ADDIS ABABA UNIVERSITY

COLLEGE OF NATURAL AND COMPUTATIONAL SCIENCES

DEPARTMENT OF MATHEMATICS

THESIS ON

NUMERICAL SOLUTIONS OF NONLINEAR HEAT EQUATIONS USING ADOMAIN DECOMPOSITION AND NATURAL DECOMPOSITION METHODS

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The undersigned hereby certify that we have read and recommend to the college of Natural Sciences for acceptance Addis Ababa University the thesis entitles "Numerical solution of nonlinear heat equations using Adomain and Natural decomposition methods" By Endalew Kebede Yeneget in partial fulfillment of the requirements for master thesis.

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Declaration

I declare that this thesis is my original work and that all source materials used for this thesis have been properly cited and acknowledged. This thesis has been submitted in partial fulfillment of the requirements for MSc degree in Mathematics at Addis Ababa University. I declare that this thesis is not submitted to any other institution anywhere for the award of any academic degree.

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Abbreviation

ODEs	Ordinary differential equations
PDE	Partial differential equation
ADM	Adomian Decomposition method
ADMT	Adomian Decomposition method with Taylor
NDM	Natural Decomposition Method
NTM	Natural Transform method
BVP	Boundary value problem
NLPDE	Nonlinear partial differential equations
NLODE	Nonlinear ordinary differential equations

Abstract

The Adomian decomposition method (ADM) is a powerful method which considers approximate solution of a non-linear equation as an infinite series which usually converges to the exact solution in the form of a rapid convergent series with easily computable components.Accrdingly, it is found to study the method to obtain solution of nonlinear heat equations. The method is a combination of Natural Transform Method (NTM) and Adomian Decomposition Method (ADM) which gives an exact solution. The exact solutions we obtained have been compared with the approximate solution of the nonlinear heat equations. ADM is able to solve this type of equations effectively and accurately without any need for discretization, perturbation, transformation and linearization.

Keywords: nonlinear heat equations, Adomain Decomposition Method, Natural Decomposition Method.

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CHAPTER ONE

INTRODUCTION

1.1 BACK GROUND OF THE STUDY

The energy connected to the random motion of atoms and molecules is known as the theory of heat. It was predicated on the caloric theory put forward by French chemist Antoine Lavoisier until the middle of the nineteenth century [1]. According to the caloric theory, heat is a fluid-like material that may be poured from one body into another. It is massless, colorless, odorless, and tasteless. A body's temperature increased with the addition of calories and reduced with the removal of calories.

The American Benjamin Thompson (Count Rumford) proved in his thesis that heat can be produced continuously [11]. The validity of the caloric hypothesis was also questioned by a number of other researchers.

One type of energy that has the capacity to perform work is heat. One type of energy can be changed into another [13]. Direct measurement of heat is not feasible. The total kinetic energy of the atoms or molecules of a body is measured as heat. Heat is therefore considered a type of energy and is expressed in Joules (J) or kilo Joules (kJ). A body's heat content is determined by its mass, temperature, and composition of materials. An object at a high temperature gives out heat energy when it moves to a lower temperature. Temperature is a measurement of a body's degree of heat or cold, or "how hot." It is independent of an object's mass or composition.

HEAT EQUATION

Heat: Heat is a type of energy that can be moved across systems when there is a temperature differential. The study of heat transfer includes both temperature variation and the calculation of these energy transfer rates. A fundamental equation for one-dimensional heat flow is the heat equation. When heat is transferred in a single direction exclusively due to a single direction variation in the medium's temperature, and when heat transmission in other directions is minor or nonexistent, the problem is considered one-dimensional heat transfer.

The heat equation is a partial differential equation involving three variables two independent variables x and t, one dependent variable u = u (x,t) is the temperature at position x and time t.
Heat equation is the temperature distribution u(x,t) in a metal road on [0, L] satisfies the heat equation $u_t(x,t) = ku_{tt}(x,t)$.
This partial differential equation tells that the rate of change of temperature at x is proportional to the second space derivative of u(x,t) at time t. The function u(x,t) is assumed to be zero at both ends of the bar and u= u(x, 0) is a given initial temperature distribution.

$u_t = ku_{xx} \quad 0 < x < L \;,\; 0 < t < \infty$, where k thermal conductivity of the material to conduct heat.

Note that, a partial differential equation is differential equation that involves two or more independent variables. The dependent variable that is an unknown function of the independent variables is u(x,t). its derivatives are frequently indicated by subscripts,such as $\frac{\partial u}{\partial x} = u_x$ and $\frac{\partial u}{\partial t} = u_t$. A PDE solution is a function of the form u(x,t) that, at least in part, satisfies the given equation in the same way across the variables x, t, and so on.

1.2 STATEMENT OF THE PROBLEM

The problem of the study focus on searching for the solutions of nonlinear heat equations using Adomain decomposition method. Linear partial differential equations can be solved by elementary methods such as separable equation, exact equation, integrating factor etc. But most nonlinear partial differential equations cannot be solved by such elementary methods. Lately, many researchers have proposed different approximation methods on how to determine solutions of nonlinear models like variation iteration method, perturbation method, linearization method etc. The study aims on how to examine the solution of non-linear heat equations using Adomain decomposition method and to make comparison of a solution obtained by Adomain decomposition method with exact solution of nonlinear heat equations.

1.3 OBJECTVE OF THE STUDY

The objective of the study is:

- To determine numerical solutions of nonlinear heat equations.
- To apply methods of solving nonlinear heat equations.
- To compare the solutions obtained by Adomain Decomposition Method with exact solutions.
- To analysis the solutions obtained by Adomain Decomposition Method and Natural Decomposition Method

1.4 SIGNIFICANCE OF THE STUDY

The finding of this study is useful on how to find solutions of nonlinear partial differential equations in different fields of science and engineering areas. Particularly, this study also used to solve nonlinear heat equations.in additional the output of the study may serves as the reference material for the students in the area of applied mathematics.

1.5 METHODOLOGY

In this thesis we had used adomain decomposition and natural decomposition methods which consists of decomposing the unknown function u(x,t) of any partial equation into a sum of an infinite number of components defined by the decomposition series of the form

$u(x,t) = \sum_{n=0}^{\infty} u_n(x,t)$, where the components, $u_{n,}(x,t), n \geq 0$ are to be determined in a recursive manner.

CHAPTER TWO

REVIEW OF LITRATURE

The solution of linear and nonlinear Klein Gordon equations which are most important mathematical models in quantum field theory with nonlinear optics and the function u(x, t) is assumed to be zero at both ends of the bar and u(x,0) = u(x,t) is a given initial temperature distribution [1]. It arises in physics, in linear and nonlinear forms and is useful in describing disperse wave phenomenon and has been solved by the method of natural decomposition method (NDM). Natural decomposition method is used to determine the approximate closed form of solution or exact solutions as the case may be for the nonlinear Klein-Gordon equations. The method is based upon the modified version of the Adomian decomposition method [14] and the Natural transform method. This new method gives exact solutions only in two iterations by computing only one Adomian polynomial thereby reducing the computational size of the work compared to other methods as demonstrate on certain nonlinear equations [3]. Also, [6] proposed natural decomposition method (NDM) for solving nonlinear two dimensional Brusselator equations with initial conditions. The method is a combination of Natural Transform Method (NTM) and the Adomian Decomposition Method (ADM) which gives an exact solution in the form of a rapid convergent series with easily computable components. The application further demonstrates the efficiency and accuracy compared to some other known techniques. The method is effective and applicable for many other NPDEs in mathematical physics.

This clearly shows that the Natural Decomposition Method (NDM) introduces significant improvement in the field over the existing technique. Natural Decomposition Method makes it very easy to solve linear and non-linear partial differential equations and approaches to the exact solutions in the form of rapid convergence series. A new computational algorithm which is called the Natural Decomposition Method for solving linear and nonlinear Schrodinger equation without any linearization, discretization, or taking some restrictive assumptions was introduced in [10]. NDM was applied for a solution of partial differential equations and become a useful tool for describing these natural phenomena of science and engineering models

[9]. Therefore, it becomes increasingly important to be familiar with all traditional and recently developed methods for solving partial differential equations, and the implementation of these

methods. Wazwaz [14] has applied the Adomian Decomposition Method to solve linear and nonlinear differential equations, which is proposed by George Adomian [4]. Adomian decomposition method can easily handle a wide class of both linear and nonlinear differential and integral equations. The Adomain decomposition method is a solution method with a wide range of applications, including the solution of algebraic and integral differential equations. In this method the solution is considered in rapidly converging infinite series, which accurately computes the series solution.

CHAPTER THREE

MATHEMATICAL FORMULATION

3.1 Analysis of The Study of Numerical Solution of Nonlinear Heat quations

The heat equation is a differential equation involving three variables two independent variables x and t, one dependent variable u = u(x,t) is the temperature at position x and time t. The partial differential heat equation is given of the form $\frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial t^2}$ where $0 < x < L$, $0 < t < \infty$, k is the thermal conductivity of the material to conduct heat. The steps that we followed in solving the problem were formulating the mathematical model of the flow problem, identifying the formulated equations linear or nonlinear in an operator form and apply the inverse operator to both sides of the equation written in an operator form. Set the unknown function u(x,t) into a decomposition series as u (x, t) = $\sum_{n=0}^{\infty} u_n (x, t)$ whose components are smartly determined. Identify the initial component $u_0(x,t)$ as the terms arising from the given conditions and from integrating the source term g(x,t), both are assumed to be known. Determine the successive components of the series solution u_k, $k \geq 1$ by applying the recursive scheme, where each component u_k can be completely determined by using the previous component u_{k-1}. Substitute the determined components into U (x, t) = $\sum_{n=0}^{\infty} u_n (x, t)$ to obtain the solution in a series form. An exact solution can be obtained in many equations if such a closed form solution exists. Then the heat equations are solved by Mat lab software using the Adomain decomposition method. The differential operators can be expressed by anyone of the following.

$$L = \frac{\partial}{\partial x}, \qquad L = \int_a^b dx$$

If the operator L is the partial derivative of u with respect to x and the integral of u with respect to x then we obtained the following.

$$L(u) = \frac{\partial u}{\partial x}, \qquad L(u) = \int_a^b (u) dx$$

If the operator L is a second order differential equation, then L is twice differentiable function. The domain of L is the twice differentiable function on an open interval. The expression L of the function u is used to describe $L(u)$ and the range of the function on I, hence $L(u)$ itself is afunction. The heat equation operator can be found from the given heat equation without sources.

The main idea of identifying linear operator is to decompose the unknown function u(x,t) of any equation in to a sum of infinite components as

$$u(x,t) = \sum_{n=0}^{\infty} u_n(x,t) = u_o(x,t) + u_1(x,t) + u_2(x,t) + u_3(x,t)+,\ldots$$

3.2 DESCRIPTION OF A DOMAIN DECOMPOSITION METHOD (ADM)

The domain decomposition method consists of decomposing the unknown function u(x,t) of any equation into a sum of an infinite number of components defined by the decomposition series $u(x,t) = \sum_{n=0}^{\infty} u_n(x,t)$ where the components, $u_n(x,t), n \geq 0$ are to be determined as a recursive manner. Any differential equation in general that can be expressed as

$$L u + N u + R u = f(x) \qquad (1)$$

where u is the unknown function, L is an easily invertible linear operator of higher order, N is a nonlinear part operator and R is the remaining part of the equation. After we defining the inverse operator of L as $L^{-1}$and by applying the invertible operator L^{-1} to both sides of the above equation (1) we have, $L^{-1}L u + L^{-1} N + L^{-1}R = L^{-1}f(x)$. From this,

$$L^{-1}L u = L^{-1}f(x) - L^{-1} N - L^{-1}R$$

It follows that $L^{-1}L u = \int_a^b u(x,t)ds = u(x,t) + \phi$, where ϕ is emerged from the integration and presented from the initial condition or from the boundary condition or both.it depends on how we choose differential operator to solve the given equation. Thus,

$$u = \phi + L^{-1}f(x) - L^{-1} N - L^{-1}R \qquad (2)$$

The final solution is an infinite series of the form $u = \sum_{n=0}^{\infty} u_n$.The nonlinear operator N(u) is decomposed in to an infinite series of Adomain polynomials of the form $N(u) = \sum_{n=0}^{\infty} A_n$ where A_n is Adomain polynomial given by $A_n = \frac{1}{n!}\frac{d^n}{d\lambda^n}[N(\sum_{i=0}^{\infty} \lambda^i u_i)]_{\lambda=0}$, . n = 0, 1, 2, 3,…

Therefore, a recurrence is used to find the remaining solution terms. Thus by substitution of the above expressions on equation (2), we have

$$\sum_{n=0}^{\infty} u_n = \phi + L^{-1}f(x) - L^{-1}\sum_{n=0}^{\infty} A_n - L^{-1}\sum_{n=0}^{\infty} R(u_n) \qquad (3)$$

Now from equation (3), we can determine the solution algorithm as follow:

$$u_o = \phi + L^{-1}f(x) \quad \text{and} \quad u_{n+1} = -L^{-1}Ru_n - L^{-1}A_n,\ n= 0, 1, 2,\ldots \qquad (4)$$

The series solution of u can be obtained by using the above equation.in this case u_o is arising from given condition and from integrating the source term f(x,t). The other successive terms of the series solution $of\ u_{n,}$ n ≥ 1 can be determined recursively by using the previous term u_{n-1}. If one of the terms of u_n is equal to zero, then the next terms are all zero.

Solution procedure: Consider the nonlinear functional equation:

$$u - N(u) = g \tag{5}$$

where N is the nonlinear operator and g is anon linear function determined after applying the inverse operator L^{-1} to the source function f. Suppose that the solution of (5) is a family of

$$u(\lambda) = \sum_{i=0}^{\infty} u_i \lambda^i \tag{6}$$

where λ is a parameter. As we have seen previously the nonlinear function N(u) can be expressed as infinite series$s$ $\quad N(u) = \sum_{i=0}^{\infty} \alpha_i u^i$ (7)

$where\ \alpha_i's$ are the coefficients of nonlinear terms. When we substitute equation (6) in equation(7),We have , $N(\sum_{n=0}^{\infty} u_n \lambda^n) = \sum_{i=0}^{\infty} \alpha_i (\sum_{n=0}^{\infty} u_n \lambda^n)^i$ (8)

Let $\sum_{i=0}^{\infty} \alpha_i (\sum_{n=0}^{\infty} u_n)^i = \sum_{i=0}^{\infty} A_i,$ then the above series is equivalent to

$$N(u) = \sum_{i=0}^{\infty} A_i \lambda^i = A_0 + A_1\lambda + A_2\lambda^2 + A_3\lambda^3 + \cdots \tag{9}$$

From above we can generalize that the values of A_i's depend only on the values of $u_i's$.

By substituting equation (9) and equation (7) in in equation (5) and let λ = 1, we obtain,

$$\sum_{n=0}^{\infty} u_n - \sum_{i=0}^{\infty} A_i = f \tag{10}$$

Consider a partial differential equation F (u(x, t)) = g(x) where F is represents a general nonlinear partial differential operator including both linear and nonlinear terms and g is a given function.

Then this can be expressed as**,** $\quad \mathrm{Lu} + \mathrm{Ru} + \mathrm{Nu} = \mathrm{g}$ $\quad$ as written in equation (1)

where L is an easily invertible operator and R is the remainder of the linear operator and Nu represents the non-linear terms of the equation.

After applying the inverse of operator L^{-1} on both sides of $\mathrm{Lu} + \mathrm{Ru} + \mathrm{Nu} = \mathrm{g}$**,** we have

$$u = \varphi_0 + L^{-1}(g) - L^{-1}(Ru) - L^{-1}(Nu)$$

where φ_0 the constant of integration that satisfies the condition $L\varphi_0 = 0$.

Now, assuming that the solution u is represented as infinite series of the form

$$u(x,t) = \sum_{n=0}^{\infty} u_n(x,t)$$

Furthermore, suppose that the nonlinear term Nu can be written as infinite series in terms of the Adomian polynomials A_n of the form $Nu = \sum_{n=0}^{\infty} A_n$ where A_n is Adomian polynomials of Nu and given by $A_n = \frac{1}{n!}\frac{d^n}{d\lambda^n}\left[N\left(\sum_{i=0}^{n} \lambda^i u_i\right)\right]_{\lambda=0}$ $n = 0, 1, 2, ...$

The first five Adomian polynomials are:

$$A_0 = N(u_0)$$

$$A_1 = u_1 N'(u_0)$$

$$A_2 = u_2 N'(u_0) + \frac{1}{2!} {u_1}^2 N''(u_0)$$

$$A_3 = u_3 N'(u_0) + u_1 u_2 N''(u_0) + \frac{1}{3!} {u_1}^3 N'''(u_0)$$

$$A_4 = u^4 N'(u_0) + \left[\frac{1}{2!} {u_2}^2 + u_1 u_3\right] N''(u_0) + \frac{1}{2!} {u_1}^2 u_2 N'''(u_0) + \frac{1}{4!} {u_1}^4 N''''(u_0)$$

After substituting we obtain $\sum_{n=0}^{\propto} u_n = \varphi_0 + L^{-1}(g) - L^{-1}(R \sum_{n=0}^{\propto} u_n) - L^{-1}(\sum_{n=0} A_n)$ where φ_0 is the initial condition.

Hence, each term of the series can be determined by using the recurrence relation as

$$\mathrm{u}_0 = \varphi_0 + \mathrm{L}^{-1}\mathrm{g}$$

$$\mathrm{u}_1 = -\mathrm{L}^{-1}\mathrm{R}\mathrm{u}_0 - \mathrm{L}^{-1}\mathrm{A}_0$$

$$\mathrm{u}_2 = -\mathrm{L}^{-1}\mathrm{R}\mathrm{u}_{1\_}\mathrm{L}^{-1}\mathrm{A}_1$$

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$$\text{generally}, \mathrm{u}_{n+1} = -\mathrm{L}^{-1}\mathrm{R}\mathrm{u}_n - \mathrm{L}^{-1}\mathrm{A}_n, \quad \mathrm{n} = 0,1,2,3, ...$$

3.3 NATURAL TRANSFORMATION METHOD

Definition: Let $f(t)$ is a function defined for all $t \geq 0$. The natural transform of the function f(t) for t∈ (0,∞) is defined by N[f(t)] = R(s,u) then $R(u,s)$ = N^{+}[f(t)] = $\int_0^{\infty} f(tu)e^{-st}$ dt,

s, u ∈ (0, ∞).[$\int_0^{\infty} f(tu)e^{-st}$dt; s, u ∈ (0, ∞)] = N^{+} [f (t)] = R^{+}(s, u),

where N [f (t)] is the natural transformation of the time function f (t) and the variables s and u are the natural transform variables.

f(t)	N[f(t)]	τ[f(t)]	S[f(t)]
1	$\frac{1}{s}$	$\frac{1}{s}$	1
T	$\frac{u}{s^2}$	$\frac{1}{s^2}$	u
N [$f'(t)$] =	$\frac{s}{u}$F **(s,** u)$-\frac{F(0)}{u}$		
N[f''(t)] =	$\frac{s^2}{u^2}$F **(s,** u)$-\frac{s}{u^2}$F **(0)**$-\frac{F'}{u}(0)$		

Table 3.1 Some special Natural transforms and the Conversion to Sumudu and Laplace

3.4 N-TRANSFORM OF ELEMENTARY NFUNCTION

Let f (t) = e^{at} be an exponential function when t ≥ 0 and a is constant. The N-Transform of this function can be written as: N (e^{at})=$\int_0^{\infty} e^{aut}e^{-st}dt$ = $\lim_{b\to\infty} \int_0^b e^{-(s-au)t}dt$**,**

$$= \lim_{b\to\infty}\left[\frac{e^{-(s-au)t}}{s-au}\right]\Bigg|_0^b = \frac{1}{s-au} = \{\frac{1}{s-a} \text{ Laplace transform if u=1}\}$$

$$= \{\frac{1}{1-au} \text{ Sumudu transform if s=1)}\}$$

3.4.1 LINEARITY PROPERTY

Theorem 3.1 If a and b are any arbitrary fixed values and , $f(t)$ and $g(t)$ are functions, then

$$N\{af(t) + bg(t)\} = aN\{f(t)\} + bN\{g(t)\}$$

Proof:

Let N{f(t)} = $\int_0^\infty f(ut)e^{-st}$dt and N{g(t)} = $\int_0^\infty g(ut)e^{-st}$dt and a and b are any constants.

Then N{af(t + bg(t)} = $\int_0^\infty e^{-st}$ {af(t) + bg(t)}dt

= a$\int_0^\infty f(ut)e^{-st}$dt + b$\int_0^\infty g(ut)e^{-st}$dt

= aN(f) + bN(g)

Example 3.1 N{2t + 4sin $2t$} = 2N (t) + 4(sin $2t$) = $2\frac{u}{s^2} + 4\frac{s}{s^2+4u^2}$

N-Transform of derivative:

We can write the natural transform of partial derivatives as

$$\mathrm{N}[\frac{\partial v}{\partial t}(x,t)] = \frac{1}{u}[S\mathrm{N}\,[v(x,t)] - v(x,0)]$$

$$\mathrm{N}[\frac{\partial^2 v}{\partial t^2}(x,t)] = \frac{1}{u^2}\,[s^2\mathrm{N}[\mathrm{v(x,t)}] - \mathrm{sv(x,0)}] - \frac{v'}{u}(x,0)$$

$$\mathrm{N}[\frac{\partial^n v}{d t^n}(x,t)] = \frac{1}{u^n}[s^n\mathrm{N}[v(x,t)] - \sum_{k=0}^{n-1}\frac{s^{n-(k+1)}}{u^{n-k}}v^k(x,0)$$

Theorem 3.2 If $N\{f(t)\} = R(s,u)$ then $N\{f'(t)\} = \frac{s}{u}R(s,u) - \frac{f(0)}{u}$.

Proof:

$$\mathrm{N}\{f'(t)\} = \int_0^\infty e^{-st} f'(ut)dt$$

$$= \lim_{p\to\infty}\int_0^p e^{-st} f'(ut)dt$$

$$= \lim_{p\to\infty}\{\left|\frac{e^{-st}f(ut)}{u}\right|_0^p + \frac{s\int_o^p e^{-st}}{u}f(ut)dt\}$$

$$= \lim_{p\to\infty}\{\left|\frac{e^{-sp}f(up)}{u} - \frac{f(0)}{u} + \frac{s\int_o^p e^{-st}}{u}f(ut)dt\}$$

$$= \frac{s\int_o^p e^{-st}}{u}f(ut)dt - \frac{f(0)}{u} = \frac{s}{u}\mathrm{R\ (s,\ u)} - \frac{f(0)}{u}$$

Theorem 3.3 If $N\{f(t)\} = R(s,u)$ then $N\{f''(t)\} = \frac{s^2}{u^2}R(s,u) - \frac{s}{u^2}f(0) - \frac{f'(0)}{u}$

Proof: N{$G'(t)$} = $\frac{s}{u}$N{G (t)}$-\frac{f(0)}{u}$, let G (t) = f'(t). Then

$$\mathrm{N}\{f''(\mathrm{t})\} = \frac{s}{u}\mathrm{N}\{f'(\mathrm{t})\} - \frac{f'}{u}(0)\frac{s}{u}\{\frac{s}{u}\mathrm{N}\{\mathrm{f(t)}\} - \frac{f(0)}{u}\} - \frac{f'(0)}{u}$$

$$= \frac{s^2\mathrm{N}\{\mathrm{f(t)}\}}{u^2} - \frac{s}{u^2}\frac{f(0)}{u} - \frac{f'(0)}{u}$$

$$\mathrm{N}\,\{f''(\mathrm{t})\} = \frac{s^2}{u^2}\mathrm{R(s,\ u)} - \frac{s}{u^2}\frac{f(0)}{u} - \frac{f'(0)}{u}$$

3.4.2 N-TRANSFORM OF INTEGRAL

Theorem 3.4 If $N\{f(t)\} = R(s,u)$ then $N\left\{\int_0^t f(p)\,dp\right\} = \frac{u}{s}R(s,u)$

Proof:

Let G (t) = $\int_0^t f(p)dp, then\ G' =$ f (t) and G (0) = 0.

After applying the N−Transform on both sides, we have

$$N\{G'(t)\} = \frac{s}{u}N\{G(t)\} - \frac{f(0)}{u}$$

$$= \frac{s}{u}N\{G(t)\} = R(s,u). \text{ this implies}$$

$$N\{G(t)\} = \frac{u}{s}R(s,u) = N\left\{\int_0^t f(p)\,dp\right\}$$

3.5 NATURAL DECOMPOSITION METHOD

Consider the general nonlinear Non-homogeneous differential equation of the form

$$\mathrm{Ly(x,t) + Ry(x,t) + Ny(x,t) = h(x,t)} \tag{12}$$

subject to the initial condition: $\mathrm{y(x,0) = f(x)}$, $y'\mathrm{(x,0) = g(x)}$

Here L represents the second order linear differential operator for $L_t = \frac{\partial^2 y}{\partial t^2}$, R is the linear differential operator its order is lower than order of L, N is the general nonlinear differential operator and h(x,t) is the source term. We apply the N-Transform on equation (12) to get:

$$N[Ly(x,t)] + N[Ry(x,t)] + N[Ny(x,t)] = N[h(x,t)] \tag{13}$$

By using the Property above the (table1) and some properties:

$$\frac{\mathrm{s}^2}{\mathrm{u}^2}\mathrm{R(x,s,u)} - \frac{\mathrm{sy(x,0)}}{\mathrm{u}^2} - \frac{\mathrm{y'(x,0)}}{\mathrm{u}} + \mathrm{N[Ry(x,t)]} + \mathrm{N[Ny(x,t)]} = \mathrm{N[h(x,t)]}$$

Substituting Equation (12) into Equation (13) yields

$$\frac{s^2}{u^2}R(x,s,u) - \frac{sf(x)}{u^2} - \frac{g(x)}{u} + N[Ry(x,t)] + N[Ny(x,t)] = N[h(x,t)] \tag{14}$$

By substituting Equation (13) in to (14) we get

$$R(x,s,u) = \frac{f(x)}{s} + \frac{ug(x)}{s^2} + \frac{u^2}{s^2}N[h(x,t)] - \frac{u^2}{s^2}N[Ry(x,t)] - \frac{u^2}{s^2}N[Ny(x,t)] \tag{15}$$

Note $H(x,t)$ is the term arising from the source term and the prescribed initial conditions. Now to deal with nonlinear term, we represent the solution in an infinite series form:

$$y(x,t) = \sum_{n=0}^{\infty} y_n(x,t) \tag{16}$$

Also, the nonlinear term can be written

$$Ny(x,t) = \sum_{n=0}^{\infty} A_n \tag{17}$$

where $A_n's$ are the polynomials of $u_0, u_1, u_2, u_3, \ldots, u_n$ and can be calculated by the following formula:

$$A_n = \frac{1}{n!}\frac{d^n}{d\lambda^n}\left[F\left(\sum_{i=0}^{n}\lambda^i y_i\right)\right], n = 0,1,2,\ldots \tag{18}$$

The general formula in equation (18) can be simplified as follows: We first assume that the nonlinear function is F (u), the Adomian polynomials are given by:

$$A_0 = F(y_0)$$

$$A_1 = u_1 F(y_0)$$

$$A_2 = u_2 F'(y_0) + \frac{1}{2!}{y_1}^2 F'(y_0)$$

And so on. Now, we substitute Equation (16) and Equation (17) into Equation (15) to get:

$$\sum_{n=0}^{\infty} y_n(x,t) = H(x,t) - N\left[\frac{u^2}{s^2}\, N\left[R\sum_{n=0}^{\infty} y_n(x,t)\right] + \frac{u^2}{s^2} N\left[\sum_{n=0}^{\infty} A_n\right]\right] \tag{19}$$

Comparing both sides of Equation (19) we conclude

$$y_0(x,t) = H(x,t)$$

$$y_1(x,t) = -N\left[\frac{u^2}{s^2}\, N[Ry_0(x,t)] + A_0\right]$$

$$y_2(x,t) = -N\left[\frac{u^2}{s^2} N[Ry_1(x,t)] + A_1\right], \ldots$$

The general recursive relation is given by:

$$y_{n+1}(x,t) = -N\left[\frac{u^2}{s^2}[Ry_n(x,t)] + A_n\right], n \geq 1$$

From this general recursive relation, we can easily compute the remaining components of $y(x,t)$ as $y_1(x,t), y_2(x,t), \ldots$ where $y_0(x,t)$ is the given initial condition and then

The exact solution is given by $y(x,t) = \sum_{n=0}^{\infty} y_n(x,t)$

3.6 APPLICATIONS ON ADOMIAN DECOMPOSITION AND NATURAL DECOMPOSITION METHODS FOR NONLINEAR HEAT EQUATIONS

Numerical Solutions of Nonlinear Heat Equations:

Example1 Solve the following nonlinear heat equation subjected to initial condition using the Adomain decomposition method $\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} - 2u^3$, with initial condition is $u(x, 0) = \frac{1+2x}{x^2+x+1}$

First, we write this equation of the form $\frac{\partial u}{\partial t} - \frac{\partial^2 u}{\partial x^2} + 2u^3 = 0$ then, we have the following.

$$L_t = \frac{\partial u}{\partial t}, \quad L_t^{-1} = \int_0^t (.)\, ds, \quad N(u) = 2u^3, \quad L_{xx} = \frac{\partial^2 u}{\partial x^2}, \quad g = 0$$

$$L_t(u) - L_{xx}(u) + N(u) = 0 \tag{20}$$

After applying L_t^{-1} on both sides equation (21), we have

$$L_t^{-1} L_t (u) = L^{-1}{}_t N(u) - L_t^{-1} L_{xx}^{-1}(u)$$

$$\int_0^t \left(L^{-1}{}_t L_t (u)\right) dt = L^{-1}{}_t L_{xx}(u) - L^{-1}{}_t N(u)$$

Then we obtain the following expression

$$u(x, t) = u_0 (x,0) - L_t^{-1} N(u) + L^{-1}{}_t L_{xx}(u)$$

Let $u(x, t) = \sum_{n=0}^{\infty} u_n(x, t) = u_0 (x,0) - L_t^{-1} N(u) + L^{-1}{}_t L_{xx}(u)$.

Thus, $u_0(x, 0) = \frac{1+2x}{x^2+x+1}$ and $u_{n+1}(x, t) = \sum_{n=0}^{\infty} R(u_n) - \sum_{n=0}^{\infty}(A_n)$,

Where $N(u) = L^{-1}{}_t \sum_{n=0}^{\infty} A_n$ and $L^{-1}{}_t L_{xx}(u) = L^{-1}{}_t \sum_{n=0}^{\infty} R(u_n)$.

Then by using the equation $u_{n+1}(x, t) = \sum_{n=0}^{\infty} R(u_n) - \sum_{n=0}^{\infty}(A_n)$

Where $A_n = \frac{1}{n!}\frac{d^n}{dh^n}[(N\sum_{i=0}^{\infty} \lambda^i u_i)]\, \lambda = 0$, $n = 0, 1, 2, 3, \ldots$

for $n = 0$, $u_1(x, t) = -L^{-1}{}_t (A_0) + L^{-1}{}_t L_{xx} u R(u_o)$

$$u_1 (x, t) = L^{-1}{}_t L_{xx} u(u_o) - L^{-1}{}_t (A_0)$$

But we write the first five Adomian polynomials as

$$A_0 = N(u_0)$$

$$A_1 = u_1 N'(u_o)$$

$$A_2 = u_2 N'(u_o) + \frac{1}{2!} u1^2 N''(uo)$$

$$A_3 = u_3 f'(u_0) + \frac{2}{2!} u_1 u_2 f''(u_0) + \frac{1}{3!} u_1{}^3 f'''(u_0)$$

$$A_4 = u_4 f'(u_0) + \left[\frac{1}{2!} u_2{}^2 + u_1 u_3\right] f''(u_0) + \frac{1}{2!} u_1{}^2 u_2 f'''(u_0) + \frac{1}{4!} u_1{}^4 f''''(u_0)$$

The first n-terms of these polynomials can be determined as follows:

$$A_0 = N(u_0),$$

$$A_1 = N'(u_0)u_1 = \frac{d}{d\lambda}N(u_0 + \lambda u_1)_{\lambda=0}$$

$$A_2 = N'(u_0)u_2 + \tfrac{1}{2!}N''(u_0){u_1}^2 = \tfrac{1}{2!}\tfrac{d^2}{d\lambda^2}N(u_0 + \lambda u_1 + \lambda^2 u_2)_{\lambda=0},$$

$$= \frac{1}{3!}\frac{d^2}{d\lambda^2}N(u_0 + \lambda u_1 + \lambda^2 u_2 + \lambda^3 u_3)_{\lambda=0}$$

Thus, $u_{o(x,t)} = \frac{1+2x}{x^2+x+1}$, and $u_{1(x,t)} = -{L^{-1}}_t\ (A_o) + L^{-1}_t L_{xx}(u_o)$

but $Ao = N(u_o) = 2{u_o}^3 = \frac{2(1+2x)^3}{(x^2+x+1)^3}$

$u_1(x, t) = \int_0^t L_{xx}(u_o)dt - \int_0^t (A_o)dt,$

where $L_x(u_o) = (\frac{1+2x}{x^2+x+1})'$

$$L_x(u_o) = \frac{2}{x^2+x+1} - \frac{(1+2x)^2}{(x^2+x+1)^2},$$

$$L_{xx}(u_o) = (\frac{2}{x^2+x+1})' - (\frac{(1+2x)^2}{(x^2+x+1)^2})'$$

Hence, $L_{xx}(u_o) = \frac{-6(1+2x)}{(x^2+x+1)^2} + \frac{2(1+2x)^3}{(x^2+x+1)^3}$

Thus, $u_1(x, t) = \int_0^t L_{xx}(u_o)dt - \int_0^t (Ao)dt$

$$= \int_0^t (\frac{-6(1+2x)}{(x^2+x+1)^2} + \frac{2(1+2x)^3}{(x^2+x+1)^3})dt - \int_0^t \left(\frac{2(1+2x)^3}{(x^2+x+1)^3}\right)dt$$

$$= \frac{-6(1+2x)}{(x^2+x+1)^2}t + \frac{2(1+2x)^3}{(x^2+x+1)^3}t - \frac{2(1+2x)^3}{(x^2+x+1)^3}t = \frac{-6(1+2x)}{(x^2+x+1)^2}t$$

$$u_1(x, t) = \frac{\mathbf{-6(1+2x)}}{\mathbf{(x^2+x+1)^2}}t$$

$u_2(x,t) = -L^{-1}_t(A_1) + L^{-1}_t L_{xx}(u_1)$ where $A_1 = u_1 N'(u_o) = 6{u_o}^2 u_1$

but $(u_o)^2 = \frac{(1+2x)^2}{(x^2+x+1)^2}$

$$A_1 = 6\frac{(1+2x)^2}{(x^2+x+1)^2}(\frac{-6(1+2x)}{(x^2+x+1)^2}t) = \frac{-36(1+2x)^3}{(x^2+x+1)^4}$$

$$L_x(u_1) = \left[\frac{-6(1+2x)}{(x^2+x+1)^2}t\right]' = \frac{-12((x^2+x+1)^2 + 6(2x+1)(2)((x^2+x+1)(2x+1)t}{(x^2+x+1)^4}$$

$$L_x(u_1) = \frac{-12}{(x^2+x+1)^2}t + \frac{12(1+2x)^2}{(x^2+x+1)^3}t$$

$$L_{xx}(u_1) = [\frac{-12}{(x^2+x+1)^2}]'t + [\frac{12(1+2x)^2}{(x^2+x+1)^3}]'t$$

$$= \frac{24(2x+1)}{(x^2+x+1)^3}t + \frac{48(2x+1)}{(x^2+x+1)^3}t - \frac{36(1+2x)^3}{(x^2+x+1)^4}t = \frac{72(2x+1)}{(x^2+x+1)^3}t - \frac{36(1+2x)^3}{(x^2+x+1)^4}t$$

$$L_{xx}(u_1) = \frac{72(2x+1)}{(x^2+x+1)^3}t - \frac{36(1+2x)^3}{(x^2+x+1)^4}t$$

$$u_2(x,t) = -L^{-}{}_t(A_1)+L^{-}{}_t(L_{xx}(u_1))$$

$$= \int_0^t \frac{(72(2x+1)}{(x^2+x+1)^3}t - \frac{-36(1+2x)^3}{(x^2+x+1)^4}t)]dt - \int_0^t \left(\frac{-36(1+2x)^3}{(x^2+x+1)^4}t\right)dt + \frac{36(2x+1)}{(x^2+x+1)^3}t^2 +$$

$$\frac{-18(1+2x)^3}{(x^2+x+1)^4}t^2 + \frac{18(1+2x)^3}{(x^2+x+1)^4}t^2$$

$$u_2(x, t) = \frac{36(2x+1)}{(x^2+x+1)^3}t^2$$

$$u_3(x, t) = -L^{-1}{}_t(A_2) + L^{-1}{}_tL_{xx}(u_2), \text{ but } A_2 = u_2N'(u_o) + \frac{(u1)^2}{2!}N''(u_o)$$

$$A_2 = 6u_o [u_ou_2+u_1{}^2] = 6(\frac{1+2x}{x^2+x+1}) [(\frac{1+2x}{x^2+x+1})(\frac{36(2x+1)}{(x^2+x+1)^3}t^2) + (\frac{-6(1+2x)}{(x^2+x+1)^2}t)^2]$$

$$A_2 = 6(\frac{1+2x}{x^2+x+1})[\frac{36(1+2x)^2}{(x^2+x+1)^4}t^2+\frac{36(1+2x)^3}{(x^2+x+1)^4}t^2],$$

$$A_2 = [\frac{432(1+2x)^3}{(x^2+x+1)^5}]t^2, \quad L_x(u_2) = (\frac{36(2x+1)}{(x^2+x+1)^3})'t^2 = 36t^2[\frac{2}{(x^2+x+1)^3} - \frac{3(1+2x)^2}{(x^2+x+1)^4}],$$

$$L_{xx}(u_2) = 36t^2[(\frac{-2(3)(1+2x)^2(2x+1)}{(x^2+x+1)^6} - (\frac{6(2x+1)(2)(x^2+x+1)^4-3(1+2x)^2(4)(x^2+x+1)^{3(2x+1)}}{(x^2+x+1)^8})$$

$$L_{xx}(u_2) = 36t^2[(\frac{-18(2x+1)}{(x^2+x+1)^4}) + \frac{12(1+2x)^3}{(x^2+x+1)^5}]$$

$$L_{xx}(u_2) = t^2[(\frac{-648(2x+1)}{(x^2+x+1)^4}) + \frac{432(1+2x)^3}{(x^2+x+1)^5}], \quad u_3(x, t) = -L^{-1}{}_t(A_2) + L_t{}^{-}L_{xx}(u_2)$$

$$u_3(x,t) = -\int_0^t([\frac{432(1+2x)^3}{(x^2+x+1)^5}]t^2)dt+\int_0^t t^2([(\frac{-648(2x+1)}{(x^2+x+1)^4}) + \frac{432(1+2x)^3}{(x^2+x+1)^5}])\,dt$$

$$u_3(x,t) = -(\left[\frac{432(1+2x)^3}{(x^2+x+1)^5}\right]\frac{t^3}{3} + \frac{t^3}{3}([(\frac{-648(2x+1)}{(x^2+x+1)^4}) + \frac{432(1+2x)^3}{(x^2+x+1)^5}\frac{t^3}{3}$$

$$u_3(x,t) = ([(\frac{-216(2x+1)}{(x^2+x+1)^4})t^3$$

Similarly,

$u_4(x,t) = \frac{1296((t^4)(1+2x)}{(x^2+x+1)^4}$, $u_5(x,t) = -\frac{7776((t^{5)(}1+2x)}{(x^2+x+1)^5}$, $u_6(x,t) = \frac{46656((t^6)(1+2x)}{(x^2+x+1)^6}$

$u_7(x,t) = -\frac{279936((t^7)(1+2x)}{(x^2+x+1)^7}$, $u_8(x,t) = \frac{1679616((t^8)(1+2x)}{(x^2+x+1)^8}$

$$u(x,t) = u_o(x,t) + u_1(x,t) + u_2(x,t) + u_3(x,t) + u_4(x,t) + u_5(x,t) + \ldots$$

$$\text{u(x,t)} = \sum_{n=0}^{\infty} u_n(x,t) = \frac{1+2x}{x^2+x+1} - \frac{6(1+2x)}{(x^2+x+1)^2}t + \frac{36(2x+1)}{(x^2+x+1)^3}t^2 - \frac{216(2x+1)}{(x^2+x+1)^4}t^3 + \frac{1296((t^4)(1+2x)}{(x^2+x+1)^4} - \frac{7776((t^{5)(}1+2x)}{(x^2+x+1)^5} + \frac{46656((t^6)(1+2x)}{(x^2+x+1)^6} - \frac{279936((t^7)(1+2x)}{(x^2+x+1)^7} + \frac{1679616((t^8)(1+2x)}{(x^2+x+1)^8} + \ldots \approx \frac{1+2x}{x^2+x+6t+1}$$

From this we can understand that for $0 < t < 1$, the infinite sum of the above approaches to the exact solution of the the given equation at position x and at time t which is expressed of the form u(x,t) = $\frac{1+2x}{x^2+x+6t+1}$

The following table shows that the comparison of the exact value, approximate value (using the first nine terms of the series), and the absolute error for each values of x from 0 to 10 at fixed time t = 0.02 obtained by using the Mat lab code.

Table 3.2 comparison of exact value, approximation value and absolute error at t=0.02

x	Exact value	Approximation value	Absolute error
0	0.8925	0.8929	0.0004
1	0.9613	0.9615	0.0002
2	0.7021	0.7022	0.0001
3	0.4985	0.4985	0.0000
4	0.3644	0.3645	0.0001
5	0.2772	0.2773	0.0001
6	0.2183	0.2183	0.0000
7	0.1771	0.1772	0.0001
8	0.1470	0.1471	0.0001
9	0.1249	0.1250	0.0001
10	0.1083	0.1083	0.0000

As we have seen, the comparison of exact solution and approximate solution obtained by using Adomain decomposition method, u(x,t) denote the solution of the given equation at different spatial points for different time steps, as the domain spatial x increases, the approximate solution converges to the exact solution. This means the error between approximation and exact solution approaches to zero. The extreme points on the interval will represent points in the solution grid. The approximation solution curve closely follows the shape of the exact solution curve.
The graph of the nonlinear heat equation is plotted using the MAT LAB code. The graph is given below.

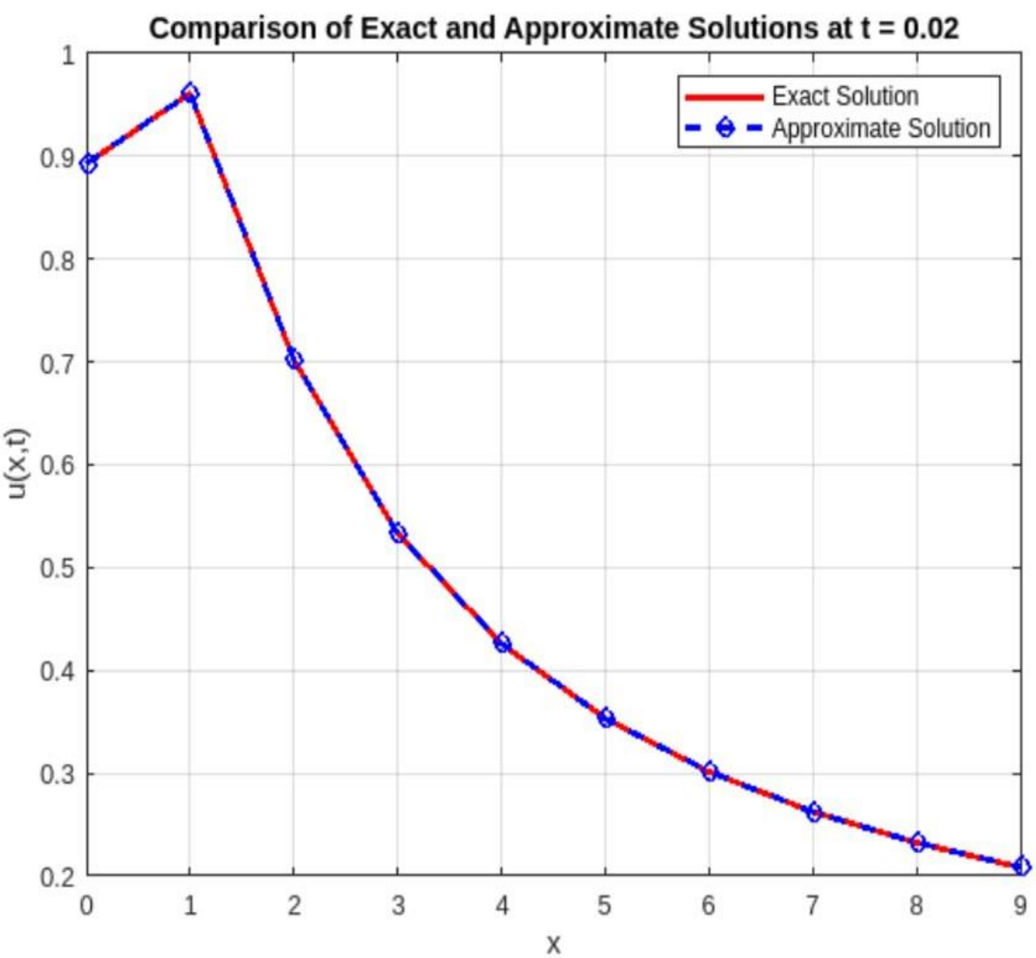

Figure 3.1 Comparison of Exact and Approximation solutions at t = 0.02

Example 2**.** The Natural decomposition method (NDM) for solving nonlinear heat equation

$$\frac{\partial u}{\partial t} + \frac{u\partial u}{\partial x} - \frac{\partial^2 u}{\partial x^2} = 0, \text{ with initial condition is } (x, 0) = 2x$$

By Taking the N-Transform in both sides of the equation we have

$$N\left(\frac{\partial u}{\partial t}\right) + N\left(\frac{u\partial u}{\partial x}\right) - N\left(\frac{\partial^2 u}{\partial x^2}\right) = N(0), \text{ with initial condition is } u(x, 0) = 2x$$

$$\frac{s}{u}u(x, s, u) - \frac{1}{u}u(x, 0) + N\left(\frac{u\partial u}{\partial x}\right) = N\left(\frac{u\partial u}{\partial x}\right)$$

$$\frac{s}{u}(x, s, u) = \frac{1}{u}u(x, 0) - N\left(\frac{u\,\partial u}{\partial x}\right) + N\left(\frac{\partial^2 u}{\partial x^2}\right)$$

$$\frac{s}{u}u(x, s, u) = \frac{2x}{u} - N\left(\frac{u\,\partial u}{\partial x}\right) + N\left(\frac{\partial^2 u}{\partial x^2}\right)$$

$$u(x, s, u) = \frac{u}{s}\left(\frac{2x}{u}\right) - \frac{u}{s}N\left(\frac{u\,\partial u}{\partial x}\right) + \frac{u}{s}N\left(\frac{\partial^2 u}{\partial x^2}\right)$$

$$Nu(x, s, u) = N\left(\frac{u}{s}\left(\frac{2x}{u}\right)\right) - N\left(\frac{u}{s}N\left(\frac{u\,\partial u}{\partial x}\right)\right) + N\left(\frac{u}{s}N\left(\frac{\partial^2 u}{\partial x^2}\right)\right)$$

$$u(x, t) = N\left(\left(\frac{2x}{s}\right)\right) - N\left(\frac{u}{s}N\left(\frac{u\,\partial u}{\partial x}\right)\right) + N\left(\frac{u}{s}N\left(\frac{\partial^2 u}{\partial x^2}\right)\right)$$

$$u(x, t) = 2x - N\left(\frac{u}{s}N\left(\frac{u\,\partial u}{\partial x}\right)\right) + N\left(\frac{u}{s}N\left(\frac{\partial^2 u}{\partial x^2}\right)\right)$$

$$u(x,t) = \sum_{n=0}^{\infty} u_n(x,t), \quad \sum_{n=0}^{\infty} u_n(x,t) = 2x - N\left(\frac{u}{s}N\left(\frac{u\,\partial u}{\partial x}\right)\right) + N\left(\frac{u}{s}N\left(\frac{\partial^2 u}{\partial x^2}\right)\right)$$

$$\sum_{n=0}^{\infty} u_n(x,t) = 2x - N\left(\frac{u}{s}N\left(\sum_{n=0}^{\infty} A_n\right)\right) + N\left(\frac{u}{s}N\left(\sum_{n=0}^{\infty}(R_{u_n}\right)\right)$$

Where A_n is Adomian polynomial then $u_o(x, t) = 2x$

$$u_1(x, t) = -N\left[\frac{u}{s}N\left[\sum_{n=0}^{\infty}(A_0)\right]\right] + N\left[\frac{u}{s}N\left(\sum_{n=0}^{\infty}(u_{n0})\right)\right]$$

$$u_2(x, t) = -N\left[\frac{u}{s}N\left[\sum_{n=0}^{\infty}(A_1)\right]\right] + N\left[\frac{u}{s}N\left(\sum_{n=0}^{\infty}(u_{n1})\right)\right]$$

$$u_3(x, t) = -N\left[\frac{u}{s}N\left[\sum_{n=0}^{\infty}(A_2)\right]\right] + N\left[\frac{u}{s}N\left(\sum_{n=0}^{\infty}(u_{n2})\right)\right]$$

$$u_4(x, t) = -N\left[\frac{u}{s}N\left[\sum_{n=0}^{\infty}(A_3)\right]\right] + N\left[\frac{u}{s}N^{+}\left(\sum_{n=0}^{\infty}(u_{n3})\right)\right]$$

$$u_5(x, t) = -N\left[\frac{u}{s}N\left[\sum_{n=0}^{\infty}(A_4)\right]\right] + N\left[\frac{u}{s}N^{+}\left(\sum_{n=0}^{\infty}(u_{n4})\right)\right]$$

$$u_6(x, t) = -N\left[\frac{u}{s}N\left[\sum_{n=0}^{\infty}(A_5)\right]\right] + N\left[\frac{u}{s}N^{+}\left(\sum_{n=0}^{\infty}(u_{n5})\right)\right]$$

$$u_7(x, t) = -N\left[\frac{u}{s}N\left[\sum_{n=0}^{\infty}(A_6)\right]\right] + N\left[\frac{u}{s}N^{+}\left(\sum_{n=0}^{\infty}(u_{n6})\right)\right]$$

$$u_8(x, t) = -N\left[\frac{u}{s}N\left[\sum_{n=0}^{\infty}(A_7)\right]\right] + N\left[\frac{u}{s}N^{+}\left(\sum_{n=0}^{\infty}(u_{n7})\right)\right]$$

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$u_{n+1}(x,t) = N\left[\ \frac{u}{s}\mathrm{N}[\sum_{n=0}^{\infty} A_n]\right] + N\left[\ \frac{u}{s}\mathrm{N}[\sum_{n=0}^{\infty} R_{u_n}]\right].$

$u_o(x,t) = 2\mathrm{x}$, $u_1(x, t) = -N[\ \frac{u}{s}\mathrm{N}[A_0]] + N[\ \frac{u}{s}\mathrm{N}(R_{u0})]$

where $A_0 = \mathrm{N}(u_0) = u_o(u_o)_{x,}$ and $A_0 = (2x\frac{\partial u}{\partial x}(2x)) = 2x\times 2 = 4x$

$A_0 = (2x\frac{\partial u}{\partial x}(2x)) = 2x\times 2 = 4x,\ A_0 = 4x$

$u_1(x, t) = -N[\ \frac{u}{s}\mathrm{N}[A_0]] + N[\ \frac{u}{s}\mathrm{N}^{+}[R_{u0}]]$

$u_1(x, t) = -N[\ \frac{u}{s}\mathrm{N}^{+}[4x]] + N[\ \frac{u}{s}\mathrm{N}[\frac{\partial^2 u}{\partial x^2}(2x)]]$

$u_1(x, t) = -N[\ \frac{u}{s}\mathrm{N}[\frac{4x}{s}]] + N[\ \frac{u}{s}\mathrm{N}[\frac{\partial}{\partial x}(2)]]$

$u_1(x, t) = -N[\ \frac{u}{s}[\frac{4x}{s}]] + N[\ \frac{u}{s}\mathrm{N}[(0)]]$

$u_1(x, t) = -N[\ \frac{u}{s}\times\frac{4x}{s}]] = -N(\frac{4xu}{s^2}) = -4xt,$

$u_1(x, t) = -4xt$

$u_2(x, t) = -N[\ \frac{u}{s}\mathrm{N}[\sum_{n=0}^{\infty}(A_1)] + N[\ \frac{u}{s}\mathrm{N}[\sum_{n=0}^{\infty}(R_{u1})]]$ where

$A_1 = u_1\mathrm{N}'(u_o) = -4xt(u_{ox}u_{ox} + u_o u_{oxx})$

$= -4xt(2\times 2 + 2x\times 0) = -4xt(4) = -16xt$

$A_1 = -16xt, u_2(x, t) = -N[\ \frac{u}{s}\mathrm{N}[A_1]] + N[\ \frac{u}{s}\mathrm{N}[R_{u1}]]$

$u_2(x, t) = N[\ \frac{u}{s}\mathrm{N}[-16xt]] + N[\ \frac{u}{s}\mathrm{N}[-4xt\left(\frac{\partial^2 u}{\partial x^2}\right)]\ ]]$

$u_2(x, t) = -N[\ \frac{u}{s}(\frac{-16xu}{s^2}) + N[\ \frac{u}{s}\mathrm{N}\ [-4t\ (\frac{\partial u}{\partial x})]\ ]]$

$u_2(x, t) = -N[\ (\frac{-16xu^2}{s^3}) + N[\ \frac{u}{s}\mathrm{N}[0]] = -N[\ (\frac{-16xu^2}{s^3}) = \frac{-16xt^2}{2!} = 8xt^2$

$u_2(x, t) = 8xt^2$, $u_3(x, t) = -16xt^3$, $u_4(x, t) = 32xt^4$, $u_5(x, t) = -64xt^5$

$u(x,t) = 2x - 4xt + 8xt^2 - 16xt^3 + 32xt^4 - 64xt^5, \dots$

$u(x, t) = 2x(1 + (-2t) + (-2t)^2 + (-2t)^3 + (-2t)^4 + (-2t)^5 + \dots) \approx \frac{2x}{1+2t}$

Thus, the infinite sum approaches to the exact solution of $u(x, t) = \frac{2x}{1+2t}$ for the given nonlinear equation whenever $0 < t < 1$

Table 3.3 comparison of exact value, approximation value and absolute error using (NDM) at t = 0.01

Position(x)	Exact value	Approximation value	Absolute error
0.000000	0.000000	0.000000	0.00
0.010101	0.020198	0.020198	0.00
0.020202	0.040388	0.040388	0.00
0.030303	0.060569	0.060569	0.00
0.040404	0.080743	0.080743	0.00

For t=0.01 the difference between exact value and approximation value approaches to zero. This result shows that the Natural Decomposition method provides closest approximation solution to the exact solution for this specific setup.

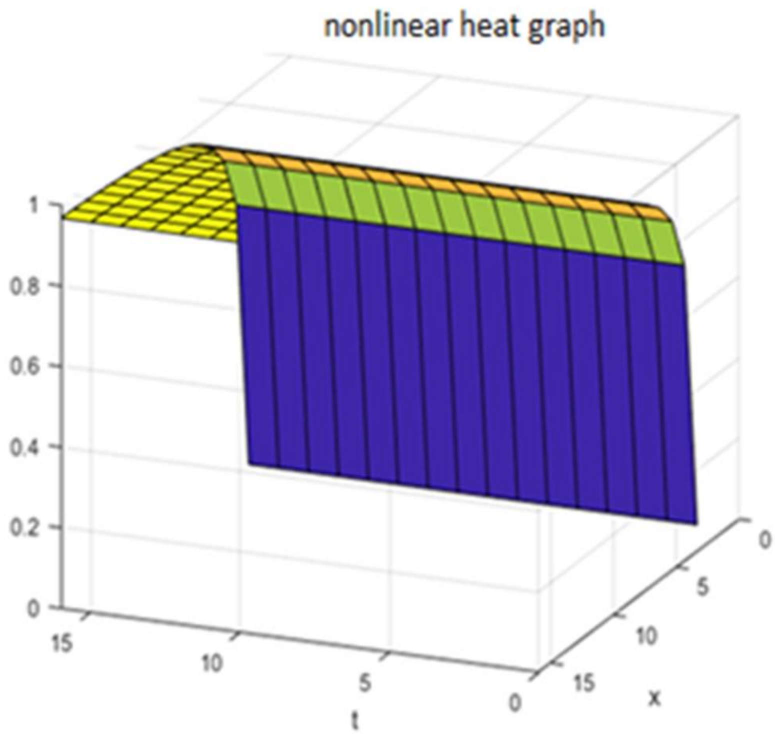

Figure 3.2 the graph of approximate solution of nonlinear heat equation. (ADM)

$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} - 2u^3$ **with initial condition is u(x, 0) =** $\frac{1+2x}{x^2+x+1}$

CHAPTER FOUR

CONCULUSION AND RECOMMENDATION

4.1 CONCLUSION

The study offers that Adomain decomposition method provides good approximate solutions that are reasonably accurate. This method is more practical for complex or large scale nonlinear heat equations. The accuracy and efficiency of approximation solutions are based on the methods implementation and the problems specific characteristics. The Adomain decomposition and natural decomposition methods replaces all partial derivatives and other terms in the PDE by approximations. The approximations of nonlinear partial differential heat equations by using Adomain decomposition and natural decomposition methods are made at discrete values of the independent variables and approximation pattern. The method is able of providing fast solutions that are highly accurate and reliable in solving this type of equations

4.2 RECOMMENDATION OF THE STUDY

Adomain Decomposition and natural decomposition methods are the best methods for further study on Numerical solutions of nonlinear partial differential equations. For the further study these methods can be extended to study how to determine numerical solutions of three and four dimensional nonlinear partial heat equations.

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Appendix

From example1 the graph of nonlinear heat mat lab cod solution (ADM)

[x,t]=meshgrid(0:15,0:15);

z=((2.*x)./(1+2.*x));

```
surf(z);
xlabel('x');
ylabel('t');
zlabel('z');
title('nonlinear heat aproximate solution')
```

```python
u_exact_vals = u_exact(X, T)
u_approx_vals = u_approx(X, T)

# Plotting the exact solution
plt.figure(figsize=(12, 6))
plt.subplot(1, 2, 1)
plt.contourf(X, T, u_exact_vals, cmap="viridis")
plt.colorbar(label="u(x, t)")
plt.title("Exact Solution")
plt.xlabel("x")
plt.ylabel("t")

# Plotting the approximate solution
plt.subplot(1, 2, 2)
plt.contourf(X, T, u_approx_vals, cmap="viridis")
plt.colorbar(label="u(x, t)")
plt.title("Approximate Solution (9 terms)")
plt.xlabel("x")
plt.ylabel("t")
```