $Q = -1$ eqn.(3.1.53) becomes

$$\begin{aligned} E_\nu^2 &= m^2 + p_z^2 - e(-1)BS + (2\nu + 1)e|-1|B \\ &= m^2 + p_z^2 + eBS + (2\nu + 1)eB \\ &= m^2 + p_z^2 + eBS + 2e\nu B + eB \end{aligned} \tag{3.1.61}$$

We can associate this energy eigen value with the relativistic form of the Landau energy levels, given by

$$E_n^2 = m^2 + p_z^2 + 2neB \tag{3.1.62}$$

Since we have two values for S eqn.(3.1.61) is two fold degenerate. When $S = +1$, eqn.(3.1.61) reduced to eqn.(3.1.62) provided $\nu = n - 1$, Similarly when $S = -1$ eqn.(3.1.61) will be equivalent to eqn.(3.1.62) only if $\nu = n$. Now when $n = 0$ the Landau energy becomes

$$E_n^2 = m^2 + p_z^2$$

So for the case in which $n = 0$, the Landau energy becomes equivalent with our energy eigenvalue only if $(2\nu + 1 + s = 0)$ which implies

$$\nu = \frac{-1}{2}(S + 1)$$

Since ν can not be negative, the $n = 0$ case can only be true if $S = -1$. Thus, the $n = 0$ state is not degenerate.

Now representing the solutions for the n^{th} Landau level by a superscript n, for $S = +1$, we can write

$$f_+^n = \begin{pmatrix} F_+(\xi) \\ 0 \end{pmatrix} = \begin{pmatrix} I_\nu(\xi) \\ 0 \end{pmatrix} = \begin{pmatrix} I_{n-1}(\xi) \\ 0 \end{pmatrix} \tag{3.1.63}$$

and for $S = -1$ we have

$$f_-^n = \begin{pmatrix} 0 \\ F_-(\xi) \end{pmatrix} = \begin{pmatrix} 0 \\ I_\nu(\xi) \end{pmatrix} = \begin{pmatrix} 0 \\ I_n(\xi) \end{pmatrix} \tag{3.1.64}$$

Here for $n = 0$, since ν can not be negative the solution f_+^n does not exist and for consistency we use

$$I_{-1}(\xi) = 0$$

When we come to the energy eigne value,

$$E_n^2 = m^2 + p_z^2 + 2neB$$

$$E_n = \pm\sqrt{m^2 + p_z^2 + 2neB} \tag{3.1.65}$$

Here the two signs of the energy corresponds to two types of solutions of the Dirac equation, the positive and negative energy solutions respectively.

3.2 Positive Energy Solutions of the Dirac Equation

For the case of positive energy $E = E_n$, $Q = -1$ and $\vec{A} = A_x\hat{i} = -yB\hat{i}$ the lower component of the spinor, χ from eqn.(3.1.22) is given by

$$\chi = \frac{\vec{\sigma}.[-i\nabla + eA_x\hat{i}]}{E_n + m}\Phi$$

where

$$\Phi = e^{i\mathbf{p}.\mathbf{X}_{\not{y}}} f(y)$$

Hence we can write

$$\begin{aligned} \chi &= \frac{\vec{\sigma}.[-i\nabla + eA_x\hat{i}]}{E_n + m} e^{i\mathbf{p}.\mathbf{X}_{\not{y}}} f(y) \\ &= \frac{\vec{\sigma}.[-i\nabla - eyB\hat{i}]}{E_n + m} e^{i\mathbf{p}.\mathbf{X}_{\not{y}}} f(y) \end{aligned}$$

But

$$\begin{aligned} \vec{\sigma}.(-i\nabla - eyB\hat{i}) &= -i(\vec{\sigma}.\nabla) - eyB(\vec{\sigma}.\hat{i}) \\ &= -i[\sigma_x \frac{\partial}{\partial x} + \sigma_y \frac{\partial}{\partial y} + \sigma_z \frac{\partial}{\partial z}] - eyB\sigma_x \end{aligned}$$

Substituting for σ_x, σ_y and σ_z this gives

$$
\begin{aligned}
\vec{\sigma}.(-i\nabla - eyB\hat{i}) &= -i\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\frac{\partial}{\partial x} - i\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}\frac{\partial}{\partial y} - i\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\frac{\partial}{\partial z} - eBy\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \\
&= \begin{pmatrix} 0 & -i\frac{\partial}{\partial x} \\ -i\frac{\partial}{\partial x} & 0 \end{pmatrix} + \begin{pmatrix} 0 & -\frac{\partial}{\partial y} \\ \frac{\partial}{\partial y} & 0 \end{pmatrix} + \begin{pmatrix} -i\frac{\partial}{\partial z} & o \\ 0 & i\frac{\partial}{\partial z} \end{pmatrix} - \begin{pmatrix} 0 & eBy \\ eBy & 0 \end{pmatrix} \\
&= \begin{pmatrix} -i\frac{\partial}{\partial z} & -i\frac{\partial}{\partial x} - \frac{\partial}{\partial y} - eyB \\ -i\frac{\partial}{\partial x} + \frac{\partial}{\partial y} - eyB & i\frac{\partial}{\partial z} \end{pmatrix} \qquad (3.2.1)
\end{aligned}
$$

Using eqn.(3.2.1) the expression for the lower components of the spinor χ becomes

$$
\begin{aligned}
\chi &= \frac{\vec{\sigma}.[-i\nabla - eyB\hat{i}]}{E_n + m} e^{i\mathbf{p}.\mathbf{X}_{\not y}} f(y) \\
&= \frac{1}{(E_n + m)} \begin{pmatrix} -i\frac{\partial}{\partial z} & -i\frac{\partial}{\partial x} - \frac{\partial}{\partial y} - eyB \\ -i\frac{\partial}{\partial x} + \frac{\partial}{\partial y} - eyB & i\frac{\partial}{\partial z} \end{pmatrix} e^{i\mathbf{p}.\mathbf{X}_{\not y}} f(y)
\end{aligned}
$$

Following we calculate two positive energy solutions which correspond to the two values of S by replacing $f(y)$ by $f_+(y)$ and $f_-(y)$ separately.

3.2.1 Positive Energy Solution for $S = +1$ (Spin Up)

Using this value of S, the expression for the lower components of the spinor becomes

$$
\begin{aligned}
\chi &= \frac{1}{(E_n + m)} \begin{pmatrix} -i\frac{\partial}{\partial z} & -i\frac{\partial}{\partial x} - \frac{\partial}{\partial y} - eyB \\ -i\frac{\partial}{\partial x} + \frac{\partial}{\partial y} - eyB & i\frac{\partial}{\partial z} \end{pmatrix} e^{i(p_x x + p_z z)} f_+(y) \\
&= \frac{1}{(E_n + m)} \begin{pmatrix} -i\frac{\partial}{\partial z} & -i\frac{\partial}{\partial x} - \frac{\partial}{\partial y} - eyB \\ -i\frac{\partial}{\partial x} + \frac{\partial}{\partial y} - eyB & i\frac{\partial}{\partial z} \end{pmatrix} e^{i(p_x x + p_z z)} \begin{pmatrix} I_{n-1}(\xi) \\ 0 \end{pmatrix} \\
&= \frac{1}{(E_n + m)} \begin{pmatrix} -i\frac{\partial}{\partial z} e^{i(p_x x + p_z z)} I_{n-1}(\xi) \\ (-i\frac{\partial}{\partial x} + \frac{\partial}{\partial y} - eyB)] e^{i(p_x x + p_z z)} I_{n-1}(\xi) \end{pmatrix} \\
&= \frac{1}{(E_n + m)} \begin{pmatrix} p_z I_{n-1}(\xi) \\ (p_x - eyB) I_{n-1}(\xi) + \frac{\partial}{\partial y} I_{n-1}(\xi) \end{pmatrix} e^{i(p_x x + p_z z)}
\end{aligned}
$$

Here $I_{n-1}(\xi)$ is a function of ξ and ξ is a function of y therefore, we can write the partial derivative with respect to y as

$$\frac{\partial I_{n-1}(\xi)}{\partial y} = \frac{\partial I_{n-1}(\xi)}{\partial \xi}\frac{d\xi}{dy} \tag{3.2.2}$$

For the case $Q = -1$ we can write ξ as

$$\xi = \sqrt{eB}(y - \frac{p_x}{eB})$$

which implies

$$\frac{\xi}{\sqrt{eB}} = (y - \frac{p_x}{eB}) \tag{3.2.3}$$

and

$$\frac{d\xi}{dy} = \sqrt{eB} \tag{3.2.4}$$

Using these we can then write

$$\begin{aligned} \frac{\partial I_{n-1}(\xi)}{\partial y} &= \frac{\partial}{\partial \xi} I_{n-1}(\xi)\sqrt{eB} \\ &= \sqrt{eB}\frac{\partial I_{n-1}(\xi)}{\partial \xi} \end{aligned} \tag{3.2.5}$$

Substituting eqn.(3.2.3)-eqn.(3.2.5) in to χ we find

$$\begin{aligned} \chi &= \frac{1}{(E_n + m)} \begin{pmatrix} p_z I_{n-1}(\xi) \\ (p_x - eyB)I_{n-1}(\xi) + \sqrt{eB}\frac{\partial}{\partial \xi} I_{n-1}(\xi) \end{pmatrix} e^{i(p_x x + p_z z)} \\ &= \frac{1}{(E_n + m)} \begin{pmatrix} p_z I_{n-1}(\xi) \\ eB(\frac{p_x}{eB} - y)I_{n-1}(\xi) + \sqrt{eB}\frac{\partial}{\partial \xi} I_{n-1}(\xi) \end{pmatrix} e^{i(p_x x + p_z z)} \end{aligned}$$

Using eqn.(3.2.3) again χ further simplifies to

$$\begin{aligned} \chi &= \frac{1}{(E_n + m)} \begin{pmatrix} p_z I_{n-1}(\xi) \\ \frac{-eB\xi}{\sqrt{eB}} I_{n-1}(\xi) + \sqrt{eB}\frac{\partial}{\partial \xi} I_{n-1}(\xi) \end{pmatrix} e^{i(p_x x + p_z z)} \\ &= \frac{1}{(E_n + m)} \begin{pmatrix} p_z I_{n-1}(\xi) \\ -\sqrt{eB}\xi I_{n-1}(\xi) + \sqrt{eB}\frac{\partial}{\partial \xi} I_{n-1}(\xi) \end{pmatrix} e^{i(p_x x + p_z z)} \end{aligned}$$

$$= \frac{1}{(E_n + m)} \begin{pmatrix} p_z I_{n-1}(\xi) \\ -\sqrt{eB}[\xi I_{n-1}(\xi) - \frac{\partial}{\partial \xi} I_{n-1}(\xi)] \end{pmatrix} e^{i(p_x x + p_z z)} \tag{3.2.6}$$

Now, consider the Hermite polynomial $H_n(\xi)$

$$H_n(\xi) = (-1)^n e^{-\xi^2} [\frac{\partial^n}{(\partial \xi)^n} e^{-\xi^2}]$$

Replacing n by $= n - 1$

$$H_{n-1}(\xi) = (-1)^{n-1} e^{\xi^2} [\frac{\partial^{n-1}}{(\partial \xi)^{n-1}} e^{-\xi^2}]$$

taking the derivative with respect to ξ we get

$$\begin{aligned} \frac{\partial}{\partial \xi} H_{n-1}(\xi) &= \frac{\partial}{\partial \xi} [(-1)^{n-1} e^{\xi^2} [\frac{\partial^{n-1}}{(\partial \xi)^{n-1}} e^{-\xi^2}] \\ &= (-1)^{n-1} (2\xi) e^{\xi^2} [\frac{\partial^{n-1}}{(\partial \xi)^{n-1}} e^{\xi^2}] + (-1)^{n-1} e^{\xi^2} [\frac{\partial^n}{(\partial \xi)^n} e^{-\xi^2}] \\ &= 2\xi [(-1)^{n-1} e^{\xi^2} \frac{\partial^{n-1}}{(\partial \xi)^{n-1}} e^{\xi^2}] - (-1)^n e^{\xi^2} [\frac{\partial^n}{(\partial \xi)^n} e^{-\xi^2}] \\ &= 2\xi H_{n-1}(\xi) - H_n(\xi) \end{aligned}$$

which implies

$$H_n(\xi) = 2\xi H_{n-1}(\xi) - \frac{\partial}{\partial \xi} H_{n-1}(\xi) \tag{3.2.7}$$

Also

$$\begin{aligned} N_n &= (\frac{\sqrt{eB}}{2^n n! \sqrt{\Pi}})^{\frac{1}{2}} \\ &= (\frac{\sqrt{eB}}{2^{n-1} 2n (n-1)! \sqrt{\Pi}})^{\frac{1}{2}} \\ &= (\frac{1}{2n} \frac{\sqrt{eB}}{(n-1)! 2^{n-1} \sqrt{\Pi}})^{\frac{1}{2}} \\ &= (\frac{1}{2n})^{\frac{1}{2}} [\frac{\sqrt{eB}}{(n-1)! 2^{n-1} \sqrt{\Pi}}]^{\frac{1}{2}} \end{aligned}$$

It then follows

$$N_n = (\frac{1}{2n})^{\frac{1}{2}} N_{n-1} \tag{3.2.8}$$

From equation (3.1.60) we know that

$$I_n(\xi) = N_n e^{\frac{-\xi^2}{2}} H_n(\xi)$$

Using eqn.(3.2.7) we can expand this to

$$I_n(\xi) = N_n e^{\frac{-\xi^2}{2}} [2\xi H_{n-1}(\xi) - \frac{\partial}{\partial \xi} H_{n-1}(\xi)]$$

Which using (3.2.8) may be rewritten as

$$\begin{aligned} I_n(\xi) &= (\frac{1}{2n})^{\frac{1}{2}} N_{n-1} e^{\frac{-\xi^2}{2}} [2\xi H_{n-1}(\xi) - \frac{\partial}{\partial \xi} H_{n-1}(\xi)] \\ &= (\frac{1}{2n})^{\frac{1}{2}} [2\xi (N_{n-1} e^{\frac{-\xi^2}{2}} H_{n-1}(\xi)) - N_{n-1} e^{\frac{-\xi^2}{2}} \frac{\partial}{\partial \xi} H_{n-1}(\xi)] \\ &= (\frac{1}{2n})^{\frac{1}{2}} [2\xi I_{n-1}(\xi) - N_{n-1} e^{\frac{-\xi^2}{2}} \frac{\partial}{\partial \xi} H_{n-1}(\xi)] \end{aligned} \tag{3.2.9}$$

To determine $N_{n-1} e^{\frac{-\xi^2}{2}} \frac{\partial}{\partial \xi} H_{n-1}(\xi)$ let us consider

$$\begin{aligned} \frac{\partial}{\partial \xi} I_{n-1}(\xi) &= \frac{\partial}{\partial \xi} [N_{n-1} e^{\frac{-\xi^2}{2}} H_{n-1}(\xi)]) \\ &= N_{n-1}(-\xi) e^{\frac{-\xi^2}{2}} H_{n-1}(\xi) + N_{n-1} e^{\frac{-\xi^2}{2}} \frac{\partial}{\partial \xi} H_{n-1}(\xi) \\ &= -\xi [N_{n-1} e^{\frac{-\xi^2}{2}} H_{n-1}(\xi)] + N_{n-1} e^{\frac{-\xi^2}{2}} \frac{\partial}{\partial \xi} H_{n-1}(\xi) \\ &= -\xi I_{n-1}(\xi) + N_{n-1} e^{\frac{-\xi^2}{2}} \frac{\partial}{\partial \xi} H_{n-1}(\xi) \end{aligned}$$

This may be rearranged

$$N_{n-1} e^{\frac{-\xi^2}{2}} \frac{\partial}{\partial \xi} H_{n-1}(\xi) = \frac{\partial}{\partial \xi} I_{n-1}(\xi) + \xi I_{n-1}(\xi) \tag{3.2.10}$$

Upon using eqn.(3.2.10), eqn.(3.2.9) becomes

$$\begin{aligned} I_n(\xi) &= (\frac{1}{2n})^{\frac{1}{2}} [2\xi I_{n-1}(\xi) - \frac{\partial}{\partial \xi} I_{n-1}(\xi) - \xi I_{n-1}(\xi)] \\ &= (\frac{1}{2n})^{\frac{1}{2}} [\xi I_{n-1}(\xi) - \frac{\partial}{\partial \xi} I_{n-1}(\xi)] \end{aligned}$$

which leads to the relation

$$\sqrt{2n}I_n(\xi) = \xi I_{n-1}(\xi) - \frac{\partial}{\partial\xi}I_{n-1}(\xi) \tag{3.2.11}$$

Substituting eqn.(3.2.11) in to eqn.(3.2.6) we finally get

$$\begin{aligned}\chi &= \frac{1}{(E_n+m)}\begin{pmatrix} p_z I_{n-1}(\xi) \\ -\sqrt{eB}\sqrt{2n}I_n(\xi)\end{pmatrix} e^{i(p_x x + p_z z)} \\ &= \begin{pmatrix} \frac{P_z}{E_n+m}I_{n-1}(\xi) \\ \\ -\frac{\sqrt{2neB}}{E_n+m}I_n(\xi)\end{pmatrix} e^{i\mathbf{p}.\mathbf{X}_{\not{y}}} \end{aligned} \tag{3.2.12}$$

In general for the case $S = +1$ and positive energy, the upper and lower components of the spinor can be given by,

$$\begin{aligned}\begin{pmatrix}\Phi \\ \chi\end{pmatrix} &= \begin{pmatrix} e^{i\mathbf{p}.\mathbf{X}_{\not{y}}} f_+(y) \\ \chi \end{pmatrix} \\ &= \begin{pmatrix} I_{n-1}(\xi)e^{i\mathbf{p}.\mathbf{X}_{\not{y}}} \\ 0 \\ \frac{p_z I_{n-1}(\xi)}{E_n+m}e^{i\mathbf{p}.\mathbf{X}_{\not{y}}} \\ \\ -\frac{\sqrt{2neB}I_n(\xi))}{E_n+m}I_n(\xi)e^{i\mathbf{p}.\mathbf{X}_{\not{y}}} \end{pmatrix} \\ &= \begin{pmatrix} I_{n-1}(\xi) \\ 0 \\ \frac{P_z}{E_n+m}I_{n-1}(\xi) \\ \\ -\frac{\sqrt{2neB}I_n(\xi)}{E_n+m}I_n(\xi) \end{pmatrix} e^{i\mathbf{p}.\mathbf{X}_{\not{y}}} \end{aligned} \tag{3.2.13}$$

3.2.2 Positive Energy Solution for $S = -1$(Spin Down)

The other value of the spinors for the case in which $S = -1$ can be determined by substituting

$$\Phi = e^{i\mathbf{p}.\mathbf{X}_{\not{y}}} f_-(y) \tag{3.2.14}$$

in to the expression for χ. With this substitution we will have

$$\chi = \frac{\vec{\sigma}.[-i\nabla + eA_x\hat{i}]}{E_n + m} e^{i\mathbf{p}.\mathbf{X}_{\not{y}}} f_-(y) \tag{3.2.15}$$

which upon using eqn.(3.2.1) expanded to

$$\begin{aligned}
\chi &= \frac{1}{(E_n+m)} \begin{pmatrix} -i\frac{\partial}{\partial z} & -i\frac{\partial}{\partial x} - \frac{\partial}{\partial y} - eyB \\ -i\frac{\partial}{\partial x} + \frac{\partial}{\partial y} - eyB & i\frac{\partial}{\partial z} \end{pmatrix} e^{i(p_x x + p_z z)} \begin{pmatrix} 0 \\ I_n(\xi) \end{pmatrix} \\
&= \frac{1}{(E_n+m)} \begin{pmatrix} [-i\frac{\partial}{\partial x} - \frac{\partial}{\partial y} - eyB] e^{i(p_x x + p_z z)} I_n(\xi) \\ i\frac{\partial}{\partial z} e^{i(p_x x + p_z z)} I_n(\xi) \end{pmatrix} \\
&= \frac{1}{(E_n+m)} \begin{pmatrix} eB(\frac{p_x}{eB} - y) I_n(\xi) - \frac{\partial}{\partial y} I_n(\xi) \\ -p_z I_n(\xi) \end{pmatrix} e^{i(p_x x + p_z z)}
\end{aligned}$$

By using eqn.(3.2.3) and eqn.(3.2.5) the above expression simplifies to

$$\begin{aligned}
\chi &= \frac{1}{(E_n+m)} \begin{pmatrix} \frac{-eB}{\sqrt{eB}} \xi I_n(\xi) - \frac{\partial}{\partial y} I_n(\xi) \\ -p_z I_n(\xi) \end{pmatrix} e^{i(p_x x + p_z z)} \\
&= \frac{1}{(E_n+m)} \begin{pmatrix} -\sqrt{eB} \xi I_n(\xi) - \frac{\partial}{\partial y} I_n(\xi) \\ -p_z I_n(\xi) \end{pmatrix} e^{i(p_x x + p_z z)} \\
&= \frac{1}{(E_n+m)} \begin{pmatrix} -\sqrt{eB} [\xi I_n(\xi) + \frac{\partial}{\partial \xi} I_n(\xi)] \\ -p_z I_n(\xi) \end{pmatrix} e^{i(p_x x + p_z z)}
\end{aligned} \tag{3.2.16}$$

Consider now

$$I_n(\xi) = N_n e^{\frac{-\xi^2}{2}} H_n(\xi)$$

Taking the derivative with respect to ξ

$$\begin{aligned}
\frac{\partial}{\partial\xi}I_n(\xi) &= N_n(-\xi)e^{\frac{-\xi^2}{2}}H_n(\xi) + N_n e^{\frac{-\xi^2}{2}}\frac{\partial}{\partial\xi}H_n(\xi) \\
&= -\xi(N_n e^{\frac{-\xi^2}{2}}H_n(\xi) + N_n e^{\frac{-\xi^2}{2}}\frac{\partial}{\partial\xi}H_n(\xi) \\
&= -\xi I_n(\xi) + N_n e^{\frac{-\xi^2}{2}}\frac{\partial}{\partial\xi}H_n(\xi) \qquad (3.2.17)
\end{aligned}$$

Here again we need to determine the value of the derivative of the Hermite polynomial. For this let us consider the generating function for the Hermite polynomials

$$\sum_{n=0}^{\infty}\frac{H_n(\xi)}{n!}Z^n = e^{(2Z\xi - Z^2)} \qquad (3.2.18)$$

Differentiating this with respect to ξ we find

$$\begin{aligned}
\sum_{n=0}^{\infty}\frac{1}{n!}\frac{\partial}{\partial\xi}H_n(\xi)Z^n &= \frac{\partial}{\partial\xi}e^{(2Z\xi-Z^2)} \\
&= 2Ze^{(2Z\xi-Z^2)} \\
&= 2Z(\sum_{n=0}^{\infty}\frac{H_n(\xi)}{n!}Z^n) \\
&= 2(\sum_{n=0}^{\infty}\frac{H_n(\xi)}{n!}Z^{n+1}) \\
&= 2(\sum_{n=0}^{\infty}\frac{H_n(\xi)}{n(n-1)!}Z^{n+1}) \\
&= 2(\sum_{n=0}^{\infty}\frac{H_{n-1}(\xi)}{(n-1)(n-2)!}Z^n)
\end{aligned}$$

$$\sum_{n=0}^{\infty}\frac{1}{n(n-1)!}\frac{\partial}{\partial\xi}H_n(\xi)Z^n = 2(\sum_{n=0}^{\infty}\frac{H_{n-1}(\xi)}{(n-1)!}Z^n)$$

This implies

$$(\sum_{n=0}^{\infty}\frac{1}{n}\frac{\partial}{\partial\xi}H_n(\xi)\frac{1}{(n-1)!})Z^n = (2\sum_{n=0}^{\infty}\frac{H_{n-1}(\xi)}{(n-1)!})Z^n$$

By comparing the terms in the bracket we can conclude

$$\frac{1}{n}\frac{\partial}{\partial\xi}H_n(\xi) \quad = \quad 2H_{n-1}(\xi)$$

or

$$\frac{\partial}{\partial\xi}H_n(\xi) = 2nH_{n-1}(\xi) \tag{3.2.19}$$

Using this we can rewrite eqn.(3.2.17) as

$$\begin{aligned}
\frac{\partial}{\partial\xi}I_n(\xi) &= -\xi I_n(\xi) + 2nN_n e^{\frac{-\xi^2}{2}} H_{n-1}(\xi) \\
&= -\xi I_n(\xi) + \frac{2n}{\sqrt{2n}} N_{n-1} e^{\frac{-\xi^2}{2}} H_{n-1}(\xi) \\
&= -\xi I_n(\xi) + \sqrt{2n} I_{n-1}(\xi)
\end{aligned}$$

which leads to the relation

$$\sqrt{2n}I_{n-1}(\xi) = \xi I_n(\xi) + \frac{\partial}{\partial\xi}I_n(\xi) \tag{3.2.20}$$

Substituting this in to eqn.(3.2.16) we get

$$\begin{aligned}
\chi = \frac{1}{(E_n+m)} &\begin{pmatrix} -\sqrt{eB}\sqrt{2n}I_{n-1}(\xi) \\ -P_z I_n(\xi) \end{pmatrix} e^{i(p_x x + p_z z)} \\
= &\begin{pmatrix} -\frac{\sqrt{2neB}}{E_n+m} I_{n-1}(\xi) \\ \\ -\frac{p_z}{E_n+m} I_n(\xi) \end{pmatrix} e^{i(p_x x + p_z z)}
\end{aligned} \tag{3.2.21}$$

With this substitution the other two components of the spinor for the case of $S = -1$ and, positive energy is given by

$$
\begin{pmatrix} \Phi \\ \chi \end{pmatrix} = \begin{pmatrix} e^{i\mathbf{p}.\mathbf{X}_{\not y}} f_-(y) \\ \chi \end{pmatrix}
$$

$$
= \begin{pmatrix} 0 \\ I_n(\xi) \\ -\frac{\sqrt{2neB}}{E_n+m} I_{n-1}(\xi) \\ -\frac{p_z(\xi)}{E_n+m} I_n(\xi) \end{pmatrix} e^{i\mathbf{p}.\mathbf{X}_{\not y}}
$$

Generally the positive energy eigne function of the Dirac equation is given by eqn.(3.1.10)

$$
\Psi(r,t) = e^{-iEt} \begin{pmatrix} \Phi \\ \chi \end{pmatrix}
$$

In using eqn.(3.2.13)this becomes

$$
\Psi_+(r,t) = e^{-iEt} \begin{pmatrix} I_{n-1}(\xi) \\ 0 \\ \frac{p_z(\xi)}{E_n+m} I_{n-1}(\xi) \\ -\frac{\sqrt{2neB}}{E_n+m} I_n(\xi) \end{pmatrix} e^{i\mathbf{p}.(\mathbf{X}_{\not y})}
$$

$$
= e^{-iEt} e^{i\mathbf{p}.\mathbf{X}_{\not y}} \begin{pmatrix} I_{n-1}(\xi) \\ 0 \\ \frac{p_z(\xi)}{E_n+m} I_{n-1}(\xi) \\ -\frac{\sqrt{2neB}}{E_n+m} I_n(\xi) \end{pmatrix}
$$

which lead to the expression

$$\Psi_{+}(r,t) = e^{(-iEt+i\mathbf{p}.(\mathbf{X}_{\not y}))} \begin{pmatrix} I_{n-1}(\xi) \\ 0 \\ \\ \frac{p_z(\xi)}{E_n+m} I_{n-1}(\xi) \\ \\ -\frac{\sqrt{2neB}}{E_n+m} I_n(\xi) \end{pmatrix} \tag{3.2.22}$$

For the case $S = +1$ and

$$\Psi_{-}(r,t) = e^{(-iEt+i\mathbf{p}.(\mathbf{X}_{\not y}))} \begin{pmatrix} 0 \\ I_n(\xi) \\ \\ -\frac{\sqrt{2neB}}{E_n+m} I_{n-1}(\xi) \\ \\ -\frac{p_z(\xi)}{E_n+m} I_n(\xi) \end{pmatrix} \tag{3.2.23}$$

for the case $S = -1$.

The positive energy eigne function can be written in a covariant form as

$$\Psi_{+}(r,t) = e^{ip_\mu.\mathbf{X}^{\mu}_{\not y}} U_{+}(y,n,p_{\not y}) \tag{3.2.24}$$

and

$$\Psi_{-}(r,t) = e^{ip_\mu.(\mathbf{X}^{\mu}_{\not y})} U_{-}(y,n,p_{\not y}) \tag{3.2.25}$$

where

$$U_{+}(y,n,p_{\not y}) = \begin{pmatrix} I_{n-1}(\xi) \\ 0 \\ \frac{p_z(\xi)}{E_n+m} I_{n-1}(\xi) \\ \\ -\frac{\sqrt{2neB}}{E_n+m} I_n(\xi) \end{pmatrix} \tag{3.2.26}$$

and

$$U_-(y,n,p_{\not{y}}) = \begin{pmatrix} 0 \\ I_n(\xi) \\ \\ -\frac{\sqrt{2neB}}{E_n+m} I_{n-1}(\xi) \\ \\ -\frac{p_z(\xi)}{E_n+m} I_n(\xi) \end{pmatrix} \tag{3.2.27}$$

Generally one can write the solutions in a covariant form as

$$\Psi_S(r,t) = e^{ip_\mu.\mathbf{X}^\mu_{\not{y}}} U_S(y,n,p_{\not{y}}) \tag{3.2.28}$$

Here $\mathbf{X}^\mu_{\not{y}}$ is the space time coordinate with out including the y coordinate.

3.3 Negative Energy Solutions of the Dirac Equation

In this section we will find the solutions of the Dirac equation for the case in which the energy $E_n = -|E_n|$, $Q = -1$ and $\vec{A} = A_x\hat{i} = -yB\hat{i}$.

From eqn.(3.1.20) and eqn.(3.1.21) when $E_n = -|E_n|$ we can write,

$$[-|E_n| - m]\Phi = \vec{\sigma}.[-i\nabla - eQ\vec{A}]\chi \tag{3.3.1}$$

and

$$[-|E_n| + m]\chi = \vec{\sigma}.[-i\nabla - eQ\vec{A}]\Phi \tag{3.3.2}$$

which implies

$$[|E_n| + m]\Phi = \vec{\sigma}.[i\nabla + eQ\vec{A}]\chi \tag{3.3.3}$$

and

$$[|E_n| - m]\chi = \vec{\sigma}.[i\nabla + eQ\vec{A}]\Phi \tag{3.3.4}$$

In this case it is easier to start with the two lower components first and then find the upper components.

Solving for Φ from eqn.(3.3.3) we find

$$\Phi = \frac{\vec{\sigma}.[i\nabla + eQ\vec{A}]}{|E_n| + m}\chi \tag{3.3.5}$$

substituting this in to eqn.(3.3.4) results in

$$[|E_n| - m]\chi = \vec{\sigma}.[i\nabla + eQ\vec{A}]\frac{\vec{\sigma}.[i\nabla + eQ\vec{A}]}{|E_n| + m}\chi \tag{3.3.6}$$

which leads to the relation

$$\begin{aligned}
[|E_n| - m][|E_n| + m]\chi &= [\vec{\sigma}.(i\nabla + eQ\vec{A})][\vec{\sigma}.(i\nabla + eQ\vec{A})]\chi \\
(|E_n|^2 - m^2)\chi &= [\vec{\sigma}.(i\nabla + eQ\vec{A})]^2\chi
\end{aligned} \tag{3.3.7}$$

The Bracketed term on the right expands to

$$\begin{aligned}
[\vec{\sigma}.(i\nabla + eQ\vec{A})]^2\chi &= [-\nabla^2 + e^2Q^2(-yB)^2 + ieQ(\frac{\partial}{\partial x}(-yB)) + 2ieQ(-yB\frac{\partial}{\partial x}) - e\sigma_z B_z]\chi \\
&= [-\nabla^2 + (eQB)^2y^2 - eQB(2iy\frac{\partial}{\partial x} + \sigma_z)]\chi
\end{aligned} \tag{3.3.8}$$

with this substitution eqn.(3.3.7) becomes

$$[|E_n|^2 - m^2]\chi = [-\nabla^2 + (eQB)^2y^2 - eQB(2iy\frac{\partial}{\partial x} + \sigma_z)]\chi \tag{3.3.9}$$

In eqn.(3.3.9) we do not have x and z dependence except through the derivative, and with this condition one can write the solution for χ as

$$\chi = e^{-i\mathbf{p}.\mathbf{X}_{\not{y}}}g(y) \tag{3.3.10}$$

where $g(y)$ is a two component matrix which can depend only on $y-$coordinate. Upon using eqn.(3.3.10), eqn.(3.3.9) becomes

$$\begin{aligned}
[|E_n|^2 - m^2]e^{-i\mathbf{p}.\mathbf{X}_{\natural}} g(y) &= [-\nabla^2 + (eQB)^2 y^2 - eQB(2iy\frac{\partial}{\partial x} + \sigma_z)]e^{-i\mathbf{p}.\mathbf{X}_{\natural}} g(y) \\
&= [p_x^2 g(y) + p_z^2 g(y) - \frac{\partial^2}{(\partial y)^2} g(y) + (eQB)^2 y^2 g(y) \\
&- 2eQByp_x g(y) - eQB\sigma_z g(y)]e^{-i\mathbf{p}.\mathbf{X}_{\natural}}
\end{aligned}$$

Eliminating the exponential terms from both sides and noting that g is only a function of y we can write

$$\begin{aligned}
[|E_n|^2 - m^2]g(y) &= p_x^2 g(y) + p_z^2 g(y) - \frac{d^2}{(dy)^2} g(y) + (eQB)^2 y^2 g(y) \\
&- 2eQByp_x g(y) - eQB\sigma_z g(y) \qquad (3.3.11)
\end{aligned}$$

Here again we have two independent solutions for $g(y)$ so that $g(y)$ may be considered as the eigne state of the Pauli matrix σ_z with eigne value $S = \pm 1$. Since $g(y)$ is a two component matrix, one can set the two values of this function which correspond to the eigne values S as

$$g_+(y) = \begin{pmatrix} G_+(y) \\ 0 \end{pmatrix} \quad , \quad g_-(y) = \begin{pmatrix} 0 \\ G_-(y) \end{pmatrix} \qquad (3.3.12)$$

With the same approach as the positive energy

$$\sigma_z g_+(y) = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix} \begin{pmatrix} G_+(y) \\ 0 \end{pmatrix} = \begin{pmatrix} G_+(y) \\ 0 \end{pmatrix} = g_+(y) \qquad (3.3.13)$$

and

$$\sigma_z g_-(y) = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} 0 \\ G_-(y) \end{pmatrix} = \begin{pmatrix} 0 \\ -G_-(y) \end{pmatrix} = -g_-(y) \qquad (3.3.14)$$

Therefore, in terms of S, one can write

$$\sigma_z g_s(y) = S g_s(y) \qquad (3.3.15)$$

In order to find the differential equation satisfied by $G(y)$ we substitute eqn.(3.3.15) in to eqn.(3.3.11)

$$\begin{aligned}
[|E_n|^2 - m^2]g_s(y) &= p_x^2 g_s(y) + p_z^2 g_s(y) - \frac{d^2 g_s}{(dy)^2} + (eQB)^2 y^2 g_s(y) \\
&- 2eQByp_x g_s(y) - eQB\sigma_z g_s(y) \\
&= p_x^2 g_s(y) + p_z^2 g_s(y) - \frac{d^2 g_s}{(dy)^2}(y) + (eQB)^2 y^2 g_s(y) \\
&- 2eQByp_x g_s(y) - SeQBg_s(y) \qquad (3.3.16)
\end{aligned}$$

In substituting the matrix $g_s(y)$ we get a second order differential equation

$$[|E_n|^2 - m^2]G_S(y) = p_x^2 G_S(y) + p_z^2 G_S(y) - \frac{d^2 G_S(y)}{(dy)^2} + (eQB)^2 y^2 G_S(y) - [2eQByp_x + eQBS]G_S(y) \qquad (3.3.17)$$

rearranging the terms,

$$[\frac{d^2}{(dy)^2} + [|E_n|^2 - m^2] - p_x^2 - p_z^2 - (eQB)^2 y^2 + 2eQByp_x + eQBS]G_S(y) = 0$$

$$\frac{1}{e|Q|B}\frac{d^2 G_S(y)}{(dy)^2} - [\sqrt{e|Q|B}(y - \frac{p_x}{eQB})]^2 G_S(y) + \frac{[|E_n|^2 - m^2 - p_z^2 + eQBS]}{e|Q|B} G_S(y) = 0 \qquad (3.3.18)$$

In order to find the solution of eqn.(3.3.18) let us define a dimensionless quantity $\tilde{\xi}$, as

$$\tilde{\xi} = \sqrt{e|Q|B}[y - \frac{p_x}{eQB}] \qquad (3.3.19)$$

$$(d\tilde{\xi})^2 = e|Q|B(dy)^2 \qquad (3.3.20)$$

Using these eqn.(3.3.18) becomes

$$[\frac{d^2}{(d\tilde{\xi})^2} - \tilde{\xi}^2 + a_S]G_S(\tilde{\xi}) = 0 \qquad (3.3.21)$$

where a_S is as defined in eqn.(3.1.42). The easiest way to solve this differential equation is to find a recursion formula by defining the following function

$$G_S(\tilde{\xi}) \;=\; e^{\frac{-\tilde{\xi}^2}{2}}\chi(\tilde{\xi}) \tag{3.3.22}$$

Differentiating this we find

$$\frac{dG_S(\tilde{\xi})}{d\tilde{\xi}} = -\tilde{\xi}e^{\frac{-\tilde{\xi}^2}{2}}\chi(\tilde{\xi}) + e^{\frac{-\tilde{\xi}^2}{2}}\frac{d\chi(\tilde{\xi})}{d\tilde{\xi}} \tag{3.3.23}$$

Differentiating again

$$\begin{aligned}\frac{d^2G_S(\tilde{\xi})}{(d\tilde{\xi})^2} &= -\tilde{\xi}(-\tilde{\xi})e^{\frac{-\tilde{\xi}^2}{2}}\chi(\tilde{\xi}) - e^{\frac{-\tilde{\xi}^2}{2}}\chi(\tilde{\xi}) - \tilde{\xi}e^{\frac{-\tilde{\xi}^2}{2}}\frac{d\chi(\tilde{\xi})}{d\tilde{\xi}} \\ &- \tilde{\xi}e^{\frac{-\tilde{\xi}^2}{2}}\frac{d\chi(\tilde{\xi})}{d\tilde{\xi}} + e^{\frac{-\tilde{\xi}^2}{2}}\frac{d^2\chi(\tilde{\xi})}{(d\tilde{\xi})^2} \\ &= e^{\frac{-\tilde{\xi}^2}{2}}\frac{d^2\chi(\tilde{\xi})}{(d\tilde{\xi})^2} - 2\tilde{\xi}e^{\frac{-\tilde{\xi}^2}{2}}\frac{d\chi(\tilde{\xi})}{d\tilde{\xi}} - e^{\frac{-\tilde{\xi}^2}{2}}(1-\tilde{\xi}^2)\chi(\tilde{\xi}) \qquad (3.3.24)\end{aligned}$$

Using these two expressions eqn.(3.3.21) may be rewritten as

$$\begin{aligned}e^{\frac{-\tilde{\xi}^2}{2}}\frac{d^2\chi(\tilde{\xi})}{(d\tilde{\xi})^2} - 2\tilde{\xi}e^{\frac{-\tilde{\xi}^2}{2}}\frac{d\chi(\tilde{\xi})}{d\tilde{\xi}} - e^{\frac{-\tilde{\xi}^2}{2}}(1-\tilde{\xi}^2)\chi(\tilde{\xi}) &- \\ \tilde{\xi}^2e^{\frac{-\tilde{\xi}^2}{2}}\chi(\tilde{\xi}) + a_Se^{\frac{-\tilde{\xi}^2}{2}}\chi(\tilde{\xi}) &= 0\end{aligned}$$

or

$$\frac{d^2\chi(\tilde{\xi})}{(d\tilde{\xi})^2} - 2\tilde{\xi}\frac{d\chi(\tilde{\xi})}{d\tilde{\xi}} + (a_S-1)\chi(\tilde{\xi}) = 0 \tag{3.3.25}$$

The above equation, eqn.(3.3.25) is just like the Hermite equation and we can assume the solution for $\chi(\tilde{\xi})$ to be some normalization constant N_ν times the Hermite polynomial, that is

$$\chi(\tilde{\xi}) \;=\; N_\nu H_\nu(\tilde{\xi})$$

where N_ν is as defined in eqn.(3.1.58).

Following the same process we followed in the case of positive energy solution, we can derive the negative energy solution for the n^{th} Landau level even in this case as

$$g_+^n = \begin{pmatrix} G_+(\tilde{\xi}) \\ 0 \end{pmatrix} = \begin{pmatrix} J_\nu(\tilde{\xi}) \\ 0 \end{pmatrix} = \begin{pmatrix} J_{n-1}(\tilde{\xi}) \\ 0 \end{pmatrix} \qquad (3.3.26)$$

for $S = +1$ and

$$g_-^n = \begin{pmatrix} 0 \\ G_-(\tilde{\xi}) \end{pmatrix} = \begin{pmatrix} 0 \\ J_\nu(\tilde{\xi}) \end{pmatrix} = \begin{pmatrix} 0 \\ J_n(\tilde{\xi}) \end{pmatrix} \qquad (3.3.27)$$

for $S = -1$. In addition $J_{-1}(\tilde{\xi}) = 0$. For the case of negative energy, $Q = -1$ and $\vec{A} = A_x\hat{i} = -yB\hat{i}$ the upper component of the spinors, Φ from eqn.(3.3.5) for $Q = -1$ is given by

$$\Phi = \frac{\vec{\sigma}.[i\nabla - eA_x\hat{i}]}{|E_n| + m}\chi$$

using eqn.(3.3.10)

$$\begin{aligned} \Phi &= \frac{\vec{\sigma}.[i\nabla - eA_x\hat{i}]}{|E_n| + m} e^{-i\mathbf{p}.\mathbf{X}_y} g(y) \\ &= \frac{\vec{\sigma}.[i\nabla + eBy\hat{i}]}{|E_n| + m} e^{-i(p_x x + p_z z)} g(y) \end{aligned}$$

But

$$\vec{\sigma}.[i\nabla + eBy\hat{i}] = \begin{pmatrix} i\frac{\partial}{\partial z} & i\frac{\partial}{\partial x} + \frac{\partial}{\partial y} + eyB \\ i\frac{\partial}{\partial x} - \frac{\partial}{\partial y} + eyB & -i\frac{\partial}{\partial z} \end{pmatrix} \qquad (3.3.28)$$

Using eqn.(3.3.28) the expression for the upper components of the spinor Φ becomes

$$\Phi = \frac{1}{(|E_n| + m)} \begin{pmatrix} i\frac{\partial}{\partial z} & i\frac{\partial}{\partial x} + \frac{\partial}{\partial y} + eyB \\ i\frac{\partial}{\partial x} - \frac{\partial}{\partial y} + eyB & -i\frac{\partial}{\partial z} \end{pmatrix} e^{-i(p_x x + p_z z)} g(y)$$

The two negative energy solutions which correspond to the two values of S can be obtained by replacing $g(y)$ by $g_+(y)$ and $g_-(y)$.

3.3.1 Negative Energy Solution for $S = +1$ (Spin Up)

For the case of negative energy and positive value of S, the upper components of the spinor becomes

$$\begin{aligned}\Phi &= \frac{1}{(|E_n| + m)} \begin{pmatrix} i\frac{\partial}{\partial z} & i\frac{\partial}{\partial x} + \frac{\partial}{\partial y} + eyB \\ i\frac{\partial}{\partial x} - \frac{\partial}{\partial y} + eyB & -i\frac{\partial}{\partial z} \end{pmatrix} e^{-i(p_x x + p_z z)} g_+(y) \\ &= \frac{1}{(|E_n| + m)} \begin{pmatrix} p_z J_{n-1}(\tilde{\xi}) \\ (eBy + p_x) J_{n-1}(\tilde{\xi}) - \frac{\partial}{\partial y} J_{n-1}(\tilde{\xi}) \end{pmatrix} e^{-i(p_x x + p_z z)}\end{aligned}$$

For the case $Q = -1$,

$$\tilde{\xi} = \sqrt{eB}(y + \frac{p_x}{eB})$$

and

$$\frac{\partial}{\partial y} J_{n-1}(\tilde{\xi}) = \sqrt{eB}\frac{\partial}{\partial \tilde{\xi}} J_{n-1}(\tilde{\xi})$$

with this substitution we get

$$\begin{aligned}\Phi &= \frac{1}{(|E_n| + m)} \begin{pmatrix} p_z J_{n-1}(\tilde{\xi}) \\ eB(\frac{p_x}{eB} + y) J_{n-1}(\tilde{\xi}) - \frac{\partial}{\partial y} J_{n-1}(\tilde{\xi}) \end{pmatrix} e^{-i(p_x x + p_z z)} \\ &= \frac{1}{(|E_n| + m)} \begin{pmatrix} p_z J_{n-1}(\tilde{\xi}) \\ \sqrt{eB}[\tilde{\xi} J_{n-1}(\tilde{\xi}) - \frac{\partial}{\partial \tilde{\xi}} J_{n-1}(\tilde{\xi})] \end{pmatrix} e^{-i(p_x x + p_z z)}\end{aligned}$$

From eq(3.2.11)

$$\sqrt{2n} J_n(\tilde{\xi}) = \tilde{\xi} J_{n-1}(\tilde{\xi}) - \frac{\partial}{\partial \tilde{\xi}} J_{n-1}(\tilde{\xi})$$

Using this the upper component of the spinor becomes

$$\begin{aligned}\Phi &= \frac{1}{(|E_n|+m)}\begin{pmatrix} p_z J_{n-1}(\tilde{\xi}) \\ \sqrt{eB}\sqrt{2n}J_n(\tilde{\xi}) \end{pmatrix} e^{-i(p_x x + p_z z)} \\ &= \begin{pmatrix} \frac{p_z}{|E_n|+m} J_{n-1}(\tilde{\xi}) \\ \\ \frac{\sqrt{2neB}}{|E_n|+m} J_n(\tilde{\xi}) \end{pmatrix} e^{-i\mathbf{p}.\mathbf{X}_{\not y}} \end{aligned} \tag{3.3.29}$$

In general for the case $S = +1$ and negative energy, the upper and lower components of the spinor can be given by,

$$\begin{aligned}\begin{pmatrix} \Phi \\ \chi \end{pmatrix} &= \begin{pmatrix} \Phi \\ e^{-i\mathbf{p}.\mathbf{X}_{\not y}} g_+(y) \end{pmatrix} \\ &= \begin{pmatrix} \frac{p_z}{|E_n|+m)} J_{n-1}(\tilde{\xi}) \\ \\ \frac{\sqrt{2neB}}{|E_n|+m)} J_n(\tilde{\xi}) \\ \\ J_{n-1}(\tilde{\xi}) \\ 0 \end{pmatrix} e^{-i\mathbf{p}.\mathbf{X}_{\not y}} \end{aligned}$$

3.3.2 Negative Energy Solution for $S = -1$ (Spin Down)

The other value of the spinors for the case $S = -1$ can be obtained by substituting

$$\chi = e^{-i\mathbf{p}.\mathbf{X}_{\not y}} g_-(y) \tag{3.3.30}$$

With this value the upper component of the negative energy spinor will be

$$\Phi = \frac{\vec{\sigma}.(i\nabla - eA_x\hat{i})}{|E_n|+m} e^{-i\mathbf{p}.\mathbf{X}_{\not y}} \begin{pmatrix} 0 \\ J_n(\tilde{\xi}) \end{pmatrix} \tag{3.3.31}$$

Upon using eqn.(3.3.28)

$$
\begin{aligned}
\Phi &= \frac{1}{(|E_n|+m)} \begin{pmatrix} i\frac{\partial}{\partial z} & i\frac{\partial}{\partial x}+\frac{\partial}{\partial y}+eyB \\ i\frac{\partial}{\partial x}-\frac{\partial}{\partial y}+eyB & -i\frac{\partial}{\partial z} \end{pmatrix} e^{-i(p_x x+p_z z)} \begin{pmatrix} 0 \\ J_n(\tilde{\xi}) \end{pmatrix} \\
&= \frac{1}{(|E_n|+m)} \begin{pmatrix} eB(y+\frac{p_x}{eB})J_n(\tilde{\xi})+\frac{\partial}{\partial y}J_n(\tilde{\xi}) \\ -p_z J_n(\tilde{\xi}) \end{pmatrix} e^{-i(p_x x+p_z z)}
\end{aligned}
$$

For the case $Q=-1$,

$$\tilde{\xi} = \sqrt{eB}(y+\frac{p_x}{eB})$$

and

$$\frac{\partial}{\partial y}J_n(\tilde{\xi}) = \sqrt{eB}\frac{\partial}{\partial\tilde{\xi}}J_n(\tilde{\xi})$$

With this substitution we get

$$
\begin{aligned}
\Phi &= \frac{1}{(|E_n|+m)} \begin{pmatrix} eB(y+\frac{p_x}{eB})J_n(\tilde{\xi})+\sqrt{eB}\frac{\partial}{\partial\tilde{\xi}}J_n(\tilde{\xi}) \\ -p_z J_n(\tilde{\xi}) \end{pmatrix} e^{-i(p_x x+p_z z)} \\
&= \frac{1}{(|E_n|+m)} \begin{pmatrix} \sqrt{eB}[\tilde{\xi}J_n(\tilde{\xi})+\frac{\partial}{\partial\tilde{\xi}}J_n(\tilde{\xi})] \\ -p_z J_n(\tilde{\xi}) \end{pmatrix} e^{-i(p_x x+p_z z)}
\end{aligned}
$$

From eqn.(3.2.20)

$$\sqrt{2n}J_{n-1}(\tilde{\xi}) = \tilde{\xi}J_n(\tilde{\xi})+\frac{\partial}{\partial\tilde{\xi}}J_n(\tilde{\xi})$$

Using this the upper component of the negative energy spinor becomes

$$\Phi = \frac{1}{(|E_n|+m)} \begin{pmatrix} \sqrt{eB}\sqrt{2n}J_{n-1}(\tilde{\xi}) \\ -p_z J_n(\tilde{\xi}) \end{pmatrix} e^{-i(p_x x+p_z z)} \tag{3.3.32}$$

$$= \begin{pmatrix} \frac{\sqrt{2neB}}{|E_n|+m}J_{n-1}(\tilde{\xi}) \\ \\ -\frac{p_z}{|E_n|+m}J_n(\tilde{\xi}) \end{pmatrix} e^{-i\mathbf{p}.\mathbf{X}_{\not{y}}} \tag{3.3.33}$$

In general for the case $S = -1$ and negative energy, the upper and lower components of the spinor can be given by,

$$
\begin{pmatrix} \Phi \\ \chi \end{pmatrix} = \begin{pmatrix} \Phi \\ e^{-i\mathbf{p}.\mathbf{X}_y} g_-(y) \end{pmatrix}
= \begin{pmatrix} \frac{\sqrt{2neB}}{|E_n|+m} J_{n-1}(\tilde{\xi}) \\ \\ -\frac{p_z}{|E_n|+m} J_n(\tilde{\xi}) \\ 0 \\ J_n(\tilde{\xi}) \end{pmatrix} e^{-i\mathbf{p}.\mathbf{X}_y}
$$

When we come to the energy eigne function, With our assumption the negative energy eigne function of the dirac equation is given by eqn.(3.1.10) and after using $E = -|E_n|$ becomes

$$
\Psi(r,t) = e^{i|E_n|t} \begin{pmatrix} \Phi \\ \chi \end{pmatrix}
$$

substituting the upper and lower components of the spinors we get,

$$
\Psi_+(r,t) = e^{iEt - i\mathbf{p}.\mathbf{X}_y} \begin{pmatrix} \frac{p_z}{|E_n|+m} J_{n-1}(\tilde{\xi}) \\ \\ \frac{\sqrt{2neB}}{|E_n|+m} J_n(\tilde{\xi}) \\ \\ J_{n-1}(\tilde{\xi}) \\ 0 \end{pmatrix} \tag{3.3.34}
$$

for the case of $S = +1$ and

$$\Psi_{-}(r,t) = e^{i|E_n|t - i\mathbf{p}.\mathbf{X}_{\not{y}}} \begin{pmatrix} \frac{\sqrt{2neB}}{|E_n|+m} J_{n-1}(\tilde{\xi}) \\ \\ -\frac{p_z}{|E_n|+m} J_n(\tilde{\xi}) \\ 0 \\ J_n(\tilde{\xi}) \end{pmatrix} \tag{3.3.35}$$

for the case of $S = -1$.

The negative energy eigne function can be written in a covariant form as

$$\Psi_{+}(r,t) = e^{ip_{\mu}.\mathbf{X}^{\mu}_{\not{y}}} V_{+}(y, n, p_{\not{y}}) \tag{3.3.36}$$

and

$$\Psi_{-}(r,t) = e^{ip_{\mu}.(\mathbf{X}^{\mu}_{\not{y}})} V_{-}(y, n, p_{\not{y}}) \tag{3.3.37}$$

where

$$V_{+}(y, n, p_{\not{y}}) = \begin{pmatrix} \frac{p_z}{|E_n|+m} J_{n-1}(\tilde{\xi}) \\ \\ \frac{\sqrt{2neB}}{|E_n|+m} J_n(\tilde{\xi}) \\ \\ J_{n-1}(\tilde{\xi}) \\ 0 \end{pmatrix} \tag{3.3.38}$$

and

$$V_{-}(y, n, p_{\not{y}}) = \begin{pmatrix} \frac{\sqrt{2neB}}{|E_n|+m} J_{n-1}(\tilde{\xi}) \\ \\ -\frac{p_z}{|E_n|+m} J_n(\tilde{\xi}) \\ 0 \\ J_n(\tilde{\xi}) \end{pmatrix} \tag{3.3.39}$$

Generally one can write

$$\Psi_{S}(r,t) = e^{ip^{\mu}.(\mathbf{X}_{\not{y}})_{\mu}} V_{S}(y, n, p_{\not{y}}) \tag{3.3.40}$$

here $\mathbf{X}^{\mu}_{\not{y}}$ is the space time coordinate with out including the y component.

3.4 Ortho-Normality of the Solutions

In this part we will check the ortho-normality of the spinors obtained for positive and negative energy. First we consider the two positive energy spinors, given by eqn.(3.2.26) and eqn.(3.2.27). From the two spinors we can see that

$$\begin{aligned}
U_+^\dagger(y,n,p_z)U_-(y,n,p_z) &= \begin{pmatrix} I_{n-1}(\xi) & 0 & \frac{p_z(\xi)}{E_n+m}I_{n-1}(\xi) & -\frac{\sqrt{2neB}}{E_n+m}I_n(\xi) \end{pmatrix} \begin{pmatrix} 0 \\ I_n(\xi) \\ -\frac{\sqrt{2neB}}{E_n+m}I_{n-1}(\xi) \\ -\frac{p_z(\xi)}{E_n+m}I_n(\xi) \end{pmatrix} \\
&= \frac{-\sqrt{2neB}p_z}{[E_n+m]^2}I_{n-1}^2(\xi) + \frac{\sqrt{2neB}p_z}{[E_n+m]^2}I_n^2(\xi) \\
&= \frac{\sqrt{2neB}p_z}{[E_n+m]^2}(I_n^2(\xi) - I_{n-1}^2(\xi)) \qquad (3.4.1)
\end{aligned}$$

Here in the limit $n \to \infty$ we can approximate $n-1 \sim n$ and with this limit, $I_{n-1}(\xi) \sim I_n(\xi)$ so that

$$U_+^\dagger(y,n,p_z)U_-(y,n,p_z) = 0 \qquad (3.4.2)$$

In the same way

$$\begin{aligned}
U_+^\dagger(y,n,p_z)U_+(y,n,p_z) &= \begin{pmatrix} I_{n-1}(\xi) & 0 & -\frac{p_z(\xi)}{E_n+m}I_{n-1}(\xi) & -\frac{\sqrt{2neB}}{E_n+m}I_n(\xi) \end{pmatrix} \begin{pmatrix} I_{n-1}(\xi) \\ 0 \\ \frac{p_z(\xi)}{E_n+m}I_n(\xi) \\ -\frac{\sqrt{2neB}}{E_n+m}I_n(\xi) \end{pmatrix} \\
&= \frac{[E_n^2+m^2+2E_nm+p_z^2]}{[E_n+m]^2}I_{n-1}^2(\xi) + \frac{2neB}{[E_n+m]^2}I_n^2(\xi)
\end{aligned}$$

In the limit $n \to \infty$ we can approximate $n-1 \sim n$ and with this limit, $I_{n-1}(\xi) \sim I_n(\xi)$. In addition, the relativistic form of the Landau energy is given by

$$E_n^2 = m^2 + p_z^2 + 2neB$$

Upon substituting these values one can get

$$\begin{aligned} U_+^\dagger(y,n,p_x)U_+(y,n,p_x) &= \frac{[E_n^2+m^2+p_z^2+2neB+2E_nm]}{[E_n+m]^2}I_n^2(\xi) \\ &= \frac{[E_n^2+E_n^2+2E_nm]}{[E_n+m]^2}I_n^2(\xi) \\ &= \frac{2E_n}{E_n+m}I_n^2(\xi) \end{aligned}$$

In the same way the product $U_-^\dagger U_-$ becomes

$$\begin{aligned} U_-^\dagger(y,n,p_x)U_-(y,n,p_x) &= \begin{pmatrix} 0 & I_n(\xi) & -\frac{\sqrt{2neB}}{E_n+m}I_{n-1}(\xi) & -\frac{p_z(\xi)}{E_n+m}I_n(\xi) \end{pmatrix} \begin{pmatrix} 0 \\ I_n(\xi) \\ -\frac{\sqrt{2neB}}{E_n+m}I_{n-1}(\xi) \\ -\frac{p_z(\xi)}{E_n+m}I_n(\xi) \end{pmatrix} \\ &= \frac{[E_n^2+m^2+2E_nm+2neB]}{[E_n+m]^2}I_n^2(\xi) + \frac{p_z}{[E_n+m]^2}I_{n-1}^2(\xi) \end{aligned}$$

In the limit $n \to \infty$ we can approximate $n-1 \sim n$ and with this limit, $I_{n-1}(\xi) \sim I_n(\xi)$. Upon substituting this together with eq (3.1.62) one can get

$$\begin{aligned} U_-^\dagger(y,n,p_x)U_-(y,n,p_x) &= \frac{[E_n^2+m^2+p_z^2+2neB+2E_nm]}{[E_n+m]^2}I_n^2(\xi) \\ &= \frac{[E_n^2+E_n^2+2E_nm]}{[E_n+m]^2}I_n^2(\xi) \\ &= \frac{2E_n}{E_n+m}I_n^2(\xi) \end{aligned}$$

The last product $U_-^\dagger U_+$ becomes

$$
\begin{aligned}
U_-^\dagger(y,n,p_y)U_+(y,n,p_y) &= \begin{pmatrix} 0 & I_n(\xi) & -\frac{\sqrt{2neB}}{E_n+m}I_{n-1}(\xi) & -\frac{p_z}{E_n+m}I_n(\xi) \end{pmatrix} \begin{pmatrix} I_{n-1}(\xi) \\ 0 \\ \frac{p_z}{E_n+m}I_{n-1}(\xi) \\ -\frac{\sqrt{2neB}}{E_n+m}I_n(\xi) \end{pmatrix} \\
&= \frac{-\sqrt{2neB}p_z}{[E_n+m]^2}I_{n-1}^2(\xi) + \frac{\sqrt{2neB}p_z}{[E_n+m]^2}I_n^2(\xi) \\
&= \frac{\sqrt{2neB}p_z}{[E_n+m]^2}[I_n^2(\xi) - I_{n-1}^2(\xi)] \qquad (3.4.3)
\end{aligned}
$$

Here in the limit $n \to \infty$ we can approximate $n-1 \sim n$ and with this limit, $I_{n-1}(\xi) \sim I_n(\xi)$ so that

$$
U_-^\dagger(y,n,p_y)U_+(y,n,p_y) = 0 \qquad (3.4.4)
$$

In General the positive energy spinors can satisfy the following ortho-normality property

$$
U_s^\dagger(y,n,p_y)U_{s'}(y,n,p_y)|_{n\to\infty} = \frac{2E_n}{E_n+m}I_n^2(\xi)\delta_{s,s'} \qquad (3.4.5)
$$

where $\delta_{s,s'}$ is the Kronecker delta.

In the same way the negative energy spinors which are given by eqn.(3.3.39) and eqn.(3.3.40) satisfy the following ortho-normality relations,

$$
\begin{aligned}
V_+^\dagger(y,n,p_y)V_-(y,n,p_y) &= \begin{pmatrix} \frac{p_z}{|E_n|+m}J_{n-1}(\tilde{\xi}) & \frac{\sqrt{2neB}}{|E_n|+m}J_n(\tilde{\xi}) & J_{n-1}(\tilde{\xi}) & 0 \end{pmatrix} \begin{pmatrix} \frac{\sqrt{2neB}}{|E_n|+m}J_{n-1}(\tilde{\xi}) \\ -\frac{p_z}{|E_n|+m}J_n(\tilde{\xi}) \\ 0 \\ J_n(\tilde{\xi}) \end{pmatrix} \\
&= \frac{\sqrt{2neB}p_z}{[|E_n|+m]^2}J_{n-1}^2(\tilde{\xi}) - \frac{\sqrt{2neB}p_z}{[|E_n|+m]^2}J_n^2(\tilde{\xi}) \\
&= \frac{\sqrt{2neB}p_z}{[|E_n|+m]^2}[J_{n-1}^2(\tilde{\xi}) - J_n^2(\tilde{\xi})]
\end{aligned}
$$

Here in the limit $n \to \infty$ we can assume $n-1 \sim n$ and with this limit, $J_{n-1}(\xi) \sim J_n(\xi)$ so that

$$
V_+^\dagger(y,n,p_y)V_-(y,n,p_y) = 0 \tag{3.4.6}
$$

In the same way

$$
\begin{aligned}
V_+^\dagger(y,n,p_y)V_+(y,n,p_y) &= \begin{pmatrix} \frac{p_z}{|E_n|+m)}J_{n-1}(\tilde{\xi}) & \frac{\sqrt{2neB}}{|E_n|+m)}J_n(\tilde{\xi}) & J_{n-1}(\tilde{\xi}) & 0 \end{pmatrix} \begin{pmatrix} \frac{p_z}{|E_n|+m}J_{n-1}(\tilde{\xi}) \\ \frac{\sqrt{2neB}}{|E_n|+m}J_n(\tilde{\xi}) \\ J_{n-1}(\tilde{\xi}) \\ 0 \end{pmatrix} \\
&= \frac{(|E_n|^2+m^2+2|E_n|m+p_z^2)}{[|E_n|+m]^2}J_{n-1}^2(\tilde{\xi}) + \frac{2neB}{[|E_n|+m]^2}J_n^2(\tilde{\xi})
\end{aligned}
$$

In the limit $n \to \infty$ we can approximate $n-1 \sim n$ which implies, $J_{n-1}(\tilde{\xi}) \sim J_n(\tilde{\xi})$. Upon substituting this and eqn.(3.1.62) we get

$$
\begin{aligned}
V_+^\dagger(y,n,p_y)V_+(y,n,p_y) &= \frac{[|E_n|_n^2+m^2+p_z^2+2neB+2|E_n|m]}{[|E_n|+m]^2}J_n^2(\tilde{\xi}) \\
&= \frac{[|E_n|^2+|E_n|^2+2|E_n|m]}{[|E_n|+m]^2}J_n^2(\tilde{\xi}) \\
&= \frac{2|E_n|}{|E_n|+m}J_n^2(\tilde{\xi})
\end{aligned}
$$

The product $V_-^\dagger V_-$ becomes

$$
\begin{aligned}
V_-^\dagger(y,n,p_{\backslash y})V_-(y,n,p_{\backslash y}) &= \begin{pmatrix} \frac{\sqrt{2neB}}{|E_n|+m)}J_{n-1}(\tilde{\xi}) & -\frac{p_z}{|E_n|+m)}J_n(\tilde{\xi}) & 0 & J_n(\tilde{\xi}) \end{pmatrix} \begin{pmatrix} \frac{\sqrt{2neB}}{|E_n|+m)}J_{n-1}(\tilde{\xi}) \\ -\frac{p_z}{|E_n|+m)}J_n(\tilde{\xi}) \\ 0 \\ J_n(\tilde{\xi}) \end{pmatrix} \\
&= \frac{[|E_n|^2+m^2+2|E_n|m+p_z^2]}{|E_n|+m}J_n^2(\tilde{\xi}) + \frac{2neB}{[|E_n|+m]^2}J_{n-1}^2(\tilde{\xi})
\end{aligned}
$$

In the limit $n \to \infty$ we can approximate $n-1 \sim n$ and with this limit, $J_{n-1}(\tilde{\xi}) \sim J_n(\tilde{\xi})$. Upon substituting this and eqn.(3.1.62) we get

$$
\begin{aligned}
V_-^\dagger(y,n,p_{\backslash y})V_-(y,n,p_{\backslash y}) &= \frac{[|E|_n^2+m^2+p_z^2+2neB+2E_nm]}{[|E_n|+m]^2}J_n^2(\tilde{\xi}) \\
&= \frac{[|E_n|^2+|E_n|^2+2|E_n|m]}{[|E_n|+m]^2}J_n^2(\tilde{\xi}) \\
&= \frac{2|E_n|}{|E_n|+m}I_n^2(\xi)J_n^2(\tilde{\xi})
\end{aligned}
$$

The last product $V_-^\dagger V_+$ becomes

$$
\begin{aligned}
V_-^\dagger(y,n,p_{\backslash y})V_+(y,n,p_{\backslash y}) &= \begin{pmatrix} \frac{\sqrt{2neB}}{|E_n|+m)}J_{n-1}(\tilde{\xi}) & -\frac{p_z}{|E_n|+m)}J_n(\tilde{\xi}) & 0 & J_n(\tilde{\xi}) \end{pmatrix} \begin{pmatrix} \frac{p_z}{|E_n|+m)}J_{n-1}(\tilde{\xi}) \\ \frac{\sqrt{2neB}}{|E_n|+m)}J_n(\tilde{\xi}) \\ J_{n-1}(\tilde{\xi}) \\ 0 \end{pmatrix} \\
&= \frac{\sqrt{2neB}p_z}{[|E_n|+m]^2}(J_{n-1}^2(\tilde{\xi}) - J_n^2(\tilde{\xi}))
\end{aligned}
$$

Here in the limit $n \to \infty$ we can assume $n-1 \sim n$ and with this limit,$J_{n-1}(\xi) \sim J_n(\xi)$ so that

$$
V_-^\dagger(y,n,p_{\backslash y})V_+(y,n,p_{\backslash y}) \;=\; 0 \qquad (3.4.7)
$$

In General the negative energy spinors satisfy the following ortho-normality property

$$V_s^\dagger(y,n,p_y)V_{s'}(y,n,p_y)|_{n\to\infty} = \frac{2|E_n|}{|E_n|+m}J_n^2(\tilde{\xi})\delta_{s,s'} \tag{3.4.8}$$

where $\delta_{s,s'}$ is one when $S = S'$ and zero otherwise.

Consider now the combined products,

$$\begin{aligned} U_+^\dagger(y,n,p_y)V_+(y,n,-p_y) &= \begin{pmatrix} I_{n-1}(\xi) & 0 & \frac{p_z(\xi)}{E_n+m}I_{n-1}(\xi) & -\frac{\sqrt{2neB}}{E_n+m}I_n(\xi) \end{pmatrix} \begin{pmatrix} -\frac{p_z}{|E_n|+m}J_{n-1}(\tilde{\xi}) \\ \frac{\sqrt{2neB}}{|E_n|+m}J_n(\tilde{\xi}) \\ J_{n-1}(\tilde{\xi}) \\ 0 \end{pmatrix} \\ &= -\frac{p_z}{|E_n|+m}I_{n-1}(\xi)J_{n-1}(\tilde{\xi}) + \frac{p_z}{E_n+m}I_{n-1}(\xi)J_{n-1}(\tilde{\xi}) \end{aligned}$$

In the limit $n \to \infty$, one can approximate $n-1 \sim n$, and it is possible to write $J_{n-1}(\tilde{\xi}) \sim J_n(\tilde{\xi})$ and $I_{n-1}(\xi) = I_n(\xi)$. Therefore, with this approximation,

$$\begin{aligned} U_+^\dagger(y,n,p_y)V_+(y,n,-p_y) &= -\frac{p_z}{|E_n|+m}I_n(\xi)J_n(\tilde{\xi}) + \frac{p_z}{E_n+m}I_n(\xi)J_n(\tilde{\xi}) \\ &= 0 \end{aligned}$$

In the same way, the following possible three products each reduces to zero.

$$U_+^\dagger(y,n,p_y)V_-(y,n,-p_y) = 0$$

$$U_-^\dagger(y,n,p_y)V_+(y,n,-p_y) = 0$$

$$U_-^\dagger(y,n,p_y)V_-(y,n,-p_y) = 0$$

From these relations we can generalize the ortho-gonality relation for the case in which $n \to \infty$, and for the combination of positive and negative energy as spinors as,

$$U_s^\dagger(y,n,p_y)V_{s'}(y,n,-p_y)|_{n\to\infty} = 0 \tag{3.4.9}$$

For the case of $n = 0$ which is the lowest Landau level, we have

$$I_{-1}(\xi) \quad = \quad 0$$

and

$$J_{-1}(\tilde{\xi}) \quad = \quad 0$$

So, for this case the positive and negative energy spinors becomes zero. That is

$$U_{+} = \begin{pmatrix} I_{n-1}(\xi) \\ 0 \\ \frac{p_z}{E_n+m} I_{n-1}(\xi) \\ -\frac{\sqrt{2neB}}{E_n+m} I_n(\xi) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \tag{3.4.10}$$

$$V_{+} = \begin{pmatrix} \frac{p_z}{E_n+m} J_{n-1}(\tilde{\xi}) \\ \frac{\sqrt{2neB}}{E_n+m} J_n(\tilde{\xi}) \\ J_{n-1}(\tilde{\xi}) \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \tag{3.4.11}$$

From this we conclude that all multiplications with these two spinors always becomes zero. Which means the $n = 0$ state represents orthogonal positive and negative energy eigenfunctions and hence is non degenerate.

Chapter 4

Discussion and Conclusion

4.1 Discussion

In solving the Dirac equation in the presence of a uniform back-ground magnetic field, by assuming the field to be along the $z-$ axis and imposing a gauge condition as given in eqn.(1.5.2) we got different spinors for the two possible positive and negative energies as given in eqn.(3.2.26), (3.2.27), (3.3.39) and (3.3.40) respectively. All the solution we got are completely dependent on the gauge condition we impose for the magnetic vector potential. However all the results we got in this work are dependent on the gauge conditions. For example if we consider the expression for the Landau energy, as in eqn.(3.1.62) one can notice that the energy of the moving particle or electron does not depend on the gauge condition we have taken.

As one can see from the expression

$$a_S = \frac{E^2 - m^2 - p_z^2 + eQBS}{e|Q|B}$$

we can not put $B \to 0$ in the final solutions, for the spinors and expect to get back the free solutions. In this limit the solution of,

$$[\frac{d^2}{(d\xi)^2} - \xi^2 + a_S]F_s(\xi) = 0$$

and

$$[\frac{d^2}{(d\tilde{\xi})^2} - \tilde{\xi}2 + a_S]G_s(\tilde{\xi}) \;\; = \;\; 0$$

becomes indeterminant so that all the spinors we got for the two energies are specific to a gauge, giving rise to a magnetic field along the $z-$axis. The choice of the background gauge does not permit us to obtain the free solutions in any limit because the free solutions belongs to another gauge orbit, namely the gauge less solutions.

From the expression

$$E_n^2 \;\; = \;\; m^2 + p_z^2 + 2neB$$

the $n = 0$ state which can be used to deal with when the magnitude of the magnetic field strength is very high, is non-degenerate and in this state we have only one solution available for the positive energy and one for the negative energy.

The forms of the orth-normality relations as done in the last section, section 3.4 are gauge dependent and are not ortho-normal, since in all the results we have the function p_x which is not gauge invariant quantity. But when the landau levels are many ($n \to \infty$) and the field strength is small, then at a certain limit, $n \sim n-1$, and only in this limit, the spinors we got for the positive and negative energy becomes ortho-normal. In most of the cases the limit for n or the maximum landau level can be calculated from the maximum attainable energy of the electron and field strength. Once we know the maximum attainable energy (E_{max}) from observational data or experimental limit, the highest landau level is given by

$$n_{max} \;\; = \;\; \frac{E_{max}^2 - m^2}{2eB}$$

Here if the field strength decreases, then n_{max} increases and the possible number of landau levels becomes very high, at this time we can say $n \to \infty$. This discussion implies only in the special cases where the magnetic field strength is very small, we will have a very large amount of Landau levels and at this level the solutions will be ortho-normal.

4.2 Conclusion

As we have seen throughout this work, the solutions of the Dirac equation strongly dependent on the gauge condition and Landau level. The energy of the electron is seen to be degenerate except the lowest level. In addition we have seen that there is no way to get the free Dirac solution from the exact solution in the presence of a uniform magnetic field, by letting the magnetic field strength to go to zero, as there is no connection between the gauge configuration given rise to a magnetic field and pure gauge fields. The solutions are seen not to be ortho-normal in general but when the number of the landau level is very high, it is seen that the solutions show ortho-normality. On the contrary the lowest Landau level is always ortho-normal, as one of the spinors do not exist. At last even if the results we got have a great application in calculations of decay-rates and scattering cross section of charged particles in the presence of a uniform magnetic field, in real situation the fields associated with most of the bodies is complicated and for such fields we can not get exact solution of the Dirac equation, only under the assumption of uniform field we can get the exact solution.

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Declaration

I hereby declare that this thesis is my original work and has not been presented for a degree in any other university. All sources of material used for the thesis have been duly acknowledged.

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