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DEPARTMENT OF MATHEMATICS



ON SOME RECURRENCE RELATIONS FOR PARTITION FUNCTIONS
A Thesis Submitted to the Department of Mathematics of Addis Ababa University in Partial Fulfillment of the requirements of the Master of Science Degree in Mathematics

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We, the under signed, hereby certify that we have read and examined this thesis, a thesis on some recurrence relations for partition functions, which is done by GOSA TEKOLA in partial fulfillment of requirement for the degree of master of science in Mathematics and recommend to the school of graduate studies for acceptance of thesis.

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ABSTRACT

Despite the definition of integer partition function is simple, many results on integer partitions can be shockingly difficult to obtain. The thesis comprises of introductory properties and topic of some recurrence relations for additive partition functions. In particular, it provides a formula for $p(z)$ to find the number of partitions of positive integer z . We proved some recurrence relations for restricted partition functions. At the end, a chart showing values of $p(z)$ for integers $1 \leq z \leq 280$ is included.

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NOTATIONS

\mathbf{Z} integer greater than or equal to 1

$\mathbf{P}(\mathbf{z})$ quantity of partitions of \mathbf{z}

$\mathbf{P}(\mathbf{z}, \mathbf{y})$ quantity of partitions of \mathbf{z} into \mathbf{y} summands

$\mathbf{A}_{\mathbf{y}, \mathbf{k}}$ the set of $\mathbf{y}-t_k$ where $t \in \mathbf{Z}$ and $0 \leq t \leq \left\lfloor \frac{\mathbf{y}}{\mathbf{k}} \right\rfloor$, and $\mathbf{y}, \mathbf{k} \in \mathbf{Z}$

$\mathbf{P}_k(\mathbf{z}, \mathbf{y})$ quantity of partitions of \mathbf{z} with largest part \mathbf{y} into given elements of $\mathbf{A}_{\mathbf{y}, \mathbf{k}}$

$\mathbf{D}_k(\mathbf{z}, \mathbf{y})$ quantity of distinct partitions of \mathbf{z} with largest part \mathbf{y} into given elements of $\mathbf{A}_{\mathbf{y}, \mathbf{k}}$

$\mathbf{B}_{\mathbf{y}, \mathbf{k}, \mathbf{z}}$ the set of $\mathbf{y}+t_k$ where $t \in \mathbf{Z}$ and $0 \leq t \leq \left\lfloor \frac{\mathbf{z}-\mathbf{y}}{\mathbf{k}} \right\rfloor$, and $\mathbf{z}, \mathbf{y}, \mathbf{k} \in \mathbf{Z}$

$\mathbf{\Psi}_k(\mathbf{z}, \mathbf{y})$ quantity of \mathbf{z} with largest part \mathbf{y} into given elements of $\mathbf{\beta}_{\mathbf{y}, \mathbf{k}, \mathbf{z}}$

$\mathbf{\delta}_k(\mathbf{z}, \mathbf{y})$ quantity of distinct partitions of \mathbf{z} with largest part \mathbf{y} into given elements of $\mathbf{\beta}_{\mathbf{y}, \mathbf{k}, \mathbf{z}}$

$\mathbf{\Omega}_k(\mathbf{z}, \mathbf{y})$ quantity of partitions of \mathbf{z} with least part \mathbf{y} into given elements of $\mathbf{\beta}_{\mathbf{y}, \mathbf{k}, \mathbf{z}}$

$\mathbf{\Delta}_k(\mathbf{z}, \mathbf{y})$ quantity of distinct partitions of \mathbf{z} with least part \mathbf{y} into given elements of $\mathbf{\beta}_{\mathbf{y}, \mathbf{k}, \mathbf{z}}$

$\mathbf{P}_e(\mathbf{z})$ quantity of partitions of \mathbf{z} onto an even number of parts

$\mathbf{P}_o(\mathbf{z})$ quantity of partitions of \mathbf{z} onto an odd number of parts

$\lfloor \mathbf{x} \rfloor$ The best integer much less than or identical to \mathbf{x}

$\mathbf{Pd}(\mathbf{z})$ range of walls of \mathbf{z} into awesome parts

$\prod(1-x^y)^{-1}$ Infinite product

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INTRODUCTION

One of the pleasures of studying mathematics lies in discovering unexpected relationships. Similarly, this thesis delves into some intriguing interactions between partition functions, Ferrers diagrams, and pentagonal numbers. A key concept in number theory involves representing a quantity z as the sum of positive integers, where the summands consist of all positive integers. The partition function $p(z)$ refers to the of possibility to express as a sum of positive integers. In this context, the order of the terms in the sum does not matter. For instance, $4+2+2+14 + 2 + 2 + 14+2+2+1$ is a partition of the number 9. Different orders of the same summands, such as $4+2+2+14 + 2 + 2 + 14+2+2+1$ and $2+2+4+12 + 2 + 4 + 12+2+4+1$, are considered identical partitions.

Determining $p(z)$ for small values of z can be challenging, and the difficulty increases as z grows. For example, $p(10)=42$, $p(20)=627$, and $p(100)=190,567,292$. This thesis focuses on a few fundamental combinatorial and algebraic aspects of partitions.

To present our main results, we use the following notations:

$A_{y,k} := \{y - tk : t \in \mathbb{Z} \text{ and } 0 \leq t \leq \lfloor \frac{y}{k} \rfloor\}$ since y and k are any positive integers.

And $\beta_{y,k,z} := \{y + tk : t \in \mathbb{Z} \text{ and } 0 \leq t \leq \lfloor \frac{z-y}{k} \rfloor\}$ since y, k and z are any positive integers of y is less than or equal to z . The objective of the thesis is to prove the following theorems.

Theorem 3.3: If $p_k(z, y)$ (respectively $d_k(z, y)$) denotes the quantity of partitions of z into elements of $A_{y,k}$ (respectively distinct elements of $A_{y,k}$). Then the following recurrence relations hold 1) $p_k(z, y) = p_k(z-y, y) + p_k(z, y-k)$ and

$$2) d_k(z, y) = d_k(z-y, y-k) + d_k(z, y-k)$$

Theorem 3.5: If $\Psi_k(z, y)$ (respectively $\delta_k(z, y)$) denotes the quantity of partitions of z into elements of $\beta_{y,k,z}$ (respectively distinct elements of $\beta_{y,k,z}$). The recurrence relations apply as follow. (1) $\Psi_k(z, y) = \Psi_k(z-y, y) + \Psi_k(z, y+k)$ and (2) $\delta_k(z, y) = \delta_k(z-y, y+k) + \delta_k(z, y+k)$

Theorem 3.7. We define $p(z)$ formula here for an easy reference

$$P(z) = \sum_{k=1}^{\infty} (-1)^{k+1} \{p(z - f(k)) + p(z - f(-k))\},$$
 where $f(k) = \frac{k(3k-1)}{2}$ and $f(-k) = \frac{-k(3(-k)-1)}{2}$ produce the sequence of pentagonal numbers.

The thesis contains three chapters. Chapter one deals with partition functions; Chapter two deals with Generating Function (GF) and Chapter three recurrence relat

CHAPTER ONE

1. PARTITIONS

In number theory, an integer partition of a positive integer z , often referred to simply as a partition, is a way of expressing z as a sum of positive integers. Two sums are considered the same partition if they contain the same integers, regardless of the order in which they appear. If the order of the summands is important, the result is known as a composition.

Definition 1.1: An integer partition of a positive integer z is a possibility of writing z as a sum of positive integers, where the sum of these integers equals z . **Example 1:** The partitions of the integers from 1 to 5 are as follows:

$$1, 2=1+1, 3=2+1=1+1+1, 4=3+1=2+2=2+1+1=1+1+1+1, 5=4+1=3+2=3+1+1=2+2+1=2+1+1+1=1+1+1+1+1$$

The following four boxes partitioned 4 in 5 different forms

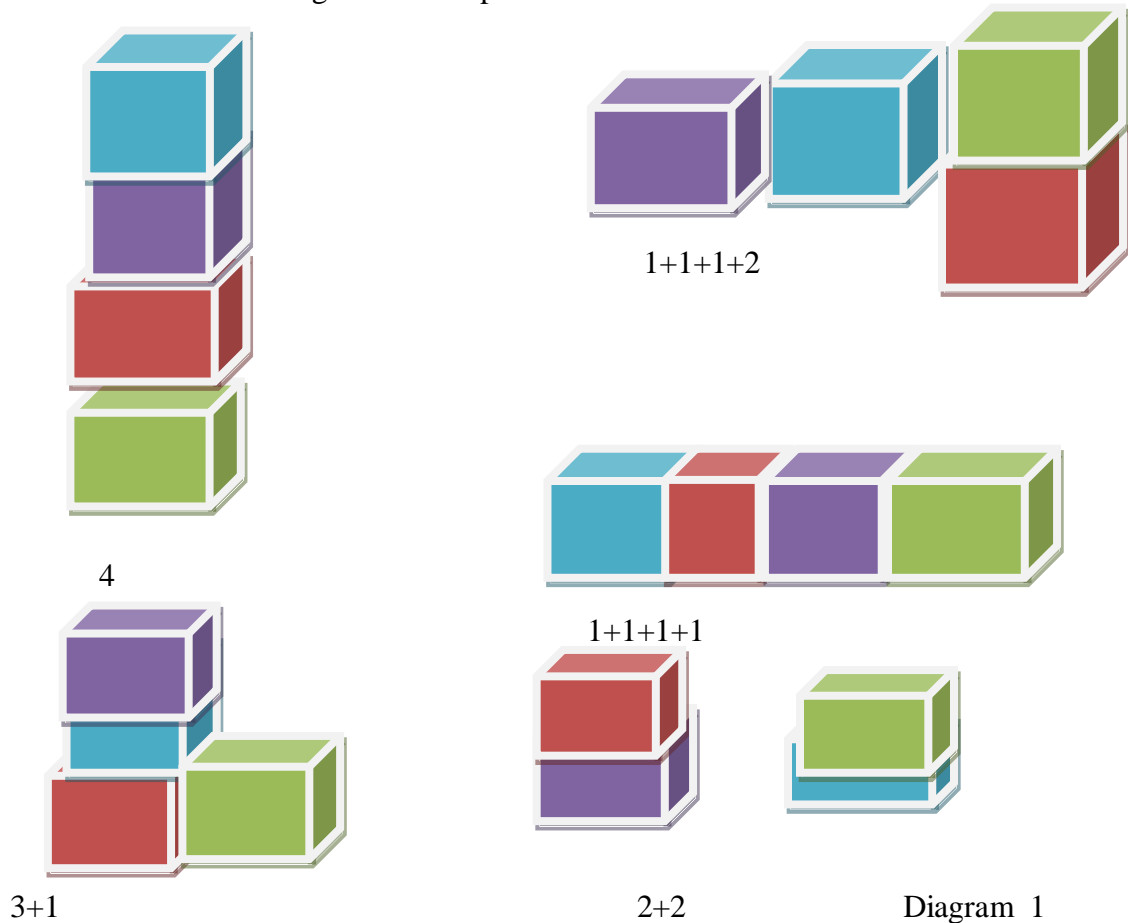


Table1: Triangle Partition P (z, k) equals quantity of partitions of z, in which the greatest summand k

z/k	1	2	3	4	5	6	7	8	9	10	11	12	13
1	1												
2	1	1											
3	1	1	1										
4	1	2	1	1									
5	1	2	2	1	1								
6	1	3	3	2	1	1							
7	1	3	4	3	2	1	1						
8	1	4	5	5	3	2	1	1					
9	1	4	7	6	5	3	2	1	1				
10	1	5	8	9	7	5	3	2	1	1			
11	1	5	10	11	10	7	5	3	2	1	1		
12	1	6	12	15	13	11	7	5	3	2	1	1	
13	1	6	14	18	18	14	11	7	5	3	2	1	1

Table 1 presents a specific type of partition function. For positive integers z and k, where z is at least k, P(z,k) denotes the number of partitions of z in which the largest part is k.

Example 2

$P(6,3)=3$ because we have $[3,1,1],[3,2,1],[3,3]$

$P(6,4)=2$ because, we have $[4,2],[4,1,1]$

$P(8,2)=4$ because, we have $[2,1,1,1,1,1],[2,2,1,1,1],[2,2,2,2],[2,2,2,1,1]$

$P(10,5)=7$ because, we have $[5,1,1,1,1],[5,5],[5,2,2,1],[5,2,1,1,1],[5,4,1],[5,3,2],[5,3,1,1]$

$P(6,2)=3$ because we have $[2,1,1,1,1],[2,2,2],[2,2,1,1]$

Theorem1.1: $\sum_{z>k}^K P(z - k, k) = P(z, k)$

Proof: If we subtract k from z, the largest summand in the resulting partition can be at most k. This confirms theorem1.1 for partitions terminated by P(z,k).

Example3: We can find $P(13, 3)$ using theorem1.1.

That is $P(13, 3)=P(13-3,1)+P(10-3,2)+P(13-3,3)$

Since, from table1 above we have $P(10,1)=1, P(10,2)=5, P(10,3)=8$ then,

$$=P(10,1)+P(10,2)+P(10,3)$$

$$=1+5+8=14$$

Hence, $P(13, 3) = 14$

Example4: Calculations for $n=13$

P(13,1)=1 because, [1,1,1,1,1,1,1,1,1,1,1,1,1]

P(13,2)=6 because, [2,1,1,1,1,1,1,1,1,1,1,1,1], [2,2,2,2,2,2,1], [2,2,2,2,2,1,1,1,1], [2,2,2,2,1,1,1,1,1,1], [2,2,2,1,1,1,1,1,1,1,1], [2,2,1,1,1,1,1,1,1,1,1,1]

P(13,3)=14 because, [3,1,1,1,1,1,1,1,1,1,1,1,1], [3,2,2,2,2,2], [3,3,3,3,1], [3,3,3,2,1,1,1], [3,3,3,1,1,1,1,1,1], [3, 3, 2, 1, 1, 1, 1, 1], [3,3,1,1,1,1,1,1,1,1,1], [3,3,3,2,2], [3,2,1,1,1,1,1,1,1,1,1], [3,3,2,2,2,1,1], [3,2,2,1,1,1,1,1,1,1,1], [3,2,2,2,2,1,1,1,1], [3,2,2,2,1,1,1,1,1], [3,3,2,2,1,1,1,1]

P(13,4)=18 because, [4,1,1,1,1,1,1,1,1,1,1,1,1], [4,4,4,1,1], [4,4,3,1,1,1], [4,4,2,1,1,1,1], [4,4,1,1,1,1,1,1,1], [4,3,1,1,1,1,1,1,1], [4,3,3,3], [4,3,2,2,2], [4,4,3,2], [4,2,1,1,1,1,1,1,1,1,1], [4,3,2,1,1,1,1,1,1,1,1], [4,2,2,2,2,1,1], [4,2,2,1,1,1,1,1,1,1,1], [4,2,2,2,1,1,1,1,1], [4,3,2,2,1,1,1], [4,3,3,2,1,1,1], [4,3,3,1,1,1,1,1,1], [4,4,2,2,2,1,1]

P(13,5)=18 because, [5,1,1,1,1,1,1,1,1,1,1,1,1], [5,2,2,2,2,2], [5,2,1,1,1,1,1,1,1,1,1], [5,2,2,2,1,1,1,1,1,1,1,1], [5,2,2,1,1,1,1,1,1,1,1,1], [5,3,3,2,1,1,1,1,1,1,1,1], [5,3,1,1,1,1,1,1,1,1,1,1,1], [5,3,2,2,2,1,1,1,1,1,1,1,1], [5,3,2,1,1,1,1,1,1,1,1,1,1], [5,3,3,1,1,1,1,1,1,1,1,1,1], [5,4,4], [5,4,1,1,1,1,1,1,1,1,1,1,1], [5,4,2,2], [5,4,3,1,1,1,1,1,1,1,1,1,1], [5,4,2,1,1,1,1,1,1,1,1,1,1], [5,5,3], [5,5,2,1,1,1,1,1,1,1,1,1,1], [5,5,1,1,1,1,1,1,1,1,1,1,1,1]

P(13,6)=14 because, [6,1,1,1,1,1,1,1,1,1,1,1,1], [6,2,2,2,1,1,1,1,1,1,1,1,1], [6,2,2,1,1,1,1,1,1,1,1,1,1], [6,2,1,1,1,1,1,1,1,1,1,1,1], [6,3,2,2], [6,3,2,1,1,1,1,1,1,1,1,1,1], [6,3,1,1,1,1,1,1,1,1,1,1,1,1], [6,3,3,1], [6,5,1,1], [6,6,1], [6,4,3], [6,4,2,1], [6,4,1,1,1,1,1,1,1,1,1,1,1], [6,4,1,1,1,1,1,1,1,1,1,1,1,1]

P(13,7)=11 because, [7,1,1,1,1,1,1,1,1,1,1,1,1], [7,2,2,2], [7,2,1,1,1,1,1,1,1,1,1,1,1], [7,2,2,1,1,1,1,1,1,1,1,1,1], [7,3,3], [7,3,2,1,1,1,1,1,1,1,1,1,1], [7,3,1,1,1,1,1,1,1,1,1,1,1,1], [7,4,2], [7,4,1,1,1,1,1,1,1,1,1,1,1,1], [7,5,1], [7,6]

P(13,8)=7 because, [8,1,1,1,1,1,1,1,1,1,1,1,1], [8,2,2,1,1,1,1,1,1,1,1,1,1], [8,2,1,1,1,1,1,1,1,1,1,1,1], [8,3,2], [8,3,1,1,1,1,1,1,1,1,1,1,1], [8,4,1], [8,5]

P(13,9)=5 because, [9,1,1,1,1,1,1,1,1,1,1,1,1], [9,2,1,1,1,1,1,1,1,1,1,1,1], [9,2,2], [9,3,1], [9,4]

P(13,10)=3 because, [10,1,1,1,1,1,1,1,1,1,1,1,1], [10,2,1], [10,3]

P(13, 11) =2 because, [11, 1, 1], [11, 2]

P (13, 13) =1=P (13, 12) because [13, 13]

Therefore, $p(13)=P(13,1)+P(13,2)+P(13,3)+P(13,4)+P(13,5)+P(13,6)+P(13,7)+P(13,8)+P(13,9)+P(13,10)+P(13,11)+P(13,12)+P(13,13)=1+6+14+18+18+14+11+7+5+3+2+1+1=101$

Therefore, $p(13) =101$

1.1. Ferrers Diagrams

Many insights into partitions can be obtained using Ferrers diagrams, which are also referred to as graphical representations or Ferrers graphs. A Ferrers diagram displays partitions as patterns of dots, where each row contains a number of dots corresponding to the terms in the partition. By rotating the Ferrers diagram along its diagonal, one can derive another type of partition known as the conjugate partition.

Example5: Conjugate of Ferrers diagram of number 14

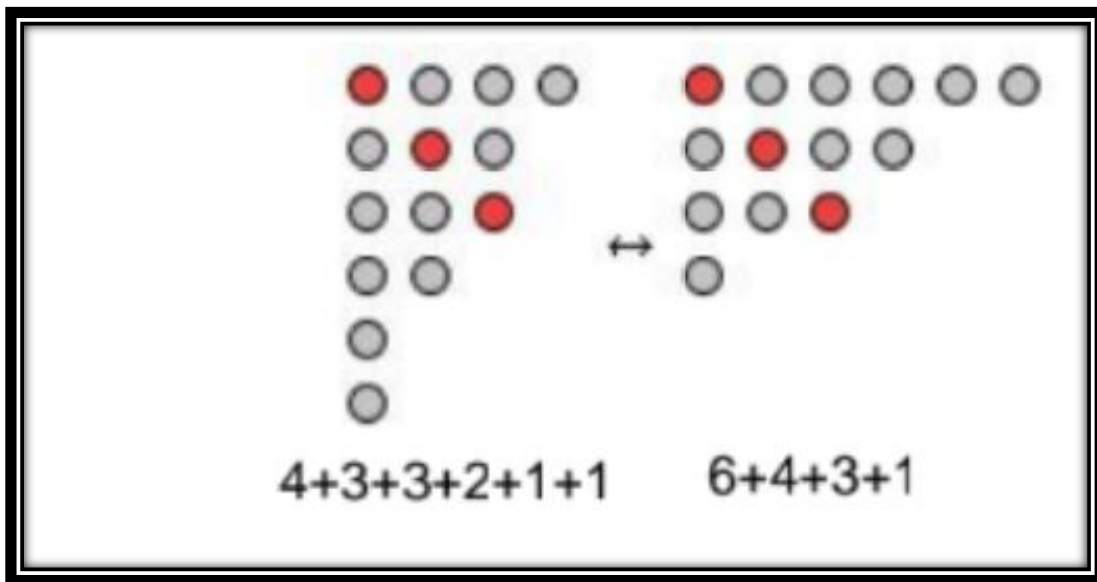


Diagram 2

Example6: Diagram 3 below illustrates the diagram of the partition of number 21

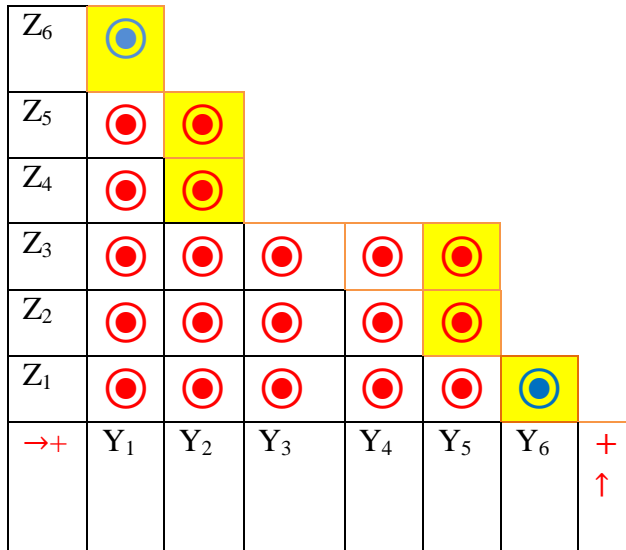


Diagram 3

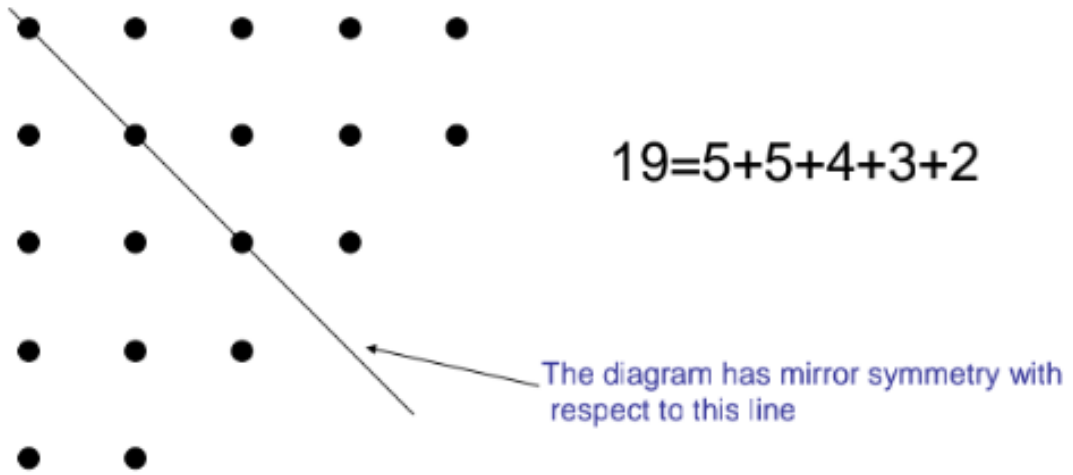
$$Z_1 + Z_2 + Z_3 + Z_4 + Z_5 + Z_6 = 6 + 5 + 5 + 2 + 2 + 1 = 21$$

$$Y_1 + Y_2 + Y_3 + Y_4 + Y_5 + Y_6 = 6 + 5 + 3 + 3 + 3 + 1 = 21$$

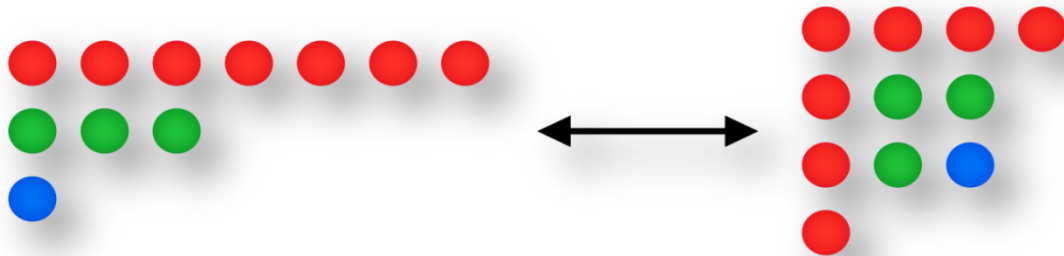
$$6 + 5 + 5 + 2 + 2 + 1 \leftrightarrow 6 + 5 + 3 + 3 + 3 + 1.$$

Given a Ferrers diagram, the conjugate Ferrers diagram is obtained by swapping rows and columns. Both the original and conjugate diagrams contain the same number of dots, meaning they represent partitions of the same integer.

Theorem 1.2: The number of partitions of z into distinct odd parts is equal to the number of self-conjugate partitions of z , where a self-conjugate partition's diagram is symmetric along the line $x=y$. This relationship stems from the observation that each odd part can be folded along the center to form a self-conjugate diagram. Consequently, the number of self-conjugate partitions matches the number of partitions with distinct odd parts.



Example 7

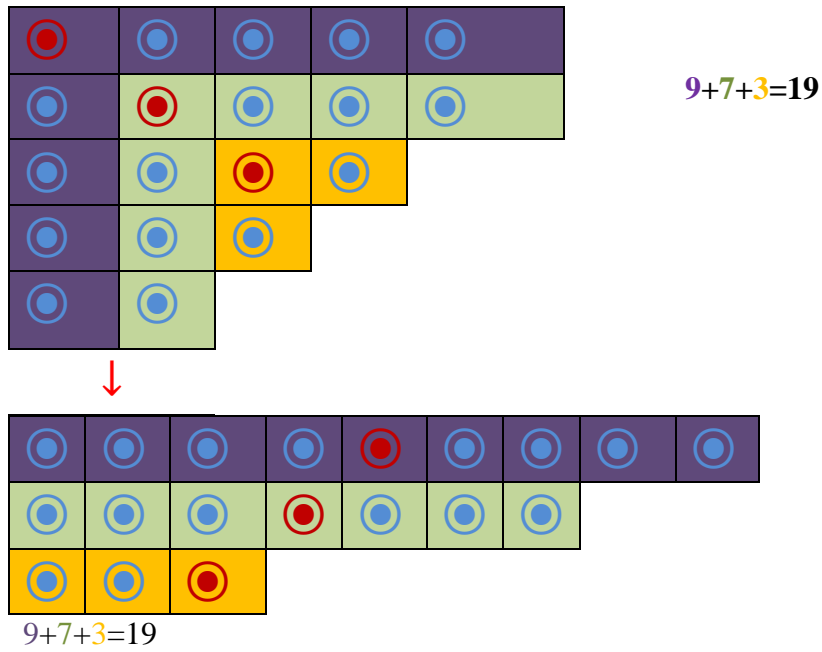


$7+3+1=11$

Diagram 5

Example8:

The diagram is symmetric with respect to the line $x=y$.



A partition is self-conjugate if and only if it consists of distinct odd parts.

Diagram 6

CHAPTER TWO GENERATING FUNCTIONS

In this section, we provide a straightforward method to compute the values of the partition function $P(z)$ for positive integers z .

2.1 Euler's Pentagonal Number Theorem

Consider the product $\prod_{k=1}^{\infty} (1-x^k)$, which is the reciprocal of the generating function for $p(z)$. We can express this as: $\prod_{k=1}^{\infty} (1-x^k) = 1 + \sum_{n=1}^{\infty} a(n) x^n$, where $a(n)$ represents the partition function. Each partition of z into distinct parts contributes a term x^z on the right-hand side with a coefficient of either $+1$ or -1 . The coefficient is $+1$ if x^z is the product of an even number of terms, and -1 otherwise. Thus, $a(n) = p_e(n) - p_o(n)$, where $p_e(n)$ (resp. $p_o(n)$) denotes the number of partitions of n into an even (resp. odd) number of distinct parts. For all n except those corresponding to a special set known as pentagonal numbers, $p_e(n) = p_o(n)$. Pentagonal numbers are partial sums of the arithmetic sequence $1, 4, 7, 10, 13, \dots, 3n+1$. If $\omega(n)$ denotes the sum of the first n terms in this sequence, then: $\omega(n) = \sum_{k=1}^n (3k+1) = \frac{3n(n+1)}{2} + n = \frac{3n^2+n}{2}$. These are known as pentagonal numbers.

2.2 Euler's Theorem

The theory of partition generating functions is described by:

$$\prod_{k=1}^{\infty} \frac{1}{1-x^k} = \sum_{z=0}^{\infty} p(z) x^z \dots^*$$

The infinite product on the left-hand side generates $p(z)$ as the coefficients in the power series expansion on the right-hand side. Expanding each factor $\frac{1}{1-x^k}$ using the geometric series yields the following:

$$\begin{aligned} \frac{1}{1-x^1} &= X^1_1 = 1+x+x^2+x^3+x^4+\dots \\ \frac{1}{1-x^2} &= X^2_2 = 1+X^2+x^4+x^6+x^8+x^{10}+\dots \\ \frac{1}{1-x^3} &= X^3_3 = 1+x^3+x^6+x^9+x^{12}+x^{15}+\dots \\ \frac{1}{1-x^4} &= X^4_4 = 1+x^4+x^8+x^{12}+x^{16}+x^{20}+\dots \\ \frac{1}{1-x^5} &= X^5_5 = 1+x^5+x^{10}+x^{15}+x^{20}+x^{25}+\dots \end{aligned}$$

By closely examining the multiplication of the series on the right side, it becomes evident that it generates the partition function. Leonhard Euler was the first to recognize that generating functions can be used to describe the number of partitions of a positive integer.

generates partition given by $\prod_{k=1}^{\infty} \frac{1}{1-x^k} = \sum_{k=0}^{\infty} p(k) x^k$ and discovery the generates the partition of positive integer of k into unequal parts. $(1+x) (1+x^2) (1+x^3) (1+x^4) \dots$ is becomes and we have

$$\checkmark (1+x)(1+x^2) = 1+x+x^2+x^3 \text{ in this multiplication up to } (1+x^2):$$

The exponent 2 implies partition of the integer 2 and the coefficient of x^2 that 1 implies the various methods of integer 2 as sum of distinct integer is 1. e. i. Only 2.

$$\checkmark (1+x)(1+x^2)(1+x^3) = 1+x+x^2+2x^3+x^4+x^5+x^6 \text{ in this multiplication of up to } (1+x^3):$$

The exponent 3 implies partition of the integer 3 and the coefficient of x^3 that 2 implies the various methods of partition integer 3 as sum of distinct integer are 2. They are 3 and 2+1.

$$\checkmark (1+x)(1+x^2)(1+x^3)(1+x^4) = 1+x+x^2+2x^3+2x^4+2x^5+2x^6+2x^7+x^8+x^9+x^{10}$$

In this multiplication up to $(1+x^4)$: the exponent 4 implies partition of the integer 4 and the coefficient of x^4 that 2 implies the various methods of partition integer 2 as sum of distinct integers are 2. They are 4 and 1+3

$$\checkmark (1+x)(1+x^2)(1+x^3)(1+x^4)(1+x^5) = 1+x+x^2+2x^3+2x^4+3x^5+3x^6+3x^7+3x^8+3x^9+3x^{10}+2x^{11}+2x^{12}+x^{13}+x^{14}+X^{15}$$

In this multiplication of upto $(1+x^5)$: the exponent 5 implies partition of the integer 5 and the coefficient of x^5 that 3 implies the various methods of partition integer 5 as sum of distinct integer are 3. They are 5, 2+3 and 1+4

$$\checkmark (1+x)(1+x^2)(1+x^3)(1+x^4)(1+x^5)(1+x^6) = 1+x+x^2+2x^3+2x^4+3x^5+4x^6+4x^7+4x^8+5x^9+5x^{10}+5x^{11}+5x^{12}+4x^{13}+4x^{14}+4x^{15}+3x^{16}+2x^{17}+2x^{18}+2x^{19}+x^{20}+x^{21}$$

In multiplication of polynomial upto $(1+x^6)$: The exponent 6 is implies partition of the integer 6 and the coefficient of x^6 that 4 implies the various methods of partition integer 6 as sum of distinct integer are 4. They are 6, 5+1, 4+2 and 3+2+1

Theorem 2.1 (Euler) (Theorem of generating function for partition function)

For $|x| < 1$ we have $\prod_{y=1}^{\infty} \frac{1}{1-x^y} = \sum_{n=0}^{\infty} p(z) x^z$, where $P(0) = 1$

Proof:

$$\text{In } \prod_{n=1}^{\infty} \frac{1}{1-x^z} = (1+x+x^2+\dots)(1+x^2+x^4+\dots) (1+x^3+x^6+\dots) \dots = 1 + \sum_{n=0}^{\infty} a(k) x^k$$

We want to show that $a(k) = p(k)$.

Suppose the factors $x^k_1 = (1+x+x^2+\dots)$, $x^{2k}_2 = (1+x^2+x^4+\dots)$, $x^{3k}_3 = (1+x^3+x^6+\dots)$ and the y^{th} factor x^{yk}_y from the where each $k_i \geq 0$,

Their product is $x^{1k_1} \cdot X^{2k_2} \cdot X^{3k_3} \dots X^{mk_m} = x^k$, where $k=k_1+2k_2+3k_3+\dots+yk_y$ and $k_1=(1+1+1+\dots+1)$, $k_2=(2+2+2+\dots+2)$, $k_3=(3+3+3+\dots+3)$, $k_y=(y+y+y+\dots+y)$ in other method it be written as follows $k=(1+1+1+\dots+1)+(2+2+2+\dots+2)+\dots+(y+y+y+\dots+y)$. This represents a partition of k into positive summands, so each term x^k arises from the corresponding partition of k . Consequently, the coefficient of x^k in $p(k)=a(x)$. Thus, we restrict x to be within certain bounds.

the interval $0 < x < 1$ and define two functions. $F_y(x) = \prod_{k=1}^y \frac{1}{1-x^k}$, and

$F(x) = \prod_{k=1}^{\infty} \frac{1}{1-x^k} = \lim_{y \rightarrow \infty} F_y(x)$. The product that defines $F(x)$ converges absolutely for $0 \leq x < 1$ because the reciprocal product $\prod(1-x)$ converges absolutely, given that the series $\sum x^k$ converges absolutely. For each fixed x , $0 \leq x < 1$ and every y , the sequence $\{F_y(x)\}$ is increasing. Because $F_{m+1} = \frac{1}{1-x^{y+1}} F_y(x) = F_y(x)$. Thus, $F_y(x) \leq F(x)$. Now $F_m(x)$ is the product of finite number of absolutely convergent series. We can write as $F_y(x) = 1 + \sum_{n=0}^{\infty} p(k, y) x^k$, here $p(k, y)$ is the number of the solution of the equation $k=k_1+2k_2+3k_3+\dots+yk_y$. In other words, $p(k, y)$ is the number of partitions of k into parts not exceeding y , if $y \geq k$ then $p(k, y) = p(k)$. Therefore, we always have $p(k, y) \leq p(k)$. With equality when $y \geq k$ in other words we have $\lim_{y \rightarrow \infty} p(k, y) = p(k)$. Now we split the series for $F(x)$ into two parts.

$$\begin{aligned} F(x) &= \sum_{k=0}^y p(k, y) x^k + \sum_{k=y+1}^{\infty} p(k, y) x^k \\ &= \sum_{k=0}^y p(k) x^k + \sum_{k=y+1}^{\infty} p(k, y) x^k \end{aligned}$$

We have $x \geq 0$ we have $\sum_{k=0}^y p(k) x^k \leq F_y(x) \leq F(x)$. This shows that the series $\sum_{k=0}^{\infty} p(k) x^k$ converges. To make clear that $p(k, y) \leq p(k)$ and become

$$\sum_{k=0}^{\infty} p(k, y) x^k \leq \sum_{k=0}^{\infty} p(k) x^k \leq F(x).$$

So, for each fixed x , the series $\sum p(k, y) x^k$ converges uniformly in y .

By letting $y \rightarrow \infty$ we get that

$$F(x) = \lim_{y \rightarrow \infty} F_y(x) = \lim_{y \rightarrow \infty} \sum_{k=0}^y p(k, y) x^k = \sum_{k=0}^{\infty} p(k) x^k \text{ which, proves Euler's}$$

identity for $0 \leq x < 1$, we extend it by analytic continuation to the unit disk $|x| < 1$.

CHAPTER THREE

3. RECURSIVE RELATIONS

3.1. Unrestricted Partition

One of the core problems in additive number theory involves unrestricted partitions, where the number of summands is not limited, repetition of summands is allowed, and the order of summands does not matter. In this context, the summands can be any positive integers, and the partition function $p(z)$ represents the number of such representations.

3.2 Restricted Partitions

3.2.1 Partitions of z into k Parts

A partition of z into exactly k parts is an unordered sum of z that uses exactly k positive integers. We denote the number of such partitions by $p(z,k)$. For example, a partition of 16 into 4 parts, such as $3+3+4+6$, involves a number 15. Subtracting 1 from each part results in a partition of $z-k$ into k parts, such as $2+2+3+5$. Partitions of z into k parts that include a 1 can be transformed into partitions of $n-1$ into $k-1$ parts by removing the 1. This establishes a one-to-one correspondence between partitions of z into k parts and the partitions of $z-k$ into k parts, along with partitions of $z-1$ into $k-1$ parts. Thus, the formula for partitions of z into k parts is given by:

$$p(z,k)=p(z-1,k-1)+p(z-k,k) \quad p(z, k) = p(z - 1, k - 1) + p(z - k, k) \quad p(z,k)=p(z-1,k-1)+p(z-k,k)$$

Example 1: For $p(5,3)$, there are 2 partitions: $5=1+2+2$ and $5=1+1+3$, which are the only ways to express 5 using 3 positive integers.

Example2: Show $p(8, 3) = 5$

$$\begin{aligned} \text{Solution } p(8, 3) &= p(8-3, 3) + p(8-1, 3-1) \\ &= p(5, 3) + p(7, 2) \\ &= 2+3=5 \end{aligned}$$

Thus the 5 partitions of 8 with 3 summands are $8=6+1+1=5+2+1=4+3+1=4+2+2=3+3+2$.

Let $p(z)$ the number of partitions of z , and let $p(z, y)$ be the number of partitions of n into m summand, thus by preceding list $p(1)=1, p(2)=2, p(3)=3, p(4)=5, p(5)=7$ and $p(5,1)=p(5,4)=p(5,5)=1, P(5)=P(5,1)+p(5,2)+p(5,3)+p(5,4)+p(5,5) = 1+2+2+1+1=7$

Clearly $P(z) = \sum_{y=1}^z p(z, y)$ and since order of the summands does not considered.

Example3: Determine all the partitions of 6 with their length

- $P(6,1)$ means Partitions of 6 with length 1 is 6 and $P(6,1)=1$
- $P(6,2)$ means partitions of 6 with length 2 are $2+4=3+3=1+5$ and $P(6,2)=3$
- $P(6,3)$ means partitions of 6 with length 3 are $1+1+4=1+2+3=2+2+2$ and $P(6,3)=3$
- $P(6,4)$ means partition of 6 with length 4 are $1+1+1+3=2+2+1+1$ and $P(6,4)=2$
- $P(6,5)$ means partition of 6 with length 5 is $1+1+1+1+2$ and $P(6,5)=1$
- $P(6,6)$ means partition of 6 with length 6 is $1+1+1+1+1+1$ and $P(6,6)=1$

Therefore $p(6)=P(6,1)+P(6,2)+P(6,3)+P(6,4)+P(6,5)+P(6,6)=1+3+3+2+1+1=11$

Table2: Different partitions form of some non negative integers

No	Number of Distinct partitions $D(z)$	Partitions without repeated summands (Distinct partitions)	Number of partitions with only Odd summands (D^O)	Number of partitions with only even summands (D^e)
1	1	(1)	(1)	-
2	1	(2)	(1+1)	2
3	2	(1+2),(3)	(1+1+1),(3)	-
4	2	(1+3),(4)	(1+1+1+1),(1+3)	(2+2),4
5	3	(2+3),(1+4),(5)	(1+1+1+1+1), (1+1+3),(5)	-
6	4	(1+2+3),(2+4), (6),(1+5),	(1+1+1+1+1+1), (3+3), (3+1+1+1), (5+1)	(2+2+2),(4+2)
7	5	(1+2+4),(3+4), (2+5), (6+1),(7)	(1+1+1+1+1+1+1) (3+1+1+1+1) (5+1+1),(3+3+1), (7)	-
8	6	(5+2+1),(3+4+1), (3+5),(7+1),(2,6), (8)	(1+1+1+1+1+1+1+1) 1 (3+1+1+1+1+1), (5+1+1+1),(7+1), (5+3), (3+3+1+1)	(4+2+2), (2+2+2+2),(4+4), (2+6),8

3.3. Recursive formula

The function $P_k(z)$ represents the number of partitions of z where the largest part is exactly k . To determine $P(z,k)$, we can use a recurrence relation to compute these partition numbers.

$p(z) = \sum_{k=1}^z p(z,k)$ and $P(z,k) = P(z-1,k-1) + P(z-k,k)$. We also have

$p(z) = p(z-1) + p(z-2) - p(z-5) - p(z-7) + p(z-12) + p(z-15) - p(z-22) - p(z-26) + p(z-35) + p(z-40) + \dots$
 Where 1, 2, 5, 7, 12, 15, 22... are pentagonal numbers. $P(-1) = p(-2) = \dots = 0$ and $p(0) = 1$

Example4: $P(5) = \sum_{k=1}^5 p(5,k) = p(5,1) + p(5,2) + p(5,3) + p(5,4) + p(5,5) = 1 + 2 + 2 + 1 + 1 = 7$

Example5: $p(4,2) = P(4-1, 2-1) + P(4-2, 2) = P(3, 1) + P(2, 2) = 2$

Thus, the partitions of 4 with 2 summands are $4=3+1=2+2$

Example6: $p(11) = p(10) + p(9) - p(6) - p(4) + p(-1) + \dots = 42 + 30 - 11 - 5 + 0 = 56$

Theorem 3.2. The number of partitions of z into distinct parts equals the number of partitions of n into odd parts. For all n , $P_d(n) = P_o(n)$

Example 7: for $z=6$ the partitions into distinct parts are $6=6, 5+1, 4+2, 3+2+1$, so $P_d(6)=4$ and partitions into odd parts are $6=5+1, 6=3+3, 6=3+1+1+1, 6=1+1+1+1+1+1$, thus, $P_o(6)=4$

3.4. Recurrence relations for restricted partition

Let Z be the set of natural numbers. A central issue in number theory is to investigate how many ways a natural number can be expressed as the sum of elements selected from a specific subset of Z . For our first theorem, we consider the scenario where the order of the summands is irrelevant. Let us define the problem as follows:

$A_{y,k} := \{y - tk : t \in \mathbb{N} \text{ and } 0 \leq t \leq \lfloor \frac{y}{k} \rfloor\}$. Where y and k are any positive integer.

Theorem3.3: If $p_k(z,y)$ (and $d_k(z,y)$) represents the number of partitions of z using elements from $A_{y,k}$ (or using distinct elements from $A_{y,k}$, respectively), then the following recurrence relations are valid:

- 1) $p_k(z,y) = p_k(z-y,y) + p_k(z,y-k)$ and
- 2) $d_k(z,y) = d_k(z-y,y-k) + d_k(z,y-k)$

Proof : (1) let us consider partitions z counted by $p_k(z,y)$. We can see that these types of partitions of z have or don't have a summand equal to y . The partitions of first type are generated by adding a summand y to each partition of $z-y$ into elements of $A_{y,k}$, and hence are $p_k(z-y,y)$ in number. The partitions of the second type are counted directly by $p_k(z,y-k)$

(2) In the same way of proof (1) above, let us consider partitions z counted by $d_k(z,y)$.

The partitions of first type are obtained by adding a summand $y-k$ to every partition of $z-y$ into elements of $A_{z,k}$, and hence are $d_k(z-y,y-k)$ in number. The partitions of the second type are simply counted by $d_k(z,y-k)$

Corollary3.4 : (i) If $P_k(z,y)$ respectively $D_k(z,y)$ denote the quantity of partitions of z into elements of $A_{y,k}$ (respectively distinct parts) with largest part y . Then the following recurrence relations hold.

$$P_k(z, y) = P_k(z-y, y) + P_k(z-k, z-k) \text{ and}$$

$$D_k(z, y) = D_k(z-y, y-k) + D_k(z-k, y-k)$$

(ii) If $p(z, y)$ respectively $d(z, y)$ denote the number of partitions of z into parts

(Respectively distinct parts) $\leq y$, then the following recurrence relations holds.

$$p(z, y) = p(z-y, y) + p(z, y-1) \text{ and}$$

$$d(z, y) = d(z-y, y-1) + d(z, y-1)$$

(iii). If $P(z, y)$ respectively $D(z, y)$ denote the number of partitions of z into parts (respectively distinct parts) parts with the largest part y , then the following recurrence relations holds.

$$P(z, y) = P(z-y, y) + P(z-1, y-1) \text{ and}$$

$$D(z, y) = D(z-y, y-1) + D(z-1, y-1)$$

Let us define $\beta_{y,k,z} := \{y + t_k; t \in \mathbb{Z} \text{ and } 0 \leq t \leq \lfloor \frac{z-y}{t} \rfloor\}$ where y, k and z are any natural numbers such that $y \leq z$. We can then state the following theorem:

Theorem 3.5: If $\Psi_k(z, y)$ (respectively $\delta_k(z, y)$ denotes the quantity of partitions of z into elements of $\beta_{y,k,z}$ (respectively distinct elements of $\beta_{y,k,z}$) then the following recurrence relations hold.

$$(1) \Psi_k(z, y) = \Psi_k(z-y, y) + \Psi_k(z, y+k) \text{ and}$$

$$(2) \delta_k(z, y) = \delta_k(z-y, y+k) + \delta_k(z, y+k)$$

Proof (1) Consider partitions of z counted by $\Psi(z, y)$. These partitions of z either include or exclude a summand equal to y . Partitions that include y are generated by adding y as a summand to each partition of $z-y$ into elements of $\beta_{y,k,z}$ resulting in $\Psi_k(z-y, y)$ such partitions. Partitions that do not include y are simply counted by $\Psi_k(z, y+k)$.

(2) Similarly, consider partitions of z counted by $\delta_k(z, y)$. Partitions of the first type are obtained by adding $y+k$ to each partition of $z-y$ into elements of $\beta_{y,k,z}$ resulting in $\delta_k(z-y, y+k)$ such partitions. Partitions of the second type are directly counted by $\delta_k(z, y+k)$

Corollary 3.6:(iv) If $\Omega_k(z, y)$ (respectively $\Delta_k(z, y)$ express the number of partitions of z into elements of $\beta_{y,k,z}$ (respectively distinct elements of $\beta_{y,k,z}$) with least part y , then the following recurrence relations hold. $\Omega_k(z, y) = \Omega_k(z-y, y) + \Omega_k(z+k, y+k)$ and

$$\Delta_k(z, y) = \Delta_k(z-y, y+k) + \Delta_k(z+k, y+k)$$

(v) If $\Psi(z, y)$ (respectively $\delta(z, y)$) $p(z, y)$ represent the number of partitions of y into parts (or distinct parts) with the smallest part at least m . Then, the following recurrence relations apply:

$$\Psi(z, y) = \Psi(z-y, y) + \Psi(z, y+1) \text{ and}$$

$$\delta(z, y) = \delta(z-y, y+1) + \delta(z, y+1)$$

(vi) If $\Omega(z, y)$ (respectively $\Delta(z, y)$ represents the number of partitions of z into parts (respectively distinct parts) with least part y , Then the following recurrence relation hold.

$$\Omega(z, y) = \Omega(z-y, y) + \Omega(z+1, y+1)$$

$$\Delta(z, y) = \Delta(z-y, y+1) + \Delta(z+1, y+1)$$

3.4.1. Recurrence relations

Theorem 3.7. We define $p(z)$ formula here for an easy reference

$$\mathbf{P}(z) = \sum_{k=1}^{\infty} (-1)^{k+1} \{p(z - f(k)) + p(z - f(-k))\}, \text{ where } f(k) = \frac{k(3k-1)}{2} \text{ and } f(-k) = \frac{-k(3(-k)-1)}{2} \text{ produce the sequence of pentagonal numbers.}$$

Proof: We want to show $\mathbf{P}(z) = \sum_{k=1}^{\infty} (-1)^{k+1} \{p(z - f(k)) + p(z - f(-k))\},$

The generating function for $p(z)$ can be expressed as:

$$P(x) = \sum_{z=0}^{\infty} p(z) x^z$$

We aim to show that this generating function satisfies the recurrence relation.

According to the pentagonal number theorem:

$$\prod_{y=1}^{\infty} (1-x^y) = \sum_{z=-\infty}^{\infty} (-1)^z x^{\frac{z(3z-1)}{2}} \cdot \sum_{z=-\infty}^{\infty} (-1)^z x^{\frac{z(3z+1)}{2}}, \text{ where } z = \frac{z(3z \pm 1)}{2}$$

The product on the left-hand side can be expanded and simplified as:

$$\prod_{y=1}^{\infty} (1-x^y) = \sum_{z=0}^{\infty} (-1)^z p(z) x^z$$

This generates the partition function with alternating signs, corresponding to the series in the given recurrence relation. Using the pentagonal number theorem, we express:

$$\mathbf{P}(z) = \sum_{k=1}^{\infty} (-1)^{k+1} \{p(z - f(k)) + p(z - f(-k))\},$$

This series representation aligns with the expansions obtained from the generating function, verifying that the recurrence relation accurately reflects the partition function behavior. The given series representation for $P(z)$ matches the pentagonal number theorem's result, thus confirming the correctness of the recurrence relation for computing the partition function.

Example 9: Find number of partitions of the following integers by using recursion relation formula.

- a. 11 c. 12

Solution

$$f(1) = \frac{1(3 \cdot 1 - 1)}{2} = 1; f(-1) = \frac{-1((3 \cdot (-1) - 1)}{2} = 2; f(2) = \frac{2(3 \cdot 2 - 1)}{2} = 5; f(-2) = \frac{-2((3 \cdot (-2) - 1)}{2} = 7$$

Where, $p(-1) = p(-2) = p(-3) = \dots = 0$ and $p(0) = 1$ where 1, 2, 5, 7 are pentagonal numbers.

$$\begin{aligned} \text{a) } p(11) &= (-1)^{1+1} \{p(11-1) + p(11-2)\} + (-1)^{2+1} \{p(11-5) + p(11-7)\} \\ &= \{p(10) + p(9)\} - \{p(6) + p(4)\} = 42 + 30 - 11 - 5 = 72 - 16 = 56 \end{aligned}$$

$$\begin{aligned} \text{b) } p(12) &= (-1)^{1+1} \{p(12-1) + p(12-2)\} + (-1)^{2+1} \{p(12-5) + p(12-7)\} + (-1)^{3+1} \{p(13-12) + p(12-15)\} \\ &= \{p(11) + p(10)\} - \{p(7) + p(5)\} + \{p(1) + p(-3)\} = (56 + 42) - (15 + 7) + (1 + 0) \\ &= (98) - (22) + 1 = 98 - 22 + 1 = 77 \end{aligned}$$

Table3: Values of the partition function, $p(z)$

Z	P(z)	Z	P(z)	Z	P(z)	Z	P(z)
1	1	71	4697205	141	16670689208	211	10085065885767
2	2	72	5392783	142	18440293320	212	10963707205259
3	3	73	6185689	143	20390982757	213	11916681236278
4	5	74	7089500	144	22540654445	214	12950095925895
5	7	75	8118264	145	24908858009	215	14070545699287
6	11	76	9289091	146	27517052599	216	15285151248481
7	15	77	10619863	147	30388671978	217	16601598107914
8	22	78	12132164	148	33549419497	218	18028182516671
9	30	79	13848650	149	37027355200	219	19573856161145
10	42	80	15796476	150	40853235313	220	21248279009367
11	56	81	18004327	151	45060624582	221	23061871173849
12	77	82	20506255	152	49686288421	222	25025873760111
13	101	83	23338469	153	54770336324	223	27152408925615
14	135	84	26543660	154	60356673280	224	29454549941750
15	176	85	30167357	155	66493182097	225	31946390696157
16	231	86	34262962	156	73232243759	226	34643126322519
17	297	87	38887673	157	80630964769	227	37561133582570
18	385	88	44108109	158	88751778802	228	40718063627362
19	490	89	49995925	159	97662728555	229	44132934884255
20	627	90	56634173	160	107438159466	230	47826239745920
21	792	91	64112359	161	118159068427	231	51820051838712
22	1002	92	72533807	162	129913904637	232	56138148670947
23	1255	93	82010177	163	142798995930	233	60806135438329
24	1575	94	92669720	164	156919475295	234	65851585970275
25	1958	95	104651419	165	172389800255	235	71304185514919
26	2436	96	118114304	166	189334822579	236	77195892663512
27	3010	97	133230930	167	207890420102	237	83561103925871
28	3718	98	150198136	168	228204732751	238	90436839668817
29	4565	99	169229875	169	250438925115	239	97862933703585
30	5604	100	190569292	170	274768617130	240	105882246722733
31	6842	101	214481126	171	301384802048	241	114540884553038
32	8349	102	241265379	172	330495499613	242	123888443077259
33	10143	103	271248950	173	362326859895	243	133978259344888
34	12310	104	304801365	174	397125074750	244	144867692496445
35	14883	105	342325709	175	435157697830	245	156618412527946
36	17977	106	384276336	176	476715857290	246	169296722391554
37	21637	107	431149389	177	522115831195	247	182973889854026

38	26015	108	483502844	178	571701605655	248	197726516681672
39	31185	109	541946240	179	625846753120	249	213636919820625
40	37338	110	607163746	180	684957390936	250	230793554364681
41	44583	111	679903203	181	749474411781	251	249291451168559
42	53174	112	761002156	182	819876908323	252	269232701252579
43	63261	113	851376628	183	896684817527	253	290726957916112
44	75175	114	952050665	184	980462880430	254	313891991306665
45	89134	115	1064144451	185	1071823774337	255	338854264248680
46	105558	116	1188908248	186	1171432692373	256	365749566870782
47	124754	117	1327710076	187	1280011042268	257	394723676655357
48	147273	118	1482074143	188	1398341745571	258	425933084409356
49	173525	119	1653668665	189	1527273599625	259	459545750448675
50	204226	120	1844349560	190	1667727404093	260	495741934760846
51	239943	121	2056148051	191	1820701100652	261	534715062908609
52	281589	122	2291320912	192	1987276856363	262	576672674947168
53	329931	123	2552338241	193	2168627105469	263	621837416509615
54	386155	124	2841940500	194	2366022741845	264	670448123060170
55	451276	125	3163127352	195	2580840212973	265	722760953690372
56	526823	126	3519222692	196	2814570987591	266	779050629562167
57	614154	127	3913864295	197	3068829878530	267	839611730366814
58	715220	128	4351078600	198	3345365983698	268	904760108316360
59	831820	129	4835271870	199	3646072432125	269	974834369944625
60	966467	130	5371315400	200	3972999029388	270	1050197489931117
61	1121505	131	5964539504	201	4328363658647	271	1131238503938606
62	1300156	132	6620830889	202	4714566886083	272	1218374349844333
63	1505499	133	7346629512	203	5134205287973	273	1312051800816215
64	1741630	134	8149040695	204	5590088317495	274	1412749565173450
65	2012558	135	9035836076	205	6085253859260	275	1520980492851175
66	2323520	136	10015581680	206	6622987708040	276	1637293969337171
67	2679689	137	11097645016	207	7206841706490	277	1762278433057269
68	3087735	138	12292341831	208	7840656226137	278	1896564103591584
69	3554345	139	13610949895	209	8528581302375	279	2040825852575075
70	4087968	140	15065878135	210	9275102575355	280	2195786311682516

CONCLUSION

- ❖ A partition of a positive integer z is a way of writing z as a sum of other positive integers.
- ❖ A partition of z of integer greater than or equal to 1 is representation of z as sum of integers greater than or equal to 1 without considering order of terms. This sum is summand of parts of partition.
- ❖ The partition is in unrestricted if in the summands repetition is allowed and order of summands is not taken into account
- ❖ By taking conjugates, the number $P(z, k)$ of partitions of z into exactly k parts is equal to the number of partitions of z in which the largest part has size k . The function $P(z, k)$ satisfies the recurrence relation $P(z, k) = P(z-k, k) + P(z-1, k-1)$. One recovers the function $P(z)$ by

$$P(z) = \sum_{k=0}^z P(z, k).$$
- ❖ Recursion Formula (RF) for $P(z)$ is a formula that relates the general term $P(z)$ of a sequence to one or more of terms that come before it.
- ❖ To Find the number of positive integer z as sum of other a positive integer z is given by $p(z) = \sum_{k=1}^{\infty} (-1)^{k+1} \{p(z - f(k)) + p(z - f(-k))\}$, where $f(k) = \frac{\pm k(3k \pm 1)}{2}$ generates the sequence of pentagonal numbers

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