ADDIS ABABA UNIVERSITY

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Department of Mathematics

GRADUATE SEMINAR REPORT

ON

Transform Methods in Differential Equations

(SUBMITTED IN PARTIAL FULFILMENT OF THE M.Sc. DEGREE IN MATHEMATICS)

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Table of contents

Contents **page**

Notations

- $\text{Supp}[\varphi]$: = Support of a function or a distribution,

  $\text{Supp}[\varphi] := \overline{\{x \in R^n : \varphi(x) \neq 0\}}$.
- D: = The whole set of infinitely differentiable functions having compact support.
- D' : = The space of all distributions.
- $D_{L^2}{}^m$:= The whole space of $f \in L^2$ which also have all derivatives of order up to m belonging to L^2.
- $f \in L^1_{loc}$: = f is locally summable $\Leftrightarrow$ a measurable function f defined on R^n is summable in a nbhd of every point in R^n.
- ε' := the space of distribution with compact support

Transform Methods in DES

Introduction

This topic introduces a powerful technique for solving differential equations, both ordinary and partial, and integral equations; involving methods quite different from the methods we are familiar. The technique involves applying an integral operator $T = Tx \rightarrow v$ to transform a function y = y(x) of the independent real variable x to a new function T (y) = T (y) (v) of the real or complex variable v. If y solves, say a differential equation (DE) and satisfies conditions allowing the integral operator to be applied the function T (y) will be found to solve an other equation TDE, perhaps also (but not necessarily) a differential equation. This gives as a new method for solving a differential equation DE.

It involves the following steps.

1) Supposing y satisfies conditions for the integral operator T to be applied, transform (DE) to (TDE).
2) Solve (TDE) to find T(y)
3) Find a function y the transform of which is T (y) which will then solve DE.

We will find that in solving differential equations, there can be at most one function y corresponding to a transformed function T(y), and hence that we will have a one- to -one correspondence between transformable y 's solving (DE) and transformed functions T(y)'s solving (TDE).

We shall limit our discussions to considerations involving the two most important and widely used transform operators, associated with the names Fourier and Laplace. In each case, we shall present some basic facts and then its application in solving the differential equation. Throughout this seminar report we shall need to integrate over an infinite interval I. f is integrable over I means;

i) f is Lebesgue integrable over I.

ii) f is piecewise continuous on every closed and bounded interval in I and the improper Riemann integrals of both f and $|f|$ exist over I.

Chapter ONE

THE LAPLACE TRANSFORMS

The Laplace transform is named for the eminent French Mathematician P.S Laplace who studied equations of the form $L\{f(t)\} = F(s) = \int_0^\infty e^{-st} f(t)dt$ in 1782. However most of the techniques of Laplace transform were not developed until a centaury or more latter. Many practical engineering problems involve mechanical or electrical systems acted on by discontinuous or impulsive forcing terms. For such problems the methods we are familiar are difficult to use. The method that especially well studied to these problems, although useful much more generally is the Laplace transform. Among the tools that are very usefully in solving LDES are integral transforms. An integral transform is a relation of the form $F(s) = \int_\alpha^\beta k(s,t)\, f(t)\, dt$ where a given function f is transformed in to another function f by means of an integral. In this case F is the transform of f and k is the kernel of transformation.

In this case the main idea is to use the relation. $F(s) = \int_\alpha^\beta k(s,t)\, f\,(t)\, dt$ to transform a problem for f in to a simpler problem for F ,to solve this simple problem and then recover the desired function f from its transform F.

1.1 Definitions

Definition1.1: - A function f is said to be piecewise continuous on $\alpha \le t \le \beta$ if the interval can be partitioned by a finite number of points $\alpha = t_0 < t_1 < ... < t_n = \beta$ such that, 1) f is continuous on each subinterval $t_{i-1} < t < t_i$

2) f approaches a finite limit as the end points of each subinterval are approached from within the subinterval.

In general if f is piecewise continuous for $t \ge a$, $\int_a^A f(t)dt$ exists for each A > a. But piecewise continuity is not enough to ensure convergence of $\int_a^\infty f(t)dt$.

Example: - Let $f(t) = \dfrac{1}{t}, t \ge 1$ which is piecewise continuous. But

$$\int_1^{\infty} \frac{dt}{t} = \lim_{A\to\infty}(\ln A) = \infty$$ which is divergent.

Definition 1.2: - Laplace transform is defined as follows.

Let f (t) be given for $t \geq 0$ and suppose that f is piecewise continuous. Then the Laplace transform of f denoted by $L\{f(t)\}$ or F(s) is defined as $L\{f(t)\} = F(s) = \int_0^{\infty} e^{-st} f(t)\, dt$.

In this case the transform kernel is $k(s,t) = e^{-st}$ which is naturally associated with linear differential equations with constant coefficients. *If* $\int_a^A f(t)\, dt$ *exists* $\forall\ A > a$ and $\lim A \to \infty$ exists we say that the improper integral converges to that limit value and other wise it diverges.

Examples

1) Let $f(t) = e^{ct}, \quad t \geq 0$

$$\Rightarrow \int_0^{\infty} e^{ct}\ dt = \lim_{A\to\infty} \int_0^A e^{ct} dt = \lim_{A\to\infty} \frac{1}{c}(e^{cA} - 1)$$

$\Rightarrow \int_0^{\infty} e^{ct} dt$ Converges if c < 0.

2) Let $f(t) = \frac{1}{t} \quad t \geq 1$

$$\Rightarrow \int_1^{\infty} \frac{dt}{t} = \lim_{A\to\infty} \int^A \frac{dt}{t} = \lim_{A\to\infty} \ln A = \infty$$

Theorem 1.1:- Suppose that;

1) f is piecewise continuous on the interval $0 \leq t \leq A$ for any positive A.

2) $|f(t)| \leq ke^{at}, t \geq M$ Where k, M, a, are real constants, k and M are necessarily positive. Then the Laplace transform $L\{f(t)\} = F(s) = \int_0^{\infty} e^{-st} f(t)\, dt$ exists for s > a. The parameter s can be either real or complex.

Proof: - It is enough to show that $F(s) = \int_0^{\infty} e^{-st} f(t) dt$ converges for s > a.

Splitting the improper integral in to two parts, we have

$$\int_0^\infty e^{-st} f(t)\, dt = \int_0^M e^{-st} f(t) dt + \int_M^\infty e^{-st} f(t) dt$$

The first integral exists by hypothesis (1) of theorem (1.1). This implies the existence of F(s) depends on the existence of the second integral. By (2) we have for

$t \ge M$, $\left|e^{-st}\right| \le ke^{-st}e^{at} = ke^{(a-s)t}$ and thus F(s) exists provided that,

$\int_M^\infty e^{(a-s)t}\, dt$ Converges by the comparison theorem.

If $f(t) = e^{ct}, t \ge 0,\ c \ne 0\ \ and\ c \in R$ then, $\int_0^\infty e^{ct} dt = \lim_{A\to\infty} \int_0^A e^{ct} dt = \lim_{A\to\infty} \frac{1}{c}(e^{cA} - 1)$

This implies the improper integral converges if c < 0 and diverges if $c \ge 0$. If we replace c by a-s, then $\int_M^\infty e^{(a-S)t}\, dt$ converges if a-s < 0 and this implies s > a which completes the proof.

Examples

1) Let f (t) = 1 then find its Laplace transform.

$$L\{1\} = \int_0^\infty e^{-st} dt = \lim_{A\to\infty} \int_0^A e^{-st} dt = \lim_{A\to\infty} (\frac{1}{s} e^{-st})\Big|_0^A = \frac{1}{s},\ s > 0$$

2) Let $f(t) = t$, then

$$L\{f(t)\} = \int_0^A e^{-st} t\ dt = \lim_{A\to\infty} \int_0^t e^{-st} t\ \ dt\ t$$

$$= \lim_{A\to\infty} \left[-\frac{1}{s} e^{-st} t \Big|_0^A + \frac{1}{s} \int_0^A e^{-st} dt \right.$$

$$= \frac{1}{s^2} \ \ for\ s > a$$

3) Let $f(t) = e^{at}, \quad t \geq 0$. Then

$$L\{f(t)\} = \int_0^\infty e^{-st} e^{at} dt = \lim_{A\to\infty} \int_0^A e^{-st} e^{at} dt = \lim_{A\to\infty} \int_0^A e^{-(s-a)t} dt$$

$$= \lim_{A\to\infty}\left(-\frac{1}{s-a} e^{-(s-a)t} \Big|_0^A \right) = \lim_{A\to\infty} \frac{-1}{s-a}\left[e^{-(s-a)A} - 1)\right]$$

$$= \frac{1}{s-a}, \; s > a.$$

4) Let $f(t) = t^n$ *where* $n \in N$.

$L\{t^n\} = \int_0^\infty e^{-st} \, t^n dt$ Then by using integration by parts,

$$\int_0^\infty e^{-st} t^n dt = \lim_{A\to\infty} (-\frac{1}{s} e^{-st} \, t^n \,)\Big|_0^A + \frac{n}{s}\int_0^\infty e^{-st} t^{n-1} dt$$

Applying this technique repeatedly we will get $L\,\{t^n\} = \dfrac{n!}{s^{n+1}}$

5) Let f(t) = Sin at

$$\Rightarrow L\{Sin\ at\ \} = \int_0^\infty e^{-st} Sin\ at\ dt$$

$$= \lim_{A\to\infty} \int_0^A e^{-st}\ Sin\ at\ dt$$

Let $u = e^{-st} \quad \Rightarrow \; du = se^{-st}, dv = Sin\ at \Rightarrow v = \dfrac{Cos\ at}{a}$

$$\Rightarrow uv - \int vdu = -\frac{e^{-st} Cos\ at}{a} - \frac{s}{a} \int e^{-st} \cos at\ dt$$

$$\Rightarrow \int_0^\infty e^{-st}\ Sinat\ dt = \lim_{A\to\infty}\left(\frac{-e^{st} \cos at}{a}\Big|_0^A \right) \quad - \frac{s}{a}\int_0^\infty e^{-st}\ Cos\ at\ dt$$

$$\Rightarrow F(s) = \frac{1}{a} - \frac{s}{a} \int_0^\infty e^{-st} Cos\ at\ dt$$

$$\Rightarrow F(s)=\frac{1}{a}-\frac{s}{a}[\lim_{A\to\infty} e^{-st}\frac{\sin at}{a}]+\frac{s}{a}\int_0^{\infty} e^{-st} Sin\ at)\ dt$$

$$\Rightarrow F(s)=\frac{1}{a}-\frac{s^2}{a^2}F(s)$$

$$\Rightarrow F(s)\left[1+\frac{s^2}{a^2}\right]=\frac{1}{a}$$

$$\Rightarrow F(s)=\frac{a}{s^2+a^2} \qquad ,s>0$$

1.2 Properties of Laplace transform

a) Linearity: - Let f_1 *and* f_2 are two functions whose Laplace transform exist for $s>a_1$ *and for* $s>a_2$ respectively. Then for $s>\max\{a_1,a_2\}$, $L\{c_1 f_1(t)+c_2 f_2(t)\}=c_1 L\{f_1(t)\}+c_2 L\{f_2(t)\}$

Proof: - $L\{c_1 f_1(t)+c_2 f_2(t)\}=c_1\int_0^{\infty} e^{-st} f_1(t)+c_2\int_0^{\infty} e^{-st} f_2(t)\,dt$

$$=c_1 L\{f_1(t)\}+c_2 L\{f_2(t)\}$$

b) **The shifting Theorem**

Let $L\{f(t)\}=F(s)$. *Then* $L\{e^{at}f(t)\}=F(s-a)$ *for* $s>a$.

Proof:-

By definition $F(s)=\int_0^{\infty} e^{-st}f(t)\,dt$

$$\Rightarrow F(s-a)=\int_0^{\infty} e^{-(s-a)t} f(t)\,dt$$

$$=\int_0^{\infty} e^{-st}(e^{at}f(t))dt$$

$$=L\{e^{at}f(t)\}$$

From this we can also get $L\{e^{-at}f(t)\}=F(s+a)$.

Example: - Find the Laplace transform of $e^{at}t^n$.

$L\{e^{at}f(t)\}=L\{e^{at}t^n\}=\frac{n!}{(s-a)^{n+1}}$, *Since* $L\{t^n\}=\frac{n!}{s^n+1}$

c) Inverse Laplace transforms

Definition: - If $L\{f(t)\}=F(s)$, then f (t) is called the inverse transform and is denoted by $L^{-1}\{F(s)\}$.

Examples: -

a) $L\{Cos\ 2x\}=\frac{s}{s^2+4} \Rightarrow L^{-1}\left\{\frac{s}{s^2+4}\right\}=Cos\ 2x$

b) Find the inverse transform of $\frac{s+7}{s^2+2s+5}$

Solution:- $F(s)=\frac{s+7}{s^2+2s+5}=\frac{(s+1)+6}{(s+1)^2+2^2}=\frac{s+1}{(s+1)^2+2^2}+3\frac{2}{(s+1)^2+2^2}$

$$\Rightarrow L^{-1}\{F(s)\}=L^{-1}\left\{\frac{s+1}{(s+1)^2+2^2}\right\}+3L^{-1}\left\{\frac{2}{(s+1)^2+2^2}\right\}$$

$$\Rightarrow f(t)=e^{-t}Cos2t+3e^{-t}Sin2t$$

$$=e^{-t}(Cos2t+3Sin2t$$

1.3 Heaviside expansion formula

The Heaviside expansion formula provides a convenient method of evaluating inverse transformation. Let F(s) be the Laplace transform of f (t) where f (t) is the quotient of two polynomials $F_1(s)$ *and* $F_2(s)$. That is $F(s)=\frac{F_1(s)}{F_2(s)}$ where the degree of $F_2(s)$ is assumed to be greater than the degree of $F_1(s)$. If $F_2(s)$ can be factored, then

$F(s)=\frac{F_1(s)}{(s-a_1)(s-a_2)...(s-a_n)}$. Now two cases will arise.

A) If $F_2(s)$ has no repeating factors, in this case F(s) can be expressed as a partial fraction.

$\Rightarrow F(s)=\frac{A_1}{s-a_1}+\frac{A_2}{s-a_2}+...+\frac{A_n}{s-a_n}$ .... (*) *where* $A_1, A_2, ..., A_n$

are to be determined. Multiplying (*) by $(s-a_r)$ we get,

$$F(s)\ (s-a_r) = A_1\ \frac{s-a_r}{s-a_1} + A_2\frac{s-a_r}{s-a_2}\ + \ldots + A_r + \ \ldots + A_n\frac{s-a_r}{s-a_n} \ldots\ldots\ldots\ldots(**)$$

If we take $\lim\limits_{s\to a_r}$ we get $\lim\limits_{s\to a_r} F(s)(s-a_r)\ = A_r$

$$\Rightarrow A_r\ = \lim_{S\to a_r}\ (s-a_r)\ \ \frac{F_1(s)}{F_2(s)}$$

$$= F_1(s)\ \lim_{S\to a_r} \frac{s-a_r}{F_2(s)}$$

Now consider $\lim\limits_{S\to a_r} \dfrac{s-a_r}{F_2(s)}$, the factor s-$a_r$ appears both in the numerator and denominator.

$$\Rightarrow It\ is\left(\frac{0}{0}\right) from\,.$$

By using L'Hospital's rule $\lim\limits_{s\to a_r}\ \dfrac{s-a_r}{F_2(s)}\ = \dfrac{1}{F_2^{'}\ (a_r)}$

$$\Rightarrow A_r = \frac{F_1(a_r)}{F_2^{'}(a_r)}$$

$$\Rightarrow F(s) = \sum_{r=1}^{n}\frac{F_1(a_r)}{F_2^{'}(a_r)}\frac{1}{s-a_r}$$

$\Rightarrow$ The inverse transform is $f(t) = \displaystyle\sum_{r=1}^{n}\frac{F_1(a_r)}{F_2'(a_r)}e^{a_r t}$

Example 1: - obtain the inverse transform of $F(s)= \dfrac{s^2+2s-3}{s(s-3)\,(s+2)}$

Solution:- $F(s)=\dfrac{F_1(s)}{F_2(S)}\ = \dfrac{s^2+2s-3}{s(s-3)\,(s+2)}$. $F_2(s)$ has three factors which are non repeating.

$a_1 = 0,\ a_2 = 3, a_3 = -2\ \ , F_1(s) = s^2+2s-3\,,$

$$F_2(s) = s(s-3)(s+2) \Rightarrow F_2'(s) = 3s^2-2s-6$$

$$\Rightarrow f(t) = \frac{F_1(0)}{F_2^{'}(0)} + \frac{F_1(3)}{F_2^{'}(3)}e^{3t} + \frac{F_1(-2)}{F_2(-2)}e^{-2t} = \frac{1}{2} + \frac{4}{5}e^{3t} - \frac{3}{10}e^{-2t}$$

The expansion for the inverse transform can also be used if the roots for $F_2(s)$ are complex.

Example 2:- Let $F(s)=\dfrac{s-3}{s^2+2s+2}=\dfrac{s-3}{(s+1+i)\ (s+1-i)}$.

Here the roots are a_1 = -1-i and a_2 = -1+i. $F_2'(s)=2s+2$

$$f(t)=\frac{F_1(a_1)}{F_2'(a_1)}e^{a_1t}\frac{F_1(a_2)}{F_2'(a_2)}e^{a_2t}$$

$$=\frac{4+i}{2i}e^{-(1+i)t}+\frac{-4+i}{2i}e^{-(1-i)t}$$

$$=\frac{e^{-2t}}{2i}\left[(4+i)e^{-it}+(-4+i)e^{it}\right]$$

$$=\frac{e^{-t}}{2i}\left[i(e^{it}+e^{-it})-4(e^{it}-e^{-it})\right]$$

$$=\frac{e^{-t}}{2i}\left[2iCost+8iS\text{int}\right]$$

$$=e^{-t}\left[Cost-4S\text{int}\right]$$

B) If $F_2(s)$ has repeating factors.

Let $F(s)=\dfrac{F_1(s)}{(s-a_1)^n+(s-a_2)^{n-1}+....(s-a_n)}$

$$\Rightarrow F(s)=\frac{A_1}{(s-a_1)^n}+...+\frac{A_n}{s-a_1}+\frac{c_2}{s-a_2}+....+\frac{c_n}{s-an}$$

$$=\sum_{r=1}^{n}\frac{A_r}{(s-a_1)^{n-r+1}}+G(s)\quad \text{Where } G(s)=\frac{c_2}{s-a_2}+...+\frac{c_n}{s-a_n}.$$

We can obtain the inverse transform of G(s) by (A).

To obtain $A_m, 1\le m\le n,$ multiply by $(s-a_1)^n$.

$$\Rightarrow F(s)\,(s-a_1)^{\,n}=\sum_{n=1}^{n}\frac{A_r}{(s-a_r)^{1-r}}+(s-a_r)\,G\,(s)$$

$$=\sum_{r=1}^{n}(s-a_1)^{r-1}+(s-a_r)G(s).$$

$$= \sum_{r=1}^{n} (s-a_1)^{r-1} A_r + (s-a_1)^n \ G(s) \ldots\ldots\ldots\ldots\ldots(*)$$

Differentiating (*) successively (m-1) times we will get,

$$\frac{d^{m-1}}{ds^{m-1}}\left[(s-a_1)^n F(s)\right] = \sum_{r=m}^{n} (r-1)(r-2)\ldots(r-m+1)(s-a_1)^{r-m} a_r + \frac{d^{m-1}}{ds^{m-1}}\left[(s-a_1)^n G(s)\right]$$

…………………………………….(**)

All terms on the right side of (**) contain the factor ($s-a_1$) raised to some power except the term with r = m. Hence $(m-1)\ \ (m-2)\ldots 3.2.1.A_m = \lim_{s\to a_1} \frac{d^{m-1}}{ds^{m-1}}(s-a_1)^n F(s)$ or

$A_m = \frac{1}{(m-1)!} \lim_{s\to a_1} \frac{d^{m-1}}{ds^{m-1}} (s-a_1)^n \ F(s), 1 \le m \le n$. Replacing r for m,

$A_r = \frac{1}{(r-1)!} \lim_{s\to a_1} \frac{d^{r-1}}{ds^{r-1}} ((s-a_1)^n)\ F(s))$. We know that $L\left\{e^{a_1 t} t^{n-t}\right\} = \frac{(n-r)!}{(s-a_1)^{n-r+1}}$

$$\Rightarrow L^{-1}\left\{\frac{1}{(s-a_1)^{n-r+1}}\right\} = \frac{1}{(n-1)!} e^{a_1 t} . t^{n-r}$$

$\Rightarrow$ The inverse transform of F(s) is $f(t) = \sum_{r=1}^{n} \frac{A_r}{(n-r)!} t^{n-r} e^{a_1 t} + g(t)$ where g(t) is the inverse transform of G(s).

Example:-Find f (t) when $F(s) = \frac{1}{s(s+2)^3}$

Solution:- $F(s) = \frac{A_1}{(s+2)^3} + \frac{A_2}{(s+2)^2} + \frac{A_3}{s+2} + \frac{C_0}{s}$

Since $\frac{1}{s}$ is the non repeating term, then its coefficient can be determined.

$$\Rightarrow F_1(s) = 1,\ F_2(s) = s(s+2)^3 \Rightarrow F_2'(s) = 4s^3 + 18s^2 + 24s + 8$$

Substituting 0 for a_r , $C_0 = \frac{F_1(0)}{F_2'(0)} = \frac{1}{8} \Rightarrow L^{-1}\left\{\frac{1}{8s}\right\} = \frac{1}{8}$.

Coefficients A_1, A_2 *and* A_3 are as follows.

$$A_1 = (s+2)^3 \frac{1}{s(s+2)^3}, at\ s=-2 \Rightarrow A_1 = \frac{-1}{2}$$

$$A_2 = \frac{d}{ds}\left[(s+2)^3 \frac{1}{s(s+2)^3}\right], at\ s=-2 \Rightarrow A_2 = \frac{-1}{(-2)^2} = \frac{-1}{4}$$

$$A_3 = \frac{1}{2}\frac{d^2}{ds^2}\left[(s+2)^3 \frac{1}{s(s+2)^3}\right], at\ s=-2, \Rightarrow A_3 = \frac{1}{2}\frac{2}{(-2)^3} = \frac{-1}{8}$$

$$\Rightarrow f(t) = \frac{e^{-2t}}{4}(t^2+t+\frac{1}{2}) + \frac{1}{8}$$

1.4 Solutions of Initial value problems

In this section we show how the Laplace transform is used to solve initial value problems for linear differential equations with constant coefficients. The transform of f' is related to the transform of f by the following theorem.

Theorem 1.2:-Suppose f is continuous and f' is piece wise continuous on any interval $0 \le t \le A$. and suppose that there exist constants k, a, and M such that, $|f(t)| \le kea^t\ for\ t \ge M$, Then $L\{f'(t)\}$ exists for s > a and $L\{f'(t)\} = sL\{f(t)\} - f(0)$.

Proof: - Consider $\int_0^A e^{-st} f'(t)\,dt$ .Let $t_1, t_2, ..., t_n$ be points in $0 \le t \le A$ where f' is discontinuous.

$$\Rightarrow \int_0^A e^{-st} f'(t)\,dt = \int_0^{t_1} e^{-st} f'(t)dt + \int_{t_1}^{t_2} e^{-st} f'(t)dt + ... + \int_{t_n}^A e^{-st} f'(t)dt$$

Integrating each terms by parts we get,

$$\int_0^A e^{-st} f'(t)\ dt = e^{-st} f(t)\Big|_0^{t_1} + e^{-st} f(t)\Big|_{t_1}^{t_2} + ... + e^{-st} f(t)\Big|_{t_n}^{A} + s\left[\int_0^{t_1} e^{-st} f(t)dt + ... + \int_{t_n}^A e^{-st} f(t)dt\right]$$

since f is continuous, $\int_0^A e^{-st} f(t)\,dt = e^{-sA} f(A) - f(0) + s\int_0^A e^{-st} f(t)dt$

As $A \to \infty, e^{-sA} f(A) \to 0$

$\Rightarrow For\ s > a, L\{f'(t)\} = sL\{f(t)\} - f(0)$. Inductively if $f, f', ..., f^{(n-1)}$ are continuous and $f^{(n)}$ is piece wise continuous, then

$$L\{f^{(n)}(t)\} = s^n L\{f(t)\} - s^{n-1} f(0) - ... - sf^{(n-2)}(0) - f^{(n-1)}(0)$$

Now let us see how the Laplace transform can be used to solve initial value problems.

Examples:-1) Consider the differential equation $y'' - y' - 2y = 0, y(0) = 1, y'(0) = 0$

Solution: - This problem can be solved simply by the methods we are familiar and the solution is $y(t) = c_1e^{-t} + c_2 e^{2t}$.

Applying the initial conditions $y = \varphi(t) = \frac{2}{3}e^{-t} + \frac{1}{3}e^{2t}$ is the particular solution of the differential equation. Now let us use the Laplace transform method and find the solution.

Let $y = \varphi(t)$ be the solution of the differential equation.

$\Rightarrow L\{y''\} - L\{y'\} - 2L\{y\} = 0$. (By using the linearity of the Laplace transform.)

Then by theorem (1.2), $s^2 L\{y\} - s\, y(0) - y'(0) - (sL\{y\} - y(0)) - 2L\{y\} = 0$

But $y(0) = 1$ *and* $y'(0) = 0$

$\Rightarrow (s^2 - s - 2)\, L\{y\} + (1 - s)\ y(0) - y'(0) = 0$

$\Rightarrow L\{y\} = \dfrac{s-1}{s^2 - s - 2} = \dfrac{s-1}{(s-2)(s+1)}$. This implies we have obtained an expression for the Laplace transform $L\{y\}$ of the solution $y = \varphi(t)$ of the given initial value problem.

Then, finding φ is finding the function whose Laplace transform is as obtained.

$$Y(s) = \frac{a}{s-2} + \frac{b}{s+1}$$

$$\Rightarrow \frac{a(s+1) + b(s-2)}{(s-2)\ (s+1)} = Y(s) = \frac{s-1}{s^2 - s - 2}$$

$\Rightarrow s - 1 = a(s+1) + b(s-2)$ is the equation that must hold for all s.

If we set s = 2, then $a = \frac{1}{3}$ and if we set s = -1, then, $b = \frac{2}{3}$

$$\Rightarrow Y(s) = \frac{\frac{1}{3}}{s-2} + \frac{\frac{2}{3}}{s+1}$$

$$\Rightarrow y = \varphi(t) = \frac{1}{3}e^{2t} + \frac{2}{3}e^{-t}$$

2) $y'' + y = Sin\ 2t$, $y(0) = 2$, $y'(0) = 1$

Solution:-Assume that the differential equation has a solution $y = \varphi(t)$. By theorem (1.2) we have $s^2 y(s) - sy(0) - y'(0) = \frac{2}{s^2+4}$, (Since the transform of Sin2t is $\frac{2}{s^2+4}$.)

$$\Rightarrow Y(s) = \frac{2s^3 + s^2 + 8s + 6}{(s^2+1)(s^2+4)}$$

$$\Rightarrow Y(s) = \frac{as+b}{s^2+1} + \frac{cs+d}{s^2+4} = \frac{(as+b)\ (s^2+4)\ +\ (cs+d)(s^2+1)}{(s^2+1)\ (s^2+4)}$$

$$\Rightarrow 2s^3 + s^2 + 8s + 6 = (a+c)\,s^3 + (b+d)\,s^2 + \ (4a+c)\,S + (4b+d)$$

$$\Rightarrow \begin{cases} a+c=2 \\ 4a+c=8 \end{cases} \quad \text{And} \quad \begin{cases} b+d=1 \\ 4a+d=6 \end{cases}$$

$$\Rightarrow a=2, c=0, b=\frac{5}{3}, d=\frac{-2}{3}$$

$$\Rightarrow Y(s) = \frac{2s}{s^2+1} + \frac{\frac{5}{3}}{s^2+1} - \frac{\frac{2}{3}}{s^2+4}$$

$\Rightarrow$ The solution is $y = \varphi(t) = L^{-1}\{Y(s)\} \Rightarrow \varphi(t) = 2Cos\,t + \frac{5}{3}S\,\text{int} - \frac{1}{3}\sin 2t.$

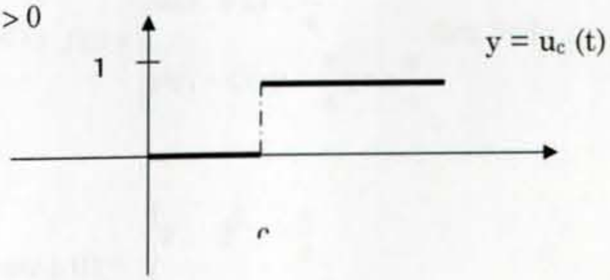
1.5 Step Functions

In this section we will see some of the most interesting applications of the transform methods occur in the solution of linear differential equation with discontinuous or impulsive forcing functions. Equations of this type frequently arise in the analysis of the flow of current in the electric circuits or in the vibration of mechanical systems.

To deal with functions having jump discontinuities, we introduce a unit step function denoted by $u_c(t)$ and defined as:

$$u_c(t) = \begin{cases} 0 & t < c \\ 1 & t \geq c \end{cases} \qquad c > 0$$

Graphically,



For a given function f, it is often necessary to consider the related function g defined by $y = g(t) = \begin{cases} 0, t < c \\ f(t-c), t \geq c \end{cases}$ which represents transformation of f a distance c in the positive direction. In terms of the unit -step function, we can write g (t) in the convenient form $g(t) = u_c(t)\, f(t-c)$

.

The Laplace transform of u_c is determined as follows.

$L\{u_c(t)\} = \int_0^\infty e^{-st} u_c(t)dt = \dfrac{e^{-cs}}{s}, s > 0$

Theorem1.3

If $F(s) = L\{f(t)\}$ *exists for* $s \geq 0$ and if c is a positive constant, then

i) $L\{u_c(t) f(t-c)\} = e^{-cs}\ L\{f(t)\} = e^{-cs} F(s)$ *for* $s > a$

ii) Conversely if $f(t) = L^{-1}\{F(s)\}$, *then* $u_c(t)\ \ f(t-c) = L^{-1}\{e^{-cs} F(s)\}$

Proof:-To prove (i) it is sufficient to compute $L\{u_c(t) f(t-c\}$

$L\{u_c(t)\ f(t-c)\} = \int_0^\infty e^{-st} u_c(t)\ f(t-c)dt = \int_0^\infty e^{-st} f(t-c)\ dt$

Let $\xi = t - c$

$\Rightarrow L\{u_c(t)\ f(t-c)\} = \int_0^\infty e^{-(\xi+c)s} f(\xi)\, d\xi$

$$= e^{-cs} \int_0^\infty e^{-s\xi} f(\xi)\, d\xi = e^{-cs}\ F(s).$$

To prove (ii) take the inverse transform on both sides.

Examples:-

1) If the function f is defined by $f(t) = \begin{cases} \sin t,\ 0 \leq t < \dfrac{\pi}{4} \\ \sin t + Cos(t - \dfrac{\pi}{4}), t \geq \dfrac{\pi}{4} \end{cases}$ then find $L\{f(t)\}$.

Solution:-

$f(t) = Sin(t) + g(t)$, where g (t) = $\begin{cases} 0 \quad if\ \ t < \dfrac{\pi}{4} \\ Cos\ \ (t - \dfrac{\pi}{4}), t \geq \dfrac{\pi}{4} \end{cases}$

$\Rightarrow g(t) = u_{\frac{\pi}{4}}(t)\ \cos(t - \frac{\pi}{4})$ and $L\{f(t)\} = L\{\sin t\} + L\left\{u_{\frac{\pi}{4}}(t)\, Cos\,(t - \frac{\pi}{4}\right\}$

$$= \frac{1}{s^2+1} + e^{\frac{-\pi s}{4}} \frac{s}{s^2+1}$$

$$\Rightarrow L\{f(t)\} = \frac{1}{s^2+1} + e^{\frac{-\pi s}{4}} \frac{s}{s^2+1}$$

2) Find the inverse transform of $F(s) = \dfrac{1-e^{-2s}}{s^2}$

Solution: - From linearity of the Laplace transform, we have

$$f(t) = L^{-1}\{F(s)\} = L^{-1}\left\{\frac{1}{s^2}\right\} - L^{-1}\left\{\frac{e^{-2s}}{s^2}\right\} = t - u_2(t)(t-2)$$

$\Rightarrow$ f may be written as f(t) = $\begin{cases} t, & 0 \le t < 2 \\ 2, & t \ge 2 \end{cases}$.

Theorem 1.4:- If F(s) $= L\{f(t)\}$ *exists for* $s > a \ge 0$, and if c is a constant, then

$L\{e^{ct} f(t)\} = F(s-c),\ s > a+c$ Conversely, if

$f(t) = L^{-1}\{F(s)\}, then\ e^{ct} f(t) = L^{-1}\{F(s-c)\}$

Proof:-The proof needs only evaluating $L\{e^{ct} f(t)\} = \int_0^\infty e^{-st} e^{ct} f(t)dt$

$$= \int_0^\infty e^{-(s-c)t} f(t)dt = F(s-c)$$

s > a+ c comes from the observation of one of our theorem.

$|f(t)| \le ke^{at}; hence\ |e^{ct} f(t)| \le k.e^{(a+c)t}$

Example: - Find the inverse transform of $G(s) = \dfrac{1}{s^2 - 4s + 5}$

Solution;- $G(s) = \dfrac{1}{(s-2)^2+1} = F(s-2)\ where\ F(s) = (s^2+1)^{-1}$

Since $L^{-1}\{F(s)\} = \sin t$, then from the theorem $g(t) = L^{-1}\{G(s)\} = e^{2t}\sin t$.

1.6 Differential equations with discontinuous forcing functions

In this section we will see some examples in which the non homogenous term, or forcing function is discontinuous.

Example: - Find the solution of the differential equation $y''+y'+\frac{5}{4}y=g(t)$, where

$$g(t)=1-u_{\pi}(t)=\begin{cases}1, & 0 \le t < \pi \\ 0, & t \ge \pi\end{cases}$$

Assume that the initial conditions are $y(0)=0,\ y'(0)=0$.

Solution: - The Laplace transform of the differential equation is,

$$s^2\,y(s)-sy(0)-y'(0)+sy(s)-y(0)+\frac{5}{4}y(s)=L\{1\}-L\{u_{\pi}(t)\}=\frac{1-e^{\pi s}}{s}$$

Inserting the initial values and solving for y(s), we will get $y(s)=\dfrac{1-e^{-\pi s}}{s(s^2+s+\frac{5}{4})}$

To find $y=\varphi(t)$, let us write $y(s)=\ 1-e^{-\pi s}\ H(s)$ where $H(s)=\dfrac{1}{s(s^2+s+\frac{5}{4})}$

Then if $h(t)=L^{-1}\{H(s)\}$, we have $y=h(t)-u_{\pi}(t)\,h(t-\pi)$.

To find $h(t)$, $H(s)=\dfrac{a}{s}+\dfrac{bs+c}{s^2+s+\frac{5}{4}}$. Upon determining the coefficients, we get

$$a=\frac{4}{5},\ b=-\frac{4}{5}\ and\ c=-\frac{4}{5}.$$

Thus $H(s)=\dfrac{\frac{4}{5}}{s}-\dfrac{4}{5}\left(\dfrac{s+1}{s^2+s+\frac{5}{4}}\right)$

$$=\frac{\frac{4}{5}}{s}-\frac{4}{5}\left(\frac{s+\frac{1}{2}+\frac{1}{2}}{(s+\frac{1}{2})^2+1}\right)$$

$$\Rightarrow h(t)=\frac{4}{5}-\frac{4}{5}(e^{\frac{-t}{2}}Cost+e^{\frac{-t}{2}}Sin\,t)$$

$\Rightarrow$ The solution is $y = h(t) - \varphi_\pi(t)\, h(t-\pi)$

Or $$\varphi(t) = \begin{cases} -(1+e^{\frac{\pi}{2}})(\frac{4}{5}e^{\frac{-t}{2}}Cost + \frac{2}{5}e^{\frac{-t}{2}}S\text{int}, & t \geq \pi \\ \frac{4}{5} - (\frac{4}{5}e^{\frac{-t}{2}}Cost + \frac{2}{5}e^{\frac{-t}{2}}S\text{int}), & t < \pi \end{cases}$$

1.7 The Unit Impulse Function

The impulse function can be approximately described by $\delta(t) = \lim_{\varepsilon \to 0} \gamma_\varepsilon(t)$

where $\gamma_\varepsilon = \begin{cases} \frac{1}{\varepsilon} & for\ 0 \leq t \leq \varepsilon \\ 0 & otherwise. \end{cases}$ which is not a distribution as $\gamma_\varepsilon(t)\ for\ 0 \leq t \leq \infty$ is not arbitrary often differentiable.

$$L\{\delta(t)\} = \int_0^\infty \left[\lim_{\varepsilon \to 0} \gamma_\varepsilon(t)\right] e^{-st} dt$$

But $\gamma_\varepsilon(t)$ can be represented as $\gamma_\varepsilon(t) = \frac{1}{\varepsilon}[u(t) - u(t-\varepsilon)]$ where u is a unit step function.

$$L\{\delta(t)\} = \lim_{\varepsilon \to 0}\left[\frac{1}{\varepsilon}\int_0^\infty u(t) - u(t-\varepsilon)\right] e^{-st} dt$$

$$= \lim_{\varepsilon \to 0}\left[\frac{1}{\varepsilon}\frac{1}{s}(1 - e^{-\varepsilon s})\right] = 1 \text{ by using L'Hospital's rule.}$$

1.8 The Convolution integral

Sometimes it is possible to identify a Laplace transform H(s) as a product of other two transforms F(s) and G(s) where they are corresponding to the known functions f and g respectively. But this doesn't mean that H(s) is the transform of f.g.

Theorem 1.5:-If $F(s) = L\{f(t)\}\ and\ G(s) = L\{g(t)\}$ both exists for $s > a \geq 0$, then,

$H(s) = F(s).G(s) = L\{h(t)\}, \quad s > a$ where;

$h(t) = \int_0 f(t-\tau)\ g(\tau)\, d\tau = \int_0 f(\tau)\, g(t-\tau) d\tau.$

The function h is known as **the convolution of f and g** and the integral is convolution integral .Conventiotionally the convolution integral is emphasized as generalized product by witting $h(t) = (f * g)(t)$. In particular $(f * g)(t) = \int_0^t f(t-\tau)\, g(\tau)\, d\tau$

1.8.1 Properties of convolution integral

$i)\ f * g = g * f$

$ii)\ f * (g_1 + g_2) = f * g_1 + f * g_2$

$iii)\ (f * g) * h = f * (g * h)$

$iv)\ f * 0 = 0 * f = 0$

$v)\ f * 1 = f$ may not be true.

Example:-If $f(t) = Cost$ then, $(f * 1)(t) = \int_0^t Cos(t-\tau)\, d\tau = -Sin(t-\tau)\Big|_0^t$

$$= -Sin(0) + S\text{int} = S\text{int}$$

$Vi)\ f * f$ can be negative.

Example:- $f(t) = \sin t$

Proof of the theorem:-Let $F(s) = \int_0^\infty e^{-s\xi} f(\xi) d\xi$ and $G(s) = \int_0^\infty e^{-s\eta} f(\eta) d\eta$

$$\Rightarrow F(s).G(s) = \int_0^\infty e^{-s\xi} f(\xi) d\xi . \int_0^\infty e^{-s\eta} f(\eta) d\eta = \int_0^\infty g(\eta)\, d\eta \int_0^\infty e^{-(\xi+\eta)} f(\xi) d\xi,$$

Since the integrand of the first integral doesn't depend on the integration variable of the second. Let $\xi = t - \eta$ for a fixed η. $\Rightarrow F(s).\ G(s) = \int_0^\infty g(\eta)\, d\eta \int_0^\infty e^{-st} f(t-\eta) dt$

Let $\eta = \tau \Rightarrow F(s).G(s) = \int_0^\infty g(\tau)\, d\tau \int_0^\infty e^{-st} f(t-\tau)\, dt$

Assuming that the order of integration can be reversed we obtain;

$$F(s)\ G(s) = \int_0^\infty e^{-st} dt \quad \int_0^t f(t-\tau)\, g(\tau)\, d\tau \text{ or}$$

$$F(s)\, G(s) = \int_0^\infty e^{-st} h(t)\, dt = L\{h(t)\}$$

Example: - Find the inverse transform of $H(s)=\dfrac{a}{s^2(s^2+a^2)}$

Solution: - Take $H(s)=\dfrac{1}{s^2}\,\dfrac{a}{s^2+a^2}$

$\dfrac{1}{s^2}$ is the transform of t, $\dfrac{a}{s^2+a^2}$ is the transform of $Sin at$.

$$\Rightarrow h(t)=\int_0^t (t-\tau)\ Sin\, a\tau\, d\tau \ =\frac{at-\sin at}{a^2}$$

CHAPTER TWO

THE FOURIER TRANSFORM

Suppose that $f:\mathbb{R}\to\mathbb{R}$ is integrable over the whole real line $\mathbb{R}$. Then the Fourier transform $\hat{f}(s)=\int_{-\infty}^{\infty} e^{-isx} f(x)dx \quad (s\in\mathbb{R})$ ....(1)

We can also denote the Fourier transform of f as $\hat{f}=F(f)$ or $\hat{f}(s)=F(f(x)(s)$ or even in the case of a function u = u (x, y) of two variables, $\hat{u}(s,y)=F_{x\to s}(u(x,y)\,(s,y))$ to indicate that it is the independent variable x , not the independent variable y, that is being transformed. Some authors define the Fourier transform as:

$$\hat{f}(s)=\int_{-\infty}^{\infty} e^{-2\pi ix\xi} f(x)\,dx \quad \text{.. (2)}$$

$$F(\lambda)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{i\lambda x} f(x)\,dx \quad \text{.. (3)}$$

From these we conclude that there are different formulas for Fourier transforms and hence their respective formulas for their inverses also vary.

2.1 Examples of Fourier transform

1) Find the Fourier transform of $f(x)=e^{-kx}\chi_{(0,\infty)}$ where

$$\chi_A(x)=\begin{cases}1, & x\in A\\ 0, & x\notin A\end{cases}$$

Solution:- $\hat{f}(s)=\int_{-\infty}^{\infty} e^{-isx} f(x)\,dx$

$$=\lim_{A\to\infty}\int_0^A e^{-isx}\,e^{-kx}\,dx$$

$$=\lim_{A\to\infty}\int_0^A e^{-(-is-k)x}dx$$

$$=\lim_{A\to\infty}\left(\frac{-e^{(is-k)x}}{is+k}\Bigg|_0^A\right)$$

$$= \lim_{A\to\infty}\left(-\frac{e^{(-is-k)A}}{is+k} + \frac{1}{is+k} \right)$$

$$= 0 + \frac{1}{is+k} = \underline{\underline{\frac{1}{k+is}}}$$

2) Find the Fourier transform of $f(x) = e^{kx}\chi_{(0,\infty)}$

Solution

$$\hat{f}(s) = \int_{-\infty}^{\infty} e^{-isx} \ f(x)dx$$

$$= \lim_{A\to\infty} \int_0^A e^{-isx}\, e^{kx}\, dx$$

$$= \lim_{A\to\infty}\left(e^{(-is+k)x}\Big|_0^A \right)$$

$$= \underline{\underline{\frac{1}{k-is}}}$$

3) Find the Fourier transform of $f(x) = e^{-k|x|}$

$$\Rightarrow f(x) = \begin{cases} e^{-kx} & if \quad x \geq 0 \\ e^{kx} & if \quad x < 0 \end{cases}$$

$$\Rightarrow \hat{f}(s) = \int_0^{\infty} e^{-isx}e^{-kx}\, dx + \int_{-\infty}^{0} e^{-isx}e^{kx}dx$$

$$= \lim_{A\to\infty}\int_0^A e^{-(-is-k)x}dx + \lim_{A\to-\infty}\int_{-\infty}^{0} e^{(-is+k)x}dx$$

$$= \frac{1}{k+is} + \frac{1}{k-is}$$

$$= \underline{\underline{\frac{k}{k^2+s^2}}}$$

4. Find the Fourier transform of $f(x) = \chi_{(-k,k)}(x)$

Solution

$$\hat{f}(s) = \int_0^k e^{-isx}dx + \int_{-k}^0 e^{-isx}dx$$

$$= -\frac{e^{-isx}}{is}\bigg|_0^k + -\frac{e^{isx}}{is}\bigg|_{-k}^0$$

$$= -\frac{e^{-iks}}{is} + \frac{1}{is} - \frac{1}{is} + \frac{e^{iks}}{is}$$

$$= \frac{1}{s}\left(\frac{e^{iks} - e^{-iks}}{i}\right)$$

$$= \frac{2Sin\, ks}{s}$$

2.2 Properties of Fourier transform

Once the Fourier integral formula has been obtained, sufficient condition for its validity can be verified directly.

i) If f(x) is piecewise continuously differentiable on each finite interval

and $\int_{\infty}^{\infty} |f(x)|^2 dx < \infty$, then $\frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-i\lambda(x-t)} f(t)\, dt\;\; d\lambda \to f(x)$ point wise at points

of continuity of f(x). At x_0 where f(x) is with jump discontinuity, the integral converges

to $\frac{1}{2}[f(x_0+) \quad + \quad f(x_0-)]$ *the average of the limit values of* $f(x)$.

ii) **Theorem 2.1:**-Suppose that $f:\mathbb{R}\to\mathbb{R}$ *and* $g:\mathbb{R}\to\mathbb{R}$ are integrable over $\mathbb{R}$ and a, b, k, are constants $\forall s \in R$. then,

a) $F(af + bg) = a\hat{f} + b\hat{g}$ **(Linearity)**

Proof:- $F(af+bg)(s) = \int_{-\infty}^{\infty} e^{-isx}(af+bg))(x)dx$

$$= \int_{-\infty}^{\infty} e^{-isx}af(x)dx + \int_{-\infty}^{\infty} e^{-isx}bg(x)dx$$

$$= a\int_{-\infty}^{\infty} e^{-isx} f(x)dx + b\int_{-\infty}^{\infty} e^{-isx} g(x)dx$$

$$= a\hat{f}(s) + b\hat{g}(s)$$

$$= F(af + bg) = a\hat{f} + b\hat{g}$$

b) $F(f(x+k))(s) = e^{iks}\hat{f}(s)$ **(Translation)**

Proof:- $F(f(x+k))(s) = \int_{-\infty}^{\infty} e^{-isx} f(x+k)dx$

$$= \int_{-\infty}^{\infty} e^{-is(x-k)} f(x)dx$$

$$= e^{isk}.\int_{-\infty}^{\infty} e^{-isx} f(x)dx$$

$$= e^{isk}\hat{f}(s)$$

If we use an other Fourier transform method, $F[f(x-h)] = e^{-2\pi ih\xi}\hat{f}(\xi)$

c) $F(f(xk))(s) = \frac{1}{|k|} f(\frac{s}{k}), k \neq 0$ **(Scaling)**

Proof:- $F(f(kx))(s) = \int_{-\infty}^{\infty} e^{-isx} f(kx)dx$

$$= \int_{-\infty}^{\infty} e^{-isx\frac{x}{k}} f\left(\frac{x}{k}\right)dx$$

$$= \int_{-\infty}^{\infty} \frac{1}{|k|} e^{--isx} f\left(\frac{x}{k}\right)dx$$

$$= \frac{1}{|k|}\int_{-\infty}^{\infty} e^{-isx} f\left(\frac{x}{k}\right)dx$$

$$= \frac{1}{|k|}.\hat{f}(\frac{s}{k})$$

d) If f' is integrable then, $F(f'(x))(s) = is\,\hat{f}(s)$

Proof:- $\int_{-\infty}^{\infty} e^{-isx} f'(x)dx = is\int_{-\infty}^{\infty} e^{-isx} f(x)dx$

$$\Rightarrow F(f'(x)(s) = is.\hat{f}(s)$$

e) $\hat{f}'(s) = \int_{-\infty}^{\infty} -ixf(x)e^{-isx}dx.$

f) **Inverse theorem**

If f is continuous and piece wise smooth, then $f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{isx}\hat{f}(s)ds$

Also if $F(\lambda) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-i\lambda x} f(x)dx$,then $f(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{i\lambda x}\ F(\lambda)d\lambda$

g) **Convolution theorem**

Let $H(\lambda) = F(\lambda).G(\lambda)$ where $F(\lambda)$ and $G(\lambda)$ are the transforms of the known functions f(x) and g(x).Then the inverse transform of $H(\lambda)$ is given by

$$H^{-1}(\lambda) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(t).g(x-t)dt$$

$\int_{-\infty}^{\infty} f(t).g(x-t)dt = f*g$ is called the **Fourier convolution integral.**

$$\Rightarrow F(f*g)(x))\ (s) = \hat{f}(s).\hat{g}(s)$$

Proof: - Let us show that $H^{-1}(\lambda) = (f*g)\ (x)$

$$H^{-1}(\lambda) = \frac{1}{\sqrt{2\pi}}\ \int_{-\infty}^{\infty} e^{-i\lambda x} F(\lambda)\ \ G(\lambda)dx$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty}\ \ \int_{-\infty}^{\infty} e^{-i\lambda(x-t)}\ f(t)G(\lambda)\ dtd\lambda$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} f(t).\ \ \int_{-\infty}^{\infty} e^{-i\lambda(x-t)}\ G\ (\lambda)\ d\lambda dt$$

$$= \frac{1}{\sqrt{2\pi}}\int_{\infty}^{\infty} f(t)g(x-t)dt$$

where we have assumed that the interchange of orders of integration is valid. Then we have shown that the Fourier transform of (f * g) (x) is equal to the product of the Fourier transform of f and the Fourier transform of g.

h) **Corollary:** - i) If $f:\mathbb{R}\to\mathbb{R}$ is integrable then $\forall s \in R, F(\hat{f}(x))(s) = 2\pi f(-s)$.

ii) If the r[th] derivative function $f^{(r)}: R \to R$ is integrable for r = 0, 1..., n, then $F(f^{(n)}(x))(s) = (is)^n \hat{f}(s)$

Notes :- a) some authors define the Fourier transform F_D of integrable f

as $\hat{f}_D(s) = k \int_{-\infty}^{\infty} e^{isx} \quad f(x)\,dx \quad (k = constant \text{ and } k \neq 0)$.

This makes essentially no change to the theory, but the inverse becomes

$$f(x) = \frac{1}{2\pi k}\int_{-\infty}^{\infty} e^{-isx}\hat{f}_D(s)ds \text{ and } F_D(f'(x))\ (s) = (-is)\hat{f}_D(s).$$

b) For integrable f, g, $\int_{-\infty}^{\infty} f(u)\, g(x-u)\, du = \int_{-\infty}^{\infty} f(x-u)\ g(u)\, du$

c) If f is not Continuous at x in the inverse theorem, then f(x) in the left is replaced by $\frac{1}{2}[f(x+) + f(x-)]$.

d) For continuous ,piece wise smooth ,integrable functions ,there can be only one f = f(x) corresponding to $\hat{f} = \hat{f}(s)$; this tells us that in these circumstances , if one can find a function f = f(x) ,the transform of which is $\hat{f} = \hat{f}(s)$, then one need look no further.

Example: - For a constant k, find $F\left(\frac{2k}{k^2+x^2}\right)$

Solution:- 1) $\hat{f}(e^{-k|x|})(s) = \frac{2k}{s^2+k^2}$

2) $F(\hat{f}(x))\ (s) = 2\pi \quad f(-s)$

$$\Rightarrow F\left(\frac{2k}{k^2+s^2}\right) = \underline{\underline{2\pi . e^{-k|s|}}}$$

2.3 Applications of Fourier transforms to ODES and PDES

We now consider several problems for ODES and PDES whose solutions are found by using Fourier transforms. In each case one of the independent variables has the infinite interval $(-\infty,\infty)$ as it s domain of definition.

Examples

1) Use the Fourier transform to find the continuous integrable function y = y(x) which satisfies the differential equation $y'+2y=e^{-|x|}$ at every non zero x in R.

Solution:- $y'+2y=e^{-|x|}$

$$\Rightarrow F(y')(s)=is\,\hat{y}\,(s)$$

$$\Rightarrow\ (is+2)\,\hat{y}\,(s)=\frac{2}{1+s^2}$$ Because k = 1.

Using partial fractions

$$\Rightarrow y(x)=e^{-x}\chi_{(0,\infty)}(x)+\frac{1}{3}e^{x}\chi_{(-\infty,0)}(x)+\frac{-2}{3}e^{-2x}\chi_{(0,\infty)}(x)$$

$$=\begin{cases} e^{-x}-\frac{2}{3}e^{-2x} & ,x>0 \\ \frac{1}{3}e^{x}, & x\le 0. \end{cases}$$

2) Assuming that $f:R\to R$ is integrable over R, use the Fourier transform to show that a solution of the differential equation $y''-y=f(x)\ \ (x\in R)$ can be expressed in the form

$y(x)=\int_{-\infty}^{\infty}k(x-u)f(u)du$ where the function k should be determined.

Solution

Applying the Fourier transform to the differential equation,

$$((is)^2-1)\,\hat{y}(s)=\hat{f}(s)$$

$\Rightarrow \hat{y}\,(s)=-(\frac{1}{1+s^2})\,\hat{f}\,(s)$. By convolution theorem and the pervious result

$\Rightarrow y(x)=\int_{-\infty}^{\infty}k(x-u)\,f(u)\,du\ (x\in R)$ is a solution where $k(x)=-e^{-|x|}\ \ (x\in R)$

3) $y''(x)-k^2\, y(x)=-f(x)\ ,\ -\infty<x<\infty$ Where k is a constant and f(x) is prescribed. We require $y(x)\,,y'(x)\to 0\ as\ |x|\to\infty$ and that f(x) has the Fourier transform $F(\lambda)$.

Solution

Let $Y(\lambda)=\ \dfrac{1}{\sqrt{2\pi}}\ \int_{-\infty}^{\infty} e^{i\lambda x}k^2 y(x)dx$

$\Rightarrow Y(\lambda)$ is the Fourier transform of y(x) where y(x) is the solution of the given differential equation. Now transforming the differential equation we will get

$$\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{i\lambda x}y'(x)dx-\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{i\lambda x}k^2 y(x)\,dx\ =-\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{i\lambda x}f(x)dx$$

$\Rightarrow(-i\lambda)^2 Y(\lambda)-k^2Y(\lambda)=-F(\lambda)$

$\Rightarrow -\lambda^2Y(\lambda)-k^2Y(\lambda)=-F(\lambda)$

$\Rightarrow Y(\lambda)=\dfrac{F(\lambda)}{\lambda^2+k^2}=F(\lambda)\ G(\lambda)$

$\Rightarrow$ The inverse transform of $\dfrac{1}{\lambda^2+k^2}\ \ is\ \ \dfrac{\sqrt{2\pi}}{k}e^{-kx}$ because,

$\dfrac{1}{\sqrt{2\pi}}\int e^{-i\lambda x}\dfrac{1}{\lambda^2+k^2}dx=\dfrac{\sqrt{2\pi}}{2k}e^{-k|x|}$ with $k>0$ using complex integration theory.

Now using the convolution $Y\,(\lambda)=F(\lambda)\,.G(\lambda)$,we obtain the solution in the form

$$y(x)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-i\lambda x}F(\lambda)\ G(\lambda)\,d\lambda=\frac{1}{2k}\ \int_{-\infty}^{\infty} e^{-k|x-t|}f(t)dt$$

4) Use the Fourier transform to find a solution u = u(x, t) of the partial differential equation $u_t=ku_{xx}\ \ (x\in R),t>0$ when subject to the initial condition

$u(x{,}0)\ =e^{x^2}\ \ (x\in R)$ and where k is a constant.

Solution

Assume that for each t the function u satisfies ,as a function of x ,the conditions for the application of the Fourier transform and that $F_{x\to s}(u_t(x,t))(s,t)=\frac{\partial}{\partial t}\hat{u}(s,t)$; that we can interchange the operation of applying the Fourier transform and partial differential equation with respect to t . Then applying the transform to the equation it gives

$$\frac{\partial}{\partial t}\hat{u}(s,t)=k(is)^2\hat{u}(s,t)$$

and hence $\hat{u}=\hat{u}(s,t)$ satisfies the first order partial differential equation $\frac{\partial \hat{u}}{\partial t}+ks^2\hat{u}=0$

$$\Rightarrow \frac{\partial}{\partial t}\left(e^{kts^2}\hat{u}\right)=0$$

$\Rightarrow \hat{u}(s,t)=e^{-kts2}g(s)$ Where g is an arbitrary function of s .However we may also apply the transform to the initial condition to get $\hat{u}(s,0)=\sqrt{\pi}e^{\frac{-s^2}{4}}$.

$$\Rightarrow \hat{u}(s,t)=\sqrt{\pi}e^{-(1+4kt)\frac{s^2}{4}}$$

$\Rightarrow$ From the table of Fourier transform $u(x,t)=\frac{1}{\sqrt{1+4kt}}e^{\frac{-x^2}{(1+4kt)}}$ because if

$f(x)=e^{-kx^2}$, then $\hat{f}(s)=\sqrt{\frac{\pi}{k}}e^{\frac{-s^2}{4}}$.

CHAPTER THREE

FOURIER TRANSFORM OF DISTRIBUTIONS

3.1 Definition and some examples of distributions

To discuss about Fourier transform of distributions, let us define what distribution is at the first place.

Definition: - For all $\varphi \in D$ (infinitely differentiable function with compact support), consider a mapping T which gives a uniquely determined finite complex value to φ where T is defined as follows.

1) T is linear functional on D. That is, for $\lambda \in \mathbb{C}$

$a)\ T(\lambda\varphi) = \lambda\, T(\varphi)$

$b)\ For\ \varphi_1, \varphi_2 \in D,\ T(\varphi_1 + \varphi_2) = T(\varphi_1) + T(\varphi_2)$

2) By a given sequence of functions $\{\varphi_j\}, \varphi_j \to 0$ where $\varphi_j \in D$,we mean that the support of each φ_j is contained in a certain compact set , and the derivatives of each degree of each $\varphi_j(x)$ tends to zero uniformly. Using our agreed notations, $\varphi_j \to 0$ means $\sup p\left[\varphi_j\right] \subset k$ (k is suitably chosen compact set), and for an arbitrary $\alpha, D^\alpha \varphi_j(x) \to 0$ uniformly. We therefore require that $\varphi_j \to 0\ implies\ T(\varphi_j) \to 0$ .Schwartz called T satisfying condition (1) and (2) "distribution". Form now on wards we say linear form instead of linear functional and write $\langle T, \varphi\rangle\ instead\ of\ T(\varphi)$ where

$\langle T, \varphi\rangle = \int T(x)\,\varphi(x)\,dx\ and\ \varphi\ is$ a test function $(\varphi \in D)$.

We also write D' for the space of all distributions.

Examples:-

1) $f \in L^1_{loc}$; $\langle f(x), \varphi(x) \rangle = \int f(x)\varphi(x)dx$

2) $\delta(Dirac's\ \delta\ measure)$; $\langle \delta, \varphi(x) \rangle = \varphi(0)$

3) $Y(x)$ ($Heavisides\ function$; $\langle Y(x), \varphi(x) \rangle = \int_0^\infty \varphi(x)\, dx \ \ (n=1)$

Definition: - (Derivatives of distributions)

For $T \in D'$, $\frac{\partial T}{\partial x_i}$ is defined as $\left\langle \frac{\partial T}{\partial x_i}, \varphi \right\rangle = -\left\langle T, \frac{\partial \varphi}{x_i} \right\rangle$.

Another way of stating the definition is to define

$$D^\alpha T \ \ for \ \ T \in D' \ as \left\langle D^\alpha T,\ \varphi \right\rangle = (-1)^{|\alpha|} \left\langle T, D^\alpha \varphi \right\rangle \quad \text{where}$$

$|\alpha| = \alpha_1 + \alpha_2 + \ldots + \alpha_n$

Example 1:- Show that $Y'(x) = \delta$

Proof:- $\left\langle Y'(x), \varphi(x) \right\rangle = -\left\langle Y(x), \varphi'(x) \right\rangle$

$$= -\int_0^\infty \varphi'(x)\, dx = \varphi(0) - \varphi(\infty)$$

$$= \varphi(0) = \langle \delta, \varphi(x) \rangle$$

Example 2:- Show that $(\log|x|)' = v.p\frac{1}{x} (n=1)$

Solution: - Here, $\log|.| \in L^1_{loc}$.

Formally,

$$\left\langle v.p\frac{1}{x}, \varphi(x) \right\rangle = \lim_{\varepsilon \to 0} \int_{|x| \ge \varepsilon} \frac{\varphi(x)}{x} dx$$

We write,

$$\langle\ (\log|x|)', \varphi(x)\ \rangle = -\langle\ (\log|x|, \varphi'(x)\ \rangle$$

$$= -\int_{-\infty}^{\infty} \log|x| \varphi'(x) dx$$

$$= -\lim_{\varepsilon \to 0}\left[\int_{-\infty}^{-\varepsilon} \log|x|\varphi'(x)dx + \int_{\varepsilon}^{\infty} \log|x|\varphi'(x)dx\right]$$

$$= \lim_{\varepsilon \to 0}\left\{\log\varepsilon\left[\varphi(\varepsilon) - \varphi(-\varepsilon)\right] + \int_{\varepsilon}^{\infty}\frac{\varphi(x)}{x}dx + \int_{-\infty}^{-\varepsilon}\frac{\varphi(x)}{x}dx\right\}$$

by integrating by parts. The first term tends to zero as $\varepsilon \to 0$,

the final expression is $v.p.\int \frac{\varphi(x)}{x}dx$.

Definition: - We say that the sequence of distributions $\{T_j\}$ tends to a distribution T when for an arbitrary $\varphi \in D$, we have $\langle T_j,\varphi\rangle \to \langle T,\varphi\rangle \quad (j \to \infty)$

To generalize the notions of Fourier's inverse formula, beyond L^2 Space we need the idea of distributions. In this section we shall show that the required properties of Fourier transform can be established in the function space S' which was introduced by schwartz.

3.2 Fourier transform in s- space

It is the whole space consisting of $\varphi \in C^{\infty}$ such that,

$(1+|x|^2)^k \; D^{\alpha}\varphi(x)$ *is bounded in* R^n).

1) The space S consists of smooth functions which together with all their derivatives decay rapidly to zero as x approaches to infinity. That is, $\varphi \in S \Leftrightarrow \varphi \in C^{\infty}$ and its all derivatives which we are allowed to multiply by any polynomial of $|x|$ tend to 0 uniformly as $|x| \to +\infty$.

2) $\varphi_j \in S \to 0 \Leftrightarrow (1+|x|^2)^k \; D^{\alpha}\varphi_j(x) \to 0$ *uniformly in* R^n.

It is very useful in Fourier analysis as it forms an easily manipulated family of functions which is mapped isomorphically on to itself by Fourier transform.

$S(R^n) = \{f \in C^{\infty}(R^n) : \forall \alpha, \beta \in N^n, Sup|x^{\alpha}\partial^{\beta} f(x)|, x \in R^n < \infty\}$

Examples:-

1) $C_0^{\infty}(R^n) \subset S(R^n)$

2) $e^{-|x|^2} \in S(R^n)$

3) For $\varepsilon > 0, e^{-(1+|x|^2 \varepsilon}$ belongs to $S(R^n)$

4) No matter how large N is $, e^{-(1+|x|^2 N}$ is not in $S(R^n)$.

Properties of S- space

- It is complete
- Closed and bounded set of S is compact
- D is dense in S.

The Fourier transform and its inverse serve to express a function f as superposition of oscillatory exponential function $e^{-i(x,\xi)}, \xi \in R^n$ according to the formula $f(x) = (2\pi)^{\frac{-n}{2}} \int e^{i(x,\xi)} \hat{f}(\xi) d\xi$.

Definition: - For $f \in S(R^n)$, the Fourier transform of f is defined by

$$\hat{f}(\xi) = (2\pi)^{\frac{-n}{2}} \int e^{-i(x,\xi)} f(x) dx$$

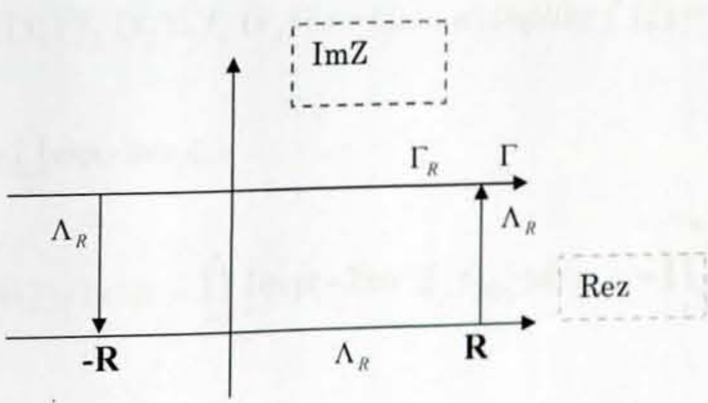
Example: - Let f(x) = $e^{-\frac{x^2}{2}} \in S(R)$

$$\hat{f}(\xi) = (2\pi)^{\frac{-1}{2}} \int e^{-ix\xi} e^{-\frac{x^2}{2}} dx$$

Complete the square in exponent to get $\frac{x^2}{2} + ix\xi = \frac{1}{2}(x^2 + 2ix\xi) = \frac{1}{2}((x+i\xi)^2 + \xi^2)$

Thus $\sqrt{2\pi}\, \hat{f} = e^{-\frac{\xi^2}{2}} \int_{-\infty}^{\infty} e^{-\frac{(x+i\xi)^2}{2}} dx$

$$= e^{-\frac{\xi^2}{2}} \int_{\Gamma} e^{-\frac{(z)^2}{2}} dz$$



Where Γ is the contour Im$Z = \xi$ traversed from left to right. Let Γ_R be the segment on Γ running from $-R + i\xi$ to $R + i\xi$. Since the integrand decays exponentially fast as $|z| \to \infty$ along Γ, the integral is equal to the limit $\lim_{R\to\infty} \int_{\Gamma_R} e^{-z^2} dz$.

By Cauchy's theorem, the integral over Γ_R is equal to the integral over Λ_R, where $\Gamma_R - \Lambda_R$ is equal to the boundary of the rectangle whose side opposite Γ_R is the interval [-R, R] on the X-axis. Thus the Fourier transform is $\lim_{R\to\infty} \int_{\Lambda_R} e^{-z^2} dz$

As $R \to \infty$, the integral over the vertical side of the rectangle $\to 0$ exponentially rapidly and we find $\hat{f}(\xi) = e^{-\frac{\xi^2}{2}} (2\pi)^{\frac{-1}{2}} \int e^{-\frac{x^2}{2}} dx = e^{-\frac{\xi^2}{2}}$.

The Fourier inverse formula in this case is $f(x) = (2\pi)^{\frac{-n}{2}} \int e^{i(x,\xi)} \hat{f}(\xi) d\xi$.

3.3 Fourier transform of several variables

Let $f \in L^1$ We set,

$\hat{f}(\xi_1, \xi_2, \ldots, \xi_n) = \int \ldots \int \exp[-2\pi i(x_1\xi_1 + \ldots + x_n\xi_n)] f(x_1, \ldots, x_n)\, dx_1, \ldots, dx_n$ or simply $\hat{f}(\xi) = \int e^{2\pi i \xi} f(x) dx$ as the Fourier transform of f(x).

In this case we write $\hat{f}(\xi) = F[f]$.

If $g \in L^1$ then, $(F^{-1}g)(x) = \int e^{2\pi x\xi} g(\xi) d\xi$ which is the inverse transform of $g(\xi)$.

3.3.1 Fundamental properties

a) $f(x) = f_1(x_1)\, f_2(x_2) \ldots f_n(x_n)$ $(i = 1, 2, \ldots, n)$ *implies* $\hat{f}(\xi) = \hat{f}_1(\xi_1) \ldots\ldots\ldots\ldots \hat{f}_n(\xi_n)$

Proof

$$\exp(-2\pi i x\xi) = \prod_{j=1}^{n} \exp(-2\pi i x_j \xi_j)$$

$$\Rightarrow \int \exp(-2\pi i x\xi) f(x) dx = \prod_{j=1}^{n} \int \exp(-2\pi i x_j \xi_j f_{j(x_j)}) dx_j \quad = \prod_{j=1}^{n} \hat{f}(\xi_j)$$

b) Let $f \in D^m$,that is f has a compact support and is m times continuously differentiable or more generally, when $|x|^{n-1}|D^\alpha f(x)| \to 0$ uniformly convergent,

($|\alpha| \le m-1$ as $x \to \infty$ and $D^\alpha f(x) \in L^1, |\alpha| \le m$),we have

$(2\pi i\xi)^\alpha \hat{f}(\xi) = \int e^{-2\pi x\xi} D^\alpha f(x)\,dx$ Where $(2\pi i\xi)^\alpha = (2\pi i\xi_1)^{\alpha_1}...(2\pi i\xi_n)^{\alpha_n}$

This can be proved by integrating (a) by parts repeatedly.

$\Rightarrow F\left[D^\alpha\ f(x)\right] = (2\pi i\xi)^\alpha\ F[f(x)],\ (|\alpha| \le m)$

c) Let $(1+|x|)^m f \in L^1$ $(m \ge 0)$ we differentiate (a) under integration sign

$D^\alpha\ \hat{f}(\xi) = F[-2\pi ix]^\alpha\ f(x) \quad (|\alpha| \le m).$

d) $F[f(x-h)] = \ e^{-2\pi ihx\xi}\ \ \hat{f}(\xi)$

Theorem 3.1:- For $f \in D^{2n}$,the following Fourier inversion formula can be established.

$F^{-1}F\ f = f \quad and \qquad FF^{-1}f = f$

3.3.2 Plancherel's theorem

Up to this point Fourier transforms have been defined only for functions belonging to L^1 and the inversion formula has been established for the functions which belongs to D^{2n}. We will generalize to the entire space of functions L^2.

More precisely, if $f \in L^2 \cap L^1$, that is f is summable and square integrable, the new generalized Fourier transform coincides with the old one, and for an arbitrary $f \in L^2$, F f $\in L^2$ and an inversion formula can also be established.

Since D^{2n} and D are dense in L^2 we can have the following.

For $f \in L^2$, we can choose $\{f_j\}$ $where$ $f_j \in D^{2n}$ and $f_j(x) \to f(x)$ in L^2.

We observe that $\{f_j\}$ is a Cauchy sequence in L^2 .

This implies since $// Ff //_{L^2} = // F^{-1} f //_{L^2} = // f //_{L^2}$ (Parseval-Plancherel)

$\left\{\hat{f}_j(\xi)\right\}$ is a Cauchy sequence in $L^2(R_\xi^{\ n})$.

Hence, $\left\{\hat{f}_j(\xi)\right\}$ has a unique limit because L^2 space is complete. We denote the limit as $\hat{f}(\xi)$.

$\hat{f}_j(\xi) \to \hat{f}(\xi)$ *in* L^2. *and we define* $\hat{f}(\xi) = F[f(x)]$

This implies $\hat{f}(\xi)$ doesn't depend on the choice of $f_j(x)$.

$\Rightarrow$ Our Fourier transform is an isometric operator from $L^2(R_x{}^n)$ *in to* $L^2(R_\xi{}^n)$.

We can generalize the Fourier transform to L^2. We note that,

$$FF^{-1}f(x) = f(x) \qquad (f \in L^2)$$
$$F^{-1}F\, g(\xi) = g(\xi) \qquad (g \in L^2)$$

Theorem (Plancherel's theorem)

For $f \in L^2$, the Fourier transform F, the Fourier inverse transform F^{-1} can be defined.

F is an isometric operator from $L^2(R_x{}^n)$ *in to* $L^2(R_\xi{}^n)$. F^{-1} is the inverse operator of F.

$$F^{-1}F = I \qquad in\ L^2\ (R_x{}^n)$$
$$F\ F^{-1} = I \qquad in\ L^2\ (R_\xi{}^n)$$

Thus the newly defined F coincides with the previous defined F when $f \in L^2 \cap L^1$

where $\hat{f}\ (\xi)\ =\ \underset{A\to\infty}{l.i.m.} \int_{|x|\le A} e^{-2\pi i x\xi} f(x)dx$ is the Fourier transform of $f \in L^2$

(the newly defined F) where l.i.m. means 'limit in mean,' that is/ limit in L^2.

3.4 Fourier transform in S' - space

S'-space is the space of linear continuous functional defined on S. Or it is the whole space of tempered distributions where a tempered distribution is a continuous linear functional on $S(\mathbb{R}^n)$,that is a continuous linear map from $S(R^n)$ to $\mathbb{C}$.

Examples:-

1) $f \in L^p(\mathbb{R}^n), 1 \le p \le \infty$, then $f \in S'$

2) If μ is a Borel measure, such that for some M, $(1+|x|^2)^{-M}\mu$ is finite measure, then the distribution defined μ is tempered distribution.

In the last sections we defined the Fourier transform Ff for

$f \in L^1$ by $\hat{f}(\xi) = \int_{-\infty}^{\infty} e^{-2\pi i x\xi} f(x)\, dx$ and $f(x) = \int_{-\infty}^{\infty} e^{2\pi i x\xi}\, \hat{f}(\xi)\, d\xi$ where f(x) is a bounded variation in a bounded finite interval. We note that in this case for an arbitrary

$\varphi \in D$, $\langle Ff, \varphi(\xi)\rangle = \langle f(x), F\varphi\rangle$ and

$$\langle F^{-1}f, \varphi(\xi)\rangle = \langle f(x), F^{-1}\varphi\rangle$$

Proof:- $\langle Ff, \varphi(\xi)\rangle = \int \varphi(\xi)d\xi \int e^{-2\pi i x\xi} f(x)dx$

From Fubini's theorem this can be written as $\int f(x)dx \int e^{-2\pi i x\xi}\varphi(\xi)d\xi$

And this is equal to $\langle f(x), F\varphi\rangle$. We obtain similar results for F^{-1}. From this we may conceive an idea how to extend Fourier transforms to distributions replacing f by T and defining $F^{-1}T$ as $\langle F^{-1}T, \varphi(\xi)\rangle = \langle T, F^{-1}\varphi\rangle$ .... (*)

And

$$\langle FT, \varphi(\xi)\rangle = \langle T, F\varphi\rangle$$

Unfortunately in this case $F^{-1}\varphi \notin D$ and $Supp\left[F^{-1}\varphi\right]$ covers the whole space. From this we see that the right hand term of (*) is meaningless. Therefore we must give up defining the Fourier transform for all distributions. This implies we will define $F^{-1}u$ for $u \in S'$.

Lemma: - A Fourier transform F is bijective and bicontinuous linear mapping from s_x to s_ξ (topologically isomorphism between s_x and s_ξ).

Definition: - Let $u_x \in S'_x$.,we define $F^{-1}u_x$ and Fu_x elements of S'_ξ as;

$$\langle Fu_x, \varphi(\xi)\rangle = \langle u_x, (F\varphi)(x)\rangle \quad (\varphi \in S_\xi)$$

$$\langle F^{-1}u_x, \varphi(\xi)\rangle = \langle u_x, (F^{-1}\varphi)(x)\rangle \quad (\varphi \in S_\xi)$$

Theorem 1.3:- Let F be a bicontinuous bijection from S'_x *to* S'_ξ and F^{-1} is the inverse of F, then $F^{-1}\,Fu = u, FF^{-1}\,u = u \qquad (u \in S')$

Proof:-From $\langle Fu, \varphi(\xi)\rangle = \langle u, (F\varphi)\,(x)\rangle$ and $u_j \to 0$ *in* S', we have

$$\langle u_j, (F\varphi)\,(x)\rangle \to 0$$

$$\Rightarrow \langle Fu_j, \varphi(\xi)\rangle \to 0$$

The set of the Fourier image of bounded sets which belong to S is also a bounded set. Therefore, F is a continuous mapping from S'_x *to* S'_ξ. A similar argument holds for F^{-1}.

$$In\ S, F^{-1}F\varphi = FF^{-1}\varphi = \varphi.$$

$$\Rightarrow \langle F^{-1}Fu, \varphi\rangle = \langle Fu, F^{-1}\varphi\rangle = \langle u, \varphi\rangle$$

That is $F^{-1}\ F = I$ *and* $FF^{-1} = I$ *in* S'

3.4.1 Fundamental properties of the Fourier transform of distributions

Let $u \in S'$, then the following equalities hold.

$a)\ F\left[D^\alpha u\right] = (2\pi i\xi)^\alpha F[u]$

$b)\ F\left[(-2\pi ix)^\alpha\ u\right] = D_\xi^{\ \alpha} F\left[u\right]$

$C)\ F\left[T_h u\right] = e^{-2\pi ih\xi}\ F[u]$

$d) F\left[e^{2\pi ihx}u\right] = T_h F\left[u\right]$

Where T_h in (c) and (d) is a distribution which is obtained from the distribution u by a parallel translation h, satisfying $(T_h f)\,(x) = f(x-h)$.

e) If $u_x \in \varepsilon'$, then we can write $F\left[u_x\right] = \langle u_x, e^{-2\pi ix\xi}\rangle$

f) $F[\delta] = 1$, $\quad F[1] = \delta$
$F^{-1}[\delta] = 1,\quad F^{-1}[1] = \delta$

Proof of f: - Let f(x) = 1.

$$\Rightarrow \langle F[1], \varphi(\xi)\rangle = \langle 1, (F\varphi)\ (x)\rangle$$

$$= \int (F\varphi)(x)\, dx \ = \int \hat{\varphi}(x)\, dx$$

$$\Rightarrow \varphi(\xi) = \int e^{2\pi i x\xi}\, \hat{\varphi}(x) dx$$

$$\Rightarrow \varphi(0) = \int \hat{\varphi}(x)\, dx$$

$$\Rightarrow \langle F[1], \varphi(\xi)\rangle = \varphi(0)$$

However for $\varphi \in D_\xi$, the functional $\varphi(\xi) \to \varphi(0)$ is the Dirac δ function.

$$\Rightarrow F[1] = \delta$$

$\Rightarrow \langle \delta, \varphi(\xi)\rangle = \varphi(0)$. Or if we use an other formula,

$$\langle F[1], \varphi(\xi)\rangle = \langle 1, (F\varphi)\ (x)\rangle$$

$$= \int (F\varphi)(\xi)\, d\xi$$

$$= (2\pi)^{\frac{n}{2}} \varphi(0)$$

$$= \left\langle (2\pi)^{\frac{n}{2}} \delta, \varphi \right\rangle$$

$$\Rightarrow F[1] = (2\pi)^{\frac{n}{2}} \delta$$

Also

$$\langle Y'(x), \varphi(x)\rangle = -\langle Y(x), \varphi'(x)\rangle = -\int_0^\infty \varphi'\ (x)\, dx = \varphi(0).$$

$$\Rightarrow Y'(x) = \delta(x)$$

$$\Rightarrow F[\delta] = \left\langle Y'(x), e^{-2\pi i x\xi}\right\rangle = \left\langle Y'(x), e^{-2\pi i x\xi}\right\rangle_x$$

$$= \left\langle \delta, e^{-2\pi i x\xi}\right\rangle_x = \underline{\underline{1}}$$

Or if we use an other formula,

$$\langle F\delta,\varphi\rangle = \langle \delta, F\varphi\rangle$$

$$= F\varphi(0) = (2\pi)^{\frac{-n}{2}} \int \varphi(x)dx$$

$$= \left\langle 2\pi)^{\frac{-n}{2}}, \varphi \right\rangle$$

$$\Rightarrow F[\delta] = (2\pi)^{\frac{-n}{2}}$$

Similarly $F^{-1}[\delta] = 1 \; and \; F^{-1}[1] = \delta.$

g) When the dimension of the base space is 1, we see that $F\left[\exp(-\pi r^2)\right] = \exp(-\pi\xi^2)$.

From this and Fourier transform of several variables, in the case of n dimension we have $F\left[\exp(-\pi(x_1^2 + x_2^2 + ... + x_n^2))\right] = \exp(-\pi\,(\xi_1^2 + ... + \xi_n^2))$.

Let $f \in S'$,and let $f, Ff \in L^1_{loc}$,i.e. f and Ff are both functions. Then for a real number $\lambda \neq 0$ we have $F[f(\lambda x)] = \dfrac{1}{|\lambda|^n} \hat{f}\,(\dfrac{\xi}{\lambda})$ when n is the dimension of the base space.

3.4.2 Concrete examples of Fourier transform

1) $F\left[v.p\,\dfrac{1}{x}\right] = \begin{cases} -\pi i & (\xi > 0) \\ \pi i & (\xi < 0) \end{cases}$ Where $v.p\,\dfrac{1}{x} = (\log|x|)'$ (Cauchy principal value) and $f(x) \in D'_{L^2}$ where $f(x) = v.p\left(\dfrac{1}{x}\right)$.

In fact $v.p\,\dfrac{1}{x} = \alpha(x)v.p\dfrac{1}{x} + (1-\alpha(x))\dfrac{1}{x}$ where α is a function taking the value 1 in the nbhd of the origin and $\alpha \in D$. $\alpha(x)v.p\,\dfrac{1}{x} \in \varepsilon' \; and \; (1-\alpha)(x)\,\dfrac{1}{x} \in L^2$

$$\Rightarrow The \;\; sum \;\; v.p\,\frac{1}{x} \in D'_{L^2}$$

$F\left[V.P\,\dfrac{1}{x}\right]$ is a function. Assume $\varphi \in S$, then we have

$$\left\langle v.p\ \frac{1}{x}\ ,\varphi \right\rangle = \lim_{\substack{\varepsilon\to 0\\ A\to\infty}} \int_{\varepsilon\le|x|\le A} \frac{\varphi(x)}{x}dx$$ .So that if we put

$$f_{\varepsilon,A}(x) = \begin{cases} \frac{1}{x} & \varepsilon \le |x| \le A \\ 0 & otherwise \end{cases}.$$

We have $v.p(\frac{1}{x}) = \lim f_{\varepsilon,A}(x)$ by the topology on S'. Therefore from the continuity of F, we have $F\left[v.p(\frac{1}{x})\right] = \lim F\left[f_{\varepsilon,A}(x)\right]$,by the topology on S'_{ξ}.

On the other hand,

$$\int_{\varepsilon\le|x|\le A} e^{-2\pi ix\xi}\frac{dx}{x} = \int_{\varepsilon\le|x|\le A} \frac{-iSin\pi x\xi}{x}dx = -2i\int_{\varepsilon}^{A}\frac{Sin2\pi x\xi}{x}dx.$$

If we assume $\xi \neq 0$, we have $\hat{f}(\xi) = -2i\lim_{A\to\infty}\int_0^A \frac{Sin2\pi x\xi}{x}dx$

$$= -2i\lim_{A\to\infty}\int_0^{2\pi A\xi}\frac{\sin x}{x}dx$$

$$= -2i\int_0^{+\infty}\frac{Sinx}{x}dx$$

$$= \begin{cases} -\pi i, & \xi > 0 \\ \pi i, & \xi < 0 \end{cases}$$

Since $\int_0^{+\infty}\frac{Sinx}{x}dx = \frac{1}{2}\pi$

2) Let Y(x) be the Heaviside function, i.e. $\langle Y,\varphi\rangle = \int_0^{\infty}\varphi(x)dx$

Then find F[Y(x)].

Solution

Since $F\left[v.p\ \frac{1}{x}\right] = \begin{cases} -\pi i, & (\xi>0) \\ \pi i, & (\xi<0) \end{cases}$, $F^{-1}\left[v.p\ \frac{1}{x}\right] = \begin{cases} \pi i, & (\xi>0) \\ -\pi i & (\xi<0) \end{cases}$

Also $F^{-}[\delta] = 1$

$$F^{-1}\left[\frac{1}{\pi i}\ v.p\frac{1}{x}+\delta\right]=2Y(\xi)$$. Since $FF^{1}=I$,

$$F[Y(x)]=\frac{1}{2\pi i}v.p\ \frac{1}{\xi}+\frac{1}{2}\delta$$

3.5 Applications in differential operators

Malagrange-EhrenpriesTheorem: - Every nonzero differential operator with constant coefficients has a Green's function. This means that if P is a polynomial

(In several variables), then the differential equation $P(\frac{\partial}{\partial x_i})u(x)=\delta(x)$ has a

distributional solution u where δ is a Dirac delta function. It can be used to show that

$P(\frac{\partial}{\partial x_i})u(x)=f(x)$ has a solution for any distribution f. The solution is not unique

in general.

If for the equation $P(D)u=f$ the convolution $\varepsilon * f$ (where ε the fundamental solution of non zero linear differential operator P (D) with constant coefficients) exists, then

$$u=\varepsilon * f$$

Proof

$$u=u*\delta$$

$$u=u*P(D)\varepsilon=P(D)(u*\varepsilon)=P(D)u*\varepsilon=f*\varepsilon=\varepsilon*f$$

Consider a nonzero linear differential operator $P(D)=\sum_{\alpha\in K}a_\alpha D^\alpha$ .... (*)

With constant coefficients where K is a finite set in the space N_0^n of multi indices, a_α's are constants such that for $\alpha_j\geq 0$ $\alpha=(\alpha_1,...\alpha_n)\in N_0^n$ $|\alpha|=(\alpha_1+...+\alpha_n)$; $\max|\alpha|=m$ and for

$D_j^{\alpha_j}=\frac{1\partial^{\alpha_j}}{i\partial x_j^{\alpha_j}}$ define $D^\alpha=D_1^{\alpha_1}\times...\times D_n^{\alpha_n}$

Now let us describe the condition of partial hypoellipticity of the operator (*) in terms of fundamental solutions.

Definition 1: - A distribution ε is called a fundamental solution to the differential operator (20) iff $P(D)\varepsilon=\delta$

Definition 2: - Let $\Omega \subset (R^n)$ $\Omega \subset (R^n)$ be an open set and let $u \in D^{'}(\Omega)$ a differential operator (*) is called elliptic if $P(D)(u) \in A(\Omega)$, then $u \in A(\Omega)$

Definition 3:- Let $\Omega \subset (R^n)$ be an open set and let $u \in D^{'}(\Omega)$ a differential operator (*) is called hypo elliptic if $P(D)(u) \in C^{\infty}(\Omega)$, then $u \in C^{\infty}(\Omega)$.

Definition 4: - Let $\Omega \subset (R^n)$ $\Omega \subset (R^n)$ be an open set and let $u \in D^{'}(\Omega)$ a differential operator (*) is called partially hypo elliptic with respect to the plane $x^{''} = 0$ if $P(D)(u) \in C^{\infty}(\Omega)$, then for any function $\psi \in D(R^m)$ the convolution $u * \psi_m \in C^{\infty}(\Omega_{\psi_m})$.(where $x^{''} \in R^{n-m}, 0 \le m < n$)

Summary of 3.5

Definition	**In terms of poly.**	**In terms of Fs.**
Differential operator (*) is Elliptic if and only if $u \in D'(\Omega), P(D)u \in A(\Omega) \Rightarrow u \in A(\Omega)$	For $\xi \neq 0$, the principal part $p_m(\xi) \neq 0$	Any Fundamental solution ε is such that $\varepsilon \in A\left([\{0\}]^c\right)$
Differential operator (*) is hypoelliptic if and only if $u \in D'(\Omega), P(D)u \in C^\infty(\Omega) \Rightarrow u \in C^\infty(\Omega)$	$\frac{p^{(\alpha)}(\xi)}{p(\xi)} \to 0 (\xi \to \infty)$	Any Fundamental solution ε is such that $\varepsilon \in C^\infty\left([\{0\}]^c\right)$
Differential operator (*) is Partially hypoelliptic Iff $u \in D'(\Omega), \psi \in D(\mathbb{R}^m)$, $P(D)u \in C^\infty(\Omega$ $\Rightarrow u * \psi_m \in C^\infty(\Omega_{\psi m})$	$\frac{p^{(\alpha)}(\xi)}{p(\xi)} \to 0 (\xi' \to \infty)$ *whereas* ξ' remains bounded	For $m, n \in \mathbb{N}_0 0 \leq m < n$ and $\forall \psi \in D(\mathbb{R}^m)$, $\in * \psi_m \in C^\infty([\sup p(\psi_m)]^c)$

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