
# Addis Ababa University

## School of Graduate Studies

## Department of Mathematics

**A Graduate project report**

**On**

# Percolation Theory

Submitted to the School of Graduate Studies
of Addis Ababa University in Partial fulfillment of the requirements for the Degree of
Master of Science in Mathematics

**By: Damtew Bewket**

**Advisor: Semu Mitiku (Ph.D)**

Addis Ababa, Ethiopia

June, 2011

# Declaration

I declare that this Project has been composed by me and that no part of the Project has formed the basis for the award of any Degree, Diploma, Associate ship, fellowship or any other similar title to me.

Author's Signature: _________________________"

# Permission

“This is to certify that this project is compiled/ a record of the research work done/ by Mr. Damtew Bewket in the Department of Mathematics, Addis Ababa University, under my supervision. I hereby also confirm that the Project can be submitted for evaluation by examiners and eventual defense.

Advisor's Signature: ________________________"

# Acknowledgment

I would like to take this opportunity to thank all people who have assisted me in the completion of this project. First of all, I would like to thank the eternal Holy God for everything that he has done throughout my life.

Next, I have immense pleasure to express my deep sense of gratitude and indebtedness to my project advisor, Dr. Semu Mitiku for his invaluable guidance, moral support and patience throughout the preparation and compilation of this project.

Moreover, I would also like to acknowledge Mr. Mebratu Adamu for his invaluable help and also I would like to sincerely thank my families for their moral and material support throughout my study.

Finally, I would like to express my deep sense of gratitude to my most affectionate and beloved wife Nigst Nigusie for her love and encouragement. I honestly could not have completed this work without her support.

# Abstract

This project is about the formulation and main theorems of percolation theory. It was introduced by Broadbent and Hamersley as a model for the flow of fluids in porous media and it is primarily concerned with the existence of open paths called open clusters. The project mainly focuses on the bond percolation, which is defined on the d-dimensional lattice. In this project, percolation probability and critical threshold are defined and some basic tools such as the FKG Inequality, the BK Inequality and the Russo's Formula are mentioned. In addition to this, the uniqueness of the infinite open cluster containing the origin and the exponential decay of the radius of the mean cluster size are also discussed.

Table of content

## Contents

# Introduction

Percolation theory was first introduced by Broadbent and Hamersley as a stochastic model for the flow of fluids in porous media with randomly blocked channels. Interpreted narrowly, it is the study of component structure of random sub graphs of graphs. Usually the underlying graph is a lattice or a lattice like graph and to obtain the random sub graph we select edges or vertices independently with the same probability $p$ [2].

Suppose that a large block of material, such as a big porous stone, is immersed under water for a long period of time. What is the probability that the water seeps in the material so that the center of the stone is wetted?

In formulating a simple stochastic model for problems of the above type, Broadbent and Hamersley gave birth to the percolation model in 1957. In the formulation of the model, they considered a two dimensional lattice $\mathrm{L}^2 = (\mathrm{Z}^2, \mathrm{E}^2)$, where $\mathrm{Z}^2$ is the set of vertices and $\mathrm{E}^2$ is the set of edges. Moreover, they considered the real number $p$ such that $0 \leq \mathrm{p} \leq 1$ and they assumed the edges to be open (present) with probability $p$ and closed (absent) otherwise, independently of all other edges. In this model, the edges represent the inner passageways of the stone and the parameter $p$ is the portion of the passages that are broad enough to allow water to flow (pass) along them. Therefore, the water will be able to flow from top to bottom or from left to right if there is an open cluster that joins the center of the stone with the periphery. Thus, Percolation theory is concerned primarily with the existence of such open paths [4].

Percolation tells us about when a system is macroscopically open to a given phenomenon. For example, it can tell us when one can have flow of traffic from one side of a town to its opposite, when electrical current can flow from one side of composite to the opposite, and how much oil one can extract from an oil reservoir [7].

The point at which the percolation transition between an open system and a closed one takes place for the first time is the percolation threshold of the system and the behavior of the system close to this point is of the prime interest and importance. Since a percolation network is created by simply blocking edges (bonds) at random, percolation is also useful as a simple model of disordered systems [7].

There are different models for percolation. These include the bond percolation model and the site percolation model. The bond percolation specifies all sites as open (present or maintained) and lets all bonds be independently either open (present or maintained) with probability $p$ or closed (deleted or absent) with probability $1-p$. On the other hand, site percolation specifies all bonds as open (present or maintained) and lets all sites be independently either open (present or maintained) with probability p or closed (deleted or absent) with probability $1-p$. However, the bond percolation model, which is defined on the d-dimensional lattice, will be described in this project.
In the first chapter of this project, we mention some basic concepts from graph theory and also we discuss the bond percolation model. In addition to this, some concepts from probability theory are also introduced.

In the second chapter, the percolation probability, the mean cluster size and the basic techniques, such as the FKG inequality, the BK inequality and the Russo's formula are mentioned. Moreover, increasing events, disjoint occurrences and pivotal edges are also discussed.

The third chapter is about the Subcritical phase, Supercritical phase, and near the critical threshold. In this chapter, the uniqueness of the infinite open cluster, the exponential decay of the radius of the open cluster and finite cluster size distribution are discussed.

# Chapter one

# Preliminary concepts and bond percolation model

## 1.1 Preliminary concepts

**Definition 1.1.1** [3]: A graph $G$ is a triple consisting of a vertex set $V(G)$, an edge set $E(G)$ and a relation that associates with each edge two vertices (not necessarily distinct) called its endpoints.

The nodes in a graph represent persons (animals, organizations, cities, countries etc) and the lines represent relationships among them.

**Definition 1.1.2** [3]: A loop is an edge whose endpoints are equal. And a graph having no loops or no two of its edges join the same pair of vertices is said to be a simple graph.

**Definition 1.1.3** [3]**:** A walk in graph $G$ is a finite non-empty sequence

$W = v_0 e_1 v_1 e_2 v_2 \dots e_k v_k$, whose terms are alternately vertices and edges such that the ends of $e_i$ are $v_{i-1}$ and $v_i$, for $1 \leq i \leq k$.

If the edges $e_1, e_2, \dots, e_k$ of a walk $W$ are distinct, $W$ is called trail. If, in addition, the vertices $v_0, v_1, v_2 \dots, v_k$ are distinct, $W$ is said to be a path.

**Definition 1.1.4** [3]: A path is a simple graph whose vertices can be ordered so that two vertices are adjacent if and only if they are consecutive in the list. A cycle is a graph with an equal number of vertices and edges whose vertices can be placed around a circle so that two vertices are adjacent if and only if they appear consecutively along the circle.

**Definition 1.1.5**: The graph theoretic distance between two vertices $u$ and $v$, in a graph $G$, is the length of the shortest path that connects them or it is the number of edges in the shortest path connecting them.

**Definition 1.1.6** [3]: A sub graph of a graph $G$ is a graph $H$ such that $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$, and the assignment of endpoints to edges in $H$ is the same as in $G$.

**Definition 1.1.7** [3]: A graph $G$ is connected if each pair of vertices in $G$ belongs to a path; otherwise, $G$ is disconnected.

**Definition 1.1.8** [3]: If $G$ is a planar graph, drawn in the plane in such a way that the edges intersect only at their common vertices, then the dual graph $G_d$ is obtained by putting a vertex in every face of $G$, and by joining two such vertices by an edge whenever the corresponding faces of $G$ share a common edge.

**Definition 1.1.9** [6]**:** Given $n$ and $p$, a random graph $G(n, p)$ is a graph with labeled vertex set $[n] = \{1, \ldots, n\}$, where each pair of vertices has an edge independently with probability p.

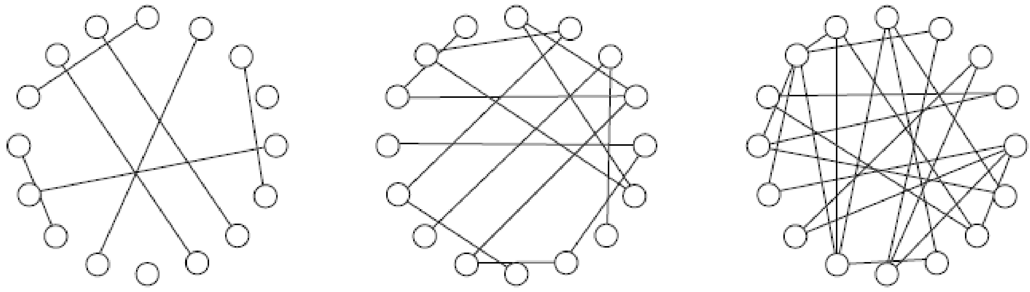

Figure 1: Three realizations of $G(16, p)$, with increasing p.

**Definition 1.1.10**: A continuous connected component of the graph of open edges is called an open cluster.

**Definition 1.1.11**: Percolation is said to occur if there is an infinite open cluster containing the origin.

**Definition 1.1.12** [6]: Let $\{X_n : n \in \mathbb{N}\}$ be a sequence of independent random variables. A tail event is an event whose occurrence or failure is determined by the values of these random variables, but which does not depend probabilistically on any finite subsequence of these random variables.

**Theorem 1.1.1 (Kolmogorov's zero-one law)** [6]: If $\{X_n : n \in \mathbb{N}\}$ is a sequence of independent variables, then any tail event $A$ satisfies $P(A) = 0$ or $P(A) = 1$.

**Proof**: Let $\mathcal{F}_1$and $\mathcal{F}_2$ be two $\sigma$ −fields of a set $\Omega$ with a common product measure $P$. Then, $\mathcal{F}_1$and $\mathcal{F}_2$ said to be independent if for every $A \in \mathcal{F}_1$and $B \in \mathcal{F}_2$, we have $P(A \cap B) = P(A)P(B)$ (i.e. every event in $\mathcal{F}_1$is independent of every event in $\mathcal{F}_2$). Now if we have two collections of events $\mathcal{A}$ and $\mathcal{B}$ such that the events of $\mathcal{A}$ are independent of the events of $\mathcal{B}$, the $\sigma$ −fields $\sigma(\mathcal{A})$ and $\sigma(\mathcal{B})$ are independent. Now let $A$ be a tail event. Then for any index $n$ it is true that $A$ is in $\mathcal{F}_{n+1} = \sigma(X_{n+1}, X_{n+2}, \dots)$. Hence $A$ must be independent of the events $X_1, X_2, \dots, X_n$, as these events are independent of the events in $\mathcal{F}_{n+1}$. Since this is true for any index $n$, $\sigma(A)$ is independent of $\sigma(X_1, X_2, \dots)$. However $A$ is in both $\sigma(A)$ and $\sigma(X_1, X_2, \dots)$. Therefore, $A$ must be independent of $A$ itself. This indicates that $P(A \cap A) = (P(A))^2$ which yields us the result since $P(A \cap A) = P(A)$.

## 1.2 Bond percolation model

Assume that $\mathrm{Z} = \{\dots, -1,0,1, \dots\}$ represents the set of all integers and suppose that $\mathrm{Z}^{\mathrm{d}}$ denotes the set of all vectors $X = (x_1, x_2, \dots, x_d)$ with integral coordinates. Then if $X \epsilon Z^d$, we generally write $x_i$ to represent the $i^{th}$ coordinate of $X$.

**Definition 1.2.1** [4]**:** The (graph theoretic) distance $\delta(X, Y)$ from $X$ to $Y$ is defined by

$$\delta(X, Y) = \sum_{i=1}^{d} |x_i - y_i|$$

and we write $|X|$ for the distance $\delta(0, X)$ from the origin to $X$.

We turn $Z^d$ into a graph, called the $d$-dimensional cubic lattice, by adding edges between all pairs $X, Y$ of points of $Z^d$ with $\delta(X, Y) = 1$ (nearest neighbors). We denote this lattice by $L^d$, and we write $Z^d$ for the set of vertices of $L^d$ and $E^d$ for the set of its edges. In graph theoretic terms we write $\mathrm{L}^{\mathrm{d}} = (\mathrm{Z}^{\mathrm{d}}, \mathrm{E}^{\mathrm{d}})$. Thus, $\mathrm{L}^{\mathrm{d}}$ is thought as a graph embedded in $\mathrm{R}^{\mathrm{d}}$, the edges being straight line segments between their end vertices [diagonal segments are not considered as an edge here].

**Definition 1.2.2** [4]**:** If $\delta(X, Y) = 1$, then we say that $X$ and $Y$ are adjacent written as $X \sim Y$. The edge from $X$ to $Y$ is represented by $\langle X, Y \rangle$. The edge $e$ is incident to the vertex $X$ if $X$ is an end vertex of $e$. Letters such as $u$,$v$,$w$, $x$, $y$ usually represent vertices, and letters such as $e$, $f$ usually represent edges. We denote the origin of $Z^d$ by 0.

Next we introduce open and closed edges.

**Definition 1.2.3** [4]**:** Let $p$ and $q$ satisfy $0 \leq p \leq 1$ and $p + q = 1$. Each edge of $L^d$ is declared to be open with probability $p$ if it is selected (maintained) and closed with probability $1 - p$ if it is not selected, independently of all other edges.

More formally, we consider the probability space.

**Definition 1.2.4** [4]**:** The probability space is a triple $(\Omega, \mathcal{F}, \mathbb{p}_p)$ where,

$$\Omega = \prod_{e \in E^d} \{0,1\}^e$$

is a sample space whose elements (points), called configurations, are represented by $\omega = (\omega(e): e \epsilon E^d)$. In this configuration, the value $\omega(e) = 0$ implies that the edge $e$ being closed, and $\omega(e) = 1$ corresponds to the edge $e$ being open; $\mathcal{F}$ is the $\sigma$-field of subsets of $\Omega$ generated by the finite dimensional cylinders; and $\mathbb{p}_p$ is the product measure with density $p$ on $(\Omega, \mathcal{F})$ given by the measure

$$\mathbb{p}_p = \prod_{e \in E^d} \mu_e$$

where $\mu_e$ is Bernoulli measure on $\{0,1\}$ which is given by $\mu_e(\omega(e) = 0) = q$,and $\mu_e(\omega(e) = 1) = p$.

We write $\mathbb{p}_p$ for this product measure and $E_p$ for the corresponding expectation operator.

**Definition 1.2.5** [4]**:** The indicator function $I_A$ of the event $A$ is defined by

$I_A(\omega) = \begin{cases} 1 \; if \; \omega \in A \\ 0 \; if \; \omega \notin A \end{cases}$ and we write $A^c$ for the complement of an event $A$.

There is a natural partial order on the set $\Omega$ of configurations, given by $\omega_1 \leq \omega_2$ if and only if $\omega_1(e) \leq \omega_2(e)$ for all $e \in E^d$.

There is a one to one correspondence between $\Omega$ and the set of subsets of $E^d$.

**Definition 1.2.6** [6]**:** For $\omega \epsilon \Omega$, we define $K(\omega) = \{e \in E^d : \omega(e) = 1\}$. Thus $K(\omega)$ is the set of open edges of the lattice when the configuration is $\omega$.

**Definition 1.2.7** [4]**:** Two sub graphs of $L^d$ are called edge-disjoint if they have no edges in common, and disjoint if they have neither edges nor vertices in common.

**Definition 1.2.8** [4]**:** Consider the random sub graph of $L^d$ containing the vertex set $Z^d$ and the open edges only. The connected components of this graph are called open clusters. We write $C(x)$ for the open cluster containing the vertex $x$, and we call $C(x)$ the open cluster at $x$.

The vertex set $C(x)$ is the set of all vertices of the lattice which are connected to $x$ by open paths, and the edges of $C(x)$ are the open edges of $L^d$ which join pairs of such vertices. By translation invariance of the lattice and of the probability measure $\mathbb{p}_p$ , the distribution of $C(x)$ is independent of the choice of $x$ [4]. The open cluster $C(0)$ at the origin is therefore typical of such clusters and we represent this cluster by the single letter $C$. We shall be interested in the size of the open cluster $C(x)$, and we denote the number of vertices in $C(x)$ by $|C(x)|$.

If $A$ and $B$ are sets of vertices of $L^d$, we shall write $'A \leftrightarrow B'$ if there exists an open path joining some vertex in $A$ to some vertex in $B$. We shall also write

$'A \leftrightarrow B\ off\ D\ '$ if there exists an open path joining some vertex in $A$ to some vertex in $B$ which uses no vertex in the set $D$.

If $A$ is set of vertices of the lattice, then we write $\partial A$ for the surface of $A$, being the set of vertices in $A$ which are adjacent to some vertex not in $A$.

# Chapter two

# Percolation probability, Mean cluster size and basic techniques

## 2.1 Percolation probability

The main quantity of interest in percolation theory is the probability that the origin belongs to a cluster with an infinite number of vertices. This probability is denoted by $\theta$ and is called the percolation probability.

**Definition 2.1.1** [4]**:** Let $C$ be the open cluster containing the origin. Then the percolation probability $\theta$ is defined by

$$\theta(\mathrm{p}) = \mathbb{p}_{\mathrm{p}}\,(|\mathrm{C}| = \infty)$$

Alternatively, we may write it as

$$\theta(\mathrm{p}) = 1 - \sum_{\mathrm{n}=1}^{\infty} \mathbb{p}_{\mathrm{p}}\,(|\mathrm{C}| = \mathrm{n}).$$

It is fundamental to percolation theory that there exists a critical value $\mathrm{p_c} = \mathrm{p_c}(\mathrm{d})$ of $p$ such that $\theta(\mathrm{p})\begin{cases} = 0 \ \text{ if } \ \mathrm{p} < p_{\mathrm{c}} \\ > 0 \ \ if \ \ p > \mathrm{p_c} \end{cases}$. Here, $\mathrm{p_c}(\mathrm{d})$ is called the critical probability and is defined formally as follows.

**Definition 2.1.2** [4]: The critical threshold or percolation threshold is defined by

$$\mathrm{p_c} = \sup\{p | \theta(p) = 0\}$$

In the one-dimensional case (i.e. $d = 1$), we observe that $\mathrm{p_c} = 1$. Indeed, if $\mathrm{p_c} < 1$, then walking along the lattice $L$ in any direction, we will almost surely meet infinitely often an open edge. This yields that all clusters are almost surely finite. However, when $d \geq 2$, it is no longer the case.

The main finding of percolation theory is that $0 < \mathrm{p_c}(\mathrm{d}) < 1$, which implies that there are two phases. These are the subcritical phase and the supercritical. The subcritical phase occurs when $\mathrm{p} < p_{\mathrm{c}}$, and it is a phase such that every vertex is almost surely in a finite open cluster. On the other hand, the supercritical phase occurs when $\mathrm{p_c} < p$ and it is a phase such that each vertex has a non zero probability of belonging to an infinite cluster. Computing the exact value of $p_c(d)$ is a challenge, and still remains an open problem for dimensions larger than 2 [6].

**Theorem 2.1.1** [1, 6]: The percolation threshold in $L^2$ is such that $\frac{1}{3} \leq p_c \leq \frac{2}{3}$.

Proof: $(i)$ First we prove that $p_c \geq \frac{1}{3}$. Let $\sigma(n)$ be the number of distinct, loop free paths of $L^2$ which have length $n$ and that begins at the origin. The exact value of $\sigma(n)$ is very difficult to compute for already moderate values of $n$, but an upper bound on $\sigma(n)$ is $4.\,3^{n-1}$. Indeed, walking from the origin, we first have four possible edges to take and then at each step up to three different edges. Let $N(n)$ be the number of such paths that are open. Since each path is open with probability $p^n$, we have

$$E_p[N(n)] = \sum_{s=1}^{\sigma(n)} E_p\left[1_{\{\text{path } s \text{ is open}\}}\right] = \sigma(n)p^n$$

.

The origin belongs to an infinite open cluster if there are open paths of all possible lengths that begin at the origin. Hence for all $n \epsilon \mathbb{N}$, we have

$$\begin{aligned}
\theta(p) &\leq \mathbb{p}_p\,(N(n) \geq 1) \\
&= \sum_{s=1}^{\sigma(n)} \mathbb{p}_p\,(N(n) = s) \\
&\leq \sum_{s=1}^{\sigma(n)} s\mathbb{p}_p\,(N(n) = s) \\
&= E_p[N(n)] \\
&= \sigma(n)p^n \\
&\leq \frac{4}{3}(3p)^n
\end{aligned}$$

By letting $n \longrightarrow \infty$, we obtain that $\theta(p) = 0$ if $p < \frac{1}{3}$. Hence we conclude that $p_c \geq \frac{1}{3}$.

(ii) Next we prove $p_c \leq \frac{2}{3}$. Let $m \epsilon \mathbb{N}$ and assume that $F_m$ is the event that there exists a closed circuit in the dual lattice $L_d^2$ that contains the box $B(m) = [-m, m]x[-m, m]$

in its interior. Moreover, let $G_m$ be the event that all edges of $B(m)$ are open. The origin belongs to an infinite cluster if $F_m$ does not occur and $G_m$ occurs, see figure 2 below.

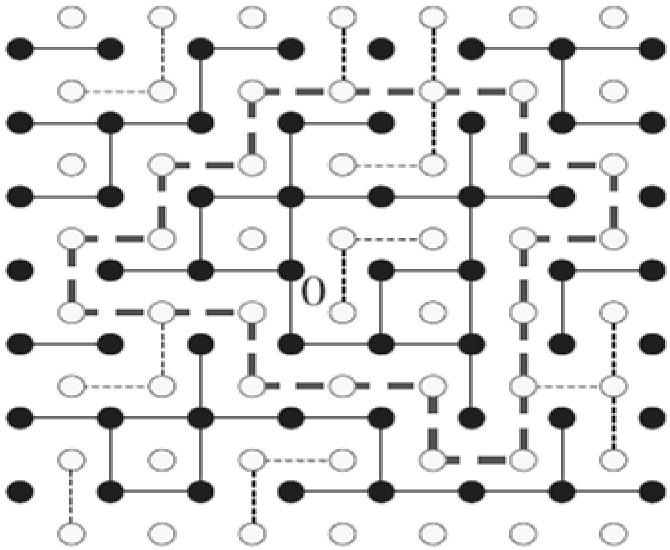


Figure 2[6]: A portion of the lattice $L^2$ (whose vertices are represented by full circles, open edges by plain lines) and its dual (whose vertices are represented by empty circles, and closed edges by dashed lines). Observe that there is a circuit of closed dual edges surrounding the origin (set in bold dashed line on the figure), which therefore belongs to a finite open cluster.

Since these events are defined on disjoint sets of edges, they are independent and therefore we have

$$\theta(p) \geq \mathbb{p}_p\,(\bar{F}_m \cap G_m) = \mathbb{p}_p\,(\bar{F}_m)\mathbb{p}_p\,(G_m) \qquad (*)$$

Now, $\mathbb{p}_p\,(G_m) > 0$ and so all we need is to show that $\mathbb{p}_p\,(\bar{F}_m) > 0$ for $p > \frac{2}{3}$.

Let $\gamma(n)$ be the number of self avoiding circuits in the dual lattice $L_d^2$ surrounding the origin and of length $n$, that consists of a single loop (in other words, the degree of every vertex of such a closed circuit is 2: we will speak of a “self-avoiding circuit”). Every such circuit must pass through a vertex of the form $(i+\frac{1}{2},\frac{1}{2})$ for some $0 \leq i \leq n-1$, because

$(a)$ to surround the origin, it has to pass through a vertex $(i+\frac{1}{2},\frac{1}{2})$ for some $i \geq 0$,

$(b)$ it can not pass through a vertex $(i+\frac{1}{2},\frac{1}{2})$ for some $i \geq n$ since it would then be at least $2n$. Such a circuit contains a self-avoiding walk of length $n-1$ starting from one of the $n$ vertices $(i+\frac{1}{2},\frac{1}{2})$ for some $0 \leq i \leq n-1$. Hence, we obtain that $\gamma(n) \leq n\sigma(n-1)$.

Now, the occurrence of the event $F_m$ requires that there is at least one such closed circuit with a length of at least $8m$ surrounding the origin. Thus, we have

$$F_m \subseteq \{there\ is\ at\ least\ one\ closed\ circuit\ of\ length\ 8m\ surrounding\ 0\}$$

$$= \bigcup_{circuit\ g\ of\ length\ at\ least\ 8m} \{g\ is\ cloed\}$$

Using the union bound, we get that

$$\mathbb{p}_{\mathrm{p}}(\mathrm{F_m}) \leq \sum_{circuit\ g\ of\ length\ at\ least\ 8m} \mathbb{p}_{\mathrm{p}}(g \text{ is closed})$$

$$= \sum_{n=8m}^{\infty} \sum_{circuit\ g\ of\ length\ n} \mathbb{p}_{\mathrm{p}}(g \text{ is closed})$$

$$= \sum_{n=8m}^{\infty} \gamma(n)(1-p)^n$$

$$\leq \frac{4(1-p)}{3} \sum_{n=8m}^{\infty} n(3(1-p))^{n-1}$$

If $p > \frac{2}{3}$, then this sum converges to some finite value, and we take $m$ to be large enough so that it is less than $\frac{1}{2}$. Therefore, from $(*)$ we get

$$\theta(p) \geq \mathbb{p}_p(\bar{F}_m)\mathbb{p}_p(G_m)$$

$$\geq \frac{\mathbb{p}_p(G_m)}{2} > 0$$ , which completes the proof of the theorem. ■

**Theorem 2.1.2** [4]: The probability that there exists an open infinite cluster satisfies

$$\psi(p) = \begin{cases} 0\ if\ \theta(p) = 0 \\ 1\ if\ \theta(p) > 0 \end{cases}.$$

**Proof**: The event that $\{L^d\ contains\ an\ infinite\ open\ cluster\}$ does not depend upon the states of any finite collection of edges. Thus, by the usual zero-one law $\psi$ takes the values 0 and 1 only. If $\theta(p) = 0$, then we get

$$\psi(p) \leq \sum_{x \in Z^d} \mathbb{p}_p\, (|C(x)| = \infty) = 0$$

On the other hand, if $\theta(p) > 0$, then $\psi(p) \geq \mathbb{p}_p\, (|C| = \infty) > 0$ so that $\psi(p) = 1$ by the zero-one law, as required. ■

The d-dimensional lattice $L^d$ may be embedded in $L^{d+1}$. Hence, the origin of $L^{d+1}$ belongs to an infinite open cluster for a particular value of $p$ whenever it belongs to an infinite open cluster of the sub lattice $L^d$. Thus $\theta(p) = \theta_d(p)$ is non-decreasing in $d$, which implies that $p_c(d+1) \leq p_c(d)$ for $d \geq 1$. (2.1)

The following theorem (Theorem 2.1.3) states that there exists a non-trivial critical phenomenon in dimensions two and more.

Before the theorem let us see some associated results and open problems. First what is the numerical value of $p_c(d)$? We know only the values $p_c(1) = 1$ and $p_c(2) = \frac{1}{2}$. $p_c(2) = \frac{1}{2}$ is far from trivial to show. It is highly unlikely that there exists a useful representation of $p_c(d)$ for any other value of $d$, although such values may be calculated with increasing degrees of accuracy with the help of larger and faster computers.

Secondly, it is not difficult to find non trivial upper and lower bounds for $p_c(d)$ when $d \geq 2$. We have seen (in theorem 2.1.1) that $\frac{1}{\lambda(2)} \leq p_c(2) \leq 1 - \frac{1}{\lambda(2)}$, and more generally $\frac{1}{\lambda(d)} \leq p_c(d) \leq 1 - \frac{1}{\lambda(d)}$ for $d \geq 3$ where $\lambda(d)$ is a connective constant defined by $\lambda(d) = \lim_{n\to\infty}(\sigma(n))^{\frac{1}{n}}$ and $\sigma(n)$ is the number of paths (or self avoiding walks) of $L^d$ having length $n$ and that begins at the origin. The exact value of $\lambda(d)$ is unknown. But it is expected that $\lambda(d) \leq 2d - 1$; to see this, note that each new step in a self avoiding walk has at most $2d - 1$ choices since it must avoid the current position, and therefore $\sigma(n) \leq 2d(2d-1)^{n-1}$.

**Theorem 2.1.3** [4]: If $d \geq 2$, then $0 < p_c(d) < 1$.

**Proof**: From (2.1) we have that $p_c(d+1) \leq p_c(d)$. Therefore, it suffices to show that $p_c(d) > 0$ for $d \geq 2$, and $p_c(2) < 1$.

First we prove that $p_c(d) > 0$ for $d \geq 2$. For this, consider bond percolation on $L^d$ when $d \geq 2$. Suppose that $\sigma(n)$ is the number of paths of $L^d$ having length $n$ and that begins at the origin. Let $N(n)$ be the number of such paths which are open. Since every such path is open with probability $p^n$, we have

$$\mathrm{E_p[N(n)] = \sigma(n)p^n}.$$

Now, the origin belongs to an infinite open cluster if there are open paths of all possible lengths that begin at the origin. So for all $n \epsilon \mathbb{N}$, we have

$$\begin{aligned} \theta(\mathrm{p}) &\leq \mathbb{p}_\mathrm{p}\,(\mathrm{N(n)} \geq 1) \\ &\leq \mathrm{E_p[N(n)]} \\ &= \sigma(\mathrm{n})\mathrm{p^n} \qquad (2.2) \end{aligned}$$

Then substituting $\sigma(\mathrm{n}) = (\lambda(\mathrm{d}) + \mathrm{o}(1))^\mathrm{n}$ as $\mathrm{n} \to \infty$, we get that

$$\theta(\mathrm{p}) \leq (\mathrm{p}\lambda(\mathrm{d}) + \mathrm{o}(1))^\mathrm{n} \qquad (2.3)$$

Now if $p\lambda(d) < 1$, then $(p\lambda(d) + \mathrm{o}(1))^\mathrm{n}$ goes to zero as $\mathrm{n} \to \infty$. This implies that $\theta(\mathrm{p}) = 0$, if $p\lambda(d) < 1$. Thus, we have

$p_c(d) \geq \frac{1}{\lambda(\mathrm{d})}$, and $\lambda(\mathrm{d}) \leq 2\mathrm{d} - 1 < \infty$.

Next we show that $p_c(2) < 1$. Consider bond percolation on $L^2$. Let $G$ be a planner graph. The planner dual of $G$ is the graph obtained from $G$ by placing a vertex in each face of $G$ (including any infinite face which may exist) and joining two such vertices by an edge whenever the corresponding face of $G$ share a boundary edge in $G$. For the sake of simplicity, we take as vertices of this dual lattice the set $\left\{X + \left(\frac{1}{2}, \frac{1}{2}\right) : X \in Z^2\right\}$ and join two such neighboring vertices by a straight line segments in $R^2$. Since each edge of $L^2$ is crossed by a unique edge of the dual, there is a one to one correspondence between the edges of $L^2$ and the edges of the dual. We declare an edge of the dual to be open or closed depending respectively on whether it crosses an open or closed edge of $L^2$. This process creates a bond percolation on the dual lattice with the same edge probability $p$.

Suppose that the open cluster at the origin of $L^2$ is finite. We see that the origin is surrounded by closed edges which are blocking of all possible routes from the origin to infinity. We may convince ourselves that the corresponding edges of the dual contain a closed circuit in the dual having the origin of $L^2$ in its interior. The converse is also true. If the origin lies in the interior of a closed circuit of the dual lattice, then the open cluster at the origin is finite. Therefore, we have $|C| < \infty$ if and only if the origin of $L^2$ lies in the interior of some closed circuit of the dual.

Let $\rho(n)$ be the number of circuits in the dual which have length $n$ and which contain in their interiors the origin of $L^2$. Then, $\rho(n)$ is estimated as follows. Each such circuit passes through some vertex of the form $\left(k+\frac{1}{2},\frac{1}{2}\right)$ for same $k$ satisfying $0 \le k < n$ because; first, it surrounds the origin and therefore passes through $\left(k+\frac{1}{2},\frac{1}{2}\right)$ for some $k \ge 0$ and, secondly, it cannot pass through $\left(k+\frac{1}{2},\frac{1}{2}\right)$ where $k \ge n$ since it would then have length at least $2n$. Thus, such a circuit contains a self-avoiding walk of length $n-1$ starting from a vertex of the form $\left(k+\frac{1}{2},\frac{1}{2}\right)$, where $0 \le k < n$. The number of such self-avoiding walk is at most $n\sigma(n-1)$ which gives us that $\rho(n) \le n\sigma(n-1)$.

Let $\gamma$ be a circuit of the dual containing the origin of $L^2$ in its interior, and let $M(n)$ be the number of such closed circuit having length $n$, then

$$\sum_{\gamma} \mathbb{P}_p\,(\gamma \text{ is closed}) \le \sum_{n=1}^{\infty} q^n\, n\sigma(n-1) \qquad (2.4)$$

$$= \sum_{n=1}^{\infty} q\, n(q\lambda(2) + o(1))^{n-1}$$

$< \infty$ if $q\lambda(2) < 1$, $q = 1-p$ and the summation is over all such $\gamma$. Moreover,

$$\sum_{\gamma} \mathbb{P}_p\,(\gamma \text{ is closed}) \to 0 \text{ as } q = 1-p \downarrow 0.$$

Hence, it may be possible to find $\pi$ with $0 < \pi < 1$ such that

$$\sum_{\gamma} \mathbb{P}_p\,(\gamma \text{ is closed}) \le \frac{1}{2} \text{ if } p > \pi.$$

From previous remarks, it follows that

$$\mathbb{P}_p\,(|C| = \infty) = \mathbb{P}_p\,(M(n) = 0 \text{ for all } n)$$
$$= 1 - \mathbb{P}_p\,(M(n) \ge 1 \text{ for some } n)$$

$$\geq 1 - \sum\nolimits_{\gamma} \mathbb{p}_{\mathrm{p}}\,(\gamma\ is\ closed)$$

$$\geq \frac{1}{2} \text{ if } p > \pi, \text{giving that } p_c(2) \leq \pi.$$

Now let us deduce that

$$p_c(2) \leq 1 - \frac{1}{\lambda(2)}.$$

For a positive integer $m$, let $F_m$ be the event that there exists a closed dual circuit containing the box $B\ (m)$ in its interior, and let $G_m$ be the event that all edges of $B\ (m)$ are open. These two events are independent, since they are defined on disjoint sets of edges. Now, similarly to $(2.4)$ we have

$$\mathbb{p}_{\mathrm{p}}\,(F_m) \leq \mathbb{p}_{\mathrm{p}}\left(\sum_{n=4}^{\infty}(M(n) \geq 1)\right)$$

$$\leq \sum_{n=4}^{\infty} q^n\, n\sigma(\mathrm{n}-1).$$

If $q < \frac{1}{\lambda(2)}$ we may find $m$ such that $\mathbb{p}_{\mathrm{p}}\,(F_m) < \frac{1}{2}$ and we choose m accordingly. Assume now that $G_m$ occurs but $F_m$ does not. The non-occurrence of $F_m$ mplies that some vertex of $B\ (m)$ lies in an infinite open path. Together with the occurrence of $G_m$ this implies that $|\ C\ | = \infty$. Therefore, using the independence of $F_m$ and $G_m$, we have

$$\theta(\mathrm{p}) \geq \mathbb{p}_p\,(\bar{F}_m \cap G_m)$$

$$= \mathbb{p}_p\,(\bar{F}_m)\mathbb{p}_p\,(\,G_m)$$

$$\geq \frac{\mathbb{p}_p\,(\,G_m)}{2}$$

$$> 0 \text{ if } q < \tfrac{1}{\lambda(2)}.$$ ■

Now consider an arbitrary infinite connected graph $G = (V, E)$. Let 0 denotes a specified vertex of $G$ which we call the origin. We define $\theta^{bond}\,(p)$ (respectively $\theta^{site}\,(p)$) to be the probability that 0 lies in an infinite open cluster of $G$ in a bond percolation (respectively site percolation) process on $G$ having parameter $p$. Here, $\theta^{bond}\,(p)$ and $\theta^{site}\,(p)$ are non decreasing functions of $p$, and the bond and site critical probabilities are given by

$p_c^{bond} = p_c^{bond}(G) = sup\{p: \theta^{bond}(p) = 0\}$ and

$p_c^{site} = p_c^{site}(G) = sup\{p: \theta^{site}(p) = 0\}$.

Now it is natural to ask whether there exists a relationship between the two critical points of given graph $G$ and the vertex degree.

**Theorem 2.1.4** [4]: Let $G = (V, E)$ be an infinite connected graph with countably many edges, origin 0; and maximum vertex degree $\Delta(< \infty)$. The critical probabilities of $G$ satisfy

$$\frac{1}{\Delta - 1} \leq p_c^{bond} \leq p_c^{site} \leq 1 - (1 - p_c^{bond})^{\Delta} \qquad (2.5)$$

**Proof**: The first inequality of (2.5) follows by counting paths, as in (2.2) and (2.3). Therefore, we turn immediately to the remaining two inequalities. In order to obtain these, we shall prove a certain stochastic inequality. Given two random subsets $X$, $Y$ of $V$ with associated expectation operator $E$; we write $X \leq_{st} Y$ and say that $X$ is stochastically dominated by $Y$, if $E(f(X)) \leq E(f(Y))$ for all bounded, measurable functions $f$ satisfying $f(A) \leq f(B)$ if $A \subseteq B \subseteq V$.

Let $C^{bond}(p)$ be a random subset of $V$ having the law of the cluster of bond percolation at the origin and let $C^{site}(p)$ be a random subset having the law of the cluster of the percolation at the origin conditional on 0 being an open vertex. We claim that

$C^{site}(p) \leq_{st} C^{bond}(p)$ (2.6) and that

$C^{bond}(p) \leq_{st} C^{site}(p')$ (2.7), where $p' = 1 - (1 - p)^{\Delta}$.

Since $\theta^{bond}(p) = \mathbb{P}_p(| C^{bond}(p) | = \infty)$, $p^{-1}\theta^{site}(p) = \mathbb{P}_p(| C^{site}(p) | = \infty)$, the remaining claims of (2.5) will follow from (2.6) and (2.7). Indeed, (2.6) and (2.7) imply that $\frac{\theta^{site}(p)}{p} \leq \theta^{bond}(p) \leq \frac{\theta^{site}(p')}{p'}$, where $p' = 1 - (1 - p)^{\Delta}$ (2.8)

which is slightly stronger than (2.5).

We construct appropriate couplings of the bond and site models in order to prove (2.6) and (2.7). Let $\omega \in \{0,1\}^E$ be a realization of a bond percolation process on $G = (V, E)$ having density $p$. We may build the cluster at the origin in the following standard manner. Let $e_1, e_2, \ldots$ be a fixed ordering of $E$. At each stage $k$ of the inductive construction, we shall have a pair $(A_k, B_k)$ where $A_k \subseteq V$ and $B_k \subseteq E$. Initially, we set $A_0 = \{0\}$, $B_0 = \emptyset$.

Having found $(A_k, B_k)$ for some $k$, we define $(A_{k+1}, B_{k+1})$ as follows. We find earliest edge $e$ in the ordering of $E$ having the following properties:

$e \in B_k$, and $e$ is incident with exactly one vertex of $A_k$, say the vertex $x$.

Now we set

$$A_{k+1} = \begin{cases} A_k & \text{if } e \text{ is closed,} \\ A_k \cup \{y\} & \text{if } e \text{ is open,} \end{cases} \quad (2.9)$$

$$B_{k+1} = \begin{cases} B_k \cup \{e\} & \text{if } e \text{ is closed,} \\ B_k & \text{if } e \text{ is open,} \end{cases} \quad (2.10), \text{ where } e = \langle x, y \rangle.$$

If no such edge $e$ exists, we declare $(A_{k+1}, B_{k+1}) = (A_k, B_k)$. The sets $A_k, B_k$ are non-decreasing, and the open cluster at the origin is given by

$$A_\infty = \lim_{k\to\infty} A_k.$$

We now augment the above construction in the following way. We color the vertex 0 red. Furthermore, to obtain the edge $e$ given above, we color the vertex y red if $e$ is open, and black otherwise. We specify that each vertex is colored at most once in the construction, in the sense that any vertex $y$ which is obtained at two or more stages is colored in perpetuity according to the first color it receives.

Let $A_\infty(\text{red})$ be the set of points connected to the origin by red paths of $G$ (that is, by paths all of whose vertices are red). We make two claims concerning $A_\infty(\text{red})$:

$(i)$ it is the case that $A_\infty(\text{red}) \subseteq A_\infty$; and all neighbors of vertices in $A_\infty(\text{red})$ which do not lie $A_\infty(\text{red})$ are black;

$(ii)$ $A_\infty(\text{red})$ has the same distribution as $C^{site}(p)$;

Claim $(i)$ is straightforward. In order to be colored red, a vertex is necessarily connected to the origin by a path of open edges. Furthermore, since all edges with exactly one end vertex in $A_\infty$ are closed, all neighbors of $A_\infty(\text{red})$ which are not themselves colored red are necessarily black.

We sketch an explanation of claim $(ii)$. Whenever a vertex is colored either red or black, it is colored red with probability $p$ independently of all earlier colorings.

The derivation of (2.7) is similar. We start with a directed version of $G$; namely the directed graph $G = (V, E)$ obtained from $G$ by replacing each edge $e = \langle x, y \rangle$ by two directed edges, one in each direction, and denoted respectively by $\{x, y\}$and $[y, x\rangle$.We now let $\overline{\omega} \in \{0,1\}^{\overline{E}}$ be a realization of (oriented) bond percolation process on $G$ having density $p$.

We color the origin green. We color a vertex $x(\neq 0)$ green if at least one edge $f$ of the form $[y, x\rangle$ satisfies $\bar{\omega}(f) = 1$; otherwise we color $x$ black.

Then $\mathbb{p}_p\ (x\ is\ green) = 1 - (1-p)^{\rho(x)} \leq 1 - (1-p)^{\Delta}$ (2.11),

where $\rho(x$ is the degree of $x$, and $\Delta = \max_x \rho(x)$.

We now build a copy $A_\infty$ of $C^{bond}\ (p)$ more or less as described in (2.9) and (2.10). The only difference is that, on considering the edge $e = \langle x, y\rangle$ where $x \in A_k$ and $y \notin A_k$, we declare $e$ to be open for the purpose of (2.9) and (2.10) if and only if $\bar{\omega}([x, y\rangle) = 1$. Finally, we let $A_\infty(\text{green})$ be the set of points connected to the origin by green paths. It may be seen that $A_\infty(\text{green}) \supseteq A_\infty$.

Furthermore, by 2.11 $A_\infty(\text{green})$ is stochastically dominated by $C^{site}\ (p')$ where

$p' = 1 - (1-p)^{\Delta}$. Inequality (2.7) follows. ■

## 2.2 Mean cluster size

The other quantity of interest in percolation theory is the mean size of an open cluster. By translation invariance, the mean size of an open cluster is the expected number of vertices in the open cluster at the origin.

**Definition 2.2.1** [6]: The mean size of an open cluster, which is denoted by $\chi$, is defined by

$$\chi(p) = \mathrm{E_p}\ [|\mathrm{C}|]$$

By expanding this expression, we obtain that

$$\chi(p) = \mathrm{E_p}\ [|\mathrm{C}|] = \sum_{\mathrm{n}=1}^{\infty} \mathrm{n}\ \mathbb{p}_{\mathrm{p}}\ (|\mathrm{C}| = \mathrm{n}) + \infty \mathbb{p}_{\mathrm{p}}\ (|\mathrm{C}| = \infty)$$

If $p > p_c$, then we have $\chi(p) = \infty$. But the converse is not obvious, and it requires quite some work to prove that if $p < p_c$, then $\chi(p) < \infty$.

In the supercritical phase, since the mean cluster size is infinite, one is more interested in the mean size of the finite clusters. It is denoted by $\chi^f\ (p)$ and is defined as the mean of $|\mathrm{C}|$ on the event that $|\mathrm{C}|$ is finite.

## 2.3 Basic techniques

In this section we will try to see some of the important tools that are repeatedly used in the study of percolation. These are the FKG Inequality, the BK Inequality and the Russo's Formula. They are discussed as follows.

## 2.3.1 Increasing Events and the FKG Inequality

The FKG inequality, which was named after Fortuin Kaseleyn and Ginibre, was first shown by Harris in 1960. It expresses the fact that two increasing events can only be positively correlated. Before directly stating the FKG Inequality, let us first see increasing events.

**Definition 2.3.1.1** [1, 4, 6]: A random variable $X$ is said to be increasing on $(\Omega, \mathcal{F})$ if

$X(\omega) \leq X(\omega')$ whenever $\omega \leq \omega'$. It is decreasing if $-X$ is increasing.

**Example** [1]: Some examples of increasing random variables:

1. The size of the open cluster containing vertex $x$.
2. The largest $k$ such that all edges in $B(k)$ are open.

**Definition 2.3.1.2** [1, 4, 6]: An event $A \in \mathcal{F}$ is increasing whenever its indicator function is increasing variable. i.e. if $I_A(\omega) \leq I_A(\omega')$ whenever $\omega \leq \omega'$.

**Example** [1]: Some examples of increasing events:

1. The origin is contained in an infinite open cluster.
2. There is an open path from $x$ to $y$.
3. The edges of $B(n)$ are open.

**Theorem 2.3.1.1 (FKG Inequality)** [4, 6]: If A and B are two increasing events, then

$$\mathbb{P}_p(A \cap B) \geq \mathbb{P}_p(A)\mathbb{P}_p(B).$$

**Proof** [6]: Let $X = I_A$ and $Y = I_B$ be the indicators of the increasing events A and B, which are increasing random variables. Then we can reformulate the FKG inequality as $E_p[XY] \geq E_p[X]E_p[Y]$. Suppose that X and Y depend on the state of edges $e_1, e_2, \ldots, e_n$ for some integer $n$. Now we prove the FKG inequality by induction on $n$ as follows.

First suppose that $n = 1$, so that X and Y are only functions of the state $\omega(e_1)$ of the edge $e_1$. Now pick any two states $\omega_1, \omega_2 \epsilon \{0,1\}$. Since both X and Y are increasing random variables, $\big(X(\omega_1) - X(\omega_2)\big)\big(Y(\omega_1) - Y(\omega_2)\big) \geq 0$, where the equality holds if $\omega_1 = \omega_2$. Therefore, we have

$$0 \leq \sum_{\omega_1=0}^{1} \sum_{\omega_2=0}^{1} \big(X(\omega_1) - X(\omega_2)\big)\big(Y(\omega_1) - Y(\omega_2)\big)\, \mathbb{p}_p(\omega_1(e_1) = \omega_1)\mathbb{p}_p(\omega_2(e_1) = \omega_2)$$

$$= \sum_{\omega_1=0}^{1} X(\omega_1)\, Y(\omega_1) \mathbb{p}_p(\omega_1(e_1) = \omega_1) + \sum_{\omega_2=0}^{1} X(\omega_2)\, Y(\omega_2) \mathbb{p}_p(\omega_2(e_1) = \omega_2)$$

$$- \sum_{\omega_1=0}^{1} \sum_{\omega_2=0}^{1} (X(\omega_1)Y(\omega_2) + X(\omega_2)Y(\omega_1)) \mathbb{p}_p(\omega_1(e_1) = \omega_1)\mathbb{p}_p(\omega_2(e_1) = \omega_2)$$

$$= 2(E_p\,[XY] - E_p\,[X]E_p\,[Y]).$$

Let $1 < k \leq n$ and suppose that the claim holds for all $m < k$, and that X and Y depend only on the states $\omega(e_1), \omega(e_2), \; . \; . \; . \; , \omega(e_k)$. Then, given $\omega(e_1), \omega(e_2), . \; .., \omega(e_{k-1})$, X and Y only depend on the state $\omega(e_k)$ of the edge $e_k$, and proceeding in similar way as above, we obtain

$$E_p\,[XY|\omega(e_1), \ldots, \omega(e_{k-1})] \geq E_p\,[X|\omega(e_1), \ldots, \omega(e_{k-1})] E_p [Y|\omega(e_1), \; \ldots, \omega(e_{k-1})]$$

and hence

$$E_p\,[XY] = E_p [E_p\,[XY|\omega(e_1), \ldots, \omega(e_{k-1})]]$$

$$\geq E_p\,[E_p\,[X|\omega(e_1), \ldots, \omega(e_{k-1})] E_p [Y|\omega(e_1), \qquad \ldots, \omega(e_{k-1})]]$$

Now, $E_p\,[X|\omega(e_1), \ldots, \omega(e_{k-1})]$ and $E_p [Y|\omega(e_1), \; \ldots, \omega(e_{k-1})]$ are increasing functions of the state of the $(k-1)$ edges $e_1, e_2, \; . \; . \; . \; , e_{k-1}$. By induction, it implies that

$$E_p\,[XY] \geq E_p\,[E_p\,[X|\omega(e_1), \ldots, \omega(e_{k-1})]] \,.\, E_p\,[E_p [Y|\omega(e_1), \; \ldots, \omega(e_{k-1})]]$$

$= E_p\,[X]E_p\,[Y]$, which completes the proof of the theorem. ■

**Example** [6]: As an example of application of the FKG inequality, consider the 2-dimensional box $B(n)$. Let $A$ be the event that there is an open path joining a vertex of the top face of $B(n)$ to the bottom face of $B(n)$ (we call such a path a TB (top-bottom) (open) crossing of $B(n)$, and let $B$ be the event that there is an open path joining a vertex of the left face of $B(n)$ to the right face of $B(n)$ (we call such a path a LR (left-right) (open) crossing of $B(n)$, as shown in Figure 3 below. Then the probability that there are both a TB and LR open crossings of $B(n)$ is at least the product of the probabilities that there is a TB open crossing and that there is a LR open crossing

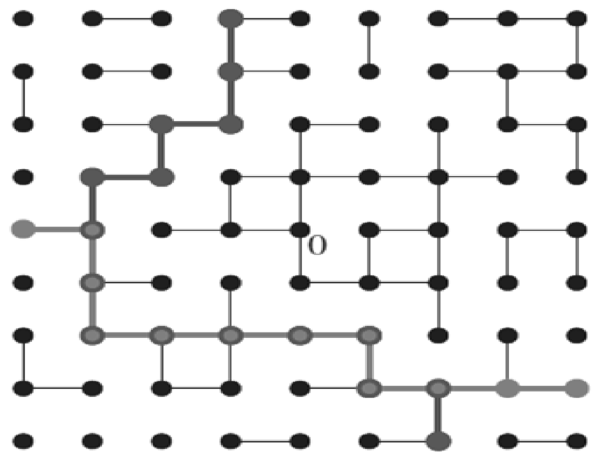


Figure 3 [6]: The box $B(5)$ with a LR and a TB open crossing.

## 2.3.2 Disjoint Occurrence and the BK inequality

The BK inequality, which was named after Van den Berg and Kesten who proved it in 1985, can be regarded as the reverse of the FKG inequality with one difference. It applies to the event $A \circ B$ that two increasing events A and B occur on disjoint sets of edges, and not to the larger event $A \cap B$ that events A and B occur on any sets of edges. A ∘ B is the set of configurations $\omega \in \Omega$ for which there are disjoint sets of open edges such that the first set guarantees the occurrence A and the second set guarantees the occurrence of B. The formal definition is given as follows.

**Definition 2.3.2.1** [4, 6]: Let $A$ and $B$ be two increasing events which depend on the states $\omega(e_1), \ldots, \omega(e_n)$ of n distinct edges $e_1, e_2, \ldots, e_n$ of $L^d$. Each of such configurations is specified uniquely by the subset $K(\omega) = \{e_i | \omega(e_i) = 1\}$ of open edges among these $n$ edges. Then $A \circ B$ is the set of ω for which there exists a subset

$H$ of $K(\omega)$ such that any $\omega'$ determined by $K(\omega') = H$ is in $A$ and any $\omega''$ determined by $K(\omega'') = K(\omega) \setminus H$ is in B.

**Theorem 2.3.2.1 (BK inequality)** [6]: If A and B are two increasing events, then

$$\mathbb{p}_p(A \circ B) \leq \mathbb{p}_p(A)\mathbb{p}_p(B).$$

**Proof** [6]: Let $G$ be a finite sub graph of $L^d$. Let A be the event that there exists an open path between vertices $u$ and $v$ and B be the event that there exists an open path between vertices $x$ and $y$. Then $A \circ B$ is the event that there exist two disjoint open paths from $u$ to $v$ and from $x$ to $y$. Now, let e be an edge of $G$. Replace e by two parallel edges $e'$ and $e''$, having the same end vertices, each of which being open with the same probability $p$, independently of each other and of all other edges. The splitting of edge e in the two edges $e'$ and $e''$ can only make our search for two disjoint open paths easier. Indeed, if in graph $G$, two paths from $u$ to $v$ and from $x$ to $y$ had to use the same edge e, they can now replace this common edge by the two distinct edges $e'$ and $e''$. The probability of finding two disjoint open paths from $u$ to $v$ and from $x$ to $y$ can therefore only increase or remain equal after this splitting. By continuing this splitting process (see figure 4 below), we replace every edge f of $G$ by two parallel edges $f'$ and $f''$. At each stage, we look for two open paths, the first one which avoids all edges marked by $''$ and the second one that avoids all edges marked by $'$. The probability of finding two such paths can only increase or remain the same at each stage. When all edges of $G$ have been split in to two, we end up with two independent copies of $G$, in the first of which we look for an open path that connects $u$ to $v$, and in the second of which we look for an open path which connects $x$ to $y$. Since such open paths occur independently in each copy of $G$, the probability that they both occur is $\mathbb{p}_p(A)\mathbb{p}_p(B)$. Thus, we get $\mathbb{p}_p(A \circ B) \leq \mathbb{p}_p(A)\mathbb{p}_p(B)$, as desired. ■

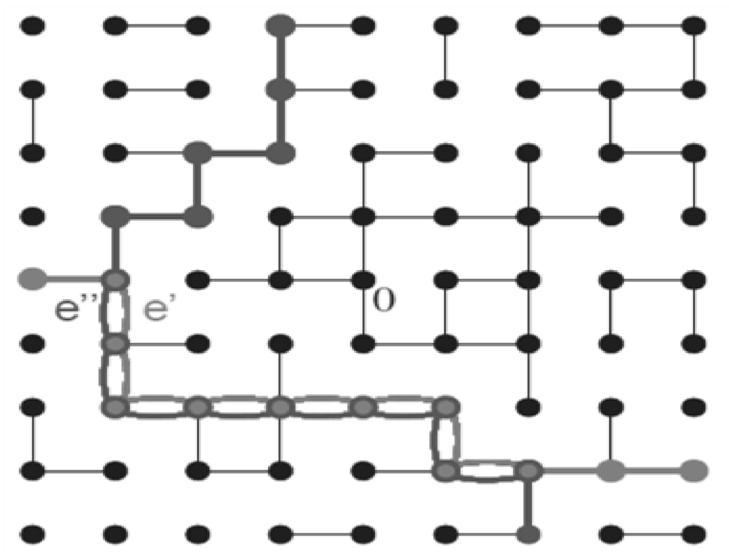


Figure 4 [6]: Construction of two independent copies of the lattice

**Example** [6]: As an example of application of the BK inequality, consider the two dimensional box $B(n)$. And let $A$ be the event that there is a top-bottom (TB) open crossing path of $B(n)$ to the bottom face of $B(n)$, and $B$ be the event that there is an a left-right (LR) open crossing, which is edge-disjoint with $A$. (This event does not occur in the example of Figure 3, but it occurs for the example in Figure 5 as shown below.) Then the probability that there are edge disjoint TB and LR open crossings of $B(n)$ is no more than the product of the probabilities that there is a TB open crossing and that there is a LR open crossing.

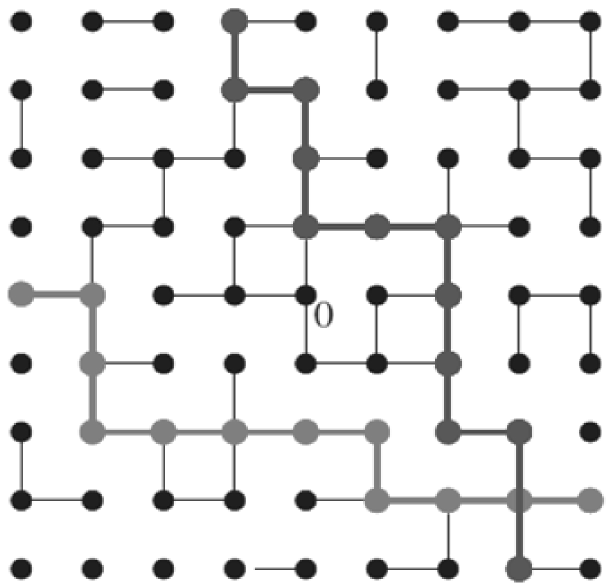


Figure 5 [6]: The box $B(5)$ with edge-disjoint LR and TB open crossings.

## 2.3.3 Pivotal edges and Russo's formula

The Russo's formula estimates the rate of change of the probability of occurrence of an increasing event $A$ as $p$ increases. Here, we first need to introduce the definition of pivotal edges. If $A$ is increasing, an edge $e$ is pivotal for $A$ if and only if $A$ occurs when $e$ is open and does not occur if $e$ is closed. A pivotal edge is thus a critical edge for the occurrence of $A$ [6].

**Definition 2.3.3.1** [6]: Let $A$ be an event and $\omega$ be a configuration. Then, edge $e$ is said to be pivotal for the pair $(A, \omega)$ if the occurrence of $A$ crucially depends on $e$. (i.e., if $I_A(\omega) \neq I_A(\omega')$ where $\omega'$ is the configuration such that $\omega'(e) = 1 - \omega(e)$ and $\omega'(f) = \omega(f)$ for all $f \epsilon E^d \setminus \{e\}$.)

The event "$e$ is pivotal for $A$" is the set of all configurations $\omega$ for which $e$ is pivotal for $(A, \omega)$. This event is independent of the state of $e$ itself, but only depends on the state of the other edges.

**Example** [6]: let $A$ be the event that there is a LR open crossing of the 2-dimensional box $B(n)$. Any edge $e$ of $B(n)$ is pivotal for $A$ if when it is removed from the graph there is no more LR open crossing of $B(n)$, but one end vertex of $e$ is joined to the left side of $B(n)$ by an open path, while the other end vertex of $e$ is joined to the right side of $B(n)$ by an other open path.

Figure 6 below shows another example of edges that are pivotal for the event that the origin is connected by an open path to the boundary $\partial S(n)$ of a diamond $S(n)$.

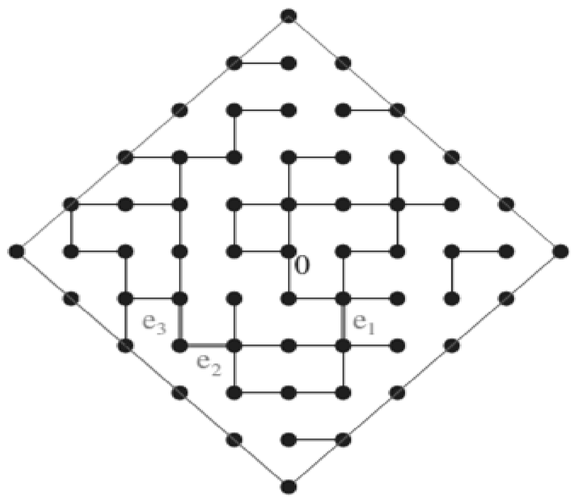


Figure 6 [6]: The three edges $e_1, e_2$ and $e_3$ are pivotal for the event $0 \leftrightarrow \partial S(5)$.

**Theorem 2.3.3.1(Russo's formula)** [6]: Let $A$ be an increasing event, which depends on the state of finitely many edges of $L^d$, and let $N(A)$ denote the number of edges that are pivotal for $A$. Then

$$\frac{d}{dp}\mathbb{P}_p(A) = E_p[N(A)]$$

.

**Proof** [6]: Let $\{X(e)|e\epsilon E^d\}$ be a collection of independent and identically distributed random variables indexed by the edge set $E^d$, uniformly distributed on [0,1]. Let $\eta_p$ be the configuration of edges defined by

$\eta_p(e) = \begin{cases} 1 & if\ X(e) < p \\ 0 & if\ X(e) \geq p \end{cases}$, for some $0 \leq p \leq 1$ and all $e\epsilon E^d$. Then, we have

$$\mathbb{P}\left(\eta_p(e) = 0\right) = \mathbb{P}(X(e) \geq p) = 1 - p$$

$$\mathbb{P}\left(\eta_p(e) = 1\right) = \mathbb{P}(X(e) < p) = p.$$

Hence, we get that $\mathbb{P}_p(A) = \mathbb{P}(\eta_p \epsilon A)$. Now, Since $A$ is an increasing event, for $\varepsilon > 0$ we get $\mathbb{P}_{p+\varepsilon}(A) = \mathbb{P}\left(\eta_{p+\varepsilon} \in A\right)$

$$= \mathbb{P}\left(\{\{\eta_{p+\varepsilon} \in A\} \cap \{\eta_p \notin A\}\} \cup \{\eta_p \epsilon A\}\right)$$

$$= \mathbb{P}(\{\eta_{p+\varepsilon} \in A\} \cap \{\eta_p \notin A\}) + \mathbb{P}(\eta_p \epsilon A)$$

$$= \mathbb{P}(\{\eta_{p+\varepsilon} \in A\} \cap \{\eta_p \notin A\}) + \mathbb{P}_p(A) \qquad ------ \qquad (**).$$

Now, if $\eta_p \notin A$ while $\eta_{p+\varepsilon} \epsilon A$, it means that there are some edges $e$ on which $A$ depends, and for which $\eta_p(e) = 0$ and $\eta_{p+\varepsilon}(e) = 1$. Or equivalently,

$$p \leq X(e) < p + \varepsilon.$$

Since $A$ depends only on the state of finitely many edges, the probability that there are more than one edge $e$ with $p \leq X(e) < p + \varepsilon$ is negligible (of the order $o(\varepsilon)$) in front of the probability that there is one such edge, when $\varepsilon \downarrow 0$.

If $e$ is the only edge for which $p \leq X(e) < p + \varepsilon$, then $e$ must be pivotal a edge for $A$, in the sense that $\eta_p \notin A$ but $\eta_{p+\varepsilon} \epsilon A$ where $\eta_{p+\varepsilon}(e) = 1 = 1 - \eta_p(e)$ and $\eta_{p+\varepsilon}(e') = \eta_p(e')$ for all other edges $e' \neq e$. Therefore,

$$\mathbb{p}(\{\eta_{p+\varepsilon} \epsilon A\} \cap \{\eta_p \notin A\}) = \sum_{e \epsilon \mathrm{E}^{\mathrm{d}}} \mathbb{p}(\{e \; is \; pivotal \; for \; A\} \cap \{p \leq X(e) < p + \varepsilon\}) + o(\varepsilon)$$

$$= \sum_{e \epsilon \mathrm{E}^{\mathrm{d}}} \mathbb{p}(e \; is \; pivotal \; for \; A)) \mathbb{p}(p \leq X(e) < p + \varepsilon) + o(\varepsilon)$$

$$= \varepsilon \sum_{e \epsilon \mathrm{E}^{\mathrm{d}}} \mathbb{p}(e \; is \; pivotal \; for \; A) + o(\varepsilon)$$

where the second equality follows from the independence of the state of an edge with the fact that $e$ is pivotal for $A$ or not.

Now, inserting this relation in (**), we get

$$\mathbb{p}_{\mathrm{p}+\varepsilon}(A) = \varepsilon \sum_{e \epsilon \mathrm{E}^{\mathrm{d}}} \mathbb{p}(e \; is \; pivotal \; for \; A) + o(\varepsilon) + \mathbb{p}_{\mathrm{p}}(A).$$

From this we obtain

$$\mathbb{p}_{\mathrm{p}+\varepsilon}(A) - \mathbb{p}_{\mathrm{p}}(A) = \varepsilon \sum_{e \epsilon \mathrm{E}^{\mathrm{d}}} \mathbb{p}(e \; is \; pivotal \; for \; A) + o(\varepsilon).$$

Dividing by $\varepsilon$, we have

$$\frac{\mathbb{p}_{\mathrm{p}+\varepsilon}(A) - \mathbb{p}_{\mathrm{p}}(A)}{\varepsilon} = \sum_{e \epsilon \mathrm{E}^{\mathrm{d}}} \mathbb{p}(e \; is \; pivotal \; for \; A) + \frac{o(\varepsilon)}{\varepsilon}$$

By taking the limit as $\varepsilon \downarrow 0$, we obtain

$$\lim_{\varepsilon \to 0} \frac{\mathbb{p}_{\mathrm{p}+\varepsilon}(A) - \mathbb{p}_{\mathrm{p}}(A)}{\varepsilon} = \sum_{e \epsilon \mathrm{E}^{\mathrm{d}}} \mathbb{p}(e \; is \; pivotal \; for \; A)$$

$$\frac{d}{dp} \mathbb{p}_{\mathrm{p}}(\mathrm{A}) = \sum_{e \epsilon \mathrm{E}^{\mathrm{d}}} \mathbb{p}(e \; is \; pivotal \; for \; A)$$

$$= \mathrm{E}_{\mathrm{p}}[\mathrm{N}(\mathrm{A})]. \qquad \blacksquare$$

**Corollary 2.3.1** [6]: Let $A$ be an increasing event, which depends on the state of finitely many edges of $L^d$, and let $N(A)$ denote the number of edges that are pivotal for $A$. Then for any $0 \leq p_1 < p_2 \leq 1$,

$$\mathbb{p}_{p_2}(A) = \mathbb{p}_{p_1}(A)\exp(\int_{p_1}^{p_2} \frac{1}{p} E_p\,[N(A)|A])dp$$

**Proof** [6]: Since the state of an edge $e$ is independent of the state of the fact that $e$ *is* pivotal for $A$ or not, by Russo's formula we have

$$\begin{aligned}\frac{d}{dp}\mathbb{p}_p(A) &= \sum_{e\epsilon E^d} \mathbb{p}(e\ is\ pivotal\ for\ A)\\ &= \frac{1}{p}\sum_{e\epsilon E^d} \mathbb{p}(\{e\ is\ pivotal\ for\ A\}\cap\{e\ is\ open\})\\ &= \frac{1}{p}\sum_{e\epsilon E^d} \mathbb{p}(\{e\ is\ pivotal\ for\ A\}\cap A)\\ &= \frac{1}{p}\sum_{e\epsilon E^d} \mathbb{p}(e\ is\ pivotal\ for\ A|A)\,\mathbb{p}_p(A)\\ &= \frac{1}{p}E_p\,[N(A)|A]\mathbb{p}_p(A).\end{aligned}$$

Thus, dividing both sides of the last equality by $\mathbb{p}_p(A)$ and integrating from $p_1 to\ p_2$ we get

$$\mathbb{p}_{p_2}(A) = \mathbb{p}_{p_1}(A)\exp\left(\int_{p_1}^{p_2} \frac{1}{p} E_p\,[N(A)|A]\right)dp \qquad \blacksquare$$

**Example** [1]**:** Suppose we take a one-dimensional graph (i.e. a line). Let A denote the event that there is an open path from 0 to n (0↔n). Since all edges between 0 and n must be open for A to be true, the value of N (A) given that A has happened is always n. Using the integral form of Russo's formula, we get:

$$\begin{aligned}\mathbb{p}_{p_2}(A) &= \mathbb{p}_{p_1}(A)\exp\left(n\int_{p_1}^{p_2} \frac{1}{p}\right)dp\\ &= \mathbb{p}_{p_1}(A)\exp\left(n\ln\frac{p_2}{p_1}\right)\end{aligned}$$

From this we obtain that

$$\frac{\mathbb{P}_{p_2}(A)}{\mathbb{P}_{p_1}(A)} = \left(\frac{p_2}{p_1}\right)^n.$$

This gives an indication of the growth rate of the likelihood of A with increase in the percolation probability. In general, for any event dependent on n edges, given that the event has happened, there can be at most n pivotal edges. So, we can generalize the above result to say that for $0 < p_1 < p_2 < 1$

$$\frac{\mathbb{P}_{p_2}(A)}{\mathbb{P}_{p_1}(A)} \leq \left(\frac{p_2}{p_1}\right)^n$$

where A is any increasing event dependent on a fixed number of edges, and this number is n.

# Chapter Three

# Subcritical phase, Supercritical phase, and near the critical threshold

The subcritical and supercritical phases of percolation are characterized respectively by the absence and presence of an infinite open cluster. The Connection probabilities decay exponentially when $p < p_c$, and there is unique infinite cluster when $p > p_c$. It is shown that $p_c = \frac{1}{2}$ for bond percolation on the square lattice.

## 3.1 Subcritical phase

In this section we will see the situation in the subcritical phase, when $p < p_c$. In this case the open cluster $C$ containing the origin is almost surely finite since we have $\theta(\mathrm{p}) = \mathbb{p}_{\mathrm{p}} (|\mathrm{C}| = \infty) = 0$.

**Definition 3.1.1** [1, 6]**:** Let $\{X_n : n \in \mathbb{N}\}$ be a sequence of independent and identically distributed variables with finite mean. Then $N$ is a stopping time for this sequence if for any $n \in \mathbb{N}$, the event $\{N = n\}$ is independent of $X_i$ with $i \geq n + 1$.

**Theorem 3.1.1 (Wald's equation)** [1, 6]: If $\{X_n : n \in \mathbb{N}\}$ is a sequence of independent and identically distributed non negative variables with finite mean $E[X]$, and if $N$ is a stopping time for this sequence, with $E[N] < \infty$, then

$$E[X_1 + X_2 + \cdots + X_N] = E[X]E[N].$$

.

**Proof** [1]: Since

$$\sum_{n=1}^{N} X_n = \sum_{n=1}^{\infty} X_n \, 1_{\{N \geq n\}}$$

by taking the expectations, we obtain that

$$E\left[\sum_{n=1}^{N} X_n\right] = E\left[\sum_{n=1}^{\infty} X_n \, 1_{\{N \geq n\}}\right]$$

$$= \sum_{n=1}^{\infty} E[X_n \, 1_{\{N \geq n\}}] \qquad (3.1)$$

where the last interchange between the expectation and the summation is valid provided that all $X_n$ are non negative. Now, since $N$ is a stopping time for the sequence $\{X_n : n \in \mathbb{N}\}$, $1_{\{N \geq n\}} = 1$ if and only if we have not stopped after having successfully observed $X_1, X_2, \dots, X_{n-1}$. The random variable $1_{\{N \geq n\}}$ is thus determined by $X_1, X_2, \dots, X_{n-1}$ and is independent of $X_n$. Therefore, (3.1) becomes

$$E\left[\sum_{n=1}^{N} X_n\right] = \sum_{n=1}^{\infty} E[X_n]E[1_{\{N \geq n\}}]$$

$$= E[X] \sum_{n=1}^{\infty} \mathbb{p}[N \geq n]$$

$$= E[X]E[N] \qquad \blacksquare$$

### 3.1.1 Exponential decay of the radius of the mean cluster size

Let $S(n)$ be the diamond of radius $n$, that is, the set of all vertices $x \in Z^d$ for which $\delta(0,x) = |x| \leq n$. Let $A_n = \{0 \leftrightarrow \partial S(n)\}$ be the event that there exists an open path connecting the origin to any vertex lying on the surface of $S(n)$, which is denoted by $\partial S(n)$. If we define the radius of $C$ by

$$rad(C) = \max_{x \in C}\{|x|\}$$

then we see that $A_n = \{rad(C) \geq n\}$.

**Theorem 3.1.2** [1, 4, 6]: If $p < p_c$, then there exists $\psi(p) > 0$ such that

$$\mathbb{p}_p(rad(C) \geq n) = \mathbb{p}_p\big(0 \leftrightarrow \partial S(n)\big)$$

$$= \mathbb{p}_p(A_n)$$

$$< \exp\big(-n\psi(p)\big).$$

**Proof**: From corollary 2.3.1, we have that

$$\mathbb{p}_{\mathrm{p}_2}(A_n) = \mathbb{p}_{\mathrm{p}_1}(A_n)\exp\left(\int_{\mathrm{p}_1}^{\mathrm{p}_2} \frac{1}{\mathrm{p}} \mathrm{E}_{\mathrm{p}}\,[\mathrm{N}(A_n)|A_n]\right)\mathrm{dp}\,.$$

By denoting $\mathbb{p}_p(A_n) = g_p(n)$, the above result can be restated as

$$g_{p_1}(n) = g_{p_2}(n) \exp\left( -\int_{p_1}^{p_2} \frac{1}{p} E_p \left[ N(A_n) | A_n \right] \right) dp . \qquad (3.2)$$

By choosing $p_1 < p_c$, we will show that the mean number of pivotal edges grows roughly linearly with $n$ when $p < p_c$ and given that $A_n$ occurs. The idea is that, since $p < p_c$, $\mathbb{P}_p(A_n) \to 0$ as $n \to \infty$. So that if $A_n$ occurs, then it must depend critically on many edges, because there can only be very few paths which connect $0$ to $\partial S(n)$. As a result, one expects that the average number of pivotal edges for $A_n$ linearly increases with $n$. Thus we need to prove that $E_p [N(A_n)|A_n]$ grows roughly linearly with $n$ when $p < p_c$.

Before computing $E_p [N(A_n)|A_n]$, we show the following lemma. Let $e_1, e_2, \ldots, e_N$ denote the (random) edges which are pivotal for $A_n$. Any path that connects the origin to $\partial S(n)$ uses one of these edges, as otherwise they could not be pivotal for $A_n$. The edges are labeled in the order of encountering when we move on such a path from the origin to $\partial S(n)$, and the first and the second end vertices of the pivotal edge $e_i$ are denoted respectively by $x_i$ and $y_i$ in the order of encountering from the origin to the surface $\partial S(n)$, (see figure 7). Hence we have $e_i = \langle x_i, y_i \rangle$. There are at least two edge disjoint paths from the origin 0 to vertex $x_1$ and from vertex $y_{i-1}$ to vertex $x_i$. Indeed, if this was not the case, there would be a pivotal edge between the origin 0 and vertex $x_1$, which is a contradiction. The open cluster appears as a set of well meshed "islands" connected to each other by pivotal edges.

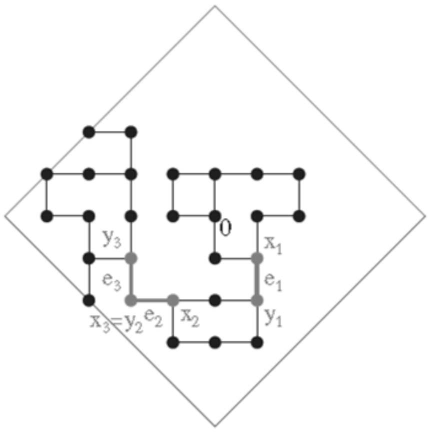


Figure 7[6]: The three edges $e_1$, $e_2$ and $e_3$ are pivotal for $A_5$. For this example

$R_1 = 2, R_2 = 2$ and $R_3 = 0$.

Consider $R_i = \delta(y_{i-1}, x_i)$ for $1 \le i \le N$, with $y_0 = 0$. Thus, the random variables $R_i$ with $1 \le i \le N$ are the distances traversed by the shortest path which connects the origin 0 to the surface $\partial S(n)$, with in the $i^{th}$ "islands" encountered as we walk along this path beginning from the origin. The distribution of the random variables $R_i$ is linked to $E_p\,[N(A_n)|A_n]$ as follows.

If $A_n$ occurs and $R_1 + R_2 + \cdots + R_k \le n - k$, then the number $N(A_n)$ of pivotal edges for $A_n$ must be at least $k$. As a result,

$$\mathbb{p}_p(R_1 + R_2 + \cdots + R_k \le n - k|A_n) \le \mathbb{p}_p(N(A_n) \ge k|A_n). \tag{3.3}$$

Therefore, we get

$$E_p\,[N(A_n)|A_n] = \sum_{k=1}^{\infty} \mathbb{p}_p(N(A_n) \ge k|A_n)$$

$$\ge \sum_{k=1}^{\infty} \mathbb{p}_p(R_1 + R_2 + \cdots + R_k \le n - k|A_n).$$

So to compute $E_p\,[N(A_n)|A_n]$, we need the distribution of the sum of the variables $R_i$. The first immediate step will enable us to replace this sum by a sum of independent and identically distributed random variables, whose distribution is easier to compute.

Let $M = \max\{k|A_k\ occurs\}$ be the radius of the largest ball whose surface is joined to the origin by an open path. The following lemma shows that the random variables $R_1, R_2, R_3, \ldots$ are jointly smaller in distribution than a sequence $M_1, M_2, M_3, \ldots$ of independent and identically distributed random variables distributed as $M$.

**Lemma 3.1.1** [4, 6]: Let $k \in \mathbb{N}$ and suppose that $r_1, r_2, \ldots, r_k \in \mathbb{N}$ such that

$\sum_{i=1}^{k} r_i \le n - k$. For $0 < p < 1$, we have

$$\mathbb{p}_p(R_k \le r_k, R_i = r_i, 1 \le i \le k-1|A_n) \ge$$
$$\mathbb{p}_p(M \le r_k)\mathbb{p}_p(R_i = r_i, 1 \le i \le k-1|A_n) \tag{3.4}$$

**Proof**: Let $k = 1$ and $r_1 \le n - 1$. If $R_1 > r_1$, then the first end-vertex, $x_1$ of the first pivotal edge $e_1$, lies either outside the ball $S(r_1 + 1)$ or on its surface $\partial S(r_1 + 1)$ (as shown in figure 8). Hence we see that $\{R_1 > r_1\} \subseteq A_{r_1+1}$. Since there are at least two edge disjoint paths between 0 and $x_1$, we obtain that $\{R_1 > r_1\} \cap A_n \subseteq A_{r_1+1} \circ A_n$.

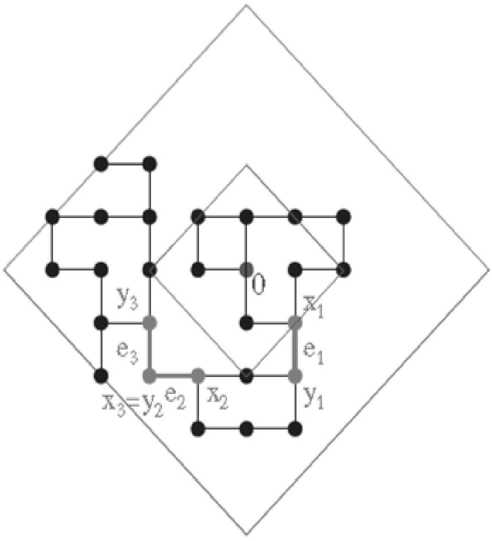


Figure 8 [6]: There are at least two edge disjointopen paths connecting 0 to $\partial S( r_1 + 1)$.

Since both $A_{r_1+1}$ and $A_n$ are increasing events, by applying the BK inequality we get

$$\mathbb{P}_p(\{ R_1 > r_1\} \cap A_n) \leq \mathbb{P}_p(A_{r_1+1} \circ A_n)$$

$$\leq \mathbb{P}_p(A_{r_1+1})\mathbb{P}_p(A_n).$$

Noting that $\mathbb{P}_p(A_{r_1+1}) = \mathbb{P}_p(M \geq r_1 + 1)$ and dividing both sides of the above inequality by $\mathbb{P}_p(A_n)$, we get that

$$\mathbb{P}_p(\{ R_1 > r_1\}|A_n) \leq \mathbb{P}_p(M \geq r_1 + 1).$$

Thus for $k = 1$, we find

$\mathbb{P}_p( R_k \leq r_k, R_i = r_i, 1 \leq i \leq k-1|A_n) \geq \mathbb{P}_p(M \leq r_k)\mathbb{P}_p( R_i = r_i, 1 \leq i \leq k-1|A_n)$.

Now suppose that $k > 1$. For any edge $e = \langle u, v\rangle$, let $G_e$ be the set of vertices that are attainable from the origin along open paths not using $e$, together with all open edges between these vertices. Let $B_e$ be the event that

$(a)$ $e$ is open,

$(b)$ $u \in G_e$ and $v \notin G_e$,

$(c)$ $G_e$ contains no vertex of $\partial S(n)$, and

$(d)$ the pivotal edges for the event $\{0 \leftrightarrow \partial S(n)\}$ are $e_1 = \langle x_1, y_1 \rangle$, $e_2 = \langle x_2, y_2 \rangle, \dots$ , $e_{k-1} = \langle x_{k-1}, y_{k-1} \rangle = e$, where $\delta(y_{i-1}, x_i) = r_i$ for all $1 \leq i \leq k-1$.

Let $B = \bigcup_e B_e$. Then for any $\omega \in A_n \cap B$, there is a unique edge $e = e(\omega)$ such that $B_e$ occurs and hence $B$ also occurs.

For $\omega \in B$, let $e = e(\omega)$ be an edge that satisfies all the conditions $(a) - (d)$ listed above. Let $G = G_e \cup \{e(\omega)\}$ be the set of vertices attainable from the origin along open path not using this edge $e$, together with all open edges between these vertices, to which the open pivotal edge $e = e(\omega)$ and its other end vertex $y_{i-1}$ , which is also denoted by $y(G)$ (See figure 9).

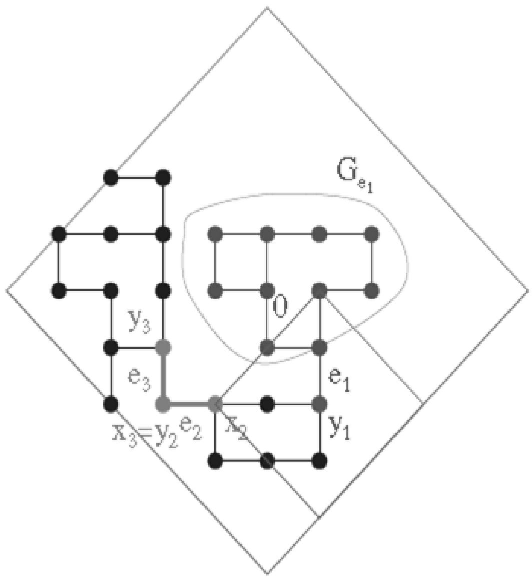


Figure 9 [6]: The circled set of vertices is the set $G_{e_1}$. At least two edge disjoint open paths connect $y(G_{e_1}) = y_1$ to $\partial S(r_2 + 1,\ y_1)$ without passing by any vertex of $G_{e_1}$ (here $k = 2$ and $r_k = r_2 = 1$).

By conditioning on $G$, we get

$$\mathbb{P}_p(A_n \cap B) = \sum_{\Gamma} \mathbb{P}_p(B, G = \Gamma) \mathbb{P}_p(A_n | B, G = \Gamma)\,,$$

where the sum is over all possible values $\Gamma$ of $G$. Now, if graph $\Gamma$ is given, $A_n$ occurs if and only if the vertex $y(\Gamma)$ is connected to $\partial S(n)$ by an open path which does not have any vertex other than $y(\Gamma)$ in common with $\Gamma$. Hence

$$\mathbb{P}_p(A_n \cap B) = \sum_{\Gamma} \mathbb{P}_p(B, G = \Gamma)\mathbb{P}_p(y(\Gamma) \leftrightarrow \partial S(n)\ off\ \Gamma). \tag{3.5}$$

In a similar way, if the graph $\Gamma$ is given, then for $\{ R_k > r_k\}$ the first end vertex $x_k$ of the $k^{th}$ pivotal edge $e_k$ lies either outside the ball $S(r_k + 1, y(\Gamma))$ of radius $r_k + 1$ centered on $y(\Gamma) = y_{k-1}$, or on its surface $\partial S(r_k + 1, y(\Gamma))$. Hence we have $\{ R_k > r_k\} \subseteq \{y(\Gamma) \leftrightarrow \partial S(r_k + 1, \Gamma)\ off\ \Gamma\}$. Since there are at least two edge disjoint open paths between $y(\Gamma) = y_{k-1}$, which avoids any edge of $\Gamma$, we have therefore, conditionally to $G = \Gamma$, that

$$\{ R_k > r_k\} \cap A_n \subseteq \{y(\Gamma) \leftrightarrow \partial S(r_k + 1, y(\Gamma))\}\circ\{y(\Gamma) \leftrightarrow \partial S(n)\ off\ \Gamma\}.$$

Now by the space invariance we see that $\mathbb{P}_p\left(y(\Gamma) \leftrightarrow \partial S\left(r_k + 1, y(\Gamma)\right)\right) = \mathbb{P}_p(A_{k+1})$. Therefore, by conditioning on $G$ and using the BK inequality, we obtain

$$\mathbb{P}_p(\{ R_k > r_k\} \cap A_n \cap B) = \sum_{\Gamma} \mathbb{P}_p(B, G = \Gamma)\mathbb{P}_p(\{ R_k > r_k\} \cap A_n | B, G = \Gamma)$$

$$\leq \sum_{\Gamma} \mathbb{P}_p(B, G = \Gamma)\mathbb{P}_p\left(\{y(\Gamma) \leftrightarrow \partial S\left(r_k + 1, y(\Gamma)\right)\}\circ\{y(\Gamma) \leftrightarrow \partial S(n)\ off\ \Gamma\}\right)$$

$$\leq \sum_{\Gamma} \mathbb{P}_p(B, G = \Gamma)\mathbb{P}_p\left(y(\Gamma) \leftrightarrow \partial S\left(r_k + 1, y(\Gamma)\right)\right)\mathbb{P}_p(y(\Gamma) \leftrightarrow \partial S(n)\ off\ \Gamma)$$

$$= \sum_{\Gamma} \mathbb{P}_p(B, G = \Gamma)\mathbb{P}_p\left(A_{r_k+1}\right)\mathbb{P}_p(y(\Gamma) \leftrightarrow \partial S(n)\ off\ \Gamma)$$

$$= \mathbb{P}_p\left(A_{r_k+1}\right)\mathbb{P}_p(A_n \cap B) \tag{3.6}$$

where the later equality follows from (3.5).

Dividing both sides of (3.6) by $\mathbb{P}_p(A_n \cap B)$, we get

$$\mathbb{P}_p( R_k > r_k | A_n \cap B) \leq \mathbb{P}_p\left(A_{r_k+1}\right).$$

And multiplying the later by $\mathbb{P}_p(B|A_n)$ we obtain

$$\mathbb{P}_p(\{ R_k > r_k\} \cap B | A_n) \leq \mathbb{P}_p\left(A_{r_k+1}\right)\mathbb{P}_p(B|A_n).$$

Now, we have

$$\mathbb{p}_p\left(A_{r_k+1}\right) = \mathbb{p}_p(M > r_k + 1)$$

$$\mathbb{p}_p(B|A_n) = \mathbb{p}_p(\, R_i = \, r_i \;\; for\; 1 \leq i \leq k-1|A_n)$$

$$\mathbb{p}_p(\{\, R_k \leq \, r_k\} \cap B|A_n) = \, \mathbb{p}_p(\, R_k \leq \, r_k \,, R_i = \, r_i \;\; for\; 1 \leq i \leq k-1|A_n).$$

from which we deduce that

$\mathbb{p}_p(\, R_k \leq r_k, R_i = r_i, 1 \leq i \leq k-1|A_n) \geq \mathbb{p}_p(M \leq r_k)\mathbb{p}_p(\, R_i = r_i, 1 \leq i \leq k-1|A_n)$. ■

From (3.4), we obtain that

$$\mathbb{p}_p(\, R_1 + \cdots + R_k \leq n-k|A_n) = \sum_{\mathrm{i}=0}^{\mathrm{n-k}} \mathbb{p}_p(\, R_1 + \cdots + R_{k-1} = i, R_k \leq n-k-i|A_n)$$

$$\geq \sum_{\mathrm{i}=0}^{\mathrm{n-k}} \mathbb{p}_p(\, R_1 + R_2 + \cdots + R_{k-1} = i|A_n)\mathbb{p}_p(M \leq n-k-i)$$

$$= \mathbb{p}_p(\, R_1 + R_2 + \cdots + R_{k-1} + M_k \leq n-k|A_n)$$

where $M_k$ is a random variable independent from the state of all edges in $S(n)$, and distributed as $M$. Iterating this operation $(k-1)$ more times, we find that

$$\mathbb{p}_p(\, R_1 + R_2 + \cdots + R_k \leq n-k|A_n) \geq \mathbb{p}(\, M_1 + M_2 + \cdots + M_k \leq n-k) \qquad (3.7)$$

where $M_1, M_2, \ldots, M_k$ is a sequence of independent and identically distributed random variables distributed as $M$. We can now find a lower bound on $\mathrm{E_p}\,[\mathrm{N}(A_n)|A_n]$ by using (3.7), and this is the second intermediate result.

**Lemma 3.1.2** [4, 6]: For $0 < p < 1$,

$$\mathrm{E_p}\,[\mathrm{N}(A_n)|A_n] \geq \frac{n}{\sum_{i=0}^{n} g_p(i)} - 1. \qquad (3.8)$$

**Proof**: From (3.3) we remember that, if $R_1 + R_2 + \cdots + R_k \leq n-k$ and if $A_n$ occurs, then the number $\mathrm{N}(A_n)$ of pivotal edges for $A_n$ must be at least $k$. Consequently, we have

$$\mathbb{p}_p[\mathrm{N}(A_n) \geq \mathrm{k}|A_n] \geq \mathbb{p}_p(\, R_1 + R_2 + \cdots + R_k \leq n-k|A_n)$$

$$\geq \mathbb{p}(\, M_1 + M_2 + \cdots + M_k \leq n-k).$$

Now, since $\mathbb{p}(\, M_i \geq r) = \mathbb{p}_p(M \geq r) = g_p(r) \to \theta(p)$ for $r \to \infty$, we make change of variable to avoid having $\mathbb{p}(\, M_i \geq r) > 0$ for $r \to \infty$ when $p_c < p$. Let

$$M_i^{'} = 1 + \min\{M_i, n\}.$$

. Then

$$\mathbb{p}( M_1 + M_2 + \cdots + M_k \le n - k) = \mathbb{p}(M_1^{'} + M_2^{'} + \cdots + M_k^{'} \le n).$$

Therefore, we can continue with these truncated random variables and hence we have

$$\mathrm{E_p}\,[\mathrm{N}(A_n)|A_n] = \sum_{\mathrm{k}=1}^{\infty} \mathbb{p}_p[\mathrm{N}(A_n) \ge \mathrm{k}|A_n]$$

$$\ge \sum_{\mathrm{k}=1}^{\infty} \mathbb{p}(M_1^{'} + M_2^{'} + \cdots + M_k^{'} \le n)$$

$$\ge \sum_{\mathrm{k}=1}^{\infty} \mathbb{p}(K \ge k + 1)$$

$$= \mathrm{E[K]} - 1$$

where $K = \min\{k|M_1^{'} + M_2^{'} + \cdots + M_k^{'} > n\}$.

Since the $M_i^{'}$s are independent and identically distributed bounded random variables, by Wald's equation we get

$$\mathrm{E}[M_1^{'} + M_2^{'} + \cdots + M_k^{'}] = \mathrm{E}[M_i^{'}]\mathrm{E[K]}.$$

Since $M_1^{'} + M_2^{'} + \cdots + M_k^{'} > n$ by definition of K, it follows that

$$\mathrm{E}[M_1^{'} + M_2^{'} + \cdots + M_k^{'}] = \mathrm{E}[M_i^{'}]\mathrm{E[K]} > n$$

whence

$$\mathrm{E[K]} > \frac{n}{\mathrm{E}[M_i^{'}]}$$

$$= \frac{n}{1 + \mathrm{E}[\min\{M_i, n\}]}$$

$$= \frac{n}{1 + \mathrm{E}[\min\{M,\ n\}]}$$

$$= \frac{n}{1 + \sum_{\mathrm{j=1}}^{\mathrm{n}} \mathrm{p}(\mathrm{M} \ge \mathrm{j})}$$

$$= \frac{n}{\sum_{\mathrm{j=0}}^{\mathrm{n}} g_{\mathrm{p}}(\mathrm{j})} \qquad \blacksquare$$

Inserting (3.8) in (3.2), we obtain that

$$g_{\mathrm{p}_1}(n) \leq g_{\mathrm{p}_2}(n)\exp\left(-\int_{\mathrm{p}_1}^{\mathrm{p}_2}\frac{1}{\mathrm{p}}\left(\frac{n}{\sum_{\mathrm{i}=0}^{\mathrm{n}} g_{\mathrm{p}}(\mathrm{i})}-1\right)\right)\mathrm{dp}\,.$$

Since this integral is difficult to compute as such, we replace functions of $p$ in the integral as follows: $\frac{1}{\mathrm{p}} \geq 1$ and $g_{\mathrm{p}}(\mathrm{i}) \leq g_{\mathrm{p}_2}(i)$ and for any $n \in \mathbb{N}$ we find that

$$g_{\mathrm{p}_1}(n) \leq g_{\mathrm{p}_2}(n)\exp\left(-(p_2-p_1)\left(\frac{n}{\sum_{i=0}^{n} g_{p_2}(i)}-1\right)\right). \tag{3.9}$$

Still we need one intermediate result, namely we need that $\sum_{i=0}^{n} g_{p_2}(i)$ is finite.

**Lemma 3.1.3** [4, 6]: For $0 < p < p_c$ and for $n \geq 1$, there exists $\delta(p) < \infty$ such that

$$g_{\mathrm{p}}(\mathrm{n}) \leq \delta(p)n^{-1/2} \qquad (3.10).$$

**Proof**: For any $n' \geq n$, (3.9) becomes

$$g_{\mathrm{p}_1}(n') \leq g_{\mathrm{p}_2}(n')\exp\left(-(p_2-p_1)\left(\frac{n'}{\sum_{i=0}^{n'} g_{p_2}(i)}-1\right)\right)$$

$$\leq g_{\mathrm{p}_2}(n)\exp\left((p_2-p_1)\left(1-\frac{n'}{\sum_{i=0}^{n'} g_{p_2}(i)}\right)\right)$$

$$\leq g_{\mathrm{p}_2}(n)\exp\left(\left(1-\frac{n'(p_2-p_1)}{\sum_{i=0}^{n'} g_{p_2}(i)}\right)\right)$$

because $g_{\mathrm{p}_2}(n) = \mathbb{p}_{\mathrm{p}_2}(A_n)$ is a decreasing function of $n$ and $n' \geq n$. Now, we can decompose the summation

$$\frac{1}{n'}\sum_{i=0}^{n'} g_{p_2}(i) = \frac{1}{n'}\left(\sum_{i=0}^{n-1} g_{p_2}(i) + \sum_{i=n}^{n'} g_{p_2}(i)\right)$$

$$\leq \frac{1}{n'}\left(n g_{p_2}(0) + (n'-n+1)g_{p_2}(n)\right)$$

$$\leq \frac{1}{n'}\left(n + n' g_{p_2}(n)\right)$$

$$\leq 3g_{p_2}(n)$$

by choosing $n^{'} = n\left\lfloor\frac{1}{g_{p_2}(n)}\right\rfloor$ from which we get $n \leq 2n^{'}g_{p_2}(n)$. Consequently, we have

$$g_{\mathrm{p}_1}(n^{'}) \leq g_{\mathrm{p}_2}(n)\exp\left(1 - \frac{(p_2 - p_1)}{3g_{\mathrm{p}_2}(n)}\right) \tag{3.11}$$

Now assume that $\mathrm{p}_2 < \mathrm{p}_\mathrm{c}$ and choose $\mathrm{p}_1$ by

$$\mathrm{p}_1 = \mathrm{p}_2 - 3g_{\mathrm{p}_2}(n)\left(1 - \ln g_{\mathrm{p}_2}(n)\right). \tag{3.12}$$

Since $g_{\mathrm{p}_2}(n)\left(1 - \ln g_{\mathrm{p}_2}(n)\right) \to 0$ for $n \to \infty$, we pick $n$ large enough so as to have the right hand side of (11) strictly positive. Substituting (3.12) in (3.11) we find that

$$g_{\mathrm{p}_1}(n^{'}) \leq (g_{\mathrm{p}_2}(n))^2 \tag{3.13}$$

Now fix $0 < p < p_c$ and use the above argument to construct a subsequence $(n_1, n_2, \dots, n_i, \dots)$ along which $g_\mathrm{p}(n_i)$ approaches 0 quickly. Pick $q$ so that $p < q < p_c$, and construct two sequences. The first one is a sequence of probabilities $\mathrm{p_i}$ beginning at $\mathrm{p}_0 = \mathrm{q}$ and the second one is a sequence of integers $n_i$ beginning at a value $n_0$ which shall be picked latter, and defined as

$$n_{i+1} = n_i\gamma_i = n_i\left\lfloor\frac{1}{g_{p_i}(n_i)}\right\rfloor \tag{3.14}$$

$$p_{i+1} = p_i - 3g_{\mathrm{p_i}}(n_i)\left(1 - \ln g_{\mathrm{p_i}}(n_i)\right) \tag{3.15}$$

where $\gamma_i = \left\lfloor\frac{1}{g_{p_i}(n_i)}\right\rfloor$. Now we find that $n_{i+1} \geq n_i$ and $p_{i+1} < p_i$. Still we need to

check that $p_i > 0$, and we will pick $n_0$ large enough to ensure it. Because of the way we constructed the sequences (3.14) and (3.15), and the discussion leading to (3.13), we find that

$$g_{\mathrm{p_{j+1}}}(n_{j+1}) \leq (g_{\mathrm{p_j}}(n_j))^2 \tag{3.16}$$

for $0 \leq j \leq i$. Now, any real sequence $\{x_j\}$ starting at a value $0 < x_0 < 1$ and defined by $x_{j+1} = {x_j}^2$ converges so quickly to 0 that the infinite sum $\sum_{j=0}^{\infty} 3x_j\left(1 - \ln x_j\right) < \infty$, and moreover converges to zero if $x_0 \to 0$ . Therefore, we may pick $x_0$ sufficiently small that this infinite sum is smaller or equal to $q - p$, and next pick $n_0$ sufficiently large that

$$\mathbb{p}_q(A_n) = g_{\mathrm{q}}\,(n_0) < x_0.$$

By using the fact that $3x(1 - lnx)$ is an increasing function, we iterate (3.15) to obtain

$$p_{i+1} = p_i - 3g_{\mathrm{p_i}}(n_i)\left(1 - \ln g_{\mathrm{p_i}}\,(n_i)\right)$$

$$= p_0 - \sum_{j=0}^{i} 3g_{\mathrm{p_j}}(n_j)\left(1 - \ln g_{\mathrm{p_j}}(n_j)\right)$$

$$\geq q - \sum_{j=0}^{\infty} 3g_{\mathrm{p_j}}(n_j)\left(1 - \ln g_{\mathrm{p_j}}(n_j)\right)$$

$$\geq q - (q - p)$$

$$= p$$

Therefore, by suitably choosing $n_0$ large enough, we guarantee that $p_i > 0$ for all $i$.

Moreover, we get that

$$\lim_{i\to\infty} p_i \geq p.$$

Now let us turn our attention to the other sequence, and expand (3.14) to obtain

$$n_{i+1} = n_0\gamma_0\gamma_1 \dots \gamma_i.$$

In addition to this, expanding (3.16), we obtain that

$$\left(g_{\mathrm{p_i}}\,(n_i)\right)^2 = g_{\mathrm{p_i}}\,(n_i) g_{\mathrm{p_i}}\,(n_i)$$

$$\leq g_{\mathrm{p_i}}\,(n_i)\left(g_{\mathrm{p_i}-1}\,(n_{i-1})\right)^2$$

$$\leq g_{\mathrm{p_i}}\,(n_i) g_{\mathrm{p_i}-1}\,(n_{i-1})\left(g_{\mathrm{p_i}-2}\,(n_{i-2})\right)^2 \leq \cdots$$

$$\leq g_{\mathrm{p_i}}\,(n_i) g_{\mathrm{p_i}-1}\,(n_{i-1}) \dots g_{\mathrm{p_1}}\,(n_1)\left(g_{\mathrm{p_0}}\,(n_0)\right)^2$$

$$\leq \left(\gamma_i\gamma_{i-1} \dots \gamma_{1\gamma_0}\right)^{-1} g_{\mathrm{p_0}}\,(n_0)$$

$$= \delta^2 {n_{i+1}}^{-1}, \text{ where } \delta^2 = n_0 g_{\mathrm{p_0}}\,(n_0).$$

Finally, we fill in the gaps in the sequence $n_1, n_2, \dots, n_i, \dots$. Let $n > n_0$, and $i$ be such that $n_{i-1} \leq n < n_i$ (since $g_{\mathrm{p_i}}(n_i) \to 0$ for $i \to \infty$, $n_{i-1} < n_i$ for large $i$). Then, since $p < \mathrm{p_i}$, we have $g_{\mathrm{p}}(\mathrm{n}) \leq g_{\mathrm{p_i}-1}(n_{i-1}) \leq \delta n_i^{-1/2} \leq \delta n^{-1/2}$. This is valid for $n < n_0$, and adjusting $\delta$ we make a similar inequality valid for $n \geq 1$. ■

A consequence of this lemma, and more precisely of (3.10), shows that there is some $\Delta(p) < \infty$ such that

$$\sum_{i=0}^{n} g_{\mathrm{p}}(\mathrm{i}) \leq \Delta(p) n^{1/2}$$

for $p < p_c$. Assume that $p_1 < p_c$, and pick $p = p_2$ so that $p_1 < p_2 = p < p_c$. By inserting this sum in (3.9) we obtain that

$$g_{\mathrm{p_1}}(n) \leq g_{\mathrm{p_2}}(n) \exp\left(-(p_2 - p_1)\left(\frac{n^{1/2}}{\Delta(p)} - 1\right)\right)$$

$$\leq \exp\left(1 - \frac{(p_2 - p_1)}{\Delta(p)} n^{1/2}\right)$$

and hence we have

$$\sum_{i=0}^{\infty} g_{\mathrm{p_1}}(n) < \infty$$

for all $p_1 < p_c$. Consequently, since $p_2 < p_c$, we find that

$$\mathrm{E_{p_2}}[M] = \sum_{i=0}^{\infty} g_{\mathrm{p_2}}(n) < \infty.$$

Now inserting this relation in (3.9), we obtain

$$g_{\mathrm{p_1}}(n) \leq g_{\mathrm{p_2}}(n) \exp\left(-(p_2 - p_1)\left(\frac{n}{\mathrm{E_{p_2}}[M]} - 1\right)\right)$$

$$\leq \exp\left(-\frac{p_2 - p_1}{\mathrm{E_{p_2}}[M]} n\right)$$

$$= \exp(-\psi(p_1) n)$$

where $\psi(p_1) = \frac{p_2 - p_1}{\mathrm{E_{p_2}}[M]} > 0$ and this completes the long proof. ■

## 3.2 Supercritical phase

In this section the situation in the supercritical phase, when $\mathrm{p} > p_c$, will be discussed. In this case there is almost surely an open cluster of infinite size. But how many are there? We will first show that there is exactly one such cluster. The next question will be evaluating the size of the other, finite cluster.

### 3.2.1 Uniqueness of the infinite open cluster

**Theorem 3.2.1**[6]: If $p > p_c$, then $\mathbb{p}_p(\textit{there exists exactly one open cluster}) = 1$.

**Proof:** Let $Y$ be the number of infinite open clusters. Since the sample space

$$\Omega = \prod_{e \in E^d} \{0,1\}$$

is a product space with a space invariant product measure $\mathbb{p}_p$, $Y$ is a translation invariant function on $\Omega$. A property of translation invariant functions is to be almost surely constant. Therefore, there exists some $k \in \mathbb{N} \cup \{\infty\}$ such that $\mathbb{p}_p(Y = k) = 1$.

Since $p > p_c$, we have $k \neq 0$. We will prove by contradiction that

(i) $k \notin [2, \infty)$ and

(ii) $k \neq \infty$, which therefore implies that $k = 1$.

Now let us see the two cases one by one.

$(i)$. Suppose first that $2 \leq k < \infty$. Let $S(n) = \{x \in Z^d : \delta(0, x) = |x| \leq n\}$ be the diamond of radius $n$ (i.e. the ball of radius $n$ with graph theoretic distance). Let $Y(0)$ be the number of infinite open clusters when all edges of $S(n)$ are closed. Since the probability that all edges of $S(n)$ are closed is strictly positive, we have

$$\mathbb{p}_p(Y(0) = k) = \frac{\mathbb{p}_p(\{Y = k\} \cap \{\textit{all edge of } S(n) \textit{ are closed}\})}{\mathbb{p}_p(\textit{all edges of } S(n) \textit{ are closed})} = 1.$$

Similarly, if $Y(1)$ is the number of infinite open clusters when all edges of $S(n)$ are open, $\mathbb{p}_p(Y(1) = k) = 1$, and therefore $\mathbb{p}_p\big(Y(0) = Y(1)\big) = 1$. We always have that $Y(0) \geq Y(1)$, but since there are only a finite number of infinite open clusters, we have $Y(0) = Y(1)$ if and only if $S(n)$ intersects exactly one such cluster. So, if $M_{S(n)}$ is the number of infinite open clusters that intersects $S(n)$, then we have

$\mathbb{P}_p\left(M_{S(n)} \geq 2\right) = 0$ for all $n \in \mathbb{N}$. Now by letting $n \to \infty$, we obtain that the diamond $S(n)$ becomes the entire lattice $L^d$ and therefore we see that

$$0 = \lim_{n\to\infty} \mathbb{P}_p\left(M_{S(n)} \geq 2\right)$$

$= \mathbb{P}_p(Y \geq 2)$, a contradiction with $\mathbb{P}_p(Y = k) = 1$ for some $2 \leq k < \infty$.

(ii)Suppose next that $k = \infty$. We use a geometric argument to get a contradiction, which is based on the following object. We call a vertex $x$ a trifurcation (see figure 10 below) if

(a) $x$ belongs to an infinite open cluster;
(b) There exist exactly three open edges incident to $x$; and
(c) The deletion of $x$ and of the three edges incident to $x$ splits the infinite open cluster containing $x$ in to exactly three disjoint infinite clusters and no finite cluster.

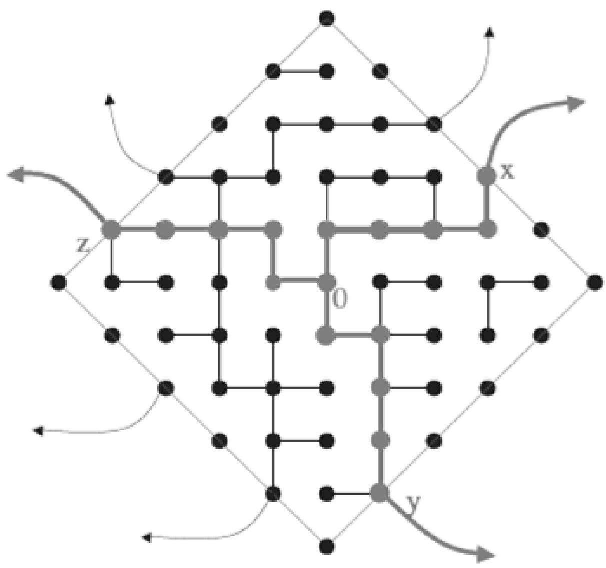


Figure 10 [6]: A sufficient condition for 0 to be a trifurcation if the three paths from $x, y$ and $z$ are open, and all other edges $S(n)$ are closed, and $x, y$ and $z$ belong to three distinct infinite open clusters. The arrows outside $\partial S(n)$ represent the connectivity to distinct infinite clusters.

Because of the space invariance of $L^d$, the probability that a vertex $x$ is a trifurcation is independent of $x$, and therefore we have

$$\mathbb{p}_p(x\ is\ a\ trifurcation\ ) = \mathbb{p}_p(0\ is\ a\ trifurcation). \qquad **$$

Let us show that this probability is non-zero. Let $M_{S(n)}(0)$ be the number of infinite open clusters that intersects $S(n)$ when all edges of $S(n)$ are closed. Then we get that $M_{S(n)}(0) \geq M_{S(n)}$. Therefore, we have

$$\mathbb{p}_p\big(M_{S(n)}(0) \geq 3\big) \geq \mathbb{p}_p\big(M_{S(n\ )} \geq 3\big) \to \mathbb{p}_p(Y \geq 3) = 1$$

as $n \to \infty$. Consequently, there is $n \in \mathbb{N}$ such that $\mathbb{p}_p\big(M_{S(n)}(0) \geq 3\big) \geq \frac{1}{2}$, and fix $n$ to this value from now on until we have shown that the probability of having a trifurcation at the origin is non-zero. If $M_{S(n)}(0) \geq 3$ then there exist three vertices $x, y, z \in \partial S(n)$ lying in three distinct infinite open clusters. Moreover, there are three paths inside $S(n)$ which join the origin to $x, y, z$ respectively, such that the origin is the unique vertex common to any two of them, and each touches exactly one vertex on $\partial S(n)$. For a configuration of edges $\omega \in \{M_{S(n)}(0) \geq 3\}$, we pick $x = x(\omega)$, $y = y(\omega)$ and $z = z(\omega)$ and the three paths as just described.

Now let $J_{x,y,z}$ be the event that all edges in these three paths are open and that all other edges in $S(n)$ are closed. Then we have

$$\mathbb{p}_p\big(J_{x,y,z}\big|M_{S(n)}(0) \geq 3\big) \geq (\min\{p, 1-p\})^{R(n)}$$

where $R(n)$ is the total number of edges in $S(n)$. Now, if $M_{S(n)}(0) \geq 3$ and if $J_{x,y,z}$ occurs, then $x$ is a trifurcation. Therefore,

$$\mathbb{p}_p(0\ is\ a\ trifurcation) \geq \mathbb{p}_p\big(J_{x,y,z}\big|M_{S(n)}(0) \geq 3\big)\mathbb{p}_p\big(M_{S(n)}(0) \geq 3\big)$$

$$\geq \frac{1}{2}(\min\{p, 1-p\})^{R(n)}$$

$$> 0\ .$$

Because of $**$, we therefore have that

$$\mathbb{p}_p(x\ is\ a\ trifurcation\ ) > 0$$

for all vertices $x \in Z^d$. Let $T(m)$ be the number of trifurcations in $S(m)$. Since

$$\mathrm{E}_p[T(m)] = |S(m)|\mathbb{p}_p(0\ is\ a\ trifurcation),$$

it implies that $T(m)$ grows in the manner of $|S(m)|$ as $m \to \infty$. The contradiction is obtained by the following geometric argument. Pick a trifurcation $t_1$ in $S(m)$, and take a vertex $x_1 \in \partial S(m)$ that is connected to $t_1$ by an open path in $S(m)$. Pick a second trifurcation $t_2$ in $S(m)$. Then by definition of a trifurcation, there must be a vertex $x_2 \in \partial S(m)$, distinct from $x_1$, such that $t_2 \leftrightarrow x_2$ in $S(m)$ (see figure 11 shown below).

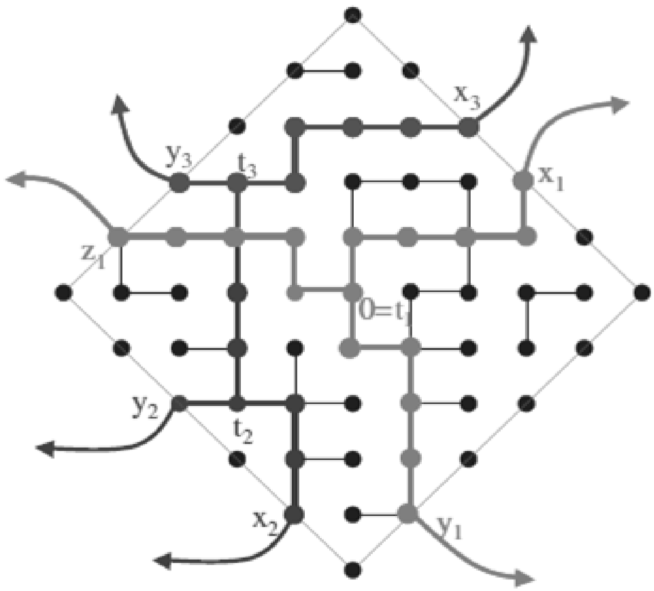


Figure 11 [6]: finding trifurcations in $S(n)$

Repeat this operation, at each stage by picking a new trifurcation $t_i$ and new vertex $x_i \in \partial S(m)$, with $t_i \leftrightarrow x_i$ in $S(m)$. There are $T(m)$ trifurcations in $S(m)$, and so we end up with $T(m)$ distinct vertices $x_i \in \partial S(m)$, which implies that

$$|\partial S(m)| \geq T(m).$$

But $T(m)$ grows in the manner of $|S(m)|$ for large $m$, which would mean that $|\partial S(m)|$ would grow in the manner of $|S(m)|$ for large $m$ as well. Hence we have reached a contradiction, because $|S(m)|$ grows in the manner of $m^d$ while $|\partial S(m)|$ grows in the manner of $m^{d-1}$.

## 3.2.2 Finite cluster size distribution

We are now interested in the size of the finite clusters which show that the size of the cluster containing the origin is exponentially decreasing with the size. We only consider the two dimensional case, where the proof is easier because of the duality. The Theorem is however valid for $d \geq 3$.

**Theorem 3.2.2** [6]: (exponential decay of the finite cluster size distribution).

If $p_c < p < 1$, then there exists $\eta(p) > 0$ such that

$$\mathbb{p}_p(|C| = n) \leq \exp(-n^{\frac{d-1}{d}}\eta(p))$$

for $n \in \mathbb{N}$.

It is also possible to find a lower bound of the form: there exists some $\gamma(p) < \infty$ such that

$$\mathbb{p}_p(|C| = n) \geq \exp\left(-n^{\frac{d-1}{d}}\gamma(p)\right).$$

**Proof**: We only prove a slightly weaker bound $\mathbb{p}_p(|C| = n) \leq n\exp\left(-\sqrt{n}\eta(p)\right)$, and only for $d = 2$ and for $\frac{2}{3} < p < 1$.

Suppose that the origin belongs to a finite cluster of size $n$. Then there exists a closed circuit in the dual lattice $L_d^2$, having the origin in its interior. Then this circuit has less than $n$ vertices. Moreover, it can be shown by using topological arguments that there is some value $\delta > 0$ such that this closed circuit contains at least $\delta\sqrt{n}$ vertices. The circuit must pass through a vertex of the form $(i + \frac{1}{2}, \frac{1}{2})$ for some $0 \leq i \leq n - 1$, because

$(a)$ to surround the origin, it has to pass through a vertex $(i + \frac{1}{2}, \frac{1}{2})$ for some $i \geq 0$,

$(b)$ it can not pass through a vertex $(i + \frac{1}{2}, \frac{1}{2})$ for some $i \geq n$ since it would then be at least $2n$. Therefore, one of these $n$ vertices must lie in a closed cluster of $L_d^2$ of size at least $\delta\sqrt{n}$ . Let us call this vertex $0_d$, and the closed cluster to which it belongs $C_d$. Now, each edge of $L_d^2$ is closed with probability $(1 - p)$, and $1 - p < \frac{1}{3} \leq p_c$ because of Theorem 2.2.1. In other words, the process of closed edges of $L_d^2$ is subcritical. Therefore, there exists $\lambda(p) > 0$ such that

$$\mathbb{p}_p\left(|C_d| \geq \delta\sqrt{n}\right) \leq \exp\left(-\lambda(p)\delta\sqrt{n}\right).$$

Since

$\mathbb{p}_p\left(\left(i+\frac{1}{2},\frac{1}{2}\right) \text{lies in a closed cluster of } L_d^2 \; of\ size\ at\ least\ \delta\sqrt{n}\right) = \mathbb{p}_p\left(|C_d| \geq \delta\sqrt{n}\right)$, we have that

$$\mathbb{p}_p(|C| = n) \leq \sum_{i=0}^{n-1} \mathbb{p}_p\left(\left(i+\frac{1}{2},\frac{1}{2}\right) \text{lies in a closed cluster of } L_d^2 \; of\ size\ at\ least\ \delta\sqrt{n}\right)$$

$$= n\mathbb{p}_p\left(|C_d| \geq \delta\sqrt{n}\right)$$

$$\leq n \exp\left(-\lambda(p)\delta\sqrt{n}\right)$$

Now by setting $\eta(p) = \lambda(p)\delta$, we obtain that

$\mathbb{p}_p(|C| = n) \leq n\exp\left(-\sqrt{n}\eta(p)\right)$, which completes the proof. ■

## 3.3 Near the critical threshold

**Critical threshold for bond percolation on the 2-dimensinal lattice**

The previous sections have equipped us with the necessary tools to eventually compute the value of $p_c$, which is proven to be equal to $\frac{1}{2}$. We begin by proving that in 2-dimensions, the percolation probability is zero when $p = \frac{1}{2}$, for which the immediate consequence is that the critical percolation threshold $p_c \geq \frac{1}{2}$. Moreover, the absence of infinite open cluster at the percolation threshold is also conjectured to hold for higher dimensions.

**Theorem 3.3.1** [5]: For bond percolation on the two-dimensional square lattice, we have $p_c \geq \frac{1}{2}$.

In order to prove this theorem, we first need to show the following lemma.

**Lemma 3.3.1 (square-root trick)** [1]: Let $A_1, A_2, \ldots, A_m$ be increasing events, all having the same probability. Then we have

$$\mathbb{p}_p(A_1) \geq 1 - \left(1 - \mathbb{p}_p\left(\bigcup_{i=1}^{m} A_i\right)\right)^{\frac{1}{\mathrm{m}}}.$$

**Proof**: Let $A_i^c$ be the complement of the event $A_i$. Then by the FKG inequality we get that

$$\begin{aligned} 1 - \mathbb{p}_p\left(\bigcup_{i=1}^{m} A_i\right) &= \mathbb{p}_p\left(\bigcap_{i=1}^{m} A_i^c\right) \\ &\geq \prod_{i=1}^{m} \mathbb{p}_p(A_i^c) \\ &= \left(\mathbb{p}_p(A_1^c)\right)^m \\ &= \left(1 - \mathbb{p}_p(A_1)\right)^m \end{aligned}$$

Now by taking the $m^{th}$ root of both sides of the equations, we obtain the result as

$$\mathbb{p}_p(A_1) \geq 1 - \left(1 - \mathbb{p}_p\left(\bigcup_{i=1}^{m} A_i\right)\right)^{\frac{1}{m}}. \qquad \blacksquare$$

**Proof of the Theorem 3.3.1:** We prove that $\theta\left(\frac{1}{2}\right) = 0$, that means almost surely there is no unbounded component at $p = \frac{1}{2}$. By the monotonicity of the percolation function this corresponds to $p_c \geq \frac{1}{2}$.

Assume that $\theta\left(\frac{1}{2}\right) > 0$ and for any $n$, define the following events. Let $A^l(n)$ be the event that there exists an infinite path that starts from some vertex on the left side of the box $B_n$, which uses no other vertex of $B_n$. Similarly, define $A^r(n), A^t(n)$ and $A^b(n)$ for the existence of analogous infinite paths starting respectively from the right, top bottom side of $B_n$ and not using any other vertex of $B_n$ beside the starting one. Notice that all these events are increasing in $p$ and that they have equal probability of occurrence. We call their union $U(n)$. Since we have assumed that $\theta\left(\frac{1}{2}\right) > 0$, we have that

$\lim_{n\to\infty} \mathbb{p}_{\frac{1}{2}}\big(U(n)\big) = 1$ . (1)

By the square-root trick lemma, we have that each single event also occurs with high probability because

$$\mathbb{p}_{\frac{1}{2}}\left(A^i(n)\right) \geq 1 - (1 - \mathbb{p}_{\frac{1}{2}}(U(n)))^{\frac{1}{4}}, for\ i = l, r, t, b \quad (2)$$

which tends to one by (1) as $n \to \infty$. Then we can choose $N$ large enough such that

$$\mathbb{p}_{\frac{1}{2}}\left(A^i(N)\right) \geq \frac{7}{8}, for\ i = l, r, t, b. \quad (3)$$

Now we shift our attention to the dual lattice. Let a dual box $B_{nd}$ be defined as all the vertices of $B_n$ shifted by $(\frac{1}{2}, \frac{1}{2})$ ; see figure 12 below.

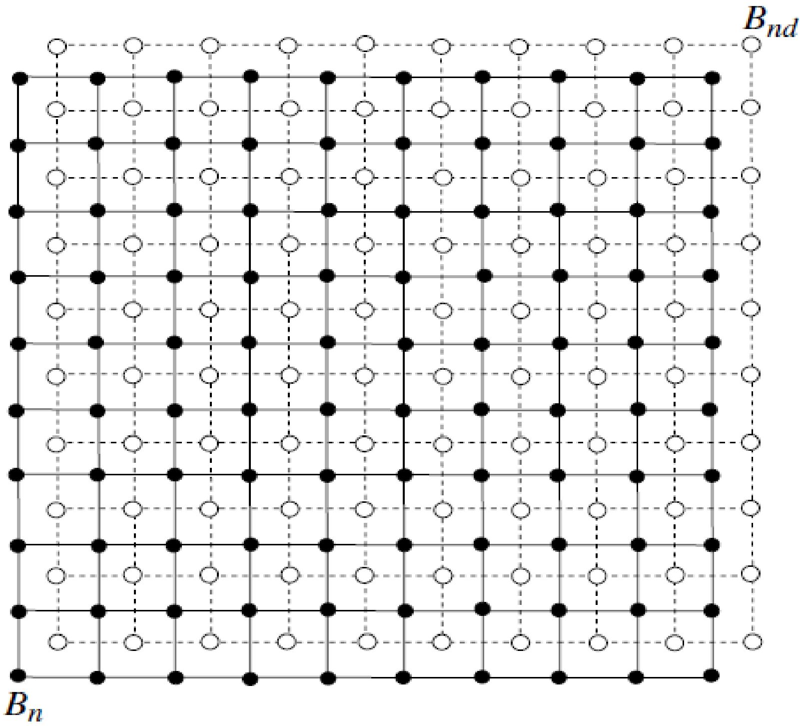


Figure 12 [5]: The box $B_n$ and its dual $B_{nd}$ , drawn with dashed lines.

Consider the events $A_d^i(n)$ that are the analogous of $A^i(n)$, but defined on the dual box. Since $p = \frac{1}{2}$ , these events have the same probability as before. Thus we can write

$$\mathbb{p}_{\frac{1}{2}}\left(A_d^i(N)\right) = \mathbb{p}_{\frac{1}{2}}\left(A^i(N)\right) \geq \frac{7}{8}, for\ i = l, r, t, b. \quad (4)$$

Now consider the event $A$ that is a combination of two events occurring on the dual lattice and two on the original lattice. This can be defined by

$$A = A^l(N) \cap A^r(N) \cap A_d^t(N) \cap A_d^b(N) \tag{5}$$

By the union bound and (4), we obtain

$$\mathbb{p}_{\frac{1}{2}}(A) = 1 - \mathbb{p}_{\frac{1}{2}}\left(A^l(N)^c \cup A^r(N)^c \cup A_d^t(N)^c \cup A_d^b(N)^c\right)$$

$$\geq 1 - (\mathbb{p}_{\frac{1}{2}}(A^l(N))^c + \mathbb{p}_{\frac{1}{2}}\left(A^r(N)\right)^c + \mathbb{p}_{\frac{1}{2}}(A_d^t(N))^c + \mathbb{p}_{\frac{1}{2}}(A_d^b(N))^c$$

$$\geq \frac{1}{2}\,. \tag{6}$$

However, the geometry of the situation and uniqueness lead to a contradiction, because they impose that $\mathbb{p}_{\frac{1}{2}}(A) = 0$; see figure 13 below. Event $A$ implies that there are infinite paths which start from opposite sides of $B_n$ that do not use any other vertex of the box. However, any two points that lie on an infinite path must be connected, as they are part of the unique infinite component. But notice that connecting $x_1$ and $x_2$ creates a barrier between $y_1$ and $y_2$ that cannot be crossed, because otherwise there would be an intersection between an edge in the dual graph and one in the original graph, which is impossible. Thus, we conclude that $y_1$ cannot be connected to $y_2$, which violates the uniqueness of the infinite cluster in the dual lattice. ■

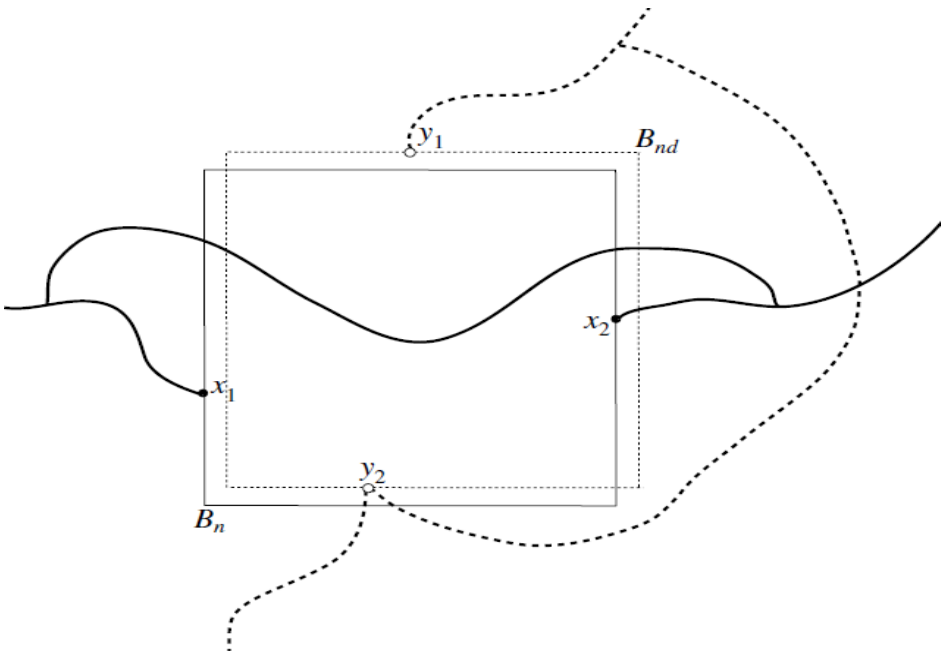


Figure 13[5]: Since there is only one unbounded component, $x_1$ must be connected to $x_2$. Similarly, in the dual graph, $y_1$ must be connected to $y_2$. This is geometrically impossible situation.

**Lemma 3.3.2** [6]: (Crossing of a square for $p = \frac{1}{2}$): Let $LR(n)$ be the event that there is a left-right open crossing of the rectangle $R(n) = [0, n+1]X[0, n]$ (that is, an open path connecting some vertex on the left side of $R(n)$ to some vertex on the right side of $R(n)$). Then we have $\mathbb{p}_{\frac{1}{2}}\big(LR(n)\big) = \frac{1}{2}$ for all $n \in \mathbb{N}$.

**Proof**: The rectangle $R(n)$ is the sub graph of $L^2$ having vertex set $[0, n+1]X[0, n]$ and edge set comprising all edges of $L^2$ joining pairs of vertices in $S(n)$, except those joining pairs $(i, j), (k, l)$ with either $i = k = 0$ or $i = k = n + 1$. Let $R_d(n)$ be the sub graph of $L_d^2$ having vertex set $\{(i + \frac{1}{2},\ j + \frac{1}{2}) | 0 \leq i \leq n,\ 1 \leq j \leq n\}$ and edge set of all edges of $L_d^2$ joining pairs of vertices in $R_d(n)$, except those joining pairs $(i, j), (k, l)$ with either $i = k = -\frac{1}{2}$ or $i = k = n + \frac{1}{2}$. The two sub graphs can be obtained from each other by a 90 degrees rotation, which relocates the vertex labeled (0,0) at the point $(n + \frac{1}{2}, -\frac{1}{2})$, see figure 14(left).

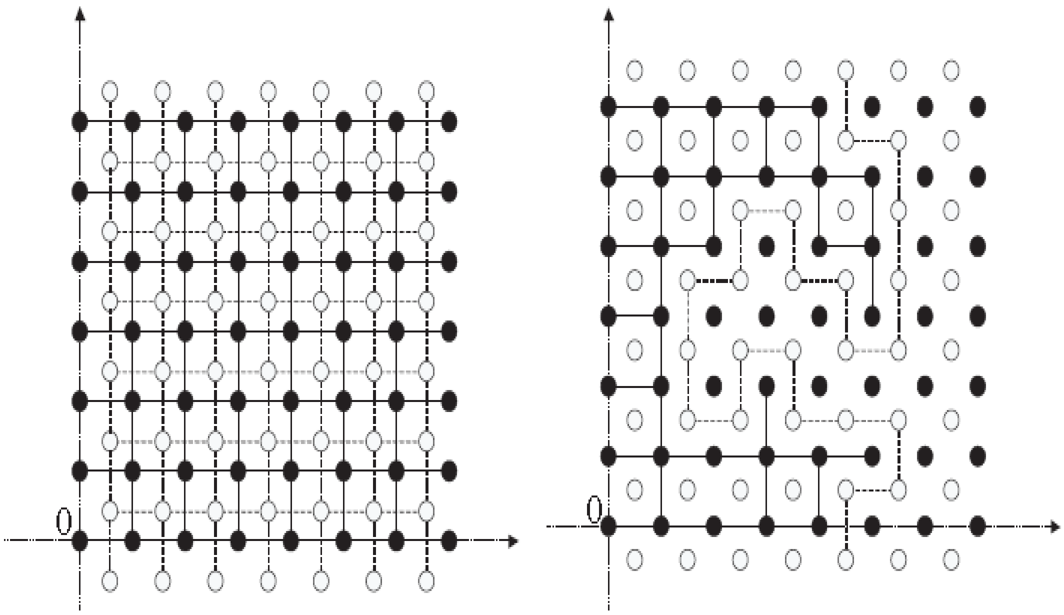


Figure 14 [6]: The box $R(n)$and its dual $R_d(n)$ for $n = 6$ (left) and an illustration of the fact that there is no left-right open crossing $R(n)$ if and only if there is a top-bottom closed crossing of $R_d(n)$ (right).

Consider the following two events. Let $LR(n)$ be the event that there exists an open path of $R(n)$ joining a vertex on the left side of $R(n)$ to a vertex on its right side, and let $TB_d(n)$ be the event that there exists an closed path of $R_d(n)$ joining a vertex on the top side of $R_d(n)$ to a vertex on its bottom side.

If $LR(n) \cap TB_d(n) \neq \emptyset$, then there is a left to right open path in $R(n)$ that crosses a top to bottom closed path $R_d(n)$. But at the crossing of these two paths, there would be an open edge of $L^2$ crossed by a closed edge of $L_d^2$, which is impossible, see figure 14(right). Hence, we find $LR(n) \cap TB_d(n) = \emptyset$. On the other hand, either $LR(n)$ or $TB_d(n)$ must occur. Let $D$ be the set of vertices that are reachable from the left side of $R(n)$ by an open path. Assume that $LR(n)$ does not occur. Then there exists a top-bottom closed path of $L_d^2$ crossing only edges of $R(n)$ contained in the edge boundary of $D$, and so $TB_d(n)$ occurs. Consequently $LR(n)$ and $TB_d(n)$ form a partition of the sample space $\Omega$, and therefore we have

$$\mathbb{p}_p\big(LR(n)\big) + \mathbb{p}_p\big(TB_d(n)\big) = 1 \qquad (*)$$

Now since $R(n)$ and $R_d(n)$ are isomorphic (they can be obtained from other by a 90 degree rotation, which relocates the vertex labeled (0,0) at the point $(n + \frac{1}{2}, -\frac{1}{2})$), flipping the polarity of each edge of $L_d^2$ yields that $\mathbb{p}_p\big(TB_d(n)\big) = \mathbb{p}_{1-p}\big(LR(n)\big)$. Now inserting this result in $(*)$, we get $\mathbb{p}_p\big(LR(n)\big) + \mathbb{p}_{1-p}\big(LR(n)\big) = 1$. Thus by taking $p = \frac{1}{2}$, we obtain that $\mathbb{p}_{\frac{1}{2}}\big(LR(n)\big) + \mathbb{p}_{\frac{1}{2}}\big(LR(n)\big) = 1$. This implies that $2\mathbb{p}_{\frac{1}{2}}\big(LR(n)\big) = 1$ from which we get $\mathbb{p}_{\frac{1}{2}}\big(LR(n)\big) = \frac{1}{2}$. Therefore we conclude that for $p = \frac{1}{2}$, we have $\mathbb{p}_p\big(LR(n)\big) = \frac{1}{2}$. ■

Now we directly deduce one of the most famous theorems in percolation theory as follows.

**Theorem 3.3.2** [1, 6]: The percolation threshold in $L^2$ is $\mathrm{p}_c = \frac{1}{2}$.

**Proof**: From theorem 3.3.1, we observe that $\mathrm{p}_c \geq \frac{1}{2}$. Assume that $\mathrm{p}_c > \frac{1}{2}$. Then the value $\mathrm{p}_c = \frac{1}{2}$ belongs to the subcritical phase and thus by theorem 3.1.2 there exists $\psi\left(\frac{1}{2}\right) > 0$ such that for all $n \in N^*$,

$$\mathbb{p}_{\frac{1}{2}}\big(0 \leftrightarrow \partial^r R(n)\big) \leq \mathbb{p}_{\frac{1}{2}}\big(0 \leftrightarrow \partial S(n)\big) < \exp\left(-n\psi\left(\frac{1}{2}\right)\right)$$

where $\{0 \leftrightarrow \partial^r R(n)\}$ is the event that the origin is connected by an open path to a vertex lying on the right side of $R(n)$, defined as

$$\partial^r R(n) = \{(n+1,k) \in Z^2 | 0 \le k \le n\},$$

and $\{0 \leftrightarrow \partial S(n)\}$ is the event that the origin is connected by an open path to a vertex lying on the perimeter of the ball of radius $n$ centered at $0$. Consequently, since $LR(n)$ is the event that there exists an open path of $R(n)$ joining a vertex on the left side of $R(n)$ to a vertex on its right side, we have

$$\mathbb{p}_{\frac{1}{2}}\big(LR(n)\big) \le \sum_{k=0}^{n} \mathbb{p}_{\frac{1}{2}}\big((0,k) \leftrightarrow \partial^r R(n)\big)$$

$$\le (n+1)\mathbb{p}_{\frac{1}{2}}\big(0 \leftrightarrow \partial^r R(n)\big)$$

$$< (n+1)\exp\left(-n\psi\left(\tfrac{1}{2}\right)\right),$$ which yields that

$\mathbb{p}_{\frac{1}{2}}\big(LR(n)\big) \to 0$ as $n \to \infty$. But this contradicts with lemma 3.3.2. Hence, we must have $p_c \le \frac{1}{2}$, which completes the proof. ■

# References


[1] Amitabha Bagchi, CSL866: Percolation and Random Graphs, IIT Delhi: 2007.

[2] Bela Bollobas, Percolation, University of Cambridge and Univesity of Memphis, 2006.

[3] Douglas B. West, Introduction to graph theory (2nd edition): University of Illinois-Urbana, 2001.

[4] Geoffrey Grimmett, Percolation (2nd edition), Berlin: Springer 1999.

[5] Massimo Franceschetti and Ronald Meester, Random Networks for Communication: Cambridge University Press, 2007.

[6] Matthias Grossglauser and Patrick Thiran, Networks out of Control: Models and Methods for Random Networks, School of Computer and Communication Sciences, Ecole Polytechnique F´ed´erale de Lausanne (EPFL), 2005.

[7] Muhammad Sahimi, Applications of percolation Theory, University of southern California, 1993.