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Addis Ababa University  
School of Graduate Studies  
Department of Mathematics

A project work on

Integration

and

Convergence Theorem

(Based on the article by W.F.Eberlein)

By

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of Addis Ababa University in Partial fulfillment of the requirements for the Degree of  
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Kibrom Gebrehiwot

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January 25, 2011

## ABSTRACT

The purpose of this study is an attempt to explore sufficient condition for the convergence of Riemann integrable functions on any positive linear functional. The Arzela dominated convergence theorem holds for the Riemann integral provided it is assumed that the limit function is Riemann integrable. Yet, one can see that a simple modification in the Arzela's dominated convergence theorem shows that the theorem can be expressed in another form which holds true for a particularly defined positive linear functional of Riemann integrable function. Consequently, we need an extended suffice convergence condition which holds true for any positive linear functional under the frame work of compact Hausdorff space and for this we obtain a theorem developed by W.F.Eberlein. To this effect, this study shows how to prove the classical convergence theorem made by W.F.Eberlein using the facts and generalizations from Sequence in particular Subsequence and Cauchy Sequence,  $L^p$  Space and unifying some well known convergence theorems.

**Key words:** Convergence of Riemann integrable functions, Positive linear functional, Arzela dominated convergence theorem, Convergence theorem by W.F.Eberlein.

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## **Declaration Letter**

**I, Kibrom Gebrehiwot, declare that this project has been composed by me and that no part of the project has formed the basis for the award of any Degree, Diploma, Associate ship, Fellowship or any other similar title to me.**

Kibrom Gebrehiwot

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## **Permission Letter**

This is to certify that this project is compiled by Mr. Kibrom Gebrehiwot in the department of Mathematics, College of Mathematics and Computational Sciences, Addis Ababa University, under my supervision.

Seid Mohammed (PhD, Asso.Prof.)

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## Mathematical Symbols and Abbreviations

### Notations

### Meaning

1.  $\int_a^{\bar{b}} f(x)dx$  The Upper Darboux/integral of  $f$  from  $a$  into  $b$ .
2.  $\int_a^{\underline{b}} f(x)dx$  The Lower integral / Darboux of  $f$  from  $a$  into  $b$ .
3.  $B[a, b]$  Bounded functions on a closed interval  $[a, b]$  .
4.  $C[a, b]$  Continuous function on a closed interval  $[a, b]$
5.  $\mathcal{R}[a, b]$  Riemann integrable functions on  $[a, b]$
6.  $C(X)$  Real valued continuous function on a compact Hausdorff space  $X$ .
7.  $C(s)$  Real valued continuous function on a compact Hausdorff space  $S$ .
8.  $S$  Compact Hausdorff space  $S$ .
9.  $\mathbb{R}$  The set of real numbers.
10.  $\mathbb{C}$  The set of complex numbers.

## Introduction

The theory of integration and convergence of integration, because of its central role in mathematics analysis and geometry, continues to afford opportunity for series investigation. The need for extending and rounding out the classical studies of Riemann integral and its convergence has stimulated considerable interest not only in a new aspect of the convergence theory but also in the simplification, modification, extension and perfection of the old ones.

In the course of these developments a remarkable result due to Arzela marked the beginning of deeper understanding of the continuity properties of the Riemann integral as a function of its integrand. The classical result of Arzela which we refer most is the so called Arzela dominated convergence theorem for Riemann integral. A number of elementary proofs for Arzela dominated convergence theorem were published by different mathematicians such as F.Riesz (1917), by L.Bieberbach and E. Landau (1918), by F. Hausdorff and H.S. Carslaw (1928), by H.A. Lauwerier (1949), by J.D. Weston (1951), and by W.F. Eberlein (1957). Despite the availability of this variety of elementary proofs for Arzela theorem, most are interdependent or completely dependent on one another. However, W.F. Eberlein proves Arzela's theorem defined on the space of real continuous functions on a compact Hausdorff space. Eberlein proof is completely different from the other proofs. It is geometric in nature in that it is based on the parallelogram law and the minimal distance property for convex sets in an inner product space.

The objective of this paper is to figure out limitation of the Arzela dominated convergence theorem; its relation and extension to W.F.Eberlein theorem and in the end provide the proof to the classical theory developed by W.F.Eberlein. The basic method for the proof stems from the work of W.F.Eberlein and further incorporating some facts and results from  $L^p$ - norms ,inner product space, sequence and Cauchy sequence in  $\mathbb{R}^p$  to a more general theory of convergence theorem working in a positive linear functional of compact Hausdorff space.

This paper comprises two chapters. In the first chapter, basic preliminary concepts and definitions on positive linear functional, vector lattice, algebra, inner product space and  $L^p$ - spaces that are needed later will be dealt on. In the second chapter, fundamental points about compact sets, sequence, Arzela dominated convergence theorem, its extension and in the end the convergence theorem developed by W.F. Eberlein, which is the main target for this paper, will be discussed.

# Chapter One

## 1. Preliminary concepts

**Introduction:** In this chapter, fundamental concepts about integral and preliminary concepts like algebra of sets, vector lattice, elementary integral and inner product spaces that are needed later will be dealt on.

### 1.1. Algebra of sets

**Definition:** - An algebra of sets on  $X$  is a non empty collection of subsets of  $X$  which is closed under finite union and closed under complement. That is,

a) If  $A_n \in \varphi$ , then  $\bigcup_{i=1}^n A_i \in \varphi$

b) If  $A \in \varphi$ , then  $A^c \in \varphi$

From (a) and (b), we can deduce the following facts as a consequence

i.  $\varphi$  is closed under finite intersections.

That is, if  $A_i \in \varphi$ , then  $\bigcap_{i=1}^n A_i \in \varphi$

**Proof:**  $\bigcap_{i=1}^n A_i = (\bigcup_{i=1}^n A_i^c)^c \in \varphi$

ii. If  $A \in \varphi$ , then,  $\emptyset \in \varphi$  (Since  $A \cap A^c = \emptyset \in \varphi$ )

iii. If  $A \in \varphi$ , then,  $X \in \varphi$  (Since  $X = A \cup A^c \in \varphi$ )

**Remark:** - Let  $X$  be any non empty set. The collection of  $\{\emptyset, X\}$  and  $P(X) = \{E : E \subseteq X\}$  are examples of algebras of subsets of  $X$ .

**Example:** - Let  $X$  be any non empty set where  $C = \{E \subseteq X : \text{either } E \text{ or } E^c \text{ is finite}\}$

Then,  $C$  is an algebra of subsets of  $X$ .

**Solution:** In the case  $X$  is a finite set, then, and since  $E \subseteq X$  we can see that  $E$  is a finite set too. Therefore, by the remark, the power set  $C = P(X)$  is an algebra of subsets of  $X$ .

Suppose  $X$  is not a finite set, then,  $E$  is either finite or infinite set and  $E^c \in C$  if  $E \in C$ .

Next, suppose  $E_i \in C$  for all  $i = 1, 2, 3, \dots$ , then, we need to show  $\bigcup_{i=1}^n E_i \in C$ .

But,  $\bigcup_{i=1}^n E_i = (\bigcap_{i=1}^n E_i^c)^c \in C$  for all  $i = 1, 2, 3, \dots$

Hence, by definition,  $C$  is algebra of subset of  $X$ .

## 1.2. Vector Lattice

**Definition:** - A non empty class  $F$  of functions  $f: X \rightarrow \mathbb{R}$  is called a vector lattice (Linear lattice) of functions if it is such that:

$$f \vee g, f \wedge g, f + g \text{ and } cf \text{ are in } F \text{ where } f, g \in F \text{ and } c \in \mathbb{R}.$$

**Remark:** -  $(f \vee g)(x) = \max\{f(x), g(x)\}$  and  $(f \wedge g)(x) = \min\{f(x), g(x)\}$

**Example 1:-** Consider  $F$  the class of all continuous functions  $f: \mathbb{R} \rightarrow \mathbb{R}$ . Then,  $F$  is a vector lattice.

**Solution:** - Let  $f, g: \mathbb{R} \rightarrow \mathbb{R}$  be a member of  $F$  and  $c \in \mathbb{R}$ , then, by the properties of continuity the following is satisfied

$$f \vee g, f \wedge g, f + g \text{ and } cf \text{ are continuous.}$$

Hence, by definition,  $F$  is a vector lattice.

**Example 2:-** Consider  $F$  the class of all functions  $f: \mathbb{R} \rightarrow \mathbb{R}$  which is continuously differentiable at each point of  $\mathbb{R}$ . Then,  $F$  is not a vector lattice.

**Solution:** - Consider, for instance,  $f(x) = x$  and  $g(x) = -x$ . Then, one can easily check that  $f$  and  $g$  are continuously differentiable in  $\mathbb{R}$ . From this, we can infer that  $f$  and  $g$  belong to  $F$ .

Now let us see the following if the conditions for vector lattice are satisfied.

$$f(x) + g(x) = x + (-x) = 0$$

$$(f \vee g)(x) = \max\{f(x), g(x)\} = \max\{x, -x\} = |x|$$

$$(f \wedge g)(x) = \min\{f(x), g(x)\} = \min\{x, -x\} = -|x|$$

$$(c.f)(x) = c.f(x) = c.x$$

Now from the properties of continuity of functions, we have  $f \vee g, f \wedge g, f + g$  and  $cf$  are continuous. However, though  $f + g$  and  $cf$  are differentiable in  $\mathbb{R}$ , one can

see that  $f \vee g$  and  $f \wedge g$  are not differentiable on  $\mathbb{R}$  because  $(f \vee g)(x) = |x|$  and  $(f \wedge g)(x) = -|x|$  are not differentiable at zero which implies  $f \vee g$  and  $f \wedge g$  do not belong to  $F$ . Hence, by definition,  $F$  is not a vector lattice.

### 1.3. Elementary integrals

**Definition:** - If  $F$  is a vector lattice of functions on  $X$  to  $\mathbb{R}$ , a function  $I: F \rightarrow \mathbb{R}$  is called an elementary integral if for  $f, g \in F$  and  $c \in \mathbb{R}$ , then, the following conditions are satisfied:

- (a).  $I(cf) = c.I(f)$
- (b).  $I(f + g) = I(f) + I(g)$
- (c). If  $f_n \rightarrow 0$  then  $I(f_n) \rightarrow 0$ .
- (d).  $I(f) \geq 0$  for  $f \geq 0$ .

The two properties (a) and (b) express the fact that  $I$  is linear functional on  $F$ . From (a), we realize that  $I(0) = 0$ . Because of property (d), we can see that  $I$  is a positive linear functional.

### 1.4. Linear Functional

**Definition:** A linear functional is a map  $T$  from  $H$  into the real or complex number. That is,  $T: H \rightarrow \mathbb{R}(\mathbb{C})$  Such that  $T(\alpha f + \beta g) = \alpha.T(f) + \beta.T(g)$  where  $\alpha, \beta \in \mathbb{R}(\mathbb{C})$  and  $f, g \in H$ .

**Definition:** - A linear functional  $T: H \rightarrow \mathbb{R}(\mathbb{C})$  is said to be bounded if there exists a positive real number  $M$  Such that  $|T(U)| \leq M. \|U\|$  for all  $U \in H$ .

**Example 1:** - Let  $T: H \rightarrow \mathbb{R}$  be given by  $T(f) = \langle f, g \rangle$  for any fixed  $g$  in  $H$ . Then,  $T$  is bounded linear functional where  $\langle ., . \rangle$  denotes an inner product space.

**Solution:** In order to show  $T$  is a linear functional, we need to show if the fore condition is satisfied.

$T(\alpha f + \beta g) = \alpha.T(f) + \beta.T(g)$  where  $\alpha, \beta \in \mathbb{R}(\mathbb{C})$  and  $f, g \in H$ .

From the given, we obtain  $T(\alpha f + \beta h) = \langle \alpha f + \beta h, g \rangle$  where  $g$  is fixed in  $H$ .

But, by definition for inner product space we have  $\langle \alpha f + \beta h, g \rangle = \langle \alpha f, g \rangle + \langle \beta h, g \rangle$

Again,  $\langle \alpha f, g \rangle + \langle \beta h, g \rangle = \alpha.\langle f, g \rangle + \beta.\langle h, g \rangle$

Then, from the given, we obtain  $\alpha.\langle f, g \rangle + \beta.\langle h, g \rangle = \alpha.T(f) + \beta.T(g)$

Accordingly, we have  $T(\alpha f + \beta g) = \alpha.T(f) + \beta.T(g)$

Hence, by definition,  $T$  is a linear functional.

Next we need to check if  $T$  is bounded. That is, we need to find  $M > 0$  and  $M \in \mathbb{R}$

Such that  $|T(f)| \leq M.\|f\|$ .

Now  $|T(f)| = |\langle f, g \rangle| \leq \|f\|.\|g\|$  (Cauchy Schwarz inequality)

But we know that  $g$  is fixed in  $H$ , without loss of generality, consider  $M = \|g\|$

Therefore,  $|T(f)| \leq M.\|f\|$  for all  $f \in H$  which, then, implies  $T$  is bounded.

From the aforementioned points, we can conclude that  $T$  is bounded linear functional.

**Example 2:** - Define  $G$  for  $f$  in  $C(I)$  by  $G(f) = 2.\int_0^{1/2} f(x)dx$ , then,  $G$  is bounded positive linear functional on  $C(I)$ , where  $C(I)$  is continuous function on a positive linear functional  $I$ .

**Solution:** - Let  $f_1$  and  $f_2$  be in  $C(I)$  and  $\alpha$  and  $\beta$  in  $\mathbb{C}$ , then,

$$\begin{aligned} \text{i. } G(\alpha.f_1 + \beta.f_2) &= 2\left(\int_0^{1/2}(\alpha.f_1 + \beta.f_2)dx\right) \\ &= 2\left(\int_0^{1/2}\alpha.f_1 dx + \int_0^{1/2}\beta.f_2 dx\right) \\ &= 2\left(\alpha.\int_0^{1/2}f_1 dx + \beta.\int_0^{1/2}f_2 dx\right) \end{aligned}$$

$$\begin{aligned}
&= \alpha \cdot \left( 2 \cdot \int_0^{1/2} f_1 dx \right) + \beta \cdot \left( 2 \cdot \int_0^{1/2} f_2 dx \right) \\
&= \alpha \cdot G(f_1) + \beta \cdot G(f_2)
\end{aligned}$$

Thus, G is linear functional.

**ii. Boundedness**

$$|G(f)| = \left| 2 \cdot \int_0^{1/2} f dx \right| \leq \int_0^{1/2} |f(x)| dx \leq 2 \cdot \|f\| \cdot \left( \frac{1}{2} - 0 \right) = 1 \cdot \|f\|$$

Take  $M = 1$  which implies  $|G(f)| \leq M \cdot \|f\|$

Thus, by definition, G is bounded.

**iii. Positivity**

Suppose that  $f(x) \geq 0$  then  $\int f(x) dx \geq 0$ . Thus, one can easily see that

$$G(f) = 2 \int_0^{1/2} f(x) dx \geq 0 \text{ provided that } f(x) \geq 0. \text{ This implies G is positive function.}$$

Therefore, from (i), (ii) and (iii), we conclude that G is positive linear functional.

## 1.5. Inner product Spaces and some basic facts

### 1.5.1. Normed Spaces

**Definition:** Let H be a vector space. A norm on H is a mapping  $\| \cdot \|: H \rightarrow [0, \infty]$  satisfying the following conditions. Let  $x, y \in H$  and  $\alpha \in \mathbb{C}$ .

- i.  $\|x\| \geq 0$  and  $\|x\| = 0$  if and only if  $x = 0$ .
- ii.  $\|\alpha \cdot x\| = |\alpha| \cdot \|x\|$
- iii.  $\|x + y\| \leq \|x\| + \|y\|$

$\mathbb{C}$ - denotes the set of complex numbers.

The pair  $(H, \| \cdot \|)$  is, then, said to be a Normed space. If H is a complete space with respect to the norm, it is called Banach space.

### 1.5.2. Inner Product Space

**Definition:** Let  $H$  be a vector space. An inner product is a mapping  $\langle ., . \rangle: H \times H \rightarrow \mathbb{C}$  satisfying the following conditions:

- i.  $\langle x, x \rangle \geq 0$  and  $\langle x, x \rangle = 0$  if and only if  $x = 0$ .
- ii.  $\langle x, y \rangle = \overline{\langle y, x \rangle}$  for all  $x, y \in H$ .
- iii.  $\langle x + y, z \rangle = \langle x, z \rangle + \langle y, z \rangle$  for all  $x, y, z \in H$ .
- iv.  $\langle \alpha x, y \rangle = \alpha \langle x, y \rangle$  for all  $x, y \in H$  and  $\alpha \in \mathbb{C}$ .

The pair  $(H, \langle ., . \rangle)$  is said to be an inner product space or Pre- Hilbert Space.

**Remark:**

1. In  $L^2[a, b]$  an inner product is defined by  $\langle f, g \rangle = \int_a^b f(x) \cdot \overline{g(x)} \cdot dx$

A norm induced from inner product is given by  $\|x\| = \sqrt{\langle x, x \rangle} = \langle x, x \rangle^{1/2}$  for all  $x \in H$ .

2. In  $L^2[a, b]$  norm is defined by

$$\|f\| = \sqrt{\langle f, f \rangle} = \sqrt{\int_a^b f(x) \cdot \overline{g(x)} \cdot dx} = \sqrt{\int_a^b |f(x)|^2 \cdot dx}$$

**Theorem 1 (Cauchy-Schwarz- Inequality)**

For all  $x$  and  $y$  belonging to a given inner product space, we have

$$|\langle x, y \rangle|^2 \leq \langle x, x \rangle \cdot \langle y, y \rangle .$$

**Proof:** If  $y = 0$ , then,  $\langle x, y \rangle = 0$  and  $\langle y, y \rangle = 0$ .

Therefore, since  $\langle x, y \rangle = 0$  and  $\langle x, x \rangle \cdot \langle y, y \rangle = 0$ , then,  $|\langle x, y \rangle|^2 \leq \langle x, x \rangle \cdot \langle y, y \rangle$  is satisfied.

If  $y \neq 0$ , then,  $\langle y, y \rangle \neq 0$ .

Now let  $\lambda = \frac{\langle x, y \rangle}{\langle y, y \rangle}$ , then, we obtain the following: -

$$\frac{|\langle x, y \rangle|^2}{\langle y, y \rangle} = \frac{\langle x, y \rangle \cdot \overline{\langle x, y \rangle}}{\langle y, y \rangle} = \frac{\langle x, y \rangle}{\langle y, y \rangle} \cdot \overline{\langle x, y \rangle} = \lambda \cdot \overline{\langle x, y \rangle} = \bar{\lambda} \cdot \langle x, y \rangle$$

$$\text{Thus, } \frac{|\langle x, y \rangle|^2}{\langle y, y \rangle} = \lambda \cdot \langle y, x \rangle = \bar{\lambda} \cdot \langle x, y \rangle = \lambda \cdot \bar{\lambda} \cdot \langle y, y \rangle = |\lambda|^2 \cdot \langle y, y \rangle = |\lambda|^2 \cdot \|y\|^2$$

Since we know that  $\langle x, x \rangle \geq 0$ , then, consider

$$\begin{aligned} 0 \leq \langle x - \lambda y, x - \lambda y \rangle &= \langle x, x \rangle - \langle x, -\lambda y \rangle + \langle -\lambda y, x \rangle + \langle -\lambda y, -\lambda y \rangle \\ &= \langle x, x \rangle - \bar{\lambda} \cdot \langle x, y \rangle - \lambda \cdot \langle y, x \rangle - |\lambda|^2 \cdot \langle y, y \rangle \\ &= \langle x, x \rangle - \frac{|\langle x, y \rangle|^2}{\langle y, y \rangle} - \frac{|\langle x, y \rangle|^2}{\langle y, y \rangle} + \frac{|\langle x, y \rangle|^2}{\langle y, y \rangle} \end{aligned}$$

$$\text{which, then, implies } 0 \leq \langle x, x \rangle - \frac{|\langle x, y \rangle|^2}{\langle y, y \rangle}$$

$$\text{Rearranging, we obtain } \frac{|\langle x, y \rangle|^2}{\langle y, y \rangle} \leq \langle x, x \rangle \text{ since } \langle y, y \rangle \geq 0.$$

$$\text{From this, we obtain the result } |\langle x, y \rangle|^2 \leq \langle x, x \rangle \cdot \langle y, y \rangle$$

**Corollary:** - For all  $x$  and  $y$  belonging to given inner product space, we have the following  $|\langle x, y \rangle| \leq \|x\| \cdot \|y\|$

**Proof:** - From Cauchy Schwarz Inequality we have

$$|\langle x, y \rangle| \leq \langle x, x \rangle \cdot \langle y, y \rangle \tag{1}$$

$$\text{But we know that } \|x\|^2 = \langle x, x \rangle \text{ and } \|y\|^2 = \langle y, y \rangle \tag{2}$$

Then, substituting (2) on (1), we have  $|\langle x, y \rangle|^2 \leq \|x\|^2 \cdot \|y\|^2$

Consequently, we obtain  $|\langle x, y \rangle| \leq \langle x, x \rangle \cdot \langle y, y \rangle$

**Theorem 2:** - Every inner product space is normed space with respect to the norm

$$\|x\| = |\langle x, x \rangle|^{1/2} \quad \text{for all } x \in H .$$

**Proof:** - Since the inner product space is a vector Space, we need to verify all the axioms of the norm:-

i.  $\|x\| \geq 0$  for all  $x$  and  $\|x\| = 0$  if and only if  $x = 0$ .

$$\|x\| = |\langle x, x \rangle|^{1/2} \quad \text{since } \langle x, x \rangle \geq 0 \quad \text{and } \langle x, x \rangle = 0 \quad \text{if and only if } x = 0 .$$

Therefore, (i) is satisfied.

ii.  $\|\alpha \cdot x\| = |\alpha| \cdot \|x\|$  for all  $x$  and  $\alpha$  real or complex.

$$\text{By definition, we have } \|\alpha x\| = [\langle \alpha x, \alpha x \rangle]^{1/2} = [\alpha \cdot \bar{\alpha} \cdot \langle x, x \rangle]^{1/2}$$

$$\text{Since } \alpha \cdot \bar{\alpha} = |\alpha|^2, \quad [\alpha \cdot \bar{\alpha} \cdot \langle x, x \rangle]^{1/2} = [|\alpha|^2 \cdot \langle x, x \rangle]^{1/2}$$

$$\text{Which is, then, implies } [|\alpha|^2 \cdot \langle x, x \rangle]^{1/2} = |\alpha| \cdot [\langle x, x \rangle]^{1/2}$$

$$\text{This implies, } |\alpha| \cdot [\langle x, x \rangle]^{1/2} = |\alpha| \cdot \|x\|$$

iii.  $\|x + y\| \leq \|x\| + \|y\|$  for all  $x, y \in X$

$$\text{We have } \|x + y\|^2 = \langle x + y, x + y \rangle = \langle x, x \rangle + \langle x, y \rangle + \langle y, x \rangle + \langle y, y \rangle$$

$$= \langle x, x \rangle + \langle x, y \rangle + \overline{\langle x, y \rangle} + \langle y, y \rangle$$

$$= \langle x, x \rangle + 2 \cdot \text{Re}\langle x, y \rangle + \langle y, y \rangle \tag{3}$$

But, from Cauchy- Schwarz Inequality

$$\text{Re}\langle x, y \rangle \leq |\langle x, y \rangle| \leq (\langle x, x \rangle)^{1/2} \cdot (\langle y, y \rangle)^{1/2} \tag{4}$$

Thus, from (3) and (4) we will have

$$\|x + y\|^2 \leq \langle x, x \rangle + 2 \cdot (\langle x, x \rangle)^{1/2} \cdot (\langle y, y \rangle)^{1/2} + \langle y, y \rangle$$

$$= [\langle x, x \rangle^{1/2} + \langle y, y \rangle^{1/2}]^2$$

Taking the square root of both sides,  $\|x + y\| \leq \langle x, x \rangle^{1/2} + \langle y, y \rangle^{1/2}$

Then, we get  $\|x + y\| \leq \|x\| + \|y\|$

Therefore, this proves that  $\|x\| = \langle x, x \rangle^{1/2}$  is a norm, and  $(X, \|\cdot\|)$  is a Normed space where  $X$  is an inner product space.

**Remark:**

1. In Cauchy Schwarz Inequality, equality holds true i.e.  $\|x + y\| = \|x\| + \|y\|$  if and only if  $x$  and  $y$  is linearly independent.
2. The norm  $\|x\| = |\langle x, x \rangle|^{1/2}$  is said to be the norm induced by an inner product.

**1.6. L<sup>p</sup>- Spaces**

Consider  $(X, \beta, \mu)$  be measure space. For  $0 < p < \infty$ , then,  $L_p(\mu) = L_p(X, \beta, \mu)$  denotes the set of all measurable functions defined on  $X$  such that  $\int_X |f|^p d\mu < +\infty$ . The space  $L_p(\mu)$  is called the space of  $P^{\text{th}}$  power integrable functions.

**Definition:** Let  $f \in L_p(X, \beta, \mu)$ . Define  $\|f\|_p$  called the  $P^{\text{th}}$  - norm of  $f$  as follows:

$$\|f\|_p = (\int |f|^p d\mu)^{1/p}$$

Now using some elementary properties of the  $L_p$  norms on  $C(S)$  in the distinguished cases where  $p = 1, 2, \infty$

Let's first set  $\|f\|_\infty = \sup |f(x)|$  where  $x$  in  $S$ .

We have  $I(f) \geq 0$  provided that  $f \geq 0$ .

Thus, we get the result  $0 \leq \int |f| \leq \int |f|$  which, then, implies  $|\int f| \leq \int |f|$  (5)

Since  $|f| \leq \|f\|_\infty$  a. e. , then, we get  $|\int f| \leq \|f\|_\infty \cdot \int 1 = \|f\|_\infty \cdot I(1)$

Now assuming that  $I(1) = 1$ , then, we obtain  $|\int f| \leq \|f\|_\infty$  (6)

Thus, from (5) and (6) we have

$$(A) \int |f| \leq \int |f| \leq \|f\|_\infty$$

Next set the following

- $\langle f, g \rangle = I(fg) = \int fg$  where  $f$  and  $g$  in  $C(s)$
- $\|f\|_2 = \langle f, f \rangle^{1/2} = \{I(f^2)\}^{1/2}$
- $\|f\|_1 = \int |f| = I(|f|)$

Then,

$$(B) |\langle f, g \rangle| \leq \|f\|_2 \cdot \|g\|_2 \quad (\text{Cauchy Schwarz Inequality})$$

**Proof:** From Cauchy Schwarz inequality for an inner product space, we have

$|\langle f, g \rangle|^2 \leq \langle f, f \rangle \cdot \langle g, g \rangle$ . Then, taking the square root of both sides, we obtain

$$|\langle f, g \rangle| \leq \langle f, f \rangle^{1/2} \cdot \langle g, g \rangle^{1/2} .$$

But, from the assumption, we have  $\|f\|_2 = \langle f, f \rangle^{1/2}$  and  $\|g\|_2 = \langle g, g \rangle^{1/2}$

Therefore, we conclude that  $|\langle f, g \rangle| \leq \|f\|_2 \cdot \|g\|_2$  which is the desired.

$$(C) \|f + g\|_2^2 + \|f - g\|_2^2 = 2\{\|f\|_2^2 + \|g\|_2^2\} \quad (\text{Parallelogram Identity})$$

**Proof:** - From the assumption, we have

$$\begin{aligned} \|f + g\|_2^2 &= \langle f + g, f + g \rangle \\ &= \langle f, f + g \rangle + \langle g, f + g \rangle \\ &= \langle f, f \rangle + \langle f, g \rangle + \langle g, f \rangle + \langle g, g \rangle \\ &= \|f\|_2^2 + \langle f, g \rangle + \langle g, f \rangle + \|g\|_2^2 \end{aligned} \quad (7)$$

Similarly,  $\|f - g\|_2^2 = \langle f - g, f - g \rangle$

$$= \langle f, f - g \rangle + \langle g, f - g \rangle$$

$$\begin{aligned}
&= \langle f, f \rangle + \langle f, -g \rangle + \langle -g, f \rangle + \langle -g, -g \rangle \\
&= \langle f, f \rangle - \langle f, g \rangle - \langle g, f \rangle + \langle g, g \rangle \\
&= \|f\|_2^2 - \langle f, g \rangle - \langle g, f \rangle + \|g\|_2^2
\end{aligned} \tag{8}$$

Adding (7) and (8) we get the required result

$$\|f + g\|_2^2 + \|f - g\|_2^2 = 2 \cdot \langle f, f \rangle + 2 \cdot \langle g, g \rangle = 2 \cdot \{ \|f\|_2^2 + \|g\|_2^2 \}$$

**(D).**  $\|f\|_1 = I(|f|) \leq \|f\|_2$

**Proof:** From the assumption, we have that  $\|f\|_1 = \int |f|$ .

Then, we have  $\int |f| = \int |f| \cdot 1 = \langle |f|, 1 \rangle$

By Cauchy Schwarz inequality and since  $\|1\|_2 = 1$ , we obtain

$$\int |f| = \int |f| \cdot 1 = \langle |f|, 1 \rangle \leq \|f\|_2 \cdot \|1\|_2 = \|f\|_2$$

Consequently, we get the result  $\|f\|_1 = I(|f|) = \langle |f|, 1 \rangle \leq \|f\|_2 \cdot \|1\|_2 = \|f\|_2$

**(E).**  $\|f\|_2^2 = \int f^2 \leq \|f^2\|_\infty = \|f\|_\infty^2$

**Proof:**  $\|f\|_2^2 = \langle f, f \rangle = \int f^2$

Since  $f^2 \geq 0$ , then, we have  $\int f^2 = \int |f^2|$

This, then, implies  $\int |f^2| \leq \|f^2\|_\infty = \|f\|_\infty^2$

Therefore, we obtain the result  $\|f\|_2^2 = \int f^2 \leq \|f^2\|_\infty = \|f\|_\infty^2$

**Chapter Two**  
**Convergence theorem on a Compact**  
**Hausdorff Space**

**2.1. Compact Sets**

**Definition:** A set  $K$  is said to be compact if, whenever it is contained in the union of a collection  $T = \{G_\alpha\}$  of open sets, then it is also contained in the union of some finite members of the sets in  $T$ .

A collection  $T$  of open sets whose union contains  $K$  is often called a covering of  $K$ . Thus, the requirement that  $K$  be compact is that every covering  $T$  of  $K$  can be replaced by a finite covering of  $K$  using only sets in  $T$ .

**Example 1:-** In  $\mathbb{R}$ , consider the subset  $H = \{x: x \geq 0\}$ , then,  $H$  is not compact set.

**Solution:** Let  $G_n = (-1, n)$  where  $n \in \mathbb{N}$ , then,  $T = \{G_n : n \in \mathbb{N}\}$  is a collection of open subsets of  $\mathbb{R}$ , whose union contains  $H$  i.e.  $\bigcup_{i=1}^n G_i \supseteq H$ . If  $\{G_{n_1}, \dots, G_{n_k}\}$  is finite sub collection of  $T$ . Let  $m = \text{Sup}\{n_1, n_2, \dots, n_k\}$ . Then,  $G_{n_j} \subseteq G_m$  for  $j = 1, 2, \dots, k$

It follows that  $G_m$  is the union of  $\{G_{n_1}, \dots, G_{n_k}\}$  i.e.  $\bigcup_{j=1}^k G_{n_j} = G_m$ . However, the real number  $m$  does not belong to  $G_m$  and hence does not belong to  $\bigcup_{j=1}^k G_{n_j}$ . Therefore, no finite union of the set  $T$  can contain  $H$ .

Hence,  $H$  is not compact set.

**Example 2:-** Let  $H = (0,1)$  in  $\mathbb{R}$ . Then,  $H$  is compact set.

**Solution:** If  $G_n = \left(\frac{1}{n}, 1 - \frac{1}{n}\right)$  for  $n > 2$ , then, the collection  $T = \{G_n: n > 2\}$  of open sets is a covering of  $H$ . If  $\{G_{n_1}, G_{n_2}, \dots, G_{n_k}\}$  is a finite sub collection of  $T$ . Let  $m = \text{sup}\{n_1, n_2, \dots, n_k\}$ . Then,  $G_{n_j} \subseteq G_m$  for  $j = 1, 2, 3, \dots, k$  and it follows that  $G_m$  is the union of  $\{G_{n_1}, G_{n_2}, \dots, G_{n_k}\}$ . That is,  $\bigcup_{j=1}^k G_{n_j} = G_m$ . However, the real number

$\frac{1}{m}$  belongs to  $H$  but does not belong to  $G_m$ . Therefore, no finite sub collection of  $T$  can form a covering of  $H$ . Hence,  $H$  is not compact set.

## 2.2. Convex sets in $\mathbb{R}^n$

**Definition:** - Let  $x_1$  and  $x_2$  be points or vectors in  $\mathbb{R}^n$  and  $\lambda$  be a scalar in  $\mathbb{R}$ . Then, the closed line segment joining  $x_1$  and  $x_2$  is the set

$$[x_1, x_2] = \{x: x = \lambda \cdot x_1 + (1 - \lambda)x_2, 0 \leq \lambda \leq 1\}.$$

A set  $T$  in  $\mathbb{R}^n$  is said to be convex if for points  $x_1$  and  $x_2$  in  $T$  their convex combination (or internal average)  $x = \lambda_1 \cdot x_1 + \lambda_2 \cdot x_2$  where  $\lambda_1 + \lambda_2 = 1$  and  $0 \leq \lambda_1, \lambda_2 \in \mathbb{R}$  is also a member of  $T$ .

Geometrically, this definition implies that the closed line segment joining  $x_1$  and  $x_2$  lies entirely in  $T$ .

**Definition:** - A convex combination of finite number of points  $x_i$  in  $\mathbb{R}^n$  is the point  $x = \sum_{i=1}^m \lambda_i \cdot x_i$  where  $\sum_{i=1}^m \lambda_i = 1$  and  $0 \leq \lambda_i \in \mathbb{R}$  for all  $i = 1, 2, \dots, m$ .

**Convex hull (convex closure)** of a set  $\varphi$ , denoted by  $\text{Co}(\varphi)$ , in  $\mathbb{R}^n$  is the collection of all convex combination of points from  $\varphi$  i.e.

$$\text{Co}(\varphi) = \{x: x = \sum_{i=1}^m \lambda_i \cdot x_i, x_i \in \varphi, \sum_{i=1}^m \lambda_i = 1; 0 \leq \lambda_i \in \mathbb{R}\} \text{ for all } i \text{ and } m \in \mathbb{Z}.$$

## 2.3. Sequences in $\mathbb{R}^n$

**Definition:** - A sequence in  $\mathbb{R}^n$  is a function  $f: N \rightarrow \mathbb{R}^n$  i.e. a function whose domain is the set of natural numbers.

Let  $f(i) = a_i$ , then, we denote  $f = (a_i)_{i=1}^{\infty} = \{a_1, a_2, \dots, a_1, \dots\}$

**Definition (Subsequence)** If  $\langle a_n \rangle_{n=1}^{\infty}$  is a sequence and if  $r_1 < r_2 < \dots < r_n < \dots$  is a strictly increasing sequence of natural numbers, then the sequence  $\langle b_n \rangle_{n=1}^{\infty}$  given by  $\{ a_{r_1}, a_{r_2}, \dots, a_{r_n}, \dots \}$  is called a subsequence of  $\langle a_n \rangle_{n=1}^{\infty}$ .

**Lemma 1:** - If a sequence  $\langle a_n \rangle_{n=1}^{\infty}$  converges to an element  $l$  in  $\mathbb{R}$ , then any subsequence of  $\langle a_n \rangle_{n=1}^{\infty}$  also converges to an element  $l$ .

**Proof:** Let  $\langle a_{n_k} \rangle_{k=1}^{\infty}$  be a subsequence of the sequence  $\langle a_n \rangle_{n=1}^{\infty}$  where

$$n_1 < n_2 < n_3 < \dots < n_k < \dots \quad \text{for all } n, k \in \mathbb{N}.$$

Then, we need to show  $\lim_{k \rightarrow \infty} a_{n_k} = l$

Let  $\varepsilon > 0$  be given, since  $\langle a_n \rangle_{n=1}^{\infty}$  converges to  $l$ , then by definition, there exists a positive integer  $N$  such that  $|a_n - l| < \varepsilon$  for all  $n \geq N$ . (9)

Now if  $k \geq N$  then  $n_k \geq N$  which, then, from (9), we obtain  $|a_{n_k} - l| < \varepsilon$  for all  $k \geq N$ .

Then, by definition,  $|a_{n_k} - l| < \varepsilon$  for all  $k \geq N$  implies  $\lim_{k \rightarrow \infty} a_{n_k} = l$

Therefore, if a sequence converges, then its subsequence converges too and it converges to the same value the sequence converges.

**Definition (Cauchy Sequence):**

A sequence  $\langle a_n \rangle$  in  $\mathbb{R}^n$  where  $n \in \mathbb{N}$  is said to be Cauchy sequence if for every  $\varepsilon > 0$  there is a natural number  $N(\varepsilon)$  such that for all  $m, n \geq N(\varepsilon)$ , then,

$$\|a_n - a_m\| < \varepsilon.$$

In other words, a sequence  $\langle a_n \rangle$  is said to be Cauchy sequence if for every  $\varepsilon > 0$  there is a natural number  $N(\varepsilon)$  such that for all  $m, n \geq N(\varepsilon)$ , then,

$$\lim_{m, n \rightarrow \infty} \|a_n - a_m\| = 0.$$

**Lemma 2:** If  $\langle a_n \rangle$  is a convergent sequence in  $\mathbb{R}^n$ , then,  $\langle a_n \rangle$  is a Cauchy sequence.

**Proof:** Since the sequence  $\langle a_n \rangle$  converges, then, without loss of generality, let  $a = \lim_{n \rightarrow \infty} a_n$ . Then, by definition, given  $\varepsilon > 0$ , there is a natural number  $N_1(\varepsilon)$  such that if  $n \geq N_1(\varepsilon/2)$ , then, we have  $\|a_n - a\| < \varepsilon$ .

Now let  $N(\varepsilon) = N_1(\varepsilon/2)$  and if for all  $m, n \geq N(\varepsilon)$ , then,

$$\|a_n - a_m\| \leq \|a_n - a\| + \|a_m - a\| \leq \varepsilon/2 + \varepsilon/2 = \varepsilon$$

Therefore, for all  $m, n \geq N(\varepsilon)$ , then,  $\|a_n - a_m\| \leq \varepsilon$ . Hence, by definition, the convergent sequence  $\langle a_n \rangle$  is a Cauchy sequence.

**Lemma 3:** - A Cauchy sequence in  $\mathbb{R}^n$  is bounded.

**Proof:** - Let  $\langle a_n \rangle$  be a Cauchy sequence in  $\mathbb{R}^n$ , then, by definition, given any  $\varepsilon > 0$  there is a positive integer  $N(\varepsilon)$  such that  $\|a_n - a_m\| < \varepsilon$  for all  $m, n \geq N$ . But, we know that  $\|a_n\| - \|a_m\| \leq \|a_n - a_m\| < \varepsilon$  for all  $m, n \geq N$ .

Then, taking  $m = N$  and Transposing, we have

$$\|a_n\| \leq \|a_N\| + \varepsilon \quad \text{for all } n \geq N.$$

Thus, for all  $n$ ,

$$\|a_n\| \leq \max \{ \|a_1\|, \|a_2\|, \dots, \|a_{N-1}\|, \|a_N\| + \varepsilon \}$$

A bound for the sequence  $\langle a_n \rangle$  is, then,

$$M = \max \{ \|a_1\|, \|a_2\|, \dots, \|a_{N-1}\|, \|a_N\| + \varepsilon \}.$$

Then, we obtain the result  $\|a_n\| \leq M$  for all  $n \geq N$  which is the required.

From this, we infer that each Cauchy sequence is bounded.

**Lemma 4:** - If a subsequence  $\langle a_{n_k} \rangle$  of Cauchy sequence  $\langle a_n \rangle$  in  $\mathbb{R}^n$  converges to an element  $l$  where  $l \in \mathbb{R}$ , then, the entire sequence  $\langle a_n \rangle$  converges to  $l$ .

**Proof:** - Since  $\langle a_n \rangle$  is Cauchy sequence. Then, by definition, given  $\varepsilon > 0$  there is a positive integer  $N(\varepsilon/2)$  such that for all  $m, n \geq N(\varepsilon/2)$

$$\|a_n - a_m\| < \varepsilon/2 \quad (10)$$

If the sequence  $a_{n_j}$  where  $j$  is an integer, converges to  $l$ , there is a natural number

$k \geq N(\varepsilon/2)$  belonging to the set  $\{n_1, n_2, n_3, \dots\}$  and such that  $\|a_k - l\| < \varepsilon/2$

Now let  $m$  be any natural number such that  $m \geq N(\varepsilon/2)$

It follows that (10) holds for this value of  $m$  and for  $n = k$

Thus,  $\|a_k - l\| \leq \|a_k - a_m\| + \|a_m - l\| < \varepsilon/2 + \varepsilon/2 = \varepsilon$  whenever  $m \geq N(\varepsilon/2)$

Therefore, the sequence  $\langle a_n \rangle$  converges to the element  $l$ , which is the limit of the subsequence  $\langle a_{n_k} \rangle$ .

Therefore, if subsequence of Cauchy sequence converges then the sequence converges and it converges to the same value as the subsequence.

## 2.4. Arzela Dominated convergence Theorem for Riemann Integral

Riemann's definition of a definite integral gave rise to a number of important developments in analysis. The geometric problem that leads to the concept of Riemann integral is the following:

Given a bounded function  $f: [a, b] \rightarrow \mathbb{R}$ , then how to define the area of the region bounded by the graph of the function and the lines  $x = a$  and  $x = b$  ?

For example, if  $f(x) \geq 0$  and  $a \leq x \leq b$  and one would like to find the area of the region  $S(f) = \{ (x, y) \in \mathbb{R}^2 : a \leq x \leq b, 0 \leq y \leq f(x) \}$ .

This can be done by approximating the required area by the union of rectangular areas from the inside and outside of  $S(f)$ . The required area is, then, captured between these approximating areas. As the number of rectangles is increased, one gets better approximations and hopes to find the required area by a limiting process.

A finite set of points  $P = \{ x_0, x_1, \dots, x_n \}$  is called a partition of  $[a, b]$  if  $\{ a = x_0 < x_1 < x_2 < \dots < \dots x_n = b \}$

For a partition  $P = \{ x_0, x_1, \dots, x_n \}$  of  $[a, b]$ ,  $\|P\| = \max_{1 \leq i \leq n} \{ (x_i - x_{i-1}) \}$  is called the norm of the partition  $P$ .

Let  $P = \{ x_0, x_1, \dots, x_n \}$  be a partition of  $[a, b]$

Let  $m_i = \inf \{ f(x) : x \in [x_{i-1}, x_i] \}$  and  $M_i = \sup \{ f(x) : x \in [x_{i-1}, x_i] \}$

Define  $L(P, f) = \sum_{i=1}^n m_i \cdot (x_i - x_{i-1})$  and  $U(P, f) = \sum_{i=1}^n M_i \cdot (x_i - x_{i-1})$

Since  $f$  is assumed to be bounded, then, each  $m_i$  and  $M_i$  for  $0 \leq i \leq n$  exists and hence each of  $L(P, f)$  and  $U(P, f)$  is well defined.

The numbers  $L(P, f)$  and  $U(P, f)$  are called, respectively, the upper sum and the lower sum of  $f$  with respect to the partition  $P$ . Geometrically,  $L(P, f)$  approximates the required area from the inside and  $U(P, f)$  approximates the required area from the outside.

**Definition:** Let  $f : [a, b] \rightarrow \mathbb{R}$  be a bounded function. The real numbers

$$\int_a^b f(x) dx = \sup \{ L(P, f) : P \text{ is a partition of } [a, b] \} \quad \text{and}$$

$\int_a^b f(x) dx = \inf \{ U(P, f) : P \text{ is a partition of } [a, b] \}$  are called the lower integral and the upper integral respectively.

The function  $f$  is said to be Riemann integrable on  $[a, b]$  if  $\int_a^b f(x) dx = \int_a^b f(x) dx$

The common value in this case is called the Riemann integral of  $f$  over  $[a, b]$  and is defined by  $\int_a^b f(x)dx = \int_{\underline{a}}^b f(x)dx = \int_a^{\bar{b}} f(x)dx$ .

**Examples:**

1. Let  $f : [0, a] \rightarrow \mathbb{R}$  be defined by  $f(x) = x$ .

Consider the partition  $P_n = \left\{ 0, a/n, 2a/n, \dots, (n-1) \cdot a/n, a \right\}$  where  $n \geq 1$

$$\begin{aligned} \text{Now, } U(P_n, f) &= \sum_{j=1}^n \left( j \cdot \frac{a}{n} \right) \cdot \frac{a}{n} \\ &= \frac{a^2}{n^2} \cdot \sum_{j=1}^n j = \frac{a^2}{n^2} \cdot \left( \frac{n(n+1)}{2} \right) = \frac{a^2}{2} \cdot \left( \frac{n+1}{n} \right) = \frac{a^2}{2} \cdot \left( 1 + \frac{1}{n} \right) \end{aligned}$$

Similarly,

$$\begin{aligned} L(P_n, f) &= \sum_{j=1}^n \left( (j-1) \cdot \frac{a}{n} \right) \cdot \frac{a}{n} \\ &= \frac{a^2}{n^2} \cdot \sum_{j=1}^n (j-1) = \frac{a^2}{n^2} \cdot \left( \frac{n(n-1)}{2} \right) = \frac{a^2}{2} \cdot \left( \frac{n-1}{n} \right) = \frac{a^2}{2} \cdot \left( 1 - \frac{1}{n} \right) \end{aligned}$$

$$\text{Then, } \sup_{n \geq 1} \{ L(P_n, f) \} = \lim_{n \rightarrow \infty} \frac{a^2}{2} \cdot \left( \frac{n-1}{n} \right) = \lim_{n \rightarrow \infty} \frac{a^2}{2} \cdot \left( 1 - \frac{1}{n} \right) = \frac{a^2}{2}$$

$$\text{Similarly, } \inf_{n \geq 1} \{ U(P_n, f) \} = \lim_{n \rightarrow \infty} \frac{a^2}{2} \cdot \left( \frac{n+1}{n} \right) = \lim_{n \rightarrow \infty} \frac{a^2}{2} \cdot \left( 1 + \frac{1}{n} \right) = \frac{a^2}{2}$$

Thus, by definition, we have got  $\int_0^a f(x)dx = \frac{a^2}{2} = \int_0^{\bar{a}} f(x)dx$

Hence,  $f$  is Riemann integrable and  $\int_0^a f(x)dx = \frac{a^2}{2}$ .

2. Let  $f : [a, b] \rightarrow \mathbb{R}$  be defined by

$$f(x) = \begin{cases} 1, & x \in \mathbb{Q} \\ 0, & x \notin \mathbb{Q} \end{cases} \quad \text{Where } a \leq x \leq b$$

Then,  $f$  is not Riemann integrable.

**Solution:** It is easy to see that for any partition  $P$  of  $[a, b]$  such that  $U(P, f) = b - a$  and  $L(P, f) = 0$ .

Then, applying the definition, we get  $\int_a^b f(x)dx = 0 \neq \int_a^{\bar{b}} f(x)dx = 1$

.

Therefore, by definition, we conclude that  $f$  is not Riemann integrable.

**Definition:** - Let  $f : [a, b] \rightarrow \mathbb{R}$  be a bounded function. Then,  $f$  is Riemann integrable if and only if for every  $\varepsilon > 0$  there exists a partition  $P$  of  $[a, b]$  such that  $U(P, f) - L(P, f) < \varepsilon$ .

**Theorem 5:** Let  $f : [a, b] \rightarrow \mathbb{R}$  be a continuous function, then,  $f$  is Riemann integrable.

**Proof:** Given  $f$  is continuous and a continuous function on a bounded interval is bounded. Therefore,  $f$  is bounded and uniformly continuous. Then, from the definition of Uniform continuity, let  $\varepsilon > 0$  be arbitrary, we choose  $\delta > 0$  such that  $|f(x) - f(y)| < \varepsilon / (b - a)$  whenever  $|x - y| < \delta$ .

Let  $P = \{x_0, x_1, \dots, x_n\}$  be any partition of  $[a, b]$  such that  $\|P\| < \delta$

Since  $f$  is continuous, it attains its maximum and minimum values  $M_i$  and  $m_i$  respectively on the interval  $[x_{i-1}, x_i]$  at some points, say  $f(y_i) = M_i$  and  $f(z_i) = m_i$  for  $1 \leq i \leq n$  where  $y_i, z_i \in [x_{i-1}, x_i]$ .

But,  $M_i - m_i = f(y_i) - f(z_i) = |f(y_i) - f(z_i)| < \varepsilon / (b - a)$  whenever  $1 \leq i \leq n$

Then, we obtain  $U(P, f) - L(P, f) = \sum_{i=1}^n (M_i - m_i) \cdot (x_i - x_{i-1})$

Which is, then, implies  $\sum_{i=1}^n (M_i - m_i) \cdot (x_i - x_{i-1}) < \varepsilon / (b - a) \cdot \sum_{i=1}^n (x_i - x_{i-1})$

Since  $\sum_{i=1}^n (x_i - x_{i-1}) = b - a$ , then, by substituting this value, we have the result

$U(P, f) - L(P, f) = \sum_{i=1}^n (M_i - m_i) \cdot (x_i - x_{i-1}) < \varepsilon / (b - a) \cdot \sum_{i=1}^n (x_i - x_{i-1}) = \varepsilon$

Which is, then, implies  $U(P, f) - L(P, f) < \varepsilon$

Therefore, by definition, we end up that  $f$  is Riemann integrable.

### Characterization of Riemann integrable functions

Let  $\mathcal{R}[a, b]$  denotes the set of all functions  $f : [a, b] \rightarrow \mathbb{R}$  which are Riemann integrable. Let  $f, g : [a, b] \rightarrow \mathbb{R}$  be bounded Riemann integrable functions and  $\alpha$  be any real number, then

i.  $f + g$  is also Riemann integrable and

$$\int_a^b (f + g)(x)dx = \int_a^b f(x)dx + \int_a^b g(x)dx$$

ii.  $\alpha \cdot f$  is Riemann integrable and  $\int_a^b (\alpha \cdot f)(x)dx = \alpha \cdot \int_a^b f(x)dx$

iii.  $f \vee g$  and  $f \wedge g$  are also Riemann integrable functions.

From these and such we can conclude that  $\mathcal{R}[a, b]$  is a vector lattice over  $\mathbb{R}$  and the map  $f \rightarrow \int_a^b f(x)dx$  is linear.

### Drawbacks of Riemann integral

Some of the main drawbacks of Riemann integrals are:

1. It is defined only for bounded functions.
2. It is defined for functions on bounded intervals only.
3. Functions, even if they are defined on a bounded interval and have a finite range i.e. are bounded functions (e.g. Dirchlet's function) are not necessarily integrable.

### Convergence of Riemann integral

Let  $\langle f_n \rangle$  be a sequence of Riemann integrable functions on an interval  $[a, b]$  and let  $f_n(x) \rightarrow f(x)$  for all  $x \in [a, b]$ . Can we say that  $f$  is Riemann integrable on  $[a, b]$  and  $\lim_{n \rightarrow \infty} \int_a^b f_n(x)dx = \int_a^b f(x)dx$  ?

**Example 1:-** Let  $f_n(x) = n \cdot e^{-nx}$  for  $x \in [0, 1]$  where  $n = 1, 2, \dots$ . Then, show that each  $f_n$  is Riemann integrable and  $\lim_{n \rightarrow \infty} f_n(x) = f(x) = 0$  for all  $x \in [0, 1]$ . However,  $\int_0^1 f_n(x)dx$  does not converge to  $\int_0^1 f(x)dx$ .

**Solution:** - (i). To show each  $f_n$  is Riemann integrable, it is suffice to show that each  $f_n$  is continuous. Clearly one can easily see that each  $f_n(x) = n \cdot e^{-nx}$  is continuous for every  $x \in [0,1]$  and  $n = 1,2, \dots$

Therefore, by Theorem 5,  $f_n$  is Riemann integrable.

(ii). To show the convergence of the Riemann integrable function, we are going to apply L'Hopitals rule. That is,  $\lim_{n \rightarrow \infty} f_n(x) = \lim_{n \rightarrow \infty} n \cdot e^{-nx} = \lim_{n \rightarrow \infty} \frac{n}{e^{nx}} = 0$  for all  $x \in [0,1]$ . As a result, we obtain the limit value equals to zero. Now, consider  $f(x) = 0$  for all  $x \in [0,1]$  and, consequently,  $f(x) = 0$  constant function is Riemann integrable.

(iii). For each  $n$ ,  $f_n$  is bounded, however, there is no positive number  $M$  such that  $|f_n(x)| \leq M$  for all  $x \in [0,1]$  and for all  $n = 1,2,3, \dots$  Thus,  $f_n(x)$  is not bounded.

Now let us integrate each and see if  $\int_0^1 f_n(x)dx$  converges to  $\int_0^1 f(x)dx$

$$\int_0^1 f_n(x)dx = \int_0^1 n \cdot e^{-nx} dx = -e^{-nx} \Big|_0^1 = -[e^{-n} - 1] = [1 - e^{-n}]$$

$$\text{Therefore, } \lim_{n \rightarrow \infty} \int_0^1 f_n(x)dx = \lim_{n \rightarrow \infty} (1 - e^{-n}) = 1$$

$$\text{Since } f(x) = 0 \text{ then } \int_0^1 f(x)dx = 0 .$$

Therefore, from this we can see that  $\int_0^1 f_n(x)dx$  does not converge to  $\int_0^1 f(x)dx$  .

**Observation 1:-** (From the above example)

$f_n$  and  $f$  are Riemann integrable and  $\lim_{n \rightarrow \infty} f_n(x) = f(x)$  for all  $x$  and  $n = 1,2, \dots$   
 However, we obtain that  $\int_0^1 f_n(x)dx$  does not converge  $\int_0^1 f(x)dx$  .

**Example2:-** Let  $f_n(x) = x^n$  where  $0 \leq x \leq 1$  and  $n = 1,2,3, \dots$

Each  $f_n$  is Riemann integrable on  $[0,1]$  and  $f_n(x)$  converges for every  $x \in [0,1]$  .

Then, can we conclude that  $\lim_{n \rightarrow \infty} \int_0^1 f_n(x)dx = \int_0^1 f(x)dx$  ?

**Solution:** Since each  $f_n$  is continuous for every  $x \in [0,1]$  . Therefore, by Theorem 5, each  $f_n$  is Riemann integrable on  $[0,1]$  .

Moreover, there exists positive number  $M$  such that  $|f_n(x)| \leq M$  for all  $n$  and for all  $x$

And,  $\lim_{n \rightarrow \infty} f_n(x) = \lim_{n \rightarrow \infty} x^n = f(x) = \begin{cases} 0, & x \in [0,1) \\ 1, & x = 1 \end{cases}$  is not Riemann integrable

This implies  $f_n$  converges uniformly to  $f(x)$  on  $[0,1]$ .

However, by definition,  $f(x) = \begin{cases} 0, & x \in [0,1) \\ 1, & x = 1 \end{cases}$  is not Riemann integrable; for the

values of  $L(P, f) = 0$  and  $U(P, f) = 1$ . Then, we obtain the result  $\int_0^1 f(x) dx = 0$  and

$\int_0^1 f(x) dx = 1$  respectively. Then, this implies  $0 = \int_0^1 f(x) dx \neq 1 = \int_0^1 f(x) dx$

Thus, we conclude that  $\int_0^1 f_n(x) dx$  does not converge to  $\int_0^1 f(x) dx$ .

**Observation 2:-**

$f_n$  is Riemann integrable, bounded and converges point wise i.e.  $\lim_{n \rightarrow \infty} f_n(x) = f(x)$

However, yet the function  $f(x)$  is not Riemann integrable and in effect  $\int_0^1 f_n(x) dx$

does not converge to  $\int_0^1 f(x) dx$ .

These and such a fore mentioned problems show the need for suffice condition which can take care of the convergence of Riemann integrable functions and for this we obtain sufficient condition by Arzela. The result of Arzela we have in mind is the so-called Arzela- Dominated convergence theorem. In particular, Arzela examined the problem of finding necessary and sufficient conditions for the integrability of the point wise limit of the sequence of integrable functions, of finding the correct mode of convergence that would preserve integrability, and of the validity of term-by-term integration of series.

**Theorem 6 :- (Arzela dominated convergence theorem)**

Let  $\langle f_n \rangle$  be a sequence of Riemann integrable functions defined on  $[a, b]$  which converges on  $[a, b]$  to a Riemann integrable function  $f$ . If there exists a constant  $M > 0$  such that  $|f_n(x)| \leq M$  for all  $x \in [a, b]$  and for all  $n = 1, 2, \dots$ . Then,

$$\lim_{n \rightarrow \infty} \int_a^b |f_n(x) - f(x)| dx = 0$$

$$\text{In particular, } \lim_{n \rightarrow \infty} \int_a^b f_n(x) dx = \int_a^b (\lim_{n \rightarrow \infty} f_n(x)) dx = \int_a^b f(x) dx$$

**Proof:** - To prove the Arzela's dominated convergence theorem, as there is no short and one proof of the theorem, it is very essential to consider the following theorem and lemmas.

**Theorem 7 (Dini's- Uniform Convergence Theorem)**

Each Monotone Sequence  $\langle f_n \rangle$  of continuous functions that converges point wise to a continuous function  $f$  on a bounded and closed interval  $[a, b]$  is uniformly convergent.

**Proof:** Let  $\varepsilon > 0$  be given. For each  $n$ , let  $g_n = f - f_n$ , and let  $E_n$  be those  $x \in [a, b]$  such that  $g_n(x) < \varepsilon$ . Plainly, each  $E_n$  is open. Since  $\{f_n\}$  is monotonically increasing, then,  $g_n$  is monotonically decreasing. It follows that the sequence  $E_n$  is ascending. Since  $f_n$  converges point wise to  $f$ , it follows that the collection  $\{E_n\}$  is an open cover of  $[a, b]$ .

By compactness, we obtain that there is some positive integer  $N$  such that  $E_N = [a, b]$ . That is, if  $n \geq N$  and  $x$  is a point in  $[a, b]$ , then,  $|f(x) - f_n(x)| < \varepsilon$  as desired.

**Lemma 5:** For each  $0 \leq f \in B[a, b]$  and for each  $\varepsilon > 0$ , there exists a continuous function  $g \in C[a, b]$  satisfying  $0 \leq g \leq f$  and  $\int_a^b f(x)dx \leq \int_a^b g(x)dx + \varepsilon$

**Proof:** From the definition of lower integral, it follows that for each  $\varepsilon > 0$  there exists a step function  $S$  on  $[a, b]$  satisfying  $0 \leq S \leq f$  and  $\int_a^b f(x)dx \leq \int_a^b S(x)dx + \varepsilon/2$

It is easy to verify that  $S$  can be transformed into a trapezoidal (trapezium)  $g$  Such that  $0 \leq g \leq S$  and  $\int_a^b S(x)dx \leq \int_a^b g(x)dx + \varepsilon/2$

Then, we obtain  $\int_a^b S(x)dx + \varepsilon/2 \leq \int_a^b g(x)dx + \varepsilon$

Hence, there exists a continuous function  $g \in C[a, b]$  satisfying  $0 \leq g \leq f$  and  $\int_a^b f(x)dx \leq \int_a^b g(x)dx + \varepsilon$

**Lemma 6:** Let  $\langle f_n \rangle$  be a decreasing sequence of bounded functions on  $[a, b]$ . If  $\lim_{n \rightarrow \infty} f_n(x) = 0$  for all  $x \in C[a, b]$ , then,  $\lim_{n \rightarrow \infty} \int_a^b f_n(x) dx = 0$ .

**Proof:** Since  $\langle f_n \rangle$  is a sequence of bounded functions on  $[a, b]$ , it follows from Lemma 5 that for  $\varepsilon > 0$  and for all  $n$ , there exists a continuous function  $g_n \in C[a, b]$  such that

$$0 \leq g_n \leq f_n \quad \text{and} \quad \int_a^b f(x) dx \leq \int_a^b g(x) dx + \varepsilon/2^n \quad (11)$$

For each  $n$ , we set  $h_n = \min \{ g_1, g_2, \dots, g_n \}$

Then,  $0 \leq h_n \leq g_n \leq f_n$  where  $h_n \in C[a, b]$  and using Squeezing theorem, we obtain that  $\lim_{n \rightarrow \infty} h_n(x) = 0$  which implies the sequence  $\langle h_n \rangle$  decreases to zero everywhere on  $[a, b]$ .

Hence, by Dini's Uniform Convergence theorem, the sequence  $\langle h_n \rangle$  converges uniformly to zero on  $[a, b]$  and consequently  $\lim_{n \rightarrow \infty} \int_a^b h_n(x) dx = 0$ .

Now the proof of the Lemma will be finished if the following inequalities are established That is, for each  $n$ ,

$$0 \leq \int_a^b f_n(x) dx \leq \int_a^b h_n(x) dx + \varepsilon \left( 1 - (1/2^n) \right) \quad (12)$$

To this end we shall first prove the following inequalities

$$\text{For each } n, \quad 0 \leq g_n \leq h_n + \sum_{i=1}^{n-1} \{ \max(g_i, \dots, g_n) - g_i \} \quad (13)$$

The inequality (13) follows easily by observing that for each  $i$  where  $1 \leq i \leq n$

$$\begin{aligned} 0 \leq g_n &= g_i + (g_n - g_i) \leq g_i + \{ \max(g_i, \dots, g_n) - g_i \} \\ &\leq g_i + \sum_{i=1}^{n-1} \{ \max(g_i, \dots, g_n) - g_i \} \end{aligned} \quad (14)$$

Since (14) holds for any arbitrary  $i, 1 \leq i \leq n$ . Then taking  $h_n = g_i$ , then (13) will hold true. That is, we obtain the desired

$$0 \leq g_n \leq h_n + \sum_{i=1}^{n-1} \{ \max(g_i, \dots, g_n) - g_i \}$$

Since  $g_n \leq f_n$  for all  $n$ , then, in effect we have

$$\max(g_i, \dots, g_n) \leq \max(f_i, \dots, f_n) = f_i \quad (\text{Since } f_n \text{ is decreasing})$$

This implies  $\{ \max(g_i, \dots, g_n) - g_i \} + g_i \leq \max(f_i, \dots, f_n) = f_i$

It follows that  $\int_a^b f_i(x)dx \geq \int_a^b \{ \max(g_i, \dots, g_n) - g_i \} dx + \int_a^b g_i(x)dx$

$$\begin{aligned} \int_a^b \{ \max(g_i, \dots, g_n) - g_i \} dx &\leq \int_a^b f_i(x)dx - \int_a^b g_i(x)dx \\ &\leq \varepsilon/2^i \quad \text{for } i = 1, 2, \dots \quad (\text{from 11 above}) \end{aligned}$$

Hence, by (13), for each  $n$ , we have

$$\begin{aligned} \int_a^b g_n(x)dx &\leq \int_a^b h_n(x)dx + \sum_{i=1}^{n-1} \varepsilon/2^i \\ &= \int_a^b h_n(x)dx + \varepsilon \left( 1 - (1/2^n) \right) \end{aligned} \quad (15)$$

Finally from (11) we have  $\int_a^b f(x)dx \leq \int_a^b g_n(x)dx + \varepsilon/2^n$  and (15) imply (12)

$$\text{i.e. } 0 \leq \int_a^b f(x)dx \leq \int_a^b h_n(x)dx + \varepsilon \left( 1 - (1/2^n) \right)$$

Taking  $n \rightarrow \infty$

We obtain the result  $0 \leq \lim_{n \rightarrow \infty} \int_a^b f_n(x)dx \leq 0$ .

Then, by squeezing theorem, we have  $\lim_{n \rightarrow \infty} \int_a^b f_n(x)dx = 0$

**Remark:**

Lemma 6 for Riemann integrable functions is already Arzela's dominated convergence theorem for monotone sequences.

Now we shall turn to the proof of Arzela's dominated convergence theorem. To this end, it is no loss in general to assume that  $0 \leq f_n(x) \leq M$  for all  $n$  and for all  $x \in [a, b]$  where  $M > 0$  is a real number. This, then, implies  $|f_n(x)| \leq M$  for all  $n$  and for all  $x \in [a, b]$  and that  $f(x) = \lim_{n \rightarrow \infty} f_n(x) = 0$  for all  $x \in [a, b]$ .

For each  $n$  and for each  $x \in [a, b]$ , set  $P_n(x) = \sup_{k \geq 0} (f_{n+k}(x))$ . Then, we obtain  $0 \leq f_n(x) \leq P_n(x)$  and the sequence  $P_n$  decreases everywhere to zero on  $[a, b]$ . Indeed,  $0 = \lim_{n \rightarrow \infty} f_n(x) = \lim_{n \rightarrow \infty} \sup \{ f_n(x) \} = \lim_{n \rightarrow \infty} P_n(x)$  for all  $x \in [a, b]$ .

Hence, by Lemma 6, we have  $\lim_{n \rightarrow \infty} \int_a^b P_n(x) dx = 0$

And so,  $0 \leq \lim_{n \rightarrow \infty} \int_a^b f_n(x) dx \leq \lim_{n \rightarrow \infty} \int_a^b P_n(x) dx = 0$

Therefore, from Squeezing (Sandwich) theorem, we obtain the desired result  $\lim_{n \rightarrow \infty} \int_a^b f_n(x) dx = 0$  and the proof is finished.

Having established lemma 6 for Riemann integrable functions, we have, in fact, proven the Arzela's theorem for monotone sequences. Now the question which we have to answer is whether we can deduce Arzela's theorem directly from its special case for monotone sequences. It is very interesting that this is indeed true. For the sake of completeness, we shall show how this can be done. In order to bring out more dramatically that Arzela's theorem is a logical consequence from the special case for monotone sequences we shall adopt the following abstract settings.

Let  $X$  be a non- empty set and  $\mathcal{L}$  be a linear space (vector space) of all real functions defined on  $X$  satisfying  $f \in \mathcal{L}$  implies  $|f| \in \mathcal{L}$ .

The latter condition implies that for every finite set of elements  $\{f_1, f_2, \dots, f_n\}$  of  $\mathcal{L}$ ,  $\max\{f_1, f_2, \dots, f_n\} \in \mathcal{L}$  and  $\min\{f_1, f_2, \dots, f_n\} \in \mathcal{L}$ .

A positive linear functional  $I$  on  $\mathcal{L}$  is called an integral whenever  $I$  has the following properties:

If  $0 \leq f_n$  for each  $n$  and the sequence  $\langle f_n \rangle$  decreases to zero everywhere on  $X$ , then  $\lim_{n \rightarrow \infty} I(f_n) = 0$  (16)

It is obvious that (16) is lemma 6 for  $I$  and  $\mathcal{L}$ . We shall now show that (15) implies the following (abstract) Arzela's type theorem.

**Theorem 8:** Let  $f \in \mathcal{L}$  be the limit of an everywhere on  $X$  convergent sequence  $\langle f_n \rangle$  of elements of  $\mathcal{L}$ . If there exists an element  $0 \leq g \in \mathcal{L}$  satisfying  $|f_n(x)| \leq M$  for all  $x \in X$  and for all  $n$ , then for every integral  $I$  on  $\mathcal{L}$ . We have  $\lim_{n \rightarrow \infty} I|f_n - f| = 0$  or in particular  $\lim_{n \rightarrow \infty} I(f_n) = I(f)$ .

**Proof:** By considering the sequence  $\{|f_n(x) - f(x)|\}$ , we have

$$\begin{aligned} |f_n(x) - f(x)| &\leq |f_n(x)| + |f(x)| \quad \dots\dots\dots \text{Triangular inequality} \\ &\leq g(x) + |f(x)| \quad \text{for all } x \in X \text{ and for all } n, \text{ where} \\ &|f| + g \in \mathcal{L}. \end{aligned}$$

We may assume without loss of generality that  $f_n(x) \geq 0$  for all  $x \in X$  and for all  $n$  and that  $f(x) = 0$  for all  $x \in X$ .

Following, we set  $g_{n,m} = \max\{f_n, f_{n+1}, \dots, f_m\}$  for each pair of indices  $m \geq n$ .

Then,  $0 \leq g_{n,m} \in \mathcal{L}$  and  $0 \leq g_{n,m} \leq g$  for all  $m \geq n$ .

Furthermore, for each  $n$ , the sequence  $\{g_{n,m}\}$  is an increasing and, consequently, the sequence  $\{I(g_{n,m})\}$  is an increasing and bounded for  $m \geq n$ .

Hence, for each  $\varepsilon > 0$  and for each  $n$ , there exists an index  $m_n \geq n$  such that  $m_n \leq m_{n+1}$  and  $0 \leq I(g_{n,k}) - I(g_{n,m}) \leq \varepsilon/2^n$  for all  $k \geq m_n$ . (17)

For the sake of simplicity we set  $U_n \leq g_{n,m_n}$ . Then,

$0 \leq \lim_{n \rightarrow \infty} \sup U_n(x) \leq \lim_{n \rightarrow \infty} \sup f_n(x) \leq \lim_{n \rightarrow \infty} f_n(x) = 0$  for all  $x \in X$  implies that  $\lim_{n \rightarrow \infty} U_n(x) = 0$  for all  $x \in X$ .

If we apply the inequalities (13) to the sequence  $\langle U_n \rangle$  then we obtain for each  $n$ ,

$$0 \leq f_n \leq U_n \leq \min\{U_i, \dots, U_n\} + \sum_{i=1}^{n-1} \{\max(U_i, \dots, U_n) - U_i\} \quad (18)$$

Since  $\max(U_i, \dots, U_n) - U_i = \max(f_i, \dots, f_{m_n}) - U_i = g_{i,m_n} - g_{i,m_i}$  and  $m_n \geq m_i$  for  $m \geq i$ .

We conclude that  $I\{\max(U_i, \dots, U_n) - U_i\} \leq \varepsilon/2^i$  for  $1 \leq i \leq n$  and so by (18),

for each  $n$ , we have

$$0 \leq I(f_n) \leq I(\min(U_1, \dots, U_n)) + \varepsilon \left(1 - \left(1/2^{n-1}\right)\right) \quad (19)$$

Since  $\lim_{n \rightarrow \infty} U_n(x) = 0$  for all  $x \in X$ , it follows that the sequence  $\{\min(U_1, \dots, U_n)\}$  decreases everywhere to zero on  $X$ . Thus, by hypothesis  $\lim_{n \rightarrow \infty} I\{\min(U_1, \dots, U_n)\} = 0$  and finally using (19) we obtain that  $\lim_{n \rightarrow \infty} I(f_n) = 0$  and the proof is finished.

As we see from Arzela dominated convergence theorem, one has to impose quite strong conditions to ensure the convergence of Riemann integrals. For example, one has to assume the limit function is also Riemann integrable.

**Example:** Let  $f_n : [0,1] \rightarrow \mathbb{R}$  be defined by  $f_n(x) = \begin{cases} 1, & \text{if } x \neq \frac{k}{2^n} \\ 0, & \text{if } x = \frac{k}{2^n} \end{cases}$  for some  $k$

where  $1 \leq k \leq 2^n - 1$  and any  $n$ .

Note that for all  $n$  and for all  $x \in [0,1]$ , we have  $0 \leq f_n(x) \leq 1$  and  $f_n \in \mathcal{R}[0,1]$  where  $n = 1, 2, 3, \dots$

Let  $f : [0,1] \rightarrow \mathbb{R}$  be defined by  $f(x) = \begin{cases} 1, & \text{if } x \neq \frac{k}{2^n} \\ 0, & \text{if } x = \frac{k}{2^n} \end{cases}$  for any  $n$  and

$1 \leq k \leq 2^n$ .

Then we can see that  $f_n(x) \rightarrow f(x)$  for all  $x \in [0,1]$  but  $f$  is not Riemann integrable. From this questions we comprehend that Arzela's dominated convergence theorem is not enough to give answer when  $f$  or  $f_n$  is not Riemann integrable. Now, let us restate the theorem in a way we clearly understand the conditions and be able to compare and contrast the theorem with other companion theorem.

### Restatement of Arzela's Dominated Convergence Theorem

Define a map  $f_n, f : [a, b] \rightarrow \mathbb{R}$

#### Conditions:-

1.  $f_n$  and  $f$  are Riemann integrable functions defined on  $[a, b]$ .
2.  $|f_n(x)| \leq M$  for all  $x$  and for all  $n$ .
3.  $\lim_{n \rightarrow \infty} f_n(x) = f(x)$  for all  $x \in [a, b]$ .

Then  $\int_a^b f(x)dx = \int_a^b \lim_{n \rightarrow \infty} f_n(x).dx = \lim_{n \rightarrow \infty} \int_a^b f_n(x)dx$

Now let us define a map  $I : \mathcal{R}[a, b] \rightarrow \mathbb{R}$  by  $I(f) = \int_a^b f(x)dx$ , then, we can show that  $I$  is a vector lattice and positive linear functional.

**Solution:** By definition,  $I$  is a vector lattice if for any  $f, g \in \mathcal{R}[a, b]$  and  $\alpha \in \mathbb{R}$ , then  $f + g, f \vee g, f \wedge g$  and  $\alpha f$  is also Riemann integrable on  $[a, b]$ . But, from integral properties  $f + g, f \vee g, f \wedge g$  and  $\alpha f$  are Riemann integrable. This implies,  $I$  is a vector lattice.

Next, let us check if  $I$  is a positive linear functional, that is, for  $f, g \in \mathcal{R}[a, b]$  and  $\alpha, \beta \in \mathbb{R}$ , then, we need to show if

$$I(\alpha f + \beta g) = \int_a^b (\alpha f + \beta g)dx = \alpha \int_a^b f(x)dx + \beta \int_a^b g(x)dx = \alpha I(f) + \beta I(g)$$

and  $I(f) \geq 0$  implies  $\int_a^b f(x)dx \geq 0$  provided that  $f \geq 0$ .

Therefore,  $I$  is a positive linear functional.

Now let us make some modification on the map of Arzela's theorem or express the Arzela's Dominated Convergence theorem in another way; keeping the other conditions intact and see what we will obtain:-

**Modified Arzela's Dominated Convergence theorem (Another way of expressing Arzela's theorem):-**

Define  $I : \mathcal{R}[a, b] \rightarrow \mathbb{R}$  by  $I(f) = \int_a^b f(x)dx$ , where  $I$  is a vector lattice and positive linear functional.

**Conditions:**

- (i)  $f_n$  and  $f$  belong to  $\mathcal{R}[a, b]$ .
- (ii) There exists  $M \in \mathbb{R}$  and  $M > 0$  such that for all  $x \in [a, b]$  and for all  $n$ .
- (iii).  $\lim_{n \rightarrow \infty} f_n(x) = f(x)$  for all  $x \in [a, b]$ .

Then  $\lim_{n \rightarrow \infty} I(f_n) = I(f)$

This modification shows Arzela's dominated convergence theorem holds true for a positive linear functional map if only the positive linear function mapping is defined with Riemann integration form. In other words, say  $I : \mathcal{R}[a, b] \rightarrow \mathbb{R}$  be a positive linear functional, then, for Arzela dominated convergence theorem to apply for the map,  $I$  must only be defined in the form of  $I(f) = \int_a^b f(x)dx$ .

We have seen that the Arzela dominated convergence theorem works for a specific positive linear functional defined only in an integral form. To this effect, we can infer that Arzela is specific suffice condition of Riemann convergence theorem. So, we need a general and extended sufficient condition of Riemann convergence theorem which holds true for any positive linear functional.

**Question of Interest:-** Can we find a sufficient condition in which the Riemann convergence holds true for any positive linear functional?

Yes, we can and this convergence of Riemann integrable functions for any positive linear functional under the frame work of compact Hausdorff Space is made possible by a great mathematician W.F.Eberlein, 1957. The classical convergence theorem developed by W.F.Eberlein is an extension of Arzela's dominated convergence theorem.

### 2.5. The Convergence theorem by W.F.Eberlein

Let  $C$  denotes the (algebra and lattice of) real valued continuous function on a compact Hausdorff Space  $S$ .

An integral over  $S$  is a positive linear functional on  $S$  i.e. a mapping  $I : C \rightarrow \mathbb{R}$  satisfying the following conditions:-

$$(a). I(\alpha f + \beta g) = \alpha I(f) + \beta I(g) \text{ where } f, g \in C \text{ and } \alpha, \beta \in \mathbb{R}.$$

$$(b). I(f) \geq 0 \text{ whenever } f \geq 0 \tag{18}$$

**Example:** Define a mapping  $I : C \rightarrow \mathbb{R}$  such that  $I(f) = \int_a^b f(x)dx$ , then,  $I$  is an integral over  $S$ .

**Solution:** We need to check the case in (18) is satisfied.

From the given, we have  $I(\alpha f + \beta g) = \int_a^b (\alpha f + \beta g) dx$

But,  $\int_a^b (\alpha f + \beta g) dx = \int_a^b (\alpha f) dx + \int_a^b (\beta g) dx$

Again,  $\int_a^b (\alpha f) dx + \int_a^b (\beta g) dx = \alpha \int_a^b f dx + \beta \int_a^b g dx$

Then, by definition,  $\alpha \int_a^b f dx + \beta \int_a^b g dx = \alpha I(f) + \beta I(g)$

Thus, we, then, end up with  $I(\alpha f + \beta g) = \alpha I(f) + \beta I(g)$  which implies  $I$  is linear functional.

$I(f) \geq 0$  implies  $\int_a^b f(x) dx \geq 0$  whenever  $f \geq 0$ . Thus,  $I(f) \geq 0$  whenever  $f \geq 0$ . Hence,  $I$  is an integral over  $S$ .

So far, we have seen that  $I$  is a positive linear functional and an integral over  $S$ . Furthermore, by Theorem 5, we know that continuous functions are Riemann integrable. So, having these points in mind let us see the classical underlying convergence theorem developed by W.F.Eberlein, its relation with Arzela's theorem and at last the proof will be given.

**Theorem 9 (W.F.Eberlein):-** Let  $\lim_{n \rightarrow \infty} f_n(x) = f(x)$  point wise ( $f_n, f$  in  $C$ ) and  $|f_n| \leq M$  for all  $n$  and a constant  $M > 0$ . Then  $\lim_{n \rightarrow \infty} I(f_n) = I(f)$ .

**Restatement (The extended theorem):** Define a map  $I: C(X) \rightarrow \mathbb{R}$  where  $I$  is a positive linear functional and  $X$  is a compact Hausdorff Space.

**Conditions:**

1.  $f_n, f \in C(X)$ . This, in the case of Arzela, is used to preserve  $f$  and  $f_n$  are Riemann integrable.
2.  $|f_n(x)| \leq M$  for all  $n$  and for all  $x$ .
3.  $\lim_{n \rightarrow \infty} f_n(x) = f(x)$  Point wise.

Then,  $\lim_{n \rightarrow \infty} I(f_n) = I(f)$

**Relationship with Arzela’s dominated convergence theorem:**

1. In case of Arzela Convergence theorem the mapping becomes  $I : \mathcal{R}[a, b] \rightarrow \mathbb{R}$ .
2.  $f_n, f \in C(X)$  Which implies  $f$  and  $f_n$  are Riemann integrable.
3.  $\lim_{n \rightarrow \infty} f_n(x) = f(x)$  Point wise.
4. Define  $I(f) = \int_a^b f(x)dx$ .
5. There exists  $M \in \mathbb{R}$  and  $M > 0$  such that  $|f_n(x)| \leq M$  for all  $x$  and for all  $n$ .

Then by using the theorem developed by W.F.Eberlein, we obtain the following result  $\int_a^b f(x)dx = I(f) = \lim_{n \rightarrow \infty} I(f_n) = \lim_{n \rightarrow \infty} \int_a^b f_n(x)dx$  which shows that Arzela’s dominated convergence theorem is satisfied and it is from this result we infer that the W.F.Eberlein theorem is an extended and general case for Arzela dominated convergence theorem.

**Proof:** To prove the classical convergence theorem, there is no short and one proof. Consequently, what we need to do is consider the following two Lemmas and other previously proved facts and gradually we will come up with the proof of the theorem.

**Lemma 7 (Stone’s theory of integration)**

If  $|f| \leq \sum_{i=1}^n |f_n|$  where  $f_n$  and  $f$  in  $C$ , then,  $I(|f|) \leq \sum_{n=1}^{\infty} I(|f_n|)$

**Proof:** Given  $\varepsilon > 0$  and  $x$  in  $S$ , then, there exists a positive integer  $N(x)$  such that  $|f(x)| < \sum_{n=1}^{N(x)} |f_n(x)| + \varepsilon$ , and the inequality persists in some neighborhood  $U(x)$  of  $x$  (because of continuity of  $C$ ).

Since  $S$  is compact, then, by definition, the open covering  $U(x)$  of  $x$  in  $S$  contains a finite sub covering  $\{U(x_j)\}$ . Let  $N$  be the largest  $N(x_j)$ .

Then, we have  $|f| \leq \sum_{n=1}^N |f_n| + \varepsilon$ .

Taking integral of both sides we obtain  $\int |f| \leq \int \sum_{n=1}^N |f_n| + \varepsilon$

From this we obtain  $\int |f| \leq \int \sum_{n=1}^N |f_n| + \int \varepsilon$

Which, then, we get  $\int |f| \leq \int \sum_{n=1}^N |f_n| + \varepsilon \int 1$

Then, it follows  $\int |f| \leq \int \sum_{n=1}^{\infty} |f_n| + \varepsilon \int 1$

We, then, get  $I(|f|) \leq \sum_{n=1}^{\infty} I(|f_n|) + \varepsilon \cdot I(1)$

This implies  $I(|f|) \leq \sum_{n=1}^{\infty} I(|f_n|) + \varepsilon$

Since  $\varepsilon > 0$  is arbitrary, then we get the desired result  $I(|f|) \leq \sum_{n=1}^{\infty} I(|f_n|)$ .

We now consider some preliminary reduction of the theorem itself. Without the loss of generality, we may assume  $f = 0$  and  $f_n \geq 0$  for all  $n$  (Otherwise replace  $f_n$  by  $|f_n - f|$ ). Since  $f_n$  is bounded, there exists a positive real number  $M$  such that  $|f_n| \leq M$ . Taking integral of both sides we obtain  $\int |f_n| \leq \int M$ . We know that  $f_n \geq 0$  and this implies  $|f_n| = f_n$ . Then, we get  $\int f_n \leq \int M$  which implies  $I(f_n) \leq I(M)$ . Since  $\overline{\lim}$  and  $\underline{\lim}$  always exists then there exists a positive real number  $M$  such that  $L = \lim_{n \rightarrow \infty} \sup I(f_n) \leq M$  exists and we need only show  $L = 0$ .

But for our case, let us assume  $\lim_{n \rightarrow \infty} I(f_n) = L$  and it is very suffice if we can show that  $L = 0$ . Then, the proposition (W.F.Eberlein theorem) is reduced into the following theorem.

**The Reduced Theorem:**

- Let (a).  $f_n$  lies in  $C$  for all  $n$ . (b).  $\sup_n \|f_n\|_{\infty} \leq M < +\infty$   
(c).  $\lim_{n \rightarrow \infty} f_n = 0$  point wise (d).  $f_n \geq 0$  for all  $n$ .  
(e).  $\lim_{n \rightarrow \infty} I(f_n) = L$

Then  $L = 0$ .

Our procedure is to establish the reduced theorem for the sequence  $\langle g_n \rangle$ . Let  $K_n$  be a convex hull of the set  $\{f_m : m \geq n\}$  i.e.  $K_n$  consists of all finite combinations  $\sum a_j \cdot f_{m_j}$  where each  $m_j \geq n$  where  $a_j \geq 0$  and  $\sum a_j = 1$ .

Now consider any sequence  $\langle g_n \rangle$  with  $g_n$  in  $K_n$ . Since each element of  $K_n$  comes from the Reduced theorem, then,  $\langle g_n \rangle$  satisfies the properties in the Reduced theorem (a) - (e). Set  $d_n = \inf\{\|g\|_2 : g \text{ in } K_n\}$  and since  $K_{n+1} \subseteq K_n$  (decreasing) it follows that  $d_n \leq d_{n+1} \leq M$  for all  $n$ . Every monotonic bounded sequence is convergent; consequently we can deduce that  $d = \lim_{n \rightarrow \infty} d_n$  exists.

Now choose  $\langle g_n \rangle$  in  $K_n$  such that  $\|g_n\|_2 \leq d_n + \frac{1}{n}$

**Question of interest 1:** Is  $\langle g_n \rangle$  Cauchy Sequence i.e. we need to show for  $m, n \geq N(\varepsilon)$  if  $\lim_{m,n} \|g_n - g_m\|_2 = 0$  ?

**Lemma 8:**  $\lim_{m,n} \|g_n - g_m\|_2 = 0$  where  $\langle g_n \rangle$  in  $K_n$ .

**Proof:** By the parallelogram identity, we have

$$\|g_n - g_m\|_2^2 = 2 \{\|g_n\|_2^2 + \|g_m\|_2^2\} - 4 \left\| \frac{1}{2}(g_n + g_m) \right\|_2^2$$

Now consider  $n \geq m$ , then by definition  $K_m = \{g_n : n \geq m\}$ . This implies,

$\frac{1}{2}(g_n + g_m)$  is in  $K_m$  because  $g_n$  and  $g_m$  are in  $K_m$ .

And  $g_m \leq \frac{1}{2}(g_n + g_m) \leq g_n$

Hence,  $\left\| \frac{1}{2}(g_n + g_m) \right\|_2 \geq \inf\{\|g\|_2 : g \text{ in } K_m\} = d_m$

Thus,

$$\begin{aligned} \|g_n - g_m\|_2^2 &= 2 \{\|g_n\|_2^2 + \|g_m\|_2^2\} - 4 \left\| \frac{1}{2}(g_n + g_m) \right\|_2^2 \\ &\leq 2 \left\{ \left( d_n + \frac{1}{n} \right)^2 + \left( d_m + \frac{1}{m} \right)^2 \right\} - 4d_m^2 \end{aligned}$$

Then taking the limit of both sides and since  $d = \lim_{n \rightarrow \infty} d_n$  exists, then, we obtain

$$\leq 2(d^2 + d^2) - 4d_m^2$$

And, since  $d = \lim_{m \rightarrow \infty} d_m$ , then we have  $2(d^2 + d^2) - 4d_m^2 \leq 2(2d^2) - 4d^2$

This implies  $\lim_{m,n} \|g_n - g_m\|_2 \leq 0$

Then, since by definition  $\| \cdot \| \geq 0$ , we obtain the desired result  $\lim_{m,n} \|g_n - g_m\|_2 = 0$

Therefore, from this we can conclude that  $\langle g_n \rangle$  is Cauchy sequence in  $L^2$ .

By Cauchy Criterion, the sequence  $\langle g_n \rangle$  converges. Hence by Lemma 1, it has a convergent subsequence.

**Remark:** - Since the sequence  $\langle g_n \rangle$  converges, then, by Lemma 1, it has a convergent subsequence and it converges to the same value the sequence converges.

Now choose  $\langle h_n \rangle$  as a subsequence of  $\langle g_n \rangle$  such that  $\sum_{m=1}^{\infty} \|h_m - h_{m+1}\|_2 < \infty$

**Question of Interest 2:-** Can we find a positive linear functional  $I$  such that the

$$\lim_{n \rightarrow \infty} I(h_n) = 0 .$$

Since  $\langle h_n \rangle$  is a subsequence of the sequence  $\langle g_n \rangle$  and  $\langle g_n \rangle$  satisfies all the properties in the reduced theorem, then, we can see that  $\langle h_n \rangle$  satisfies all the properties (a) to (e) too.

We, then, conclude by showing that  $L = \lim_{n \rightarrow \infty} I(f_n) = 0$

Now, since  $\langle h_n \rangle$  satisfies the reduced theorem properties (a) to (e). Then, by property (c), we obtain  $\lim_{n \rightarrow \infty} h_n = 0$  point wise. Then, by Lemma 1, we have  $\lim_{n \rightarrow \infty} g_n(x) = 0$ .

Next, we need to show that if  $\lim_{n \rightarrow \infty} I(h_n) = 0$ .

Consider,  $h_n = \sum_{m=1}^{\infty} (h_m - h_{m+1})$

Then, taking absolute value for both sides, we have

$$|h_n| = |\sum_{m=1}^{\infty} (h_m - h_{m+1})| \leq \sum_{m=1}^{\infty} |h_m - h_{m+1}|$$

Then, by Stone's lemma, we found  $I(|h_n|) \leq \sum_{m=1}^{\infty} I(|h_m - h_{m+1}|)$

And, by property (D),  $I(|h_m - h_{m+1}|) \leq \|h_m - h_{m+1}\|_2$

Now, from the definition of positive linear functional, we have  $0 \leq |I(h_n)| \leq I(|h_n|)$ . This, then, gives us the result

$$0 \leq |I(h_n)| \leq I(|h_n|) \leq \sum_{m=1}^{\infty} I(|h_m - h_{m+1}|) \leq \sum_{m=1}^{\infty} \|h_m - h_{m+1}\|_2$$

Since, from Lemma 8,  $\lim_{m,n} \|g_n - g_m\|_2 = 0$  and  $\langle h_n \rangle$  is a subsequence of  $\langle g_n \rangle$ , then, we have  $\lim_{n \rightarrow \infty} \sum_{m=1}^{\infty} \|h_m - h_{m+1}\|_2 = 0$ . That is, it converges to zero when  $n \rightarrow \infty$ .

$$0 \leq |I(h_n)| \leq I(|h_n|) \leq \sum_{m=1}^{\infty} I(|h_m - h_{m+1}|) \leq \sum_{m=1}^{\infty} \|h_m - h_{m+1}\|_2 = 0$$

Then, by Sandwich (Squeezing) theorem, we get  $\lim_{n \rightarrow \infty} I(|h_n|) = 0$ .

Now, since the subsequence  $\langle h_n \rangle$  satisfies the reduced theorem properties (a) to (e), then, by property (d), we have  $h_n \geq 0$  for all  $n$ . From this, it follows that  $I(|h_n|) = I(h_n)$ . As a result, we get  $\lim_{n \rightarrow \infty} I(|h_n|) = \lim_{n \rightarrow \infty} I(h_n) = L = 0$ .

We know that  $\langle h_n \rangle$  is a subsequence of the sequence  $\langle g_n \rangle$  and  $\lim_{n \rightarrow \infty} I(h_n) = L = 0$ . Consequently, we can deduce that  $\lim_{n \rightarrow \infty} I(g_n) = L = 0$ .

**Question of interest 3:-** Is  $\lim_{n \rightarrow \infty} I(f_n) = L = 0$  ?

We know that  $\langle g_n \rangle$  is a sequence in  $K_n$  where  $K_n$  is a convex hull of  $\{f_m : m \geq n\}$  and  $\lim_{n \rightarrow \infty} I(g_n) = L = 0$ . Consequently, we obtain that  $\lim_{n \rightarrow \infty} I(f_n) = L = 0$ . Moreover, since by definition  $I(0) = 0$ , then,  $I(0) = 0 = I(f)$  provided that  $f = 0$ . But, from reduced theorem assumption, we know that  $f = 0$ . Then, this will give us the desired result  $\lim_{n \rightarrow \infty} I(f_n) = L = 0 = I(0) = I(f)$ .

Thus, for any positive linear functional  $I$  if  $\lim_{n \rightarrow \infty} f_n(x) = f(x)$  point wise and there exists a positive real number  $M$  such that  $|f_n(x)| \leq M$  for all  $n$  and for all  $x$  where  $f$  and  $f_n$  in  $C$ , then,  $\lim_{n \rightarrow \infty} I(f_n) = I(f)$ .

**Example:** Define  $I: C(X) \rightarrow \mathbb{R}$  such that  $I(f) = k \cdot f$  where  $k > 0$ .

$I$  is positive linear functional. Since for any  $\alpha, \beta \in \mathbb{R}$  and  $f, g \in C(X)$ , then we have

By definition,  $I(\alpha f + \beta g) = k \cdot (\alpha f + \beta g)$

But,  $k.(\alpha f + \beta g) = k.(\alpha f) + k.(\beta g)$  .Then, this is again equals,

$$k.(\alpha f + \beta g) = k.(\alpha f) + k.(\beta g) = \alpha.(k.f) + \beta.(k.g)$$

$$\text{From the given, } \alpha.(k.f) + \beta.(k.g) = \alpha.I(f) + \beta.I(g)$$

$$\text{Thus, we obtain } I(\alpha f + \beta g) = \alpha.I(f) + \beta.I(g)$$

Moreover,  $I(f) \geq 0$  implies  $k.f \geq 0$  . But, this is possible only if  $f \geq 0$  . Therefore,  $I(f) \geq 0$  provided that  $f \geq 0$  .

Consider now the conditions  $f$  and  $f_n$  in  $C$  and  $\lim_{n \rightarrow \infty} f_n(x) = f(x)$  point wise.

Moreover, if for all  $n$  and for all  $x$ ,  $|f_n(x)| \leq M$  where  $M$  is a positive real number.

Then, by the theorem, we can infer that  $\lim_{n \rightarrow \infty} I(f_n) = I(f)$ .

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