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On

STRUCTURE OF SEMIPRIME GOLDIE RINGS

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CHAPTER ONE

CHAPTER TWO

STRUCTURE OF SEMIPRIME GOLDIE RINGS

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INTRODUCTION

The first major step in a structure theory for non commutative Noetherian ring is Goldie΄ s theorem. But this seminar report explores some consequences and extensions of Goldie΄ s theorem. The first chapter of this report includes definitions and facts that are already familiar and are useful to understand new concepts which are discussed in this material.

The second chapter of the material contains four sections. Section 2.1 deals on the phenomenon and connections of different rings that have the same right quotient rings with earlier results. section 2.2 focus on minimal prime ideals. Here it is shown that a semiprime right Goldie ring R is equivalent to a direct sum of prime right Goldie rings, namely $\oplus R/P_i$, where the P_i are the minimal prime ideals and each prime right Goldie ring is equivalent to a matrix ring over right ore domain.

The connections between right ideals of R and of Q where Q is a right quotient ring of R, with special interest in uniform and essential ideals, and the concept 'essential right ideals are generated by regular elements' are discussed in section 2.3. Finally in theorem 2.3.1 of section 2.3 it was shown that the endomorphism ring of a uniform right ideal of R is a right ore domain. This result will now be extended to describe endomorphism rings of all right ideals and certain modules, in section 2.4.

NOTATIONS

In this seminar report, if N is the submodule of an R-module M the notation $N \lhd M$ is used. If N is an essential submodule of M we write $N \lhd_e M$. Analogously if I is an ideal of a ring R we use the notation $I \lhd R$, in particular if I is a right ideal of R and I is an essential ideal of R we write $I \lhd_r R$ and $I \lhd_e R$ respectively.

CHAPTER ONE

PRELIMINARIES

The objective of this chapter is to remind concepts such as Noetherian and Artinian modules and rings, semiprime rings, Goldie rings, essential submodules (essential ideals),... and so on, which are useful to understand the new concepts in the rest chapter of this material.

Definition: Let R be a ring, and M be an abelian group. Then M is called a *left R-module* if there exists a scalar multiplication $\mu: R \times M \longrightarrow M$ denoted by $\mu(r, m) = rm$, for all $r \in R$ and all $m \in M$, such that for all $r, r_1, r_2 \in R$ and all $m, m_1, m_2 \in M$,

(i) $r(m_1+m_2) = rm_1+rm_2$

(ii) $(r_1+r_2)m = r_1m+r_2m$

(iii) $r_1 (r_2m) = (r_1r_2)m$

(iv) $1m = m$

The fact that the abelian group M is a left right R-module will be denoted by $M = {}_RM$, $M = M_R$ respectively .

Example: let M be the set of m x n matrices over a ring R. Then M is a module over R, because M is an additive abelian group under the usual addition of matrices and the usual scalar multiplication (ra_{ij}) of the matrix $(a_{ij}) \in M$ by the element $r \in R$ satisfies the above four conditions for a module .

Definition: A non-empty subset N of an R-module M is called an *R-submodule* (or simply submodule)of M if

(i) $a - b \in N$ for all $a, b \in N$

(ii) $ra \in N$ for all $a \in N$, $r \in R$.

M and 0 are R-submodules, called trivial submodules.

Example : If M is an R module and $x \in M$, then the set $Rx = \{ rx \mid r \in R \}$ is an R-submodule of M , for

$$r_1x - r_2x = (r_1 - r_2)x \in Rx$$

$$r_1(r_2x) = (r_1r_2)x \in Rx, \text{ for all } r_1, r_2 \in R.$$

Definition: Let M be a left R-module

(i) The module M is said to be *Noetherian* if every ascending chain $M_1 \subseteq M_2 \subseteq \dots$ of submodules of M must terminate after a finite number of steps.

(ii) The module M is said to be *Artinian* if every descending chain $M_1 \supseteq M_2 \supseteq \dots$ of submodules of M must terminate after a finite number of steps.

There is a corresponding definition for rings

Definition: Let R be any ring

(i) The ring R is said to be *left Noetherian* if the module ${}_RR$ is Noetherian

(ii) The ring R is said to be *left Artinian* if the module R_R is Artinian

(iii) If R satisfies the condition for both right and left ideals, then it is simply said to be *Noetherian* or *Artinian.*

Example : 1. Because the ring of integers is a principal ideal ring, any ascending chain of ideals of $\mathbb{Z}$ is of the form $(n) \subset (n_1) \subset (n_2) \subset \dots$where $n, n_1, n_2, \dots$ are in $\mathbb{Z}$. Because $(n_i) \subseteq (n_{i+1})$ implies n_{i+1}/n_i , any ascending chain of ideals in $\mathbb{Z}$ starting with n can have at most n distinct terms . This shows that $\mathbb{Z}$ as a $\mathbb{Z}$-module is Noetherian. But $\mathbb{Z}$ as a $\mathbb{Z}$ -module has an infinite properly descending chain $(n) \supset (n^2) \supset (n^3) \supset \dots$, showing that $\mathbb{Z}$ is not Artinian as a $\mathbb{Z}$-module.

2. Let $M = M_n(D)$ be the n x n matrix ring over a division ring D. Then M is an n^2- dimensional vector space over D, and each left ideal as well as aright ideal of M is a sub space over D. Thus any ascending or descending chain of left (as well as right) ideals can not contain more than $n^2 + 1$ terms. Thus $M = M_n(D)$ is both Noetherian and Artinian ring.

Definition: (i) A non zero module M is called *simple* (irreducible) if its only submodules are (0) and M .

(ii) A *simple ring* is any non zero ring R such that the only ideals of R are (0) and R.

(iii) The *socle of a module M* is the sum of all simple submodules of M and is denoted by soc(M).

(iv) A *semisimple module* is any module M such that soc(M) = M.

Fact (1): For any ring R, the following are equivalent

(i) All right(left) R-modules are semisimple ;

(ii) R_R and $_RR$ are semisimple ;

(iii) Either R is the zero ring or R ≃ $M_{n1}(D)$ x ....x $M_{nk}(D)$ for some positive integers n and some division ring D.

Definition: A ring satisfying the conditions of fact 1 above is called a *semisimple ring*.

Fact (2): 1.If A is a semisimple ring, then

a) There is a finite set of pair wise orthogonal idempotents such that (i) $1 = \sum_{i=1}^{n} e_i$, (ii) $A_A = \oplus_{i=1}^{n} Ae_i$, (iii) Ae_i is a minimal left ideal of A for each
$i = 1, \ldots, n$.

b) A is left Artinian and left Noetherian.

c) A has only finitely many minimal two sided ideals and equals to their direct sum.

2. A ring A is semisimple iff it is equal to a direct sum of finitely many two sided ideals each of which is a simple ring. This decomposition is unique.

Definition: A *prime ideal* in a ring R is any proper ideal P of R such that whenever I and J are ideals of R with IJ ⊆ P, either I ⊆ P or J ⊆ P.

Example: 1. $p\mathbb{Z}$ is a prime ideal of $\mathbb{Z}$. Where p is a prime number.

2. Let R = $M_2(\mathbb{Z})$ and P = $M_2(p\mathbb{Z})$ where p is prime number. Let A = $M_2(n\mathbb{Z})$ and B = $M_2(m\mathbb{Z})$ for some $m, n \in \mathbb{Z}$ such that AB ⊆ P, implies $(n\mathbb{Z})(m\mathbb{Z}) \subseteq p\mathbb{Z}$, so either $n\mathbb{Z} \subseteq p\mathbb{Z}$ or $m\mathbb{Z} \subseteq p\mathbb{Z}$ since $p\mathbb{Z}$ is prime ideals of $\mathbb{Z}$. Thus either A ⊆ P or B ⊆ P. Hence P is prime ideal of R.

Definition: A *prime ring* is a ring in which 0 is a prime ideal.

Example: 1. $\mathbb{Z}$ is a prime ring.

2. Since $\mathbb{Z}$ is prime ring then $M_n(\mathbb{Z})$ is also prime ring .

Definition: (i) A *semiprime ideal* in a ring R is any ideal of R which is an intersection of prime ideals of R.

(ii) A *semiprime ring* is any ring in which (0) is a semiprime ideal.

Example: Let R = $\begin{pmatrix} \mathbb{Z} & \mathbb{Z} \\ 0 & \mathbb{Z} \end{pmatrix}$, the prime ideals of R are of the form $\begin{pmatrix} p\mathbb{Z} & \mathbb{Z} \\ 0 & \mathbb{Z} \end{pmatrix}$ or $\begin{pmatrix} \mathbb{Z} & \mathbb{Z} \\ 0 & q\mathbb{Z} \end{pmatrix}$ where p and q are prime integers, then the intersection of these prime ideals is $\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$ therefore, $\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$ is a semiprime ideal in R. Since the intersection of prime ideals in R is $\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$, then R is a semiprime ring .

Fact (3): 1. For any ring R the following are equivalent

(i) R is right Artinian and semiprime;

(ii) R is left Artinian and semiprime;

(iii) R is semisimple.

2. For a ring the following are equivalent

(i) R is prime and right Artinian ;

(ii) R is prime and left Artinian;

(iii) R is simple and right Artinian;

(iv) R is simple and left Artinian;

(v) R is simple and semisimple;

(vi) R ≅ $M_n(D)$ for some positive integer n and some divisor ring D .

Definition: (i) A module M is uniform if M ≠ 0 and also each non zero submodule of M is an essential submodule.

(ii) A module M is said to have finite uniform dimension if it contains no infinite direct sum of nonzero submodules.

(iii) A ring R is called right Goldie ring if R has finite uniform dimension and R satisfies the ascending chain condition on right annihilators.

Example: R = $\begin{pmatrix} \mathbb{Z} & \mathbb{Q} \\ 0 & \mathbb{Q} \end{pmatrix}$ is right Noetherian ring. Hence it is right Goldie ring.

Remark: Every right Notherian ring is right Goldie.
Recall that for X a subset of a ring R we have the annihilator of X,
ann(X) = { r ϵ R| xr = 0 , for all x ϵ X } which is an ideal. The ideal A of a ring R is called an annihilator ideal if A =ann(X) for some X ⊆ R.

1.3: Essential Right Ideals

Definition: (i) Let R be a ring. A submodule of R_R is called *right ideal.*

(ii) Suppose that N is a submodule of M such that, for all non zero submodules X of M, one has $N \cap X \neq 0$. Then N is called an *essential submodule of* M, denoted by $N \lhd_e M$.

(iii) If a right ideal I is an essential submodule of a ring R, then it is called an *essential right ideal.*

Note that, $\zeta(R) = \{ a \in R \mid aE = 0$ for some E an element of the set of essential right ideals of R$\}$, $\zeta(R)$ is an ideal known as the right singular ideal.

Definition: A *regular element* in a ring R is any non zero divisor, i.e any element $x \in R$ such that $\mathrm{rann}_R(x) = 0$ and $\mathrm{lann}_R(x) = 0$. The set of regular elements of R denoted by $C_R(0)$.

Fact (4): (1) (G0ldie theorem) Let R be a ring, then the following are equivalent

(i) R is semiprime right Goldie;

(ii) R is semiprime , $\zeta(R) = 0$ and $\mathrm{rudim} R < \infty$;

(iii) R has right quotient ring Q which is semisimple Artinian. Further, R is prime if and only if $\mathbb{Q}$ is simple .

(2) Let R be a semiprime ring of finite right uniform dimension with $\zeta(R) = 0$, and let $E \lhd_r R$.

(i) E contains an element c such that $(\mathrm{rann} c) \cap E = 0$;

(ii) E is essential if and only if E contains a regular element of R.

(3) Let R be a semiprime ring. Consider the ring $\mathrm{End}(R_R)$, and note that, since R is semiprime, $R \hookrightarrow \mathrm{End}R_R$ with each element of R acting via left multiplication. Also there is a 1 in $\mathrm{End}R_R$. Let R^1 denote the sub ring of $\mathrm{End}R_R$ generated by R and 1. This then the ring used here.

(i) $R \lhd R^1$ and R is an essential right ideal of R^1;

(ii) R^1 is a semiprime ring;

(iii) R^1 satisfies the ascending chain condition on right ideals which share the same intersection with R;

(iv) If R is semiprime right Goldie, then so too is R^1.

Definition: Let S be a (non-empty) multiplicative closed subset of a ring R, and let ass S = { r ∈ R | rs = 0 for some s ∈ S }.

Then *a right quotient ring of* R with respect to S is a ring Q together with a homomorphism $\theta : R \to Q$ such that

(i) For all s ∈ S, θ(s) is a unit in Q;

(ii) For all q ∈ Q, $q = \theta(r)\,\theta(s)^{-1}$ for some r ∈ R, s ∈ S; and

(iii) Ker θ = ass S .

If further, ass S = 0, one can identify R with its image under θ, and then each q ∈ Q takes the form rs^{-1} .

Example: consider a skew polynomial ring $R = K[x, \alpha]$, where α is an authomorphism of the ring K, and set $S = \{ 1, x, x^2, \ldots \}$. Then the skew Laurent ring K[x, α]is both right and left quotient ring for R = [x, α] with respect to S.

Definition: A *classical quotient ring* for a ring R is a right ring of fraction (right quotient ring) for R with respect to the set of all regular elements in R.

Note that, every commutative ring has a classical quotient ring.

Definition: (i) A multiplicative closed subset S of R is said to satisfy the *right Ore condition* if for each r ∈ R and s ∈ S there exist r'∈ R, s' ∈ S such that rs' = sr' .

(ii) A multiplicative closed subset S of a ring R which satisfies the right Ore condition is said to be a *right Ore set*.

(iii) An integeral domain R is called a *right Ore domain* if $C_R(0)$ is a right ore set .

(iv) Let X be a multiplicative set in a ring R. Then X is *right reversible* if and only if X satisfies condition (a) in fact(5(i)a) below . A right denominator set is any right reversible Ore set .

Example: Let R = K[x, y] with xy = y(x+1), which is a Noetherian integral domain .

Remark: Any right Noetherian integral domain R is aright Ore domain.

Fact (5): (i) Let X be a multiplicative set in a ring R, and assume that there exists right ring of fractions with $\phi : R \to S$ with respect to X.

a) If r ∈ R and x ∈ X such that xr = 0, then there exists x' ∈ X such that rx' = 0.

b) X is a right Ore set in R.

(ii) a) If I is an ideal in a semiprime ring R, then

lann(I) = rann(I)

b) If R is a semiprime right Goldie ring and $P_1, P_2, \ldots, P_n$ are its minimal primes, then the set of regular elements of R equals $C(P_1) \cap \ldots \cap C(P_2)$.

Fact (6): Let S be any right denominator set in a ring R, and let $\mathbb{Q} = R_S$,

(i) If $B \lhd_r \mathbb{Q}$, then $B \cap R \lhd_r R$ and $B = (B \cap R)\mathbb{Q}$.

(ii) If $I \lhd R$ and $\mathbb{Q}$ is a right Noetherian, then $I\mathbb{Q} \lhd \mathbb{Q}$.

(iii) Any finite set $q_1, q_2, q_3, \ldots, q_n$ of elements of $\mathbb{Q}$ has a common denominator; i.e there exist $r_1, r_2, r_3, \ldots, r_n \in R$ and $s \in S$ such that $q = r_i s^{-1}$ for each i, i =1, 2, ..., n.

Fact (7): Let S be a right Ore set of regular elements of a ring R and let $\mathbb{Q} = R_S$. Let $A \lhd_r R$ and $B \lhd_r Q$. Then

(i) $A \lhd_e R \iff A\mathbb{Q} \lhd_e \mathbb{Q}$;

(ii) $\mathrm{rudim}\mathbb{Q} = \mathrm{rudim}R$;

(iii) $\mathrm{udim}A_R = \mathrm{udim}A\mathbb{Q}_Q = \mathrm{udim}A\mathbb{Q}_R$;

(iv) $\mathrm{udim}B_Q = \mathrm{udim}B_R = \mathrm{udim}(B \cap R)_R$;

(iv) $B \lhd_e \mathbb{Q} \iff B \cap R \lhd_e R$;

Fact (8): Let R be a semiprime ring and A an ideal, then

(i) rannA = lannA (= annA);

(ii) annA is the unique complement ideal to A in R ;

(iii) annA is an intersection of those minimal prime ideals of R which do not contain A;

(iv) ${}_RA_R$ is uniform if and only if annA is a minimal prime ideal;

(v) $A \lhd_e {}_RR_R$, if and only if annA = 0;

(vi) If A is not contained in any minimal prime of R, then $A \lhd_e R_R$.

1.5: Tenssor Product

Definition: Given modules M_R and $_RN$ over ring R and an abelian group A, a function $\beta : M \times N \longrightarrow A$ is said to be R-*biliniar* if

(i) $\beta(x_1 + x_2, y) = \beta(x_1, y) + \beta(x_2, y)$;

(ii) $\beta(x, y_1 + y_2) = \beta(x, y_1) + \beta(x, y_2)$;

(iii) $\beta(xr, y) = \beta(x, ry)$ for all $x, x_1, x_2 \in M$; $y, y_1, y_2 \in N$ and $r \in R$.

Definition: A *tensor product* of the modules M_R and $_RN$ is an abelian group $M \otimes_R N$, together with an R-biliniar map τ: $M \times N \longrightarrow M \otimes_R N$ such that for any abelian group A and any R-biliniar map β: $M \times N \longrightarrow A$ there exists a unique $\mathbb{Z}$ -homomorphism $f : M \otimes_R N \longrightarrow A$ such that $f\tau = \beta$. For $x \in M$, $y \in N$ the image $\tau(x, y)$ is denoted by $x \otimes y$. Thus the diagram commutes.

$$\begin{tikzcd}
M \times N \arrow[r] \arrow[dr] & M \otimes_R N \arrow[d, dashed] \\
 & A
\end{tikzcd}$$

The tensor product of M and N is constructed as follows. Let F be the free $\mathbb{Z}$-module with generators $\{ (x, y) \mid x \in M \text{ and } y \in N \}$, and let i: $M \times N \longrightarrow F$ be the inclusion mapping. Let K be the submodule of F generated by all elements of the form

$(x_1 + x_2, y) - (x_1, y) - (x_2, y)$, $(x, y_1 + y_2) - (x, y_1) - (x, y_2)$ or $(xr, y) - (x, ry)$ where $x, x_1, x_2 \in M$; $y, y_1, y_2 \in N$ and $r \in R$.

Let π: $F/K \longrightarrow F$ be the natural projection. If we let τ: $M \times N \longrightarrow F/K$ be the composition πi, then F/K satisfies the definition of $M \otimes_R N$.

Example: Let A be any finite abelian group. We will show that $\mathbb{Q} \otimes zA = (0)$. If $q \in \mathbb{Q}$ and $a \in A$, then a has finite order, and so there exists a positive integer n with $na = 0$. Since we have taken the tensor product over $\mathbb{Z}$, we can freely move integers across the tensor product symbol $\otimes$.

Thus $q \otimes a = (qn^{-1})n \otimes a = qn^{-1} \otimes na$

$= qn^{-1} \otimes 0$

$= 0$

Showing that every element of $\mathbb{Q} \otimes zA$ must be zero. Therefore $\mathbb{Q} \otimes zA = (0)$

Note that tensor products are unique up to isomorphism. Each element of $M \otimes_R N$ can be expressed as $\sum_j^k x_j \otimes y_j$ where $x \in M$ and $y \in N$ for $1 \le j \le k$.

CHAPTER TWO

STRUCTURES OF SEMIPRIME GOLDIE RINGS

2.1 ORDERS IN QUOTIENT RING

Definition: A ring Q is called a *quotient ring* if every regular element of Q is a unit.

Example: The m × m matrix ring over a field is a quotient ring. Let $\mathbb{Q}$ be the set of rational numbers, Then Q = $M_2(\mathbb{Q})$ is a quotient ring.

Let $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in Q$ be regular where a, b, c, and d are elements of $\mathbb{Q}$, then

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} \frac{d}{ad-bc} & \frac{b}{bc-ad} \\ \frac{-c}{ad-bc} & \frac{-a}{bc-ad} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$, where

$\begin{pmatrix} \frac{d}{ad-bc} & \frac{b}{bc-ad} \\ \frac{-c}{ad-bc} & \frac{-a}{bc-ad} \end{pmatrix} \in Q$. Therefore $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in M_2(\mathbb{Q})$ is a unit

Proposition 2.1.1: If Q is a right Artinian ring, then Q is a quotient ring. Indeed every right regular element is a unit.

Proof: Let Q be right Artinian ring. If s ∈ Q is a right regular then consider the descending chain $\{s^nQ\}$ that stabilizes with say $s^nQ = s^{n+1}Q$ and thus $s^n = s^{n+1}q$ for some q .
Since s is right regular so too is s^n, and yet $s^n(sq-1) = s^nsq - s^n = s^{n+1}q - s^n = 0$. Thus $sq - 1 = 0$ implies sq = 1, and moreover, s is left regular. Finally $s(qs-1)s = 0$,so $q = s^{-1}$.

Definition: Given a quotient ring Q, a sub ring R, not necessarily containing 1, is called a *right order* in Q if each q ∈ Q has the form rs^{-1} for some s, r ∈ R. A left order is defined analogously, and a ring which is both a left and a right order is called an order.

Example: $M_2(\mathbb{Z})$ is a left order in $M_2(\mathbb{Q})$, where $\mathbb{Z}$ and $\mathbb{Q}$ are the set of integers and the set of rational numbers respectively.
The sub ring R = $M_2(\mathbb{Z})$ of matrices with integer entries is a left order in the ring Q = $M_2(\mathbb{Q})$ of 2 × 2 matrices with rational entries . To show this given any 2 × 2 matrix q ∈ Q, we can fined a common denominator for the rational entries of q. If c is the scalar matrix determined by the common denominator of the entries, then cq ∈ R = $M_2(\mathbb{Z})$, thus R = $M_2(\mathbb{Z})$ is a left order in Q .

Let $q = \begin{pmatrix} \frac{1}{2} & \frac{2}{3} \\ \frac{5}{4} & \frac{3}{2} \end{pmatrix} \in M_2(\mathbb{Q})$, then $q = \begin{pmatrix} \frac{1}{2} & \frac{2}{3} \\ \frac{5}{4} & \frac{3}{2} \end{pmatrix} = \begin{pmatrix} \frac{6}{12} & \frac{8}{12} \\ \frac{15}{12} & \frac{18}{12} \end{pmatrix} = \begin{pmatrix} 6 & 8 \\ 15 & 18 \end{pmatrix} \begin{pmatrix} \frac{1}{12} & 0 \\ 0 & \frac{1}{12} \end{pmatrix}$

$$= \begin{pmatrix} 6 & 8 \\ 15 & 18 \end{pmatrix} \begin{pmatrix} 12 & 0 \\ 0 & 12 \end{pmatrix}^{-1}.$$

Therefore $q = rs^{-1}$, where $r = \begin{pmatrix} 6 & 8 \\ 15 & 18 \end{pmatrix}$ and $s = \begin{pmatrix} 12 & 0 \\ 0 & 12 \end{pmatrix}$ which are elements of $M_2(\mathbb{Z})$.

Lemma 2.1.1: Let R_i be a sub ring not necessarily with 1, of the quotient ring Q_i ,for $i = 1, 2, \ldots, n$. Then $\oplus R_i$ is a right order in $\oplus Q_i$ if and only if each R_i is a right order in Q_i.

Proof: suppose each R_i is a right order in Q_i. Let $q_i \in Q_i$ for $i = 1, 2, \ldots, n$; then $q_i = r_i s_i^{-1}$ for some $s_i, r_i \in R_i$, $i = 1, 2, \ldots, n$. Then $q \in \oplus Q_i$ can be written as

$$q = (q_1, q_2, \ldots, q_n)$$

$$= (r_1 s_1^{-1}, r_2 s_2^{-1}, \ldots, r_n s_n^{-1})$$

$$= (r_1, r_2, \ldots, r_n)\,(s_1^{-1}, s_2^{-1}, \ldots, s_n^{-1})$$

$= (r_1, r_2, \ldots, r_n)\,(s_1, s_2, \ldots, s_n)^{-1}$ letting $r = (r_1, r_2, \ldots, r_n)$ and $s = (s_1, s_2, \ldots, s_n)$, then we get

$$q = rs^{-1}.$$

Therefore $\oplus R_i$ is a right order in $\oplus Q_i$.

Conversely, suppose $\oplus R_i$ is a right order in $\oplus Q_i$, this implies every $q \in \oplus Q_i$ can be written as $q = rs^{-1}$ where $r, s \in \oplus R_i$.

Let $q = (q_1, q_2, \ldots, q_n)$, $r = (r_1, r_2, \ldots, r_n)$, $s = (s_1, s_2, \ldots, s_n)$, then

$$(q_1, q_2, \ldots, q_n) = (r_1, r_2, \ldots, r_n)\,(s_1^{-1}, s_2^{-1}, \ldots, s_n^{-1})$$

$$= (r_1 s_1^{-1}, r_2 s_2^{-1}, \ldots, r_n s_n^{-1})$$

Which implies $q_i = r_i s_i^{-1}$ for all $i = 1, 2, \ldots, n$. Therefore R_i is a right order in Q_i, for all $i = 1, 2, \ldots, n$.

Proposition 2.1.2: Let R be a sub ring (with 1) of a ring Q and let S = {units of Q} ∩ R.

(i) If Q is the right quotient ring of R, then Q is a quotient ring; R is a right order in Q and $S = C_R(0)$;

(ii) If Q is a quotient ring and R is a right order in Q, then $Q = R_S$. if further, either R is also a left order in Q or Q is a right Artinian, then $S = C_R(0)$ and Q is the right quotient ring of R .

Proof: (i) If $q \in Q$ is regular, with $q = rs^{-1}$ say, where $r, s \in R$, then since r is a multiple of regular element q, $r = qs \in C_R(0)$ and also in a finite ring every regular element is a unit, hence r is a unit of Q . Therefore q is a unit and Q is a quotient ring. Since $r \in R$, where R is a sub ring of Q and r is regular which is a unit in Q, {units in Q} ∩ R = $C_R(0)$ implies $S = C_R(0)$. Every $q \in Q$ written as $q = rs^{-1}$, where $r, s \in R$ for a quotient ring Q, then R is a right order in Q.

(ii)The first claim is an immediate consequence of the definition as noted above $C_R(0) \subseteq C_Q'(0)$. If R is also a left order, then $C_R(0) \subseteq {}'C_Q(0)$ and thus $C_R(0) = S$, Therefore Q is the right quotient ring of R. In other case when Q is right Artinian $C_Q'(0) = C_Q(0)$ by proposition(2.1.1). Hence $S = C_R(0)$ and Q is the right quotient ring of R .

Remark: There is a distinction between the phrases 'R is a right order in Q' and ' Q is the right quotient ring of R', over and above the convention that rings have a 1 but right orders need not. This definition vanishes when R is also a left order or Q is right Artinian. In particular, the above proposition shows that when Q is semisimple Artinian the distinction disappears, so ' semiprime Goldie ring' is synonymous with 'right order with 1, in a semisimple Artinian ring '.

Corollary 2.1.1: R is semiprime right Goldie with right quotient ring Q if and only if $M_n(R)$ is a semiprime right Goldie ring with right quotient ring $M_n(Q)$.

Proof: Suppose that R is semiprime right Goldie with right quotient ring Q. By Goldie theorem Q is semisimple Artinian, so too is $M_n(Q)$. We will show that $M_n(R)$ is a right order in $M_n(Q)$. If $x \in M_n(Q)$, then taking a common denominator we can write x in the form $(a_{ij}\, c^{-1})$, where $a_{ij}, c \in R$. Using the standard embedding of Q in $M_n(Q)$, this means that $x = ac^{-1}$ with $a, c \in M_n(R)$. Thus $M_n(R)$ is a right order in $M_n(Q)$ and so is semiprime right Goldie. Conversely, suppose $M_n(R)$ is semiprime right Goldie ring with right quotient ring $M_n(Q)$. This implies each $x \in M_n(Q)$ can be written as $x = ac^{-1}$, where $a, c \in M_n(R)$.
Let $y = (b_{ij}) \in M_n(Q)$, where $b_{ij} \in Q$, then we can write b_{ij} as $b_{ij} = (n.\frac{c}{d})c^{-1}$, where $\frac{nc}{d}, c \in R$ such that n is the numerator of b_{ij}, d is the denominator of b_{ij} and c is the common denominator of $b_{ij}'s$. Hence Q is semiprime Goldie ring with right quotient ring Q.

Lemma 2.1.2: Let R be a right order in a quotient ring Q and let S be a sub ring of Q(not necessarily with 1). Suppose further that there are units a, b of Q such that $aRb \subseteq S$. Then S is also a right order in Q.

Proof: Given $q \in Q$, consider the element $a^{-1}qa$ of Q. By definition $a^{-1}qa = rt^{-1}$ for some $r, t \in R$. But then $q = art^{-1}a^{-1} = arbb^{-1}t^{-1}a^{-1} = (arb)(atb)^{-1}$. Since arb, atb $\in aRb \subseteq S$, then S is a right order in Q.

Corollary 2.1.2: (i) if R is a right order in a quotient ring Q and S is a ring (not necessarily with 1) such that $R \subseteq S \subseteq Q$. Then S is a right order in Q.

(ii) If R is a prime right Goldie ring, $0 \neq A \lhd R$, and S is a subring of R with $A \subseteq S \subseteq R$ then S is a prime right Goldie ring, and has the same right quotient ring as R.

Proof: (i) Let a, b, are units of Q such that a = b = 1, then $aRb = 1R1 = R \subseteq S$. Hence S is a right order in Q.

(ii) We know that if R is prime ring and I is a non zero ideal then $I \lhd_e R_R$. From this we have $A \lhd_e R_R$ and therefore by fact (4(2)(ii)) A contains a regular element, c say of R. Then c is a unit of the right quotient ring Q of R and $cR \subseteq S$.

Theorem 2.1.1: A ring R, not necessarily with 1, is a right order in a semisimple Artinian ring Q if and only if R is semiprime right Goldie.

Proof: suppose R is semiprime right Goldie. By fact (4(3)(iv)) so too is the ring R^1, described there . Therefore, by Goldie theorem, R^1 has a semisimple Artinian right quotient ring Q. Thus R^1 is a right order in Q. Now by fact (4(3)(i)) R is an essential right ideal of R^1 and thus by fact (4(2)(ii)) it contains regular element, a say of R^1. Since $aR^1 \subseteq R$, the above lemma (2.1.2) shows that R is a right order in Q.
For the converse, suppose R is a right order in a semisimple Artinian ring Q. Then by the remark next to the proof of proposition 2.1.2 R is semiprime right Goldie ring, thus R has a semisimple Artinian quotient ring Q, Then by Goldie theorem R is semiprime right Goldie.

Corollary 2.1.3: Let R be a semiprime right Goldie ring. A an essential right ideal and B a left ideal containing a regular element. Then A and AB are semiprime right Goldie rings (with out 1) having the same right quotient ring as R.

Proof: Suppose the hypothesis of the statement holds. This implies A contains a regular element a say, and let $b \in B$ a regular element of B. Then $aRb \subseteq AB \subseteq A \subseteq R$
From this we know that AB and A are right order in a right quotient ring as R. Hence by the above theorem AB and A are semiprime right Goldie rings.

Definition: Let R_1 and R_2 are right orders in Q and if there are units $a_1, a_2, b_1, b_2 \in Q$ such that $a_1R_1b_1 \subseteq R_2$ and $a_2R_2b_2 \subseteq R_1$, then R_1 and R_2 are termed as *equivalent right orders*.

EXAMPLE: The rings R, A and AB described in the above corollary are equivalent.

Note that: The study of equivalent orders is facilitated by the notion of fractional ideal.

Definition: Suppose that R is a right or left order in a quotient ring Q. Then a *fractional right R-ideal* is a submodule I of Q_R, such that $aI \subseteq R$ and $bR \subseteq I$ for some units $a, b \in Q$. Fractional left ideal is defined similarly.

If I is both a fractional right R-ideal and a fractional left S-ideal for some other order S, then I is called a fractional (S, R) ideal.

More specific example; $\frac{1}{2}\mathbb{Z}$ is a fractional $\mathbb{Z}$-ideal and $M_2(\mathbb{Z})$ is a fractional ideal over $\begin{pmatrix} \mathbb{Z} & 2\mathbb{Z} \\ \mathbb{Z} & \mathbb{Z} \end{pmatrix}$.

Definition: The *right order* and *left order* of a fractional left (right) R-ideal I are defined respectively to be $O_l(I) = \{ q \in Q \mid qI \subseteq I \}$

$$O_r(I) = \{ q \in Q \mid Iq \subseteq I \}.$$

Example: $I = \begin{pmatrix} 2\mathbb{Z} & 2\mathbb{Z} \\ \mathbb{Z} & \mathbb{Z} \end{pmatrix} \subseteq Q = M_2(\mathbb{Q})$, then

$$O_r(I) = M_2(\mathbb{Z}) \text{ and } O_l(I) = \begin{pmatrix} \mathbb{Z} & 2\mathbb{Z} \\ \frac{1}{2}\mathbb{Z} & \mathbb{Z} \end{pmatrix}.$$

For if let $q \in M_2(\mathbb{Z})$ say $q = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ where $a, b, c, d \in \mathbb{Z}$, then

$$Iq = \begin{pmatrix} 2\mathbb{Z} & 2\mathbb{Z} \\ \mathbb{Z} & \mathbb{Z} \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 2(\mathbb{Z}a + \mathbb{Z}c) & 2(\mathbb{Z}b + \mathbb{Z}d) \\ \mathbb{Z}a + \mathbb{Z}c & \mathbb{Z}b + \mathbb{Z}d \end{pmatrix} = \begin{pmatrix} 2\mathbb{Z}(a+c) & 2\mathbb{Z}(b+d) \\ \mathbb{Z}(a+c) & \mathbb{Z}(b+d) \end{pmatrix}$$

$$\subseteq \begin{pmatrix} 2\mathbb{Z} & 2\mathbb{Z} \\ \mathbb{Z} & \mathbb{Z} \end{pmatrix}$$

$$= I$$

Therefore $Iq \subseteq I$, which shows $O_r(I) = M_2(\mathbb{Z})$.

Let $q = \begin{pmatrix} a & 2b \\ \frac{1}{2}c & d \end{pmatrix} \in M_2(\mathbb{Q})$, then

$$qI = \begin{pmatrix} a & 2b \\ \frac{1}{2}c & d \end{pmatrix} \begin{pmatrix} 2\mathbb{Z} & 2\mathbb{Z} \\ \mathbb{Z} & \mathbb{Z} \end{pmatrix} = \begin{pmatrix} 2\mathbb{Z}(a+c) & 2\mathbb{Z}(b+d) \\ \mathbb{Z}(a+c) & \mathbb{Z}(b+d) \end{pmatrix}$$

$$\subseteq \begin{pmatrix} 2\mathbb{Z} & 2\mathbb{Z} \\ \mathbb{Z} & \mathbb{Z} \end{pmatrix} = I$$

Therefore $qI \subseteq I$, which shows $O_l(I) = \begin{pmatrix} \mathbb{Z} & 2\mathbb{Z} \\ \frac{1}{2}\mathbb{Z} & \mathbb{Z} \end{pmatrix}$.

Lemma 2.1.3: Let R be a right order in Q and let I be a fractional right or left R-ideal. Then $O_r(I)$ and $O_l(I)$ are right orders in Q and are equivalent to R.

Proof: Suppose I is a fractional right R-ideal. Let a, b are units in Q such that $aI \subseteq R$ and $bR \subseteq I$, note that $abO_r(I) \subseteq R \subseteq O_r(I)$. Here $O_r(I)$ is a sub ring of Q, letting a = b = 1, aRb = 1R1 = $R \subseteq O_r(I)$ which implies that $O_r(I)$ is a right order in Q by lemma (2.1.2), similarly for $O_l(I)$. Thus $aO_l(I)b \subseteq aO_l(I)I \subseteq aI \subseteq R$. Therefore we have $aO_l(I)b \subseteq R$ ....(1)
and also we have that $RaI \subseteq R$ since $I \lhd R$ implies $bRaI \subseteq bR \subseteq I$ then $bRa \subseteq O_l(I)$.............(2)
Therefore from (1) and (2) we have that $O_l(I) \sim R$, similarly for $O_r(I)$.

Proposition 2.1.3: Let R be a right order in a quotient ring Q, and let $I, J \lhd Q_R$. Then

(i) $I \hookrightarrow IQ \simeq I \otimes Q$

(ii) $\mathrm{Hom}(I, J) \hookrightarrow \mathrm{Hom}_Q(IQ, JQ)$ via $\alpha \mapsto \alpha \otimes 1$, this giving the unique extension of α to IQ and JQ.

(iii) under this embedding, $\mathrm{Hom}_R(I, J) = \{ \beta \in \mathrm{Hom}(IQ, JQ) \mid \beta I \subseteq J \}$.

(iv) If further, I is a fractional right R-ideal, then $\mathrm{Hom}_R(I, J) \simeq \{ q \in Q \mid qI \subseteq J \}$ and $\mathrm{End}I \simeq O_l(I)$.

Proof : (i) Let $I \lhd Q_R$ by proposition(2.1.1)(ii) we have that $Q = R_S$, then by the fact

($M \longrightarrow M$, $m \longmapsto \bar{m}$, where $\bar{m} \in \bar{M} = M/\mathrm{ass}S$, is universal with respect to R-homomorphisms from M to R_S -module M_S. Hence $M \otimes R_S \simeq M_S$). This implies that $I \otimes_R Q_R \simeq IQ$ again this implies that $I \hookrightarrow IQ \simeq I \otimes Q$.

(ii) by(i) we have, $I \hookrightarrow I \otimes Q$ and $J \hookrightarrow J \otimes Q$. Let $\alpha : I \longrightarrow J$ then $\alpha \otimes 1 : I \otimes Q \longrightarrow J \otimes Q \Longrightarrow \mathrm{Hom}\ (I , J) \hookrightarrow \mathrm{Hom}(IQ, JQ)$.

(iii) Let $\alpha \in \mathrm{Hom}_R(I, J)$, we want to show that $\alpha \in \{ \beta \in \mathrm{Hom}(IQ ,JQ) \mid \beta I \subseteq J \}$ α is a map $\alpha : I \dashrightarrow J \Rightarrow \alpha : I1_Q \rightarrow J1_Q$ since $I ,J \lhd Q_R \Rightarrow \alpha : I \otimes 1_Q \rightarrow J \otimes 1_Q$ $\Rightarrow \alpha = \beta \otimes 1_Q : I \otimes 1_Q \rightarrow J \otimes 1_Q$, then $\beta(I) \otimes 1_Q \subseteq J \otimes 1_Q \Rightarrow \beta(I) \subseteq J \Rightarrow \beta(I) \otimes 1_Q \subseteq J$, thus $\alpha = \beta \otimes 1_Q (I) \subseteq J$. Therefore $\alpha \in \{ \beta \in \mathrm{Hom}(IQ, JQ) \mid \beta I \subseteq J \}$. Hence $\mathrm{Hom}_R(I, J) \subseteq \{ \beta \in \mathrm{Hom}(IQ, JQ) \mid \beta I \subseteq J \}$ ....(i)

Let $\alpha \in \{ \beta \in \mathrm{Hom}(IQ, JQ) \mid \beta I \subseteq J \}$, we want to show that $\alpha \in \mathrm{Hom}_R (I, J)$ Let $\alpha : IQ \longrightarrow JQ$ with $\alpha I \subseteq J$ which implies $\alpha: I \otimes Q \longrightarrow J \otimes Q$, take $1_Q \in Q$ we get $\alpha: I \otimes 1_Q \longrightarrow J \otimes 1_Q$ which implies, $\alpha: I1_Q = I \longrightarrow J1_Q = J$, then $\alpha: I \longrightarrow J$, this means that $\alpha \in \mathrm{Hom}_R(I, J)$.

Hence $\{\beta \in \mathrm{Hom}(IQ, JQ) \mid \beta I \subseteq J \} \subseteq \mathrm{Hom}_R(I, J)$

Therefore by (i) and (ii) we concluded that $\mathrm{Hom}_R(I, J) = \{ \beta \in \mathrm{Hom}(IQ, JQ) \mid \beta I \subseteq J \}$.

(iv) Let I be a fractional right R-ideal;

$\Rightarrow bR \subseteq I$ for some unit b in Q

$\Rightarrow bRQ \subseteq IQ$

$\Rightarrow Q \subseteq IQ$ since R is a sub ring in Q. Let $g \in IQ \Rightarrow g = iq$ for some $i \in I$, $q \in Q \Rightarrow g = iq \in Q$ since $I \lhd Q$ and $i, q \in Q \Rightarrow IQ \subseteq Q$. Therefore $IQ = Q$. Thus $\mathrm{Hom}(IQ, JQ) = \mathrm{Hom}(Q, JQ)$.

Since any such homomorphism is realized by left multiplication by some element of Q, then $\mathrm{Hom}(Q, IQ) \simeq \{ q \in Q \mid qI \subseteq J \}$. Therefore $\mathrm{Hom}_R(I, J) \hookrightarrow \mathrm{Hom}(IQ, JQ) = \mathrm{Hom}(Q, IQ)$

$\simeq \{q \in Q \mid qI \subseteq J\}$ by (ii)

Therefore $\mathrm{Hom}_R(I, J) \simeq \{ q \in Q \mid qI \subseteq J \}$.

2.2 MINIMAL PRIMES

Definition: A *minimal prime ideal* in a ring R is any prime ideal of R that does not properly contain any other prime ideals.

Example: 1 . $2\mathbb{Z}/4\mathbb{Z}$ is a minimal prime ideal of a ring $\mathbb{Z}/4\mathbb{Z}$.

2 . $\begin{pmatrix} 2\mathbb{Z}/4\mathbb{Z} & 0 \\ 2\mathbb{Z}/4\mathbb{Z} & 0 \end{pmatrix}$ is a minimal prime ideal of $M_2(\mathbb{Z}/4\mathbb{Z})$.

The notion below will be kept fixed in this section.

Let R be semiprime right Goldie ring and Q be the right quotient ring of R. By Goldie theorem, Q is semisimple; therefore, $Q = \oplus Q_i$ with each Q_i being a simple Artinian ring generated as an ideal of Q by a central idempotent, e_i say. Note that $e_i = 1_{Q_i}$. For each i, the ideal $M_i = \sum\{Q_j \mid j \neq i\}$ is a maximal ideal; and this gives all the maximal (=minimal primes) ideals of Q .

Let $P_i = M_i \cap R$, $A_i = Q_i \cap R$ and $A = \oplus A_i$. It will be seen that these ideals are of special interest.

Proposition 2.2.1: (i) The ideal P_i are the minimal prime ideal of R;

(ii) The ideals of the form $I \cap R$, with $I \lhd Q$, are precisely the annihilator ideals of R;

(iii) The A_i are the minimal non zero annihilator ideal;

(iv) The ideal $A = \oplus_{i=1}^{k} A_i$ is an essential right ideal of R, and is a right order in Q equivalent to R;

(v) A_i is a right order in Q_i.

Proof : (i) Suppose $X, Y \lhd R$ with $XY \subseteq P_i$ then by fact (6(ii)) $XQ, YQ \lhd Q$; so $XQYQ = XYQ \subseteq P_iQ \subseteq M_i$, which implies $XQ \subseteq M_i$ or $YQ \subseteq M_i$ (this is because M_i is maximal and hence prime), thus $X \subseteq P_i$ or $Y \subseteq P_i$ this shows that P_i is prime. Since R is semiprime ring its prime radical is zero. Therefore $\cap P_i = 0$, for i = j, $P_i \nsubseteq P_j$ which implies that P_i's are the minimal prime ideals.

(ii) If $I \lhd Q$, then I is an intersection of some of the M_i. Therefore $I \cap R$ is an intersection of some P_i. By fact (8(iii)) the intersection of minimal ideals is annihilator ideal so is $I \cap R$.

(iii) Let $Q_i \lhd Q$, then Q_i is an intersection of some of the M_i. Therefore $Q_i \cap R$ is an intersection of some P_i. By (ii) the intersection of minimal prime ideals is annihilator ideals so $Q_i \cap R = A_i$ is minimal non zero annihilator ideals.

(iv) Since $Q_i \lhd_r R$ and by fact (6(i)) $A_iQ = (Q_i \cap R)Q = Q_i$. So

$$
\begin{aligned}
AQ &= (\oplus A_i)Q \\
&= \oplus A_iQ \\
&= \oplus Q_i \text{, since } A_iQ = Q_i .
\end{aligned}
$$

Hence $AQ = Q$.

This implies $AQ \lhd_e Q$ ($AQ = Q$ is a submodule of $Q = AQ$, let X be any non zero submodule of $Q = AQ$ then $AQ \cap X \neq 0$ this implies that $AQ \lhd_e Q = AQ$), therefore by fact (7(i)) $A \lhd_e R_R$. By corollary (2.1.3) A is semiprime right Goldie ring and by theorem (2.1.1) A is a right order in Q.

(v) From (iv) above we get $A = \oplus A_i$ is a right order in Q, then by lemma (2.1.1) A_i is a right order in Q_i.

Corollary 2.2.1: R contains a finite direct sum of prime right Goldie rings (not necessarily with 1) having the same right quotient ring.

Proof: Let A_i is a right order in Q_i, where Q_i is simple Artinian ring. By theorem (2.1.1) A_i is prime Goldie ring and by lemma (2.1.1) $\oplus A_i$ is a right order in $\oplus Q_i$, where $\oplus Q_i = Q$ is a simple Artinian ring . Hence by theorem (2.1.1) $\oplus A_i$ is a prime right Goldie ring having the same right quotient ring $Q = \oplus Q_i$.

The next result shows that R is also equivalent as a right order, to a direct sum of prime rings with 1, namely the rings R/P_i, with P_i a minimal prime ideal of R.

Proposition 2.2.2: (i) $R/P_i \cong e_iR$;

(ii) $R \subseteq R' = \oplus_{i=1}^{k} e_iR$;

(iii) R' is a semiprime right Goldie ring and $R' \sim R$;

(iv) e_iR is a prime right Goldie ring and $e_iR \sim A_i$;

(v) $C_R(0) = \cap_{i=1}^{k} C_R(P_i)$

Proof: (i) Let $\theta : Q \longrightarrow e_iQ$ defined by $q \mapsto e_iq$ and let $M_i = \ker \theta$, so $P_i = M_i \cap R$ is the kernel of the restriction to R. Hence by fundamental theorem of homomorphism $R/p_i \cong e_iR$.

(ii) Since $e_i = 1_{Q_i}$ we have $R \subseteq \oplus_{i=1}^{k} e_iR = R'$,

(iii) $AR' = A \subseteq R$ since $A_i = A_ie_i$. By proposition (2.2.1) above, A contains a unit, say a of Q. Then $aR' \subseteq R \subseteq R$ and so $R' \sim R$ which shows that R′ and R are equivalent right orders. This means that R' and R have the same quotient ring Q which is a semisimple Arinian ring. Thus R' is a semiprime right Goldie ring by Goldie theorem.

(iv) $R' = \oplus_{i=1}^{k} e_iR$ is a semiprime right Goldie ring in a semisimple Artinian ring Q, then by lemma (2.1.1) e_iR is a prime right Goldie ring in a simple Artinian ring Q_i.
By(i) we have $R/p_i \cong e_iR$. Since R/p_i is a prime ring e_iR is prime ring and since $R' = \oplus_{i=1}^{k} e_iR$ is a right Goldie ring, e_iR is right Goldie ring. Hence $e_iR \sim A_i$.

(v) If $c \in C_R(0)$, then for each I, e_ic is a unit of e_iQ and hence is regular in e_iR. Therefore, $c \in \cap_{i=1}^{k} C_R(P_i)$by (i). This implies $C_R(0) \subseteq \cap_{i=1}^{k} C_R(P_i)$
For the reverse direction, if $c \in \cap_{i=1}^{k} C_R(P_i)$, then $\mathrm{rann}_R(c) \subseteq P_1 \cap P_2 \cap \ldots \cap P_k = 0$ and similarly $\mathrm{lann}(c) = 0$. Hence c is regular which implies $c \in C_R(0)$. Therefore $\cap_{i=1}^{k} C_R(P_i) \subseteq C_R(0)$. Thus $C_R(0) = \cap_{i=1}^{k} C_R(P_i)$.

We can also show $C_R(0) \subseteq \cap_{i=1}^{k} C_R(p_i)$in other way. If $x \in C(P_1) \cap C(P_2) \cap \ldots \cap C(P_n)$, then $\mathrm{rann}_R(x) \subseteq P_1 \cap P_2 \cap \ldots \cap P_n = 0$ and similarly $\mathrm{lann}(x) = 0$.
Hence x is regular. Conversely, let x be a regular element of R, thus by fact(5(ii)(a))each $P_i = \mathrm{lann}_R(I_i) = \mathrm{rann}_R(I_i)$for some ideal I_i. If $r \in R$ and $xr \in P_i$, then $xrI_i = 0$, whence $rI_i = 0$. Thus by fact (5(ii)(b))we have $C_R(0) \subseteq \cap_{i=1}^{k} C_R(P_i)$.

Corollary 2.2.2: R contained in a direct sum of prime right Goldie rings (with1) having the same right quotient ring.

Proof: We have e_iR is a prime right Goldie ring with right quotient ring Q_i and also we have $R \subseteq R' = \oplus e_iR$. Hence the corollary follows.

Proposition 2.2.3: Let R be a semiprime ring with finitely many minimal prime ideals $P_1, P_2, \ldots, P_k$. Then R is right Goldie if and only if R/P_i is a right Goldie for each i.

Proof: If R is right Goldie, then by proposition (2.2.2) $e_iR \cong R/P_i$ which shows that R/P_i is right Goldie for each i.
Conversely if each R/P_i is right Goldie, then $R \hookrightarrow S = \oplus R/P_i$, a semiprime right Goldie ring with quotient ring $Q = \oplus_{i=1}^{k} Q_i$ say.
If $A_i = Q_i \cap R$ then $A = \oplus_{i=1}^{k} A_i$ is an ideal of R and also an essential right ideal of S. By corollary (2.1.3) $A \sim S$. So by corollary (2.1.6) R is right order in Q, then by theorem (2.1.1) R is right Goldie. To see that R contains an equivalent right order which is a matrix ring over a right Ore domain, but not necessarily with 1, first let us see some terminology ;
Suppose that $S \cong M_n(T)$ for rings S, T with 1. A set of matrix units of S is a subset $M=\{e_{ij} \mid i, j = 1, 2, \ldots, n\}$ of S such that $\sum e_{ij} = 1$ and $e_{ij}e_{kl} = \delta_{jk}e_{il}$,where δ_{jk} is the kronecker delta symbol. If $T' = \{ s \in S \mid se_{ij} = e_{ij}s \text{ for all } i, j \}$.The centralizer of M in S, it follows that S

$\cong M_n(T')$, with M corresponds to the standard set of matrix units. Note that, if T is division ring, then $T \cong T'$.

Theorem 2.2.1: (Faith-Utumi theorem)

A semiprime rightGoldie ring R contains an equivalent right order which is a direct sum of matrix rings over right Ore domains (not necessarilywith1) .

Proof: It is enough to consider the case when R is prime right Goldie (but not with1) and $Q \cong M_n(D)$for some division ring D where Q is a right quotient ring. The first step is to prove that a set of matrix units M of Q can so be chosen that $cM \subseteq R$ for some regular element $c \in R$. By fact (6(iii)), there will be a regular element c with $Mc \subseteq R$ .The set $M' = \{ c^{-1}e_{ij}c \mid e_{ij} \in M\}$ is a set of matrix units and $cM' \subseteq R$. Thus M can be rechoosen as desired and we may assume that D is its centralizer in Q. For some other regular element b, it is true that $Mb \subseteq R$. Now consider the sets $C = \{x \in R \mid xM \subseteq R\}$ and $B = \{x \in R \mid Mx \subseteq R\}$. Clearly $C \lhd_l R$ and $B \lhd_r R$. The multiplicative properties of M show that $CM = (CM)M$; hence $CM = C$. Similarly $MB = M(MB) = B$. Consider the sub ring BC of R. Since $c \in C$ and $b \in B$ are regular elements and $bRc \subseteq BC \subseteq R$, BC is a right order in Q equivalent to R. Furthermore, since $BC = MBCM$, then $BC = M_n(K)$ where $K = \{ m \in M \mid me_{ij} = e_{ij}m \text{ for all } i , j \}$ is the centralizer of M in BC. So $K \subseteq D$ and hence an integral domain. From fact (7(ii)) $\text{rudim} M_n(K) = \text{rudim} Q = n$. Therefore $\text{rudim} K = 1$, and so K is a right Ore domain, with right quotient division ring say D'.

2.3: Right ideals in semiprime rings

Through out this section R denotes a semiprime right Goldie ring with right quotient ring. The connection between right ideals of R and of Q are described .

Definition: An ideal I_1 and I_2 of a ring R are said to be *complementary* to each other if $R = I_1 + I_2$ and $I_1 \cap I_2 = 0$ or equivalently $R = I_1 \oplus I_2$.
In Q every right ideal has the form eQ, with $e = e^2$,
$eQ \cap (1-e)Q = eQ \cap (Q-eQ) = eQ \cap Q \setminus eQ = (0)$, and $eQ + (1-e)Q = eQ + (Q-eQ) = Q$. This shows that eQ is a complement right ideal (complementary to (1-e)Q)and since for a semi prime ring R and an ideal I, ann(I) is the unique complement ideal to I in R, therefore eQ is an annihilator right ideal of (1-e)Q .

Proposition 2.3.1: (i) A right ideal I of R is a complement right ideal iff $I = J \cap R$ for $J \lhd_r Q$.

(ii) An annihilator right ideal I of R has the form $J \cap R$ for $J \lhd_r Q$.

(iii) If, further R is a left Goldie ring and $J \lhd_r Q$, then $J \cap R$ is an annihilator right ideal of R .

Proof: (i) If $K \lhd_r R$ with $I \cap K$, then

$$\begin{aligned}(IQ \cap R) \cap K &= IQ \cap (R \cap K)\\ &= (IQ \cap K) \cap R\\ &= 0 \cap R\\ &= 0\end{aligned}$$

So if I is a complement right ideal then $IQ \cap R = I$. Thus letting $J = IQ$ for $I\,Q \lhd_r Q$ we have $I = J \cap R$ for $J \lhd_r Q$.
Conversely since $J \lhd_r Q$, $J = eQ$ for $e = e^2$, thus $I = J \cap R = eQ \cap R = eQ$, which shows that I is a complement right ideal .

(ii) Suppose $J \lhd_r Q$, then by proposition (2.2.1) the annihilator right ideals of R has the form $J \cap R$.

(iii) let J = rannk with K = lannJ . Then $K = Q(K \cap R)$, so

$J = \mathrm{rann}_Q(K \cap R)$ and thus $J \cap R = \mathrm{rann}_R(K \cap R)$.

Lemma 2.3.1 : (i) A right ideal U of R is uniform if and only if UQ is a minimal right ideal of Q ;

(ii) If U a uniform right ideal of R and $0 \neq u \in U$, then

a) udim(rannu) = udim R_R -1;
b) If $I \lhd_r R$ with rannu $\subseteq I$, then $I \lhd_e R$.

Proof: (i) suppose U be uniform right ideal of R ,then by fact (7(iii)) $\text{udim}U_R = \text{udim}UQ_R = \text{udim}UQ_Q$,hence UQ is uniform right ideal of Q . Since uniform right ideals of Q are minimal, UQ is minimal right ideal of Q.

(ii) a) Here UQ is a minimal right ideal of Q and so rannU is a maximal right ideal, hence udim(rannu) = udimR_R -1.

b) Suppose $I \lhd_r R$ with rannUI and by proposition (2.3.1) the annihilator ideals of R are of the form $J \cap R$ for $J \lhd_r Q$ which is different from zero, then $I \lhd_e R$.

Proposition 2.3.2: Let U be a uniform right ideal of R

(i) Any non zero $\alpha \in \text{Hom}(U, R)$ is a monomorphism;

(ii) If $I \lhd_r R$, then the following are equivalent

a) $\text{Hom}(U, I) \neq 0$;
b) I contains a right ideal isomorphic to U;
c) $IU \neq 0$.

Proof : (i) By proposition(2.1.3) $\text{Hom}(U, R) \hookrightarrow \text{Hom}(UQ, RQ) = \text{Hom}(UQ, Q)$thus $\alpha \in \text{Hom}(U, R)$ is a monomorphism .

(ii) (a$\Longrightarrow$b) If $0 \neq \alpha \in \text{Hom}(U, I)$, then $U \simeq \alpha U \subseteq I$.

(b$\Longrightarrow$c) $0 \neq \alpha(u^2) = \alpha(U)U \subseteq IU \Longrightarrow IU \neq 0$.

(c$\Longrightarrow$a) If $IU \neq 0$, then $xU \neq 0$ for some $x \in I$. Therefore the map $U \longrightarrow xU$ is a monomorphism. So $xU \simeq U \subseteq I$, thus $(U, I) \neq 0$.

Corollary 2.3.1: If U, V are uniform right ideals of R, the following are equivalent

(i) $UV \neq 0$;
(ii) $VU \neq 0$;
(iii) U contains isomorphic copy of V ;
(iv) V contains an isomorphic copy of U.

Proof: (i $\Longrightarrow$ii) Let UV $\neq$ 0, we want to show that VU $\neq$ 0,

Suppose VU = 0 $\Longrightarrow$ $(UV)^2 = (UV)(UV) = U(VU)V = 0$

$\Longrightarrow$ UV = 0, a contradiction.

Therefore, VU $\neq$ 0.

(ii$\Longrightarrow$iii) Let VU $\neq$ 0, we want to show that UV $\neq$ 0

Suppose UV = 0 $\Longrightarrow$ $(VU)^2 = (VU)(VU) = V(UV)U = 0$

$\Longrightarrow$ VU = 0, a contradiction.

Therefore UV $\neq$ 0.

(ii $\Longrightarrow$ iii,iv) follows from the above proposition c $\Longrightarrow$ b .

(iii $\Longrightarrow$iv) follows from the above proposition b $\Longrightarrow$ c .

The condition that U, V contains an isomorphic copy of V, U respectively are summed up by saying that U, V are sub isomorphic.

Lemma 2.3.2: (i) The number of sub isomorphic classes of uniform right ideals of R equals the number of minimal prime ideals;

(ii) If R is prime, then all pair of uniform right ideals are sub isomorphic;

(iii) If R is prime and has a minimal right ideal, then R is simple Artinian.

Proof: (i) Both numbers are equal to the number of isomorphism classes of minimal right ideals of the right quotient ring of R.

(ii) Let R be prime, by (i) the number of minimal prime ideals are equal to the number of uniform right ideals which are sub isomorphic.

(iii) Since R has finite uniform dimension it contains an essential right ideal, say E, which is a direct sum of uniform right ideals. By (ii) these must all be minimal and isomorphic. Hence EndE $\simeq M_n(D)$ for some n and some division ring D. By proposition (2.1.3(iv)) ,$O_l(E) \simeq$ EndE which shows that $O_l(E) \simeq M_n(D)$, so is its own quotient ring ; yet by lemma (2.1.3) R is an equivalent right order, then R = $O_l(E)$ is simple Artinian .

Theorem 2.3.1: If U is a uniform right ideal of R, then EndU is a right order in the division ring $End_Q UQ$, and so EndU is a right Ore domain.

Proof: By lemma (2.3.1) UQ is a minimal right ideal of Q. Hence EndUQ is a division ring and as noted in proposition (2.1.3), EndU is a sub ring. Let $0 \neq \alpha \in$ EndUQ and consider V = U $\cap$ αU since UQ_R is uniform and $V \lhd_r R$, V is non zero.

Hence $UV \supseteq V^2 \neq 0$ and so by corollary (2.3.1) U and V are sub isomorphic.

Choose an isomorphic copy of U inside V, and let W be its inverse image under α. Then $W \subseteq U$ and there is an isomorphism β: $U \longrightarrow W$; so $\beta \in \mathrm{End}U$. Note that $\gamma = \alpha\beta \in \mathrm{End}U$. This shows that $\alpha = \gamma\beta^{-1}$ which shows EndU is a right order in EndUQ. So EndU is a right Ore domain.

Proposition 2.3.3: Let $I \lhd_r R$ and let $b \in R$. Then there exists $d \in I$ such that $\mathrm{udim}(b+d)R = \mathrm{udim}(bR+I)$.

Proof: If $\mathrm{udim}(bR+I) = \mathrm{udim}bR$; we can take $d = 0$. Otherwise there exist $U \subseteq I$, a uniform right ideal such that $bR \cap U = 0$. By induction it is enough to fined $u \in U$ such that $\mathrm{udim}(b+u)R = \mathrm{udim}(bR\oplus u) = \mathrm{udim}(bR) + 1$.

Next we show that $\mathrm{rann}(b) \nsubseteq \mathrm{rann}U$. For this suppose that $\mathrm{rann}(b) \subseteq \mathrm{rann}(u) = A$ say, and let $A' = \mathrm{lann}A$. Now $A \lhd R$ and A' is the complement to A (since R is semiprime, $\mathrm{rann}A = \mathrm{lann}A = \mathrm{ann}A$ and annA is the unique complement ideal to A).

Therefore $A' \simeq bA' \subseteq A'$ and so $\mathrm{udim}bA' = \mathrm{udim}A'$ and thus $bA' \lhd_e A'$. However $U \subseteq A'$ and so $0 \neq bA' \cap U \subseteq bR \cap U$ which contradicts to $bR \cap U = 0$. Hence $\mathrm{rann}(b) \nsubseteq \mathrm{rann}(U)$. Therefore there is some $u \in U$ with $\mathrm{rann}(b) \nsubseteq \mathrm{rann}(U)$, and this is the required element. To see this, note that $\mathrm{rann}(b) + \mathrm{rann}(u)$ is an essential right ideal, by lemma (2.3.1(ii)(b)) and by fact (4(2)(ii)) it contains a regular element, c say with $c = x + y$ and $bx = uy = 0$. Note that $bc = (b+u)y$ and $uc = (b+u)x$. Choose any element $br + us \in bR \oplus uR$. The right Ore condition gives elements $r', s' \in R$, $c' \in C_R(0)$ such that $rc' = cr'$ and $sc' = cs'$.

But then $(br + us)c' = bcr' + ucs' = (b+u)(yr' + xs') \in (b+u)R$.

Therefore $\mathrm{udim}(b+u)R = \mathrm{udim}(bR \oplus uR) = \mathrm{udim}(bR) + \mathrm{udim}(uR) = \mathrm{udim}(bR) + 1$.

Lemma 2.3.3: (i) A right ideal E of R is essential if and only if $EQ = Q$;

(ii) A principal right ideal cR is essential if and only if $c \in C_R(0)$;

(iii) If R is also left Goldie, then a right ideal E of R is essential if and only if $\mathrm{lann}E = 0$.

Proof : (i) suppose E is right ideal of R and $E \lhd_e R$, since the only essential right ideal of Q is Q by fact (7(i)) $EQ = Q$. Suppose $EQ = Q$, we have $Q = EQ \lhd_e Q$ which implies that $E \lhd_e Q$ by fact (7(i)).

(ii) If $cR \lhd_e R$ then $cQ = Q$ by (i). Thus c is a unit of Q and so by definition c is regular in R. Conversely if $c \in C_R(0)$ by definition c is a unit in Q which implies $cQ = Q$ again by (i) $cR \lhd_e R$.

(iii) Suppose $E \lhd_e R$, then by (i) $EQ = Q$

$\mathrm{lann}(E) = \{ r \in R \mid Er = 0 \}$, but $Er = 0$, if $r = 0$.

Therefore lann$E = 0$.

Conversely, If E is not essential then $EQ \neq Q$ and so lann$EQ \neq 0$. It follows lann$E \neq 0$ by the proof of prop (2.3.1(iii)).

Corollary 2.3.2: Each essential right ideal E of R is generated by regular elements.

Proof: By fact (4(2)(ii)) E contains a regular element, c say. By the above proposition, given any $b \in E$, there exists $d \in cR$ such that udim$(b + d)R$ =udim$(bR + cR)$ = rudim(R). Therefore $(b + d)R \lhd_e R$ and, by the above lemma $b + d \in C_R(0)$. Since $b \in (b + d)R + cR$ and $b \in E$ then E is generated by regular elements .

2.4 ENDOMORPHISM RINGS

Again in this section R denotes a semiprime right Goldie ring with quotient ring Q. in the theorem on section 2.3 it was shown that the endomorphism ring of a uniform right ideal of R is a right Ore domain and is semiprime right Goldie. This result will now extended to describe endomorphism rings of all right ideals and certain modules.

Consider EndM where M is finitely generated module over Q. This is easily described, for M decomposes as a finite direct sum of simple modules which can be grouped together in isomorphism classes. Evidently then EndM is a direct sum of matrix rings over division rings $\overset{k}{\underset{i=1}{\oplus}} M_{ni}(D_i)$ say .

Now recall that if R be an integral domain and M be R-module, an element $m \in M$ is said to be a *torsion element* if $\text{ann}(m) \neq (0)$. And if the set of torsion elements of M, $\tau(M) = 0$, then M is called a torsion free module. Also when $\tau(M) = M$, we call M is a *torsion module*. A module M_S over any ring S is *torsionless* if given any $0 \neq m \in M$ there exist $\alpha \in M^* = \text{Hom}(M, S)$ such that $\alpha(M) \neq 0$, also for any module M_S the left S-module ${}_SM^*$ is torsionless since if $0 \neq \alpha \in M^*$ and $m \in M$ is such that $\alpha(m) \neq 0$. Then the map $M^* \longrightarrow S$ via $\beta \mapsto \beta(m)$ is as required.

When S = R, M_R being torsion free. Which means that its torsion submodule with respect to $C_R(0)$ is zero. This is equivalent to saying that the map $M \longrightarrow M \otimes_R Q$ is an embedding, since the torsion submodule is the kernel. Note that a torsion free module need not be torsionless.

Example: $\mathbb{Q}_\mathbb{Z}$ is torsion free but it is torsionless where $\mathbb{Q}$ and $\mathbb{Z}$ is the set of rational numbers and the set of integers respectively.

Proposition 2.4.1: Let M_R be a torsionless module. Then

- (i) M_R is torsion free and $\text{udim}(M_R) = \text{udim}(M \otimes Q_R) = \text{udim}(M \otimes Q_Q)$;
- (ii) If M_R is finitely generated, then udim $M_R < \infty$;
- (iii) If udim$M < \infty$,then $M \hookrightarrow R^n$ for some n ;
- (iv) If udim$M$ = t, then $M \hookrightarrow R^t$.

Proof: (i) Suppose M_R be torsion less module, thus if given $0 \neq m \in M$ there exists $\alpha \in \text{Hom}(M, R)$ such that $\alpha(m) \neq 0$. Let $c \in C_R(0)$, then $\alpha(mc) = \alpha(m)c \neq 0$, thus $mc \neq 0$ which shows that M_R is torsion free. This means that its torsion submodule with respect to $C_R(0)$ is zero which is equivalent to saying that $M \hookrightarrow M \otimes_R Q$, then by fact (7(iii)) $\text{udim}M_R = \text{udim}M \otimes Q_R = \text{udim}M \otimes Q_Q$.

(ii) Since $M \otimes Q_Q$ is finitely generated and Q is semisimple, $M \otimes Q_Q$ has finite uniform dimension, hence by (i) $\text{udim}M_R = \text{udim}M \otimes Q_Q$.

(iii) Among all maps from M to free R-modules of finite rank choose one, say $\alpha \in \mathrm{Hom}(M, R^n)$, so that udim($\ker\alpha$) is as small as possible. Suppose $\ker\alpha \neq 0$, choose $0 \neq U \subseteq \ker\alpha$ with U uniform, and pick $0 \neq u \in U$. There exists $\beta \in \mathrm{Hom}(M, R)$ such that $\beta(u) \neq 0$.
Now define $\gamma : M \longrightarrow R^{n+1}$ by $\gamma(m) = (\alpha(m), \beta(m))$. Clearly $\ker\gamma = \ker\alpha \cap \ker\beta$.
Consider $\ker\beta \cap U = V$ say. If $V \neq 0$, then V is uniform and essential in U. Therefore for each $u \in U$ there exist an essential right ideal E of R with $UE \subseteq V$, pick $c \in E \cap C_R(0)$. Then $0 = \beta(uc) = \beta(u)c$,and so $\beta(u) = 0$, a contradiction . Thus $\ker\beta \cap U = 0$; but then udim($\ker\gamma$) $<$ udim($\ker\alpha$)which is also a contradiction , hence $\ker\alpha = 0$ which shows that α is monomorphism ,thus $M \hookrightarrow R^n$ for some n .

(iv) By part (iii) we may suppose that $M \subseteq R^n = \sum_{i=1}^{n} e_i R$ say. If $n > t$, note that $M \cap e_i R = 0$ for some i. Otherwise $n \leq t$, udimM $\geq$ udim($M \cap e_iR$) $\geq n$. But then $M \hookrightarrow R^n/e_iR \simeq R^{n-1}$ and applying induction i.e Since $M \hookrightarrow R$ we have
$M \hookrightarrow R^{n-1} \oplus R^1 = R^n \subseteq R^t \implies M \hookrightarrow R^t$. n

Proposition 2.4.2: Let M be torsionless of finite uniform dimension, and S = EndM.

(i) If $N \lhd M$, then undim(N_R) = udim(NM^*) S ;

(ii) If $I \lhd S$, then udimI_S = udimIM_R ;

(iii) $S \hookrightarrow \mathrm{End}_Q(M \otimes Q)$.

Proof : (i) and (ii) are proved in a similar manner as follows .
Note that , if $I \neq 0$ then $IM \neq 0$, since $I \subseteq$ EndM. Similarly if $NM^* = 0$ then $(M^*N)^2 = (M^*N)(M^*N) = M^*(NM^*)N = 0$ and so, R being semiprime and in a semiprime ring the only nilpotent right or left ideal is zero thus $M^*N = 0$. However, M is torsion less and so N = 0. If $N_1 \oplus N_2 \oplus \ldots \oplus N_k$ is a direct sum of non zero sub modules of N, then $N_1M^* \oplus N_2M^* \oplus \ldots \oplus N_kM^*$ is a direct sum of non zero right ideals of S. Thus udim(N_R) = udim$(NM^*)_s$ and udimI_S = udimIM_R

(iii) If $\alpha \in S$, then $\alpha \otimes 1 \in \mathrm{End}(M \otimes Q)$ is a unique extension since $M \hookrightarrow M \otimes Q$, then EndM = S $\hookrightarrow$ End($M \otimes Q$) via the map $\alpha \mapsto \alpha \otimes 1$.

Theorem 2.4.1: Let M_R be a torsionless module of finite uniform dimension and S = EndM. Then S is semiprime right Goldie and rudimS = udimM_R.

Proof: suppose $I \lhd S$ with $I^2 = 0$, then $(M^*IM)^2 = M^*IM\, M^*IM \subseteq M^*IM = 0$ and so M^*IM is a nilpotent ideal of R. Since R is semiprime the only nilpotent ideal is 0, therefore $M^*IM = 0$ and so $IM = 0$, since M is torsionless. However $I \subseteq S$, so $I = 0$. This shows that S is semiprime since it has no non zero nilpotent ideals.

By proposition (2.4.2(i)and(ii)) S has finite right uniform dimension equal to that of M_R, also by proposition (2.4.2(iii)), $S \hookrightarrow \text{End}(M \otimes Q)$ by proposition(2.4.1), $(M\otimes Q)_Q$ is finitely generated and since we consider End $(M \otimes Q)_Q$ where $(M \otimes Q)_Q$ is finitely generated module over Q and Q is semisimple Artinian . Therefore its sub ring S satisfies an ascending chain condition on right annihilators, this shows that S is semiprime right Goldie.

Proposition 2.4.3: Let M_R be torsion less of finite uniform dimension and S = EndM. If $M' \lhd_e M$, then there exists $c \in M'M^* \subseteq S$ such that c is a unit of End$(M \otimes Q)$.

Proof: By proposition (2.4.2(i) and (ii)) we have $M'M^* \lhd_e S$, and since S is semiprime right Goldie by theorem (2.4.1) $c \in M'M^*$ for some $c \in C_S(0)$. Let K be the kernel of c acting on $M \otimes Q$; So $c(K \cap M) = 0$. then $c(K \cap M)M^* = 0$ and since $(K \cap M)M^* \subseteq S$, it follows that $(K \cap M)M^* = 0$. By proposition (2.4.2) $K \cap M = 0$, and therefore, using proposition (2.4.1(i)) $K = 0$. Thus c is a monomorphism on $M\otimes Q$ and so is an isomorphism, that is c is a unit in End$(M\otimes Q)$.

Corollary 2.4.1: Let M_R be torsionless of finite uniform dimension.

(i) EndM is a right order in the semisimple Artinian ring End$(M \otimes Q)$.
(ii) If R is prime, then EndM is prime.
(iii) If $N \lhd_e M$, then EndN $\sim$ EndM.

Proof: (i) Let $\alpha \in \text{End}(M \otimes Q)$. Let M_0 be the inverse image of M under α, then $M_0 \lhd_e (M \otimes Q)_R$. Now let $M' = M_0 \cap M$ and choose $c \in M'M^*$ for some $c \in C_S(0)$. $cM \subseteq M'$ and so $\alpha c = \gamma : M \longrightarrow M$. There fore $\alpha = \gamma c^{-1}$ and both $\gamma, c \in S$.

(ii) Let R be prime, then End$(M\otimes Q)$ is simple. So by (i) , End(M) is prime.

(iii) Choose c, a unit of End$(M\otimes Q)$, so that $cM \subseteq M$. Note that since $N \lhd_e M$ implies udimN = udimM, then $N\otimes M = M\otimes Q$. So there is an embedding $N \hookrightarrow M \otimes Q$ and EndN $\hookrightarrow$ End$(M \otimes Q)$. Then (EndN)c $\subseteq$ EndM and c(EndM) $\subseteq$ N which shows that EndM $\sim$ EndN.

Proposition 2.4.4: Let R be a semiprime Goldie ring

(i) If M_R is finitely generated torsion free, then $M \hookrightarrow R^n$ and so is torsionless.

(ii) If M_R is torsionless of finite uniform dimension, then EndM is semiprime Goldie.

Proof: (i) $M \hookrightarrow M \otimes Q \hookrightarrow Q^n$ for some n. Now choose a common denominator c for the generators in Q^n of M_R. Then $M \simeq cM \simeq R^n$.

(ii) $M \otimes Q_Q$ has finite uniform dimension, so too has the left Q-module $\mathrm{Hom}(M \otimes Q, Q)$. Since R is also a left order in Q, then $\mathrm{udim}_R(\mathrm{Hom}(M \otimes Q, Q)) < \infty$. However $M^* \hookrightarrow \mathrm{Hom}(M \otimes Q, Q)$. Therefore ${}_RM^*$ has finite uniform dimensiom and is torsionless which shows that $\mathrm{End}M^*$ is semiprime left Goldie ring by theorem (2.4.1) .
Note that there is a natural embedding $\mathrm{End}M \hookrightarrow \mathrm{End}\, M^* \hookrightarrow \mathrm{End}(M \otimes Q)$. By the above proposition, there is a unit $c \in \mathrm{End}(M \otimes Q)$ with $c \in MM^*$. However ; $c\mathrm{End}M^* \subseteq MM^*(\mathrm{End}\, M^*) \subseteq MM^* \subseteq \mathrm{End}M$. Thus $\mathrm{End}M \sim \mathrm{End}M^*$ and hence is left as well as right Goldie.
Remark: Every right ideal I of R is torsionless and of finite uniform dimension, then by the above proposition EndI is semiprime right Goldie ring with rudimEndI = udimI. If I is uniform then rudimEndI = 1, and so EndI is a right ore domain.

Theorem 2.4.2: R is equivalent to a direct sum of matrix rings over right Ore domain.

Proof: Using proposition (2.3.2) and lemma (2.3.2), choose an essential right ideal I of R which is a direct sum $\oplus_{i=1}^{t} U_i$ of uniform right ideals so that, for each i, j either $U_i \simeq U_j$ or else $\mathrm{Hom}(U_i, U_j) = 0 = \mathrm{Hom}(U_j, U_i)$. It shows that EndI is a direct sum of matrix rings over right Ore domain. Then by corollary (2.4.1) $\mathrm{End}I \sim \mathrm{End}(R_R) \simeq R$. Hence R is equivalent to a direct sum of matrix rings over right Ore domain.

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