COMPARISON OF SEVERAL ESTIMATORS FOR THE COMMON MEAN OF DIFFERENT NORMAL POPULATIONS: A MONTE CARLO APPROACH.

by

Gebre-Egziabher Kiros

A THESIS

SUBMITTED IN (PART)

FULFILLMENT FOR THE [illegible]E OF

MASTER OF SCIENCE IN STATISTIC[illegible] THE

ADDIS ABABA UNIVERSITY

June, 1990

Addis Ababa

ADDIS ABABA UNIVERSITY

SCHOOL OF GRADUATE STUDIES

TITLE OF RESEARCH Comparison of Several Estimators for the Common Mean of Different Normal Populations: A Monte Carlo Approach.

NAME OF CANDIDATE Gebre-Egziabher Kiros

DEPARTMENT Statistics

FACULTY Science

APPROVED BY

	Advisor & Chairman	External Examiner	Internal Examiner
Name	Dr. Abebe Tessera	Dr.F.J.Sichona	Dr. I. Litvin
Sign.			

Date ____________

TO MY FATHER

ABSTRACT

In this paper several estimators of the common unknown mean of possibly different normal populations when the variances are unknown are compared through an extensive MonteCarlo study. When the σ_i^2s are equal, it is observed that the unweighted mean estimator is optimal. However, when varying the σ_i^2s, its efficiency is substantially less than the other estimators. For the range of the parameters considered in this study, it is found that the weighted mean, the maximum likelihood, the Neyman-Scott and the Kalbfleish estimators have high efficiency. Comparing their precision with the computer time they require to estimate μ, the weighted mean estimator is recommended except in some cases.

ACKNOWLEDGEMENTS

I would like to express my gratitude and record my indebtedness to my advisor Dr. Abebe Tessera, Associate Professor and chairman of the department of statistics, for suggesting this problem to me and for his effort as a research advisor and for his valuable guidance. I would also like to thank my institute, National Urban Planning Institute (N.U.P.I.) for allowing me to use all computer facilities.

I am greatful to the many people who assisted me in this work. It is a pleasure for me to express my gratitude to Ato Gebrenegus Ghilagaber (Lecturer in Asmara University) who is on study leave for his Ph.D. in the University of Uppsala, Sweden for the very useful reprints he sent me. My sincere gratitudes go to my colleagues in the N.U.P.I. computing center for their day to day assistance. Finaly, I am indepted to my friends Ato Abebaw Taddessie and Ato Abebe Kebede for their help and encouragment throughout the paper work.

Table of Contents

CHAPTER 1

INTRODUCTION

In a variety of agricultural, biological, medical, etc. situations, the problem of combining several estimates into one optimal estimate often arises. This paper is concerned with the comparison of different methods of such combinations.

Suppose we have $k(\geq 3)$ normal populations having a common unknown mean μ and unknown variances σ_i^2 possibly unequal and consider the problem of estimating the common mean on the basis of independent samples from these populations. For estimating the common unknown mean of several normal populations with known variances, the best estimator is the weighted estimator with weights proportional to the inverse of their population variances. But when the variances σ_i^2's are unknown, there are several competing estimators. Some of these are: the unweighted mean, the weighted mean, the maximum likelihood, the Neyman-Scott and the Kalbfleish-Sprott.

The problem has been studied by many researchers. Among them are Neyman & Scott(1948), Levy(1970), Kalbfleish & Sprott(1970), J.N.K. Rao & Subrahmaniam(1971), J.N.K. Rao(1973,1980), Levy & Mantel(1974), Norwood & Hinkelmann(1977), Shinazoki(1978) and Bhattacharya(1979). The details of their contributions are discussed in the next chapter.

All the studies done so far mainly focus on the asymptotic behavior or small sample property of the estimators. However, in real life we usually face situations in which the n_i's are not so large ($20 \le n_i \le 30$) and k is small ($3 \le k \le 10$). In this study emphasis is given to the comparison of these estimators for the above values of k and n_i.

All the estimators compared in the study are unbiased estimators of μ. The comparison of the estimators is made through the frequency distribution of the estimates, their empirical variances and their efficiency relative to the optimal estimator(when the variances are known). For the study, an extensive simulation is performed on a powerful computer, **HP-UX** 9000/500.

In chapter 2, the theoretical basis of the estimators and contributions of previous researchers to this problem is given. Chapter 3 is devoted to the simulation procedures used. In chapter 4, the results of the Monte Carlo experiment are analysed. These results are presented graphically and in table form. A summary of the study and its relationships with previous studies are discussed in chapter 5. The computer program written to obtain the simulation results is given in Appendix I. The frequency distributions of the estimates under different circumstances are tabulated in Appendix II.

CHAPTER 2

LITERATURE REVIEW

2.1 Introduction

Suppose that independent samples are observed from k (≥ 3) normal populations with a common unknown mean μ but possibly different unknown variances $\sigma_1^2, \sigma_2^2, \ldots, \sigma_k^2$. Let x_{ij} denote the j^{th} observation in the i^{th} sample; $j=1,2,\ldots,n_i$, $i=1,2,\ldots,k$, where $n_i (\geq 1)$ is the number of observations from $N(\mu, \sigma_i^2)$.

Let $\bar{x}_i = \frac{1}{n_i} \sum_{j=1}^{n_i} x_{ij}$ $\quad i=1,2,\ldots,k.$

Clearly $\bar{x}_i$ is distributed as $N\left(\mu, \frac{\sigma_i^2}{n_i}\right)$ and $\bar{x}_1, \bar{x}_2, \ldots, \bar{x}_k$ are independent and unbiased estimators of μ. Moreover, in samples from a normal distribution, the sample mean is the best estimator of the population mean.

The problem of combining the $\bar{x}_i$'s to a single estimator of μ with best precision often arises in many areas of statistical practice. For instance data of this type may occur when k laboratories made separate determinations $\bar{x}_i$ of the same physical or chemical quantity, each with an estimated standard error. Because of personal biases or local conditions of experimentation these estimates of μ may not be the same and it is desired to average their results in the best manner possible.

This problem has been treated at length in the statistical literature and several researchers have studied it and various methods of combining the $\bar{x}_i$'s to one estimator are developed. Out of the several developed methods, a choice of a method which gives the best precision in a given condition is necessary.

Neyman and Scott (1948) proposed an estimator of μ ($\hat{\mu}_{KS}$), which is asymptotically more efficient than the maximum likelihood estimator ($\hat{\mu}_{ML}$) when the n_i's differ , and the n_i's and σ_i^2 's are uniformly bounded as the number of estimators to be combined (k) approaches infinity. Cochran and Carroll (1953), Meier (1953) and others found the weighted mean ($\hat{\mu}_{WT}$)with weights inversely proportional to their sample variances to be more efficient than the unweighted mean ($\hat{\mu}_{UW}$) except in the case of small n_i (≤ 5) or small heterogeneity in the σ_i^2.

By considering equal σ_i^2 and $1 \le n_i \le 10$, Levy (1970) investigated by simulation that $\hat{\mu}_{ML}$ has higher precision than $\hat{\mu}_W$. Kalbfleish and Sprott (1970) derived the conditional likelihood function of μ, and obtained an estimator ($\hat{\mu}_{KS}$) which maximizes the conditional likelihood function.

C.R.Rao(1970) developed a new method of estimating heteroscedastic variances for a general linear model called MINQUE (MInimum Norm Quadratic Unbiased Estimation). J.N.K.Rao and Subrahmanian (1971), and J.N.K.Rao(1973,1980) applied this principle to combine the $\bar{x}_i$'s and demonstrated through Monte Carlo study that this method with some modifications is superior to some other estimators when n_i is small (≤ 4), k is relatively large and the heterogeneity in the σ_i^2 is small to moderate. Another important result which was shown is that MINQUE may not lead to substantial gains in efficiency when $n_i \geq 8$ and the heterogeneity in the σ_i^2 increases.

Levy and Mantel (1974) studied the efficiencies of some of the commonly used estimators of μ by means of an empirical study. Their result suggests that for moderate diversity in the σ_i^2, $\hat{\mu}_{UW}$ could be superior to either $\hat{\mu}_{WT}$ or $\hat{\mu}_{ML}$. Their sampling design was restricted to the case k=6, $1 \leq n_i \leq 11$ and σ_i^2's increase in geometric progression in such a way that the average of the σ_i^2's is one.

Norwood and Hinkelmann (1977) gave a necessary and sufficient condition in which the weighted mean has uniformly smaller variance than any of the $\bar{x}_i$'s. Shinozaki (1978) suggested a class of unbiased estimators of μ and also offered a necessary and sufficient condition for the estimator to have smaller variance than every sample mean. Bhattacharya (1979) offered a simplification of the proof given earlier by Shinozaki.

However, not much work is done so far on the more general problem of selecting among different methods of estimating the common mean, especially when k is small ($3 \leq k \leq 10$), and the n_i's are not large ($20 \leq n_i \leq 30$) for every i=1,2,...,k.

Avoiding more complex procedures of estimating μ such as the Bayesian method which requires apriori information the following estimating methods are considered :

i) The Unweighted Mean

ii) The Weighted Mean

iii) The Maximum Likelihood Estimator

iv) The Neyman-Scott Estimator

v) The Kalbfleish-Sprott Estimator.

2.2 The Estimators Considered

2.2.1 The Unweighted Mean

The unweighted mean estimator of μ is given by

$$\hat{\mu}_{UW} = \frac{\sum_{i=1}^{k} n_i \bar{x}_i}{n} \tag{1}$$

where $n = \sum_{i=1}^{k} n_i$.

This estimator is the simplest of all other alternate estimators of μ. If all n_i's are equal,

$$\hat{\mu}_{UW} = \frac{\sum_{i=1}^{k} \bar{x}_i}{k}$$

which is the arithmetic mean of k objects. This estimator is used as an initial approximation to those methods which require the Newton-Raphson iteration

2.2.2 The Weighted Mean

When the σ_i^2's are known, the best linear unbiased estimator of μ which also attains the Cramér Rao lower bound is

$$\hat{\mu}_{BL} = \frac{\sum_{i=1}^{k} n_i \sigma_i^{-2} \bar{x}_i}{\sum_{i=1}^{k} n_i \sigma_i^{-2}} \tag{2}$$

If the σ_i^2's are unknown, the well-known unbiased estimator of μ is

$$\hat{\mu}_{WT} = \frac{\sum_{i=1}^{k} n_i S_i^{-2} \bar{x}_i}{\sum_{i=1}^{k} n_i S_i^{-2}} \tag{3}$$

where

$$S_i^2 = \frac{\sum_{j=1}^{n_i} (x_{ij} - \bar{x}_i)^2}{n_i - 1} \qquad i=1,2,\ldots,k.$$

are the sample variances which are unbiased estimators of σ_i^2's. Eq(3) is referred to as the weighted mean estimator of μ, with weights proportional to the inverse of the sample variances.

2.2.4 The maximum Likelihood Estimator

The probability density function of X_{ij} is given by

$$f_X(x_{ij};\mu,\sigma_i^2) = \frac{1}{\sqrt{2\pi}} \exp\left\{-\frac{1}{2\sigma_i^2}(x_{ij}-\mu)^2\right\}$$

where $-\infty < \mu < \infty$ and $\sigma_i^2 > 0$ for every $i=1,2,\ldots,k$; $j=1,2,\ldots n_i$.

The likelihood function of X_{ij}'s is

$$L = \prod_{i,j}^{k} f_{X_{ij}}(x_{ij};\mu,\sigma_i^2)$$

$$= \prod_{i=1}^{k} (2\pi\sigma_i^2)^{-\frac{n_i}{2}} \exp\left\{-\frac{1}{2\sigma^2}\sum_{j=1}^{k}(x_{ij}-\mu)^2\right\}$$

$$= (2\pi)^{\left[-\frac{1}{2}\sum_{i=1}^{k} n_i\right]} \prod_{i=1}^{k} (\sigma_i^2)^{-\frac{n_i}{2}} \exp\left\{-\frac{1}{2}\sum_{i=1}^{k}\sum_{j=1}^{n_i}\frac{(x_{ij}-\mu)^2}{\sigma_i^2}\right\}$$

The log-likelihood function of L is

$$l=\ln L = -\frac{1}{2}(\ln 2\pi)\sum_{i=1}^{k} n_i - \frac{1}{2}\sum_{i=1}^{k} n_i(\ln \sigma_i^2) - \frac{1}{2}\sum_{i=1}^{k}\sum_{j=1}^{n_i}\frac{(x_{ij}-\mu)^2}{\sigma_i^2}$$

The most convenient way to obtain the maximum likelihood estimator is to examine all the local maxima of l. Differentiating l with respect to μ and σ_i^2, and equating the results to zero we get the following:

$$\frac{\partial l}{\partial \mu} = \sum_{i=1}^{k} \sum_{j=1}^{k} \frac{(x_{ij}-\mu)^2}{\sigma_i^2} = \sum_{i=1}^{k} \frac{n_i(\bar{x}_i-\mu_{ML})}{\sigma_i^2} = 0 \qquad (4)$$

$$\frac{\partial l}{\partial \sigma_i^2} = -\frac{n_i}{2\sigma_i^2} + \frac{1}{2\sigma_i^4} \sum_{j=1}^{n_i} (x_{ij}-\hat{\mu}_{ML})^2 = 0 \ , \ i=1,2,\ldots,k.$$

which implies that

$$\sigma_i^2 = \frac{1}{n_i} \sum_{j=1}^{n_i} (x_{ij} - \hat{\mu}_{ML})^2 \ , \ i=1,2,\ldots,k.$$

But $\sum_{j=1}^{n_i} (x_{ij} - \hat{\mu}_{ML})^2$ can be written as

$$\sum_{j=1}^{n_i} \left[(x_{ij} - \bar{x}_i) + (\bar{x}_i - \hat{\mu}_{ML}) \right]^2 = \sum_{j=1}^{n_i} (x_{ij} - \bar{x}_i)^2 + n_i(\bar{x}_i-\hat{\mu}_{ML})^2$$

$$= (n_i-1)S_i^2 + n_i(\bar{x}_i - \hat{\mu}_{ML})^2.$$

Therefore

$$\sigma_i^2 = \frac{1}{n_i} \left[(n_i-1)S_i^2 + n_i(\bar{x}_i-\hat{\mu}_{ML})^2 \right] \qquad (5)$$

Substituting eq(5) into eq(4) one obtains an estimating equation

$$\sum_{i=1}^{k} \frac{n_i \left[n_i(\bar{x}_i-\hat{\mu}_{ML}) \right]}{(n_i-1)S_i^2 + n_i(\bar{x}_i-\hat{\mu}_{ML})^2} = 0 \qquad (6)$$

The only unknown parameter in eq(6) is μ and this equation can be considered as a non-linear equation in μ. Denoting the maximum

$$\Delta = \frac{\displaystyle\sum_{i=1}^{k} \frac{n_i^2(\bar{x}_i-\mu^t)}{(n_i-1)S_i^2 + n_i(\bar{x}_i-\mu^t)^2}}{\displaystyle\sum_{i=1}^{k} \frac{n_i^2\left[(n_i-1)S_i^2 - n_i(\bar{x}_i-\mu^t)^2\right]}{\left[(n_i-1)S_i^2 + n_i(\bar{x}_i-\mu)^2\right]^2}} \qquad (9)$$

From the above equation we can rewrite eq(8) as

$$\mu^{t+1} = \mu^t - \Delta(\mu^t) \qquad (10)$$

2.7.4 The Neyman - Scott Estimator

Neyman and Scott (1948) considered the more general form of eq(8). After substituting an arbitrary weight w_i instead of n_i they studied the more general estimating equation

$$\sum_{i=1}^{k} \frac{w_i\left[n_i(\bar{x}_i-\mu)\right]}{(n_i-1)S_i^2 + n_i(\bar{x}_i-\mu)^2} = 0 \qquad (11)$$

They showed that, as k approaches infinity, the estimator of μ obtained from eq(11) has a great precision if w_i is chosen to be n_i-2. Replacing w_i by n_i-2, the Neyman-Scott estimator is obtained as a solution of

$$\sum_{i=1}^{k} \frac{(n_i-2)\left[n_i(\bar{x}_i-\mu)\right]}{(n_i-1)S_i^2 + n_i(\bar{x}_i-\mu)^2} = 0 \qquad (12)$$

The value of μ obtained from eq(12) is called the Neyman-Scott estimator and is denoted by $\hat{\mu}_{NS}$.

They also showed that the asymptotic variance of $\hat{\mu}_{NS}$ is always less than that of $\hat{\mu}_{ML}$, except in the case when equal observations are taken from each sample. $\hat{\mu}_{NS}$ can also be obtained with the same procedure as $\hat{\mu}_{NS}$. As it can be seen easily from eq(12), this estimating equation has the peculiar property that samples of size 2 contribute nothing, which intuitively seems incorrect.

2.2.5 The Kalbfleish - Sprott Estimator

For every fixed μ , the statistics

$$T_i^2 = \sum_{i=1}^{k} (x_{ij} - \mu)^2, \quad i=1,2,\ldots,k$$

are jointly sufficient for the nuisance parameters σ_i^2. Noting this fact , Kalbfleish and Sprott (1970), derived the conditional distribution of the observations given T_i^2 which is independent of σ_i^2 and is proportional to

$$\prod_{i=1}^{k} T_i^{-(n_i - 2)} \prod_{i,j} dx_{ij} \Big/ \prod_{i=1}^{k} dT_i^2 . \tag{13}$$

Let J be the $k \times n$ Jacobian matrix, where $n = \sum_{i=1}^{k} n_i$.

The entries of the matrix J are the partial derivatives

$$\frac{\partial T_i}{\partial x_{mj}} = \begin{cases} 2(x_{mj}-\mu) & \text{if } m = i,\ j=1,\ldots,n_i \\ 0 & \text{otherwise} \end{cases}$$

The conditional likelihood is therefore proportional to

$$C(\mu) = f(x_{11},\ldots,x_{kn} \mid T_1^2,\ldots,T_k^2) \Big/ \left| JJ' \right|^{1/2}.$$

In this case $\left| JJ' \right| = 4^k \prod_{i=1}^{k} T_i^2$.

$$\therefore \quad C(\mu) = \prod_{i=1}^{k} T_i^{-(n_i-2)} \Big/ \left| JJ' \right|^{1/2}$$

$$C(\mu) = 2^{-k} \prod_{i=1}^{k} \left\{ \sum_{j=1}^{n_i} (x_{ij}-\mu)^2 \right\}^{-(n_i-1)/2} \tag{14}$$

Taking the logarithm of $C(\mu)$, differentiating with respect to μ and equating to zero, one can arrive at the equation

$$\sum_{i=1}^{k} \frac{(n_i-1)\left[n_i(\bar{x}_i-\mu)\right]}{(n_i-1)S_i^2 + n_i(\bar{x}_i-\mu)^2} = 0 \qquad (15)$$

Here again, the solution of eq(15) may be found by the Newton-Raphson method. The approximation to μ obtained from the above equation is the Kalbfleish - Sprott estimator ($\hat{\mu}_{KS}$). Comparing equations (12) and (15), one observes that the Kalbfleish and Sprott estimator is to be preferred to the Neyman and Scott estimator, because it takes into account samples of size 2. But $\hat{\mu}_{KS}$ does not take into account samples of size 1. The maximum likelihood estimator is free of these limitations.

The next chapter is devoted on how the data was obtained and the procedures followed to compare the five estimating methods explained in this chapter. The sampling design of the experiment, the computer program which is used to compare the precision of the estimators and all the necessary steps which are relevant for the study are clearly discussed.

CHAPTER 3

SIMULATION PROCEDURES

3.1 The Monte Carlo Method

The Monte Carlo method is a numerical method of solving mathematical problems by means of random sampling. It consists of solving various problems of computational mathematics by means of the construction of some random process for each such problem, with the parameters of the process equal to the required quantities of the problem. The method may briefly be described as the device of studying artificial stochastic model of a physical or mathematical process. For a model to be useful it is essential that, given a reasonably limited set of descriptors, all its relevant behavior and properties can be determined by deriving the model with certain random inputs and observing the corresponding outputs. The Monte Carlo method can be used to access the behavior of models, and also it can be used to gauge the performance of various techniques. Here, the method is applied to compare several estimators of the common mean of different normal populations.

The following are some possible reasons for the widespread popularity of the method :

1) Most complex, real-world systems with stochastic elements cannot be accurately described by a mathematical model which can be evaluated analytically. Thus, simulation is often the only type of investigation possible.

2) Monte Carlo allows one to estimate the performance of an existing system under some projected set of operating conditions.

3) Alternative proposed system designs can be compared via simulation to see which best meets a specified requirment.

4) In a Monte Carlo we can maintain much better control over experimental conditions than would generally be possible when experimenting with the system itself.

Monte Carlo method is not without its drawbacks. In particular, the following are some of its disadvantages:

1) Monte Carlo models are often expensive and time-consuming to develop.

2) On each run a stochastic simulation model produces only estimates of a model's true characterstics for a particular set of input parameters. Thus, several independent runs of the model will be required for each set of input parameters to be studied.

3.2 Computer Generation of Random Variables

3.2.1 Generating Uniform Random Variables

The great majority of random-number generators in use today are linear congruential generators. The i^{th} integer Z_i in the pseudo-random sequence is computed from Z_{i-1} by the recursion

$$Z_i = (a Z_{i-1} + c) \bmod m , \quad i = 1, 2, \ldots$$

where m (the modulus), a (the multiplier), c (the increment), and Z_0 (the seed or starting value) are non-negative integers such that $m > 0$, $a < m$, $c < m$, and $Z_0 < m$. The desired uniform random numbers on (0,1) are obtained by letting

$$U_i = Z_i/m \quad i = 1, 2, \ldots$$

The parameters a, c, m, and Z_0 determine the statistical quality of the generator. By careful choice of these four parameters the U_i's can be made independently and identically distributed (i.i.d.) random variables over the interval (0,1).

The data sets used in the Monte Carlo study are generated using **DRAND48** , a subroutine available on HP 9000/500 HP-UX. The function **DRAND48** generates pseudo-random numbers using the above linear congruential algorithm and a 48-bit integer arthimetic. It returns non-negative, double precision, floating point values uniformly distributed over the interval (0,1).

3.2.2 Normal Random Number Generation

There are several ways of generating normal random variables. Some of the methods are exact and some are approximate methods. The generating techinique used in this study is the Marsaglia (Polar) method, which is an improvement of the Box and Muller method. The Marsaglia method is chosen because of its simplicity for programming, being fast and exact. The Box and Muller, and Marsaglia (Polar) methods are described here under.

i) The Box and Muller (Sine-Cosine) Method

The Box and Muller method is used to obtain a pair of independently and identiacally distributed (i.i.d.) exact standard normal random variables by means of one-to-one transformation of two i.i.d. Uniform(0,1) random variables.

Let U_1 and U_2 be independent Uniform(0,1) random variables.

Define the transformations

$$X_1 = \sqrt{-2 \ln U_1} \quad \cos 2\pi U_2$$

$$X_2 = \sqrt{-2 \ln U_1} \quad \sin 2\pi U_2 \tag{1}$$

Squaring both sides of the above equations we get

$$X_1^2 = -2 \ln U_1 \cos^2 2\pi U_2$$

$$X_2^2 = -2 \ln U_1 \sin^2 2\pi U_2 \tag{2}$$

Adding these two expresions of (2), we have

$$X_1^2 + X_2^2 = -2 \ln U_1$$

which implies that

$$U_1 = \exp\left\{ - \frac{1}{2} (X_1^2 + X_2^2) \right\}$$

Also

$$\frac{X_1}{X_2} = \frac{\sin 2\pi U_2}{\cos 2\pi U_2} = \tan 2\pi U_2$$

which gives

$$U_2 = \frac{1}{2\pi} \arctan (X_2/X_1) \quad .$$

The Jacoboan of transformation is

$$J = \frac{\partial(u_1, u_2)}{\partial(x_1, x_2)} = \begin{vmatrix} \frac{\partial u_1}{\partial x_1} & \frac{\partial u_1}{\partial x_2} \\ \frac{\partial u_2}{\partial x_1} & \frac{\partial u_2}{\partial x_2} \end{vmatrix}$$

$$= -\frac{1}{2\pi} \exp \left\{ -\frac{1}{2}(x_1^2 + x_2^2) \right\}$$

The absolute value of J is

$$\left| J \right| = \frac{1}{2\pi} \exp \left\{ -\frac{1}{2} (x_1^2 + x_2^2) \right\}$$

$$= \frac{1}{\sqrt{2\pi}} \exp\left\{ -\frac{1}{2} x_1^2 \right\} \frac{1}{\sqrt{2\pi}} \exp \left\{ -\frac{1}{2} x_2^2 \right\}$$

Hence,

$$f_{X_1, X_2}(x_1, x_2) = f_{X_1}(x_1) \cdot f_{X_2}(x_2) \qquad (3)$$

From eq(3) it follows, the joint probability density function of X_1 and X_2 is the product of their marginal probability density functions which implies X_1 and X_2 are independent standard normal variables .

Although the method has the advantage of being easy to program, it is slow in execution as it requires the calculation of a log, a square root and trignometric functions. This disadvantage is removed by Marsaglia method as outlined below.

ii) The Polar Marsaglia Method

This method avoids the use of trignometric functions. The method is a rejection method which can be summarized as follows : Let U be a random variable distributed as Uniform(0,1). Then 2U is distributed as Uniform(0,2) and V = 2U-1 is distributed as Uniform(-1,1).

If we select two independent Uniform(-1,1) random variables, say V_1 and V_2, then these are random points in the square whose vertices are (-1,1), (-1,-1), (1,-1) and (1,1).

Let $$W = V_1^2 + V_2^2$$

and

$$Y = \left[(-2 \ln W)/ W \right]^{1/2}.$$

Then W conditional on $V_1^2 + V_2^2 < 1$ is uniformly distributed on (0,1) and independently of V_1/V_2. This W can be used for U_1 in (1), and replacing

$$\cos 2\pi U_2 \text{ by } V_1/\sqrt{V_1^2 + V_2^2}$$

and

$$\sin 2\pi U_2 \text{ by } V_2/\sqrt{V_1^2 + V_2^2}$$

yields

$$X_1 = V_1 Y , \quad X_2 = V_2 Y$$

which are i.i.d. N(0,1) random variables. This algorithm is appreciately faster than the Box and Muller method.

3.3 Goodness of Fit Test

Kolmogrov-Simirnov test was done for various sample sizes. The test was performed for the generator (DRAND48) and to the transformed normal random variables. The main interest here is, to examine whether the generator was correctly used, and the transformation from uniform to normal random variables was correctly done. The output of the test and its tabulated value for α=0.05 is given here under.

Table 3.1 Kolmogrov-Simirnov Test

	Absolute difference		
Sample size	Uniform	Normal	Tabulated value
5	0.548	0.407	0.563
10	0.128	0.369	0.409
20	0.196	0.225	0.294
25	0.193	0.105	0.264
30	0.123	0.083	0.242
50	0.115	0.109	0.190
100	0.071	0.073	0.136
200	0.043	0.057	0.096
500	0.051	0.024	0.061
1000	0.017	0.020	0.043
2000	0.018	0.029	0.030
4000	0.011	0.016	0.025

In all the sample sizes tested, the results show no indication of deviation from their theoretical distribution for both uniform and normal distributions.

3.4 Design of the Experiment

The study is designed to investigate the precision of the estimators discussed in chapter 2. In this section the values of k (the number of estimates to be combined), the sample sizes considered, and the variance patterns chosen are dicussed. For simplicity, it is assumed that μ=0. The number of populations chosen are k=3,6,8 and 10.

3.4.1 Choices of Sample Sizes

i) Equireplicated Case

In order to see the effect of increase in the σ_i^2's alone, the case of equal sample sizes is studied. In this case, equal sample observations are taken from each population. The sample sizes considered here are n_i= 20, n_i=25, and n_i=30 for all i=1,2,...,k.

ii) Unequal n_i's

Here, four sets of n_i values are used. These four sets are denoted by case A, case B, case C, and case D. Case A consists of values ranging from 20 to 24, which are small relative to the other set of values. In case B, the values vary from 26 to 30, which are relatively large. Case C consists of sample sizes which varies from 20 to 30, in ascending order. In case D the values are the reverse of case C, the sample sizes vary in descending order. These four cases together with the chosen k's are tabulated in table 2.

Table 2: **The sample sizes considered**

k	case A			case B			case C			case D		
3	20	22	24	26	28	30	20	25	30	30	25	20
6	20	20	22	26	26	28	20	20	25	30	30	25
	22	24	24	28	30	30	25	30	30	25	20	20
8	20	20	20	26	26	28	20	20	20	30	30	30
	22	22	22	28	28	30	25	25	30	25	25	20
	24	24		30	30		30	30		20	20	
10	20	20	20	26	26	26	20	20	20	30	30	30
	20	22	22	28	28	28	20	25	25	30	25	25
	22	24	24	30	30	30	25	30	30	25	20	20
	24			30			30			20		

3.4.2 The Variance Patterns Considered

Four σ_i^2 patterns are chosen. These vary from small to large heterogeneity. To examine the effect of the changes in the n_i's only, the case when $\sigma_i^2=1$ for $i=1,2,\ldots,k$ is also investigated on the precision of the estimators. This condition was studied by Levy(1970) for $1\le n_i\le 10$.

The σ_i^2 patterns are denoted by I, II, III, and IV. The values in each pattern are given in table 3 below.

Table 3 Variance patterns considered

k	σ_i^2 patterns I			II			III			IV		
3	0.33	0.66	1.0	1	2	3	5	10	15	10	20	30
6	0.16	0.33	0.5	1	2	3	5	10	15	10	20	30
	0.66	0.83	1.0	4	5	6	20	25	30	40	50	60
8	0.125	0.25	0.375	1	2	3	5	10	15	10	20	30
	0.5	0.625	0.75	4	5	6	20	25	30	40	50	60
	0.875	1.0		7	8		35	40		70	80	
10	0.1	0.2	0.3	1	2	3	5	10	15	10	20	30
	0.4	0.5	0.6	4	5	6	20	25	30	40	50	60
	0.7	0.8	0.9	7	8	9	35	40	45	70	80	90
	1.0			10			50			100		

Then for every combination of (k, n_i, σ_i^2), 1000 independent sample runs are made. For all possible combinations, a total of 136,000 independent sample runs are done.

3.5 The Computer Program

An interactive program which accomplishes the desired simulation is written in Pascal (the printout of the program is given in Appendix I) . The program is compiled and executed on HP 9000/500. The main program has several subroutines and functions, which facilitates the heavy computations involved.

i) The Input

To run the program the following parameters are required:

1. k , the number of normal populations from which samples are to be taken.

2. Seed value, the initalizing point for the uniform random number generator.

3. n_i's, the sample sizes from each group.

4. σ_i^2's, population variances of each group.

ii) Computation of Estimates

After the program receives the input parameters sequentally and correctly, DRAND48 is invoked several times for generating independent uniform random variables on (0,1). These are transformed to standard normal by Marsaglia method. Again, these standard normal random variables are multiplied by the square root of the population variances to get the necessary normal samples distributed as $N(0,\sigma_i^2)$. After that, the sample means($\bar{x}_i$) and sample variances (S_i^2) are computed. The above process continues for all i=1,2,,...,k. Then, the different estimators are calculated according to the formulas given in chapter 2.

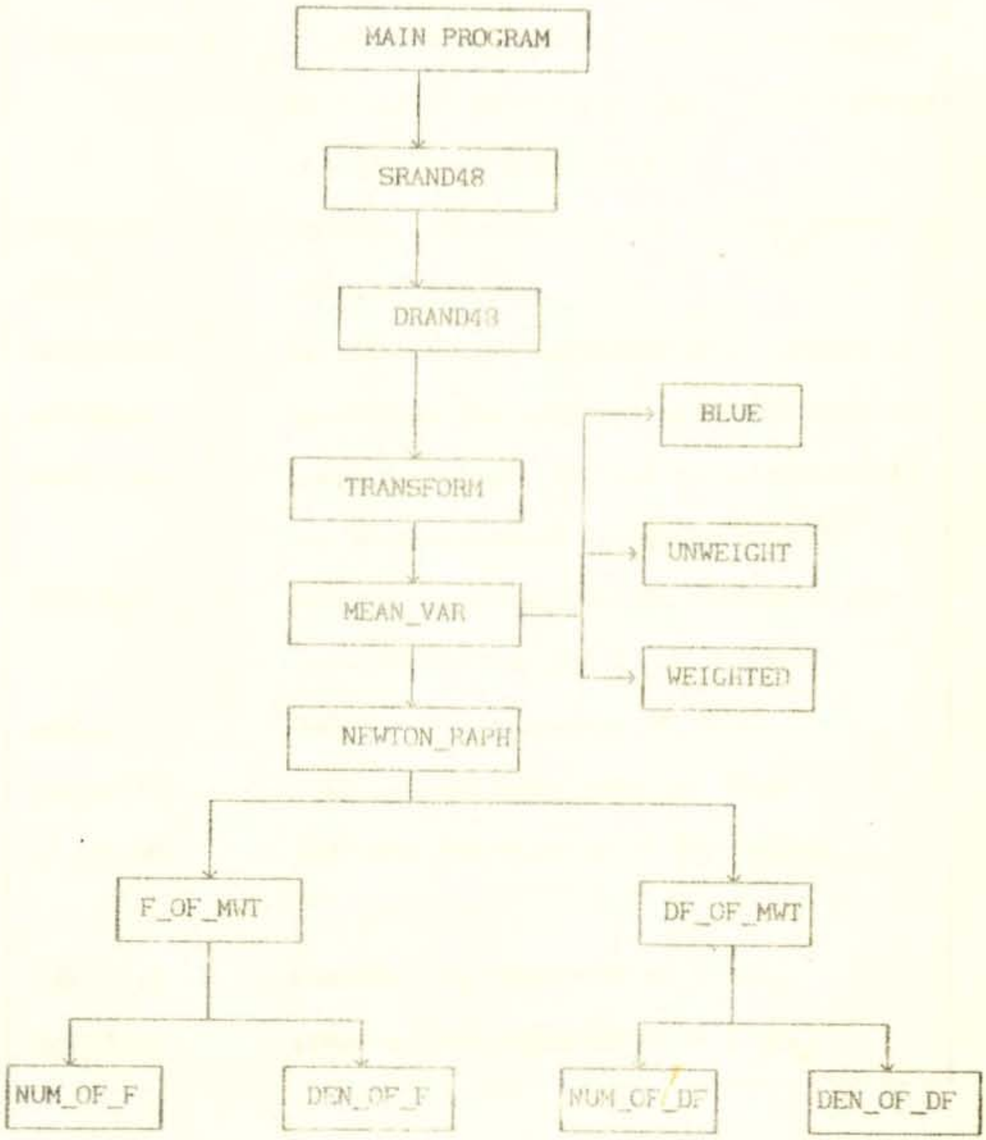
The maximum likelihood, the Neyman-Scott and the Kalbfleish estimators are obtained using the Newton-Raphson algorithm. For these estimators, the unweighted mean is used as an initial point of approximation. The iteration stops if the difference of two successive estimates in absolute value is less than ε=1.0x10^{-10}. In all runs convergence was attained before 15 iterations. The computer time needed to accomplish the whole task for a given combination is directly proportional to the n_i's and varies from 80 seconds to 250 seconds.

iii) The Output

The output of the computer program consists of all the input parameters, the estimated means including the estimate resulting from the best linear unbiased estimator which attains the Cramér Rao lower bound, and the serial number which indicates the run number which goes from 1 to 1000. The estimates of μ are printed with eight significant digits.

Another small independent program was also written to compute the mean and variance of the estimates. The inputs for this program are the estimates resulted by the big program. It also computes the efficiency of the estimators relative to the BLUE.

Logical Flow of Subroutines in the Program



Description of the Subprograms

Subprogram	Purpose
SRAND48	Provides a seed value to DRAND48.
DRAND48	Generates uniform pseudo-random numbers.
TRANSFORM	Transforms uniform random variables to standard normal and then to the desired standard random variable.
MEAN_VAR	Computes the sample means and variances.
BLUE	Computes the BLUE.
UNWEIGHTED	Calculates the unweighted mean estimator.
WEIGHTED	Calculates the weighted mean estimator
NEWTON_RAP	Computes the ML, NS, and KS estimators by the Newton-Raphson iteration method.
F_OF_MWT	A function subprogram to calculate the value of $f(\mu)$ at the t^{th} iteration.
NUM_OF_F	Computes the numerator of $f(\mu)$.
DEN_OF_F	Computes the denominator of $f(\mu)$.
DF_OF_MWT	Evalutes the value of $f'(\mu)$ at the t^{th} iteration.
NUM_OF_DF	Computes the numerator of $f'(\mu)$.
DEN_OF_DF	Computes the denominator of $f'(\mu)$.

This chapter was mainly concentrated on the data used to compare the performance of the estimators. The simulation procedures used, the choices of sample sizes, choices of σ_i^2 patterns and the computer program written is broadly discussed. The next chapter will focus mainly on the comparison of the estimators. The results of the simulation study will be tabulated and discussed briefly. The behavior and precision of the estimators will be compared graphically, with their empirical variances. The efficiency of the estiamtors relative to the best linear unbiased estimator will be compared empirically and displayed graphically.

CHAPTER 4

COMPARISON OF ESTIMATORS

An extensive Monte Carlo study has been done to evaluate the behavior of the different estimators. Over all, a total of 136,000 independent runs were run. In this chapter, the simulation results are tabulated and graphically presented. The estimators will be compared through their frequency distributions, their empirical variaances, and their efficiency relative to the best linear unbiased estimator(BLUE).

Table 4.1: Abbrivation of estimators used in the Monte Carlo study

Estimator	Label	Formula
Unweighted Mean	UW	$\sum_{i=1}^{k} n_i \bar{x}_i \Big/ \sum_{i=1}^{k} n_i$
Weighted Mean	WT	$\sum_{i=1}^{k} n_i S_i^{-2} \bar{x}_i \Big/ \sum_{i=1}^{k} n_i S_i^{-2}$
Maximam Likelihood	ML	$\sum_{i=1}^{k} \frac{n_i^2 (\bar{x}_i - \hat{\mu}_{ML})}{(n_i - 1) S_i^2 + n_i (\bar{x}_i - \hat{\mu}_{ML})^2} = 0$
Kalbfleish-Sprott	KS	$\sum_{i=1}^{k} \frac{n_i (n_i - 1)(\bar{x}_i - \hat{\mu}_{KS})}{(n_i - 1) S_i^2 + n_i (\bar{x}_i - \hat{\mu}_{KS})^2} = 0$
Neyman-Scott	NS	$\sum_{i=1}^{k} \frac{n_i (n_i - 2)(\bar{x}_i - \hat{\mu}_{NS})}{(n_i - 1) S_i^2 + n_i (\bar{x}_i - \hat{\mu}_{NS})^2} = 0$

4.1 Graphical Display of the Estimators

The frequency distribution of the estimates resulted by the different estimators show the number of cases falling into different class intervals. The frequency distribution of all the estimators is tallied and tabulated in Appendix II. Here, interest lies on the number of cases that fall in the interval containing the true parameter (μ=0) and to examine the distribution of the estimates as a whole. To observe the behavior of the estimators under different circumstances, the percentages of the estimates that fall into the given class intervals are plotted.

As the scope of the Monte Carlo study considered is vast, graphical represntation of each and every case requires a large number of pages. In order to save space and resources, from the sample space of investigation, the extreme cases are plotted for all parameters. The number of populations (k) selected for plotting are k=3 and k=10. The variance patterns chosen for plotting are when the σ_i^2's are equal, less diversed (pattern I) and more diversed (pattern IV). Likewise, the sample sizes selected in this section for plot are, in the case of equal sample sizes, m=20 and m=30, and for the unequal sample sizes case, cases C & D.

The graphs are displayed in Figures 1 to 20. By looking at these figures and the frequency distributions tabulated in appendix II, the following results are observed.

a) Equal sample sizes case

In this case the estimators reduce to three, because ML=NS=KS. For k=3 and 10, three variance patterns are considered and dicussed below:

i) Equal variance case

Figures 1,2,3 and 4 are graphs when the σ_i^2's are equal. Under this circumstance, the plots are close to each other. Comparing Figures 1 & 2, the percentage of the estimates that fall in the interval containing the parameter to be estimated increases from 9.2% to 18.3 perecent. This indicates that as k increases, the estimates are close to the true mean(μ=0), and the shape of the curves become close to the normal one. As the sample sizes increase these four figures show that their shapes become more closer to the shape of a normal curve. The UW seems to have a shape which is more closer to the normal one than the others. This is due to the fact that, when the σ_i^2's are identical UW=BLUE.

ii) Variance pattern I

Figures 5, 6, 7 and 8 are the plots of the estimates when the σ_i^2's are less diversed(pattern I). When k=3 and m=20, (Figure 5) the curves are flatter and positively skewed. But as k increases to 10 (Figure 6), the graphs approach the normal curve. The percentage of the values that fall in the interval containing μ=0 increases from 12% to 30 %. When the sample sizes increase, the value of the estimators become close to the true mean. If both k and m increase the shape of the WT and ML become peaked, which shows the estimates are more closer to the parameter of interest. With this variance pattern, the WT & ML have the same shape, but UW has a different shape (flatter than the others). Comparing Figures 6 & 8, as the sample sizes increase from 20 to 30, the percentage of the estimated values that lie in the interval containing μ=0 increases from 30% to 38.4%.

iii) Variance pattern IV

Figures 9 to 12 are the graphs for k=3, m=20; k=10, m=20; k=3, m=30; k=10, m=30 ; when the σ_i^2's are highly diversed (pattern IV). In all these graphs UW is flatter than the rest of estimators. This indicates that the number of estimates produced by the unweighted mean estimator which are close to μ=0 are less in number than the estimates produced by the other estimators. WT and ML have the same shape. This indicates the outputs produced by both estimators are close to each other. As k and/or m increases the curves become more peaked. The percentage of values that lie in the interval containing μ=0 increases from 9% to 16%.

Figure 1 Graphical display of the estimators when k=3, m=20, equal variances

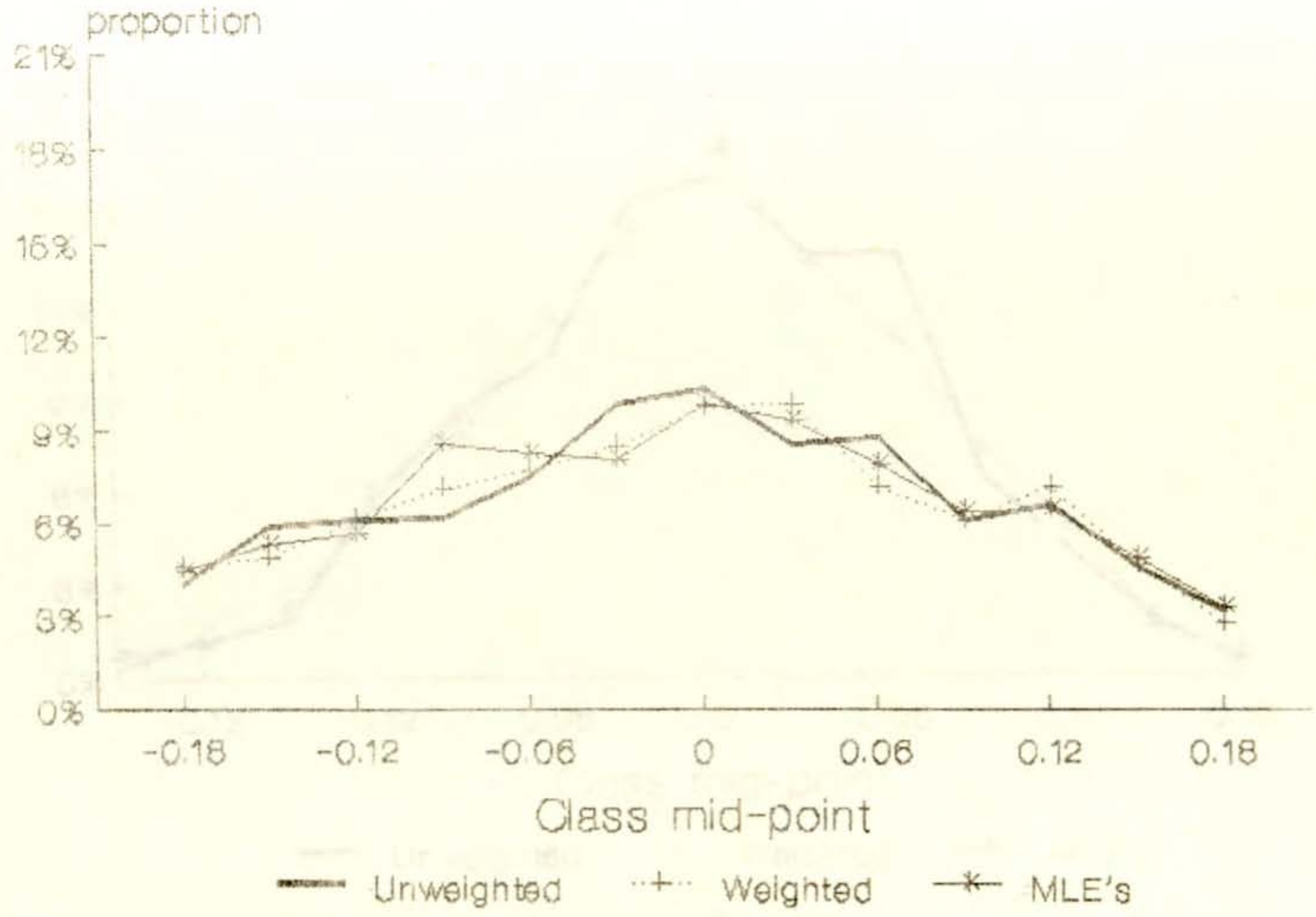


Figure 2 Graphical display of the estimators
k=10, m=20, equal variances

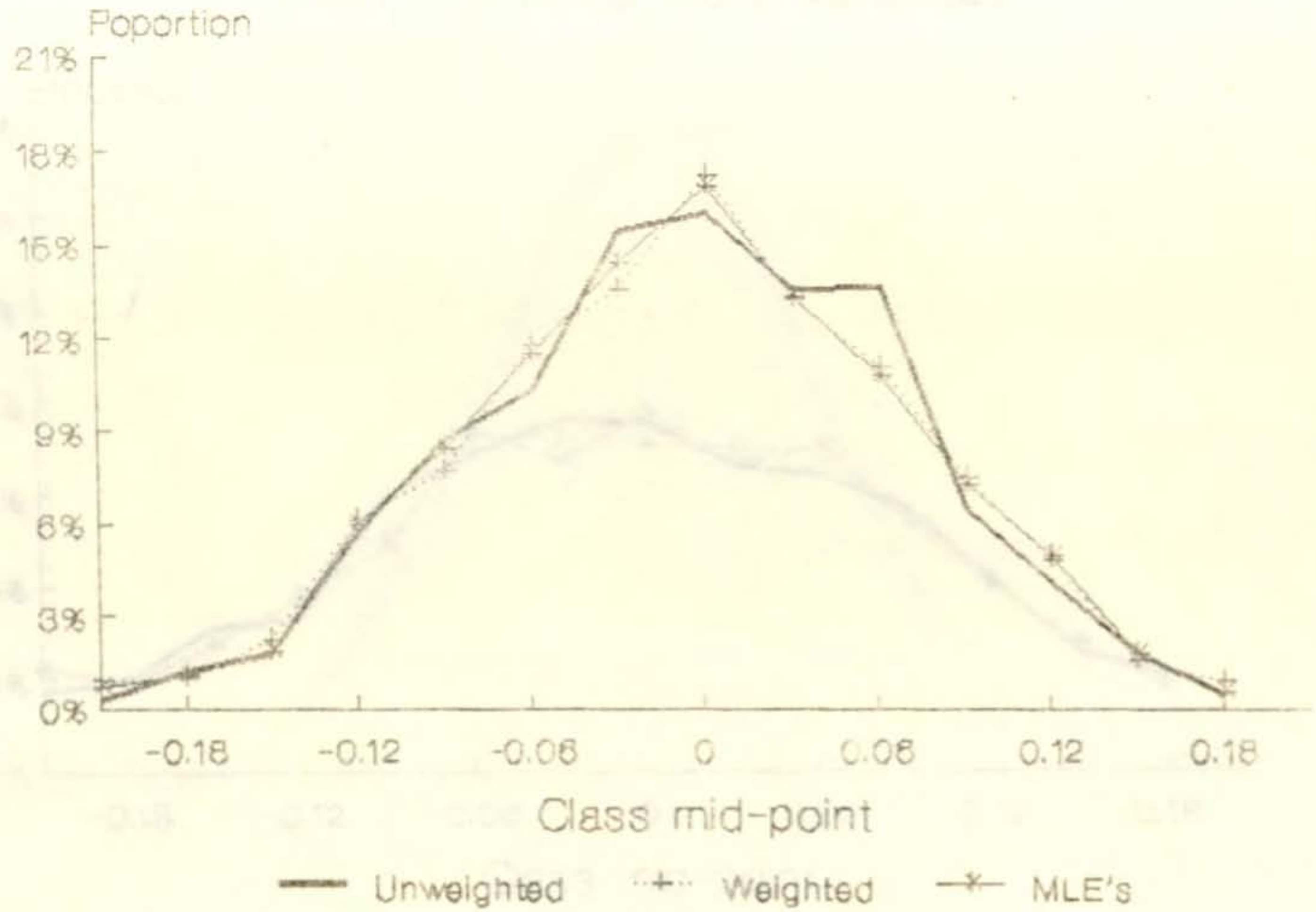


Figure 3 Graphical display of the etimators when k=3, m=30, equal variances

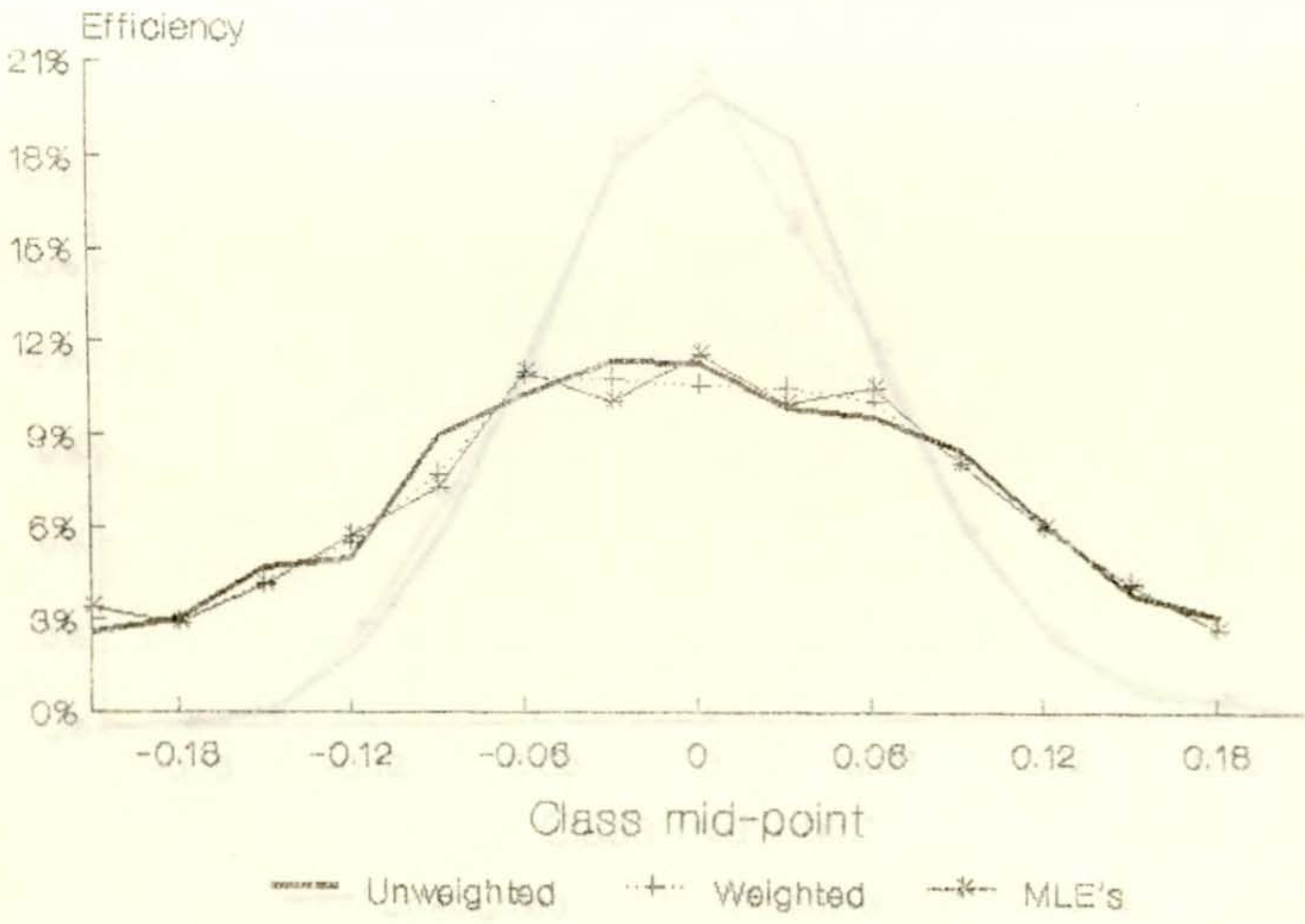


Figure 4 Graphical display of the estimators when k=10,m=30,equal variances

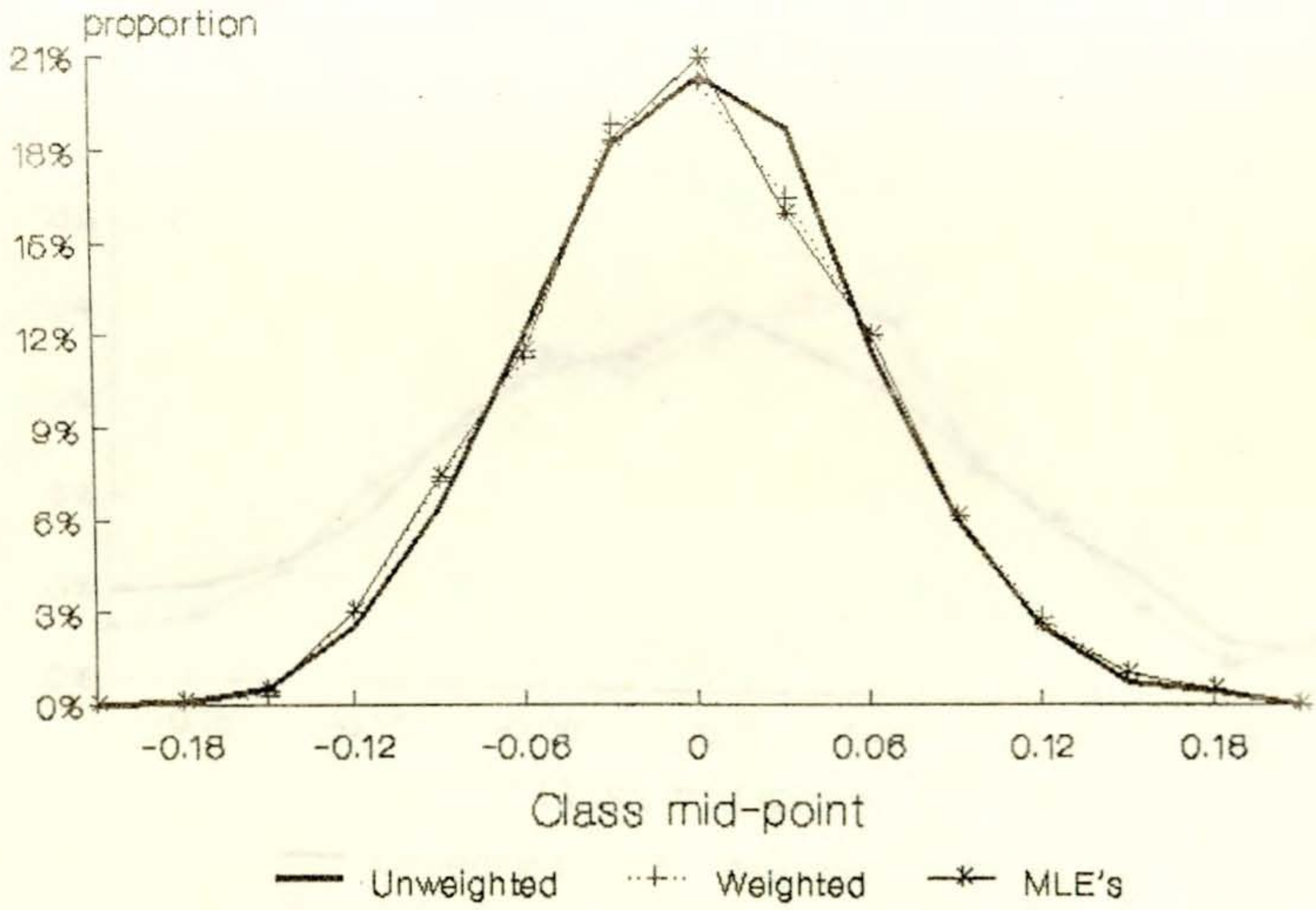


Figure 5 Graphical display of the estimators when k=3, m=20, varinace pattern I

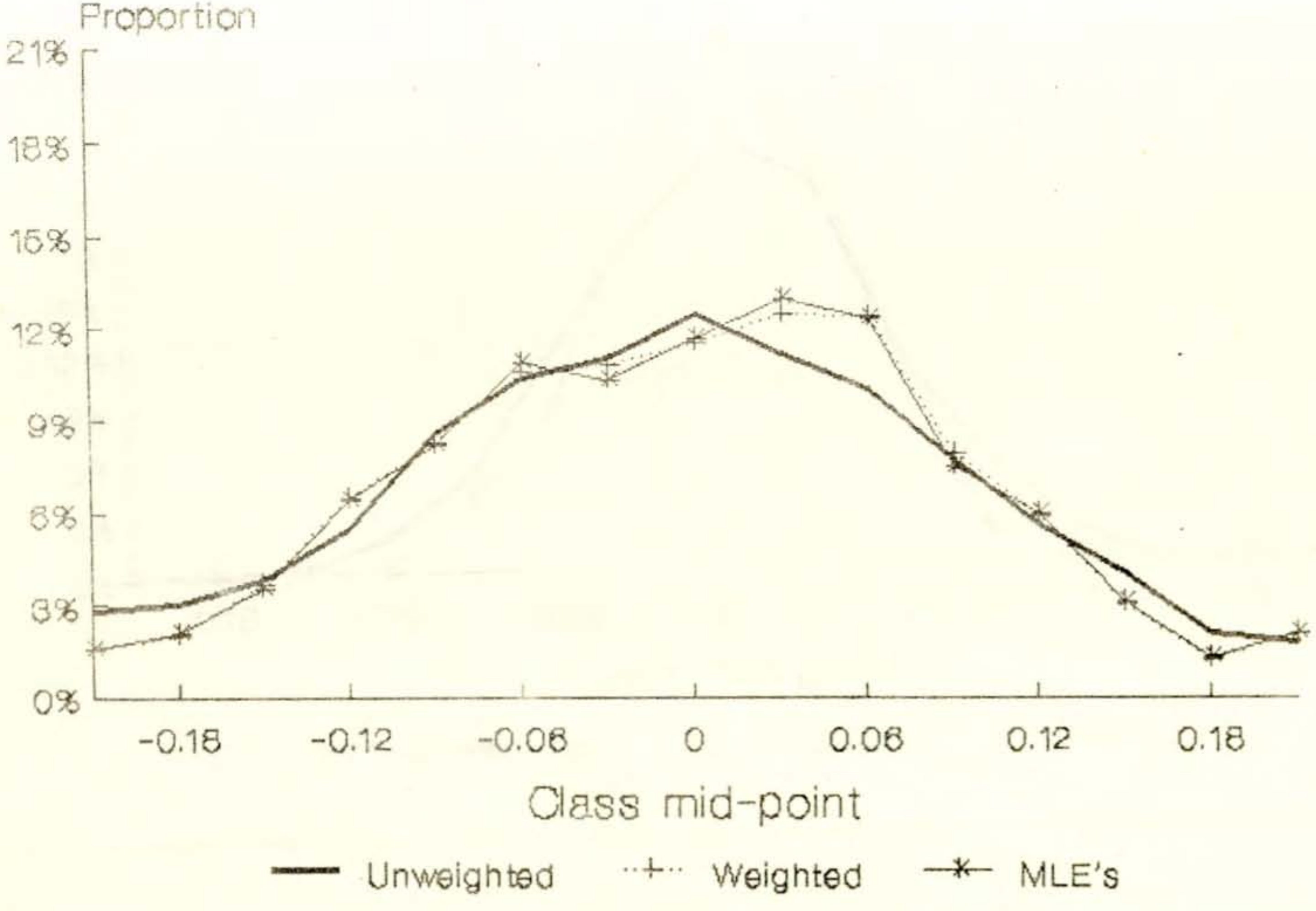


Figure 6 **Graphical display of the estimators when k=10, m=20, variance pattern I**

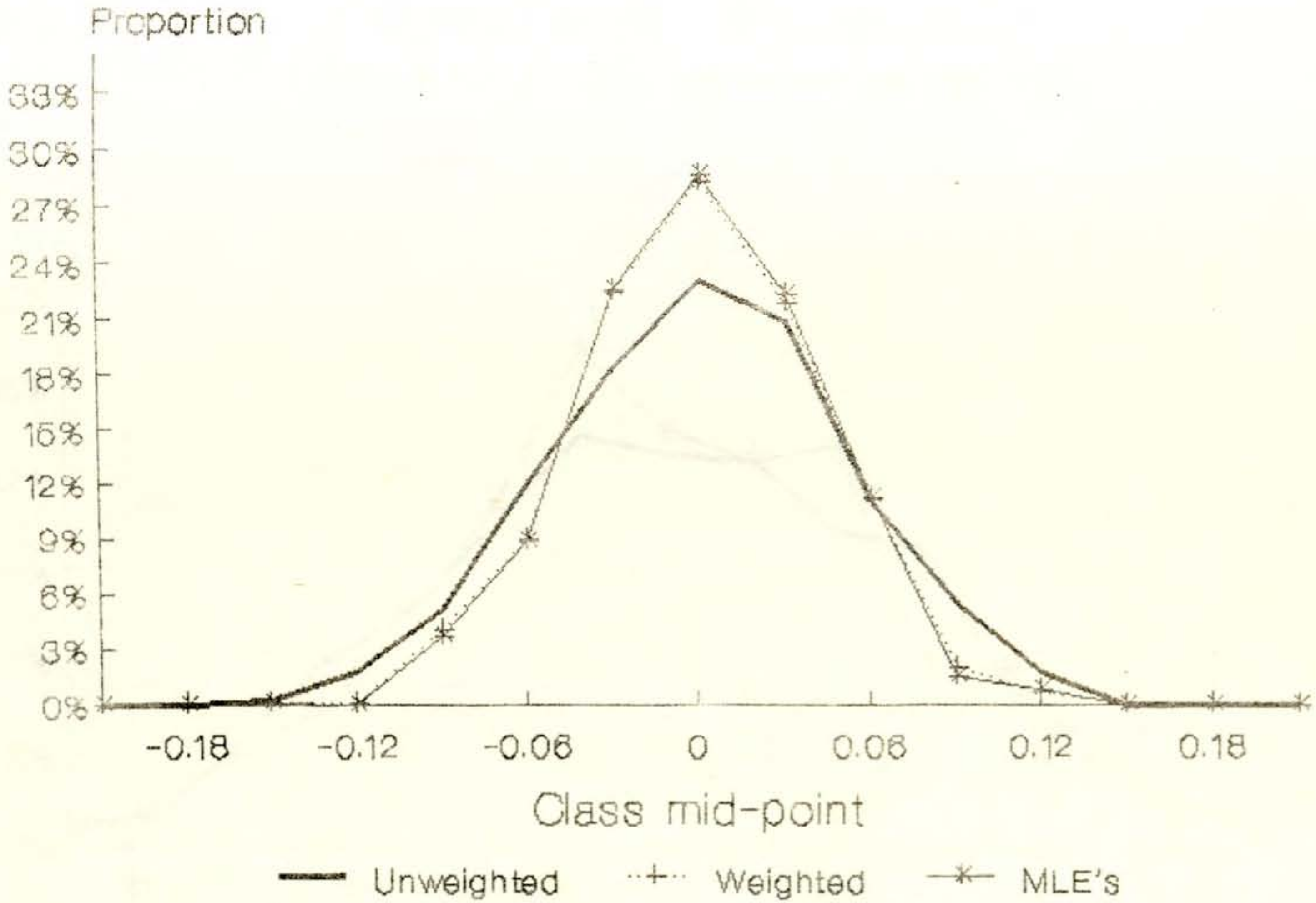


Figure 7 Graphical display of estimators when k=3, m=30, variance pattern I

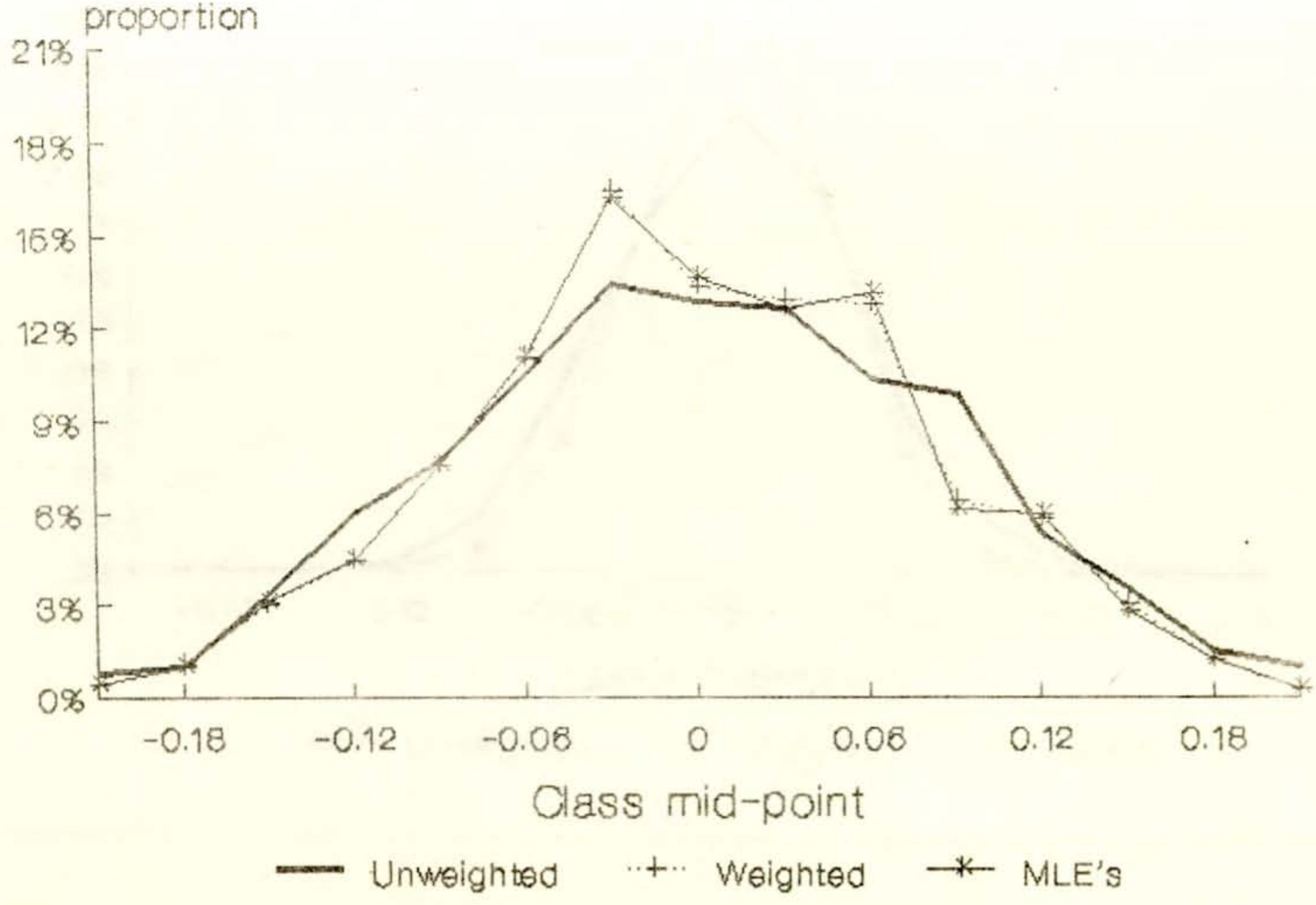


Figure 8 **Graphical display of the estimators when k=10, m=30, variance patterns I**

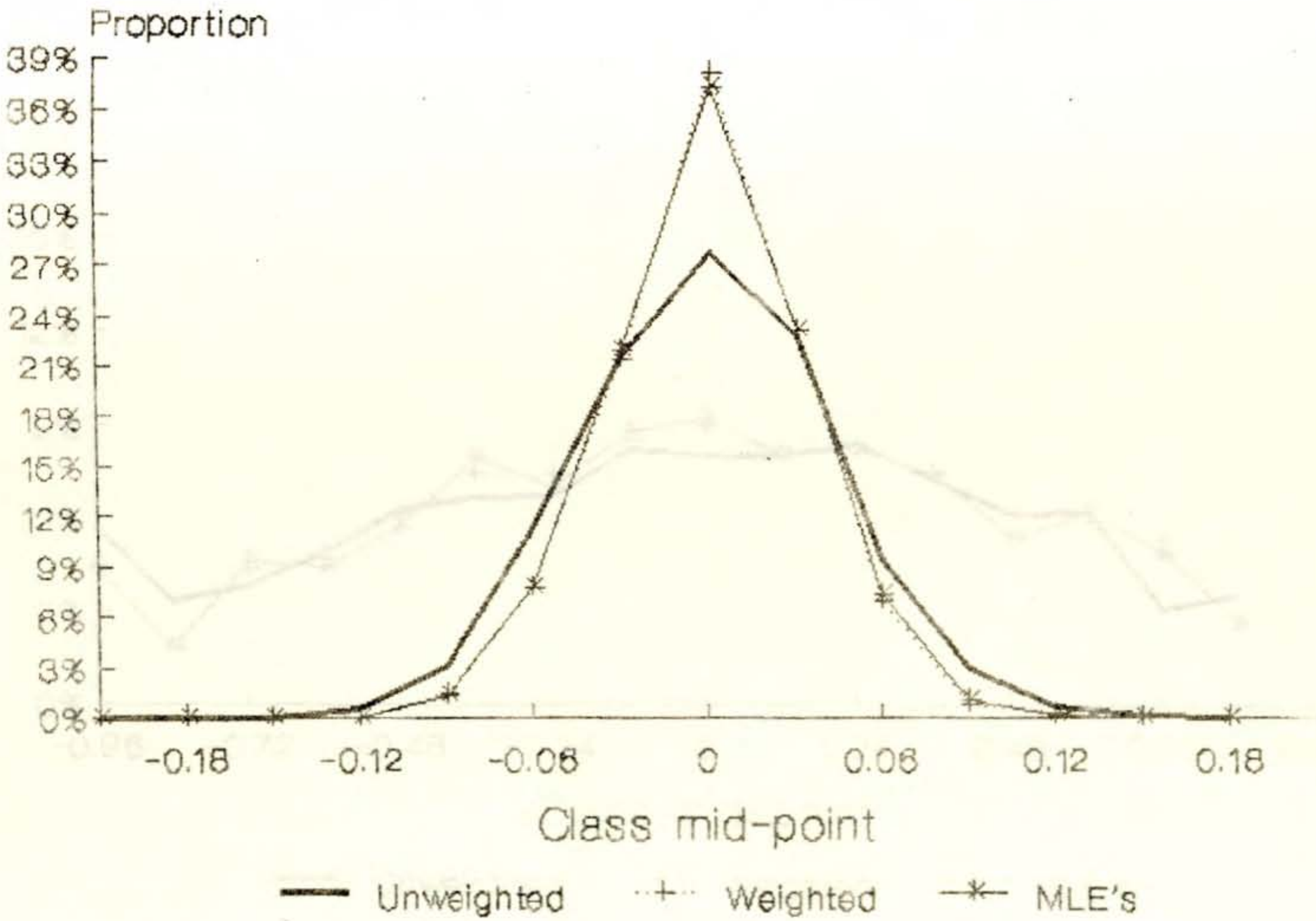


Figure 9 **Graphical display of the estimators when k=3, m=20, variance pattern IV**

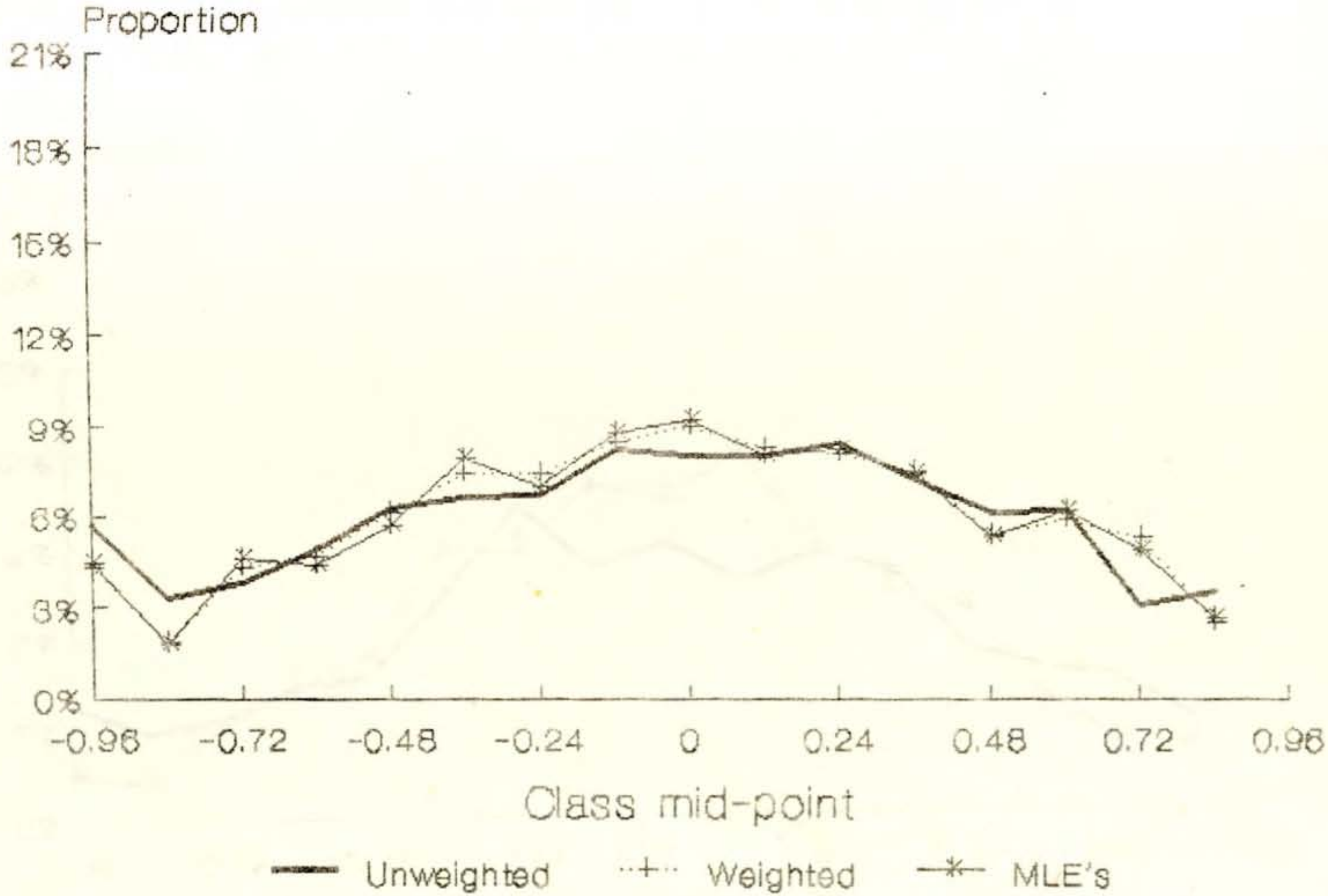


Figure 10 Graphical display of the estimators when k=10, m=20, variance pattern IV

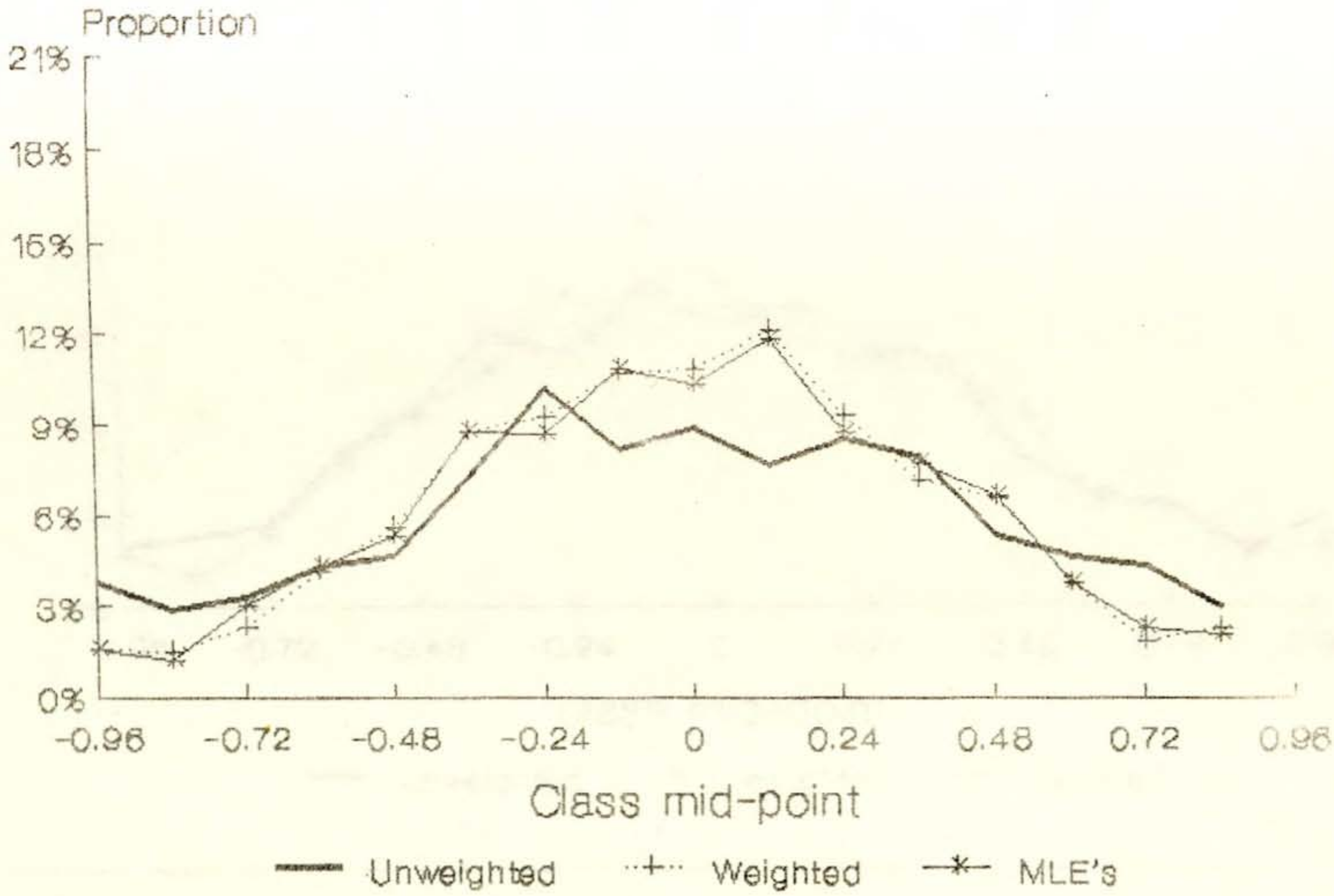


Figure 11 Graphical display of the estimators when k=3, m=30, variance pattern IV

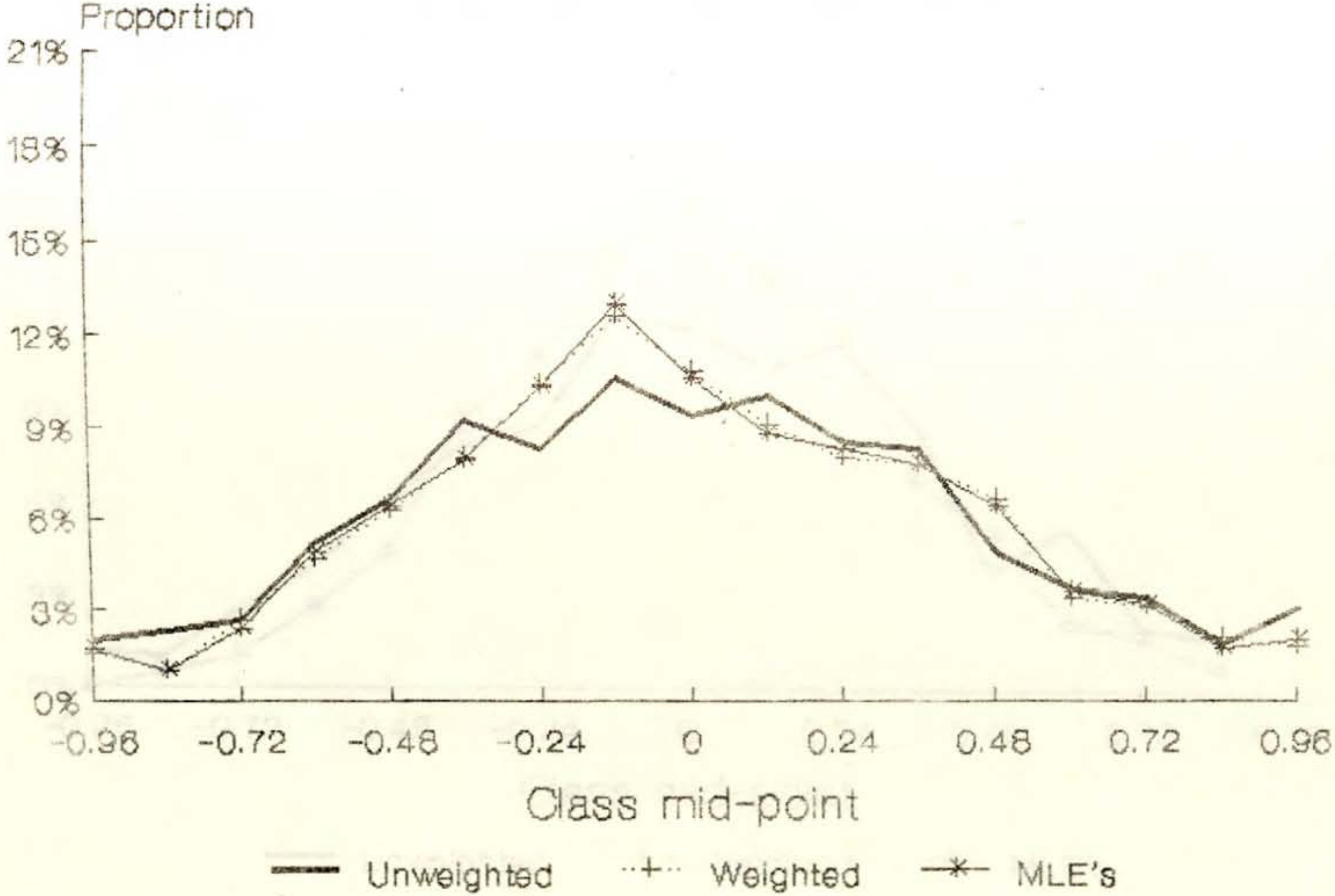
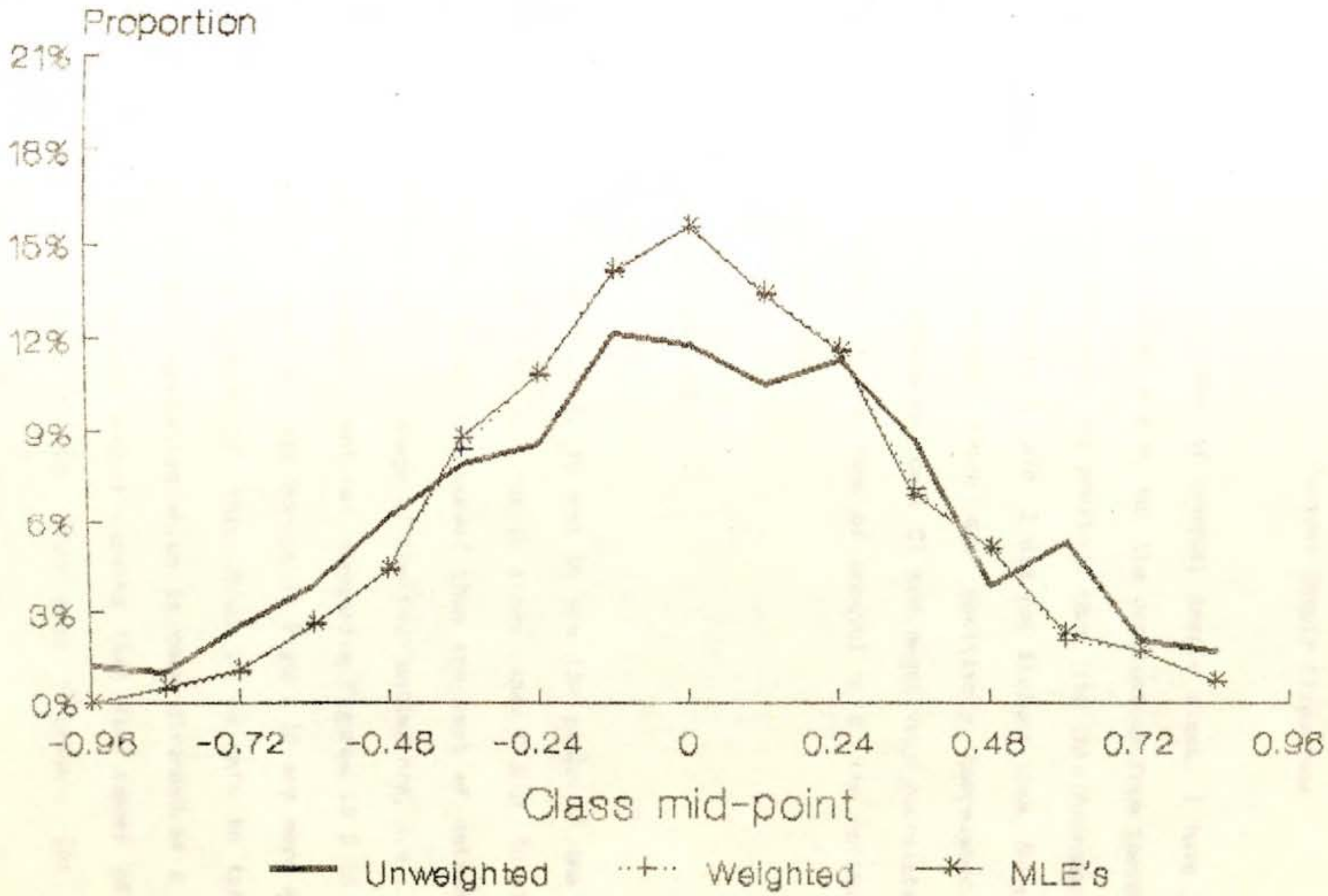


Figure 12 **Graphical display of the estimators when k=10, m=30, variance pattern IV**



b) Unequal Sample Sizes Case

Under the case of unequal sample sizes, I have plotted the graphs for cases A & B, but the conclusions from these graphs are not different from the previous ones, for this reasonand the large number of figures I have, I did not include them. Here, the cases when the sample sizes are positively correlated with the population variances (case C) and negatively correlated (case D) are plotted. In the case of unequal n_i's, the estimators to be compared are five.

i) Variance pattern I

Figures 13, 14, 15 and 16 are the plots of the estimators when k=3 and k=10, and sample sizes cases C & D. In all figures, the UW is flatter and skewed than the rest of estimators. From these figures , the shape of the four estimators, i.e., WT, ML, NS and KS are almost identical. Comparing Figures 13 & 15 (k=3, case C and k=3, case D), the curves in Figure 15 are more skewed than that are in Figure 13. This shows it is safe to take a large sample from a population which is more diversed.As k increases, the curves become peaked showing that the number of estimates which are close to the true mean increases. The percentage increases from 13% to 34%.

ii) Variance pattern IV

Figures 17, 18 , 19 and 20 are the graphical displays of the estimators for k=3 and k=10. The sample sizes considered here are cases C and D. From the frequancy distribution of the estimates and these four graphs we have the following results:

As the heterogeneity of the n_i's and the σ_i^2's increases, the curves become more skewed and flatter. As k increases the curves are little skewed. With the same k and σ_i^2's pattern, if the sample sizes taken are reversed (case D) one can easily see a significant change among the curves. The curves are more skewed in case D than in case C. Comparing Figures 18 and 20, if the samples observed from a population with high varinace are not large, the estimates are not closer to the true mean as in the case where large samples are observed.

Figure 13 **Graphical display of the estimators when k=3, case C, variance pattern I**

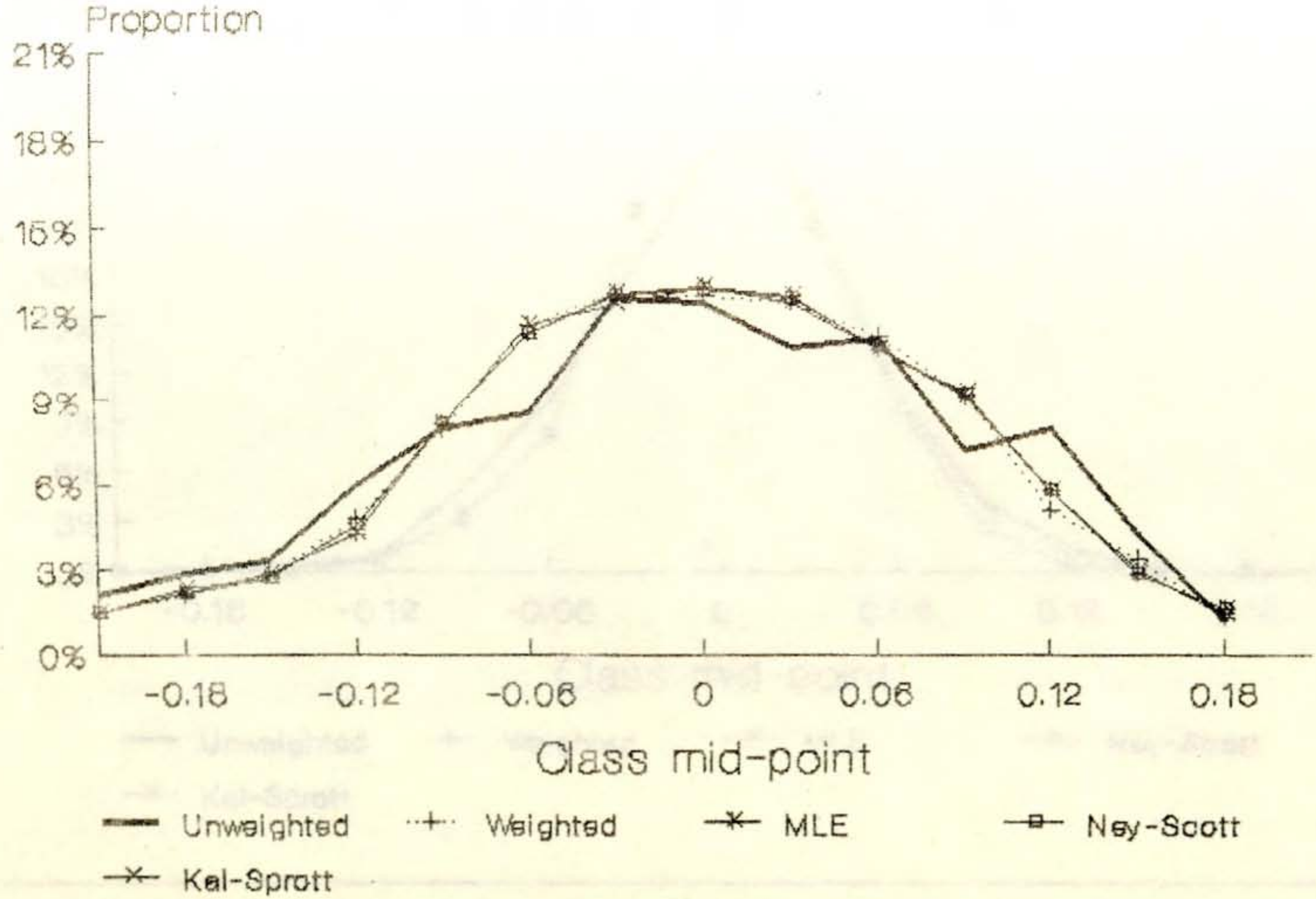


Figure 14 **Graphical display of the estimators when k=10, case C, variance pattern I**

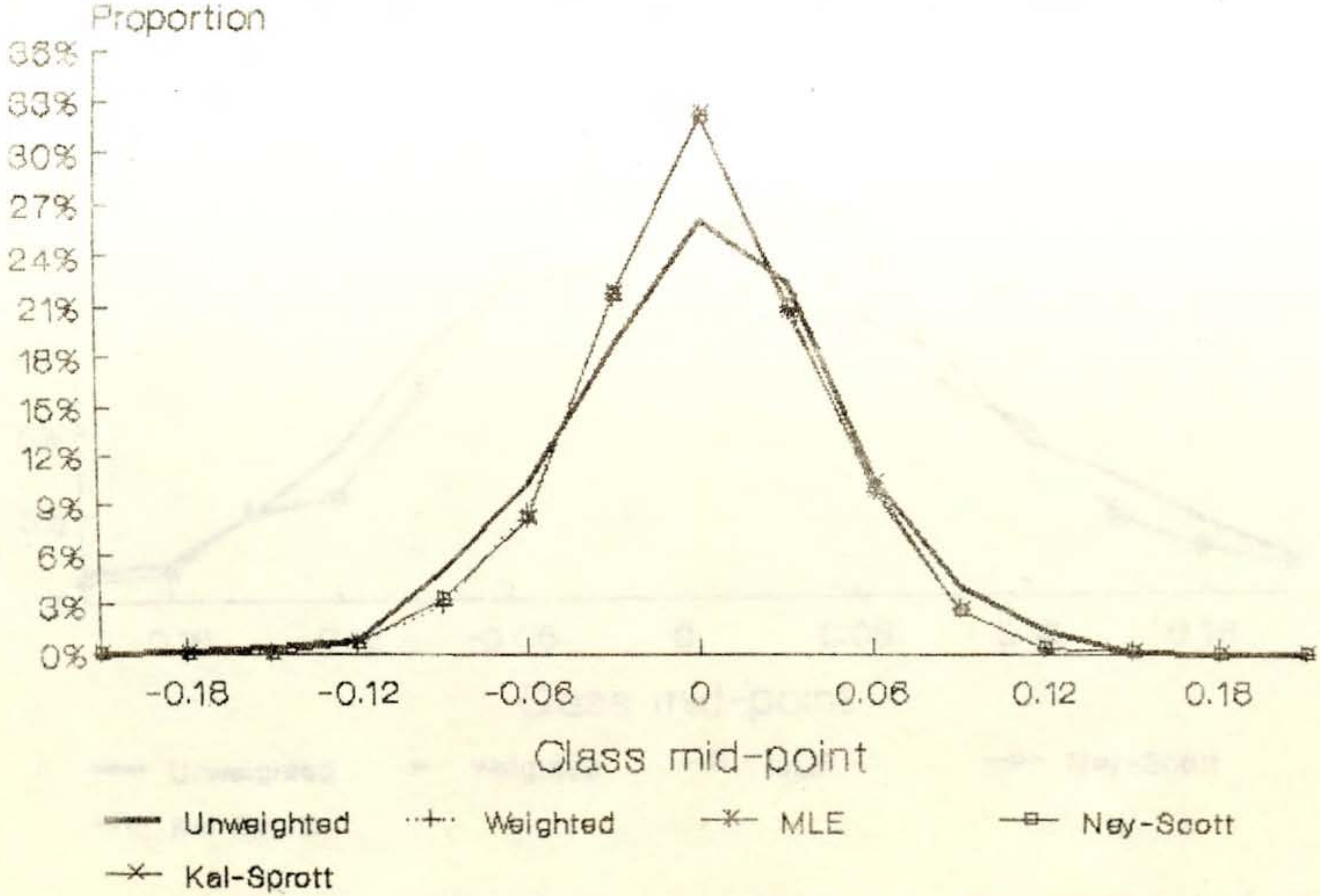


Figure 15 Graphical display of the estimators when k=3, case D, variance pattern I

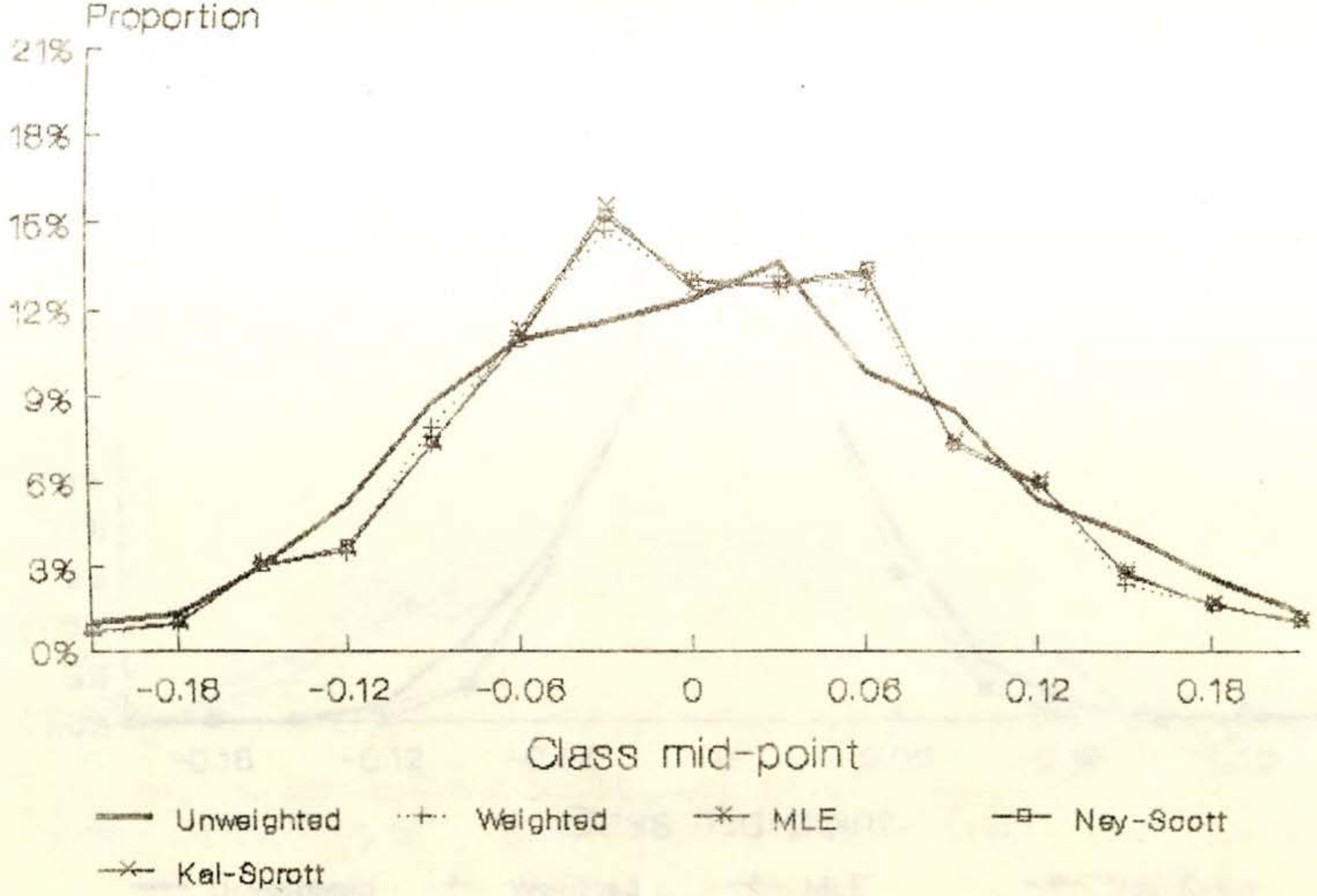


Figure 16 **Graphical display of the estimators when k=10, case D, variance pattern I**

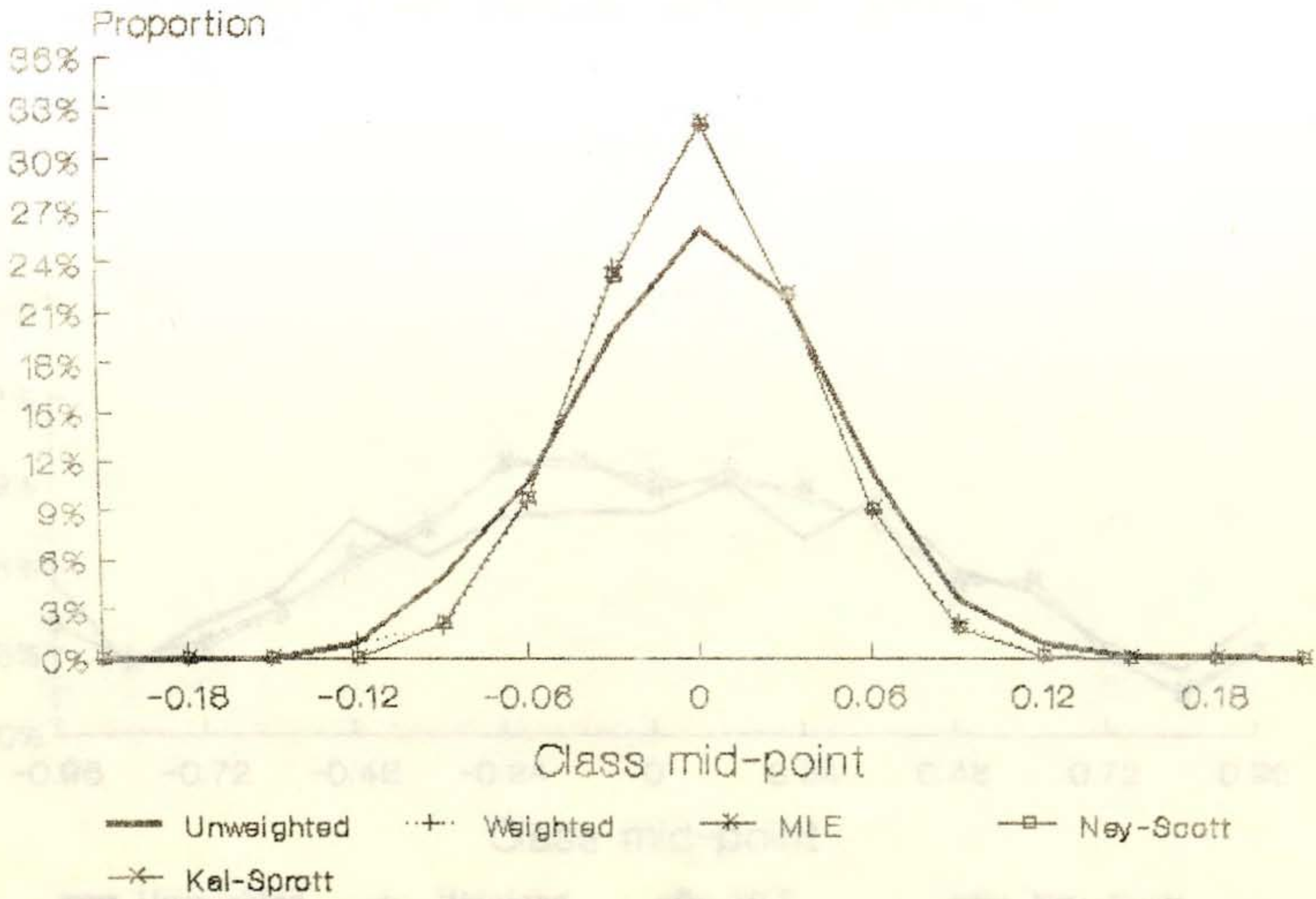


Figure 17 Graphical display of the estimators when k=3, case C, variance pattern IV

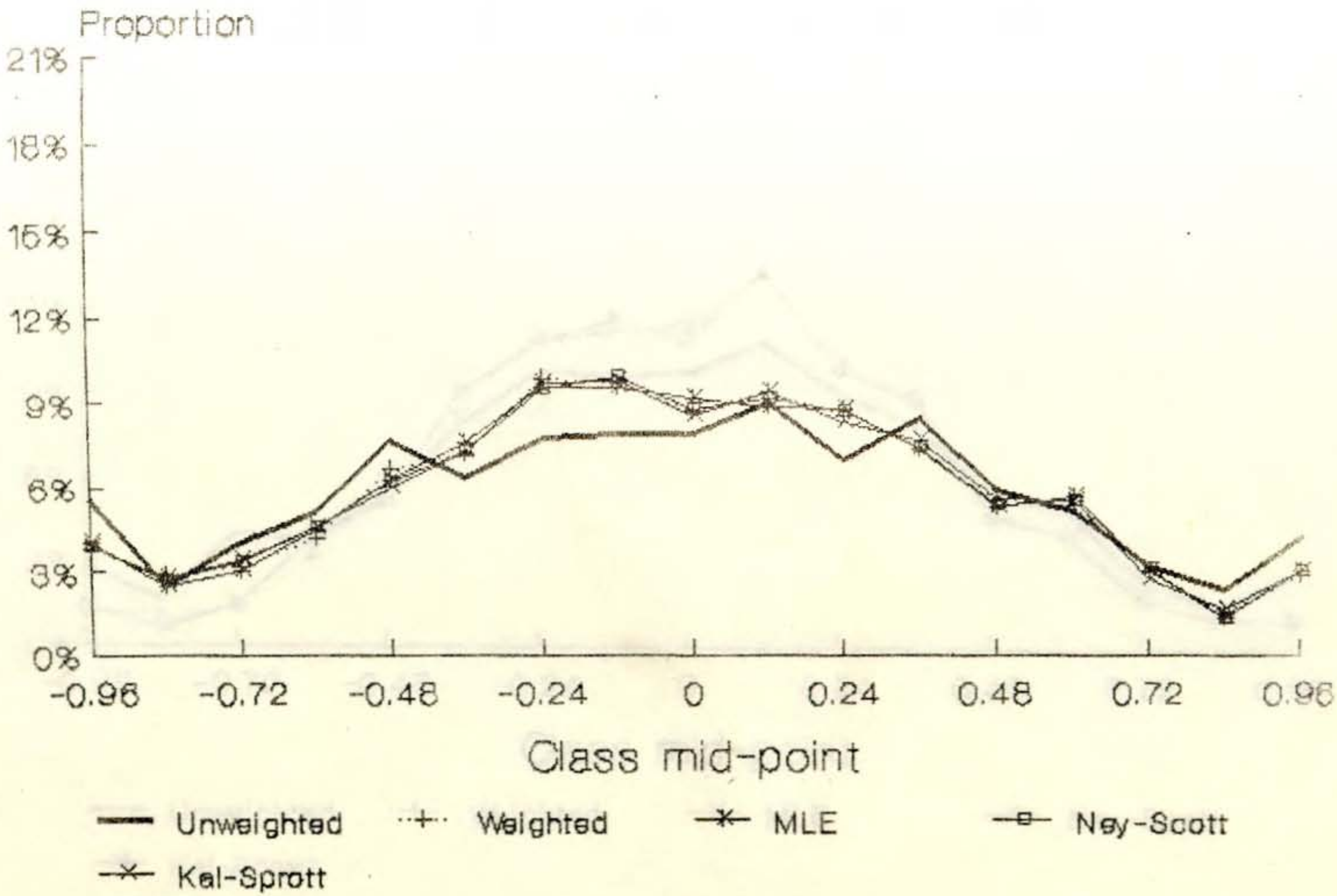
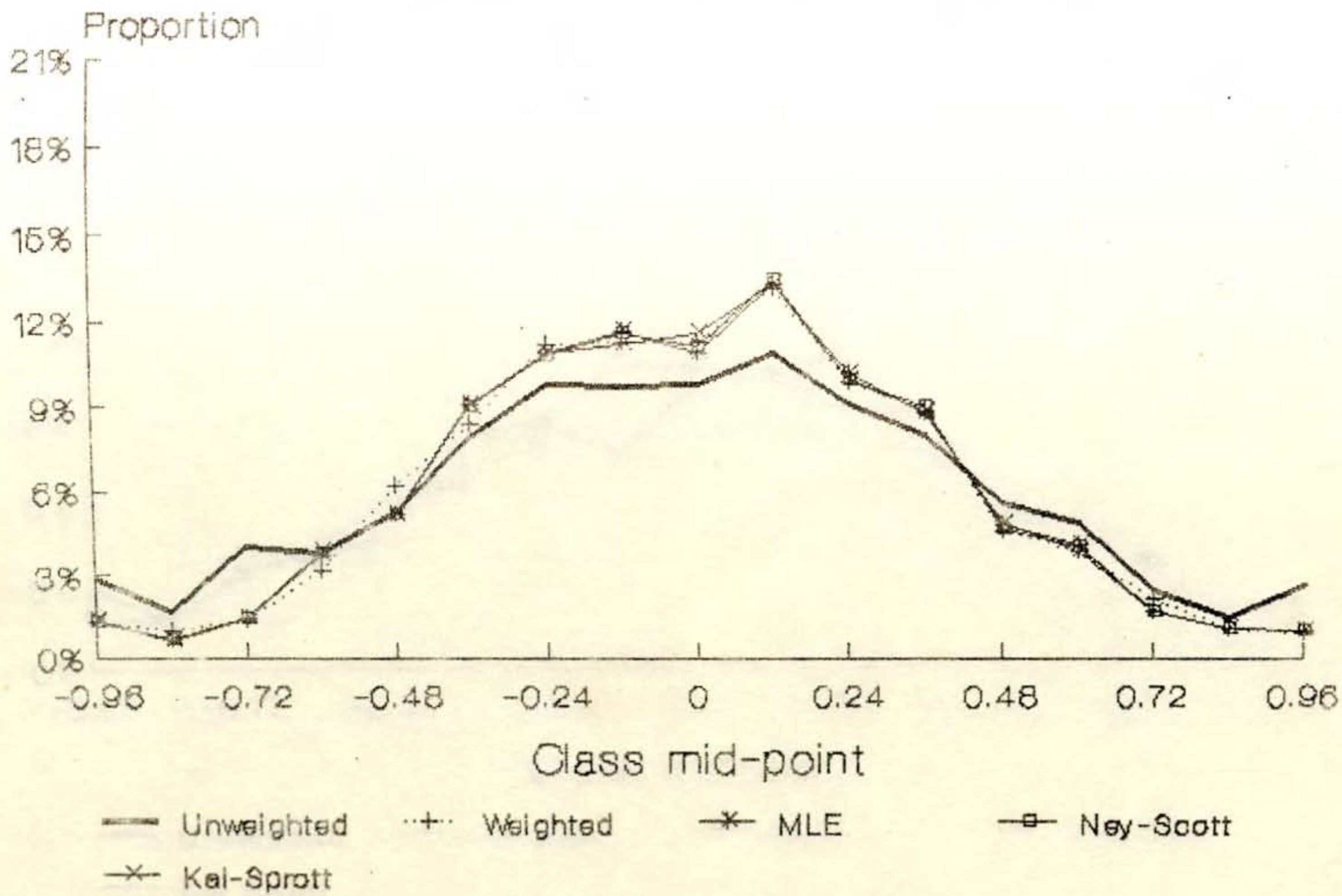
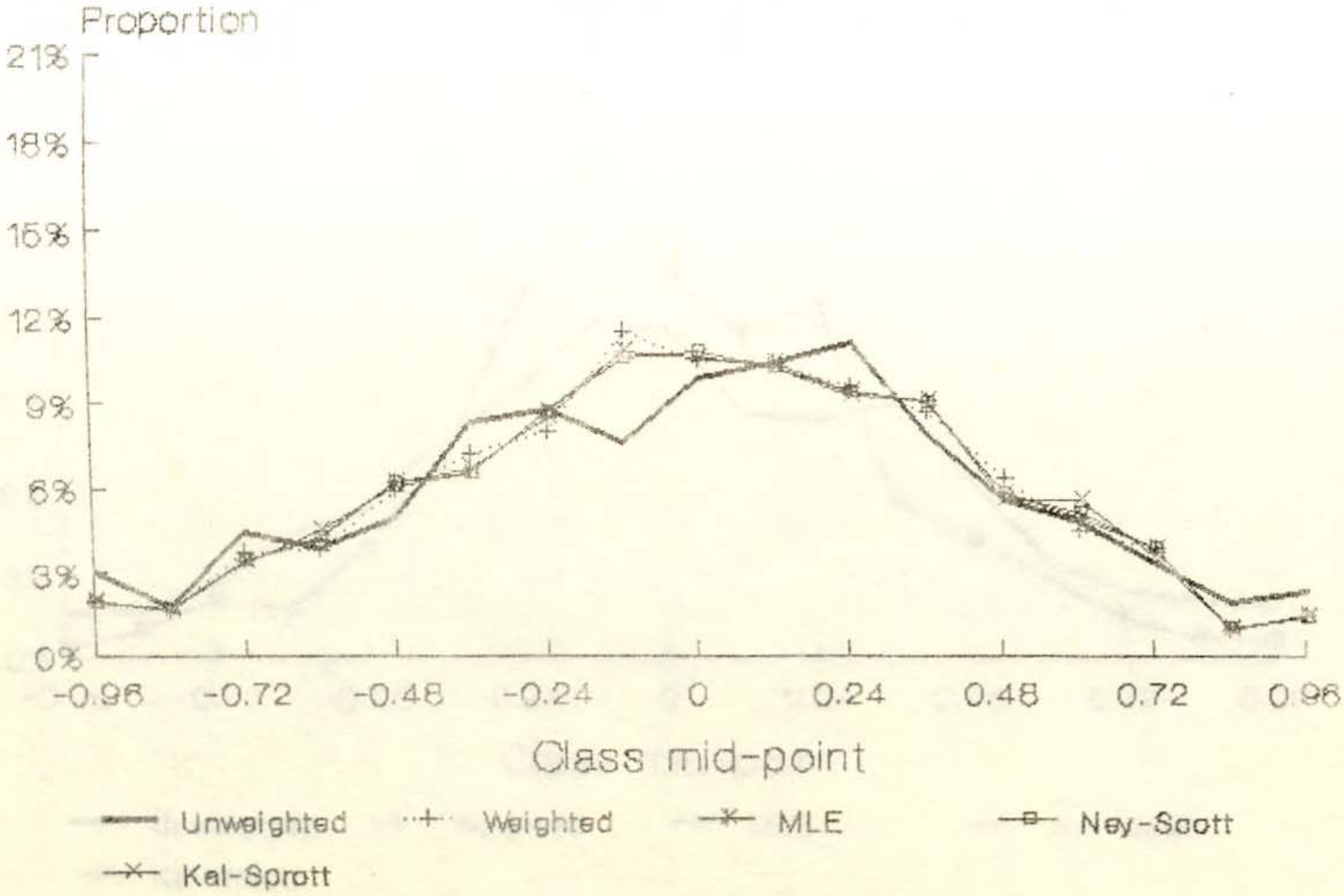


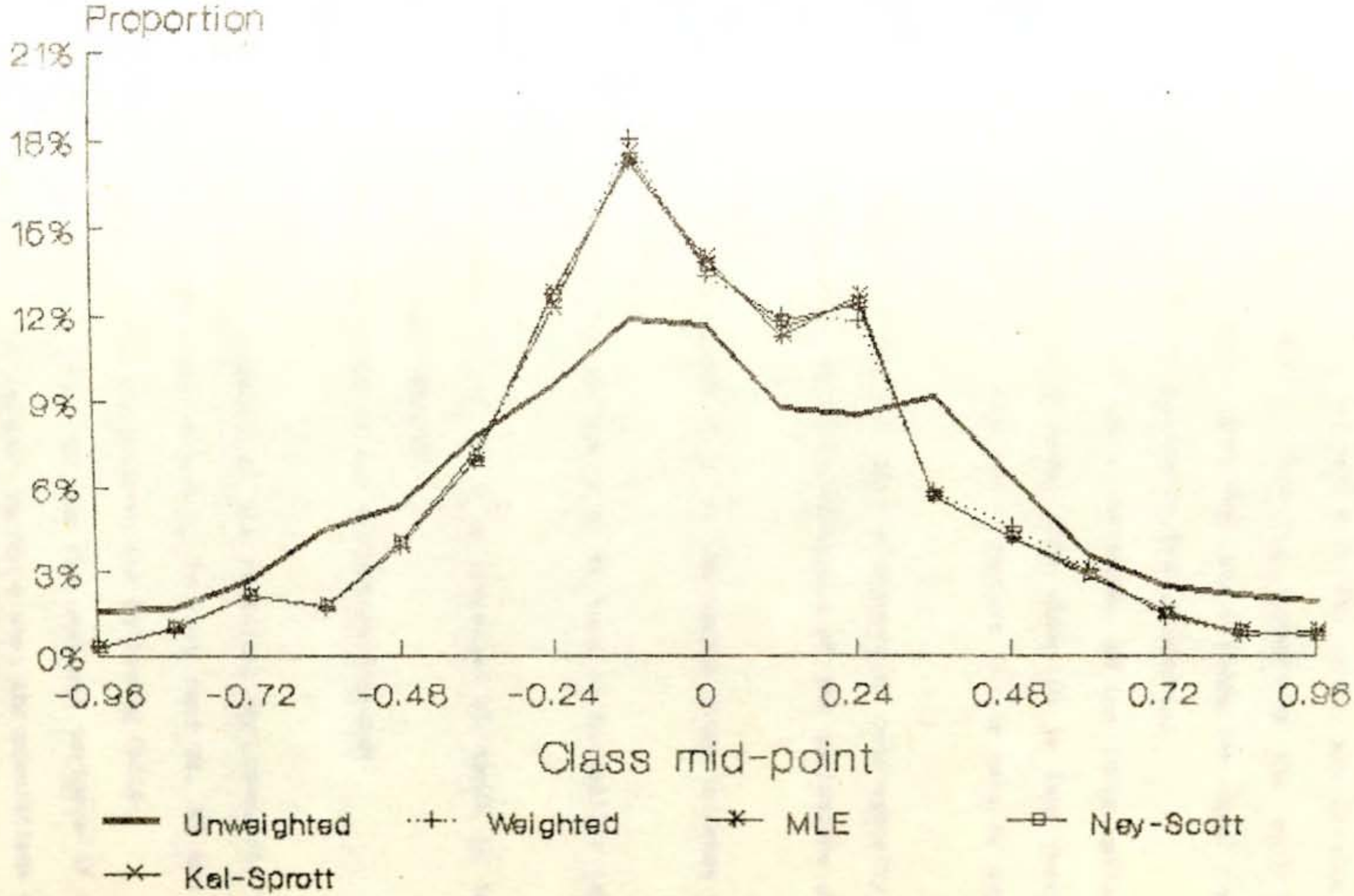
Figure 18 **Graphical display of the estimators when k=10, case C. variance pattern IV**



'igure 19 Graphical display of the estimators when k=3, case D, variance pattern IV



gure 20 Graphical display of the estimates when k=10, case D, variance pattern IV



4.2 Empirical Variances

The empirical variances of the estimates are summarized and tabulated in Tables 4.2, 4.3 and 4.4. The values are printed to eight significant digits. Table 4.2 summarizes the empirical variance of the estimates when the sample sizes are equal (m=20, m=25, m=30). Table 4.2 gives the following results:

1. Except in the case of equal variances, UW has large variance compared with the other methods. This shows UW is less precise than the other methods, with the exception of the case of equal varinaces.

2. For all k=3, 6, 8 and 10, as the degrees of heterogeneity of the σ_i^2's increase the empirical variances of the estimators also increase.

3. For all possible values of k, as the sample sizes increase the variances decrease.

4. As k increases, the variances of ML tend to be smaller than those of WT.

5. As the hetrogeneity of the σ_i^2's increases ML tends to have slightly small variance than WT.

6. As k increases variance of all estimators decrease.

Table 4.3 is the summary of the empirical variances for the unequal sample sizes cases A and B. In this case ML, NS and KS vary. From Table 4.3 one can observe the following facts:

1. In equal variances case, UW has the smallest variance of all other estimators but the largest variance when the populations are heterogeneous.

2. For all k=3, 6, 8 and 10, and sample sizes cases A and B, as the σ_i^2's become more diversed, the variances become larger.

3. Comparing the columns of cases A and B, the estimators have less variance in case B than case A.

4. As k increases the empirical variances of all estimators decrease.

5. The three estimators ML, NS, and KS have approximately equal empirical variances.

6. As k and/or the diversity of the σ_i^2's increase ML, NS, and KS haveslightly smaller variance than WT.

7. Except in very few cases, when k=3 and the σ_i^2's are less diversed, WT has slightly smaller variance than the other estimators.

Table 4.4 consists of empirical variances of the estimators when the sample sizes are positively associated with the σ_i^2's and viceversa (cases C and D). From Table 4.4, the following results are observed.

1. UW has large variances than the rest estimators except when the population variances are equal.

2. For all values of k and cases C and D, as the heterogeneity of the σ_i^2's increases the variance of all estimators increase.

3. The estimators have less variance in case D than in case C.

4. As k increases and/or the σ_i^2's are more diversed, ML, NS, and KS have less variances than WT.

5. As k increases the variances of all estimators decrease.

6. In all cases, ML, NS, and KS are equally precise.

Table 4.2: Empirical Variances of the Estimates, Equal Sample Size Case

Variance Pattern	k	m = 20 UW	m = 20 WT	m = 20 ML	m = 25 UW	m = 25 WT	m = 25 ML	m = 30 UW	m = 30 WT	m = 30 ML
1	3	0.01007687	0.00892965	0.00898909	0.00907136	0.00777241	0.0077905	0.007402001	0.00631132	0.00630659
	6	0.00467096	0.0334713	0.00335519	0.00394609	0.00278481	0.00278143	0.0031601	0.00229003	0.00229855
	8	0.00345508	0.00261272	0.00260224	0.0031886	0.00223366	0.00222501	0.00245346	0.00171509	0.00169341
	10	0.00261943	0.00174653	0.00171948	0.00229346	0.00146192	0.00144932	0.00179054	0.00115982	0.00115264
	3	0.01660604	0.01782912	0.01797339	0.01302334	0.01369854	0.01379632	0.01036989	0.01099612	0.01103202
	6	0.00786334	0.00893083	0.0088824	0.00617942	0.00670791	0.00667671	0.0051721	0.00555776	0.00546653
	8	0.00576894	0.00656958	0.00658019	0.00483904	0.00520591	0.00515664	0.00419943	0.00456222	0.00454605
	10	0.00546894	0.00598512	0.00584313	0.00415998	0.00458124	0.00456886	0.00332243	0.00361299	0.00360229
[illegible]	3	0.03218611	0.02837102	0.02837682	0.02705769	0.02275879	0.02282333	0.02216952	0.01805092	0.01808779
	6	0.02739526	0.02051499	0.02036038	0.02371528	0.01824752	0.01832634	0.01858165	0.01351079	0.0134472
	8	0.02648024	0.01979391	0.01975302	0.02274897	0.01637485	0.016323	0.01756151	0.01273156	0.01269689
	10	0.02862339	0.01921853	0.01905535	0.02121049	0.01391304	0.01374112	0.01673933	0.01150068	0.01146368
[illegible]	3	0.16752476	0.14965194	0.14935693	0.13827608	0.011735552	0.11691068	0.11119018	0.09699225	0.09676498
	6	0.15383194	0.11717351	0.11617717	0.11834332	0.08865499	0.08807985	0.10106934	0.07117342	0.07107032
	8	0.1415714	0.9876441	0.09849959	0.10917897	0.07620089	0.07579119	0.09237043	0.06586402	0.06536314
	10	0.13984476	0.1024638	0.1021063	0.11025725	0.06877145	0.06898031	0.09374065	0.06019159	0.06006403
[illegible]	3	0.33433489	0.28804475	0.2898183	0.284446217	0.2276109	0.22885572	0.22928459	0.19648612	0.19772385
	6	0.28197518	0.21221327	0.21213133	0.22723717	0.16817038	0.16809616	0.1856323	0.13244192	0.13231459
	8	0.27865417	0.19893962	0.19655282	0.2265155	0.16753496	0.16703285	0.18503335	0.12677494	0.12712292
	10	0.30008015	0.1892075	0.18808422	0.21258566	0.14499152	0.14334325	0.17611993	0.11100773	0.11064976

Table 4.3: Empirical Variances of the Estimates, Unequal Sample Size
Cases A & B

iance tern	k	Case A UW	Case A WT	Case A ML	Case A NS	Case A KS	Case B UW	Case B WT	Case B ML	Case B NS	Case B KS
al	3	0.01499015	0.01624495	0.01629848	0.01629766	0.01629722	0.01139424	0.01180854	0.01186529	0.0118652[illegible]	0.00118651[illegible]
	6	0.00756[illegible]13	0.00814086	0.00800934	0.00800576	0.00800207	0.00591917	0.0065239	0.00650667	0.00650[illegible]	0.00650[illegible]
	8	0.005792[illegible]	0.00624675	0.00615928	0.00615719	0.00615505	0.00460231	0.00484429	0.00481191	0.00481184	0.00481181
	10	0.00476591	0.00518085	0.00518083	0.00518072	0.00518076	0.00355495	0.00383193	0.0037777	0.00377[illegible]24	0.00377886
I	3	0.01088436	0.00917596	0.00922201	0.00922278	0.00922393	0.00186579	0.00689985	0.00690546	0.00690623	0.00690[illegible]
	6	0.00460216	0.00346803	0.00346277	0.00346324	0.00346385	0.00342477	0.00260384	0.00260879	0.00260875	0.0026087[illegible]
	8	0.00318038	0.00237402	0.00234281	0.00234221	0.0023416	0.0024357	0.00177206	0.00177696	0.00177676	0.00177656
	10	0.00279823	0.00172564	0.00170183	0.00170266	0.00170362	0.00193347	0.0012541	0.0012508	0.00125079	0.00125078
I	3	0.03197762	0.02744411	0.02742042	0.02742701	0.02743504	0.0258425	0.02140796	0.02135199	0.02135[illegible]	0.021366[illegible]
	6	0.02730526	0.02002711	0.01998744	0.01998946	0.01999228	0.02250556	0.0162194	0.01630487	0.01631051	0.016316[illegible]
	8	0.02688646	0.01900537	0.01888861	0.0188954	0.01890335	0.01913529	0.0136436	0.0136774	0.01367684	0.01367636
	10	0.002628628	0.01774395	0.01765705	0.01765687	0.01765713	0.01977979	0.01431865	0.04120624	0.01420535	0.0142045[illegible]
I	3	0.15347041	0.13841144	0.13861098	0.13862399	0.13864212	0.11525261	0.09671447	0.09664918	0.09665396	0.0966602[illegible]
	6	0.13909313	0.10165325	0.10191606	0.10195843	0.10200819	0.10069805	0.08100552	0.08071669	0.080707	0.08069727
	8	0.13229647	0.09501199	0.09546637	0.09547232	0.095481	0.10401716	0.07051518	0.07082415	0.01083673	0.07085099
	10	0.13090337	0.0900043	0.09048145	0.09050411	0.09053116	0.1031894	0.06624202	0.0657842	0.06578579	0.0657882[illegible]
V	3	0.29940833	0.26644647	0.26853695	0.26836077	26817446	0.24054447	0.20859918	0.21037007	0.21034954	0.21032947
	6	0.25228663	0.19420087	0.19377136	0.19375226	0.19373672	0.21735124	0.1691289	0.16857008	0.1685746[illegible]	0.1685809[illegible]
	8	0.25372787	0.17067312	0.16939551	0.16942232	0.16945604	0.20596502	0.14141021	0.1411881	0.14118997	0.1411[illegible]
	10	0.25761717	0.18032616	0.17960688	0.17962115	0.17964131	0.19266372	0.13404122	0.1334985	0.1334894[illegible]	0.13348111

Table 4.4: Empirical Variances of the Estimates, Unequal Sample Size
Cases C & D

nce r'n	k	Case C UW	Case C WT	Case C ML	Case C NS	Case C KS	Case D UW	Case D WT	Case D ML	Case D NS	Case D KS
	3	0.01356813	0.01411513	0.01399951	0.01399804	0.01399791	0.01356813	0.01411513	0.01399951	0.01399804	0.01399791
	6	0.00709651	0.0074173	0.00741761	0.00741693	0.00741701	0.00709651	0.0074173	0.00741761	0.00741693	0.00741701
	8	0.00504324	0.00556007	0.00554916	0.00555005	0.00555167	0.00504324	0.00556007	0.00554916	0.00555005	0.00555167
	10	0.00399727	0.004486	0.00439928	0.00439784	0.00439682	0.00399727	0.004486	0.00439928	0.00439784	0.00439682
	3	0.00931404	0.0078036	0.00783159	0.00783708	0.00784408	0.00819415	0.0069365	0.00691342	0.00690937	0.00690557
	6	0.00374003	0.00287847	0.00287402	0.00287305	0.00287235	0.00352135	0.00250514	0.00250459	0.0025364	0.0025028
	8	0.00289417	0.00222963	0.00222842	0.00222899	0.00222875	0.00255624	0.00178335	0.00178716	0.00178678	0.00178649
	10	0.0023647	0.00164933	0.00165543	0.001656	0.0016568	0.00214121	0.00137401	0.00137651	0.00137655	0.00137667
	3	0.03057186	0.02628152	0.02622657	0.02623838	0.02625442	0.02627517	0.02151892	0.02143154	0.02141133	0.02139137
	6	0.02457816	0.01899488	0.01880251	0.01879942	0.01879845	0.02241873	0.01605257	0.01604347	0.01603582	0.01602884
	8	0.02333715	0.01588883	0.01585129	0.01585864	0.01586894	0.01912187	0.01345427	0.01322191	0.01321924	0.01321741
	10	0.022392	0.01621444	0.01603795	0.01602948	0.01602192	0.01992807	0.01288795	0.01273695	0.01273434	0.01273233
	3	0.014151255	0.12500431	0.12625616	0.12623292	0.12622244	0.12383119	0.10789944	0.10783734	0.10784604	0.10786419
	6	0.12180079	0.09789507	0.0976641	0.09764696	0.09763974	0.09799361	0.07229392	0.07203059	0.07206415	0.07210674
	8	0.10742859	0.08300046	0.08330144	0.08325663	0.08321872	0.09616621	0.06844056	0.06803363	0.06804898	0.06807097
	10	0.12189141	0.08309064	0.08192329	0.08196687	0.08202391	0.10025737	0.07141594	0.07100431	0.07101857	0.07103798
	3	0.31206907	0.25503182	0.25459822	0.25505513	0.2555803	0.24083107	0.20685975	0.20784916	0.20777263	0.20770717
	6	0.25238249	0.20075134	0.19990156	0.19985927	0.19983519	0.21980172	0.17497696	0.17455112	0.17461679	0.17469974
	8	0.23331433	0.17604234	0.17488118	0.17482132	0.17477575	0.22423337	0.1522894	0.15052377	0.15042557	0.15033061
	10	0.22907821	0.15687466	0.15547253	0.15544257	0.15542743	0.20040834	0.12568122	0.12543435	0.12539127	0.12535276

4.3 Relative Efficiency of the Estimators

We use the variance of an estimator to measure its performance for a given distribution. However, to compare estimators we use the notion of relative efficiency. The relative efficiencies of the five estimators are computed and tabulated in Tables 4.5 and 4.6. They are calculated as

$$\frac{\text{Empirical variance of the BLUE}}{\text{Empirical variance of the estimator}} \times 100.$$

The relative efficiencies of the estimators for k=3 and k=10 are plotted for various sample sizes case and σ_i^2 patterns. These plots are on Figures A to P. A carefull comparison of these graphs lead to the results dicussed below.

i) Equal n_i's

When the σ_i^2's are equal, UW is 100% efficient. But for variance patterns I to IV, UW is substantially less efficient than the alternate estimators. Comparing Figures A to D, WT is slightly efficient than ML when k is small. As k increases ML is slightly efficient than WT. Also as k increases UW becomes less and less efficient than WT, and ML. For instance comparing Figures A and D, the efficiency decreases from 80% to 57%.

ii) Uneqaul n_i's

Figures E, F, G and H are plots for k=3 and k=10, for cases A and B. In this case the estimators ML, NS, and KS are not the same, but they have the same precision. When the σ_i^2's vary UW has least efficiency, even when the diversity is less. For k=3, WT is slightly more efficient than ML, NS, and KS. As k increases the reverse is true. Figures I, J, K and L are the graphs of the relative efficiencies for cases C and D. The same result as above is observed except the efficiencies of ML, NS, and KS slightly increase.

The last four figures are plotted to examine the performance of the estimators when the sample sizes vary with a fixed variance pattern. Figures M, N, O and P give the following results:

1. When k=3, WT is slightly efficient than ML in all choices of the sample sizes.
2. When k=10, ML is slightly efficient than UW in all choices of the sample sizes.
3. In all sample sizes chosen UW has less efficiency, its efficiency reduces when the diversities of σ_i^2's increase.

In chapter 5 the results of the simulation study will be summarized and discussed. Moreover, these results will be discussed with regard to the previous related studiès.

Table 4.5: Efficiency of the Estimates Relative to the BLUE, Equal Sample Sizes Case

K	Equal			I			II			III			IV		
	UW	WT	ML	UW	WT	ML	UW	WT	ML	UW	WT	ML	UW	WT	ML
3	100.00	93.14	92.39	83.80	94.56	93.98	85.02	96.46	96.44	84.15	94.20	94.38	80.28	93.19	92.61
6	100.00	88.05	88.53	66.36	92.69	92.39	69.04	92.19	92.89	72.27	94.88	95.69	70.66	93.89	93.92
8	100.00	88.03	87.89	68.28	90.29	90.65	66.85	89.43	89.61	63.54	91.07	91.32	63.79	89.36	90.42
10	100.00	91.38	93.60	60.25	90.40	91.82	60.31	89.82	90.59	66.13	90.25	90.57	57.66	91.45	92.00
3	100.00	95.07	94.40	79.92	93.27	93.06	79.65	94.69	94.42	80.07	94.35	94.70	78.18	97.70	97.17
6	100.00	92.12	92.55	65.44	92.73	92.85	71.32	92.69	92.29	70.58	94.22	94.83	67.98	91.85	91.89
8	100.00	92.95	93.84	65.36	93.30	93.67	65.76	91.35	92.64	64.64	92.61	93.11	68.71	92.90	93.18
10	100.00	90.60	91.05	56.97	89.38	90.16	62.46	95.22	96.41	59.10	94.75	94.46	63.44	93.01	94.08
3	100.00	94.31	94.00	82.82	97.14	97.21	77.54	95.23	95.04	82.35	94.41	94.63	82.14	95.85	95.25
6	100.00	93.06	94.61	68.82	94.97	94.61	69.80	96.00	96.46	67.19	95.41	95.55	70.35	98.61	98.70
8	100.00	92.05	92.38	65.83	94.17	95.38	69.15	95.38	95.65	67.15	94.18	94.90	63.96	93.35	93.09
10	100.00	91.96	92.23	60.62	93.59	94.17	64.25	93.51	93.81	60.28	93.88	94.08	58.92	93.49	93.79

Table 4.6f Efficiency of the Estimates Relative to the BLUE, Equal Sample Sizes Case

K	Equal UW	Equal WT	Equal ML	Equal NS	Equal KS	I UW	I WT	I NL	I NS	I KS	II UW	II WT	II NL	II NS	II KS	III UW	III WT	III ML	III NS	III KS	IV UW	IV WT	IV ML	IV NS	IV KS
3	100.00	92.28	91.97	91.98	91.98	79.98	94.87	94.40	94.39	94.38	82.12	95.68	95.77	95.74	95.72	83.98	93.11	92.98	92.97	92.96	88.38	99.31	98.58	98.60	98.67
6	100.00	93.13	94.66	94.70	94.74	68.46	90.85	90.98	90.97	90.96	69.47	94.57	94.76	94.75	94.74	68.26	93.40	93.16	93.12	93.08	71.52	92.22	93.12	93.13	93.14
8	100.00	92.73	94.04	94.08	94.11	68.12	91.26	92.47	92.50	[illegible]	65.64	92.82	93.50	93.56	93.52	67.46	93.94	93.49	93.49	93.48	62.35	92.59	93.39	93.37	[illegible]
10	100.00	91.99	91.99	91.99	91.99	58.57	94.97	[illegible]	96.75	[illegible]	[illegible]	92.99	93.45	93.46	93.45	63.00	91.63	91.15	91.13	91.10	64.84	92.63	93.00	93.00	92.99
3	100.00	96.49	96.05	96.03	96.03	84.07	95.83	95.76	97.75	[illegible]	79.10	95.49	95.74	95.71	95.67	80.62	96.08	96.14	96.14	96.15	85.80	98.08	98.11	98.12	98.13
6	100.00	90.73	90.97	90.98	91.00	72.87	95.85	95.67	95.67	95.67	67.01	92.99	92.50	92.41	92.43	76.20	94.72	95.06	95.07	95.08	[illegible]	93.07	93.58	93.38	[illegible]
8	100.00	95.00	95.64	95.65	95.65	69.86	96.03	95.76	95.77	95.78	67.54	94.73	94.49	94.50	94.50	64.27	94.81	94.59	94.58	94.56	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]
10	100.00	92.77	94.70	94.09	94.01	61.19	94.33	94.58	94.58	94.58	67.60	92.94	93.57	93.68	93.68	69.25	92.29	92.94	92.93	92.93	64.51	[illegible]	[illegible]	92.81	[illegible]
3	100.00	96.12	96.92	96.93	96.93	80.24	95.77	95.33	95.37	95.28	82.08	95.48	95.88	95.63	95.58	84.97	96.19	95.24	95.26	95.27	75.52	92.81	92.51	92.80	[illegible]
6	100.00	95.68	95.67	95.68	95.68	73.70	95.76	95.91	95.94	95.96	72.39	93.67	94.63	94.65	95.65	73.28	91.85	92.06	92.08	92.09	72.77	91.89	91.88	91.90	[illegible]
8	100.00	90.70	90.88	90.87	90.84	73.33	95.18	95.91	95.21	95.27	63.06	92.65	92.87	92.83	92.77	70.85	91.70	91.37	91.42	91.46	69.92	92.07	93.39	93.32	[illegible]
10	100.00	89.11	90.86	90.89	90.91	64.14	91.95	91.61	91.58	91.54	66.49	91.82	92.84	92.88	92.93	61.58	90.33	91.62	91.57	91.51	61.48	89.78	90.50	90.61	[illegible]
3	100.00	96.12	96.92	96.93	96.93	81.70	96.51	96.83	96.89	96.94	79.85	97.50	97.90	97.99	98.08	83.46	95.79	95.84	95.83	95.82	93.35	96.30	96.38	96.38	[illegible]
6	100.00	95.68	95.67	95.68	95.68	66.71	93.77	93.79	93.83	93.86	66.48	92.85	92.90	92.95	92.99	71.12	96.40	96.76	96.71	96.65	74.54	93.68	93.87	93.83	[illegible]
8	100.00	90.70	90.88	90.87	90.84	65.87	92.42	94.22	94.28	94.26	65.24	92.73	94.36	94.38	94.39	66.28	93.13	93.68	93.66	93.63	63.91	94.10	95.21	95.27	[illegible]
10	100.00	89.11	90.86	90.89	90.91	60.61	94.45	94.28	94.28	94.27	60.57	93.66	94.77	94.79	94.81	66.04	92.71	93.25	93.23	93.20	58.66	93.53	93.72	93.75	[illegible]

Figure A Relative efficiency of the estimators to the BLUE ; k=3, m=20

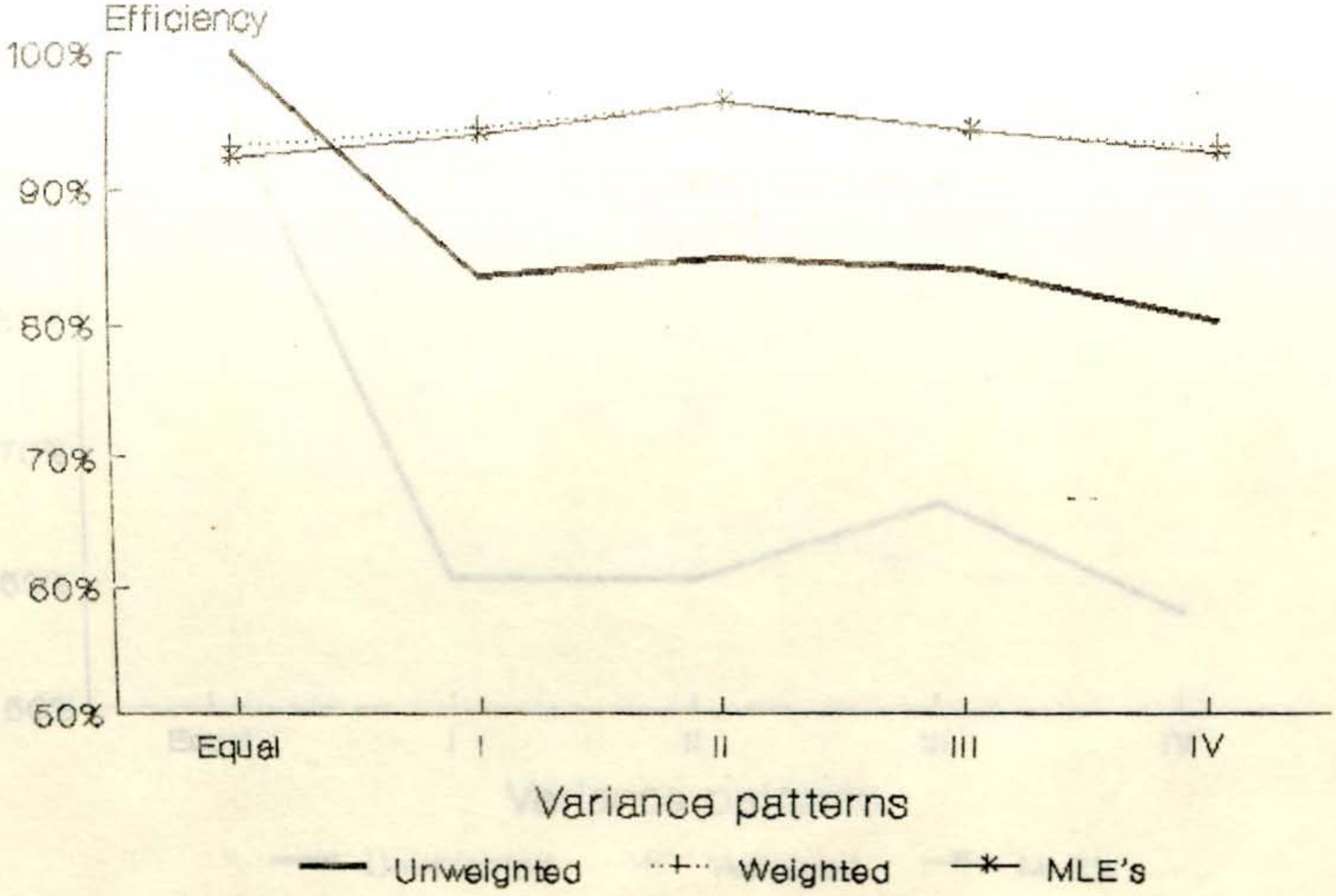


Figure B Relative efficiency of the estimators to the BLUE; k=10, m=20

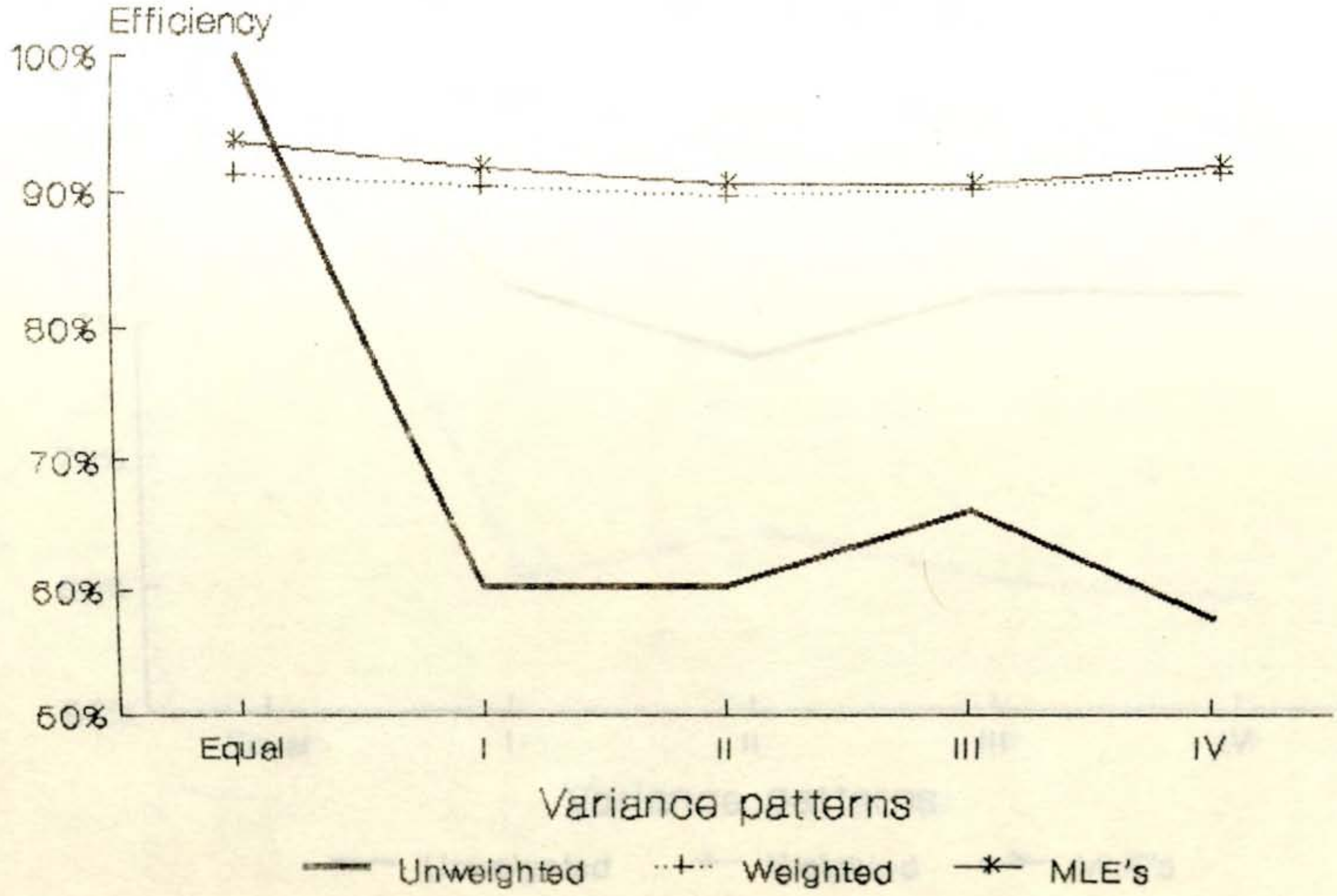


Figure C **Relative efficiency of the estimators to the BLUE; k=3, m=30**

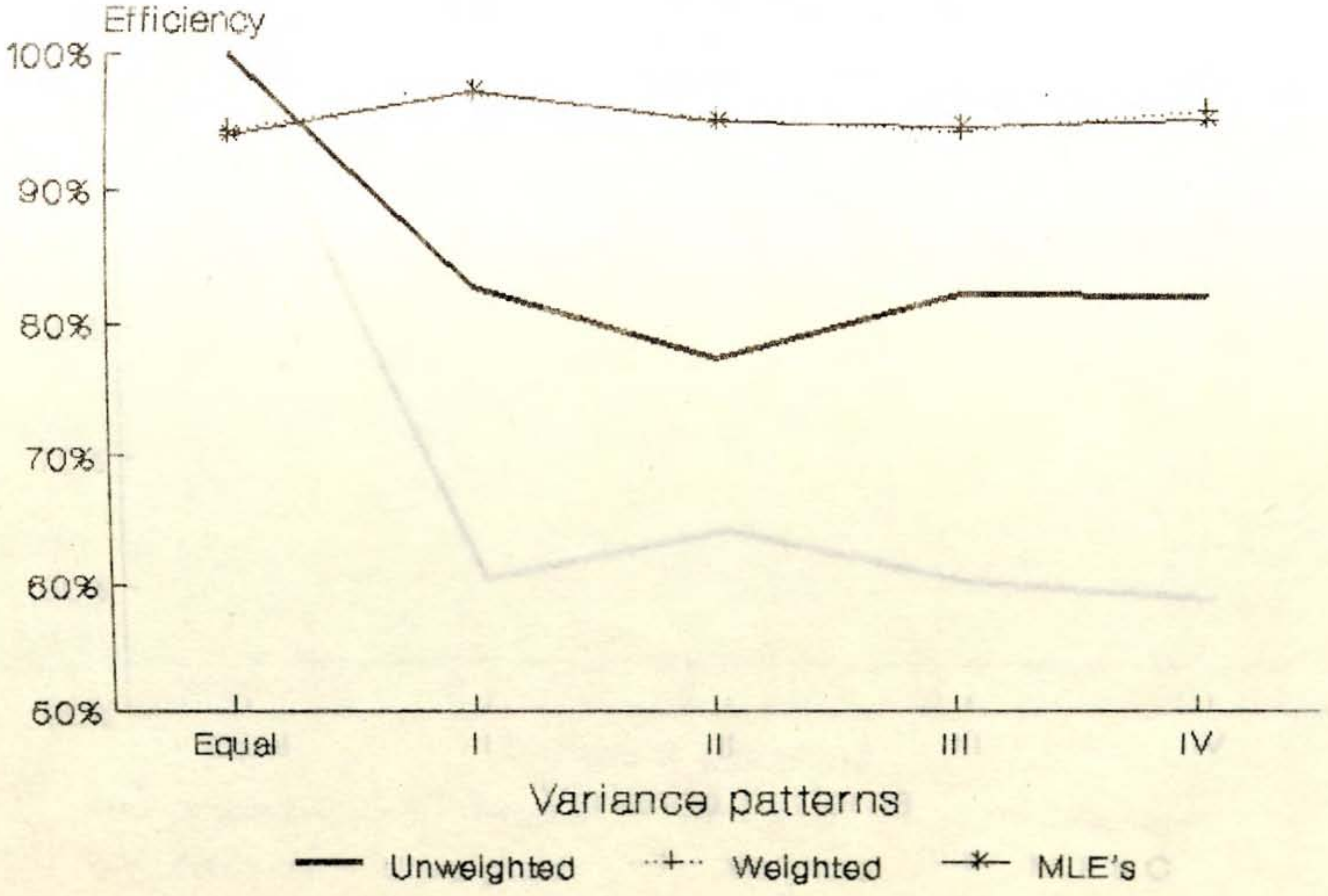


Figure D Relative efficiency of the estimators to the BLUE; k=10, m=30

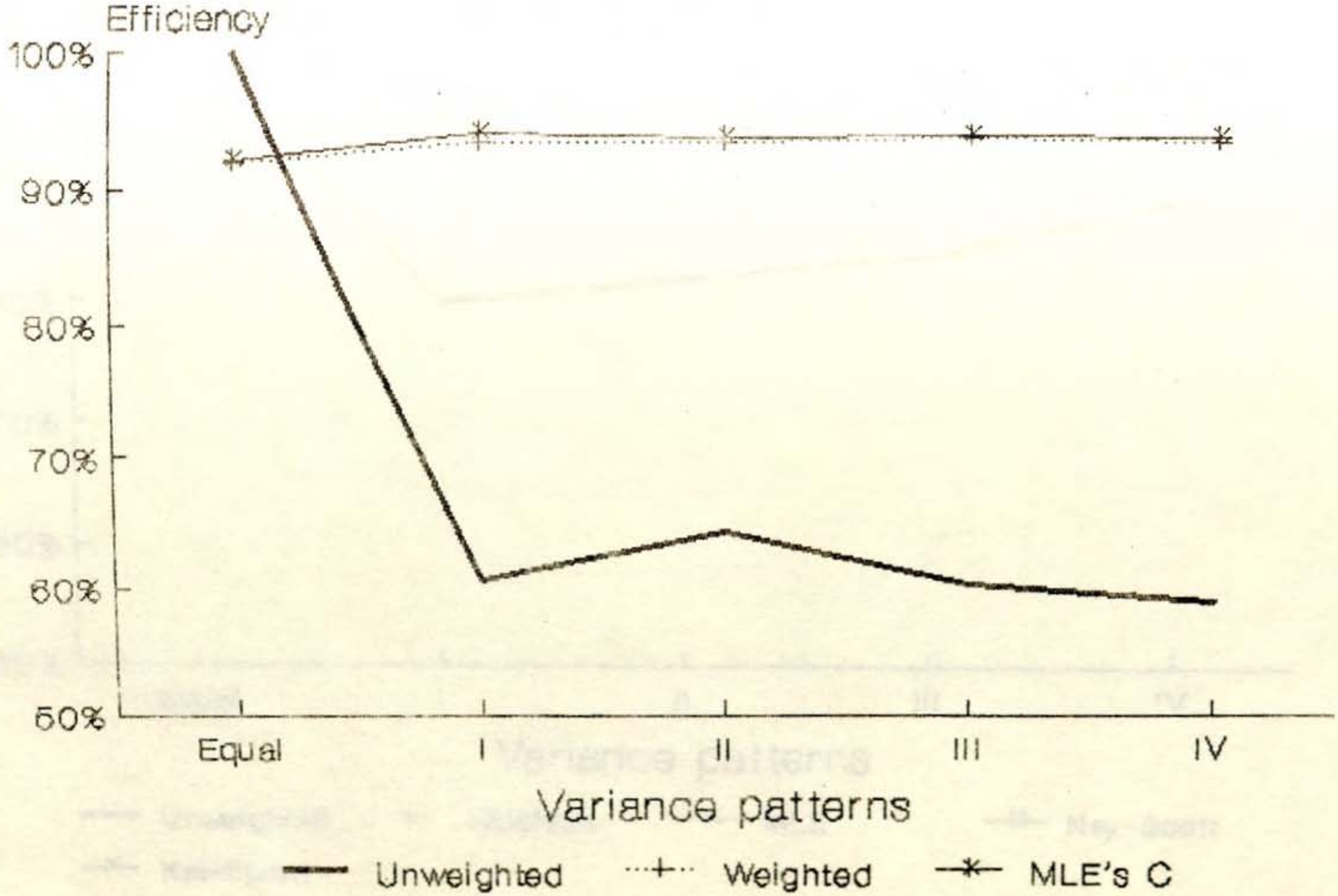


Figure E **Relative efficiency of the Estimators to the BLUE; k=3, case A**

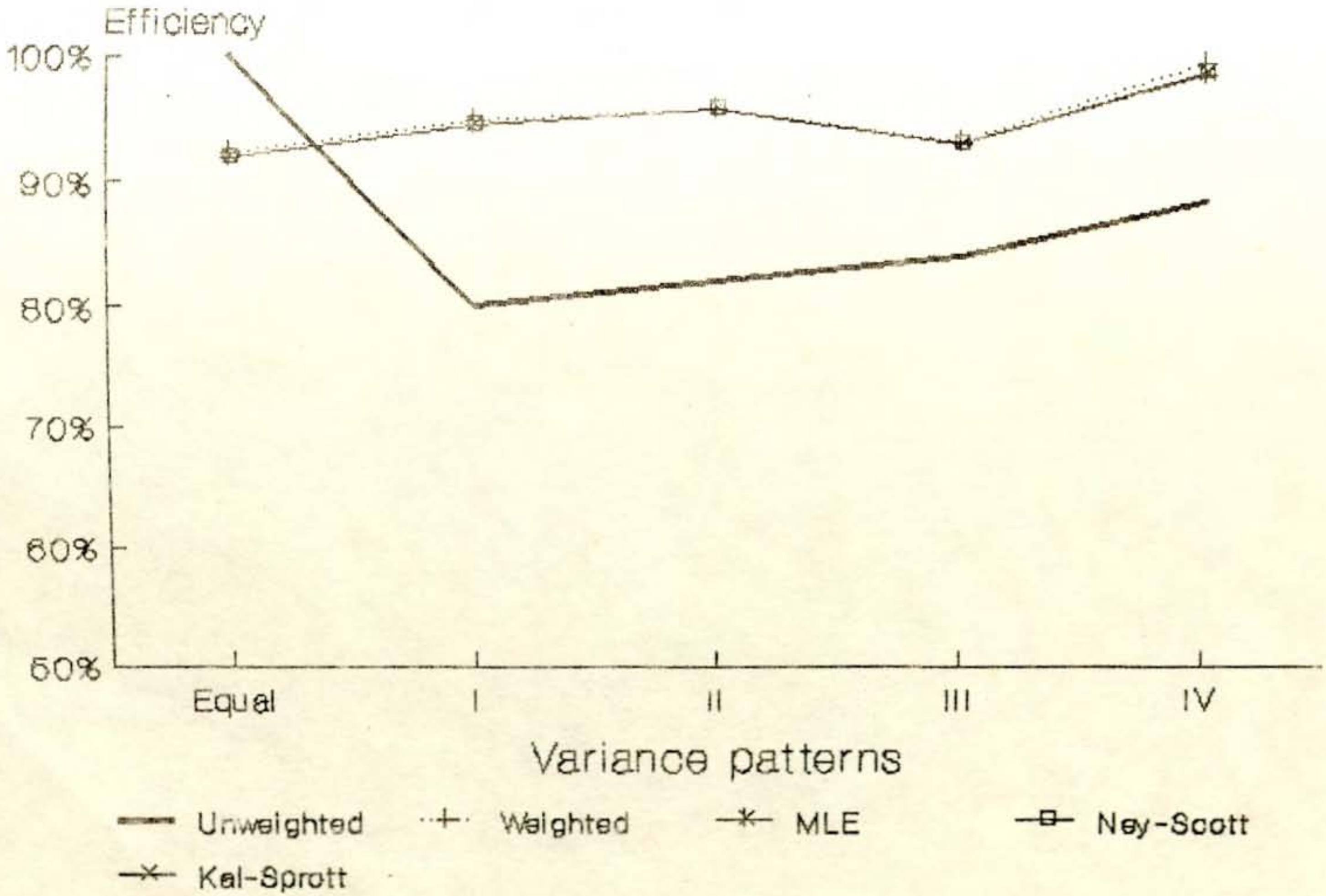


Figure F Relative efficiency of the estimators to the BLUE; k=10, case A

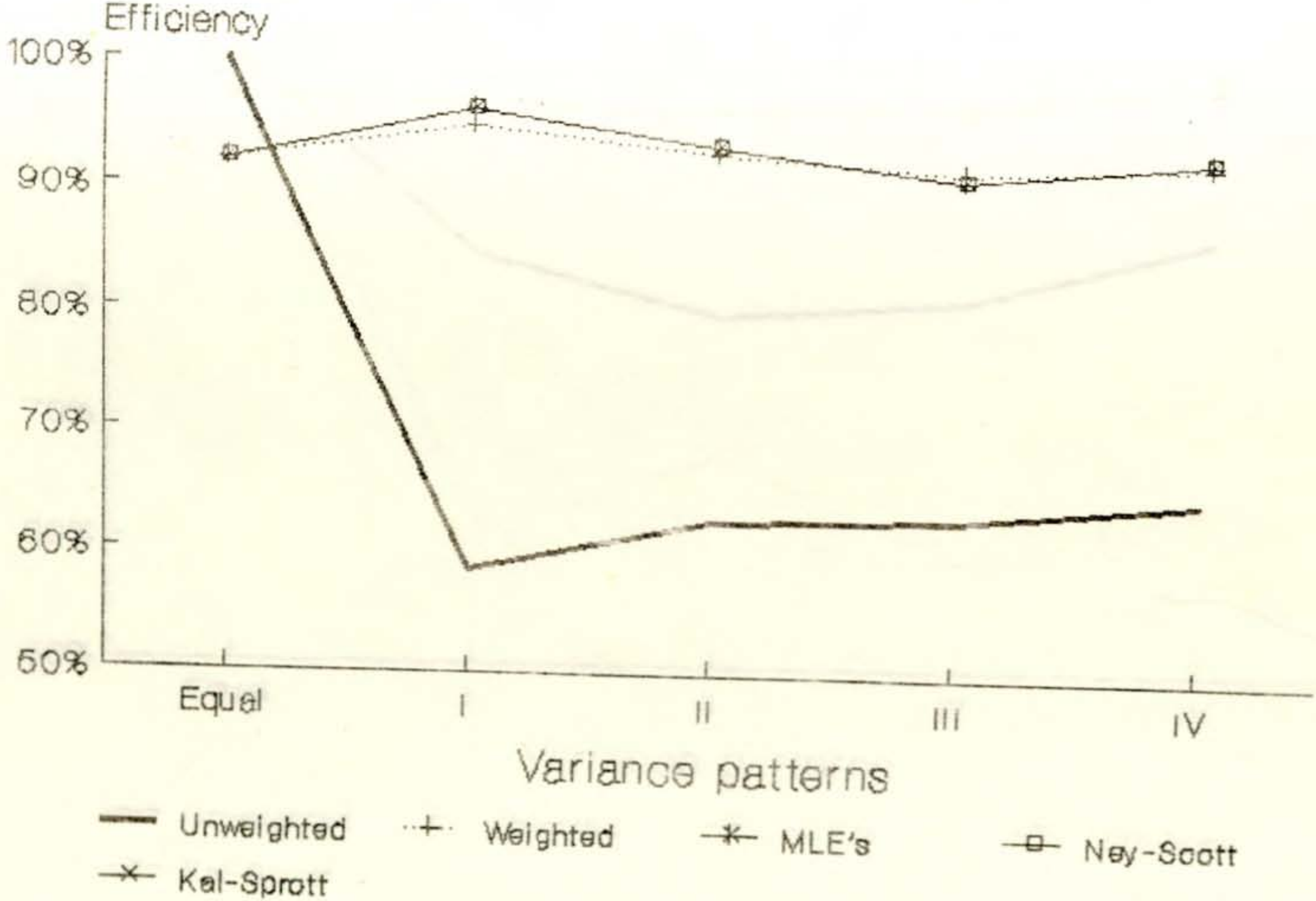


Figure G Relative efficiency of the estimators to the BLUE; k=3, case B

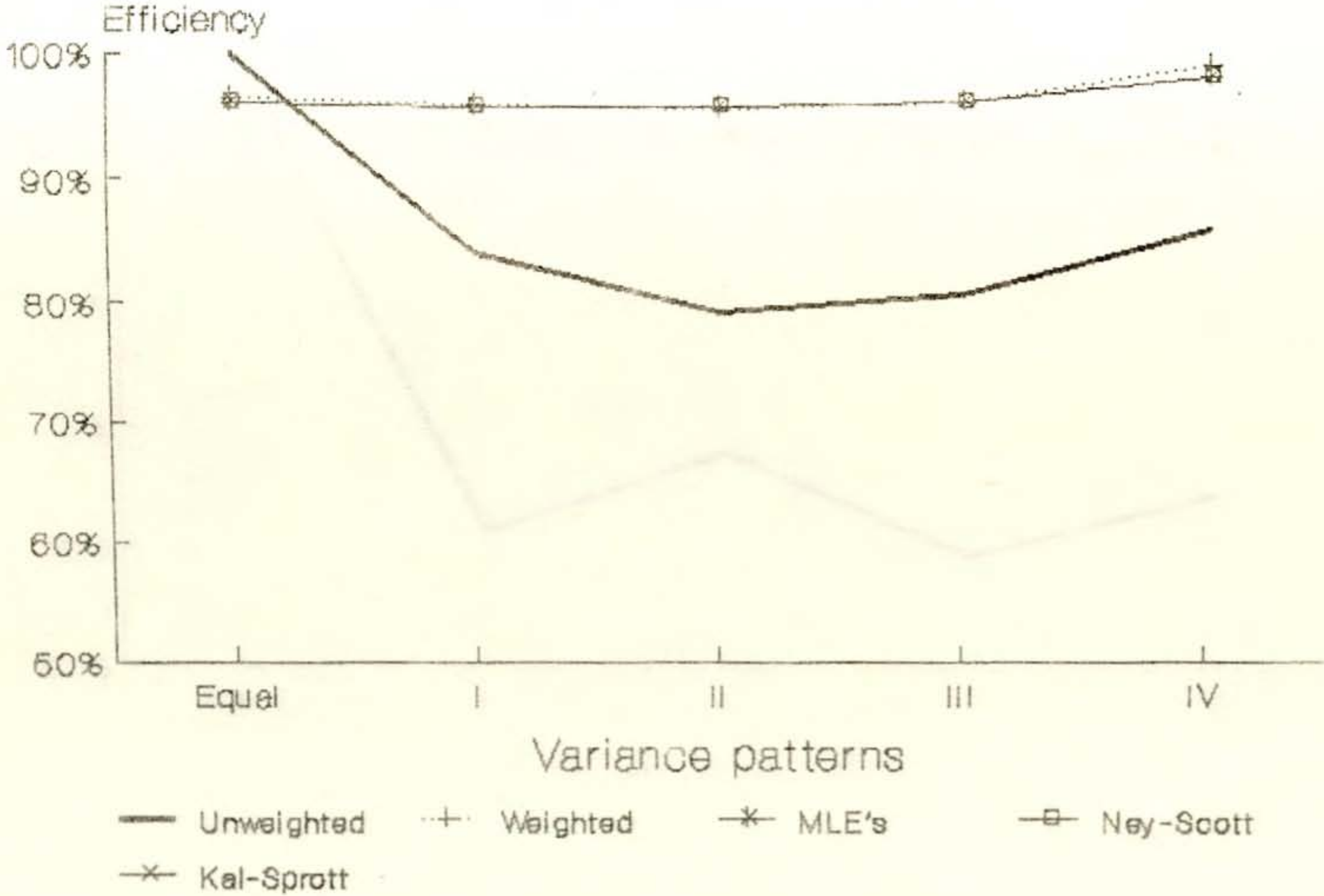


Figure H Relative efficiency of the estimators to the BLUE; k=10, case B

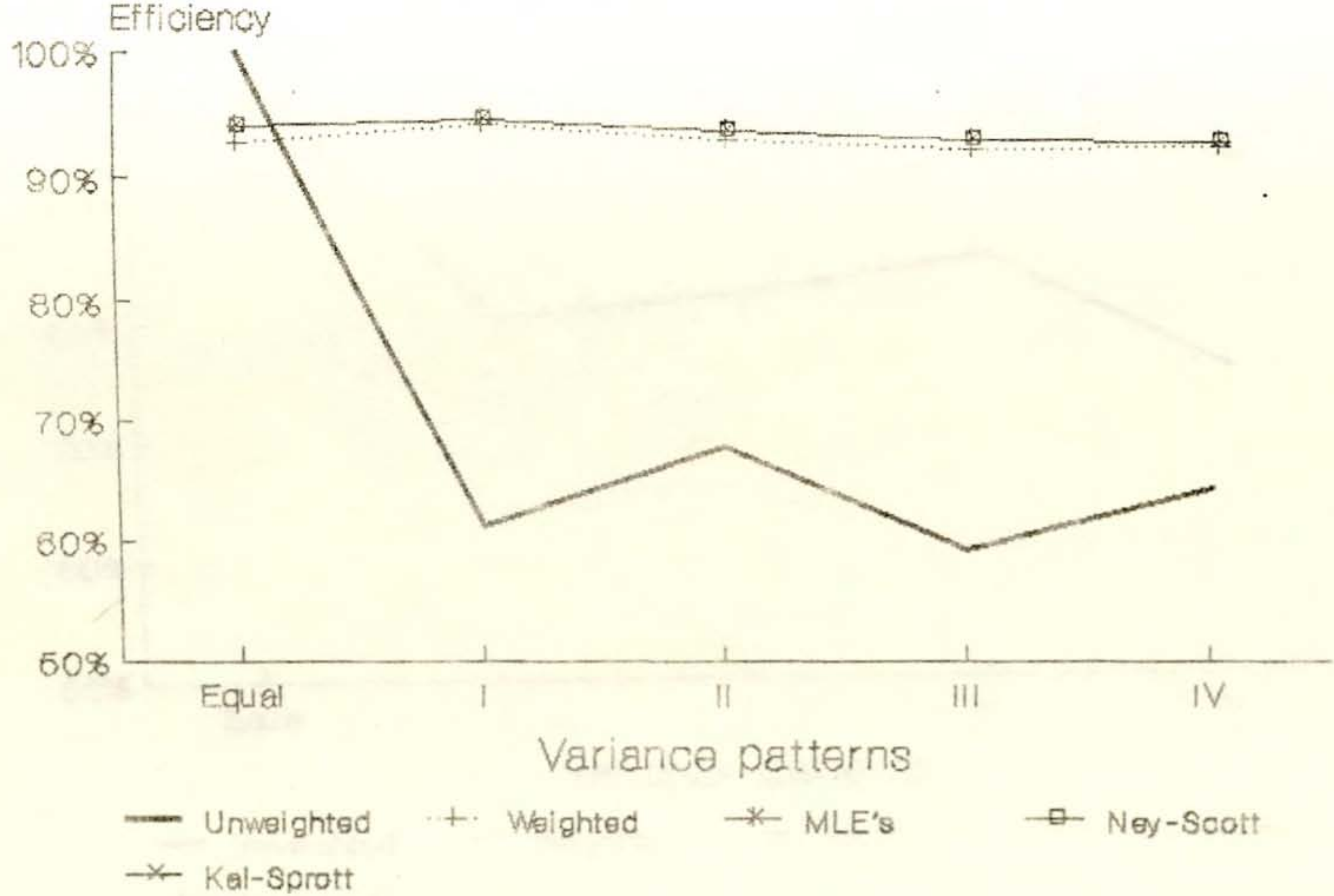


Figure I Relative efficiency of the estimators to the BLUE; k=3, case C

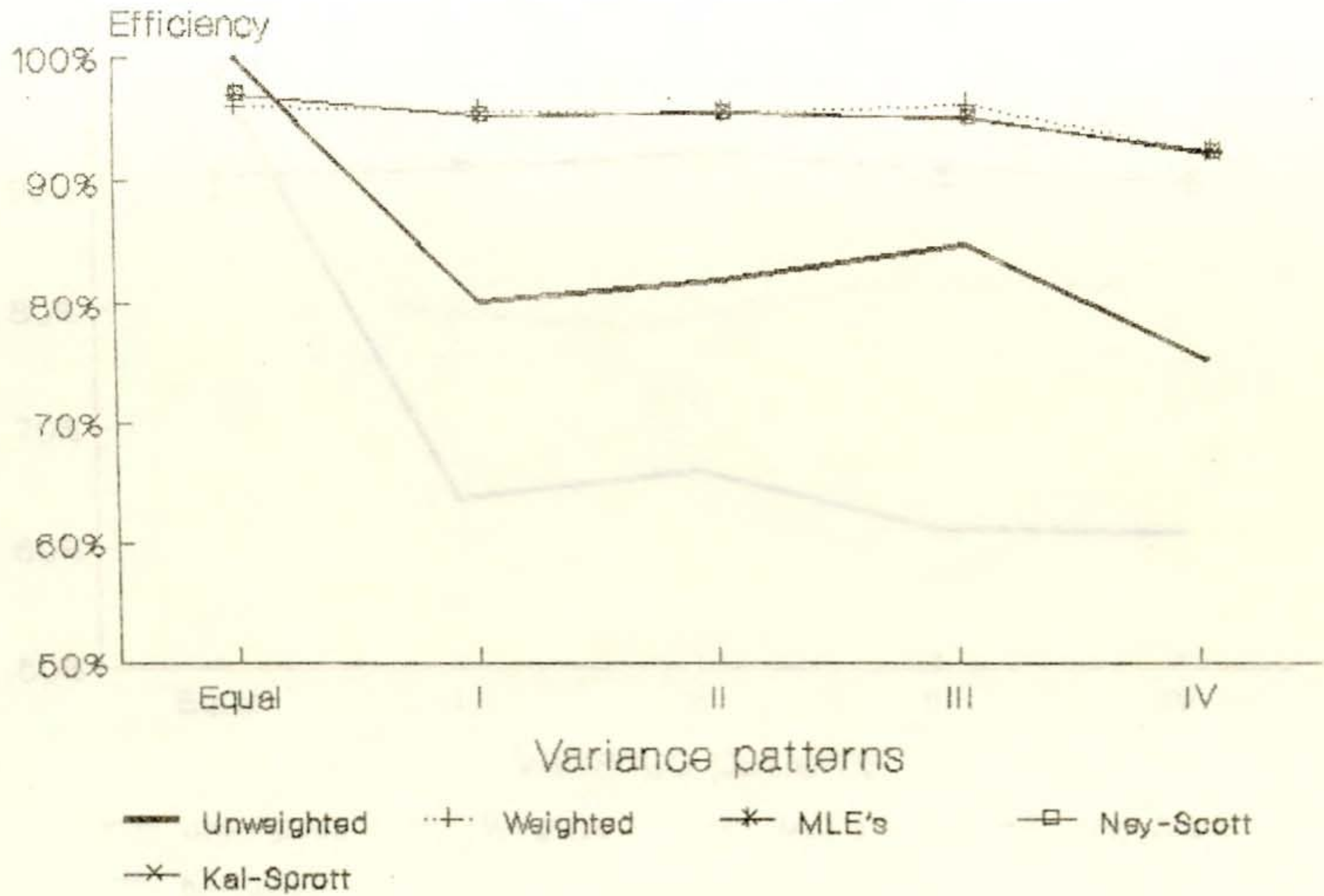
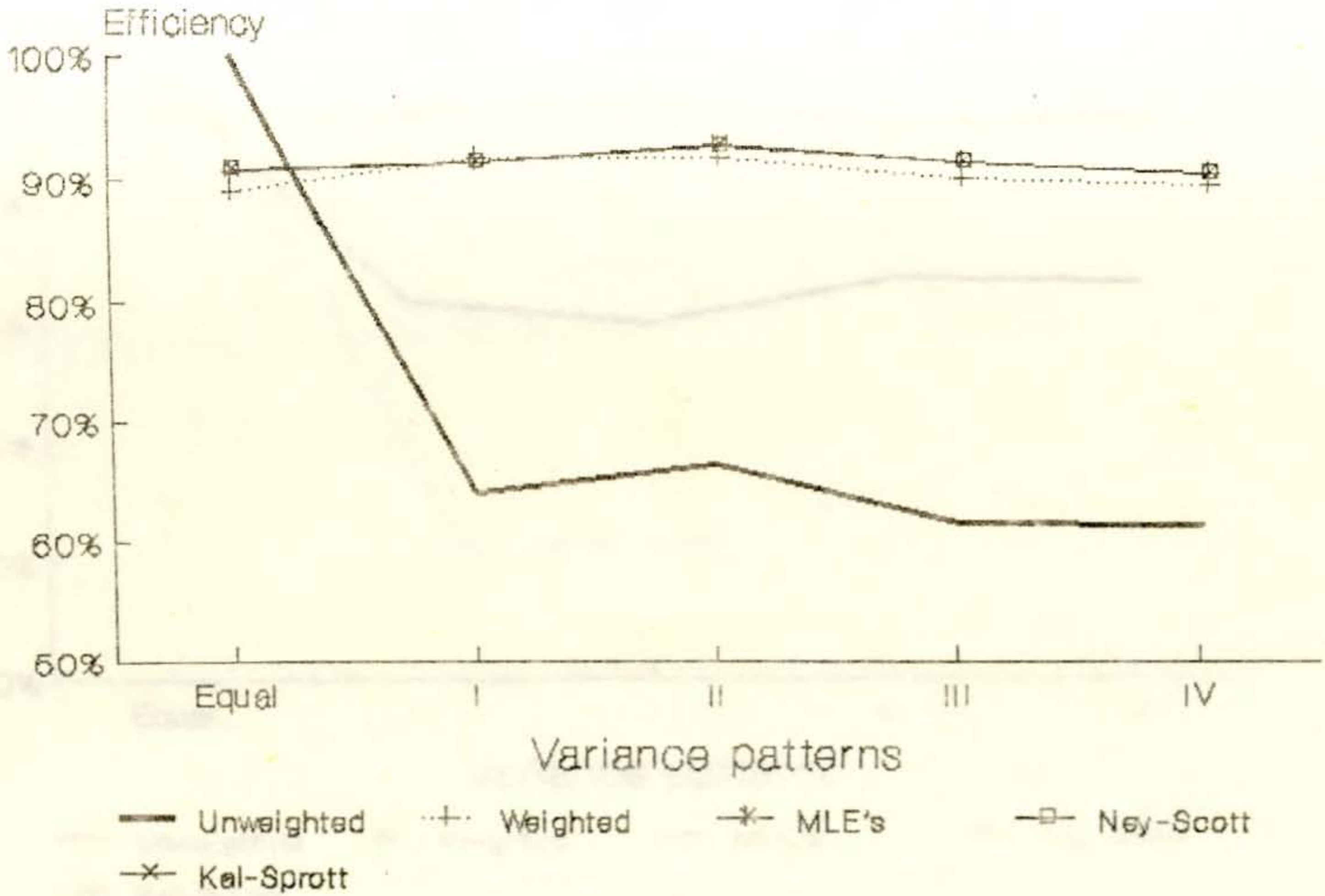


Figure J

Relative efficiency of the estimators to the BLUE; k=10, case C



gure K Relative efficiency of the estimators to the BLUE; k=3, case D

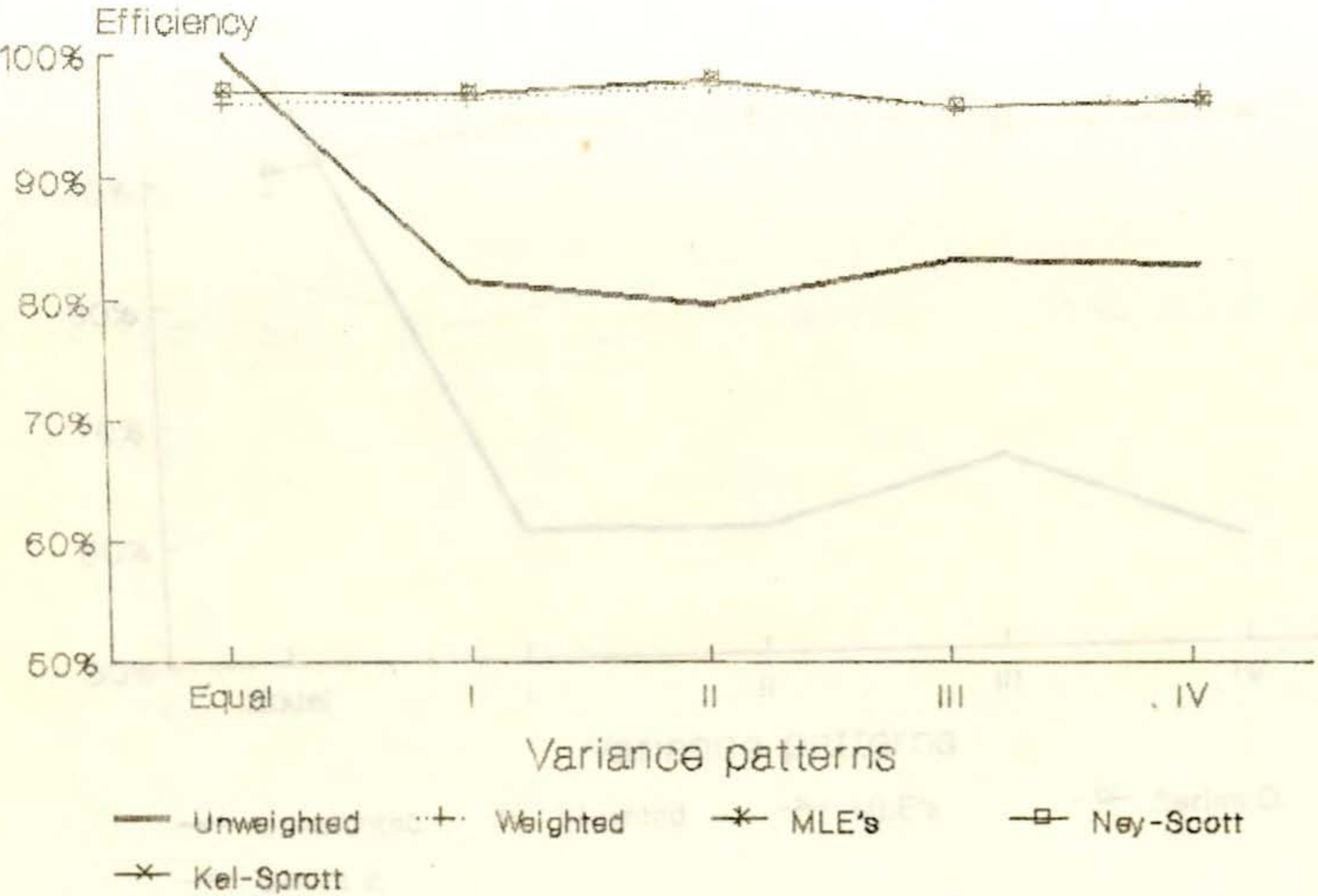


Figure L Relative efficiency of the estimators to the BLUE; k=10, case D

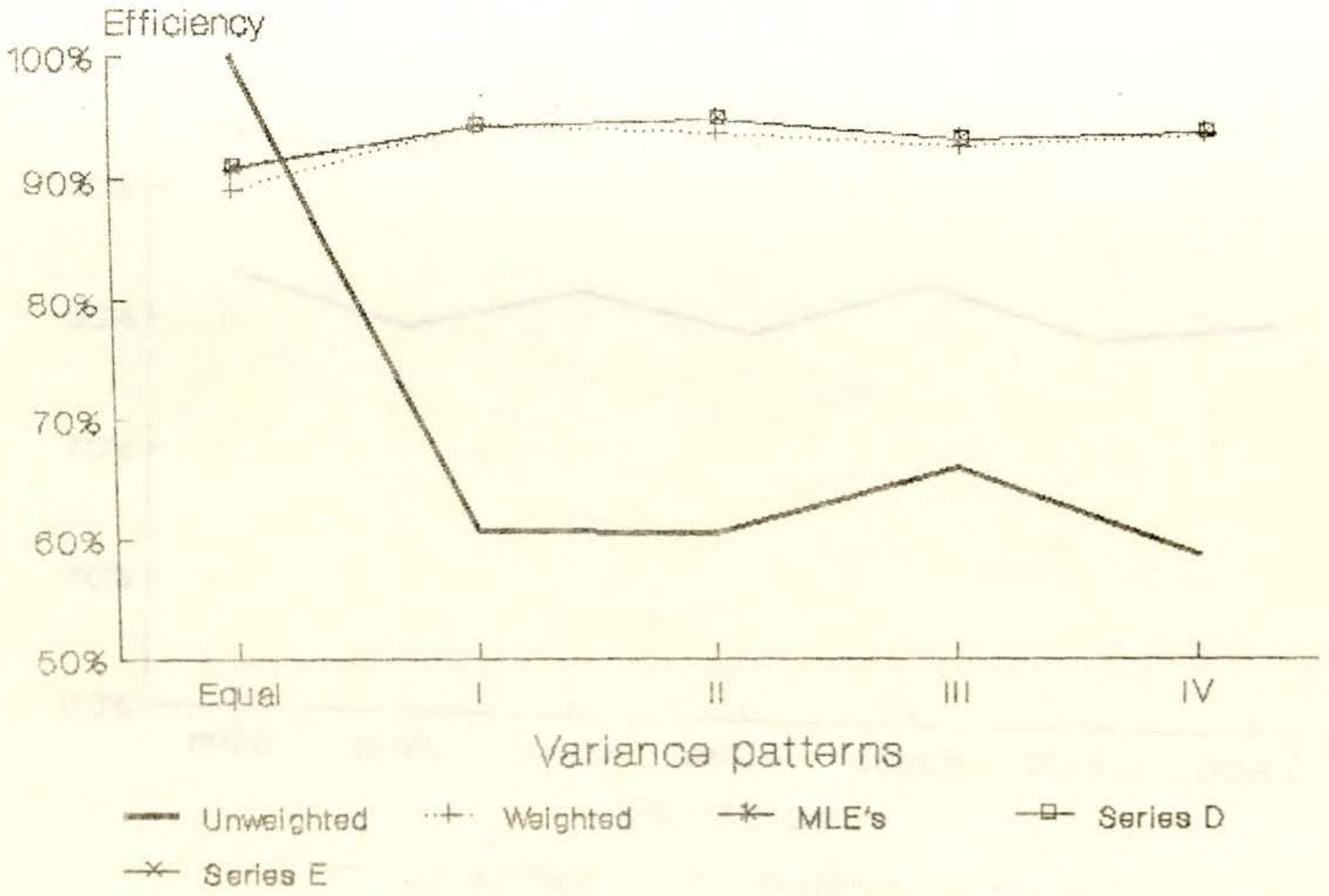


Figure M **Relative efficiency of the estimators to the BLUE; k=3, variance pattern I**

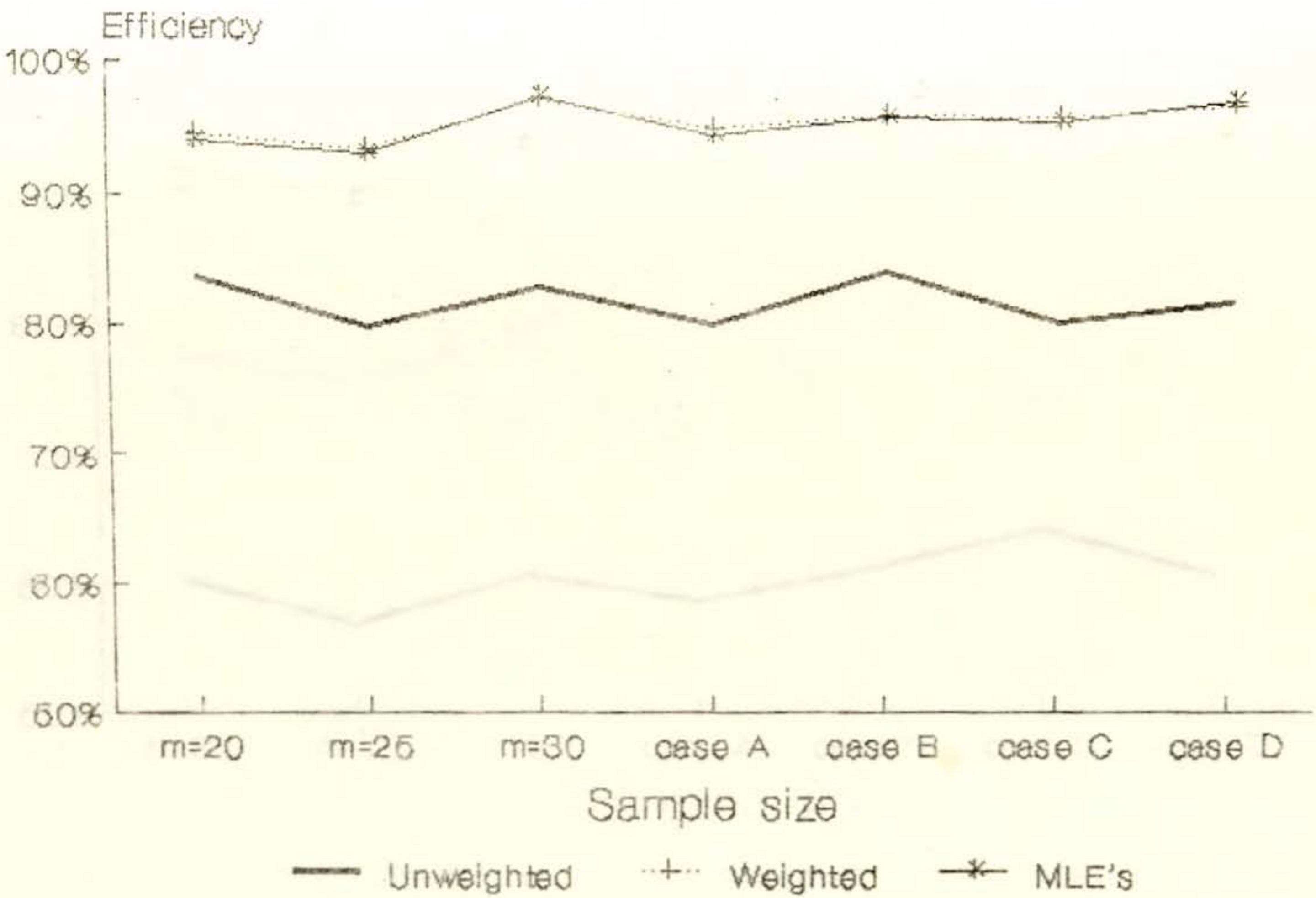


Figure O Relative efficiency of the estimators to the BLUE; k=3, variance pattern IV

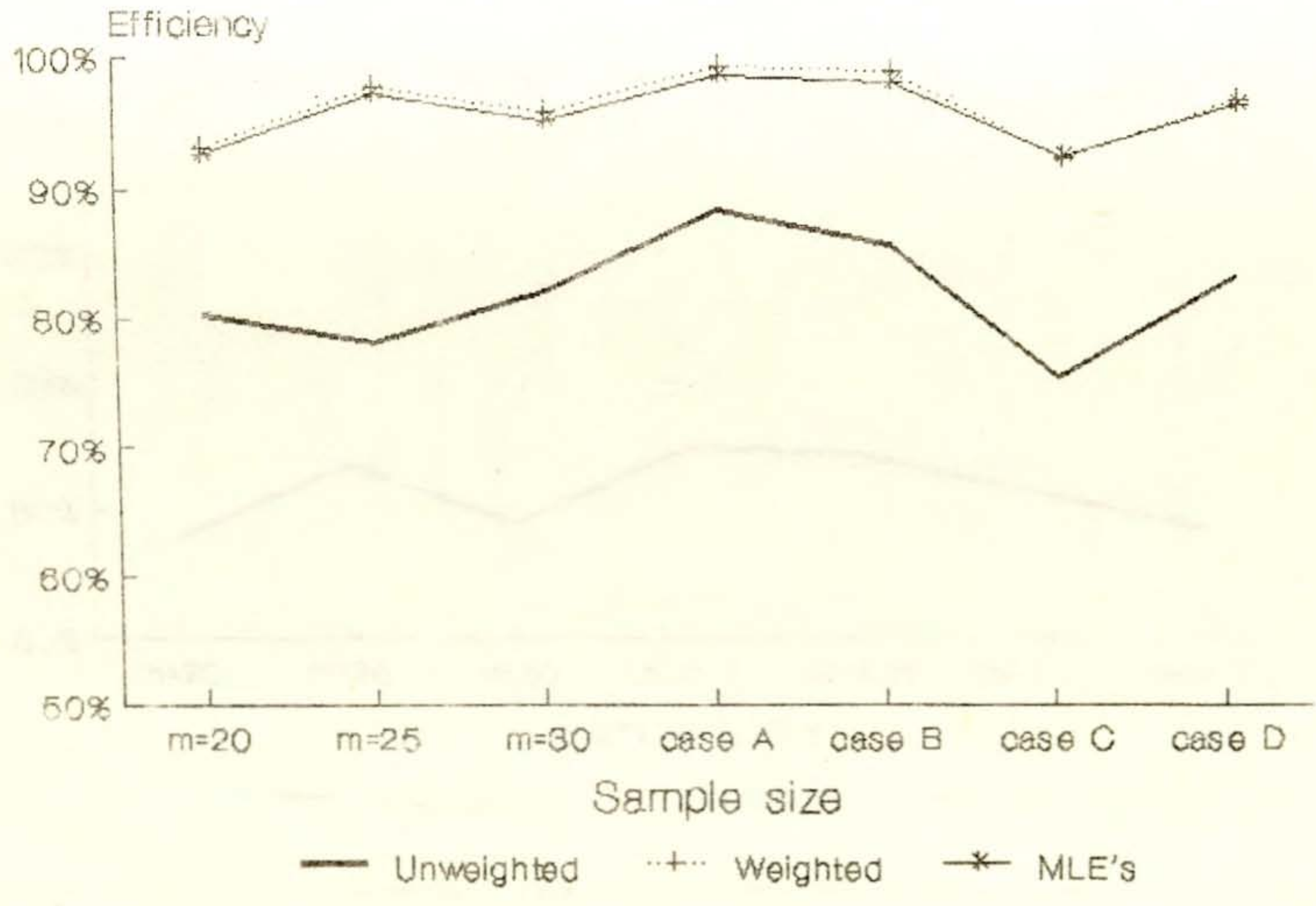
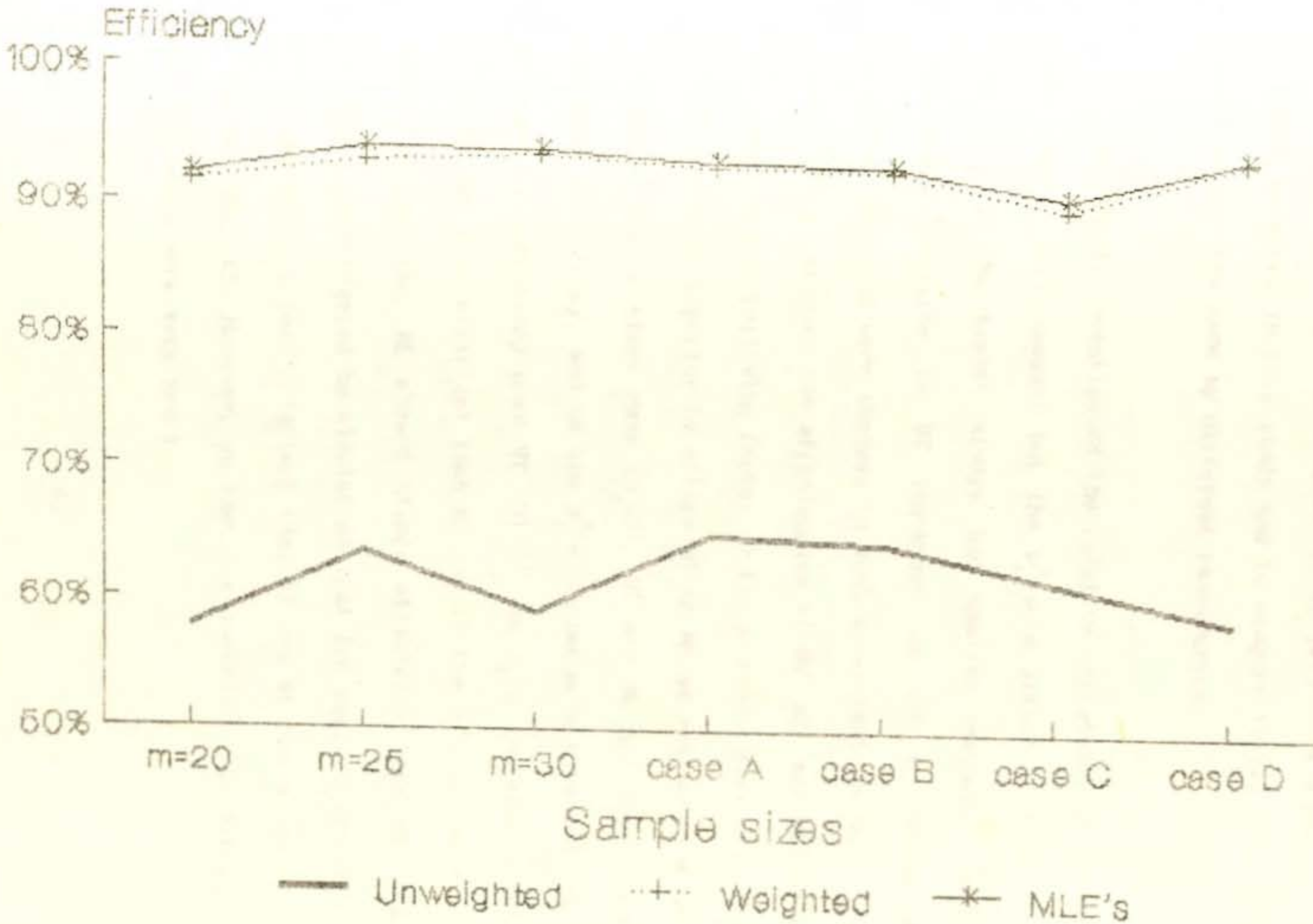


Figure P Relative efficincy of the estimators to the BLUE; k=10, variance pattern IV



CHAPTER 5

SUMMARY AND CONCLUSIONS

The purpose of this chapter is to provide a summary of the results obtained in this study and to compare these results with previous studies made by different researchers.

Levy(1970) investigated the relative efficiency of WT and ML when the n_i were unequal but the σ_i^2 were infact uniform ($\sigma_i^2=1$). For K > 3, ML almost always had smaller variance, and its efficiency relative to WT increased as the n_i became more diversed. The n_i's were choosen in such a way that $1 \le n_i \le 11$. Levy & Mantel (1974) studied the efficiencies of UW, WT, and ML for k=6, and observed the following facts: 1) For moderate diversity in the σ_i^2, UW could be superior to either WT or ML as estimator of μ. 2) For equal sample sizes case (n_i=6), WT and ML had approximately the same efficiency, and as the σ_i^2's become more diversed, ML had an edge on efficiency over WT. 3) if the n_i's increase with the σ_i^2's, WT was more efficient than ML and if the n_i's decrease while the σ_i^2's increase, ML showed higher efficiency than WT. J.N.K. Rao(1980) investigated by simulation that for small n_i ($2 \le n_i \le 6$), ML was significantly less efficient than KS and NS was slightly more efficient than KS. However, as the n_i's increase, the differences in efficiency were very small.

Regarding the precision of the estimators the results obtained in this study lead us to the following conclusions.

1. When the σ_i^2's, $i = 1,\ldots,k$ are the same UW is fully efficient and it performs better than any other estimator. But when the σ_i^2's are different UW is the least efficient estimator. As k and/or the diversity of the σ_i^2's increases, the efficiency of UW decreases.

2. In the case of equal n_i's , ML = NS = KS. If the n_i's are not equal, these estiamtors are unequal, but the estimates are very close to each other. This can be justified by comparing equations (7), (12), and (15) in chapter 2. Their weights differ by n_i, n_i-1 and n_i-2. For large values of n_i's these weights are approximately equal. Their difference in precision may be substantial for small sample sizes. In our study the sample sizes range from 20 t0 30 and the estimators have almost the same precision. Thus it does not matter much whether we use ML, NS, or KS.

3. When k=3, WT is slightly more efficient than the other estimators. But, when k increases, ML, NS and KS are slightly more efficient than WT. It is also to be noted that when the n_i's vary ML, NS and KS are superior to WT.

4. Except in few cases, the efficiency of all estimators increases, when the n_i's and σ_i^2's are combined as in case D (small n_i's correspond to large σ_i^2's) than in case C(large n_i's correspond to small σ_i^2's).

In general when the sample sizes are not too large (i.e. $20 \le n_i \le 30$) and the number of populations (k) is small ($3 \le k \le 10$), **WT**, **ML**, **NS** and **KS** have high efficiency, and their efficiences are not significantly different. The estimators **ML**, **NS**, and **KS** can not be computed directly as **UW** or **WT**. They are obtained by some iteration methods, such as the Newton-Raphson method and such iterative methods require more computer time. In conclusion, based on the the simulation results and considering the computer time required to calculate the estimators there appears to be support for using the **WT** rather than the other methods. But it is worthwhile to use **ML** or **NS** or **KS** when k is large and/or the n_i's are diversed as in case D.

APPENDIX

The pascal program written is given below.

```
{ main program }

program SIMULATION(input,output);
{ This program generates normal random variables
  from N(μ,σ_i^2) calculates sample means and variances
  and computes different estimates of μ according
  their formula. }
const
     run_no  = 1000;    { number of independent runs }
     m_loop  = 10;      { maximum no. of populations }
     m_ni    = 30;      { maximum no. of sample size considered }
     eps     = 1.0e-10; { constant used as index of convergence }

type
     si_type = array[1..m_loop] of real; { an array of S_i^2's }
     ni_type = array[1..m_loop] of integer; { an array of n_i's }
     xi_raw  = array[1..m_ni] of real;{ samples from N(0,σ_i^2)}
     xi_type = array[1..m_loop] of xi_raw;{ matrix of x_ij's }
var        { global variables }
     p, k, m, seed, type_f         : integer;
     m_u                           : real;
     n                             : ni_type;
     x_bar, s, sigma               : si_type;
     x                             : xi_type;
     ok_tag, flag                  : boolean;

function DRAND48: longreal; external;{ generates uniform r. v }.
procedure SRAND48(seed: integer); external;{ Initializes DRAND48 }
```

```
procedure MEAN_VAR(var x_bar, s:si_type);

   { This procedure calculates sample means & variances }
var   i, j          : integer;
      sum           : si_type;
      x_rowdata     : xi_type;
procedure TRANSFORM(var x_raw : xi_raw; i :integer);

   { Transforms Uniform r. v. to Normal using the polar Marsaglia
      method }
var  v1, v2, r, y, xtemp    : real;
     y1, y2                 : real;
     cheker, j              : integer;
begin    { TRANSFORM }
     cheker := 0;
     for j := 1 to n[i] do
         begin
            if(cheker = 0) then
              begin
                repeat
                      v1 := 2.0*DRAND48 - 1.0;
                      v2 := 2.0*DRAND48 - 1.0;
                      r  := sqr(v1) + sqr(v2);
                until(( eps < r) and (r < 1));
                y  := sqrt((-2.0*ln(r))/r);
                y1 := v1*y;
                y2 := v2*y;
                x_raw[j] := y1*sqrt(sigma[i]);
                xtemp    := y2*sqrt(sigma[i]);
                cheker := 1
             end
           else bigin
             x_raw[j] := xtemp;
             cheker   := 0 end
     end
end;  { end of TRANSFORM }
begin  { beginning of MEAN_VAR }
     for i := 1 to k do sum[i] := 0;
     for i := 1 to k do
```

```
        begin
            TRANSFORM(x_rowdata, i);
            for j := 1 to n[i] do x[i,j] := x_rowdata[j];
            for j := 1 to n[i] do
               sum[i] := sum[i] + x[i,j];
            x_bar[i]  := sum[i]/n[i]
        end;
        for i := 1 to k do
        begin
            s[i] := 0.0;
            for j := 1 to n[i] do
              s[i] := s[i] + sqr(x[i,j] - x_bar[i]);
            s[i] := s[i]/(n[i] - 1)
        end
end;    { MEAN_VAR }
procedure BLUE(var m_u: real);
  { Calculates the BLUE assuming variances are known }
var i          : integer;
    w          : si_type;
    mnum, mden :real;
begin      { BLUE }
     mnum := 0.0 ; mden := 0.0;
     for i:= 1 to k do
       begin
          w[i] := n[i]/sigma[i];
          mnum := mnum + w[i]*x_bar[i];
          mden := mden + w[i]
       end;
     m_u := mnum/mden
end;         { BLUE }
function UNWEIGHT : real;
  { This function computes the unweighted mean }
var   i          : integer;
    mnum, mden : real;
begin        { UNWEIGHT }
    mnum := 0.0; mden := 0.0;
    for i := 1 to k do
```

```
        begin
          mnum := mnum + n[i]*x_bar[i];
          mden := mden + n[i]
        end;
      UNWEIGHT := mnum/mden
end;        { UNWEIGHT }
procedure WIEGHTED(var m_u : real);
  { Computes the weighted mean }
var i          : integer;
    w          :si_type
    mnum mden  :real;
begin        { WEIGHTED }
     for i := 1 to k do
       begin
          w[i] := n[i] / s[i];
          mnum := mnum + w[i] * x_bar[i];
          mden := mden + w[i]
       end;
     m_u := mnum / mden;
end;         { WEIGHTED }
procedure NEWTON_RAPH(var m_u : real; type_f :integer;
            ok_tak, flag : boolean);
  { This procedure is used to obtain the maximum likelihood,
    Neyman-Scott & Kalbfleish-Sprott estimators. }
const   max_itt = 100;
type  mw_type : array[1..max_itt] of real;
var
    mw         : mw_type;
    i, l       : integer;
    round      : longreal;
    conv_ok    : boolean;
function F_Of_MWT(mwt :real); real;
  { This funcion evaluates the estimating equation f(μ) }
var  i :integer ; sum :real;
function NUM_OF_F(ni :integer; xi, mwt : real) : real;
  { Evaluates the numerator of f(μ) }
```

```
begin      { NUM_OF_F }
    NUM_OF_F := (ni - type_f) * ni * (xi - mwt)
end;       { NUM_OF_F }
function DEN_OF_F(ni :integer ; si, xi, mwt :real) : real;
  { Evaluates the denumerator of f(μ) }
begin      { DEN_OF_F }
    DEN_OF_F := (ni -1) * si + ni * sqr(xi - mwt)
end;       { DEN_OF_F }
begin      { F_OF_MWT}
    sum := 0.0;
    for i := 1 to k do
      sum := sum + (NUM_OF_F(n[i],x_bar[i]),mwt) /
              DEN_OF_F(n[i],s[i],x_bar[i],mwt);
    F_OF_MWT := sum
end;       { F_OF_MWT }
function DF_OF_MWT(mwt : real) : real;
  { Evaluates the derivative of f(μ) }
var  i :integer ; sum : real;
function NUM_OF_DF(ni :integer; si, xi, mwt :real) :real;
  { Evaluates the numerator of f'(μ) }
begin    { NUM_OF_DF }
    NUM_OF_DF := (ni - type_f) * ni * ((ni - 1) * si - ni * sqr(xi - mwt))
end;     { NUM_OF_DF }
function DEN_OF_DF(ni : integer; si, xi, mwt : real) : real;
  { Evaluates the denumerator of f'(μ) }
begin    { DEN_OF_DF }
    DEN_OF_DF := sqr((ni - 1) * si + ni * sqr(xi - mwt));
end;     { DEN_OF_DF }
begin    { DF_OF_MWT }
    sum := 0.0;
    for i := 1 to k do
      sum := sum + (NUM_OF_DF(n[i],s[i],x_bar[i],mwt)/
            DEN_OF_DF(n[i],s[i],x_bar[i],mwt);
    df_of_mwt := sum
end;     { DF_OF_MWT }
```

```
begin   { NEWTON_RAPH }
    mw[1] := UNWEIGHT;
    l := 1;
    conv_ok := false;
    while (not conv_ok) and (l < max_itt) do
      begin
        if df_of_mwt(mw[l]) < eps then
          begin
            round := df_of_mwt(mw[l]) + eps;
            mw[l+1] := mw[l] - f_of_mwt(mw[l]) / round;
            ok_tag  := true;
          end else
            mw[l+1] := mw[l] - f_of_mwt(mw[l]) / df_of_mwt(mw[l]);
          l := l + 1;
          if abs(mw[l] - mw[l-1]) < eps then  conv_ok := true;
      end;
      if conv_ok then m_u := mw[l];
      else flag := false;
end;    { NEWTON_RAPH }


begin   { beggining of the main program }
    writeln('How many groups are to combined ....enter k');
    readln(k); writeln(k);
    writeln('Enter the seed value ....seed');
    readln(seed); writeln(seed);
    writeln('Enter the sample sizes ....ni');
    for p := 1 to k do
    begin
        read(n[p]); writeln([p]:3);
    end; writeln;
    writeln('Enter the population variances ....sigmas');
    for p := 1 to k do
    begin
        read(sigma[p]); write(sigma[p]:6:2);
    end; writeln;
    ok_tag  := false;
    flag    := true;
    SRAND48(seed);
```

```
for m := 1 to run_no do
begin
    MEAN_VAR(x_bar,s);
    write(m, '  ');
    BLUE(m_u);
    write('  ',m_u:11:8);
    write('  ',UNWEIGHT:11:8);
    WEIGHTED(m_u);
    write('  ',m_u:11:8);
    for p := 1 to 3 do
    begin
        type_f := p - 1;
        NEWTON_RAPH(m_u, type_f, ok_tag, flag);
        write('  ',m_u:11:8);
    end;
    writeln;
end;
end. { end of the program }
```

This program calculates the mean, variance and efficiency of the estimates relative to the BLUE. The inputs for this program are the outputs of SIMULATION (the main program).

```
program MEAN_VAR_ESTIMATOR(input,output);
const
    row      = 1000;
    column   = 6;
    m_c      = 50;
type file_nm_type = packed array[1..m_c];
var
    sum, mean, v, ss, reff  : array[1..column] of real;
    est                     : array[1..row,1..column] of real;
    file_name               : file_nm_type;
    d_f, out_f              : text;
    j, k                    : integer;
    interact                : boolean;
    exit                    : boolean;
    exit_s                  : integer;
function parametri $alias 'pas.parameters'$
   (pos : integer ; var str :file_nm_type ; len : integer ) :
   integer; external;
begin

    exit := false;
    while not exit do
      begin
        i := m_c;
        k := parametri(1, file_name, i);
        if k <=0 then begin
          writeln('Name of the data file ?');
          readln(file_name);
          interact := true;
        end else begin
          interact := false;
```

```
end;
if interact then writeln('I am computing the means and var.');
reset(d_f,file_name);
for i := 1 to column do
begin
     mean[i] := 0.0;
     ss[i]   := 0.0;
     sum[i]  := 0.0;
     v[i]    := 0.0;
     reff[i] := 0.0;
end;
i := 1;
while not eof(d_f) do
  begin
     read(d_f,ser);
     for j := 1 to column do
       begin
          read(d_f,est[i,j]);
          sum[j] := sum[j] + est[i,j];
       end;
       readln(d_f);
       i := i + 1;
  end;
 for j := 1 to column do mean[j] := sum[j]/ser;
 for j := 1 to column do
   begin
     for i := 1 to row do ss[j] := ss[j] +
            sqr(est[i,j] - mean[j]);
     end;
 append(out_f,'mean_var.out');
 for i := 1 to column do
   begin
     v[i]    := ss[i] / (ser - 1);
     reff[i] := (v[1]/v[i]) * 100.0;
   end;
 writeln(out_f,file_name);
 for i := 1 to column do
  writeln(out_f,i:2,mean[i]:12:8,v[i]:12:8,reff[i]:8:2);
```

```
        close(t_f);
        close([illegible]);
        writeln('Want to exit? .... if yes enter 0 otherwise 1 ');
        readln(exit_s);
        if exit_s = 0 then exit := true;
    end;
end.
```

APPENDIX II

For the cases of equal variance, patterns I and II, the class boundaries used to tabulate the frequency distributions are given below.

Class Interval	Value
≤ -0.198277625	1
-0.198277625 - -0.167773375	2
-0.167773375 - -0.137269125	3
-0.137269125 - -0.106764875	4
-0.106764875 - -0.076260625	5
-0.076260625 - -0.045756375	6
-0.045756375 - -0.015252125	7
-0.015252125 - 0.015252125	8
0.015252125 - 0.045756375	9
0.045756375 - 0.076260625	10
0.076260625 - 0.106764875	11
0.106764875 - 0.137269125	12
0.137269125 - 0.167773375	13
0.167773375 - 0.198277625	14
≥ 0.198277625	15

For variance patterns III & IV, the following class intervals were used:

Class Interval	Value
≤ -0.937518825	1
-0.937518825 - -0.812516315	2
-0.812516315 - -0.687513805	3
-0.687513805 - -0.562511295	4
-0.562511295 - -0.437508785	5
-0.437508785 - -0.312506275	6
-0.312506275 - -0.187503765	7
-0.187503765 - -0.062501255	8
-0.062501255 - 0.062501255	9
0.062501255 - 0.187503765	10
0.187503765 - 0.312506275	11
0.312506275 - 0.437508785	12
0.437508785 - 0.562511295	13
0.562511295 - 0.687513805	14
0.687513805 - 0.812516315	15
0.812516315 - 0.937518825	16
≥ 0.937518825	17

Table 1 Frequency Distribution of the Estimates when k = 3, equal sample sizes, equal variance

m = 20			m = 25			m = 30		
UW	WT	ML	UW	WT	ML	UW	WT	ML
60	65	66	48	47	49	26	34	34
40	46	25	28	30	29	30	49	29
59	49	43	37	47	25	47	42	41
61	61	57	71	72	76	50	55	57
62	71	86	78	71	68	89	77	73
75	77	83	83	80	83	103	109	110
99	85	81	112	109	104	114	108	101
104	98	99	125	124	129	113	106	116
86	99	94	111	99	95	99	105	100
88	72	79	72	74	75	96	101	105
61	61	64	60	73	69	85	81	81
66	72	65	71	66	69	61	59	60
46	47	49	46	38	39	38	42	41
32	28	33	27	31	28	31	27	27
61	69	66	31	39	42	18	25	25
1000	1000	1000	1000	1000	1000	1000	1000	1000

Table 2 Frequency Distribution of the Estimates
when k = 3, unequal sample sizes, equal variance

clas	m = 20					m = 25					m = 30				
int.	UW	NT	ML	NS	KS	UW	NT	ML	KS	KS	UN	WT	ML	NS	KS
1	57	62	60	61	61	26	29	30	30	30	46	48	49	47	45
2	30	25	28	27	26	21	20	18	18	18	24	40	37	40	41
3	40	50	48	47	49	44	45	49	48	50	54	46	49	47	48
4	64	51	52	55	54	54	53	51	53	51	71	58	55	58	61
5	64	75	76	74	74	70	76	72	72	72	89	71	872	70	66
6	91	92	59	92	91	94	82	86	84	85	70	101	98	100	102
7	95	87	94	91	93	96	103	99	100	99	97	110	109	107	108
8	105	104	100	191	101	126	113	123	123	123	115	88	93	93	93
9	107	98	101	99	98	101	113	105	108	108	93	93	97	95	93
10	87	87	82	82	82	111	95	94	94	94	89	84	80	83	83
11	79	70	70	72	73	92	92	90	91	91	79	89	85	85	84
12	51	62	62	61	59	52	63	64	62	63	65	59	64	63	64
13	39	45	44	44	45	49	50	50	51	49	41	44	43	43	44
14	31	28	31	31	31	30	28	26	25	26	28	26	26	26	25
15	60	64	63	63	63	34	38	40	41	41	39	43	43	43	43
Tot.	1000	1000	1000	1000	1000	1000	1000	1000	1000	1000	1000	1000	1000	1000	1000

Table 3 Frequency Distribution of the Estimates
when $k = 6$, equal sample sizes, equal variance

clas int.	m = 20 UW	WT	ML	m = 25 UW	WT	ML	m = 30 UW	WT	ML
1	11	20	22	5	4	4	1	3	2
2	14	18	16	10	17	16	10	10	9
3	41	35	31	28	30	28	13	18	20
4	57	64	66	53	53	60	39	40	40
5	78	77	81	73	76	73	83	80	83
6	118	115	112	112	122	120	121	118	115
7	118	110	112	124	104	109	151	146	151
8	122	117	124	155	153	146	170	158	157
9	131	115	104	165	157	156	159	164	159
10	124	125	131	105	101	102	113	123	123
11	74	82	85	89	88	90	68	64	64
12	50	52	45	45	52	53	41	38	40
13	34	35	35	19	24	24	18	22	22
14	18	18	19	13	12	13	9	9	8
15	10	17	17	4	7	6	4	7	7
T	1000	1000	1000	1000	1000	1000	1000	1000	1000

Table 4 Frequency Distribution of the Estimates when k = 6, unequal sample sizes, equal variance

Case A					Case B					Case C				
UW	NT	ML	NS	KS	UW	NT	ML	KS	KS	UN	WT	ML	NS	KS
9	10	10	10	10	6	5	5	5	5	6	9	9	9	9
13	14	16	16	15	6	18	18	18	18	16	15	15	15	15
34	39	35	35	36	25	25	25	25	25	31	25	28	28	28
58	68	62	62	64	69	59	58	58	58	52	60	62	61	60
87	86	92	90	89	80	70	71	69	69	62	70	59	60	62
111	102	101	104	103	99	111	113	115	115	110	97	106	104	103
130	128	134	133	133	151	134	136	136	136	136	141	140	144	141
143	126	131	131	131	163	164	162	162	163	142	131	130	126	129
112	127	116	116	118	153	136	133	133	133	141	150	146	147	150
100	86	91	90	88	110	122	122	121	119	111	96	105	106	103
88	93	93	94	94	66	64	69	70	71	85	98	91	92	91
54	45	51	52	54	41	44	39	39	39	64	48	46	47	47
36	28	32	31	29	26	28	30	30	30	23	35	39	37	37
17	28	28	28	28	11	13	12	12	12	9	12	12	13	14
8	10	. 8	8	8	3	7	7	7	7	12	13	12	11	11
1000	1000	1000	1000	1000	1000	1000	1000	1000	1000	1000	1000	1000	1000	1000

able 5: Frequency Distribution of the Estimates
when k = 8 , equal sample size, equal variance

clas int.	m = 20 UW	WT	ML	m = 25 UW	WT	ML	m = 30 UW	WT	ML
1	6	13	11	2	5	4	3	2	2
2	14	11	13	4	4	5	1	6	6
3	19	25	26	23	18	22	13	13	11
4	34	38	38	29	45	44	31	35	36
5	76	71	75	81	82	76	80	73	73
6	110	121	106	128	107	110	132	128	128
7	162	137	148	154	161	160	164	160	171
8	153	161	153	177	183	187	172	169	159
9	147	137	129	174	141	139	170	158	156
10	116	108	106	93	114	111	117	122	124
11	88	82	86	78	72	78	76	83	85
12	38	54	30	30	39	40	28	37	35
13	24	25	18	20	21	15	6	8	8
14	10	11	15	5	5	6	7	4	4
15	5	6	6	2	3	3	[illegible]	2	2
Tot.	1000	1000	1000	1000	1000	1000	1000	1000	1000

6 Frequency Distribution of the Estimates
when k = 8, unequal sample size, equal variances

m = 20					m = 25					m = 30				
UW	NT	ML	NS	KS	UW	NT	ML	NS	KS	UW	NT	ML	NS	KS
11	11	9	9	9	1	2	2	2	2	3	5	3	3	3
10	9	11	11	11	7	8	7	7	7	7	10	14	14	14
16	22	20	20	20	18	17	17	17	17	18	22	18	19	19
43	49	47	48	48	32	42	43	44	44	34	36	36	35	35
95	94	98	98	99	78	71	72	71	71	80	80	86	84	84
115	132	125	125	122	111	114	114	115	115	102	105	105	107	105
165	146	143	143	146	167	167	167	165	164	151	148	150	148	150
167	154	160	162	164	185	176	176	175	176	165	152	147	150	150
138	128	129	130	129	166	147	145	147	147	162	168	167	166	167
92	105	103	102	103	107	128	134	133	133	130	117	120	121	117
71	68	72	73	72	81	79	73	74	74	88	94	85	84	87
46	42	39	38	37	27	26	16	26	26	36	38	45	44	44
23	24	22	23	24	11	11	12	12	12	15	15	13	14	14
5	15	15	15	15	5	8	9	9	9	3	6	6	6	6
3	1	1	1	1	4	4	3	3	3	6	4	5	5	5
1000	1000	1000	1000	1000	1000	1000	1000	1000	1000	1000	1000	1000	1000	1000

e 7 Frequency Distribution of the Estimates
when k = 10, equal sample size, equal variance

m = 20			m = 25			m = 30		
UW	WT	ML	UW	WT	ML	UW	WT	ML
3	9	7	1	2	2	0	0	0
12	10	11	4	3	2	1	1	1
18	22	19	10	22	19	5	3	4
57	61	60	36	27	33	25	30	30
87	77	84	65	76	74	64	73	4
103	118	115	128	113	112	121	113	115
155	136	145	172	168	170	182	189	184
161	173	169	169	171	166	204	202	210
136	133	134	117	172	131	187	165	160
137	111	108	119	126	64	115	120	120
65	75	73	66	66	33	60	61	61
41	48	50	36	33	16	25	28	26
18	16	18	12	15	4	7	10	10
5	9	6	4	4	2	4	5	5
2	2	2	1	2		0	C	0
1000	1000	1000	1000	1000	1000	1000	1000	1000

e 8 Frequency Distribution of the Estimates
k = 10, unequal sample size, equal variance

Case A					Case B					Case C				
UW	NT	ML	NS	KS	UW	NT	ML	NS	KS	UW	NT	ML	NS	KS
0	2	3	3	3	0	0	0	0	0	3	3	3	3	3
7	7	3	3	3	5	7	6	6	6	5	9	8	8	8
16	13	17	17	17	10	10	12	12	12	14	20	18	18	18
27	41	36	36	36	11	16	14	14	14	25	29	31	31	31
75	68	73	73	73	49	57	56	56	57	72	66	67	65	68
101	121	122	122	124	135	126	32	132	131	115	112	112	115	113
194	170	166	166	164	207	191	182	182	182	185	194	192	189	188
170	160	161	160	160	185	193	192	193	191	209	196	197	201	201
143	147	148	149	149	182	175	179	179	179	149	162	164	163	162
117	111	111	110	109	104	106	108	107	109	107	89	85	85	86
78	81	83	83	83	69	68	70	70	70	51	67	72	72	71
48	50	47	48	49	31	35	33	33	33	37	34	34	33	34
17	18	17	17	17	8	13	13	13	13	9	13	11	11	11
4	6	9	9	9	2	1	1	1	1	3	5	5	5	5
4	5	4	4	4	2	2	2	2	2	0	1	1	1	1
1000	1000	1000	1000	1000	1000	1000	1000	1000	1000	1000	1000	1000	1000	1000

ble 9 Frequency distribution of the estimates
when k=3, equal sample sizes, variance patterns I & II

m=20						m=25						m=30			
I			II			I			II			I			
UWT	WT	MLE	UWT	WT	MLE	UWT	WT	MLE	UWT	WT	MLE	UWT	WT	MLE	UWT
28	16	16	134	116	120	12	8	8	107	86	84	8	4	4	98
30	20	21	29	33	31	21	10	11	38	50	52	10	11	10	31
38	37	35	48	46	45	37	35	34	50	49	51	33	30	31	54
55	65	64	57	65	63	62	56	55	55	59	60	60	45	45	50
86	83	82	60	68	73	81	83	87	74	56	55	77	76	76	74
104	106	109	68	70	68	95	111	109	73	75	74	105	110	111	64
110	108	103	75	69	68	130	129	132	69	88	84	135	165	163	74
125	115	117	71	66	68	128	123	116	77	81	85	129	134	137	91
112	125	130	67	70	69	111	133	142	70	75	75	127	130	121	83
100	124	123	73	71	70	114	112	106	69	64	66	104	128	132	71
77	79	75	45	59	60	77	86	83	47	-63	64	99	64	61	81
56	59	60	55	54	56	53	49	53	54	60	54	53	58	60	55
40	30	31	48	54	47	37	39	38	50	57	57	35	30	28	49
21	12	13	34	44	45	23	12	12	38	35	35	15	12	12	40
18	21	21	146	115	117	19	14	14	129	102	104	10	3	3	85
1000	1000	1000	1000	1000	1000	1000	1000	1000	1000	1000	1000	1000	1000	1000	1000

Table 10 Frequeny distribution of the estimates
when k=3, equal sample sizes, variance patterns III & IV

Cl. int	III			IV			III			IV			III			
	UWT	WT	MLE	UWT	WT	MLE	UWT	WT	MLE	UWT	WT	MLE	UWT	WT	MLE	U[illegible]
1	8	3	3	57	43	45	7	6	5	22	30	23	2	3	2	[illegible]
2	10	14	14	33	19	18	11	8	9	27	18	26	5	3	4	[illegible]
3	18	12	13	38	43	46	12	12	12	25	37	25	15	9	8	[illegible]
4	32	29	28	49	47	44	23	23	25	34	47	32	28	13	15	[illegible]
5	61	54	56	63	61	57	57	43	39	50	49	56	56	53	52	[illegible]
6	87	90	91	66	74	79	71	63	70	86	71	81	94	108	105	[illegible]
7	98	112	105	67	74	70	93	109	103	99	90	101	113	96	101	[illegible]
8	138	136	145	82	85	88	131	136	134	101	101	100	143	140	149	[illegible]
9	104	125	119	80	90	92	124	139	146	109	104	111	146	165	158	[illegible]
10	112	105	101	80	83	80	124	155	150	107	101	101	133	149	149	[illegible]
11	97	93	101	84	81	82	133	117	116	93	99	94	101	107	110	[illegible]
12	80	89	86	72	75	74	80	87	89	81	77	83	71	70	69	[illegible]
13	69	58	56	61	54	53	62	48	51	58	46	60	52	0	46	[illegible]
14	30	35	37	62	59	62	34	31	30	44	44	42	20	0	20	[illegible]
15	26	19	20	31	53	49	25	11	9	25	34	26	8	8	8	[illegible]
16	15	16	15	35	25	27	9	10	9	21	30	21	8	4	4	[illegible]
17	15	10	10	40	34	34	4	2	3	18	22	18	3	0	0	[illegible]
Tot	1000	1000	1000	1000	1000	1000	1000	1000	1000	1000	1000	1000	1000	1000	1000	100[illegible]

Table 11 Frequeny distribution of the estimates
when k=3, unequal sample sizes variance pattern I & II

cl. int.	Case A															Case B				
	I					II										II				
	UWT	WT	MLE	NS	KS	UWT	WT	MLE	NS	KS	UWT	WT	MLE	NS	KS	UWT	WT	MLE	NS	KS
1	27	23	24	24	24	132	116	114	114	114	12	8	6	6	6	108	91	88	88	89
2	25	18	18	17	18	55	41	42	41	41	25	16	17	17	18	51	48	51	51	50
3	41	42	43	43	42	43	52	54	55	55	38	25	26	26	25	53	51	47	46	46
4	77	63	61	63	63	54	41	43	43	43	46	58	58	58	58	52	59	62	63	64
5	70	75	76	75	75	65	65	65	66	59	78	84	83	82	82	71	68	66	67	67
6	80	89	80	89	90	58	73	68	67	66	91	98	98	100	100	87	77	80	79	78
7	95	122	116	116	115	68	82	73	72	79	129	120	122	121	120	62	95	97	96	96
8	123	120	125	124	124	59	73	74	75	72	143	147	144	145	145	88	81	78	78	78
9	121	122	121	122	123	58	67	64	64	64	131	152	153	152	153	67	73	78	80	79
10	94	100	98	98	98	65	62	71	71	72	112	104	105	105	105	60	74	73	72	73
11	83	99	102	102	100	51	70	71	79	68	94	94	94	94	94	60	69	65	66	66
12	63	54	56	55	57	60	49	52	52	52	44	44	45	45	45	58	49	50	49	49
13	51	40	36	37	36	53	70	40	42	43	28	30	27	27	27	48	43	44	44	45
14	25	25	24	24	24	51	56	53	52	52	16	13	15	15	15	36	39	37	39	38
15	26	10	11	11	11	128	114	111	111	111	13	7	7	7	7	99	83	83	82	82
Tot	1000	1000	1000	1000	190	1000	1000	1000	1000	1000	1000	1000	1000	1000	1000	1000	1000	1000	1000	1000

Table 12 Frequency Distribution of the Estimates, when k = 3, unequal sample size, cases C and D variance patterns I and II

class int.	Case C										Case D									
	I					II					I					II				
	UWT	WT	MLE	NS	KS	UWT	WT	MLE	NS	KS	UWT	WT	MLE	NS	KS	UWT	WT	MLE	NS	KS
1	21	15	15	15	15	130	116	112	112	111	10	6	7	7	7	108	91	92	93	93
2	29	21	22	22	23	30	37	42	43	40	13	9	9	10	10	37	42	40	39	38
3	33	28	29	27	27	39	52	49	47	51	30	32	31	30	30	45	42	53	41	41
4	60	48	43	46	46	85	51	54	55	54	52	13	36	37	36	68	43	42	51	54
5	80	79	82	82	81	68	66	68	68	69	88	41	73	74	74	78	81	53	80	77
6	86	117	114	114	117	63	74	74	73	70	110	145	113	109	110	66	80	77	83	86
7	126	128	128	127	124	58	72	69	72	73	116	188	152	152	155	78	85	74	86	81
8	125	127	130	129	130	61	83	80	78	82	124	227	130	130	127	84	82	86	80	84
9	109	125	127	126	125	65	64	77	74	72	136	211	128	129	129	64	77	81	80	79
10	112	113	108	109	108	73	65	56	61	60	99	111	133	134	132	56	66	76	66	68
11	72	92	91	92	93	57	54	56	53	56	86	37	73	73	75	56	62	64	63	61
12	80	51	58	58	58	48	58	59	61	58	54	17	59	60	61	64	70	72	71	70
13	43	34	29	29	30	58	58	47	47	49	42	5	29	28	27	46	48	49	48	49
14	12	13	16	16	14	40	45	51	50	49	26	1	16	16	17	35	36	30	32	33
15	12	9	8	8	9	125	105	106	106	106	14	0	11	11	10	115	85	87	87	86
Tot.	1000	1000	1000	1000	1000	1000	1000	1000	1000	1000	1000	1000	1000	1000	1000	1000	1000	1000	1000	1000

Table 13 Frequency Distribution of the Estimates, when k = 3, unequal sample size, cases C and D variance patterns I and II

class int.	Case C										Case D									
	I					II					I					II				
	UWT	WT	MLE	NS	KS	UWT	WT	MLE	NS	KS	UWT	WT	MLE	NS	KS	UWT	WT	MLE	NS	KS
1	8	5	5	5	5	36	34	34	34	33	3	1	1	1	1	30	22	23	23	23
2	11	7	6	6	7	25	18	20	20	21	9	4	2	2	3	18	27	26	26	25
3	19	20	20	21	20	37	35	33	33	34	12	7	9	9	8	37	25	25	25	26
4	26	22	24	23	23	50	41	42	42	41	21	17	18	18	18	47	34	32	32	32
5	40	25	53	53	53	53	68	64	64	64	40	46	41	43	44	49	50	56	54	54
6	87	55	75	73	74	67	74	80	80	80	83	74	77	75	74	71	86	81	83	83
7	114	70	110	113	112	81	76	74	74	74	110	116	124	124	124	90	99	101	101	101
8	118	118	113	114	114	95	94	97	98	100	139	155	144	144	144	101	101	100	99	98
9	132	145	147	143	141	103	188	85	84	83	170	169	169	170	170	104	109	111	113	113
10	102	128	130	131	134	100	117	116	116	113	131	140	141	140	140	101	107	101	100	101
11	117	98	96	99	97	76	84	86	87	89	97	108	113	117	113	99	93	94	95	96
12	69	88	88	86	81	52	80	75	73	72	85	76	74	74	74	77	81	83	82	81
13	60	60	56	55	55	67	58	62	62	64	50	49	51	51	51	46	58	60	60	59
14	48	43	49	50	50	41	38	32	33	32	25	21	19	19	19	44	44	42	42	43
15	29	12	11	11	11	33	32	35	35	34	17	10	10	10	10	34	25	26	25	25
17	12	11	12	12	12	30	25	25	25	26	6	4	4	4	4	30	21	21	22	22
18	8	6	5	5	5	54	38	40	40	40	2	3	3	3	3	22	18	18	18	18
Tot.		1000	1000	1000	1000	1000	1000	1000	1000	1000	1000	1000	1000	1000	1000	1000	1000	1000	1000	1000

able 14 Frequency Distribution of the Estimates,
when k = 3, unequal sample size, cases C and D
variance patterns III and IV

ass nt.	Case C										Case D									
	III					IV					III					IV				
	UWT	WT	MLE	NS	KS	UWT	WT	MLE	NS	KS	UWT	WT	MLE	NS	KS	UWT	WT	MLE	NS	KS
1	10	7	10	10	10	55	38	40	39	39	6	0	0	0	0	30	19	20	19	19
2	8	5	2	2	2	25	29	25	27	28	11	5	6	6	5	18	16	16	17	17
3	16	15	15	15	15	40	32	33	33	34	17	12	10	10	11	44	37	34	34	33
4	38	32	29	29	31	51	42	45	46	46	28	33	37	37	37	39	38	42	43	45
5	63	49	55	54	52	77	67	63	62	60	43	47	46	46	46	50	59	62	62	61
6	81	84	80	83	83	64	72	76	73	73	83	87	87	86	86	83	72	67	65	65
7	106	105	102	101	103	78	99	96	95	97	136	130	126	126	127	88	80	85	87	87
8	126	128	137	138	137	79	97	96	100	98	125	115	115	115	115	76	115	108	106	108
9	135	139	130	128	129	79	87	92	88	86	122	150	158	159	157	99	105	106	108	106
10	118	157	160	159	156	90	94	89	92	94	136	140	136	135	137	104	104	103	103	103
11	104	99	101	100	100	70	86	88	87	83	117	129	128	128	124	111	96	94	114	93
12	100	87	85	87	88	85	74	74	75	77	83	65	65	66	69	79	87	90	87	91
13	58	38	37	37	37	59	55	53	54	56	42	44	44	44	43	56	64	58	65	56
14	23	23	26	25	25	51	54	57	55	55	29	26	24	24	25	48	45	51	43	56
15	23	22	19	20	20	32	30	30	31	27	12	13	14	14	14	33	39	39	21	36
16	6	7	9	9	9	23	15	13	13	16	8	3	3	3	3	19	9	10	7	10
17	5	3	3	3	3	42	29	30	30	30	2	1	1	1	1	23	15	14	15	14
Tot.	1000	1000	1000	1000	1000	1000	1000	1000	1000	1000	1000	1000	1000	1000	1000	1000	1000	1000	1000	1000

Table 15 Frequency Distribution of the Estimates,
when k = 6, equal sample sizes, variance patterns I and II

class int.	m = 20						m = 25						m = 30					
	I			II			I			II			I			II		
	UW	WT	ML	UW	WT	ML	UW	WT	ML	UW	WT	ML	UW	WT	ML	UW	WT	ML
1	4	1	1	125	86	85	1	0	0	91	75	74	1	0	0	71	50	52
2	5	0	1	38	32	28	3	0	0	41	30	38	0	1	1	34	26	26
3	18	6	4	48	54	53	8	4	3	56	55	52	8	2	2	44	33	30
4	28	20	21	51	55	61	33	18	19	71	67	64	23	9	9	68	61	61
5	61	60	59	58	69	72	80	49	50	77	59	61	62	51	51	69	72	74
6	117	101	96	75	76	72	106	124	121	70	92	87	127	136	134	77	101	106
7	163	187	195	77	88	81	171	191	197	70	81	83	182	203	210	84	103	100
8	175	207	208	57	88	96	199	236	231	71	96	94	228	244	235	92	92	93
9	161	183	176	79	86	83	171	188	187	81	86	89	172	199	204	90	104	105
10	133	132	138	57	74	79	118	116	119	63	88	88	110	108	106	82	91	89
11	77	67	62	64	59	57	69	46	44	62	62	59	66	40	41	77	79	79
12	34	22	26	54	43	44	24	22	21	50	62	63	16	6	6	51	65	61
13	15	12	11	45	53	58	13	6	8	41	36	36	3	1	1	48	54	54
14	8	2	0	56	44	36	3	0	0	51	32	32	2	0	0	34	30	30
15	1	0	0	116	93	95	1	0	0	105	79	80	0	0	0	79	39	40
Tot.	1000	1000	1000	1000	1000	1000	1000	1000	1000	1000	1000	1000	1000	1000	1000	1000	1000	1000

16 Frequency Distribution of the Estimates,
hen k = 6, equal sample sizes, variance patterns III and IV

m = 20						m = 25						m = 30					
III			IV			III			IV			III			IV		
UW	WT	ML	UW	WT	ML	UW	WT	ML	UW	WT	ML	UW	WT	ML	UW	WT	ML
10	2	2	36	25	24	2	0	0	25	12	12	1	0	0	17	3	3
8	12	6	24	27	27	7	3	3	16	15	15	7	0	0	8	9	9
17	14	18	50	23	24	18	2	3	32	19	17	10	2	2	26	15	13
37	24	25	48	37	42	25	23	24	50	40	41	28	14	14	35	31	35
79	50	52	53	76	65	35	57	55	65	57	52	45	40	38	61	59	61
78	81	78	73	80	95	88	58	56	72	87	94	73	71	78	72	59	55
96	113	112	83	90	79	122	110	108	79	95	98	131	147	144	93	108	110
125	157	159	84	96	99	135	162	161	109	115	113	153	168	173	113	120	121
127	142	142	89	103	98	144	169	173	92	130	120	146	169	167	117	149	145
119	126	124	98	101	101	140	163	163	100	106	118	145	150	148	111	140	141
106	103	108	101	104	102	97	112	115	93	94	94	106	126	125	110	112	110
71	76	82	72	80	91	88	72	68	91	83	79	75	67	67	89	83	85
47	44	37	59	63	61	35	36	36	63	59	58	44	27	29	51	52	52
36	39	35	41	32	34	32	23	27	40	47	44	22	10	10	33	31	30
25	7	10	22	32	25	22	3	2	27	19	23	12	5	4	27	14	15
8	8	8	25	14	16	8	7	6	21	12	12	2	1	1	13	7	7
11	2	2	40	17	17	2	0	0	26	10	10	0	0	0	18	8	8
1000	1000	1000	1000	1000	1000	1000	1000	1000	1000	1000	1000	1000	1000	1000	1000	1000	1000

Table 17 frequeny distribution of the estimates when k=6 unequal sample size, Cases A & B, variance patterns I & II

cl. int	Case A I UWT	WT	MLE	NS	KS	Case A II UWT	WT	MLE	NS	KS	Case B I UWT	WT	MLE	NS	KS	Case B II UWT	WT	MLE	NS	KS
1	2	1	1	1	1	104	74	72	71	70	0	0	0	0	0	93	67	67	67	67
2	1	1	0	0	0	133	43	42	42	40	2	0	0	0	0	24	21	21	21	21
3	13	8	10	10	10	45	51	56	56	51	5	4	5	5	5	58	46	43	43	43
4	36	17	16	16	16	58	65	65	66	54	30	13	12	12	12	55	43	50	50	51
5	86	77	75	75	74	59	61	64	64	69	67	41	39	39	39	70	84	85	84	83
6	131	126	123	122	123	82	69	67	68	70	117	145	145	145	145	72	78	71	73	73
7	149	169	175	176	175	67	82	86	86	73	200	188	199	199	199	76	90	91	90	90
8	175	209	220	219	220	76	79	79	76	82	183	227	218	217	217	81	121	110	111	112
9	169	186	175	176	175	70	94	78	80	72	193	211	212	212	212	80	74	90	89	88
10	116	100	104	104	104	63	73	82	83	60	117	111	104	105	107	76	88	85	85	84
11	59	67	59	59	60	59	68	63	62	56	53	37	42	42	40	61	84	82	82	83
12	34	32	35	35	36	62	68	74	73	58	23	17	19	19	19	49	51	54	54	53
13	25	5	5	5	4	49	50	54	55	49	7	5	4	4	4	60	49	47	47	48
14	4	1	1	1	1	42	44	36	36	49	3	1	1	1	1	45	37	37	37	37
15	0	1	1	1	1	131	79	82	82	106	0	0	0	0	0	100	67	67	67	67
Tot	1000	1000	1000	1000	1000	1000	1000	1000	1000	1000	1000	1000	1000	1000	1000	1000	1000	1000	1000	1000

able 18 frequeny distribution of the estimates
when k=6 unequal sample size, Cases C & D,
variance patterns I & II

1. nt	Case A I UWT	WT	MLE	NS	KS	Case A II UWT	WT	MLE	NS	KS	Case B I UWT	WT	MLE	NS	KS	Case B II UWT	WT	MLE	NS	KS
1	1	1	1	1	1	100	78	75	75	75	1	0	0	0	0	89	53	54	55	55
2	3	1	1	1	1	45	28	29	29	28	1	0	0	0	0	38	35	37	38	38
3	13	3	4	4	4	55	34	37	38	38	5	2	2	2	2	50	49	45	45	45
4	24	22	21	21	21	43	67	62	60	60	38	16	17	17	17	50	50	57	56	53
5	58	43	43	43	43	60	70	80	83	85	57	50	52	52	51	68	80	72	76	77
6	124	130	133	133	132	75	77	75	73	73	120	124	119	120	121	94	84	84	78	78
7	198	207	205	204	204	71	82	81	81	80	171	191	196	197	196	69	105	105	110	110
8	195	211	209	210	212	72	78	79	81	80	207	235	231	230	233	75	85	88	87	89
9	177	190	188	189	189	80	92	91	89	92	174	221	222	221	218	84	90	91	89	88
10	112	112	116	115	114	71	90	83	83	82	137	105	104	104	106	79	79	72	74	73
11	51	57	54	55	55	63	61	72	73	69	51	38	38	38	37	58	83	88	88	89
12	27	18	19	19	19	67	64	59	59	64	27	13	14	14	14	53	49	48	47	47
13	12	5	6	5	5	46	53	49	49	48	9	5	5	5	5	46	41	43	42	42
14	4	0	0	0	0	41	50	54	53	52	1	0	0	0	0	44	49	53	53	51
15	1	0	0	0	0	111	76	74	74	74	1	0	0	0	0	103	68	63	64	65
Tot	1000	1000	1000	1000	1000	1000	1000	1000	1000	1000	1000	1000	1000	1000	1000	1000	1000	1000	1000	1000

Table 19 frequeny distribution of the estimates
when k=6 unequal sample size, Cases A & B,
variance patterns III & IV

cl. int	Case A										Case B									
	III					IV					III					IV				
	UWT	WT	MLE	NS	KS	UWT	WT	MLE	NS	KS	UWT	WT	MLE	NS	KS	UWT	WT	MLE	NS	KS
1	5	0	0	0	0	18	19	19	19	28	2	0	0	0	0	27	15	15	15	15
2	11	5	5	5	5	18	16	16	16	15	2	1	1	1	1	21	8	9	9	9
3	14	8	10	10	10	26	26	26	27	31	8	9	10	9	9	28	20	20	20	20
4	32	22	21	22	23	36	36	36	36	52	26	11	10	11	11	43	39	35	35	35
5	65	47	46	45	43	52	54	55	55	54	49	34	34	34	34	67	60	61	61	60
6	83	89	92	93	93	85	87	83	82	81	81	78	84	84	34	62	77	73	73	75
7	113	121	117	116	117	86	83	87	86	80	117	132	123	123	122	75	99	105	105	104
8	120	152	153	152	151	108	106	106	107	85	145	161	161	162	164	111	102	99	99	99
9	117	130	137	138	140	113	115	114	112	96	142	159	165	165	164	111	140	143	143	143
10	132	158	150	150	151	116	121	113	115	126	146	156	151	150	150	114	92	89	89	89
11	102	111	115	115	112	84	91	91	88	73	115	122	123	123	123	95	113	113	113	113
12	84	66	62	62	63	92	82	81	85	70	90	74	78	78	78	73	81	88	88	81
13	62	50	49	49	49	68	80	80	78	69	40	40	36	36	36	59	71	64	64	65
14	31	26	28	28	28	43	39	39	39	48	23	18	19	19	19	47	39	44	44	43
15	15	8	8	8	8	19	19	19	20	32	10	3	3	3	3	30	22	20	20	21
16	12	5	5	5	5	17	19	20	20	28	1	1	1	1	1	16	7	7	7	7
17	2	2	2	2	2	19	16	15	15	32	3	1	1	1	1	21	5	15	15	15
Tot	1000	1000	1000	1000	1000	1000	1000	1000	1000	1000	1000	1000	1000	1000	1000	1000	1000	1000	1000	1000

Table 20 frequency distribution of the estimates,
when k=6 unequal sample size, Cases C & D,
variance patterns III & IV

cl. int	Case B III UWT	III WT	III MLE	III NS	III KS	Case B IV UWT	IV WT	IV MLE	IV NS	IV KS
1	6	3	2	2	2	29	16	15	15	15
2	6	2	2	2	2	18	18	21	20	20
3	10	8	10	10	9	35	28	25	25	26
4	34	24	24	25	26	37	45	44	44	43
5	52	33	33	32	32	59	51	50	51	52
6	67	82	79	79	79	79	68	71	70	67
7	104	109	110	112	111	74	89	85	85	91
8	133	163	158	158	157	97	108	113	113	108
9	122	146	158	154	155	102	125	123	127	128
10	152	136	135	138	137	96	83	86	85	87
11	135	134	130	130	133	100	104	115	109	109
12	81	73	77	76	74	62	97	88	92	91
13	45	49	43	43	44	74	49	56	54	53
14	62	25	26	26	26	43	43	40	43	42
15	18	7	7	7	7	33	26	29	28	29
16	4	4	4	4	4	25	14	15	14	14
17	5	2	2	2	2	37	26	24	25	25
Tot	1000	1000	1000	1000	1000	1000	1000	1000	1000	1000

cl. int	Case D II UWT	II WT	II MLE	II NS	II KS	Case D IV UWT	IV WT	IV MLE	IV NS	IV KS
1	1	0	0	0	0	26	15	15	15	15
2	5	2	2	2	2	19	15	15	15	15
3	11	2	2	2	2	37	25	25	25	25
4	22	19	17	17	17	34	33	36	36	36
5	43	31	36	37	37	65	53	45	45	45
6	86	80	79	78	78	76	100	106	106	106
7	114	115	110	112	113	86	102	101	105	105
8	152	168	165	164	163	122	120	117	113	111
9	157	189	197	195	193	98	107	102	99	102
10	131	157	158	158	160	102	115	123	124	125
11	121	116	110	111	110	80	88	92	93	90
12	77	73	75	75	75	99	82	82	85	86
13	44	33	34	34	35	58	60	56	54	53
14	25	12	11	11	11	39	36	36	36	37
15	9	3	3	3	3	24	23	23	23	23
16	2	0	1	1	1	15	17	18	18	18
17	0	0	0	0	0	2	9	8	8	8
Tot	1000	1000	1000	1000	1000	1000	1000	1000	1000	1000

Table 21 Frequency distribution of the estimates
when k=8, equal sample sizes, variance patterns I and II

cl. int.	m=20						m=25						m=30					
	I			II			I			II			I			II		
	UWT	WT	MLE	UWT	WT	MLE	UWT	WT	MLE	UWT	WT	MLE	UWT	WT	MLE	UWT	WT	MLE
1	0	0	0	97	80	0	0	0	0	88	63	61	0	0	0	75	43	45
2	4	2	2	38	30	1	1	0	0	39	33	38	0	0	0	33	31	28
3	6	6	6	56	41	6	6	2	2	47	38	35	2	0	0	46	50	52
4	20	13	12	68	69	23	23	11	12	61	60	61	12	2	2	62	56	54
5	17	54	51	77	79	55	55	40	39	65	71	64	46	29	28	71	86	90
6	121	116	122	74	93	123	123	98	100	73	74	79	111	96	100	88	93	89
7	181	200	193	72	84	179	179	202	200	68	89	93	208	227	227	113	106	104
8	236	226	233	89	80	217	217	282	286	67	102	104	249	306	302	85	108	108
9	160	205	204	59	83	185	185	187	179	105	102	94	193	196	201	89	106	104
10	111	116	118	64	67	108	108	123	130	71	81	85	122	106	106	71	79	88
11	50	48	44	47	65	75	75	38	38	64	82	80	38	32	32	67	81	80
12	30	8	10	60	52	22	22	16	13	59	63	61	11	6	5	64	55	52
13	10	6	5	43	50	4	4	1	1	43	43	45	7	0	0	32	41	43
14	0	0	0	35	32	2	2	0	0	49	31	29	1	0	0	33	31	27
15	0	0	0	121	95	0	0	0	0	101	68	71	0	0	0	71	34	36
Tot	1000	1000	1000	1000	1000	1000	1000	1000	1000	1000	1000	1000	1000	1000	1000	1000	1000	1000

e 22 Frequency distribution of the estimates
when k=8, equal sample sizes, variance patterns III and IV

m=20						m=25						m=30					
III			IV			III			IV			III			IV		
UWT	WT	MLE	UWT	WT	MLE	UWT	WT	MLE	UWT	WT	MLE	UWT	WT	MLE	UWT	WT	MLE
8	2	2	45	20	18	5	0	0	30	11	9	1	0	0	10	5	5
9	3	2	24	12	13	2	1	2	18	17	18	1	2	1	11	3	3
19	9	10	34	26	27	17	7	6	23	17	20	9	3	3	23	15	13
35	24	23	47	49	49	26	18	15	42	32	30	21	9	10	47	27	27
69	44	41	64	56	56	42	27	31	62	58	56	35	28	28	52	54	53
71	78	81	60	86	82	73	72	75	62	89	81	76	60	64	90	79	84
104	145	134	92	93	89	113	130	124	105	103	106	101	105	101	99	121	118
115	139	151	98	94	98	125	146	139	90	97	89	178	191	196	106	132	132
136	153	154	107	129	143	165	177	188	103	126	136	164	207	202	118	147	147
119	136	131	81	117	111	134	171	175	98	116	119	140	152	154	112	132	132
119	106	109	79	90	85	130	125	177	103	107	110	107	125	125	84	98	99
82	87	89	75	74	75	85	78	83	76	69	63	91	68	72	84	75	80
53	33	30	57	55	55	44	28	26	69	71	70	43	34	31	58	54	46
28	27	29	43	39	39	25	14	13	43	36	34	19	11	10	41	22	24
21	9	10	35	25	21	8	2	3	34	27	29	9	5	5	27	17	18
7	5	4	26	13	17	4	3	2	23	16	15	4	0	0	20	13	14
5	0	0	33	22	21	2	1	1	19	8	9	1	0	0	18	6	5
1000	1000	1000	1000	1000	1000	1000	1000	1000	1000	1000	1000	1000	1000	1000	1000	1000	1000

ble 23 Frequencyy distribution of the estimates
when k=8 unequal sample size, Cases A & B,
variance patterns I & II

	Case A										Case B									
	I					II					I					II				
t	UWT	WT	MLE	NS	KS	UWT	WT	MLE	NS	KS	UWT	WT	MLE	NS	KS	UWT	WT	MLE	NS	KS
1	0	0	0	0	0	110	79	77	78	78	0	0	0	0	0	78	39	44	43	42
2	0	0	0	0	0	651	34	34	33	33	0	0	0	0	0	23	44	36	36	37
3	2	0	0	0	0	41	55	56	57	58	5	0	0	0	0	44	34	33	34	35
4	26	9	8	8	8	47	52	56	54	52	14	8	8	8	8	50	55	54	54	53
5	49	46	43	44	44	60	72	69	70	71	36	35	36	36	34	74	70	75	75	75
6	120	116	124	122	122	64	81	77	77	75	107	105	105	106	108	70	70	69	70	70
7	179	191	189	190	190	80	76	71	72	75	204	187	189	188	188	90	86	88	87	87
8	193	244	239	238	238	89	81	82	82	84	242	298	293	293	293	81	107	109	109	110
9	199	215	218	219	219	73	90	97	97	92	208	237	235	235	236	89	119	113	115	114
10	127	105	111	110	110	72	81	85	85	96	123	99	102	102	101	84	98	103	102	101
11	56	54	48	48	48	51	75	72	72	71	44	28	29	29	29	78	81	84	85	86
12	30	16	18	18	18	52	63	68	69	70	14	3	3	3	3	67	66	60	60	59
13	9	3	2	2	2	52	65	65	63	62	1	0	0	0	0	53	40	41	41	42
14	0	0	0	0	0	45	28	20	21	22	0	0	0	0	0	28	39	42	42	41
15	0	0	0	0	0	113	68	71	71	71	0	0	0	0	0	89	46	47	47	48
Tot	1000	1000	1000	1000	1000	1000	1000	1000	1000	1000	1000	1000	1000	1000	1000	1000	1000	1000	1000	1000

ble 24 Frequency distribution of the estimates
when k=8, unequal sample size Case C & D,
variance patterns I & II

	Case C										Case D									
	I					II					I					II				
t	UWT	WT	MLE	NS	KS	UWT	WT	MLE	NS	KS	UWT	WT	MLE	NS	KS	UWT	WT	MLE	NS	KS
1	0	0	0	0	0	89	63	64	64	63	0	0	0	0	0	71	45	46	46	46
2	1	0	0	0	0	37	29	31	31	28	0	0	0	0	0	43	36	34	34	34
3	0	1	1	1	1	53	46	39	39	43	4	0	0	0	0	54	50	49	49	48
4	24	11	9	9	9	64	68	70	69	69	16	4	5	5	5	50	63	64	65	65
5	55	43	47	47	47	59	69	75	74	74	50	35	34	33	32	68	63	68	67	69
6	128	105	112	112	112	72	95	98	99	98	92	88	89	90	90	78	98	92	92	93
7	188	198	202	202	204	83	95	85	87	89	198	207	204	203	204	71	95	96	97	97
8	235	269	268	269	268	92	98	101	102	104	244	284	284	285	285	73	104	109	108	107
9	182	210	207	206	205	68	87	90	87	84	203	241	241	242	241	96	112	112	113	1110
10	106	97	96	97	96	71	74	78	78	78	131	103	105	104	105	96	93	90	88	92
11	52	43	45	45	46	61	78	73	74	74	48	28	29	30	30	76	77	76	77	76
12	26	11	12	11	11	53	67	68	68	69	12	10	9	8	8	62	60	61	61	61
13	2	2	1	1	1	53	42	42	42	41	2	0	0	0	0	53	44	47	46	46
14	1	0	0	0	0	48	39	37	37	38	0	0	0	0	0	35	25	21	22	21
15	0	0	0	0	0	97	49	49	48	48	0	0	0	0	0	74	35	35	35	35
Tot	1000	1000	1000	1000	190	1000	1000	1000	1000	1000	1000	1000	1000	1000	1000	1000	1000	1000	1000	1000

Table 25 Frequey distribution of the estimates, when k=8, unequal sample size, Case A & B, variance patterns III & IV

cl. int.	Case A III UWT	WT	MLE	NS	KS	Case A IV UWT	WT	MLE	NS	KS	Case B III UWT	WT	MLE	NS	KS	Case B IV UWT	WT	MLE	NS	KS
1	10	1	1	1	1	28	7	8	8	8	1	0	0	0	0	19	8	8	8	8
2	5	2	1	1	1	28	15	14	14	14	2	0	0	0	0	13	9	7	7	8
3	18	15	16	16	16	28	28	28	28	28	14	5	5	5	5	24	15	15	15	14
4	33	17	18	17	17	48	30	28	28	28	21	12	12	12	12	46	28	29	29	29
5	42	44	43	44	44	70	63	63	63	63	52	37	36	36	35	67	59	61	62	62
6	79	75	77	77	77	79	98	99	100	100	78	70	73	74	76	80	79	79	78	77
7	115	104	108	108	108	86	110	110	109	109	109	111	109	109	106	86	115	112	112	113
8	138	175	169	169	167	92	119	121	121	121	133	160	160	159	161	81	121	128	128	128
9	135	178	173	173	174	93	110	113	113	114	159	186	180	181	182	117	137	132	132	132
10	119	134	138	138	138	90	110	107	107	106	154	180	183	180	179	104	121	126	125	125
11	102	115	119	120	120	95	88	89	89	88	111	104	111	114	113	123	100	96	97	97
12	85	60	55	54	55	77	78	81	80	83	88	87	81	80	81	74	89	85	85	85
13	62	39	41	42	42	62	56	50	51	49	26	31	33	33	33	54	57	61	61	61
14	35	25	27	26	26	40	29	30	30	30	35	14	11	11	11	49	29	28	28	28
15	10	12	9	9	9	21	28	32	32	32	14	3	6	5	6	29	13	12	12	12
16	10	1	2	2	2	23	18	15	15	15	2	0	0	0	0	14	12	12	12	12
17	2	3	3	3	3	40	13	12	12	12	1	0	0	0	0	21	8	9	9	9
Tot	1000	1000	1000	1000	190	1000	1000	1000	1000	1000	1000	1000	1000	1000	1000	1000	1000	1000	1000	1000

Table 26 Frequency distribution of the estimates when k=8, unequal sample size Case C & D, variance patterns III and IV

cl.	Case C										Case D									
	III					IV					III					IV				
int	UWT	WT	MLE	NS	KS	UWT	WT	MLE	NS	KS	UWT	WT	MLE	NS	KS	UWT	WT	MLE	NS	KS
1	3	0	0	0	0	32	22	22	22	23	2	0	0	0	0	24	8	8	8	8
2	5	3	3	4	4	20	18	19	19	18	0	0	0	0	0	26	12	11	11	11
3	10	8	8	7	6	33	25	22	21	21	6	1	1	1	1	31	23	25	25	25
4	26	15	14	14	16	45	35	34	35	35	19	13	11	11	12	53	31	28	27	27
5	42	28	30	29	28	64	59	70	70	71	44	39	40	40	39	63	56	58	59	59
6	61	60	64	65	63	8181	75	67	67	65	90	70	65	65	65	67	89	88	89	88
7	121	117	116	117	121	90	103	101	98	100	111	112	123	124	122	96	93	96	96	97
8	136	166	160	162	157	93	114	115	118	117	154	168	157	156	158	91	138	132	131	131
9	153	188	186	181	184	103	118	122	121	124	153	176	186	184	183	102	117	119	118	117
10	149	122	126	128	127	105	116	109	110	104	132	184	186	188	188	115	125	132	124	125
11	105	131	133	135	136	85	104	113	113	115	118	124	116	115	117	111	99	103	103	103
12	84	91	91	89	88	69	87	84	85	85	80	72	74	75	75	58	77	79	77	77
13	53	39	38	38	39	69	50	52	51	51	54	23	23	23	22	66	60	59	63	63
14	32	24	22	22	22	46	28	25	25	27	27	14	14	14	14	31	28	32	30	30
15	13	7	8	8	8	30	28	28	27	25	8	1	2	2	2	17	22	17	18	18
16	6	1	1	1	1	18	12	10	11	12	2	3	2	2	2	27	12	12	12	12
17	1	0	0	0	0	17	6	7	7	7	2	0	0	0	0	22	10	10	9	9
Tot	1000	1000	1000	1000	190	1000	1000	1000	1000	1000	1000	1000	1000	1000	1000	1000	1000	1000	1000	1000

7 Frequency Distribution of the Estimates,
hen k = 10, equal sample sizes, variance patterns I & II

m = 20						m = 25						m = 30					
I			II			I			II			I			II		
UW	WT	ML	UW	WT	ML	UW	WT	ML	UW	WT	ML	UW	WT	ML	UW	WT	ML
0	0	0	122	81	79	0	0	0	82	50	45	0	0	0	64	36	33
0	0	0	37	37	44	0	0	0	26	27	30	0	0	0	41	25	24
2	1	1	45	53	50	0	0	0	45	27	27	0	0	0	43	33	38
18	1	0	69	53	47	8	1	1	52	55	61	5	0	0	51	64	65
52	41	37	63	64	65	55	21	19	74	71	72	31	15	13	64	77	76
120	90	89	65	82	80	115	103	107	67	87	88	113	78	77	85	105	99
184	225	226	65	88	98	213	232	232	86	95	89	216	214	219	108	84	94
231	284	278	69	85	73	240	319	316	83	110	114	277	382	374	95	124	120
209	219	223	66	101	109	205	221	225	78	102	91	228	231	232	107	101	102
111	112	113	75	76	82	104	79	78	80	105	113	93	70	73	67	106	101
55	20	16	57	72	68	47	20	18	81	78	76	30	8	11	69	76	77
18	7	8	62	60	61	11	4	4	64	61	60	6	2	1	62	61	66
0	0	0	56	44	41	2	0	0	45	57	61	1	0	0	38	46	45
0	0	0	33	34	35	0	0	0	49	30	31	0	0	0	43	30	29
0	0	0	116	70	68	0	0	0	88	45	42	0	0	0	63	32	31
1000	1000	1000	1000	1000	1000	1000	1000	1000	1000	1000	1000	1000	1000	1000	1000	1000	1000

e 28 Frequency Distribution of the Estimates,
when k = 10, equal sample sizes, variance patterns III and IV

	m = 20						m = 25						m = 30					
s	III			IV			III			IV			III			IV		
	UW	WT	ML	UW	WT	ML	UW	WT	ML	UW	WT	ML	UW	WT	ML	UW	WT	ML
	6	2	2	38	16	16	1	0	0	18	7	7	1	0	0	12	0	0
2	6	3	2	29	15	12	4	1	1	25	9	9	4	1	1	10	4	5
3	18	12	12	33	23	30	13	3	5	17	16	15	5	3	3	25	10	11
4	40	25	26	43	42	43	34	11	10	45	40	44	21	10	9	39	27	26
5	46	34	32	47	56	53	52	31	29	63	58	58	33	21	17	61	45	44
6	74	62	70	73	88	88	67	68	74	91	88	79	81	58	63	79	84	88
7	99	106	104	102	93	87	121	113	113	86	98	105	102	113	110	86	109	108
8	123	152	142	82	107	109	137	172	170	107	130	134	146	171	176	122	142	141
9	151	175	180	89	109	104	160	195	190	95	138	128	177	126	210	118	156	156
0	122	153	153	77	122	119	138	174	114	91	115	120	142	175	178	105	135	134
11	105	102	103	86	94	88	100	115	118	106	102	99	119	118	117	113	117	116
12	98	78	84	80	72	78	92	60	57	86	74	78	81	73	76	87	71	69
13	54	76	42	54	66	67	34	35	37	53	52	51	56	29	29	39	51	51
14	27	31	33	47	38	38	30	17	17	31	40	40	14	11	11	53	21	23
15	13	15	12	44	19	23	9	5	5	28	11	14	10	1	0	21	17	17
	6	1	1	30	23	21	3	0	0	24	18	16	5	0	0	17	7	7
	12	2	2	46	17	14	5	0	0	24	4	3	3	0	0	13	4	4
t.	1000	1000	1000	1000	1000	1000	1000	1000	1000	1000	1000	1000	1000	1000	1000	1000	1000	1000

equency Distribution of the Estimates,
= 10, unequal sample size, cases A and B
ce patterns I and II

	Case A									Case B									
	I				II					I					II				
	WT	MLE	NS	KS	UWT	WT	MLE	NS	KS	UWT	WT	MLE	NS	KS	UWT	WT	MLE	NS	KS
	0	0	0	0	115	66	61	61	61	0	0	0	0	0	79	41	42	42	42
	0	0	0	0	43	34	31	32	32	0	0	0	0	0	42	49	47	47	47
	0	0	0	0	53	62	69	68	68	0	0	0	0	0	54	45	42	42	42
	6	6	6	6	66	65	67	67	67	6	0	1	1	1	45	52	60	59	59
	24	25	25	25	51	68	66	65	66	42	15	14	14	14	63	80	75	76	75
	107	104	104	104	69	83	91	93	93	104	95	94	94	94	80	86	90	90	92
3	206	212	212	212	79	78	76	76	75	196	206	207	207	207	81	89	88	88	38
5	301	296	294	294	79	93	39	87	88	282	332	323	323	324	82	96	101	102	100
9	212	220	222	222	79	79	73	74	74	214	249	260	261	259	92	112	105	104	105
7	103	101	100	100	58	82	83	81	82	121	87	85	84	85	75	102	96	96	96
8	35	29	30	30	58	80	85	86	86	29	15	15	15	15	79	74	73	72	71
3	6	7	7	7	60	58	62	63	62	4	1	1	1	1	65	56	62	63	64
0	0	0	0	0	54	55	44	44	44	1	0	0	0	0	49	49	51	51	51
2	0	0	0	0	36	32	41	63	40	1	0	0	0	0	39	24	25	25	25
0	0	0	0	0	100	65	62	63	63	0	0	0	0	0	75	45	43	43	43
0	1000	1000	1000	1000	1000	1000	1000	1000	1000	1000	1000	1000	1000	1000	1000	1000	1000	1000	1000

Frequency Distribution of the Estimates,
k = 10, unequal sample size, cases C and D
ance patterns I and II

Case C										Case D									
I					II					I					II				
WT	WT	MLE	NS	KS	UWT	WT	MLE	NS	KS	UWT	WT	MLE	NS	KS	UWT	WT	MLE	NS	KS
0	0	0	0	0	102	72	70	70	70	0	0	0	0	0	82	35	36	36	36
1	0	0	0	0	32	31	33	33	33	0	0	0	0	0	31	30	31	30	29
3	1	0	0	0	46	50	47	48	47	0	0	0	0	0	41	42	32	32	32
8	7	6	6	6	83	48	47	46	45	9	1	0	0	0	71	53	62	63	61
80	29	33	33	33	57	73	78	76	80	49	18	21	21	21	59	84	79	78	79
103	86	81	82	82	81	84	86	87	87	106	94	96	97	96	77	77	81	80	80
187	215	217	216	216	84	95	90	91	89	198	237	231	231	233	68	92	95	98	99
261	322	322	321	322	76	105	105	105	105	259	319	321	319	320	81	115	106	105	108
224	209	207	211	212	70	88	91	91	90	219	220	220	221	220	97	102	11	112	110
107	99	104	101	99	76	77	79	78	78	113	89	91	91	90	90	111	101	99	100
41	28	26	26	26	62	84	80	78	80	36	21	19	19	19	68	85	90	90	90
14	2	3	3	3	49	53	59	63	63	9	1	1	1	1	58	55	59	61	60
1	2	1	1	1	46	49	47	45	44	1	0	0	0	0	50	51	48	47	47
0	0	0	0	0	37	33	30	31	31	1	0	0	0	0	49	28	32	33	33
0	0	0	0	0	99	58	58	58	58	0	0	0	0	0	78	40	37	36	36
1000	1000	1000	1000	1000	1000	1000	1000	1000	1000	1000	1000	1000	1000	1000	1000	1000	1000	1000	1000

Frequency Distribution of the Estimates,
n k = 10, unequal sample size, cases A and B,
iance patterns III and IV

Case A										Case B									
III					IV					III					IV				
UWT	WT	MLE	NS	KS	UWT	WT	MLE	NS	KS	UWT	WT	MLE	NS	KS	UWT	WT	MLE	NS	KS
8	0	0	0	0	30	15	19	19	20	2	0	0	0	0	15	6	6	6	6
7	4	6	6	6	17	12	13	13	12	5	0	0	0	0	16	3	1	1	1
12	8	5	5	5	33	19	20	20	20	8	0	0	0	0	33	13	16	17	17
40	25	27	27	27	60	36	37	37	36	18	9	10	10	10	44	38	35	33	33
65	54	54	54	53	61	54	56	58	59	52	35	33	33	33	58	57	64	65	65
90	80	77	77	79	81	85	79	77	78	91	71	66	67	67	77	95	88	87	87
115	125	128	127	125	77	88	87	86	85	117	120	130	129	129	87	102	107	109	107
128	154	152	153	154	855	110	108	110	109	141	194	192	193	193	101	137	128	127	129
127	155	162	162	159	102	122	122	123	123	142	168	163	161	161	125	110	113	114	113
108	150	143	144	148	80	118	115	114	115	146	176	179	180	179	109	129	126	125	126
110	111	110	110	109	95	97	98	97	97	111	119	118	120	121	97	115	129	129	129
87	74	78	77	77	83	86	90	90	90	90	65	65	64	64	84	81	71	72	72
56	38	36	36	36	56	66	71	70	70	39	26	26	25	25	56	52	54	53	53
32	14	15	15	15	51	35	32	33	33	21	11	13	13	13	44	35	35	35	35
8	6	5	5	5	33	23	23	23	23	13	3	2	2	2	29	14	15	15	15
3	2	2	2	2	25	17	16	16	16	2	2	2	2	2	9	8	7	7	7
4	0	0	0	0	31	14	14	14	14	2	1	1	1	1	16	5	5	5	5
1000	1000	1000	1000	1000	1000	1000	1000	1000	1000	1000	1000	1000	1000	1000	1000	1000	1000	1000	1000

ency Distribution of the Estimates,
0, unequal sample size, cases C and D,
atterns III and IV

	Case C								Case D									
	III			IV					III					IV				
	MLE	NS	KS	UWT	WT	MLE	NS	KS	UWT	WT	MLE	NS	KS	UWT	WT	MLE	NS	KS
1	1	1	1	29	13	13	13	14	0	0	0	0	0	16	4	3	3	3
1	1	1	1	17	10	7	7	6	2	0	0	0	0	17	9	10	10	10
3	2	2	2	40	14	15	15	15	10	4	4	4	4	27	22	22	22	22
6	17	15	15	38	32	37	38	39	21	9	9	9	9	45	17	18	18	18
4	41	44	44	53	62	52	52	51	51	32	36	36	37	54	39	39	40	40
9	73	71	70	79	84	91	90	90	89	89	79	79	79	78	70	69	69	72
0	113	114	114	98	112	109	108	109	99	111	116	116	115	96	123	128	127	122
2	149	147	147	97	113	117	116	112	160	1175	182	182	179	119	181	173	174	176
1	162	165	168	98	113	109	111	116	162	174	172	172	175	117	134	140	137	138
7	169	169	164	109	132	133	135	133	145	166	165	165	169	88	120	113	118	116
37	130	129	133	91	99	102	100	100	90	120	115	115	110	86	118	127	124	125
74	80	79	78	80	89	88	90	89	87	65	67	67	69	92	58	57	57	57
39	36	37	37	56	46	48	47	49	41	36	37	37	36	63	46	42	42	42
20	22	22	21	49	39	41	40	39	25	14	13	13	13	36	31	30	29	28
22	0	0	1	25	22	17	17	17	9	3	4	4	4	25	13	14	15	16
2	2	2	2	15	11	11	11	11	7	1	0	0	0	22	7	8	8	7
2	2	2	2	26	9	10	10	10	2	1	1	1	1	19	8	7	7	8
00	1000	1000	1000	1000	1000	1000	1000	1000	1000	1000	1000	1000	1000	1000	1000	1000	1000	1000

BIBLIOGRAPHY

1. Atkinson, A. C. and Pearce, M. C.(1976). The Computer Generation of Beta, Gamma and Normal Random Variables. Journal of the Royal Statistical Society, Ser. A, 139,431-61.

2. Bhattacharya, C.G.(1979). A Note on Estimating the Common Mean of k-Normal Populations. Sankhyā, 40, Ser. B, 272-5.

3. Box, G.E.P. and Muller, M.E.(1958). A Note on the Random Normal Deviates. Annals of Mathematical Statistics, 29, 610-1.

4. Chaubey, Y.P.(1983). A Non-negative Estimator of Variance Component Closest to MINQUE. Sankhyā, Ser. A, 45, 201-11.

5. Cochran, W.G.(1937). Problems Arising in the Analysis of a Series of Experiments. Journal of the Royal Statistical Society, Supplement 4, 102-08.

6. Cochran, W.G., & Caroll, S. P.(1953). A Sampling Investigation of the Efficiency of Weighting Inversely as the

7. Cox, D.R., & Hinkley, D.V.(1974). Theoretical Statistics. Weiley, New York.

8. Graybill, F.A., & and Deal, R.B.(1959). Combinig Unbiased Estimators. Biometrics, 15, 543-50.

9. Hoaglin, D.C., Mosteller, F. and Tukey, J.W.(1983). Understanding Robust and Exploratory Data Analysis. Wiley, New York.

10. Kalbfleish, J.D., & Sprott, D.A.(1970). Application of Likelihood Models Involving Large Numbers of Parameters: (With Discussion). Journal of the Royal Statistical Society, Ser. B, 32, 175-208.

11. Kelton, W.D. and Law A.M.(1982). Simulation, Modelling and Analysis. McGraw-Hill, New York.

12. Kinderman, A.J., & Ramage, J.G.(1976). Computer Generation of Normal Random Variables. Journal of the American Statistical Association, 71, 893-6.

13. Kubokawa, T.(1987). Admissible Minmax Estimation of a Common Mean of Two Normal Populations. The Annals of Statistics, 15, 3, 1245-56.

14. Lewis, T.G. and Smith, B.J.(1979). Computer Principles of Modelling and Simulation. Houghton Miffin Company, Boston.

15. Levy, P.S.(1970). Combining Independent Estimators - an Empirical Study. Technometrics, 12, 162-5.

16. Levy, P.S.(1974). Combining Unbiased Estimates - a Further Examination of Some Old Estimators. Journal of Statistical Computation and Simulation, 3, 147-60.

17. Meier, P.(1953). Variance of a Weighted Mean. Biometrics, 9, 59-73.

18. Meyer, H.A.(1954). Symposium on Monte Carlo Methods. Wiley, New York.

19. Morgan, B.J.T.(1984). Elements of Simulation. Chapman and Hill, New York.

20. Neyman, J., & Scott, E.(1948). Consistent Estimations Based on Partially Consistent Observations. Econometrics, 16, 1-32.

21. Norwood, T.E. & Hinkelmann, K.(1977). Estimating the Common Mean of Several Normal Populations. The Annals of Statistics, 5, 1047-50.

22. Ohatni, K.(1987). On th Use of Graybill-Deal Estimator When Two Normal Population Means May Not be Common. Communications in Statistics Simulation and Computation, 16, 3, 855-70.

23. Press, W.H., Flanney, B.P., Teukolsky, S.A. & Vetterling, W.T.(1986). Numerical Recipes, Cambridge University Press, Cambridge.

24. Rao, C.R.(1970). Estimation of Hetroscedastic Variances in Linear Models. Journal of the American Statistical Association, 65, 161-72.

25. Rao, C.R.(1973). Linear Statistical Inference and Applications. 2nd Edition, Wiley, New York.

26. Rao, C.R. & Kleffe, J.(1988). Estimation of Variance Components and Applications. North-Holland, Amsterdam.

27. Rao, J.N.K.(1973). On the Estimation of Heteroscedastic Variances. Biometrics, 29, 11-24.

28. Rao, J,N.K.(1980). Estimating the Common Mean of Possibly Different Normal Populations: A Simulation Study. Journal of the American statistical Association, 75, 447-53.

29. Rao, J.N.K. & Subrahmaniam(1971). Combining Independent Estimators and Estimation in Linear Regression with Unequal Variances. Biometrics, 27, 971-90.

30. Shinozaki, N.(1978). A Note on Estimating the Common Mean of k Normal Distributions and the Stein Problem. Communications in Statistics Theory and Methods, 15, 1421-32.

31. Sinha, B. K.(1979). Is the Maximum Likelihood Estimate of the Common Mean of Several Normal Populations Admissible? Sankhyā, 40, Ser. B, 192-6.

32. Sinha, S.K.(1983). Bayesian Estimation of the Mean of a Normal Distribution When the Coefficients of Variation is Known. The Statistcian, 32, 339-45.

33. Shreider, Y.A.(1966). The Monte Carlo, Method of Statitical Trials. Pergamon Press, London.

34. Voinov, V.G(1984). Variance and Its Unbiased Estimator for the Common Mean of Several Normal Populations. Sankhyā. Ser. B, 46, 291-300.

35. Wichmann, B.A., & Hill, I.D.(1982). An efficient and Portable Random Number Generator. Applied Statistics, 31, 2, 188-90.

36. Zacks, S. (1966). Unbiased Estimation of the Common Mean of Two Normal Distributions Based on Small Samples of Equal Size. Journal of the American Association, 61, 467-76.

DECLARATION

The thesis is my original work and has not been presented for a degree in any other university.

Gebre-Egziabher Kiros

This thesis has been submitted for examination with my approval as university advisor.

Dr. Abebe Tessera