Fractional Bernoulli Numbers and Polynomials

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Declaration

I, with student ID number $GSR/2553/06$, hereby declare that this thesis is my own work and that it has not been previously submitted for assessment or completion of any post graduate qualification to another university or for another qualification.

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Certificate

I hereby certify that I have read this dissertation prepared by Eyerusalem Woldeyohannes under my supervision and recommended that, it should be accepted as fulfilling the dissertation requirement.

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This is to certify that the thesis prepared by Eyerusalem Woldeyohannes Tato, entitled: **Fractional Bernoulli Numbers and Polynomials** and submitted in partial fulfillment of the requirements for the Degree of Doctor of Philosophy (in Mathematics) compiles with the regulations of the University and meets the accepted standards with respect to originality and quality.

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Dedication

This Thesis is Dedicated To:

My father **Woldeyohannes Tato,**
My Mother **Addisalem Abafogi**
&
My daughter **Afomia Yonas**

Abstract

In this thesis, we investigate the fractional Bernoulli numbers $B_{a,m}$ and polynomials $B_{a,m}(z)$ for all positive fractions $a = \frac{\alpha}{\beta}, \quad \beta \neq 0$. We establish several properties of these numbers and polynomials which are analogous to the classical as well as hypergeometric Bernoulli numbers and polynomials. We also develop the higher order fractional Bernoulli numbers $B_{a,m}^{(t)}$ and polynomials $B_{a,m}^{(t)}(z)$ and their connection to other identity.

Contents

INTRODUCTION

The Swiss mathematician Jacob Bernoulli (1654-1705) introduced the now famous numbers and polynomials that bear his name Bernoulli numbers B_m and polynomials $B_m(z)$. In 2008, A. Hassen and H. D. Nguyen studied on the generalization of these numbers and polynomials that they referred to as hypergeometric Bernoulli numbers $B_m(N)$ and polynomials $B_m(N, z)$, see [14]. Again in 2017, Hassen and Nguyen introduced the (error) Bernoulli numbers $B_{\frac{1}{2},m}$ with half subscript, [12]. Later in 2018, Geleta, Hunduma Legesse [21] considered the error Bernoulli polynomials $B_{\frac{1}{2},m}(z)$ and he discussed some of their properties and established the relations between $B_{\frac{1}{2},m}(z)$ and the Hermite polynomials $H_m(z)$.

The Bernoulli numbers and polynomials are defined, respectively, by the generating function

$$\frac{x}{e^x - 1} = \sum_{m=0}^{\infty} B_m \frac{x^m}{m!}, \quad |x| < 2\pi$$

and

$$\frac{xe^{xz}}{e^x - 1} = \sum_{m=0}^{\infty} B_m(z) \frac{x^m}{m!}, \quad |x| < 2\pi,$$

where the variables x and z can be real or complex. These numbers and polynomials first appeared in a list of formulas for summing the pth powers of the first m positive integers, for $p = 1$ to $p = 10$, see [2], [3]

In [16], F. T. Howard generalized these numbers and polynomials using the notations A_m and $A_m(z)$ by

$$\frac{x^2/2!}{e^x - 1 - x} = \sum_{m=0}^{\infty} A_m \frac{x^m}{m!}$$

and

$$\frac{x^2 e^{xz}/2!}{e^x - 1 - x} = \sum_{m=0}^{\infty} A_m(z) \frac{x^m}{m!},$$

respectively. More recently, A. Hassen and H. D. Nguyen in [13] and [14] considered a further generalizations of Bernoulli numbers and polynomials given by

$$\frac{x^N/N!}{e^x - T_{N-1}(x)} = \sum_{m=0}^{\infty} B_m(N) \frac{x^m}{m!}$$

and

$$\frac{x^N e^{xz}/N!}{e^x - T_{N-1}(x)} = \sum_{m=0}^{\infty} B_m(N,z)\frac{x^m}{m!},$$

where $N \in \mathbb{N}$ and $T_N(x) = \sum_{m=0}^{N} \frac{x^m}{m!}$ is the Taylor polynomial of order N of the exponential function e^x.
Now in this thesis we study what we call fractional Bernoulli numbers and polynomials defined in the form

$$\frac{x^{2a}e^{-x^2}}{a\gamma(a,x^2)} = \sum_{m=0}^{\infty} B_{a,m}\frac{x^m}{m!},$$

and

$$\frac{x^{2a}e^{xz-x^2}}{a\gamma(a,x^2)} = \sum_{m=0}^{\infty} B_{a,m}(z)\frac{x^m}{m!}.$$

Actually the term fractional Bernoulli numbers was born in the study of fractional hypergeometric zeta function [22]. In this thesis we study several properties of these numbers and polynomials and further study their connection to the Hermite polynomials. We organized as follows:

In the first chapter of this dissertation, we shall review the properties and recurrence formulas of the classical and the hypergeometric Bernoulli numbers and polynomials. We shall also discuss some properties and recurrence relations of higher order classical (as well as hypergeometric) Bernoulli numbers and polynomials.

In chapter two, we develop the generalization of *error Bernoulli numbers* $B_{\frac{1}{2},m}$ *and polynomials* $B_{\frac{1}{2},m}(z)$ to what we call the *fractional Bernoulli numbers* $B_{a,m}$ *and polynomials* $B_{a,m}(z)$, which are given by

$$\frac{x^{2a}e^{-x^2}}{a\gamma(a,x^2)} = \sum_{m=0}^{\infty} B_{a,m}\frac{x^m}{m!}$$

and

$$\frac{x^{2a}e^{xz-x^2}}{a\gamma(a,x^2)} = \sum_{m=0}^{\infty} B_{a,m}(z)\frac{x^m}{m!},$$

respectively, where $a = \frac{\alpha}{\beta}$, $\beta \neq 0$ and $\gamma(a,x) = \int_0^x e^{-t}t^{a-1}dt$ is the lower incomplete gamma function. Note that with $a = \frac{1}{2}$, we get the error Bernoulli numbers $B_{\frac{1}{2},m}$ and polynomials $B_{\frac{1}{2},m}(z)$.

We also establish some properties and recurrence formulas for the fractional Bernoulli numbers and polynomials which are analogous to the classical and hypergeometric Bernoulli numbers and polynomials. The main results in this chapter appear in Theorems 2.6 and 2.12.

In the third chapter, we discuss our result regarding the relationship between fractional Bernoulli polynomials and the Hermite polynomials using the determinant representation of fractional Bernoulli polynomials.

In the last chapter, we introduce *higher order fractional Bernoulli numbers and polynomials* and we investigate their recursion formulas in Theorem 4.4 and Lemma 4.6. We also develop the differential equation of higher order fractional Bernoulli polynomials in Theorem 4.9 which is given by

$$\frac{B_{a,m}}{m!}\frac{d^m y}{dz^m}+\frac{B_{a,m-1}}{(m-1)!}\frac{d^{m-1}y}{dz^{m-1}}+\cdots+\left(\frac{B_{a,2}}{2!}+\frac{1}{a}\right)\frac{d^2y}{dz^2}-\frac{z}{2at}\frac{dy}{dz}+\frac{m}{2at}y=0,$$

where $y=B^{(t)}_{a,m}(z)$.

Finally, we establish the relation between higher order fractional Bernoulli polynomials and $\varphi_{m,k}(a)$, where

$$\varphi_{m,k}(a)=\frac{k!}{a^{m+1-k}}\sum_{j=0}^{m}\sum_{i=0}^{\alpha-1}\binom{m}{j}\omega_a^{-i(k-j)},\qquad \omega_a=e^{\frac{2\pi i}{a}}.$$

Chapter 1

Preliminaries

In this chapter, we discus the basic definitions and properties of classical (as well as, hypergeometric) Bernoulli numbers and polynomials.

1.1 Bernoulli Numbers and Polynomials

The Swiss mathematician Jacob Bernoulli (1654-1705) introduced the now famous numbers and polynomials that bear his name, Bernoulli numbers and polynomials, in his book *Ars Conjectandi*, published posthumously (Basel, 1713). The Bernoulli numbers first appeared in a list of formulas for summing the p^{th} powers of the first m positive integers, for $p = 1$ to $p = 10$. The following are the first three examples of Jacob Bernoulli's formulas:

$$\sum_{k=1}^{m} k = \frac{1}{2}m(m+1)$$

$$\sum_{k=1}^{m} k^2 = \frac{1}{6}m(m+1)(2m+1)$$

$$\sum_{k=1}^{m} k^3 = \frac{1}{4}m^2(m+1)^2.$$

Each of these sum can be generalized in the following way:

$$\sum_{k=1}^{m-1} k^p = \frac{B_{p+1}(m) - B_{p+1}}{p+1}, \quad p \geq 1, \quad m \geq 2.$$

The $B_m(z)$ are called Bernoulli polynomials, and are given by

$$B_m(z) = \sum_{k=0}^{m} \binom{m}{k} B_k z^{m-k}, \qquad m \geq 2, \tag{1.1.1}$$

where B_k are rational numbers called Bernoulli numbers. They can be defined recursively as follows:

$$B_0 = 1, \quad \text{and} \quad \sum_{k=0}^{m-1} \binom{m}{k} B_k = 0 \quad \text{for } m \geq 2. \tag{1.1.2}$$

Using the recurrence formula (1.1.2), with $m \geq 2$, the first few Bernoulli numbers are

$$B_1 = -\frac{1}{2}, \quad B_2 = \frac{1}{6}, \quad B_3 = 0,\ B_4 = -\frac{1}{30}, \quad B_5 = 0,\ B_6 = \frac{1}{42}, \quad B_7 = 0,$$
$$B_8 = -\frac{1}{30}, \quad B_9 = 0, \quad B_{10} = \frac{5}{66}.$$

From equation (1.1.1), we have $B_m = B_m(0)$ for $m \geq 0$. The sum in (1.1.2) can also be written in the form

$$B_m = \sum_{k=0}^{m} \binom{m}{k} B_k, \quad m \geq 2. \tag{1.1.3}$$

Combining (1.1.1) and (1.1.3) yields

$$B_m = B_m(1), \quad m \geq 2.$$

Bernoulli numbers with even subscripts ≥ 2 alternate in sign, and those with odd subscripts ≥ 3 are zero (see [3] for the details).

From (1.1.1) and (1.1.3), we can compute Bernoulli polynomials. Here are the first few:

$$B_0(z) = 1, \quad B_1(z) = z - \frac{1}{2}, \quad B_2(z) = z^2 - z + \frac{1}{6}, \quad B_3(z) = z^3 - \frac{3}{2}z^2 + \frac{z}{2},$$
$$B_4(z) = z^4 - 2z^3 + z^2 - \frac{1}{30}, \quad B_5(z) = z^5 - \frac{5}{2}z^4 + \frac{5}{3}z^3 - \frac{z}{6},$$
$$B_6(z) = z^6 - 3z^5 + 5z^4 - \frac{z^2}{2} + \frac{1}{42}.$$

One of the most remarkable properties of Bernoulli numbers has to do with their connection to the Riemann zeta function $\zeta(s)$, defined for $Re(s) > 1$ by the infinite series

$$\zeta(s) = \sum_{k=1}^{\infty} \frac{1}{k^s}.$$

Leonhard Euler (1707-1783) discovered that when s is an even integer, the sum can be expressed in terms of Bernoulli numbers by the formula

$$\zeta(2m) = (2\pi)^{2m} \frac{|B_{2m}|}{2(2m)!}.$$

Euler also demonstrated that the value of the classical zeta function $\zeta(s)$ at negative integers is expressible in terms of Bernoulli numbers:

$$\zeta(-m) = -\frac{B_{m+1}}{m+1}.$$

There are alternative methods for introducing Bernoulli numbers and polynomials (see [23], [19], [2]). We shall briefly discuss three of them below.

Generating Function

The generating function is one of the most useful methods. It was conceived by Euler [2], who observed that Bernoulli numbers and polynomials occur as coefficients in the following power series expansions:

$$\frac{x}{e^x - 1} = \sum_{m=0}^{\infty} B_m \frac{x^m}{m!}, \quad |x| < 2\pi \tag{1.1.4}$$

and

$$\frac{xe^{zx}}{e^x - 1} = \sum_{m=0}^{\infty} B_m(z) \frac{x^m}{m!}, \quad |x| < 2\pi, \tag{1.1.5}$$

where the variables x and z can be real or complex. The functions on the left are called generating functions for the Bernoulli numbers and polynomials, respectively.

Generating functions lead to simple and direct proofs of many basic properties of Bernoulli numbers and polynomials. The following are some of the basic properties deduced from the generating functions. (Proofs can be found in Apostol [2]). One of our objectives is to establish similar properties for the fractional Bernoulli numbers and polynomials.

Difference equation:

$$B_m(z+1) - B_m(z) = mz^{m-1} \text{ if } m \geq 1.$$

Symmetry relation:

$$B_m(1-z) = (-1)^m B_m(z), \quad m \geq 0.$$

Addition formula:

$$B_m(z_1 + z_2) = \sum_{k=0}^{m} \binom{m}{k} B_k(z_1) z_2^{m-k}, \quad m \geq 0.$$

Raabe's multiplication formula

$$B_m(nz) = n^{m-1} \sum_{k=0}^{n-1} B_m\left(z + \frac{k}{n}\right), \quad m \geq 0, \quad n \geq 1.$$

Derivative formulas:

$$\text{For } m \geq 1, \quad B'_m(z) = mB_{m-1}(z),$$
$$B''_m(z) = m(m-1)B_{m-2}(z), \ldots, B^{(k)}_m(z) = k!\binom{m}{k}B_{m-k}(z).$$

Integration formulas:

$$\int_{z_1}^{z_2} B_m(t)dt = \frac{B_{m+1}(z_2) - B_{m+1}(z_1)}{m+1}, \quad m \geq 1$$
$$\int_{z}^{z+1} B_m(t)dt = z^m, \quad m \geq 1$$
$$\int_{0}^{1} B_m(t)dt = 0, \quad m \geq 1.$$

Appell Sequence

The Appell sequence is the other approach to define Bernoulli numbers and polynomials:

$$B_0(z) = 1, \tag{1.1.6}$$
$$B'_m(z) = mB_{m-1}(z), \tag{1.1.7}$$
$$\int_0^1 B_m(z)dz = \begin{cases} 1, & m = 0 \\ 0, & m > 0. \end{cases} \tag{1.1.8}$$

The recurrence formula (1.1.1), the generating function (1.1.5), and the Appell sequence (1.1.6) − (1.1.8) are equivalent statements for polynomials $B_m(z)$ (see [19]).

Fourier Series

Another approach to define the Bernoulli polynomial is Fourier series. For $m > 0$,

$$B_m(z) = -\frac{m!}{(2\pi i)^m}\sum_{r=-\infty}^{\infty}\frac{e^{2\pi irz}}{r^m} \qquad (r \neq 0, \quad 0 < z < 1).$$

In [23] D.H. Lehmer proved that the above Fourier series is equivalent to the following functional equation:

$$\frac{1}{n}\sum_{k=0}^{n-1} B_m\left(z + \frac{k}{m}\right) = n^{-m}B_m(nz). \tag{1.1.9}$$

He used this equation as a definition of the classical Bernoulli polynomials. Again in [23] Lehmer showed that (1.1.9) is equivalent to the generating function (1.1.5).

We close this section by introducing a generalization of the Bernoulli numbers and polynomials to what are commonly known as *higher order Bernoulli numbers* $B_m^{(t)}$ *and polynomials* $B_m^{(t)}(z)$. The higher order Bernoulli polynomials are defined by

$$\left(\frac{x}{e^x-1}\right)^t e^{xz} = \sum_{m=0}^{\infty} B_m^{(t)}(z)\frac{x^m}{m!}, \quad t \in \mathbb{N}_0,\ |x| < 2\pi.$$

If $z = 0$, then we have the higher order Bernoulli numbers $B_m^{(t)} = B_m^{(t)}(0)$. For $t = 1$, $B_m^{(1)}(z) = B_m(z)$ which is the classical Bernoulli polynomial and $B_m^{(1)}(0) = B_m$, the classical Bernoulli numbers.

The higher order Bernoulli numbers and polynomials satisfy the following recurrences:

(i) $B_0^{(0)}(z) = 1,\ B_m^{(0)}(z) = z^m \quad \text{for} \quad m \in \mathbb{N}$,

(ii) $B_m^{(t)}(z) = \sum_{k=0}^{m} \binom{m}{k} B_k(z) B_{m-k}^{(t-1)}$,

(iii) $B_0^{(t+1)} = 1$,

(iv) $B_m^{(t+1)} = \left(1 - \frac{m}{t}\right) B_m^{(t)} - m B_{m-1}^{(t)}, \quad m \in \mathbb{N}$,

(v) $B_{m+1}^{(t)}(z) = \left(z - \frac{t}{2}\right) B_m^{(t)}(z) - t \sum_{k=0}^{m-1} \binom{m}{k} \frac{B_{m-k+1}}{m-k+1} B_k^{(t)}(z)$.

For proofs of these and other properties see [17].

The higher order Bernoulli polynomials $B_m^{(t)}(z)$ also satisfy the differential equation

$$\frac{B_m}{m!}\frac{d^m y}{dz^m} + \frac{B_{m-1}}{(m-1)!}\frac{d^{m-1}y}{dz^{m-1}} + \cdots + \frac{B_2}{2!}\frac{d^2y}{dz^2} - \left(\frac{z}{t} - \frac{1}{2}\right)\frac{dy}{dz} + \frac{m}{t}y = 0,$$

where $y = B_m^{(t)}(z)$.

Note that letting $t = 1$ in the above equation, we obtain a differential equation satisfied by the classical Bernoulli polynomials $y = B_m(z)$:

$$\frac{B_m}{m!}\frac{d^m y}{dz^m} + \frac{B_{m-1}}{(m-1)!}\frac{d^{m-1}y}{dz^{m-1}} + \cdots + \frac{B_2}{2!}\frac{d^2y}{dz^2} - \left(z - \frac{1}{2}\right)\frac{dy}{dz} + my = 0.$$

1.2 Hypergeometric Bernoulli Numbers and Polynomials

A generalization of Bernoulli numbers and polynomials that will be of interest to our work in this thesis involves modifying the generating function. This was first introduced by F. T. Howard in [16], where he used A_m and $A_m(z)$ when generalizing the Bernoulli numbers and polynomials, respectively:

$$\frac{x^2/2!}{e^x-1-x}=\sum_{m=0}^{\infty}A_m\frac{x^m}{m!} \quad \text{and} \quad \frac{x^2e^{xz}/2!}{e^x-1-x}=\sum_{m=0}^{\infty}A_m(z)\frac{x^m}{m!}.$$

In recent years, A. Hassen and H. D. Nguyen in [13] and [14] studied the generalization of Bernoulli numbers and polynomials (that they referred to as hypergeometric Bernoulli numbers and polynomials).

Definition 1.1. *For any integer $N \geq 1$, the hypergeometric Bernoulli polynomials of order N, $B_m(N,z)$, are defined as:*

$$\frac{x^N e^{xz}/N!}{e^x-T_{N-1}(x)}=\sum_{m=0}^{\infty}B_m(N,z)\frac{x^m}{m!}, \tag{1.2.1}$$

where $T_{N-1}(x)$ is the Taylor polynomial of order $N-1$ for the exponential function.

In particular, if we put $N=1$ in (1.2.1), it reduces to the classical Bernoulli polynomials $B_m(z)$ and when $N=2$ it represents the polynomials $A_m(z)$ considered by Howard.

For $z=0$, we then have the rational numbers $B_m(N)=B_m(N,0)$, called hypergeometric Bernoulli numbers, and they are generated by

$$\frac{x^N}{N!(e^x-T_{N-1}(x))}=\sum_{m=0}^{\infty}B_m(N)\frac{x^m}{m!}.$$

The motivation for naming the numbers $B_m(N)$ and polynomials $B_m(N,z)$ as hypergeometric Bernoulli numbers and polynomials is that the generating function $f(x)=\frac{x^N/N!}{e^x-T_{N-1}(x)}$ can be expressed in terms of the confluent hypergeometric function:

$$\frac{x^N/N!}{e^x-T_{N-1}(x)}=\frac{1}{{}_1F_1(1,N+1;x)},$$

where

$${}_1F_1(b,c;x)=\sum_{m=0}^{\infty}\frac{(b)_m}{(c)_m}\frac{x^m}{m!}, \quad \text{for} \quad b,c\in\mathbb{R}$$

and $(b)_m$ is the Pochhammer symbol:

$$(b)_m = \begin{cases} b(b+1)\cdots(b+m-1), & m \geq 1 \\ 1, & m = 0. \end{cases}$$

Analogous to the classical Bernoulli numbers and polynomials, hypergeometric Bernoulli numbers and polynomials also admit equivalent definitions. For example, we could define the polynomials by the following recurrence formula:

$$B_m(N,z) = \sum_{k=0}^{m} \binom{m}{k} B_k(N) z^{m-k}. \tag{1.2.2}$$

Another approach to defining hypergeometric Bernoulli polynomials is in terms of Appell sequence with zero mean:

$$B_0(N,z) = 1 \tag{1.2.3}$$

$$B'_m(N,z) = mB_{m-1}(N,z) \tag{1.2.4}$$

$$\int_0^1 (1-z)^{N-1} B_m(N,z)dz = \begin{cases} 1/N, & m = 0 \\ 0, & m > 0. \end{cases} \tag{1.2.5}$$

The three approaches, that is, the generating function (1.2.1), the recurrence formula (1.2.2), and the Appell sequence (1.2.3) – (1.2.5), of hypergeometric Bernoulli polynomials are equivalent. (For the proofs, see [14] and [19].)

In [19], it was further proved that the hypergeometric Bernoulli polynomials satisfy following properties.

(i) Addition formula:

$$B_m(N, z_1 + z_2) = \sum_{k=0}^{m} \binom{m}{k} B_k(N, z_1) z_2^{m-k}.$$

(ii) Difference equation:

$$B_m(N, z+1) - \sum_{k=0}^{N-1} \binom{m}{k} B_{m-k}(N,z) = \binom{m}{N} z^{m-N}. \tag{1.2.6}$$

(iii) For all $m > N$

$$\sum_{k=0}^{m-N} \binom{m}{k} B_k(N) = 0. \tag{1.2.7}$$

To prove (1.2.7), put $z = 0$ in the equation (1.2.6) to get

$$B_m(N, 1) = \sum_{k=0}^{N-1} \binom{m}{k} B_{m-k}(N).$$

Letting $z = 1$ in equation (1.2.1), we obtain

$$\begin{aligned} B_m(N, 1) &= \sum_{k=0}^{m} \binom{m}{k} B_k(N) \\ &= \sum_{k=0}^{N-1} \binom{m}{k} B_{m-k}(N) + \sum_{k=0}^{m-N} \binom{m}{k} B_k(N). \end{aligned}$$

Therefore

$$\sum_{k=0}^{m-N} \binom{m}{k} B_k(N) = 0.$$

A property of hypergeometric Bernoulli numbers that parallels the classical Bernoulli numbers is that they are connected with a generalization of the Riemann zeta function. More precisely, for negative integers $m < 2 - N(N = 1, 2, ...)$, the value of ζ_N is given by

$$\zeta_N(-m) = (-1)^{m-N+1} \binom{m+1}{N}^{-1} B_{m+1}(N), \tag{1.2.8}$$

where $\zeta_N(s)$ is *hypergeometric zeta function* defined by

$$\zeta_N(s) = \frac{1}{\Gamma(s + N - 1)} \int_0^\infty \frac{x^{s+N-2}}{e^x - T_{N-1}(x)} dx.$$

When $N = 1$, we note that equation (1.2.8) reduces to Euler's result:

$$\zeta(-m) = -\frac{B_{m+1}}{m + 1}.$$

Remark 1.2. *It is known that the classical Bernoulli numbers with odd subscripts* B_{2k+1}, $(k = 1, 2, ...)$ *are zero. But this doesn't hold in the case of hypergeometric Bernoulli numbers. For instance, for* $N = 2$ *each of* $B_3(2), B_5(2)$ *and* $B_7(2)$ *are nonzero. Also* $B_m(1) = B_m(0) = B_m$*, however,* $B_m(2, 1) \neq B_m(2, 0) = B_m(2)$*, which is different from the classical case.*

Definition 1.3. *The higher order hypergeometric Bernoulli polynomials are defined by the generating function*

$$\left(\frac{x^N/N!}{e^x - T_{N-1}(x)}\right)^t e^{xz} = \sum_{m=0}^{\infty} B_m^{(t)}(N,z)\frac{x^m}{m!},$$

where $T_N(x) = \sum_{m=0}^{N} \frac{x^m}{m!}$ *is the* N^{th} *Taylor polynomial of* e^x*, and the higher order hypergeometric Bernoulli numbers are defined by* $B_m^{(t)}(N) = B_m^{(t)}(N,0)$*.*

For $t = 1$, then $B_m^{(1)}(N,z) = B_m(N,z)$, the hypergeometric Bernoulli polynomials, and $B_m^{(1)}(N,0) = B_m(N)$, the hypergeometric Bernoulli numbers.

Theorem 1.4. *The following results appeared in [8] and [17]:*

(i) $B_m^{(t+1)}(N,z) = \frac{1}{N}\left(N - \frac{m}{t}\right)B_m^{(t)}(N,z) + \frac{1}{N}\frac{m}{t}(z-t)B_{m-1}^{(t)}(N,z)$*, for* $m \geq 1$,

(ii) $B_{m+1}^{(t)}(N,z) = \left(z - \frac{t}{N+1}\right)B_m^{(t)}(N,z) - tN\sum_{k=0}^{m-1}\binom{m}{k}\frac{B_{m-k+1}(N)}{m-k+1}B_k^{(t)}(N,z).$

(iii) $(m-N+1)(m-N+2)\cdots mB_{m-N}^{(t-1)}(N,z) = B_m^{(t)}(N,z+1) - \sum_{j=0}^{N-1}\binom{m}{j}B_{m-j}^{(t)}(N,z),$

(iv) $tNB_m^{(t)}(N,z) + (z-t)mB_{m-1}^{(t)}(N,z) - \frac{t}{(N-1)!}B_{m+1}^{(t)}(N,z) = mB_m^{(t)}(N,z),$

(v) $tNB_m^{(t)}(N) - tmB_{m-1}^{(t)}(N) - \frac{t}{(N-1)!}B_{m+1}^{(t)}(N) = mB_m^{(t)}(N).$

For details see Theorems 1.8 and 2.1 and Lemma 2.3.

Theorem 1.5. *In [17] (Theorem 1.6), it is shown that the higher order hypergeometric Bernoulli polynomials satisfies the differential equation which is given by*

$$\frac{B_m(N)}{m!}\frac{d^m y}{dz^m} + \frac{B_{m-1}(N)}{(m-1)!}\frac{d^{m-1}y}{dz^{m-1}} + \cdots + \frac{B_2(N)}{2!}\frac{d^2 y}{dz^2} - \left(\frac{z}{tN} - \frac{1}{N(N+1)}\right)\frac{dy}{dz} + \frac{m}{tN}y = 0,$$

where $y = B_m^{(t)}(N,z)$*.*

For $N = 1$, the above theorem reduces to the differential equation of the higher order classical Bernoulli polynomials, and also for $N = t = 1$, it reduces to the differential equation of the classical Bernoulli polynomials.

Chapter 2

Fractional Bernoulli Numbers and Polynomials and Some of their Properties

In this chapter, we introduce the fractional Bernoulli numbers and polynomials and some of their properties which are analogous to the classical (hypergeometric) Bernoulli numbers and polynomials.

2.1 Introduction

As mentioned before, Hassen and Nguyen introduced the error Bernoulli numbers in [12] and the error Bernoulli polynomials were introduced by Geleta in [21]. In this chapter, we discuss the fractional Bernoulli numbers and polynomials by extending the subscript $a = \frac{1}{2}$ that appeared in the error Bernoulli numbers and polynomials to a positive fraction $a = \frac{\alpha}{\beta}, \quad \beta \neq 0$.

We state and prove some basic properties of fractional Bernoulli numbers and polynomials which are analogous to those of the hypergeometric Bernoulli numbers and polynomials given in the first chapter. Also we establish some of their recurrence formulas. The main results of this chapter are stated in Theorem 2.6 and Theorem 2.12.

2.2 Fractional Bernoulli Numbers and Polynomials

Hassen and Nguyen in [12] defined the (error) Bernoulli numbers $B_{\frac{1}{2},m}$ by

$$\frac{2xe^{-x^2}}{\sqrt{\pi}\operatorname{erf}(x)} = \sum_{m=0}^{\infty} B_{\frac{1}{2},m} \frac{x^m}{m!}, \tag{2.2.1}$$

where $\operatorname{erf}(x)$ is the error function given by

$$\operatorname{erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-t^2} dt.$$

It was shown by the authors that for m even, $B_{\frac{1}{2},m}$ satisfies the recurrence relation

$$B_{\frac{1}{2},0} = 1, \quad \sum_{k=0}^{m} \frac{(-1)^k m! B_{\frac{1}{2},2k}}{(2k)!(m-k)!(2(m-k)+1)} = 1 \ \ (m \geq 1).$$

We note that $B_{\frac{1}{2},m} = 0$ for m is odd since $xe^{-x^2}/\mathrm{erf}(x)$ is an even function. The first few values are:

$$B_{\frac{1}{2},0} = 1, \quad B_{\frac{1}{2},2} = -\frac{4}{3} \quad B_{\frac{1}{2},4} = \frac{64}{15}, \quad B_{\frac{1}{2},6} = -\frac{256}{21},$$
$$B_{\frac{1}{2},8} = -\frac{4096}{45}, \quad B_{\frac{1}{2},10} = \frac{81920}{33}.$$

In [21], Geleta studied the error Bernoulli polynomials $B_{\frac{1}{2},m}(z)$ defined by

$$\frac{2xe^{xz-x^2}}{\sqrt{\pi}\mathrm{erf}(x)} = \sum_{m=0}^{\infty} B_{\frac{1}{2},m}(z)\frac{x^m}{m!}. \tag{2.2.2}$$

For $z = 0$, we have the rational number $B_{\frac{1}{2},m} = B_{\frac{1}{2},m}(0)$, which reduce to the coefficients of (2.2.1), consider by Hassen and Nguyen.

Legesse also established the recurrence formula of the error Bernoulli polynomials by

$$B_{\frac{1}{2},m}(z) = \sum_{k=0}^{m} \binom{m}{k} B_{\frac{1}{2},m-k} \ z^k.$$

Here are the first few values of error Bernoulli polynomials:

$$B_{\frac{1}{2},0}(z) = 1, \quad B_{\frac{1}{2},1}(z) = z, \quad B_{\frac{1}{2},2}(z) = z^2 - \frac{4}{3}, \quad B_{\frac{1}{2},3}(z) = z^3 - 4z,$$
$$B_{\frac{1}{2},4}(z) = z^4 - 8z^2 + \frac{64}{15}, \quad B_{\frac{1}{2},5}(z) = z^5 - \frac{40}{3}z^3 + \frac{64}{3}z.$$

In this thesis, we extend the subscript $N = \frac{1}{2}$ to a positive fraction $a = \frac{\alpha}{\beta}, \ \beta \neq 0$.

Definition 2.1. *For $b, c \in \mathbb{R}$, the Kummer function $\Phi_{b,c}(x)$ is defined by*

$$\Phi_{b,c}(x) = \ _1F_1(b, b+c; x),$$

where $_1F_1(b, c; x)$ is the confluent hypergeometric function.

In particular, $\Phi_{1,1}(x) = \frac{e^x-1}{x}$ is the reciprocal of the generating function for the classical Bernoulli numbers B_m. Also for $a \in \mathbb{N}$, we have

$$\frac{1}{\Phi_{1,a}(x)} = \frac{1}{{}_1F_1(1, a+1; x)} = \frac{x^a}{a!(e^x - T_{a-1}(x))} = \sum_{m=0}^{\infty} B_m(a) \frac{x^m}{m!},$$

and hence the reciprocal of $\Phi_{1,a}(x)$ is the generating function for the hypergeometric Bernoulli numbers. Note that for all real positive a,

$$\begin{aligned} {}_1F_1(1, a+1; x) &= e^x x^{-a}(\Gamma(a+1) - a\Gamma(a, x)) \\ &= a e^x x^{-a}(\Gamma(a) - \Gamma(a, x)) \\ &= a e^x x^{-a} \gamma(a, x), \end{aligned}$$

where $\gamma(a,x) = \int_0^x e^{-t} t^{a-1} dt$ and $\Gamma(a,x) = \int_x^\infty e^{-t} t^{a-1} dt$ are the lower and upper incomplete gamma functions, respectively.

Definition 2.2. *For any positive real values a, we define the fractional Bernoulli numbers $B_{a,m}$ by the generating function*

$$\frac{x^{2a} e^{-x^2}}{a\gamma(a, x^2)} = \sum_{m=0}^{\infty} B_{a,m} \frac{x^m}{m!}. \tag{2.2.3}$$

The generating function in the left hand side of equation (2.2.3) is the reciprocal of the Kummer function $\Phi_{1,a}(x^2)$. Observe that, while a can be an integer in the above definition, the resulting numbers are different from the hypergeometric Bernoulli numbers of Hassen and Nguyen. This is due to the fact that our generating function is a function of x^2.

Proposition 2.3. *The fractional Bernoulli numbers satisfies the following recursion formula.*

$$\sum_{k=0}^{[m/2]} \frac{a(-1)^k B_{a,m-2k}}{k!(k+a)(m-2k)!} = \begin{cases} \frac{(-1)^{\frac{m}{2}}}{(m/2)!}, & \text{if } m \text{ is even} \\ 0, & \text{if } m \text{ is odd.} \end{cases}$$

Proof. From equation (2.2.3), we obtain

$$e^{-x^2} = a\gamma(a, x^2) x^{-2a} \sum_{m=0}^{\infty} B_{a,m} \frac{x^m}{m!}$$

$$= \left(\sum_{k=0}^{\infty} \frac{a(-1)^k x^{2k}}{k!(k+a)}\right)\left(\sum_{m=0}^{\infty} B_{a,m}\frac{x^m}{m!}\right)$$

$$= \sum_{m=0}^{\infty}\sum_{k=0}^{[m/2]} \frac{a(-1)^k B_{a,m-2k}}{(k+a)k!(m-2k)!}x^m,$$

where in the second equality, we have used $\gamma(a,x) = \sum_{k=0}^{\infty} \frac{(-1)^k x^{k+a}}{k!(k+a)}, \quad a \neq 0, -1, -2, ...$ Therefore,

$$\sum_{m=0}^{\infty}(-1)^m \frac{x^{2m}}{m!} = \sum_{m=0}^{\infty}\sum_{k=0}^{[m/2]} \frac{a(-1)^k B_{a,m-2k}}{k!(k+a)(m-2k)!}x^m.$$

We compare the coefficients of powers of x to obtain the required result. □

Analogous to the classical Bernoulli numbers, the fractional Bernoulli numbers with odd subscripts are zero. To prove this, from Proposition 2.3, we note that

$$B_{a,m} = -m!\sum_{k=1}^{[m/2]}\left(\frac{a(-1)^k}{k!(k+a)(m-2k)!}\right)B_{a,m-2k}.$$

Clearly $B_{a,1} = 0$. For m odd, $m-2k$ is odd, and hence we apply induction to conclude that $B_{a,m} = 0$ for all odd m. The first few nonzero fractional Bernoulli numbers are

$$B_{a,0} = 1,\ B_{a,2} = -\frac{2}{1+a},\ B_{a,4} = \frac{24}{(1+a)^2(2+a)},\ B_{a,6} = \frac{120(-1+a)}{(1+a)^3(2+a)(3+a)},$$

$$B_{a,8} = \frac{40320(2-6a-a^2+a^3)}{(1+a)^4(2+a)^2(3+a)(4+a)},\ B_{a,10} = \frac{3628800(-2+16a-11a^2-4a^3+a^4)}{(1+a)^5(2+a)^2(3+a)(4+a)(5+a)}.$$

In this section, we connect the fractional Bernoulli numbers $B_{a,m}$ with the fractional zeta function $\zeta_{b,c}(s)$, where the fractional zeta function is defined as follows.

Definition 2.4. *For b, c positive real numbers, the fractional zeta function is defined by*

$$\zeta_{b,c}(s) = \sum_{k=1}^{\infty} \frac{1}{x_{b,c;k}^s}, \tag{2.2.4}$$

where $Re(s) > 1$ and $x_{b,c;k} \neq 0$ are the complex zeros of $\Phi_{b,c}(x)$.

A. Byrnes, L. Jiu and V. H. Moll in [5] observed that the Kummer function $\Phi_{b,c}(x)$ and the fractional zeta function $\zeta_{b,c}(x)$ are related by

$$\frac{\Phi_{b,c+1}(x)}{\Phi_{b,c}(x)} = 1 + \frac{b+c}{c}\sum_{k=1}^{\infty}\zeta_{b,c}(k+1)x^k.$$

They also described the values of the fractional zeta function at p, $p \in \mathbb{N}$, by

$$\zeta_{b,c}(p) = -\frac{cB(b+p-1,c)}{(b+c)(b+c+p-1)(p-2)!B(b,c)} - \sum_{r=1}^{p-2}\frac{B(b+r,c)}{B(b,c)r!}\zeta_{b,c}(p-r),$$

where $B(b,c)$ is the beta function given by

$$B(b,c) = \int_0^1 x^{b-1}(1-x)^{c-1}dx = \frac{\Gamma(b)\Gamma(c)}{\Gamma(b+c)}.$$

Here are the first few values

$$\begin{aligned}
\zeta_{b,c}(2) &= -\tfrac{bc}{(b+1)^2(1+b+c)} \\
\zeta_{b,c}(3) &= \tfrac{bc(b-c)}{(b+c)^3(b+c+1)(b+c+2)} \qquad (2.2.5) \\
\zeta_{b,c}(4) &= -\tfrac{bcP_4(b,c)}{(b+c)^4(b+c+1)^2(b+c+2)(b+c+3)},
\end{aligned}$$

where

$$P_4(b,c) = b^2 + b^3 - 4bc - 2b^2c + c^2 - 2bc^2 + c^3.$$

The following theorem appeared as Theorem 2.6 in [5].

Theorem 2.5. *The Kummer function $\Phi_{b,c}(x)$ satisfies the following factorization*

$$\Phi_{b,c}(x) = e^{bx/(b+c)}\prod_{k=1}^{\infty}\left(1 - \frac{x}{x_{b,c;k}}\right)e^{x/x_{b,c;k}}, \qquad (2.2.6)$$

where $\{x_{b,c;k}\}$ is the sequence of nonzero complex zeros of the function $\Phi_{b,c}(x)$.

We now prove a relation between our fractional Bernoulli numbers and the fractional zeta functions.

Theorem 2.6. *The fractional Bernoulli numbers* $B_{a,m}$ *are expressed in terms of fractional zeta function as*

$$B_{a,m} = \begin{cases} 1 & \textit{for} \quad m = 0 \\ 0 & \textit{for} \quad m = 2k+1, \quad k = 0, 1, 2, ... \\ -\frac{2}{1+a} & \textit{for} \quad m = 2 \\ \frac{(2k)!}{a}\zeta_{1,a}(k) & \textit{for} \quad m = 2k, \quad k \geq 2. \end{cases}$$

Proof. Using (2.2.6)

$$\Phi_{1,a}(x^2) = e^{\frac{x^2}{1+a}} \prod_{k=1}^{\infty} \left(1 - \frac{x^2}{x_{1,a;k}}\right) e^{\frac{x^2}{x_{1,a;k}}}.$$

Then, we have

$$\begin{aligned} \frac{\Phi'_{1,a}(x^2)}{\Phi_{1,a}(x^2)} &= \frac{2x}{1+a} + 2x \sum_{k=1}^{\infty} \left[\frac{1}{x_{1,a;k}} - \frac{1}{x_{1,a;k}(1 - x^2/x_{1,a;k})}\right] \\ &= \frac{2x}{1+a} + 2x \sum_{k=1}^{\infty} \frac{1}{x_{1,a;k}} \left(1 - \sum_{i=0}^{\infty} \left(\frac{x^2}{x_{1,a;k}}\right)^i\right). \end{aligned}$$

Thus,

$$\frac{\Phi'_{1,a}(x^2)}{\Phi_{1,a}(x^2)} = \frac{2x}{1+a} - 2\sum_{i=1}^{\infty} x^{2i+1}\zeta_{1,a}(i+1). \tag{2.2.7}$$

Again since

$$\Phi_{1,a}(x^2) = a x^{-2a} e^{x^2} \gamma(a, x^2).$$

By applying logarithmic differentiation on this expression, we obtain

$$\frac{\Phi'_{1,a}(x^2)}{\Phi_{1,a}(x^2)} = -\frac{2a}{x} + 2x + \frac{2a}{x\Phi_{1,a}(x^2)}. \tag{2.2.8}$$

Combining (2.2.7) and (2.2.8) yields

$$\frac{1}{\Phi_{1,a}(x^2)} = 1 - \frac{x^2}{1+a} - \frac{1}{a}\sum_{i=2}^{\infty} x^{2i}\zeta_{1,a}(i).$$

Therefore, we find that

$$\sum_{m=0}^{\infty} B_{a,m}\frac{x^m}{m!} = 1 - \frac{x^2}{1+a} - \frac{1}{a}\sum_{i=2}^{\infty} x^{2i}\zeta_{1,a}(i).$$

We compare the coefficients of powers of x to obtain the result. □

Definition 2.7. *The fractional Bernoulli polynomials, denoted by $B_{a,m}(z)$, are defined by the generating function*

$$\alpha(z,x) = \frac{x^{2a}e^{xz-x^2}}{a\gamma(a,x^2)} = \sum_{m=0}^{\infty} B_{a,m}(z)\frac{x^m}{m!}. \tag{2.2.9}$$

Theorem 2.8. *The fractional Bernoulli polynomials can be expressed in terms of the fractional Bernoulli numbers*

$$B_{a,m}(z) = \sum_{k=0}^{m} \binom{m}{k} B_{a,k} z^{m-k}. \tag{2.2.10}$$

Furthermore, these polynomials $B_{a,m}(z)$ satisfy the recurrence formula

$$\sum_{k=0}^{[m/2]} \frac{(-1)^k}{k!(m-2k)!}\left[\frac{a}{k+a}B_{a,m-2k}(z) - z^{m-2k}\right] = 0. \tag{2.2.11}$$

Proof. We combine (2.2.9) and (2.2.3) to get

$$\begin{aligned}\sum_{m=0}^{\infty} B_{a,m}(z)\frac{x^m}{m!} &= \frac{x^{2a}e^{-x^2}}{a\gamma(a,x^2)}e^{xz} \\ &= \left(\sum_{m=0}^{\infty} B_{a,m}\frac{x^m}{m!}\right)\left(\sum_{m=0}^{\infty} z^m\frac{x^m}{m!}\right) \\ &= \sum_{m=0}^{\infty}\sum_{k=0}^{m}\binom{m}{k} B_{a,k}z^{m-k}\frac{x^m}{m!}.\end{aligned}$$

Comparing coefficients yields (2.2.10). To prove (2.2.11), we multiply (2.2.9) by the denominator $a\gamma(a,x^2)\,x^{-2a}$ and expand both sides of the resulting equation into power series to get

$$\left(\sum_{m=0}^{\infty} z^m\frac{x^m}{m!}\right)\left(\sum_{k=0}^{\infty}(-1)^k\frac{x^{2k}}{k!}\right) = \left(\sum_{k=0}^{\infty}\frac{a(-1)^kx^{2k}}{k!(k+a)}\right)\left(\sum_{m=0}^{\infty} B_{a,m}(z)\frac{x^m}{m!}\right).$$

Carrying out the multiplication, we obtain

$$\sum_{m=0}^{\infty}\sum_{k=0}^{[m/2]}\frac{(-1)^kz^{m-2k}}{k!(m-2k)!}x^m = \sum_{m=0}^{\infty}\sum_{k=0}^{[m/2]}\frac{a(-1)^kB_{a,m-2k}(z)}{(k+a)k!(m-2k)!}x^m.$$

We now compare the coefficients to get the recursive formula of the fractional Bernoulli polynomials.

□

Here are the first few values of fractional Bernoulli polynomials:

$$B_{a,0}(z) = 1, \quad B_{a,1}(z) = z, \quad B_{a,2}(z) = z^2 - \frac{2}{1+a}, \quad B_{a,3}(z) = z^3 - \frac{6z}{1+a},$$

$$B_{a,4}(z) = z^4 - \frac{12z^2}{1+a} + \frac{24}{(1+a)^2(2+a)}, \quad B_{a,5}(z) = z^5 - \frac{20z^3}{1+a} + \frac{120z}{(1+a)^2(2+a)}.$$

Proposition 2.9. *We have*

$$B_{a,m}(z_1 + z_2) = \sum_{j=0}^{m} \binom{m}{j} B_{a,j}(z_1) z_2^{m-j}.$$

Proof. Using the fact that $\binom{n}{m}\binom{m}{k} = \binom{n}{k}\binom{n-k}{m-k}$,

$$\begin{aligned}
B_{a,m}(z_1 + z_2) &= \sum_{k=0}^{m} \binom{m}{k} B_{a,k} \ (z_1 + z_2)^{m-k} \\
&= \sum_{k=0}^{m} \binom{m}{k} B_{a,k} \sum_{j=0}^{m-k} \binom{m-k}{j} z_1^j z_2^{m-k-j} \\
&= \sum_{k=0}^{m} \sum_{j=k}^{m} \binom{m}{k} \binom{m-k}{j-k} B_{a,k} z_1^{j-k} z_2^{m-j} \\
&= \sum_{0 \le k \le j \le m} \binom{m}{j} \binom{j}{k} B_{a,k} z_1^{j-k} z_2^{m-j} \\
&= \sum_{j=0}^{m} \binom{m}{j} \sum_{k=0}^{j} \binom{j}{k} B_{a,k} z_1^{j-k} z_2^{m-j} \\
&= \sum_{j=0}^{m} \binom{m}{j} B_{a,j}(z_1) z_2^{m-j}.
\end{aligned}$$

□

Remark 2.10. *We note that*

(i) $B_{a,m}(0) = B_{a,m}$ *and* $B_{a,0}(z) = 1$,

(ii) $\frac{d}{dz}B_{a,m}(z) = mB_{a,m-1}(z), \quad m = 1, 2, \ldots,$

(iii) $\mathsf{B}_{a,m}(z+1) - B_{a,m}(z) = \sum_{k=1}^{m} \binom{m}{k} B_{a,m-k}(z)$, *and*

(iv) $B_{a,m}(0) \neq B_{a,m}(1)$.

Proposition 2.11. *Let a be a positive half odd integer. Then the polynomials $B_{a,m}(z)$ satisfy*

$$B_{a,m}(1-z) = \sum_{k=0}^{m} \binom{m}{k} (-1)^{k+1} B_{a,k}(z).$$

Proof. Using equation (2.2.9), we obtain

$$\begin{aligned}
\sum_{m=0}^{\infty} B_{a,m}(1-z)\frac{x^m}{m!} &= \frac{x^{2a}e^{x(1-z)-x^2}}{a\gamma(a,x^2)} \\
&= -\frac{(-x)^{2a}e^{-zx-x^2}}{a\gamma(a,x^2)}e^{x} \\
&= -\left(\sum_{m=0}^{\infty} B_{a,m}(z)\frac{(-x)^m}{m!}\right)\left(\sum_{m=0}^{\infty}\frac{x^m}{m!}\right) \\
&= -\sum_{m=0}^{\infty}\sum_{k=0}^{m}\binom{m}{k} B_{a,k}(z)(-1)^k\frac{x^m}{m!}.
\end{aligned}$$

Comparing the coefficients yields the lemma. □

Theorem 2.12. *For $n \in \mathbb{N}$, we have*

$$\sum_{j=0}^{n-1} B_{a,m}\left(z + \frac{j}{n}\right) = \sum_{k=0}^{m+1} \binom{m+1}{k} \frac{n^{k-m}}{m+1} B_{a,k}(z)(B_{m+1-k}(n) - B_{m+1-k}),$$

where B_m and $B_m(z)$ are the classical Bernoulli numbers and polynomials respectively.

Proof. We have

$$\sum_{m=0}^{\infty}\left[\sum_{j=0}^{n-1} B_{a,m}\left(z+\frac{j}{n}\right)\right]\frac{x^m}{m!}$$

$$= \sum_{m=0}^{\infty} B_{a,m}(z)\frac{x^m}{m!} + \sum_{m=0}^{\infty} B_{a,m}(z+1/n)\frac{x^m}{m!} + \cdots + \sum_{m=0}^{\infty} B_{a,m}\left(z+\frac{n-1}{n}\right)\frac{x^m}{m!}$$

$$= \frac{x^{2a}e^{xz-x^2}}{a\gamma(a,x^2)} + \frac{x^{2a}e^{x(z+1/n)-x^2}}{a\gamma(a,x^2)} + \cdots + \frac{x^{2a}e^{x(z+\frac{n-1}{n})-x^2}}{a\gamma(a,x^2)}$$

$$= \frac{x^{2a}e^{xz-x^2}}{a\gamma(a,x^2)}\left[1+e^{x/n}+e^{2x/n}+\cdots+e^{\frac{n-1}{n}x}\right]$$

$$= \frac{x^{2a}e^{xz-x^2}}{a\gamma(a,x^2)}\frac{e^x-1}{e^{x/n}-1}$$

$$= \frac{x^{2a}e^{xz-x^2}}{a\gamma(a,x^2)}\frac{n}{x}\left[\frac{\frac{x}{n}e^{\frac{x}{n}n}}{e^{x/n}-1} - \frac{x/n}{e^{x/n}-1}\right]$$

$$= \frac{n}{x}\left(\sum_{m=0}^{\infty} B_{a,m}(z)\frac{x^m}{m!}\right)\left(\sum_{m=0}^{\infty} B_m(n)\frac{(x/n)^m}{m!}\right) - \frac{n}{x}\left(\sum_{m=0}^{\infty} B_{a,m}(z)\frac{x^m}{m!}\right)\left(\sum_{m=0}^{\infty} B_m\frac{(x/n)^m}{m!}\right)$$

$$= \sum_{m=0}^{\infty}\sum_{k=0}^{m+1}\binom{m+1}{k} B_{a,k}(z)\frac{n^{k-m}}{m+1}(B_{m+1-k}(n)-B_{m+1-k})\frac{x^m}{m!}.$$

Therefore,

$$\sum_{j=0}^{n-1} B_{a,m}\left(z+\frac{j}{n}\right) = \sum_{k=0}^{m+1}\binom{m+1}{k}\frac{n^{k-m}}{m+1}B_{a,k}(z)(B_{m+1-k}(n)-B_{m+1-k}).$$

□

Putting $n = 1$ in the theorem, we obtain

Corollary 2.13.

$$B_{a,m}(z) = \frac{1}{m+1}\sum_{k=0}^{m+1}\binom{m+1}{k} B_{a,k}(z)(B_{m+1-k}(1)-B_{m+1-k}).$$

Remark: Using (2.2.9)

$$\alpha(z,x)=\frac{x^{2a}e^{xz-x^2}}{a\gamma(a,x^2)},$$

we see that

$$x\frac{\partial\alpha}{\partial x}-(2a+zx-2x^2)\alpha(z,x)+2a\alpha(z,x)\alpha(0,x)=0.$$

We substitute the power series expansion of (2.2.9) and note that $\alpha(0,x)=\sum_{m=0}^{\infty}B_{a,m}\frac{x^m}{m!}$ to get

$$\begin{aligned}\textstyle\sum_{m=0}^{\infty}(m-2a)B_{a,m}(z)\frac{x^m}{m!} \quad &- \quad z\textstyle\sum_{m=0}^{\infty}B_{a,m}(z)\frac{x^{m+1}}{m!}+2\sum_{m=0}^{\infty}B_{a,m}(z)\frac{x^{m+2}}{m!}\\ &+ \quad 2a\textstyle\sum_{m=0}^{\infty}\sum_{k=0}^{m}\binom{m}{k}B_{a,k}(z)B_{a,m-k}\frac{x^m}{m!}=0.\end{aligned}$$

We now compare coefficients to get

$$\begin{aligned}(m+1)B_{a,m+1}(z) \quad &- \quad (m+1)zB_{a,m}(z)+2m(m+1)B_{a,m-1}(z)\\ &+ \quad 2a\textstyle\sum_{k=0}^{m}\binom{m+1}{k}B_{a,k}(z)B_{a,m-k+1}=0, \quad m=1,2,\ldots.\end{aligned} \tag{2.2.12}$$

One can also use this relation to calculate the fractional Bernoulli polynomials step by step, starting from $B_{a,0}(z)=1$, $B_{a,1}(z)=z$.

Setting $z=0$ in (2.2.12) yields the recurrence formula for the fractional Bernoulli numbers as follows:

$$(m+1)B_{a,m+1}+2m(m+1)B_{a,m-1}+2a\sum_{k=0}^{m}\binom{m+1}{k}B_{a,k}B_{a,m-k+1}=0, \quad m=1,2,.... \tag{2.2.13}$$

As we have seen in the fractional Bernoulli polynomials, we can also find the fractional Bernoulli numbers from (2.2.13) step by step, starting from $B_{a,0}=1$, and $B_{a,1}=0$.

Similarly, the identity

$$\frac{\partial\alpha}{\partial z}-x\alpha(z,x)=0$$

leads to

$$\sum_{m=0}^{\infty}\frac{d}{dz}B_{a,m}(z)\frac{x^m}{m!}-\sum_{m=0}^{\infty}mB_{a,m-1}(z)\frac{x^m}{m!}=0.$$

This implies

$$\frac{d}{dz}B_{a,m}(z)=mB_{a,m-1}(z), \quad m=1,2,\cdots.$$

This formula allows us to express the derivative of fractional Bernoulli polynomials in terms of another fractional Bernoulli polynomials.

Theorem 2.14. *Let* $p \in \mathbb{N}$ *and*

$$g(z,x,p) = \frac{x^{2a}e^{xz-x^2}}{(a\gamma(a,x^2))^p} = \sum_{m=0}^{\infty} \beta_{a,p,m}(z)\frac{x^m}{m!}.$$

Then

$$\frac{\partial g}{\partial x} = (2ax^{-1} + z - 2x)g(z,x,p) - 2apx^{-1}g(z,x,p)\alpha(x,1) \tag{2.2.14}$$

and

$$\begin{aligned} & 2ap\sum_{k=0}^{m+1}\binom{m+1}{k}\beta_{a,p,k}(z)\beta_{a,1,m-k+1} \qquad (2.2.15) \\ = \; & (2a-m-1)\beta_{a,p,m+1}(z) + (m+1)z\beta_{a,p,m}(z) - 2(m+1)m\beta_{a,p,m-1}(z), \end{aligned}$$

where $\alpha(x,1) = g(0,x,1)$ *and* $\beta_{a,p,m} = \beta_{a,p,m}(0)$.

Proof. The first one is clear. To show (2.2.16) multiplying both sides of (2.2.14) by x to get

$$\sum_{m=0}^{\infty} m\beta_{a,p,m}(z)\frac{x^m}{m!} = 2ag(z,x,p)+zxg(z,x,p)-2x^2g(z,x,p)-2apg(z,x,p)\alpha(x,1).$$

From definition of $g(z,x,p)$, we have

$$\begin{aligned} & \sum_{m=0}^{\infty} m\beta_{a,p,m}(z)\frac{x^m}{m!} \\ = \; & 2a\sum_{m=0}^{\infty}\beta_{a,p,m}(z)\frac{x^m}{m!} + z\sum_{m=0}^{\infty}\beta_{a,p,m}(z)\frac{x^{m+1}}{m!} - 2\sum_{m=0}^{\infty}\beta_{a,p,m}(z)\frac{x^{m+2}}{m!} - \\ & 2ap\sum_{m=0}^{\infty}\sum_{k=0}^{m}\binom{m}{k}\beta_{a,p,k}(z)\beta_{a,1,m-k}\frac{x^m}{m!} \end{aligned}$$

Comparing the coefficients yields (2.2.16). □

Remark 2.15. *Note that when* $p = 1$, *we have* $\beta_{a,1,m}(z) = B_{a,m}(z)$.

Chapter 3

The Connection of Fractional Bernoulli Polynomials with Hermite Polynomials

In this chapter, we discuss the determinant representation of the fractional Bernoulli polynomials to express the fractional Bernoulli polynomials in terms of Hermite polynomials.

3.1 Determinant representation of Fractional Bernoulli polynomials

The determinant representation of the classical Bernoulli numbers were introduced by J. W. L. Glaisher in 1875 (see [11]). He also expressed the Cauchy and Euler numbers using matrix determinant. More recently, R. Booth and H. D. Nguyen in [4] discussed a determinant formula of Bernoulli polynomials by considering a square version of Pascal's triangle as follows:

$$B_m(z) = \frac{(-1)^m}{(m-1)!} \begin{vmatrix} 1 & 1 & 0 & 0 & \cdots & 0 \\ z & \frac{1}{2} & 1 & 0 & \cdots & 0 \\ z^2 & \frac{1}{3} & 1 & 2 & \cdots & 0 \\ z^3 & \frac{1}{4} & 1 & 3 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ z^m & \frac{1}{m+1} & \binom{m}{0} & \binom{m}{1} & \cdots & \binom{m}{m-2} \end{vmatrix}_{m+1} \tag{3.1.1}$$

$$= \frac{(-1)^m}{(m-1)!} |b_{ij}|,$$

where $m+1$ is the dimension of the matrix and for $i=1,2,...,m+1$

$$b_{ij} = \begin{cases} z^{i-1}, & j=1 \\ \frac{1}{i}, & j=2 \\ \binom{i-1}{j-3}, & j>2. \end{cases}$$

The authors set $z=0$ in (3.1.1) and expand the determinant along the first column and performed the following row and column operations on the expanded matrix: multiply row i by $i+1$ and divide column j, beginning with the third column, by $j-1$. From this they obtain the Bernoulli numbers by the matrix determinant as follows:

$$B_m = \frac{(-1)^{m-1}}{(m+1)!} \begin{vmatrix} 1 & 2 & 0 & 0 & \cdots & 0 \\ 1 & 3 & 3 & 0 & \cdots & 0 \\ 1 & 4 & 6 & 4 & \cdots & 0 \\ 1 & 5 & 10 & 10 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \binom{m+1}{0} & \binom{m+1}{1} & \binom{m+1}{2} & \binom{m+1}{3} & \cdots & \binom{m+1}{m-1} \end{vmatrix}_m .$$

R. Booth and H. D. Nguyen in [4] also presents the extension of this formula to a class of generalized Bernoulli polynomials by

$$B_m(N,z) = \frac{(-1)^m (N!)^m 1!2!3! \cdots (m-N-1)!}{1!2!3! \cdots (m-1)!1!2!3! \cdots N!} |b_{ij}|,$$

where

$$b_{ij} = \begin{cases} z^{i-1}, & j=1 \\ \binom{i-j+N+1}{i-1}^{-1}, & 2 \le j \le N+2 \\ \binom{i-1}{j-N-2}, & j \ge N+2. \end{cases}$$

for $i,j=1,2,...,m+1$.

In this section, we show that the fractional Bernoulli polynomials also admit a determinant representation that shows a relation between a polynomial of degree m and those with lower degrees. To this end, we consider

$$f(x)=\sum_{m=0}^{\infty} c_m x^m, \qquad g(x)=\sum_{m=0}^{\infty} a_m x^m,$$

and their quotients:

$$\frac{f(x)}{g(x)}=\sum_{m=0}^{\infty} A_m x^m.$$

Now equating the coefficients of the quotient yields:

$$c_m=\sum_{k=0}^{m} a_k A_{m-k}.$$

As in [4], solving the above equation using Crammer's Rule, we obtain

$$A_m=(-1)^m\frac{1}{a_0^m}\begin{vmatrix} c_0 & a_0 & 0 & 0 & \cdots & 0\\ c_1 & a_1 & a_0 & 0 & \cdots & 0\\ c_2 & a_2 & a_1 & a_0 & \cdots & 0\\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots\\ c_{m-1} & a_{m-1} & a_{m-2} & a_{m-3} & \cdots & a_0\\ c_m & a_m & a_{m-1} & a_{m-2} & \cdots & a_1 \end{vmatrix}. \tag{3.1.2}$$

From the definition of the fractional Bernoulli polynomials, we can view the equation (2.2.9) as a division of two power series as follows:

$$\frac{e^{xz-x^2}}{ax^{-2a}\gamma(a,x^2)}=\frac{f(x)}{g(x)}=\sum_{m=0}^{\infty} A_m x^m,$$

where

$$f(x)=\sum_{m=0}^{\infty}\sum_{k=0}^{[m/2]}\frac{z^{m-2k}(-1)^k}{k!(m-2k)!}x^m \quad \text{and} \quad g(x)=\sum_{m=0}^{\infty}\frac{a(-1)^m}{m!(m+a)}x^{2m}.$$

Thus, with $A_m=\frac{1}{m!}B_{a,m}(z)$, c_m and a_m defined by

$$c_m=\sum_{k=0}^{[m/2]}\frac{z^{m-2k}(-1)^k}{k!(m-2k)!}, \quad \text{and} \quad a_m=\begin{cases}\frac{a(-1)^{m/2}}{(\frac{m}{2})!(\frac{m}{2}+a)}, & \text{if } m=\text{even}\\ 0, & \text{if } m=\text{odd},\end{cases}$$

(2.2.9) yields determinant formula for the fractional Bernoulli polynomials:

$$
\begin{aligned}
B_{a,m}(z) &= m!(-1)^m a^m \\
&\times \begin{vmatrix}
1 & \frac{1}{a} & 0 & 0 & \cdots & 0 \\
z & 0 & \frac{1}{a} & 0 & \cdots & 0 \\
\frac{z^2}{2!}-1 & -\frac{1}{1+a} & 0 & \frac{1}{a} & \cdots & 0 \\
\frac{z^3}{3!}-z & 0 & -\frac{1}{1+a} & 0 & \cdots & 0 \\
\frac{z^4}{4!}-\frac{z^2}{2!}+\frac{1}{2} & \frac{1}{2!(2+a)} & 0 & -\frac{1}{1+a} & \cdots & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
\sum_{k=0}^{[m/2]} \frac{z^{m-2k}(-1)^k}{k!(m-2k)!} & 0 & \frac{(-1)^{\left[\frac{m-1}{2}\right]}}{\left[\frac{m-1}{2}\right]!\left(\left[\frac{m-1}{2}\right]+a\right)} & 0 & \cdots & 0
\end{vmatrix}_{m+1} \\
&= m!(-1)^m a^m |b_{ij}|,
\end{aligned}
\tag{3.1.3}
$$

where

$$
b_{ij} = \begin{cases}
0, & i+1<j \text{ and } i=j+2k, j\neq 1, k=0,1,... \\
\sum_{k=0}^{\left[\frac{i-1}{2}\right]} \frac{z^{i-1-2k}(-1)^k}{k!(i-1-2k)!}, & j=1 \\
\frac{(-1)^{\left[\frac{i-j+1}{2}\right]}}{\left[\frac{i-j+1}{2}\right]!\left[\frac{i-j+1}{2}+a\right]}, & j=i+1 \text{ and } i=j+k, j\neq 1, k=\text{odd}
\end{cases}
$$

for $i,j=1,2,...,m+1$.

Finally, we put $z=0$ in (3.1.3), and we find the determinant formula of the fractional Bernoulli numbers:

$$B_{a,m} = m!(-1)^m a^m \begin{vmatrix} 1 & \frac{1}{a} & 0 & 0 & \cdots & 0 \\ 0 & 0 & \frac{1}{a} & 0 & \cdots & 0 \\ -1 & -\frac{1}{1+a} & 0 & \frac{1}{a} & \cdots & 0 \\ 0 & 0 & -\frac{1}{1+a} & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \frac{(-1)^{[\frac{m}{2}]}}{[m/2]!} & 0 & \frac{(-1)^{\left[\frac{m-1}{2}\right]}}{\left[\frac{m-1}{2}\right]!\left(\left[\frac{m-1}{2}\right]+a\right)} & 0 & \cdots & 0 \end{vmatrix}_{m+1}.$$

3.2 The Connection of Fractional Bernoulli Polynomials with Hermite Polynomials

In 1859 Chebyshev introduced the Hermite polynomials $\mathcal{H}_m(z)$ by the formula

$$\mathcal{H}_m(z) = (-1)^m e^{z^2} \frac{d^m}{dz^m} e^{-z^2}, \quad m = 0, 1, 2, ... \tag{3.2.1}$$

They are an important and applicable class of polynomials, especially in mathematical physics in problems involving the integration of Laplace's equation and Helmholtz' equation in parabolic coordinates, in quantum mechanics, etc. (see [20]). Now using equation (3.2.1), the first few Hermite polynomials are

$$\mathcal{H}_0(z) = 1, \quad \mathcal{H}_1(z) = 2z, \quad \mathcal{H}_2(z) = 4z^2-2, \quad \mathcal{H}_3(z) = 8z^3-12z, \quad \mathcal{H}_4(z) = 16z^4-48z^2+12,$$

and in general,

$$\mathcal{H}_m(z) = \sum_{k=0}^{[m/2]} \frac{(-1)^k m!}{k!(m-2k)!} (2z)^{m-2k},$$

where $[\nu]$ denotes the largest integer $\leq \nu$.

The Hermite polynomials multiplied by the constant factor $\frac{1}{m!}$ are the coefficients in the expansion

$$e^{2zx-x^2} = \sum_{m=0}^{\infty} \frac{\mathcal{H}_m(z)}{m!} x^m, \quad |x| < \infty,$$

and hence the equation in the left hand side is called the generating function of the Hermite polynomials.
In [21], Geleta described the relation of error Bernoulli polynomials $B_{\frac{1}{2},m}(z)$

to the Hermite polynomial by

$$\mathcal{H}_{2m}(z/2)=\sum_{k=0}^{m}\frac{(-1)^k(2m)!B_{\frac{1}{2},2m-2k}(z)}{(2m-2k)!k!(2k+1)},$$

and

$$\mathcal{H}_{2m+1}(z/2)=\sum_{k=0}^{m}\frac{(-1)^k(2m+1)!B_{\frac{1}{2},2m+1-2k}(z)}{(2m+1-2k)!k!(2k+1)}.$$

Now in this section, we discus the relationship between the fractional Bernoulli polynomials and Hermite polynomials. From equation (2.2.9), we find that

$$\begin{aligned} e^{xz-x^2} &= ax^{-2a}\gamma(a,x^2)\sum_{m=0}^{\infty}B_{a,m}(z)\frac{x^m}{m!} \\ &= \left(\sum_{k=0}^{\infty}\frac{(-1)^k a x^{2k}}{k!(k+a)}\right)\left(\sum_{m=0}^{\infty}B_{a,m}(z)\frac{x^m}{m!}\right) \\ &= \sum_{m=0}^{\infty}\sum_{k=0}^{[m/2]}\frac{a(-1)^k B_{a,m-2k}(z)}{k!(k+a)(m-2k)!}x^m. \end{aligned}$$

We thus have

$$\mathcal{H}_m(z)=\sum_{k=0}^{[m/2]}\frac{a(-1)^k B_{a,m-2k}(2z)m!}{k!(k+a)(m-2k)!}. \tag{3.2.2}$$

From (3.2.2), we obtain

$$\mathcal{H}_0(z)=B_{a,0}(2z),\quad \mathcal{H}_1(z)=B_{a,1}(2z),\quad \mathcal{H}_2(z)=B_{a,2}(2z)-\tfrac{2!a}{1+a}B_{a,0}(2z),$$

$$\mathcal{H}_3(z)=B_{a,3}(2z)-\tfrac{3!a}{1+a}B_{a,1}(2z),\quad \mathcal{H}_4(z)=B_{a,4}(2z)-\tfrac{4!a}{2!(1+a)}B_{a,2}(2z)+\tfrac{4!a}{2!(2+a)}B_{a,0}(2z).$$

To express the fractional Bernoulli polynomials $B_{a,m}(z)$ in terms of Hermite polynomials $\mathcal{H}_m(z)$, we first note that

$$\frac{e^{xz-x^2}}{ax^{-2a}\gamma(a,x^2)}=\frac{\sum_{m=0}^{\infty}\mathcal{H}_m(z/2)\frac{x^m}{m!}}{\sum_{m=0}^{\infty}\frac{(-1)^m a}{(m+a)}\frac{x^{2m}}{m!}}=\sum_{m=0}^{\infty}B_{a,m}(z)\frac{x^m}{m!}.$$

Using (3.1.2) with $c_m=\frac{1}{m!}\mathcal{H}_m(z/2)$ and

$$a_m=\begin{cases}\frac{a(-1)^{m/2}}{(m/2)!(\frac{m}{2}+a)}, & \text{if } m=\text{even}\\ 0, & \text{if } m=\text{odd},\end{cases}$$

we have

$$B_{a,m}(z) = (-1)^m m! \begin{vmatrix} \mathcal{H}_0(z/2) & 1 & 0 & 0 & \cdots & 0 \\ \mathcal{H}_1(z/2) & 0 & 1 & 0 & \cdots & 0 \\ \frac{1}{2!}\mathcal{H}_2(z/2) & \frac{-a}{1+a} & 0 & 1 & \cdots & 0 \\ \frac{1}{3!}\mathcal{H}_2(z/2) & 0 & \frac{-a}{1+a} & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \frac{1}{m!}\mathcal{H}_m(z/2) & 0 & \frac{a(-1)^{\frac{m-1}{2}}}{(\frac{m-1}{2})!(\frac{m-1}{2}+a)} & 0 & \cdots & 0 \end{vmatrix}_{m+1}.$$

Starting with the first row, we factor $\frac{1}{(i-1)!}$ from row i, to obtain

$$B_{a,m}(z) = \frac{(-1)^m}{1!2!3!\cdots(m-1)!} \times$$

$$\begin{vmatrix} \mathcal{H}_0(z/2) & 1 & 0 & 0 & \cdots & 0 \\ \mathcal{H}_1(z/2) & 0 & 1 & 0 & \cdots & 0 \\ \mathcal{H}_2(z/2) & \frac{-2!a}{1+a} & 0 & 2! & \cdots & 0 \\ \mathcal{H}_2(z/2) & 0 & \frac{-3!a}{1+a} & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \mathcal{H}_{m-1}(z/2) & \frac{am!(-1)^{\frac{m-1}{2}}}{(\frac{m-1}{2})!(\frac{m-1}{2}+a)} & 0 & \frac{am!(-1)^{\frac{m-3}{2}}}{(\frac{m-3}{2})!(\frac{m-3}{2}+a)} & \cdots & (m-1)! \\ \mathcal{H}_m(z/2) & 0 & \frac{am!(-1)^{\frac{m-1}{2}}}{(\frac{m-1}{2})!(\frac{m-1}{2}+a)} & 0 & \cdots & 0 \end{vmatrix}_{m+1}.$$

Expanding this matrix with respect to the first column yields

$$B_{a,m}(z) = \frac{(-1)^m}{1!2!3!\cdots(m-1)!}\left[\mathcal{H}_0(z/2)|b_{11}| - \mathcal{H}_1(z/2)|b_{21}| + \cdots + (-1)^m\mathcal{H}_m(z/2)|b_{m+1\ 1}|\right],$$

where b_{k1} is the m by m matrix obtained by deleting the k^{th} row and the first column. Therefore,

$$B_{a,m}(z) = \frac{(-1)^m}{1!2!3!\cdots(m-1)!}\sum_{k=0}^{m}(-1)^k\mathcal{H}_k(z/2)|b_{k+1,1}|. \tag{3.2.3}$$

Here are the first few examples:

$$B_{a,0}(z) = \mathcal{H}_0(z/2), \quad B_{a,1}(z) = \mathcal{H}_1(z/2), \quad B_{a,2}(z) = \mathcal{H}_2(z/2) + \tfrac{2!a}{1+a}\mathcal{H}_0(z/2),$$

$$B_{a,3}(z) = \mathcal{H}_3(z/2) + \tfrac{3!a}{1+a}\mathcal{H}_1(z/2),$$

$$B_{a,4}(z) = \mathcal{H}_4(z/2) + \tfrac{4!a}{2!(1+a)}\mathcal{H}_2(z/2) + \left[\tfrac{4!a^2}{(1+a)^2} - \tfrac{4!a}{2!(2+a)}\right]\mathcal{H}_0(z/2),$$

$$B_{a,5}(z) = \mathcal{H}_5(z/2) + \tfrac{5!a}{3!(1+a)}\mathcal{H}_3(z/2) + \left[\tfrac{5!a^2}{(1+a)^2} - \tfrac{5!a}{2!(2+a)}\right]\mathcal{H}_1(z/2).$$

Chapter 4

Higher Order Fractional Bernoulli Numbers and Polynomials

In this chapter, we investigate the recursion formulas of higher order fractional Bernoulli numbers and polynomials. We also establish the connection of higher order fractional Bernoulli polynomials with the identity

$$\varphi_{m,k}(a) = \frac{k!}{a^{m+1-k}} \sum_{j=0}^{m} \sum_{i=0}^{\alpha-1} \binom{m}{j} \omega_a^{-i(k-j)},$$

where $a = \frac{\alpha}{\beta}$, $\beta \neq 0$ is a positive nonzero fraction and $\omega_a = e^{\frac{2\pi i}{a}}$.

4.1 Higher Order Fractional Bernoulli Numbers and Polynomials and Some of Their Properties

In this section, we introduce the higher order fractional Bernoulli numbers and polynomials and investigate some of their basic properties. In particular, we develop the differential equation satisfied by the higher order fractional Bernoulli polynomials.

Definition 4.1. *For $0 \neq t \in \mathbb{R}$, the higher order fractional Bernoulli polynomials defined by the generating function*

$$\left(\frac{x^{2a} e^{-x^2}}{a\gamma(a, x^2)} \right)^t e^{xz} = \sum_{m=0}^{\infty} B_{a,m}^{(t)}(z) \frac{x^m}{m!}, \tag{4.1.1}$$

and the higher order fractional Bernoulli numbers are defined by $B_{a,m}^{(t)} = B_{a,m}^{(t)}(0)$. In particular, $B_{a,m}^{(1)}(z) = B_{a,m}(z)$, the fractional Bernoulli polynomials, and $B_{a,m}^{(1)}(0) = B_{a,m}$, the fractional Bernoulli numbers.

Remark 4.2. *The higher order fractional Bernoulli polynomials satisfy the following properties:*

(i)

$$B_{a,m}^{(t)}(z) = \sum_{k=0}^{m} \binom{m}{k} B_{a,k}^{(t)} z^{m-k}, \tag{4.1.2}$$

(ii)

$$B^{(t)}_{a,m}(z_1+z_2)=\sum_{j=0}^{m}\binom{m}{j}B^{(t)}_{a,j}(z_1)z_2^{m-j}. \tag{4.1.3}$$

Proof. Exactly similar to Theorem 2.8 and Proposition 2.9, respectively. □

Proposition 4.3. *Let a be positive half integer and $0\neq t\in\mathbb{Z}$,*

$$B^{(t)}_{a,m}(1-z)=\sum_{k=0}^{m}\binom{m}{k}B^{(t)}_{a,k}(z)(-1)^{k+t}.$$

Proof.

$$\begin{aligned}\sum_{m=0}^{\infty}B^{(t)}_{a,m}(1-z)\frac{x^m}{m!}&=\left(-\frac{(-x)^{2a}e^{-x^2}}{a\gamma(a,x^2)}\right)^{t}e^{(-x)z}e^{x}\\&=(-1)^t\left(\sum_{m=0}^{\infty}B^{(t)}_{a,m}(z)\frac{(-x)^m}{m!}\right)\left(\sum_{m=0}^{\infty}\frac{x^m}{m!}\right)\\&=(-1)^t\sum_{m=0}^{\infty}\sum_{k=0}^{m}\binom{m}{k}B^{(t)}_{a,k}(z)(-1)^k\frac{x^m}{m!}.\end{aligned}$$

Comparing the coefficients gives the required result. □

Theorem 4.4. *We have*

$$\begin{aligned}&(m+1)B^{(t)}_{a,m+1}(z)-(m+1)zB^{(t)}_{a,m}(z)+2tm(m+1)B^{(t)}_{a,m-1}(z)+\\&\quad 2at\sum_{k=0}^{m}\binom{m+1}{k}B^{(t)}_{a,k}(z)B_{a,m-k+1}=0,\quad m=1,2,...\end{aligned}\tag{4.1.4}$$

and

$$(m+1)B^{(t)}_{a,m+1}+2tm(m+1)B^{(t)}_{a,m-1}+2at\sum_{k=0}^{m}\binom{m+1}{k}B^{(t)}_{a,k}B_{a,m-k+1}=0,\quad m=1,2,.... \tag{4.1.5}$$

Proof. Differentiating both sides of (4.1.1) with respect to x and then multiplying by x, we obtain

$$\begin{aligned}&2ta\sum_{m=0}^{\infty}B^{(t)}_{a,m}(z)\frac{x^m}{m!}-2t\sum_{m=0}^{\infty}B^{(t)}_{a,m}(z)\frac{x^{m+2}}{m!}+z\sum_{m=0}^{\infty}B^{(t)}_{a,m}(z)\frac{x^{m+1}}{m!}-\\&\quad 2at\sum_{m=0}^{\infty}\sum_{k=0}^{m}\binom{m}{k}B^{(t)}_{a,k}(z)B_{a,m-k}\frac{x^m}{m!}=\sum_{m=0}^{\infty}mB^{(t)}_{a,m}(z)\frac{x^m}{m!}.\end{aligned}$$

Comparing the coefficients, then we have the recurrence

$$(m+1)B_{a,m+1}^{(t)}(z)-(m+1)zB_{a,m}^{(t)}(z)+2tm(m+1)B_{a,m-1}^{(t)}(z)+$$
$$2at\sum_{k=0}^{m}\binom{m+1}{k}B_{a,k}^{(t)}(z)B_{a,m-k+1}=0,\quad m=1,2,....$$

Putting $z=0$ in (4.1.4) yields the recurrence formula of the higher order fractional Bernoulli numbers (4.1.5). $\square$

Remark 4.5. *For* $t=1$, (4.1.4) *and* (4.1.5) *become the recurrence formulas of the fractional Bernoulli polynomials* (2.2.12) *and the fractional Bernoulli numbers* (2.2.13)*, respectively.*

Theorem 4.6. *For* $m>0$,

$$B_{a,m+1}^{(t+1)}(z)=(2at)^{-1}(2at-m-1)B_{a,m+1}^{(t)}(z)+(2at)^{-1}(m+1)zB_{a,m}^{(t)}(z)-ma^{-1}(m+1)B_{a,m-1}^{(t)}(z).$$

Proof. Let

$$\alpha(a,x)=\frac{x^{2a}e^{-x^2}}{a\gamma(a,x^2)}.$$

Then from (4.1.1) we have

$$\alpha^t(a,x)e^{xz}=\sum_{m=0}^{\infty}B_{a,m}^{(t)}(z)\frac{x^m}{m!}. \tag{4.1.6}$$

Differentiating both sides of (4.1.6) with respect to x, we have

$$2t(ax^{-1}-x)\alpha^t(a,x)e^{xz}-2atx^{-1}\alpha^{t+1}(a,x)e^{xz}+z\alpha^t(a,x)e^{xz}=\sum_{m=0}^{\infty}mB_{a,m}^{(t)}(z)\frac{x^{m-1}}{m!}.$$

Then multiplying these by x and using (4.1.6), we find that

$$2ta\sum_{m=0}^{\infty}B_{a,m}^{(t)}(z)\frac{x^m}{m!}-2t\sum_{m=0}^{\infty}B_{a,m}^{(t)}(z)\frac{x^{m+2}}{m!}-2ta\sum_{m=0}^{\infty}B_{a,m}^{(t+1)}(z)\frac{x^m}{m!}+$$
$$z\sum_{m=0}^{\infty}B_{a,m}^{(t)}(z)\frac{x^{m+1}}{m!}=\sum_{m=0}^{\infty}mB_{a,m}^{(t)}(z)\frac{x^m}{m!}.$$

Then we get the result by comparing the coefficients. $\square$

Theorem 4.7. *For $t \in \mathbb{N}$,*

$$B^{(t)}_{a,m}(z_1 + z_2 + \cdots + z_t) = \sum_{\substack{k_1 + \cdots + k_t = m \\ k_1, ..., k_t \geq 0}} \binom{m}{k_1, ..., k_t} B_{a,k_1}(z_1) \cdots B_{a,k_t}(z_t),$$

where $\binom{m}{k_1, ..., k_t}$ *is the multinomial coefficients and* $[B_a(z)]^k = B_{a,k}(z)$.

Proof. Using equation (4.1.1),

$$\begin{aligned}
&\sum_{m=0}^{\infty} B^{(t)}_{a,m}(z_1 + z_2 + \cdots + z_t)\frac{x^m}{m!} \\
&= \left(\frac{x^{2a}e^{-x^2}}{a\gamma(a, x^2)}\right)^t e^{(z_1+z_2+\cdots+z_t)x} \\
&= \left(\sum_{m=0}^{\infty} B_{a,m}(z_1)\frac{x^m}{m!}\right)\left(\sum_{m=0}^{\infty} B_{a,m}(z_2)\frac{x^m}{m!}\right)\cdots\left(\sum_{m=0}^{\infty} B_{a,m}(z_t)\frac{x^m}{m!}\right) \\
&= \left(\sum_{m=0}^{\infty} B^m_a(z_1)\frac{x^m}{m!}\right)\left(\sum_{m=0}^{\infty} B^m_a(z_2)\frac{x^m}{m!}\right)\cdots\left(\sum_{m=0}^{\infty} B^m_a(z_t)\frac{x^m}{m!}\right) \\
&= e^{[B_a(z_1)+\cdots+B_a(z_t)]x} \\
&= \sum_{m=0}^{\infty}(B_a(z_1) + \cdots + B_a(z_t))^m\frac{x^m}{m!} \\
&= \sum_{m=0}^{\infty} \sum_{\substack{k_1 + \cdots + k_t = m \\ k_1, ..., k_t \geq 0}} \left[\binom{m}{k_1, ..., k_t} B_{a,k_1}(z_1) \cdots B_{a,k_t}(z_t)\right]\frac{x^m}{m!},
\end{aligned}$$

where the last equality is using multinomial expansion and the power of each term, in the expansion, is degraded to subscripts; i.e., $[B_a(z)]^k = B_{a,k}(z)$. Hence the theorem follows. □

Lemma 4.8. *We have*

$$\frac{d^k}{dz^k}B^{(t)}_{a,m}(z) = \frac{m!}{(m-k)!}B^{(t)}_{a,m-k}(z), \quad m \geq k.$$

Proof. Since

$$\frac{d^k}{dz^k}\alpha^t(a,x)e^{xz} = \alpha^t(a,x)\frac{d^k}{dz^k}e^{xz} = \sum_{m=0}^{\infty}\frac{d^k}{dz^k}B^{(t)}_{a,m}(z)\frac{x^m}{m!},$$

we find

$$\alpha^t(a,x)e^{xz}x^k = \sum_{m=0}^{\infty}\frac{d^k}{dz^k}B^{(t)}_{a,m}(z)\frac{x^m}{m!}.$$

Therefore,

$$\sum_{m=0}^{\infty}B^{(t)}_{a,m-k}(z)\frac{x^m}{(m-k)!} = \sum_{m=0}^{\infty}\frac{d^k}{dz^k}B^{(t)}_{a,m}(z)\frac{x^m}{m!}.$$

Comparing the coefficients yields the lemma. □

Theorem 4.9. *The higher order fractional Bernoulli polynomials satisfies the differential equation:*

$$\frac{B_{a,m}}{m!}\frac{d^m y}{dz^m} + \frac{B_{a,m-1}}{(m-1)!}\frac{d^{m-1}y}{dz^{m-1}} + \cdots + \left(\frac{B_{a,2}}{2!}+\frac{1}{a}\right)\frac{d^2y}{dz^2} - \frac{z}{2at}\frac{dy}{dz} + \frac{m}{2at}y = 0,$$

where $y = B^{(t)}_{a,m}(z)$.

Proof. By (4.1.4), we have

$$(m+1)B^{(t)}_{a,m+1}(z) - (m+1)zB^{(t)}_{a,m}(z) + 2tm(m+1)B^{(t)}_{a,m-1}(z) +$$
$$2at\sum_{k=0}^{m}\binom{m+1}{k}B^{(t)}_{a,k}(z)B_{a,m-k+1} = 0, \quad m = 1,2,....$$

Now replacing m by $m+1$, we find

$$mB^{(t)}_{a,m}(z) - zmB^{(t)}_{a,m-1}(z) + 2tm(m-1)B^{(t)}_{a,m-2}(z) + 2at\sum_{k=0}^{m-1}\binom{m}{k}B^{(t)}_{a,k}(z)B_{a,m-k} = 0. \tag{4.1.7}$$

Since $y = B^{(t)}_{a,m}(z)$, then by Lemma 4.8,

$$B^{(t)}_{a,m-k}(z) = \frac{(m-k)!}{m!}\frac{d^k y}{dz^k}, \quad k = 1,2,...,m.$$

Therefore,

$$B^{(t)}_{a,k}(z) = \frac{k!}{m!}\frac{d^{m-k}y}{dz^{m-k}}, \quad k = 0,1,...,m.$$

Substituting this in (4.1.7) and multiplying both sides by $\frac{1}{2at}$, we get

$$\frac{m}{2at}y - \frac{z}{2at}\frac{dy}{dz} + \frac{1}{a}\frac{d^2y}{dz^2} + \sum_{k=0}^{m-1}\frac{B_{a,m-k}}{(m-k)!}\frac{d^{m-k}y}{dz^{m-k}} = 0.$$

Therefore,

$$\frac{B_{a,m}}{m!}\frac{d^m y}{dz^m}+\frac{B_{a,m-1}}{(m-1)!}\frac{d^{m-1}y}{dz^{m-1}}+\cdots+\frac{B_{a,2}}{2!}\frac{d^2y}{dz^2}+B_{a,1}\frac{dy}{dz}+\frac{m}{2at}y-\frac{z}{2at}\frac{dy}{dz}+\frac{1}{a}\frac{d^2y}{dz^2}=0.$$

Hence the theorem follows. □

By taking $t = 1$ in the above theorem, we can get the differential equation of the fractional Bernoulli polynomials.

Corollary 4.10. *For $t = 1$ and $y = B_{a,m}(z)$, we have*

$$\frac{B_{a,m}}{m!}\frac{d^m y}{dz^m}+\frac{B_{a,m-1}}{(m-1)!}\frac{d^{m-1}y}{dz^{m-1}}+\cdots+\left(\frac{B_{a,2}}{2!}+\frac{1}{a}\right)\frac{d^2y}{dz^2}-\frac{z}{2a}\frac{dy}{dz}+\frac{m}{2a}y=0,$$

which is the differential equation of the fractional Bernoulli polynomials.

4.2 The Connection of Higher Order Fractional Bernoulli Polynomials with the Identity $\varphi_{m,k}$

For the positive integer N C. R. Ernst and A. Hassen in [9] defined $\varphi_{m,k}$ by

$$\varphi_{m,k}(N)=\frac{k!}{N^{m+1-k}}\sum_{j=0}^{m}\sum_{i=0}^{N-1}\binom{m}{j}\omega_N^{N-i(k-j)},$$

where $\omega_N = e^{\frac{2\pi i}{N}}$ is the N^{th} root of unity.

The authors connect this function to the hyperbolic lacunary Bernoulli polynomials $B_m(N,k,z)$ by

$$\sum_{j=0}^{m}\binom{m+k}{j}B_j(N,k,z)\varphi_{m+k-j,k}(N)=\binom{m+k}{k}k!z^m,$$

where the *hyperbolic lacunary Bernoulli polynomials* is given by

$$\frac{x^k e^{zx-x/N}}{N^{k-1}k!\sum_{m=0}^{N-1}\omega_N^{N-m}e^{\frac{\omega_N^m x}{N}}}=\sum_{m=0}^{\infty}B_m(N,k,z)\frac{x^m}{m!}.$$

As we have seen in the first and second chapter, the hyperbolic lacunary Bernoulli polynomials also satisfies the addition and recurrence properties (see [9] for the details).

Addition formula:

$$B_m(N,k,z_1+z_2)=\sum_{j=0}^{m}\binom{m}{j}B_j(N,k,z_1)z_2^{m-j},$$

Recurrence formula:

$$B_m(N,k,z)=\sum_{j=0}^{m}\binom{m}{j}B_j(N,k)z^{m-j},$$

where $B_m(N,k)=B_m(N,k,0)$ is the hyperbolic lacunary Bernoulli numbers, and for N even integer and $j>0$,

$$B_{Nj+k}(N,k)=0.$$

In particular, if $N=2,\ k=1$, then

$$B_{2j+1}=0.$$

In this section, we redefine the identity $\varphi_{m,k}(a)$ for positive nonzero fractions $a=\frac{\alpha}{\beta},\ \ \beta\neq 0$ and establish the relationship between the identity $\varphi_{m,k}$ and higher order fractional Bernoulli polynomials $B^{(t)}_{a,m}(z)$.

Definition 4.11. *For a positive nonzero fraction* $a=\frac{\alpha}{\beta},\ \ \beta\neq 0$ *and* $\omega_a=e^{\frac{2\pi i}{a}}$. *We define* $\varphi_{m,k}$ *by*

$$\varphi_{m,k}(a)=\frac{k!}{a^{m+1-k}}\sum_{j=0}^{m}\sum_{i=0}^{\alpha-1}\binom{m}{j}\omega_a^{-i(k-j)},$$

where m *and* k *are positive integers.*

Lemma 4.12. *We have*

$$\varphi_{m,k}(a)=\begin{cases}\frac{\beta k!}{a^{m-k}}\binom{m}{k}, & \text{if } m\geq k\\[2ex] 0, & \text{if } m<k.\end{cases}$$

Proof. Let $m<k$. Now since the sum of powers of complex numbers of modulus one is zero, then

$$\sum_{s=0}^{\alpha-1}\omega_a^{-s(k-j)}=0, \tag{4.2.1}$$

whenever $-s(k-j)\not\equiv 0(mod\ a)$. Therefore, $\varphi_{m,k}(a)=0$ because $j\leq m<k$. If $m\geq k$, by equation (4.2.1) all terms are zero except for $j=k$. Then we have,

$$\varphi_{m,k}(a)=\frac{\alpha k!}{a^{m+1-k}}\binom{m}{k}=\frac{\beta k!}{a^{m-k}}\binom{m}{k}.$$

For $m = k$,

$$\varphi_{k,k}(a) = \beta k!.$$

$\square$

Theorem 4.13. *We have*

$$\sum_{j=0}^{\alpha-1} \omega_a^{-jk} B_{a,m}^{(t)} \left(\frac{\omega_a^j + 1}{a} \right) = \begin{cases} \beta a^{1-k} \binom{m}{k} B_{a,m-k}^{(t)} \left(\frac{1}{a} \right), & \text{if } m \geq k \\ 0, & \text{if } m < k. \end{cases}$$

Proof. Using equation 4.1.2,

$$\begin{aligned} \sum_{j=0}^{\alpha-1} \omega_a^{-jk} B_{a,m}^{(t)} \left(\frac{\omega_a^j + 1}{a} \right) &= \sum_{j=0}^{\alpha-1} \omega_a^{-jk} \left[\sum_{i=0}^{m} \binom{m}{i} B_{a,i}^{(t)} \left(\frac{\omega_a^j + 1}{a} \right)^{m-i} \right] \\ &= \sum_{i=0}^{m} \binom{m}{i} B_{a,i}^{(t)} \frac{1}{a^{m-i}} \sum_{h=0}^{m-i} \sum_{j=0}^{\alpha-1} \binom{m-i}{h} \omega_a^{-j(k-h)} \\ &= \sum_{i=0}^{m} \binom{m}{i} B_{a,i}^{(t)} \frac{a^{1-k}}{k!} \frac{k!}{a^{m-i+1-k}} \sum_{h=0}^{m-i} \sum_{j=0}^{\alpha-1} \binom{m-i}{h} \omega_a^{-j(k-h)} \\ &= \frac{a^{1-k}}{k!} \sum_{i=0}^{m} \binom{m}{i} B_{a,i}^{(t)} \, \varphi_{m-i,k}(a). \end{aligned}$$

If $m < k$, by the above lemma $\varphi_{m-i,k}(a) = 0$ for all $i = 0, 1, ..., m$. Then the sum is equal to zero.
If $m \geq k$, since $m - i \geq k \ \ \forall i = 0, 1, ..., m - k$, then again using 4.1.2, we obtain

$$\begin{aligned} \sum_{j=0}^{\alpha-1} \omega_a^{-jk} B_{a,m}^{(t)} \left(\frac{\omega_a^j + 1}{a} \right) &= \frac{a^{1-k}}{k!} \sum_{i=0}^{m-k} \binom{m}{i} B_{a,i}^{(t)} \, \varphi_{m-i,k}(a) \\ &= \frac{a^{1-k}}{k!} \sum_{i=0}^{m-k} \binom{m}{i} B_{a,i}^{(t)} \, \frac{\beta k!}{a^{m-i-k}} \binom{m-i}{k} \\ &= \beta a^{1-k} \sum_{i=0}^{m-k} \binom{m-k}{i} \binom{m}{k} B_{a,i}^{(t)} \, (1/a)^{m-k-i} \\ &= \beta a^{1-k} \binom{m}{k} B_{a,m-k}^{(t)}(1/a). \end{aligned}$$

$\square$

Corollary 4.14. *We have*

$$\sum_{j=0}^{\alpha-1} \omega_a^{-jk} B_{a,m}\left(\frac{\omega_a^j+1}{a}\right) = \begin{cases} \beta a^{1-k}\binom{m}{k} B_{a,m-k}\left(\frac{1}{a}\right), & \textit{if } \ m \geq k \\ \\ 0, & \textit{if } \ m < k. \end{cases}$$

Proof. Put $t = 1$ in the above theorem. □

Theorem 4.15. *We have*

$$\sum_{j=0}^{\alpha-1} \omega_a^{-jk} \mathsf{B}_{a,m}^{(t)}\left(z + \frac{\omega_a^j+1}{a}\right) = \begin{cases} \beta a^{1-k}\binom{m}{k} B_{a,m-k}^{(t)}(z+\frac{1}{a}), & \textit{if } \ m \geq k \\ \\ 0, & \textit{if } \ m < k. \end{cases}$$

Proof. By equation 4.1.3,

$$\begin{aligned} \sum_{j=0}^{\alpha-1} \omega_a^{-jk} \mathsf{B}_{a,m}^{(t)}\left(z + \frac{\omega_a^j+1}{a}\right) &= \sum_{j=0}^{\alpha-1} \omega_a^{-jk} \sum_{i=0}^{m} \binom{m}{i} B_{a,i}^{(t)}\left(\frac{\omega_a^j+1}{a}\right) z^{m-i} \\ &= \sum_{i=0}^{m} \binom{m}{i} z^{m-i} \sum_{j=0}^{\alpha-1} \omega_a^{-jk} B_{a,i}^{(t)}\left(\frac{\omega_a^j+1}{a}\right). \end{aligned}$$

If $m < k$, then by Theorem 4.13 the sum becomes zero.
If $m \geq k$, then we obtain

$$\begin{aligned} \sum_{j=0}^{\alpha-1} \omega_a^{-jk} \mathsf{B}_{a,m}^{(t)}\left(z + \frac{\omega_a^j+1}{a}\right) &= \beta a^{1-k} \sum_{i=0}^{m} \binom{m}{i}\binom{i}{k} B_{a,i-k}^{(t)}(1/a) z^{m-i} \\ &= \beta a^{1-k} \sum_{i=k}^{m} \binom{m}{k}\binom{m-k}{i-k} B_{a,i-k}^{(t)}(1/a) z^{m-i} \\ &= \beta a^{1-k} \binom{m}{k} \sum_{i=0}^{m-k} \binom{m-k}{i} B_{a,i}^{(t)}(1/a) z^{m-k-i} \\ &= \beta a^{1-k} \binom{m}{k} B_{a,m-k}^{(t)}(z + \frac{1}{a}). \end{aligned}$$

Hence the theorem follows. □

Put $t = 1$ in Theorem 4.15 to obtain the following corollary.

Corollary 4.16. *We have*

$$\sum_{j=0}^{\alpha-1} \omega_a^{-jk} \mathsf{B}_{a,m}\left(z + \frac{\omega_a^j + 1}{a}\right) = \begin{cases} \beta a^{1-k} \dbinom{m}{k} B_{a,m-k}(z + \frac{1}{a}), & \textit{if} \;\; m \geq k \\ 0, & \textit{if} \;\; m < k. \end{cases}$$

Next we give one of our results regarding the connection of higher order fractional Bernoulli polynomials and the identity $\varphi_{m,k}$.

Theorem 4.17. *We have*

$$\sum_{j=0}^{m} \binom{m+k}{j} B_{a,j}^{(t)}(z) \varphi_{m+k-j,k}(a) = \frac{\beta(m+k)!}{m!} B_{a,m}^{(t)}\left(z + \frac{1}{a}\right).$$

Proof. By Theorem 4.15 and using the definition of $\varphi_{m,k}(a)$ and also since $m \geq 0$, we have

$$\begin{aligned}
&\sum_{j=0}^{m} \binom{m+k}{j} B_{a,j}^{(t)}(z) \varphi_{m+k-j,k}(a) \\
&\quad = \sum_{j=0}^{m+k} \binom{m+k}{j} B_{a,j}^{(t)}(z) \frac{k!}{a^{m+k-j+1-k}} \sum_{i=0}^{m+k-j} \sum_{h=0}^{\alpha-1} \binom{m+k-j}{i} \omega_a^{-h(k-i)} \\
&\quad = k! a^{k-1} \sum_{h=0}^{\alpha-1} \omega_a^{-hk} \sum_{j=0}^{m+k} \binom{m+k}{j} B_{a,j}^{(t)}(z) \left(\frac{\omega_a^h + 1}{a}\right)^{m+k-j} \\
&\quad = k! a^{k-1} \sum_{h=0}^{\alpha-1} \omega_a^{-hk} B_{a,m+k}^{(t)}\left(z + \frac{\omega_a^h + 1}{a}\right) \\
&\quad = \beta k! \binom{m+k}{k} B_{a,m}^{(t)}\left(z + \frac{1}{a}\right) \\
&\quad = \frac{\beta(m+k)!}{m!} B_{a,m}^{(t)}\left(z + \frac{1}{a}\right).
\end{aligned}$$

□

Corollary 4.18. *We have*

$$B_{a,m}^{(t)}\left(z + \frac{1}{a}\right) - B_{a,m}^{(t)}(z) = \frac{m!}{\beta(m+k)!} \sum_{j=0}^{m-1} \binom{m+k}{j} B_{a,j}^{(t)}(z) \varphi_{m+k-j,k}(a).$$

Proof. From the above theorem, we obtain

$$\binom{m+k}{m} B^{(t)}_{a,m}(z)\varphi_{k,k}(a)+\sum_{j=0}^{m-1}\binom{m+k}{j} B^{(t)}_{a,j}(z)\varphi_{m+k-j,k}(a)=\frac{\beta(m+k)!}{m!}B^{(t)}_{a,m}\left(z+\frac{1}{a}\right).$$

Therefore,

$$B^{(t)}_{a,m}\left(z+\frac{1}{a}\right)-B^{(t)}_{a,m}(z)=\frac{m!}{\beta(m+k)!}\sum_{j=0}^{m-1}\binom{m+k}{j} B^{(t)}_{a,j}(z)\varphi_{m+k-j,k}(a).$$

□

Corollary 4.19. *We have*

$$B^{(t)}_{a,m}\left(\frac{1}{a}\right)=\frac{m!}{\beta(m+k)!}\sum_{j=0}^{m}\binom{m+k}{j} B^{(t)}_{a,j}\ \varphi_{m+k-j,k}(a).$$

Proof. Put $z = 0$ in Corollary 4.18 to obtain the required result. □

The following corollary describe that there is a relationship between fractional Bernoulli polynomials and the identity $\varphi_{m,k}$.

Corollary 4.20. *We have*

$$\sum_{j=0}^{m}\binom{m+k}{j} B_{a,j}(z)\varphi_{m+k-j,k}(a)=\frac{\beta(m+k)!}{m!}B_{a,m}\left(z+\frac{1}{a}\right).$$

Proof. Setting $t = 1$ in Theorem 4.17 yields the corollary. □

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