**ADDIS ABABA UNIVERSITY**

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ON

# THE LENGTH OF MODULES ON THE WEYL ALGEBRA

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# CHAPTER ONE

## Introduction

This chapter and the next are devoted to a review of the basic concepts of ring and module theory . Because we are assuming the reader already has some familiarity with these topics , only brief description of the basic concepts and results that are needed in this paper is presented.

## 1. Prerequisites and preliminaries

**Definition:-** A system $(R,+,\cdot)$ were R is a non-void set, + and · are two binary operations defined on the set R, is called a ring if it satisfies the following conditions.

1. (R, +) be abelian group.
2. (a, b) .c = a.(b.c) for any $a,b,c \in R$
3. a.(a+c) = a.b + a.c (left distributive law)

   (b + c).a= b.a + c.a (right distributive law)

Example. The set of integers $\mathbb{Z}$ under usual addition and multiplication is a ring.

**Definition**:- A ring with unity in which all non-zero elements form a group under multiplication is called a divisor ring.

Note:-Let R be a ring

- R is commutative if ab=ba for every $a,b \in R$.
- $e \in R$ is an identity of R if ae=ea=a for every $a \in R$.
- $a \in R$ is a zero divisor of R if ab=ba=0, for some non-zero $b \in R$.

- 0 is called a trivial zero – divisor.

**Definition**:-A subring of a ring R is a subset of R which by itself is a ring with respect to the same set of binary operations. Thus $S \subset R$ is a subring of $(R,+,\cdot)$ if $(S,+,\cdot)$ is a ring.

Examples. The set of integer $\mathbb{Z}$ is a subring of a set of rational numbers under usual addition and multiplication of numbers.

**Definition**:-Let R be a ring and I be a non-void subset of R, then I is an ideal of R.

If i) $a,b, \in I \Rightarrow a - b \in I$ and

ii) $a \in I, r \in R \Rightarrow ar \in I$ and $ra \in I$

Example. The set E of even integers is an ideal of the ring $\mathbb{Z}$ of integers.

**Definition**:- Let R be a ring and I be any ideal of R then the system $(R/I,+,\cdot)$ where addition and multiplication are defined by $(r+I)+(s+I)=(r+s)+I$ and $(r+I).(s+I)=rs+I$ for all $r,s \in R$, is a ring.This ring is called the quotient ring of R. with respect to the ideal I.

Example Let $I=\{6n/n \in \mathbb{Z}\}$,then I is an ideal of $\mathbb{Z}$

Hence $(\mathbb{Z}/I,+,\cdot)$ is a quotient ring of $\mathbb{Z}$.

**Definition**:- A ring R is called simple if it has no ideals other than (0) and R.

**Theorem 1.1.1** If R is a division ring then it is a simple ring.

**Proof** Let I ($\neq 0$) be an ideal of a division ring R.

Since $I \neq (0)$, $\exists a(\neq 0) \in I$, Again as R is a division ring a is a unit in R, i.e $\exists b \in R$ such that ab=1,hence $1 \in I$ since $a \in I$. For any $r \in R$.we get $r = r.1 \in I$ consequently R =I, Hence R has no proper ideal. Therefore R is a simple ring.

Definition:- Let R be a commutative division ring. Then R is a field.

**Theorem 1.1.2** Let K be a field the polynomial over K denoted by $K[X] = K[x_1, \ldots x_n]$ is a ring with usual addition and multiplication defined as follows.

Let $f = (a_0, a_1, \ldots)$ and $g=(b_0,b_1,\ldots)$ where $a_i, b_i \in K$

$f+g = (a_0+b_0, a_1+b_1, \ldots)$

$fg = (c_0,c_1,c_2, \ldots)$ where $c_k = \sum_{i+j=k} a_i b_j$ for every $k \geq 0$

**Proof** Let $a_n$ and $b_m$ be coefficients of f and g respectively such that $a_i=0, \forall i \geq n+1$ and $b_j=0$, $\forall_j \geq m+1$, if $s = \max\{m,n\}$, then $a_t=b_t=0$, $\forall t>s$, so $a_t + b_t=0$ $\forall t>s$ hence $f + g \in K[X]$

Again consider $k>m + n$, then if $i+ j=k$ either $i>m$ or $j>n$ hence $a_ib_j =0$, $\forall i$ and j satisfying $i+j =k$, for $k>m+n$, This gives that $c_k=0$, $\forall k \geq m+n+1$, hence $fg \in K[x]$

This shows the closurity of addition and multiplication

Since (K, +) is abelian group, so is (K[X],+),with (0,0….)is the additive identity and for any $f=(a_0,a_1\ldots.) \in K[X]$ , $-f = (-a_0,-a_1,\ldots\ldots) \in K[X]$ is the additive inverse of f.

Finally it can be easily checked that "$\cdot$" is distributive on left as well as on right over +.

Hence (K[X], +, $\cdot$) is a ring.

**Definition**:- A ring R is said to satisfy the ascending chain condition (ACC) on ideals if for every chain $I_1 \subset I_2 \subset I_3 \subset \ldots\ldots$ of ideals of R, there is an integer m such that $I_i=I_m$ for all $i \geq m$.

**Definition**:- A ring R is said to satisfy the descending chain condition (DCC) on ideals, if for every chain $I_1 \supset I_2 \supset I_3 \supset \ldots\ldots$ of ideals of R ,there is an integer n such that $I_i=I_n$ for all $i \geq n$.

Example. The ring $\mathbb{Z}$ satisfies the ascending but not the descending chain condition on ideals.

Since $\mathbb{Z}$ is a principal ideal domain it satisfies ACC. But $2\mathbb{Z} \supset 2^2\mathbb{Z} \supset \ldots\ldots \supset 2^i\mathbb{Z} \supset 2^{i+1}\mathbb{Z} \supset \ldots\ldots$ is a strictly descending chain of ideals of $\mathbb{Z}$, since for all $i \in \mathbb{N}$, we have $2^{i+1}=2^i 2 \in 2^i\mathbb{Z}$ and also $2^i \notin 2^{i+1}\mathbb{Z}$ since an equation $2^i = 2^{i+1}r$ for some $r \in \mathbb{Z}$ would contradict the fact that $\mathbb{Z}$ is a UFD.

**Definitions**

- A ring R is left (resp.right) Noetherian if R. satisfies the ascending chain condition on left (resp.right) ideals.
- R is said to be Noetherian if R is both left and right Noetherian.
- A ring R is left (resp. Right) Artinian if R satisfies the descending chain condition on left (resp.right) ideals.
- R is said to be Artinian if R is both left and right Artinian.

Example. A division ring R is both Noetherian and Artinian since the only left or right ideals are R and 0.

**Theorem 1.1.3** A commutative ring R is Noetherian if and only if every ideal of R is finitely generated.

**Proof** ($\Rightarrow$) Let I be an ideal of R and let S= {N | N is a finitely generated ideal of R such that $N \subseteq I$ }.

Since (0) is finitely generated, $(0) \in S$ implies $S \neq \emptyset$.

Since R is noetherian maximum condition holds, it follows that S has a maximal element. Say M.

Since $M \in S$ it is finitely generated. We shall show that I=M.

As $M \in S$, $M \subseteq I$ . If possible let

$\Rightarrow$As $M \in S$, $M \subseteq I$ ,If possible let $I \neq M$ ,then there exists an element $a \in I$, $a \in M$, consider the ideal $M+(a)_r$ ,$M+(a)_r$ is finitely generated ideal of R.[It is generated by $\theta \cup \{a\}$ where ( $\theta$ )=M].since $a \in I$ , $M+(a)_r \subseteq I$ .This gives that $M+(a)_r \in S$, But $a \notin M \Rightarrow M \subset M+(a)_r$,This violates the maximality of M. Hence I=M As a consequence I is finite generated.

Conversely;Let $I_1 \subseteq I_2 \subseteq \ldots$.be an infinite ascending chain of ideals of R consider $I = \cup_{i \geq 0} I_i$ ,I is an ideal of R. Now each ideal is finitely generated Let $S=\{a_1,a_2,\ldots\ldots,a_n\}$. Be a generating set of I i.e. $I=(a_1,a_2,\ldots,a_n)$, For each $j \in \{1,2,\ldots.n\}, a_j \subseteq I$

Implies the existence of some positive integer $i_j$ such that $a_j \in I_{i_j}$

Let $k=\max\{i_1,\ldots..,i_n\}$, Then $a_j \in I_k, \forall_j=1,2,\ldots.,n$,This gives that $I_k \subseteq I$ but $I_k \subseteq I$ ,hence $I=I_k$. Now for all $m \geq k$, $I_m \subseteq I=I_k$ gives $I_m=I_k$, hence R is noetherian.

**Theorem 1.1.4 (Hilbert Basis Theorem)** If R is a commutative noetherian ring with identity then so is $R[x_1,\ldots,x_2]$.

**Proof**. Clearly it suffice to show that R[x] is noetherian . suppose that R[x] is not noetherian .let I be an ideal of R[x] that is not finitely generated ,we shall constract inductively an infinite ascending chain of ideals in R ,there by achieving a contradiction .choose $f_1 \in I$ of smallest possible degree .Assume by induction that $f_1,\ldots f_k$ have been chosen ,let $f_{k+1}$ be the polynomial of smalles possible degree in $I/(f_1,\ldots,f_k)$ .since I is not finitely generated ,this construction produces an infinite sequence of polynomials $f_1, f_2,\ldots \in I$ .let $n_i$ be the degree and $a_i$ the leading coefficient of $f_i$ .since R is noetherian the ascending chain of ideals

$$(a_1) \subseteq (a_1, a_2) \subseteq \ldots$$

Must be stationary ,say $(a_1,\ldots,a_k) = (a_1,\ldots,a_{k+1})$,Hence

$a_{k+1} = \sum_1^k b_i a_i$ for some $b_i \in R$ , put

$$g = f_{k+1} - \sum_1^k b_i x^{n_{k+1}-n_i} f_i$$

Since ,by construction the degrees satisfy $n_1 \le n_2 \le \ldots$ it follows that g is indeed a polynomial .Note that $g \in I$, *but* $g \notin (f_1,\ldots,f_k)$. However g has smaller degree than $f_{k+1}$ .a contradiction ,Thus I must be finitely generated .

## CHAPTER TWO

### 2.1. Modules over Rings

Modules over ring are a generalization of abelian groups (which are modules over $\mathbb{Z}$) They are basic in the further study of algebra .This chapter is mostly devoted to carrying over to modules various concepts and results of group theory. Also we shall show how some of the techniques we have developed for handling modules over a ring so far in this chapter can be put good use to provide proofs of some classical theorems, These theorems concern module over the Weyl Algebra.

**Definition**:- Let R be a ring ,A (left) R-module is an additive ablian group M together with a function $R \times M \to M$ given by $(r, m) \mapsto rm$ ,such that for all r,s$\in$ R and m,n$\in$ M.

1) r(m+n)=rm+rn

2) (r+s)m=rm+sm

3) r(sm)=(rs)m

If R has an identity element $1_R$ and

4. $1_R$m=m for all m$\in$ M.

Then M is said to be a unitary R-module .If R is a division ring then a unitary R-module is called a(left) vector space.

A(unitary) right R-module is defined similarly via a function $M \times R \to M$ *denoted by* $(m, r) \mapsto mr$ and satisfying the obvious analogues of (1)-(4)

Example Let R be a commutative ring ,and let I be an ideal of R .

A very important example of an R-module is R it self ,R is of course ,an abelian group, the multiplication in R gives us a mapping,

$\therefore R \times R \to R$ and the ring axioms ensure that this "scalar multiplication" turns R in to an R-module.

ii) Since I is closed under addition and multiplication by arbitrary elements of R it follows that I too is an R-module under the addition and multiplication of R.

iii) We show next that the residue class ring R/I can be viewed as an R-module.

Of course R/I has a natural abelian group structure. We need to provide it with a scalar multiplication by elements of R, To this end ,let r,s∈ R be such that r + I = s + I in R/I ,and let a∈ R ,Thus r-s∈ I and so ar – as = a(r-s) ∈ I ,hence ar + I = as + I,It follows that we can define a mapping $R \times R/I \to R/I$ *by* $(r, s+I) \mapsto rs+I$ , and it is easy to check that R/I becomes an R-module with respect to this scalar multiplication.

**Definition**:- Let M be a module over the commutative ring R ,and let N be a subset of M,we say that N is a sub-module of M .or an R-submodule of M precisely when N is it self an R-module with respect to the operations for M.

### 2.2.The Submodule Criterion

Let R be a commutative ring and let N be a subset of the R-module M .Then N is a submodule of M if and only if the following conditions hold.

1. $N \neq \phi$
2. Whenever a,b∈ N and r,s∈ R ,then ra+sb∈ N.

## 2.3 .Module Homomorphsim

**Definition**:- Let M and N be modules over the commutative ring R A mapping $f : M \to N$ is said to be a homomorphism of R-modules precisely when

f(rm+sn)=rf(m)+sf(n) for all m,n $\in$ M and r,s $\in$ R A homomorphism.

- f is said to be a monomorphsim if it is injective
- f is said to be an epimorphsim if it is surjective
- f is said to be an isoomorphsim if it is bijective

If $f_i : M \to N$ $(fori = 1,2)$ are R-homomorphsim then the mapping $f_1 + f_2 : M \to N$ *defined by* $(f_1 + f_2)(m) = f_1(m) + f_2(m), \forall m \in M$ *is* also an R-homomorphsim.

Example Let R be a ring and $f : M \to N$ be an R-module homomorphism then kerf is a submodule of M.

Example For any module the zero map $0 : M \to N$ given by m $\mapsto$ 0, $\forall$ m $\in$ M is a module homomorphsim

Example If R is a ring the map $R[x] \to R[x]$ given by f $\mapsto$ xf is an R-module homomorphism( for example (x+3) $\mapsto$ x(x+3))

Remark: if $f : M \to N$ and $g : N \to P$ are homomorphism of modules over the commutative ring R then the composition $gof : M \to P$ is also an R-module homomorphism.

Further if f and g are isomorphism ,then so is gof .

**Definition**:- Let N be a module over a commutative ring R. we say that N is a simple R-module ,precisely when $N \neq 0$ and the only submodules of N are 0 and N it self .

**Lemma 2. 3.1** Let N be a module over a commutative ring R. then N is simple if and only if N is isomorphic to an R-module of the form R/M for some maximal ideal M of R.

**Proof** ($\Leftarrow$) Let M be a maximal ideal of R, Then R/M is a field so it has exactly two ideals, namely itself and its zero ideal. Hence the R-module R/M has exactly two submodules ,namely itself and its zero submodules.

Therefore R/M is simple and hence N is simple.

($\Rightarrow$) Suppose that N is a simple R-module since $N \neq 0$,there exists $n \in N$ with $n \neq 0$,Hence $0 \neq Rn \subseteq N$,Frist we have to show that $Rn=\{rn : r \in R\}$ is a submodule of N.

Let $a,b \in Rn$, Then a=rn and b=sn for some $r,s \in R$ ,then for $x,y \in R$ we have

xa=xrn and yb=ysn thus xa+yb= xrn+ysn=(xr+ys)n=kn $\in$ Rn for some $k \in R$

Therefore Rn is a submodule of N.

Since N is simple and $Rn \neq 0$,we must have Rn=N ,it follows that N is a cyclic R-module.

Also N is a subset of R implies that the map $f : R \to N$ given by f(r)=rn, $\forall r \in R$ is an epimorphsim.

Let I=kerf then I is an ideal of R .

Define $g : R/I \to N$ by $g(\bar{r})$=f(r) , $\forall \bar{r} = r+I \in R/I$ .

Now for any $\bar{r}, \bar{s} \in R/I$, we have

$$\bar{r} = \bar{s} \Leftrightarrow r\text{-}s \in I = \ker f$$

$$\Leftrightarrow f(r\text{-}s) = 0$$

$$\Leftrightarrow f(r) = f(s)$$

$$\Leftrightarrow g(\bar{r}) = g(\bar{s})$$

This shows that g is well defined as well as one to one

Also $g(\bar{r} + \bar{s}) = g(\overline{r+s}) = f(r+s) = f(r) + f(s) = g(\bar{r}) + g(\bar{s})$

$g(a\bar{r}) = f(ar) = af(r) = ag(\bar{r})$

implies that g is monomorphsim

Finally $g(R/I) = f(R) = N$ implies g is also onto.

Hence g is an isomorphism and, hence $R/I \cong N$

### 2. 4 .Definitions and Remarks

Let G be a module over a commutative ring R. A strict chain of submodules of G is a finite strictly increasing chain

$G_O \subset G_1 \subset \ldots \subset G_{n-1} \subset G_n$ of submodules of G such that

$G_o = 0$ *and* $G_n = G$.

The **length** of the strict chain is the number of links,that is one less than the number of terms (so that the displayed strict chain above has length n).

Let $G = G_o \supset G_1 \supset \ldots \supset G_n$ be a chain of submodules of G ,A one step refinement of this chain is any chain of the form

$G = G_O \supset ...G_i \supset N \supset G_{i+1} \supset .... \supset G_n$ or $G = G_o \supset ..... \supset G_n \supset N$ ,where N is a submodule of $G_i$ and (if i>n) $G_{i+1}$ is submodule of N.

A refinement of a submodule chain S is any chain of submodules obtained from S by a finite sequence of one step refinement. A refinement of S is said to be proper if its length is larger than the length of S.

We consider 0 to be a strict chain of of length 0 of submodules of the zero R-module.

A strict chain of submodules of G given by

$0=G_O \subset G_1 \subset ... \subset G_{n-1} \subset G_n$ =G is said to be a **composition series** for G precisely when $G_i/G_{i-1}$ is a simple R-module for each i =1,2,…,n.

Let G be a module over a commutative ring R . we say that G has finite length precisely when G has a composition series .When this is the case the length of G ,denoted by $l(G)$ is defined to be the length of any composition series for G.

When G does not have finite length that is when G does not have a composition series we shall sometimes indicate this by writing $l(G) = \infty$.

**Proposition 2.4.1** Let G be a module over a commutative ring R,Then G has finite length if and only if G is both Noeterian and Artinian ,that is if and only if G satisfies both the ascending and descending chain conditions on submodules.

**Proof** ($\Rightarrow$) Assume that G has finite length $l(G)$. Then it follows that any ascending chain of submodules of G can not have more than $l(G)$ of its inclusion strict and so must be eventually stationary. Similarly any descending chain of submodules of G must be eventually stationary.

($\Leftarrow$) Assume that G is both Noeterian and Artinian.so that it satisfies both the maximal and minimal conditions for submdules, we suppose that G does not have a composition series and look for a contradiction.Then

S:={M:Mis a submodules of G and $l(M)=\infty$}is not empty ,since $G\in S$,Hence by the minimal condition S has a minimal member say H.Now $H\neq 0$,since the zero submodule of G certainly has a composition series .Thus by the maximal condition the set of proper submodules of H has at least one maximal member $H'$ say By choice of H and the fact that $H'\subset H$ we see that $H'$ has a composition series , Let

$H_o'\subset H_1'\subset ....\subset H_t'$be a composition series for $H'$, so that $H_o'=0$ , $H_t'=H'and\ \ t=l(H')$.

Since $H'$ is a maximal proper submodule of H it follws that H/$H'$ has exactly two submodule and so is simple .Hence

$H_o'\subset H_1'\subset ....\subset H_t'$ is a composition series for H ,this is a contradiction

Therefore G must have a composition series.

**Theorem 2.4.2 (Jordan - Hölder Theorem).** Let G be a non-zero module of finite length over the commutative ring R ,then every pair of composition series for G are isomorphic .

**Proof** Since $G\neq 0$, Let n:=$l(G)\geq 1$,we use induction on n .The claim is clear when n=1 and so we assume that n>1 and that the result has been proved for smaller values of n.

Let $G_O\subset G_1\subset ...\subset G_{n-1}\subset G_n$ and

$G_o' \subset G_1' \subset ... \subset G_{n-1}' \subset G_n'$ be two composition series for G (so that $G_o = G_o' = 0$ *and* $G_n = G_n' = G$).

The argument for the inductive step splits in to two cases .the frist case is where $G_{n-1} = G_{n-1}'$, Then we have $G_n / G_{n-1} = G_n' / G_{n-1}'$ and both $G_O \subset G_1 \subset ... \subset G_{n-1}$ and $G_o' \subset G_1' \subset ... \subset G_{n-1}'$ are composition series for $G_{n-1} = G_{n-1}'$ since $l(G_{n-1})$ =n-1 we apply the inductive hypothesis to these two composition series for $G_{n-1}$ and the desired result in this case follows easily .

The second case is where $G_{n-1} \neq G_{n-1}'$, Then we set H=$G_{n-1} \cap G_{n-1}'$ ,now $G_n / G_{n-1} \cong G_{n-1}' / H$ *and* $G_n' / G_{n-1}' \cong G_{n-1} / H$ so that all four of these modules are simple .thus if H=0 (so that both $G_{n-1}$ and $G_{n-1}'$ are simple and n=2) the desired coclusion has been obtained .Thus we assume that $H \neq 0$ , In this case

$0 \subset H \subset G_{n-1} \subset G_n$ is a strict chain of submodules of G=$G_n$ and both $G_n/G_{n-1}$ and $G_{n-1}$/H are simple. Now the above strict chain can be extended by the insertion of extra terms to a composition series for G since such a composition series for G must have length n. it follows that $l(H)$ =n-2 .

In particular ,we obtain a composition series ,

$H_O \subset H_1 \subset ... \subset H_{n-3} \subset H_{n-2}$ for H ( so that $H_o$=0 and $H_{n-2}$ =H),The comments in the preceding paragraph now show that the two composition series

$H_O \subset H_1 \subset ... \subset H_{n-3} \subset H_{n-2} \subset G_{n-1} \subset G_n$ and

$H_O \subset H_1 \subset ... \subset H_{n-3} \subset H_{n-2} \subset G_{n-1}' \subset G_n'$ for G are isomorphic. But we can now use the inductive hypothesis (on two composition series for $G_{n-1}$)to see that the two composition series.

$G_O \subset G_1 \subset ... \subset G_{n-1} \subset G_n$ and $H_O \subset H_1 \subset ... \subset H_{n-3} \subset H_{n-2} \subset G_{n-1} \subset G_n$ for G are isomorphic.

Similarly the composition series

$H_O \subset H_1 \subset ... \subset H_{n-3} \subset H_{n-2} \subset G'_{n-1} \subset G'_n$ and $G'_o \subset G'_1 \subset ... \subset G'_{n-1} \subset G'_n$ are isomorphic ,and so we can complete the inductive step.

The theorem is therefore proved by induction.

Example Let G=<x> be a cyclic group of order 24, then o(x)=24 since G is cyclic it is abelian group ,Hence it is Z-module.Consider two series

$G=<x> \supset <x^2> \supset <x^6> \supset <x^{12}> \supset <0>$ .....(1)

$G=<x> \supset <x^3> \supset <x^6> \supset <x^{12}> \supset <0>$ .....(2)

These are two chain of submodules for G ,The factor modules of (1) are $<x>/<x^2>$, $<x^2>/<x^6>$,$<x^6>/<x^{12}>$ and $<x^{12}>$ which are of orders 2,3,2,2 respectively ,since these are of prime orders they are all simple . Similarly the factor modules of (2) are $<x>/<x^3>$, $<x^3>/<x^6>$,$<x^6>/<x^{12}>$ and $<x^{12}>$ which are of orders 3,2,2,2 respectively.These are again simple.So (1) and (2) both are a composition series of G ,one can see that both of these series are of some length 4 .Further as two cyclic groups of the same order are isomorphic ,we have

$<x>/<x^2> \cong <x^3>/<x^6>$,$<x^2>/<x^6> \cong <x>/<x^3>$,$<x^6>/<x^{12}> \cong <x^6>/<x^{12}>$,

$<x^{12}> \cong <x^{12}>$. Thus there is a one to one correspondence between the factor module of (1) and those of (2) such that the corresponding factor module are isomorphic.

## 2.5. Algebra

**Definition**. Let K be a commutative ring with identity. A K-algebra (or algebra over K) M is a ring M such that:

1. (M, +) is a unitary (left) K-module.
2. $k(mn)=(km)n=m(kn)$ for all $k \in K$ and $m,n \in M$

Example Every ring R is an additive abelian group and hence a $\mathbb{Z}$-module. It is easy to see that R is actually a $\mathbb{Z}$ -algebra.

**Proposition 2.5.1** If K is a commutative ring with identity then the polynomial ring $K[x_1,x_2,\ldots,x_n]$ is a K-algebra with the respective K-module structure given in the usual way.

**Proof** To show K[X] is a K-algebra we need to show that

(K[X],+) is a unitary (left) K-module.

$k(fg)=(kf)g=f(kg)$. for all $k \in K$ and $f,g \in K[X]$

Now, since K [X] is a ring with unity (K[X],+)is a belian group.

Let the map $K \times K[X] \rightarrow K[X]$ be the action of K on K[X] be given by $(k,f) \mapsto kf$, where $k \in K$ and $f \in K[X]$.

For $k,k' \in K$ and $f,g \in K[X]$ ,where $f=(a_0,a_1,\ldots)$ , $g=(b_0,b_1,\ldots)$ $a_i, b_i \in K$

We have.

1. $k.(f+g) = k.(a_0+b_o, a_1+b_1, \ldots)=(k(a_0+b_0),k(a_1+b_1), \ldots)$

   $= (ka_0,kb_0, ka_1+kb_1,\ldots\ldots$(as <.> is left distributive)

   $= (ka_0,a_1,\ldots)+(kb_0,kb_1,\ldots)$

$= k(a_0,a_1,\ldots) + k(b_0,b_1,\ldots)$

$= kf + kg.$

2. $(k+k').f = [(k+k').a_0, (k+k').a_1,\ldots)]$. . . as <.> is left distributive.

   $= (ka_0 + k'a_0, ka_1 + k'a_1,\ldots)$ as <.> is right distributive.

   $= (ka_0,ka_1,\ldots) + (k'a_0, k'a_1\ldots)$

   $= kf + k'f$

3. $(k\,k').f = [(k\,k').a_0, (k\,k').a_1,\ldots)]$

   $= [k(k'a_0),k(k'a_1),\ldots)]$. . . as (K[X],.) is semi-group.

   $= k(k'a_0, k'a_1,\ldots)$

   $= k[k'(a_0,a_1,\ldots)]$

   $= k(k'f)$

4. $1.f = (1.a_0,1.a_1,\ldots)$

   $= (a_0,a_1,\ldots)$ as 1 is a unity of K.

   $= f.$

Therefore K[X] is a left K-module.

Moreover since K[X] is a unitary left K-module it is a left K-vector space.

Finally let $f,g \in K[X]$,where $f = a_0,a_1,\ldots)$ and $g = (b_0,b_1,\ldots)$

Then for $k \in K$ we have

$k.(f.g) = k.[(a_0,a_1,\ldots)\cdot(b_0,b_1,\ldots)]$

$= [k.(a_0,a_1,\ldots)] \cdot (b_0,b_1,\ldots)$ since <.> is associative

$= (ka_0,ka_1,\ldots) \cdot (b_0,b_1,\ldots)$

$= (a_0k,a_1k,\ldots)\ (b_0,b_1,\ldots)$since K is commutative ring.

$= [(a_0,a_1,\ldots)\ k](b_0,b_1,\ldots)$

$= (a_0,a_1,\ldots)[k(b_0,b_1,\ldots)]$

$= f(kg)$

Therefore K[X] is a K-algebra.

**Proposition 2.5.2** Let K[X] be a vector space over a field K. then the set of endomorphism of K[X] on to K[X] denoted by $End_K(K[X])$ is K-algebra.

**Proof** Show first $(End_K(K[X],+,\cdot)$ ) is a ring. Where'+' and '·' are defined as follows.

For all $\psi$ , $\varphi \in End_K(K[X])$. $\psi - \varphi$ is the mapping from K[X] to K[X] given by $(\psi+\varphi)(f)) = \psi(f) + \varphi(f)$ , $\forall$ $f \in K[X]$. and

$\psi\varphi$ is the composite mapping of K[X] in to K[X] i.e $\psi\varphi(f)=\psi(\varphi(f))$. $\forall$ $f \in K[X]$,

Since K[X] is a vector space the mapping $\varphi$ from K[X] to K[X] is a linear operator of K[X].

Closurity of (+) and (·)

Let $\psi$ , $\varphi \in End_K(K[X])$ and $f,g \in K[X]$

$(\psi + \varphi)(f + g) = \psi(f+g) + \varphi(f+g)$, by definition

$= [\psi(f) + \psi(g)] + [\varphi(f) + \varphi(g)]$ by linearity.

$= [\psi(f) + \varphi(f) + [\psi(g) + \varphi(g)]$ as $(K[X],+)$ is abelian

$= (\psi + \varphi)(f) + (\psi + \varphi)(g)$

$\Rightarrow \psi + \varphi \in End_K(K[X])$

Again $(\psi \varphi)(f+g) = \psi(\varphi(f+g))$

$= \psi(\varphi(f) + \varphi(g))$

$= \psi(\varphi(f)) + \psi(\varphi(g))$

$= \psi\varphi(f) + \psi\varphi(g)$

$\Rightarrow \psi \varphi \in End_K(K[X])$

Since $(\psi + \varphi) + \theta = \psi + (\varphi + \theta)$ and $\psi + \varphi = \psi + \varphi \; \forall, \psi, \varphi, \theta \in End_K(K[X])$

We have that '+' is both associative and commutative.

The map $Z:K[X] \to K[X]$ defined by $Z(f) = 0$, $\forall f \in K[X]$ is the zero element of $End_K(K[X])$

For any $\varphi \in End_K(K[X])$, the map $(-\varphi): K[X] \to K[X]$ given by $(-\varphi)(f) = -\varphi(f)$, $\forall f \in K[X]$ is the additive inverse of $\varphi$.

For any $f \in K[X]$, the mapping $I : K[X] \to K[X]$ given by $I(f) = f$ is the identity element of $End_K(K[X])$

Therfore $End_K(K[X],+)$ is abalian group.

Moreover $(\psi + \varphi)\theta = \psi\theta + \varphi\theta$ and $\theta(\psi + \varphi) = \theta\psi + \theta\varphi$

And $(\psi\varphi)\theta = \psi(\varphi\theta)$ for all $\psi, \varphi, \theta \in End_K(K[X])$

Therfore $(End_K(K[X],+,\cdot)$ is a ring with unity.

Let K x $End_K(K[X]) \to End_K(K[X])$ be the action of K on $End_K(K[X])$ given by

$(k, \varphi) \to k\varphi$. $\forall k \in K$ *and* $\forall \varphi \in End_K(K[X])$

Closurity

Let $f,g \in K[X]$, $a, b \in K$, then

$(k\varphi)(af+bg) = k\varphi(af+bg) = k(a\varphi(f) + b\varphi(g))$ by linearity.

$= (ka)\varphi(f) + (kb)\varphi(g)$

$= (ak)\varphi(f) + (bk)\varphi(g)$ since K is commutative

$= a(k\varphi(f)) + b(k\varphi(g))$

$\Rightarrow k\varphi$ is linear

$\Rightarrow k\varphi \in End_K(K[X])$

Moreover for $f \in K[X]$, $\psi, \varphi \in End_K(K[X])$ and $k,k' \in K$, we have

$[k(\psi + \varphi)](f) = k[(\psi+\varphi)(f)] = k(\psi(f)+\varphi(f))$

$= (k\psi)(f) + (k\varphi)(f)$

$= (k\psi + k\varphi)(f)$

$\Rightarrow k(\psi + \varphi) = k\psi + k\varphi$

$[(k+k')\varphi](f) = (k+k')\varphi(f)$

$= \varphi(k+k')f) = \varphi(kf + k'f)$

$= \varphi(kf)+\varphi(k'f)$

$= k\varphi(f)+k'\varphi(f)$

$= (k\varphi + k'\varphi)(f)$

$\Rightarrow (k+k')\varphi = k\varphi + k'\varphi$

$[(kk')\varphi](f) = (kk')\varphi(f)$

$= \varphi((kk')f)$

$= \varphi(k(k'f))$

$= k\varphi(k'f)$

$= k(k'\varphi(f))$

$= k(k'\varphi)(f)$

$\Rightarrow (kk')\varphi = k(k'\varphi)$

Therefore $End_K(K[X])$ is a left K-module.

Since iv) $(I\varphi)(f) = I\varphi(f) = \varphi(f)$

$\Rightarrow I\varphi = \varphi$

We have $End_K(K[X])$ is a unitary left K-module and hence it is a left K-vector space.

Finally for $\psi, \varphi \in End_K(K[X])$, $k \in K$ and $f \in K[X]$, we have

$[\psi(k\varphi)](f) = \psi[(k\varphi)(f)] = \psi(k\varphi(f))$

$= k\psi(\varphi(f)) = [(k\psi)\varphi](f)$

$\Rightarrow \psi(k\varphi) = (\psi k)\varphi = k(\psi\varphi)$

$\therefore End_K(K[X])$ is an algebra over K.

**Definition**:- Let R be a K-algebra, we say that R is graded if there are K-vector subspaces $R_i$, $i \in \mathbb{N}$, such that

1) $R=\Theta_{i\in\mathbb{N}} R_i$

2) $R_i . R_j \subseteq R_{i+j}$

The $R_i$ are called the homogeneous components of R. The elements of $R_i$ are the homogeneous elements of degree i.

If $R_i=0$ when $i<0$, then we say that the grading is positive.

Example The K-algebra $K[x_1,\ldots,x_n]$ is a graded algebra. The monomials

$$x^{k_1} x^{k_2} \ldots x^{k_n}$$

with $k_1+\ldots+k_n= m$ form a basis of the homogeneous component of degree m.

Let $R = \Theta_{i\geq 0}R_i$ and $S = \Theta_{i\geq 0}S_i$ be graded algebras over a field K .A homomorphism of K-algebra $\phi: R \to S$ is graded if $\phi(R_i) \subseteq S_i$ . Thus a graded homomorphism is one that preserves the degree.

A graded algebra admits a special kind of module, Let $R = \Theta_{i\geq 0}R_i$ be a graded K-algebra. A left R –module M is a graded module if there exist K-vector spaces $M_i$ for $i\geq 0$, such that.

1) $M= \Theta_{i\geq 0} M_i$

2) $R_i M_j \subseteq M_{i+j}$

The $M_i$ are the homogeneous components of degree i of M

**Definition**:- Let R by a K-algebra. A family $F=\{F_i\}_{i\geq 0}$ of K-vector spaces is a filtration of R. if

1) $F_0 \subseteq F_1 \subseteq \ldots \subseteq R$

2) $R = \cup_{i\geq 0} F_i$

3) $F_i . F_j \subseteq F_{i+j}$

If an algebra has a filtration it is called a filtered algebra. It is convenient to use the convention that $F_j=\{0\}$ if $j<0$

If $R$ as a filtered algebra then the *associated graded algebra* $gr\Gamma_R$ is defined as follows:

As a vector space $gr\Gamma_R = \oplus_{i\geq 0} G_i$ where, $\Gamma$ is the filterations of R,

1) $G_0 = \Gamma_0$, and $\forall i > 0, \Gamma_i/\Gamma_{i-1}$,

2) the multiplication is defined by $(x+\Gamma_i)(y+\Gamma_j) = x\cdot y+\Gamma_{i+j}$

**Definition**:-Let M be a left R-module. A family $\Gamma=\{\Gamma_i\}_{i\geq 0}$ of K-vector space of M is a filtration of M if it satisfies.

1) $\Gamma_0 \subseteq \Gamma_1 \subseteq \ldots \subseteq M$

2) $\cup_{i\geq 0} \Gamma_i = M$

3) $R_i \Gamma_j \subseteq \Gamma_{i+j}$, i=0 implies that each $\Gamma_j$ is a K-vector space.

4) $\Gamma_i$ is a K-vector space of finite dimension.

The convention that $\Gamma_j=\{0\}$ if $j<0$ remains in force.

Remark:-Let M be a left R-module and $\Gamma$ be a filtration of M with respect to the filtration $F$ of R .If $\operatorname{gr}\Gamma_M$ is finitely generated then we say that $\Gamma$ is a good filtration of M.

**Definition**:- Let K be a commutative ring with identity and M be K-algebra. Then subalgebra of M is a subring of M that is also a K-submodule of M.

Example Let E be the set of even Integers, then E is a subalgebra of $\mathbb{Z}$.

## 2. 6. Weyl algebra

Now we are in position to define the weyl algebra. Let K be a filed ,$K[X]=K[x_1,x_2,x_3,\ldots]$be a vector space of infinite dimension over K.

let $\hat{x}_i$ *and* $\partial_i$ be the operators of K[X] which are defined on a polynomial $f\in K[X]$ by the formula $\hat{x}_i(f) = x_i \cdot f$ *and* $\partial_i(f) = \partial f/\partial x_i$ .Then these are linear operators of K[X], and denote the set of all operators from K[X] to K[X] by $\mathrm{End}_K(K[X])$, so that the Weyl algebra is defined as a subalgebra of $\mathrm{End}_K(K[X])$.

**Definition**:- The n-th Weyl algebra $A_n$ is the K-subalgebra of $\mathrm{End}_K(K[X])$ generated by the operators, $\hat{x}_1, \hat{x}_2, \ldots, \hat{x}_n$, *and* $\partial_1, \partial_2, \ldots, \partial_n$

**Proposition 2.6.1** The algebra $A_n$ is not commutative.

**Proof** Let $\hat{x}_i, \partial_i \in A_n$, since $A_n$ is a ring $\partial_i \cdot \hat{x}_i \in A_n$ Now for $f \in K[X]$, by using the rule for the differentiation of a product, we have

$\partial_i . \hat{x}_i(f)=\partial_i(\hat{x}_i.f)$

$=\partial_i(\hat{x}_i)f + \hat{x}_i\partial_i(f))$

$=f + \hat{x}_i . \partial_i(f)$

$=(1+\partial_i.\hat{x}_i)(f)$

$\Rightarrow \partial_i.\ \hat{x}_i = 1 + \hat{x}_i\ .\ \partial_i$

Or $\partial_i.\ \hat{x}_i - \hat{x}_i.\partial_i = 1 = 0$

$\Rightarrow A_n$ is not commutative.

Note. If a,b, $\in A_n$ then their commutator is the operator [a,b]=ab-ba

Thus $\partial_1 \hat{x}_i - \hat{x}_i\ \partial_i = [\partial_i, \hat{x}_i] = 1$

Similarly we have

$[\partial_i,\ \hat{x}_j] = \delta_{ij}\ .\ 1$

$[\partial_i\ .\ \partial_j] = [\ \hat{x}_i\ .\ \hat{x}_j] = 0$ , where $1 \leq i,j \leq n$ and $\delta_{ij}$ the **kronecker delta** symbol it equals 1 if i=j and zero otherwise.

## 2.7 Canonical form

In this section we construct a basis for the Weyl algebra as a K-vector space. This basis is known as the canonical basis. If an element of $A_n$ is written as a linear combination of canonical basis, then we say that it is in canonical form.

The multi-index $\alpha$ is an element of $\mathbb{N}^n$, and $\alpha = (\alpha_1, \alpha_2, ..., \alpha_n)$

Now $x^\alpha$ means the monomial $x_1^{\alpha_1} x_2^{\alpha_2} \ldots x_n^{\alpha_n}$

The degree of this monomial is the length $|\alpha|$ of the multi-index $\alpha$ , namely

$|\alpha| = \alpha_1 + \alpha_2 + ... + \alpha_n$

Notice that a pair $(\alpha, \beta)$ of multi-indices in $\mathbb{N}^n$ is itself a multi-index in $\mathbb{N}^{2n}$ ,

so it makes sense to talk of its length .

**Note:** The factorial of a multi – index $\beta \in \mathbb{N}^n$ is defined by $\beta! = \beta_1!\beta_2!...\beta_n!$

**Lemma 2.7.1:** Let $\delta, \beta \in \mathbb{N}^n$ and assume that $|\delta| \leq |\beta|$ . Then $\partial^\beta(x^\delta) = \beta!$ if $\delta = \beta$ , and other wise zero .

Proof: if $\delta = \beta$ . Then, $\partial^\beta(x^\delta) = \partial^\beta(x^\beta)$

$$= \partial^{(\beta_1,\ldots,\beta_n)}(x^{(\beta_1,\ldots,\beta_n)})$$

$$= (\partial_1^{\beta_1}\partial_2^{\beta_2}...\partial_n^{\beta_n})(x_1^{\beta_1}x_2^{\beta_2}...x_n^{\beta_n})$$

$$= (\partial_1^{\beta_1}\partial_2^{\beta_2}...\partial_{n-1}^{\beta_{n-1}})(x_1^{\beta_1}x_2^{\beta_2}...x_{n-1}^{\beta_{n-1}})\beta_n!$$

$$= (\partial_1^{\beta_1}\partial_2^{\beta_2}...\partial_{n-2}^{\beta_{n-2}})(x_1^{\beta_1}x_2^{\beta_2}...x_{n-2}^{\beta_{n-2}})\beta_{n-1}!\,\beta_n!$$

$$= (\partial_1^{\beta_1}\partial_2^{\beta_2}...\partial_{n-3}^{\beta_{n-3}})(x_1^{\beta_1}x_2^{\beta_2}...x_{n-3}^{\beta_{n-3}})\beta_{n-2}!\beta_{n-1}!\,\beta_n!$$

$$\vdots$$

$$= \beta_1!\,\beta_2!\,\beta_3!...\beta_{n-2}!\,\beta_{n-1}!\,\beta_n!$$

$$= \beta!$$

If $\delta \neq \beta$ and $|\delta| < |\beta|$ , $\partial^\beta(x^\delta) = 0$.

**Proposition 2.7.2:** The set B = { $x^\alpha\partial^\beta$ : $\alpha, \beta \in \mathbb{N}^n$ } is a basis of $A_n$ as a vector space over K.

**Proof** i, Let $c_{\alpha\beta} \in K$ for $\alpha, \beta \in \mathbb{N}^n$ such that $\sum c_{\alpha\beta}\, x^\alpha\partial^\beta = 0$, where 0 is the zero operator.

Let $f \in$ K[X] be a monomial of degree $\beta$ ,

i.e. $f = x^\beta = x_1^{\beta_1}x_2^{\beta_2}...x_n^{\beta_n}$ where $|\beta| = \beta_1 + \beta_2 + ... + \beta_n$ and $x = (x_1, x_2, ..., x_n)$

Now, $\sum c_{\alpha\beta}\, x^\alpha\partial^\beta(f) = 0(f) = 0$

$\Rightarrow \beta!\sum c_{\alpha\beta}\, x^\alpha = 0$ (by Lemma 2.2.1)

$\Rightarrow \sum c_{\alpha\beta}\, x^\alpha = 0$

Since $x^\alpha \neq 0$, we have $c_{\alpha\beta} = 0 \;\; \forall \alpha, \beta \in \mathbb{N}^n$

Hence B is linearly independent.

ii, Let D$\in$ $A_n$ and D = $\sum c_{\alpha\beta} x^{\alpha}\partial^{\beta}$ be a finite linear combination , $c_{\alpha\beta} \in K$.

We need to show that if some $c_{\alpha\beta} \neq 0$, then D $\neq$ 0.

But D is a linear operator of K[X].

• Suppose D = 0

$\Rightarrow c_{\alpha\beta} = 0, \forall \alpha, \beta \in \mathbb{N}^n$ , since B is linearly independent .

This follows $c_{\alpha\beta} \neq 0 \Rightarrow$ D $\neq$ 0.

• Suppose D $\neq$ 0

$\Rightarrow \exists 0 \neq f \in K[X]$ Such that D (f) $\neq$ 0 (otherwise D (f) = 0, $\forall f \in K[X] \Rightarrow$ D = 0)

Choose $\delta$ be the index such that $c_{\alpha\delta} \neq 0$ for some index $\alpha$ , but $c_{\alpha\beta} = 0$ , for all indices $\beta$ such that $|\beta| < |\delta|$.

Now, D ($x^{\delta}$) = $\sum c_{\alpha\delta} x^{\alpha}\partial^{\delta}(x^{\delta}) = \delta! \sum c_{\alpha\delta} x^{\alpha}$ (by Lemma 2.2.1)

$\Rightarrow$ D ($x^{\delta}$) $\neq$ 0, since at least one of the coefficients $c_{\alpha\delta} \neq 0$ by the choice of $\delta$.

Hence D $\neq$ 0 $\Rightarrow$ D = $\sum c_{\alpha\beta} x^{\alpha}\partial^{\beta}$ for some $c_{\alpha\beta} \neq 0$ .$\rightarrow\leftarrow$

Hence B is a basis of $A_n$.

**Examples**: 1. Find the canonical form of $\partial_2^2 x_1 + \partial_1 x_1$ in $A_3$

Solution:- $\partial_2^2 x_1 + \partial_1 x_1 = \partial_2 \partial_2 x_1 + \partial_1 x_1$

$= \partial_2 x_1 \partial_2 + (1 + x_1 \partial_1)$

$= (x_1 \partial_2)\partial_2 + 1 + x_1 \partial_1$

$= 1 + x_1 \partial_1 + x_1 \partial_2^2$

Hence $1 + x_1 \partial_1 + x_1 \partial_2^2$ is a canonical form of $\partial_2^2 x_1 + \partial_1 x_1$ in $A_3$.

2. Find the canonical form of $\partial_1^2 x_1^2$ in $A_3$.

Solution:- $\partial_1^2 x_1^2 = \partial_1 \partial_1 x_1 x_1$

$= \partial_1 (1 + x_1 \partial_1) x_1$

$= \partial_1 x_1 + \partial_1 x_1 \partial_1 x_1$

$= 1 + x_1 \partial_1 + \partial_1 x_1 ( 1 + x_1 \partial_1 )$

$= 1 + x_1 \partial_1 + \partial_1 x_1 + \partial_1 x_1 x_1 \partial_1$

$= 1 + x_1 \partial_1 + (1 + x_1 \partial_1 ) + (1 + x_1 \partial_1 ) x_1 \partial_1$

$= 2 + 2 x_1 \partial_1 + x_1 \partial_1 + x_1 \partial_1 x_1 \partial_1$

$= 2 + 3 x_1 \partial_1 + x_1 (1 + x_1 \partial_1) \partial_1$

$= 2 + 3 x_1 \partial_1 + x_1 \partial_1 + x_1 \ x_1 \partial_1 \partial_1$

$= 2 + 4 x_1 \partial_1 + x_1^2 \partial_1^2$

Hence $2 + 4 x_1 \partial_1 + x_1^2 \partial_1^2$ is a canonical form of $\partial_1^2 x_1^2$ in $A_3$.

## 2.8. The Degree of an Operator

**Definition**:- Let $D \in A_n$. The degree of D is the largest length of the multi-indices $(\alpha, \beta) \in \mathbb{N}^n \times \mathbb{N}^n$ for which $x^\alpha \partial^\beta$ appears with non zero coefficient in the canonical form of D. and it is denoted by deg (D)

Note - We use the convention that the zero operator has degree - $\infty$

- If D, D′ ∈ $A_n$, then we have

- deg (D+ D′ ) ≤ max{deg(D), deg(D′ )}
- deg(D D′ )= deg(D) + deg(D′ )
- deg[D, D′ ]=deg(D) + deg(D′ ) – 2

Example Let $D = x_1^3 x_2^2 \partial_1 - 3 x_1^2 x_2 x_3^4$. Then deg(D)=7

**Theorem 2.8.1** The algebra $A_n$ is simple.

**Proof** Let $0 \neq I$ be two-sided ideal of $A_n$.

Let $0 = D \in I$ be an element with smallest degree in I.

Case-1 $\deg(D)=0$ $=D$ is a constant

$\Rightarrow D$ is a unit in $A_n$

$\Rightarrow \exists\ D' \in A_n$ such that $D\,D' = 1$, where 1 is the identity operator.

Since I is an ideal , $D' \in A_n$, $D \in I$ implies $1 = D\,D' \in I$

$\Rightarrow 1 \in I$

Now $P \in A_n$, $1 \in I \Rightarrow P = P.1 \in I$

$\Rightarrow A_n \subseteq I$

Again as I is an ideal of $A_n$ we have $I \subseteq A_n$

Therefore $I = A_n$

Case -2 Let $D = \sum c_{\alpha\beta}\, x^{\alpha} \partial^{\beta}$ ,where $0 \neq c_{\alpha\beta} \in K$ for some $\alpha, \beta \in \mathbb{N}^n$

$\deg(D) \neq 0 \Rightarrow \alpha + \beta > 0$

Now $x_i \in A_n$, $D \in I \Rightarrow x_i D$, $Dx_i \in I$

$\beta_i = 0 \Rightarrow x_i D - Dx_i = 0$ in I

But $\deg(x_i D - Dx_i) = \deg[x_i, D]$, since $[x_i, D] = x_i D - Dx_i$

$\leq \deg(x_i) + \deg(D) - 2$

$= 1 + \alpha + \beta - 2$

$= \alpha + \beta - 1$

This contradicts the choice of D.

Thus $\beta_i=0$ for all i=1,. . ., n

$\Rightarrow \beta=0$

Since $\alpha+\beta>0$, we must have $\alpha_i=0$ for some i.

$\Rightarrow 0\neq \partial_i D- D\partial_i = [\partial_i, D] \in I$

$\Rightarrow \deg(\partial_i D - D\partial_i)=\deg[\partial_i,D] \leq \alpha+\beta-1$

Again this contradicts the choice of D.

Hence $\alpha_i=0$ for all $i=1, \ldots n$.

$\Rightarrow \alpha=0$

$\Rightarrow$D is constant.

$\Rightarrow$ I contain only the constant element

Hence by case-1 it follows that $I=A_n$

$\Rightarrow A_n$ does not contain two – sided proper ideal.

Therefore $A_n$ is simple.

## 2.9. Modules over the Weyl algebra

The Weyl algebra was constructed as a subring of an endomorphism ring. If $\mathbb{C}[X]$ is a polynomial ring $\mathbb{C}[x_1,\ldots,x_n]$ then we have that $A_n(\mathbb{C})$ is a sub ring of $End_{\mathbb{C}}(\mathbb{C}[X])$.

One deduces form this that the polynomial ring is a left $A_n$-module. Thus the action of $x_i$ on $\mathbb{C}[X]$ is by straightforward multiplication whilst $\partial_i$ act by differentiation with respect to $x_i$.

**Proposition 2.9.1.** $\mathbb{C}[x]$ is a left $A_1$ – module with the action of x and $\partial_x$ on $\mathbb{C}[x]$ is given by $x(f)=xf$ and $\partial_x(f)=\partial f/\partial_x$ for all $f \in \mathbb{C}[x]$.

**Proof** Since $(\mathbb{C}[x],+,\cdot)$ is a ring we have that $(\mathbb{C}[x],+)$ is a belian group.

Since x is a map from $\mathbb{C}[x]$ on to $\mathbb{C}[x]$ for any $f\in\mathbb{C}[x]$, $xf \in \mathbb{C}[x]$. Thus multiplication is closed.

Now let $D,D' \in A_1$ and $f, g \in \mathbb{C}[x]$. Let consider the case that the elements are monomials. i.e.

Let $D=x^{\alpha}\partial_x^{\beta}$ , $D' = x^{\gamma}\partial_x^{\delta}$ , $f=x^n$ , $g=x^m$ for $\alpha,\beta,\gamma,\delta,n,m \in \mathbb{N}$

1) $$\begin{aligned} D(f+g) &= x^{\alpha}\partial_x^{\beta}(x^n+x^m) = x^{\alpha}(\partial_x^{\beta}(x^n)\partial_x^{\beta}(x^m)) \\ &= x^{\alpha}(\partial_x^{\beta}(x^n)) + x^{\alpha}(\partial_x^{\beta}(x^m)) \qquad \text{by linearity} \\ &= x^{\alpha}\partial_x^{\beta}(x^n) + x^{\alpha}\partial_x^{\beta}(x^m) = Df + Dg \end{aligned}$$

Similarly

2) $(D+D')f = Df + D'f$

3) $(DD')f = D(D'f)$

4) $1(f)=1.f=f$

Moreover

1) $[\partial_x, x]f = (\partial_x x - x\partial_x)f = 1 \cdot f = f$
2) $[\partial_x, \partial_x]f = (\partial_x^2 - \partial_x^2)f = 0 \cdot f = 0$
3) $[x, x]f = (x^2 - x^2)f = 0 \cdot f = 0$

Therefore $\mathbb{C}[x]$ is a left $A_1$-module.

**Proposition 2.9.2** $\mathbb{C}[x]$ is simple $A_1$-module.

**Proof** Let $0 \neq f \in \mathbb{C}[x]$ and consider the submodule $A_1 f$ of $\mathbb{C}[x]$ generated by

$f \in \mathbb{C}[x]$.

**Claim** $A_1 f = \mathbb{C}[x]$.

Let $f = a_0 + a_1 x + \ldots + a_n x^n$ with $a_n \neq 0$, where $a_i \in \mathbb{C}$ and $n \in \mathbb{N}$

For $\partial_x^n \in A_1$ we have

$\partial_x^n(f) = n!$

Since $A_1 f$ is a submodule, $f \in \mathbb{C}[x]$, $\partial_x^n \in A_1$ implies $n! = \partial_x^n(f) \in A_1 f$

$\Rightarrow A_1 f$ contains a constant

$\Rightarrow \mathbb{C}[x] \subseteq A_1 f$

But by hypothesis we have $A_1 f \subseteq \mathbb{C}[x]$

Hence $A_1 f = \mathbb{C}[x]$

Therefore $\mathbb{C}[x]$ Does not contain a proper submodule and hence

$\mathbb{C}[x]$ is simple.

**Proposition 2.9.3** $\mathbb{C}[x,y]$ is a simple left $A_2$-module .

**Proof** To show that $\mathbb{C}[x,y]$ is a left $A_2$–module it suffices to show that

$[\partial_i, x_j]f = \delta_{ij} \cdot f$ *and* $[x_i, x_j]f = [\partial_i, \partial_j]f = 0$

For every $x_i, \partial_i \in A_2$, $f \in \mathbb{C}[x,y]$

$$\begin{aligned} \textit{Now}\ [\partial_i, x_j]f &= \left(\partial_i x_j - x_j\partial_i\right)f = \partial_i x_j(f) - x_j\partial_i(f) \\ &= \partial_i(x_j)f + x_j\partial_i(f) - x_j\partial_i(f) \\ &= \partial_i(x_j)f \end{aligned}$$

$\Rightarrow$ if $i = j$ we have $\partial_i(x_j)f = 1.f = f$

Otherwise $\partial_i(x_j)f = 0.f = 0$

Thus $[\partial_i, x_j]f = f$ if $i=j$ and

$[\partial_i, x_j]f = 0$ other wise

Again $[\partial_i, \partial_j]f = (\partial_i\partial_j - \partial_j\partial_i)f = 0.f = 0$

$[x_i, x_j]f = (x_ix_j - x_jx_i)f = 0.f = 0$

Therefore $\mathbb{C}[x,y]$ is a left $A_2$-module

The prove for simplicity is similar to that of $\mathbb{C}[x]$

Therefore $\mathbb{C}[x,y]$ is a simple $A_2$- module.

## 2. 10. Localization

**Definition:-** Let R be a commutative ring and let p be a prime ideal of R, let S:=R\P, a multiplicatively closed subset of R.The ring of fractions $S^{-1}R$ is denoted by $R_p$ it is a ring with only one maximal ideal is called the localization of R at P with maximal ideal $\{x \in R_P : x=\frac{a}{s}$ for some $a \in P, s \in S\}$

Example $2\mathbb{Z}$ is a prime ideal of $\mathbb{Z}$ and so we can form the localization $\mathbb{Z}_{2\mathbb{Z}}$,This localization can be identified with the subring of $\mathbb{Q}$ consisting of all rational numbers which can be expressed in the form m/n with $m,n \in \mathbb{Z}$ and n odd.

**Example** Let the polynomial x be choosen in $\mathbb{C}[x]$,then the set of rational functions of the form $f/x^r$ where f is a polynomial in $\mathbb{C}[x]$ denoted by $\mathbb{C}[x]_x$ is the localization of $\mathbb{C}[x]$ at x.

**Proposition 2.10.1** $\mathbb{C}[x]_x$ is a left $A_1$-module of $\mathbb{C}(x)$(field of rational function) where the actions of x,and $\partial_x$ on $\mathbb{C}[x]_x$ is given by $x(f/x^r) = xf / x^r$ and $\partial_x(f / x^r) = \frac{\partial_x(f)x^r - f\partial_x(x^r)}{x^{2r}}$

**Proof** since the set of rational functions are a ring we have $(\mathbb{C}[x]_x, +)$ is ablian group.

Let $f,g \in \mathbb{C}[x]$ for $x, \partial_x \in A_1$ we have

i) $x(f/x^n + g/x^m) = x(f/x^n) + x(g/x^m)$ by linearity.

$$= \frac{xf}{x^n} + \frac{xg}{x^m}$$

And $\partial_x(f/x^n + g/x^m) = \partial_x(f/x^n) + \partial_x(g/x^m)$

$$= \frac{x^n \partial_x(f) - f\partial_x(x^n)}{x^{2n}} + \frac{x^m \partial_x(g) - g\partial_x(x^m)}{x^{2m}}$$
$$= \frac{x\partial_x(f) - nf\partial(x)}{x^{n+1}} + \frac{x\partial_x(g) - mg\partial_x(x)}{x^{m+1}}$$
$$= \partial_x(f/x^n) + \partial_x\left(g/x^m\right)$$

Similarly

ii) $(x+\partial_x)f/x^n = xf/x^n + \partial_x(f/x^n)$

iii) $(\partial_x \,.\, x)\, f/x^n = \partial_x(x \cdot f/x^n)$

iv) $1.(f/x^n)=1.f/x^n = f/x^n$

Therefore $\mathbb{C}[x]_x$ is a left $A_1$- module.

## CHAPTER THREE

### 3.1. Length of Modules over the Weyl Algebra

In the introductory chapter about modules was concerned with the very basic principles of the theory of modules over rings. In particular over the Weyl Algebra, and, indeed one could take the view that Chapter-2 although it contains important and fundamental mathematics for our purposes.

This chapter introduces the main topic of the seminar "length of modules over the Weyl algebra". First we develop the theory of holonomic modules which is the most important sort of module for studying length of modules over the Weyl Algebra.

For this we start by introducing the dimension and multiplicity of $A_n$-module using the Hilbert polynomial.

### 3.2. Dimension and Meltiplicity

**Definitions and notations**

- A numerical polynomial is a polynomial p(t) of Q[t] such that $p(n) \in \mathbb{Z}$ for all integers n>>0

- Put $\binom{x}{r} = x(x-1)...(x-r+1)/r!$.if r is a positive integer and x is a variable .

  if r=0 we put $\binom{x}{0} = 1$.

- Any polynomial f(x) of degree d in Q[x] can be written as

f(x) $= c_d\binom{x+d}{d} + c_{d-1}\binom{x+d-1}{d-1} + ... + c_0\binom{x}{0}, c_i \in Q$

Example. $f(x)=\frac{2}{3}x^3+\frac{1}{2}x+2=4\binom{x+3}{3}-8\binom{x+2}{2}+\frac{31}{6}\binom{x+1}{1}+\frac{5}{6}\binom{x}{0}$

- $\binom{x+r}{r}-\binom{x+r-1}{r}=\binom{x+r-1}{r-1}$

- The difference function of a function $f:\mathbb{C}\to\mathbb{C}$ is defined by

$$\Delta f(x)=f(x)-f(x-1)=c_d\binom{x+d-1}{d-1}+\ldots c_1\binom{x}{0}.$$

- It follows that by induction on deg(f)=d, that if f(n) $\in \mathbb{Z}$ for n>>0.

We have $c_i \in \mathbb{Z}$ for all i(and so f(n) $\in \mathbb{Z}$ for all n$\in \mathbb{Z}$ ). It follows that

If F(n) is a numerical function such that; F(n) – F(n-1)=f(n) , for $n>n_0$, then

$$F(n)=c_d\binom{n+d+1}{d+1}+\ldots+c_0\binom{n+1}{1}+const \text{ for } n>n_0$$

Remark:- Let $M=\oplus_{i\geq 0} M_i$ be a finitely generated graded module over the polynomial ring $K[x_1,\ldots,x_n]$.There exist a polynomial $h(t)\in\mathbb{Q}[t]$ and a positive integer N such that

$$\sum_0^t \dim(M_i)=h(t) \text{ for every } t\geq N$$

The polynomial $h(t)$ is known as the Hilbert polynomial of M.

- Bernstein filtration is the filtration defined using the degree of operators in $A_n$. Denote by $B_k$ the set of all operators $A_n$ of degree ≤ k. These are vector subspaces of $A_n$.

The Bernstein filtration has a very special feature; each $B_k$ is a vector space of finite dimension.

A basis for $B_k$ is determined by the monomials $x^{\alpha}\partial^{\beta}$ with $|\alpha|+|\beta|\le k$.

In particular $B_0$=K(=field) and $\{1, x_1,...,x_n,\partial_1,...,\partial_n\}$ is a basis of $B_1$.

Now we are ready to define the dimension and multiplicity of a module over $A_n$.

**Definition**:- Let M is a finitely generated left $A_n$-module. Suppose that $\Gamma$ is a good filtration of M with respect to the Bernstein filtration $B$. Denote by $h(t,\Gamma,M)$ the Hilbert polynomial of the graded module $gr\Gamma_M$ over the polynomial ring $S_n$ (a polynomial ring in 2n variable over $A_n$), then we have for t ≥ 0

$$h(t,\Gamma,M) = \sum_{0}^{t}\dim_K(\Gamma_i/\Gamma_{i-1}) = \dim_K(\Gamma_t)$$

i) The dimension d(M) of M is the degree of $h(t,\Gamma,M)$

ii) The multiplicity of M is $m(M) = d!c_{d(M)}$, where $c_d$(M) is the leading coefficient of $h(t,\Gamma,M)$ and d is the degree of $h(t,\Gamma,M)$

Examples

Let M be the left $A_2$ – module$A_2$ he Bernstein filtration $B$ is a good filtration of M and it is possible to calculate $h(t,B,M)$. For this we must determine the dimension of $B_t$. But the monomials $x^{\alpha}\partial^{\beta}$ with $|\alpha|+|\beta|\le t$ form a basis of $B_t$ as a K- vector space. So it is enough to count the elements of this basis. To do this we must count the non-negative solutions of the equation

$\alpha_1+\alpha_2+\beta_1+\beta_2\le t$, since $\alpha_i\ge 0 and \beta_i\ge 0$, in particular if t=1

i.e $\alpha_1+\alpha_2+\beta_1+\beta_2\le 1$, Thus there are 5 solutions

Hence $h(1,B,M) = 5 = \binom{5}{4} = \binom{1+4}{4}$

Form this we get that $\alpha_1+\alpha_2+\beta_1+\beta_2 \le t$ has $\binom{t+4}{4}$ solutions

Hence $h(t,B,M)=\binom{t+4}{4}$ , As a polynomial in $t$ it has degree 4 and leading coefficient 1/4! Thus

$d(A_2)=4$ and $m(A_2)=1$

In general if M is a left $A_n$ – module then $h(t,B,M)=\binom{t+2n}{2n}$

And d $(A_n)=2n$ and $m(A_n)=1$

Consider the $A_2$ -module $\mathbb{C}[x,y]$, define a filtration $\Gamma$ of $\mathbb{C}[x,y]$, such that $\Gamma_i$ is the space of all polynomials of degree $\le$ t. Since $B_i$ contains the polynomials in x, y of degree i, We have that $B_i\Gamma_i=\Gamma_{i+i}$ it follows that $\Gamma$ is good filtration of $\mathbb{C}[x,y]$, and it is possible to calculate $h(t,\Gamma,\mathbb{C}[x,y])$ by counting the non-negative solutions of the equation n +m $\le$t where n and m are degree of x and y respectively. Thus there are $\binom{t+2}{2}$ such solution

Hence $\dim\Gamma_i=\binom{t+2}{2}$ Thus $h(t,\Gamma,\mathbb{C}[x,y])=\binom{t+2}{2}$ is a polynomial of degree 2 and leading coefficient 1/2! .Hence $d(\mathbb{C}[x,y])=2$, $m(\mathbb{C}[x,y])=1$

In general for an $A_n$ – module $\mathbb{C}[x_1,x_2,\ldots,x_n]$ ,we have

$h(t,\Gamma,\mathbb{C}[x,y])=\binom{t+n}{n}$ , $d(\mathbb{C}[x_1,x_2,\ldots,x_n])=n$ and $m(\mathbb{C}[x_1,x_2,\ldots,x_n])=1$

### 3. 3. Basic Properties

Let M be a finitely generated left $A_n$ - module and $\Gamma$ a good filtration of M with respect to *B*. Let N be a submodule of M. Denote by $\Gamma'$ *and* $\Gamma''$ the filtrations induced by $\Gamma$ in N and M/N. respectively.

It follows that we have an exact sequence of $S_n$ - modules namely.

$$0 \rightarrow \mathrm{gr}\,\Gamma'_N \rightarrow \mathrm{gr}\,\Gamma_M \rightarrow \mathrm{gr}\,\Gamma''_{M/N} \rightarrow 0$$

Since $\Gamma$ is good. . $\mathrm{gr}\,\Gamma_M$ is finitely generated, But $S_n$ is a noetherian ring . Hence $\mathrm{gr}\,\Gamma'_N$ and $\mathrm{gr}\,\Gamma''_{M/N}$ are also finitely generated , There fore both $\Gamma'$ *and* $\Gamma''$ are good filtration.

On the other hand. Since the sequence of vector spaces.

$$0 \rightarrow \Gamma'_k/\Gamma'_{k-1} \rightarrow \Gamma_k/\Gamma_{k-1} \rightarrow \Gamma''_k/\Gamma''_{k-1} \rightarrow 0$$

Is exact we have that.

$$\dim\left(\Gamma'_k/\Gamma'_{k-1}\right) + \dim\left(\Gamma''_k/\Gamma''_{k-1}\right) = \dim\left(\Gamma_k/\Gamma_{k-1}\right)$$

Summing these terms for k=0,1, . . . ,s and assuming that s>>0 one obtains

$$h(s,\Gamma',N) + h(s,\Gamma'',M/N) = h(s,\Gamma,M) \ldots\ldots (2^*)$$

**Theorem 3.3.1** Let M be a finitely generated left $A_n$-module and N a submodule of M.

1) d(M) = max {d(N) , d(M/N)}

2) If d(N) = d (M/N ) then m(M)= m(N) + m (M/N)

Proof since M is finitely generated it admits a good filtration $\Gamma$ . Let $\Gamma'$ *and* $\Gamma''$ be the good fitrations of N and M/N induced by $\Gamma$ , Thus (2*) holds for an infinite number of values of s. Hence we have an equality of polynomials.

$h(s,\Gamma',N) + h(s,\Gamma'',M/N) = h(s,\Gamma,M)$

Since d (m) corresponds to the degree of $h(s,\Gamma,M)$ , we have that

$d(M) \leq \max\{d(N), d(M/N)\}$

But the leading coefficients of these polynomials are positive. Thus we must have equality in the formula above and (1) is proved.

Now, if d (M/N) = d (N), then all these polynomials have the same degree. Thus the leading term of $h(s,\Gamma,M)$ is the sum of the leading terms of $h(s,\Gamma',N)$ and $h(s,\Gamma'',M/N)$ and (2) follows immediately.

### 3.4. Bernstein's Ineqality

**Lemma 3.4.1** Let M be a finitely generated left $A_n$- module with filtration $\Gamma$ with respect to $B$. Suppose that $\Gamma_o \neq 0$. The K- linear transformation

$\phi: B_i \rightarrow \mathrm{Hom}_K(\Gamma_i, \Gamma_{2i})$ , which maps a $\in B_i$ to the linear transformation $\phi_a(u)=au$ is injective.

**Proof** The statement of the lemma is equivalent to a $\Gamma_i \neq 0$, when ever $0 \neq a \in B_i$ we will prove this by induction on i ,if i=0 , then $B_0 = K$ and the statement above is equivalent to $\Gamma_o \neq 0$ , this is true by hypothesis.

Suppose that if $0 \neq b \in B_{i-1}$ then $b\Gamma_{i-1} \neq 0$. Let a be a non- zero element of $B_i$. if $a\Gamma_i = 0$ , then $a \notin K$ hence the canonical form of a has a term $c\,x^{\alpha}\partial^{\beta}$ with $c \in K/\{0\}$ and $|\alpha|+|\beta|>0$ . Suppose that $\alpha_i \neq 0$ for this monomial. Then $\partial_i c x^{\alpha-\epsilon_i}\partial^{\beta}$ is a sum and in the canonical form of $[a, \partial_i]$ .Thus $[a, \partial_i]$ is a non-zero element of $B_{i-1}$ .since $a\Gamma_i = 0$ , we conclude that

$$[a,\partial_i]\Gamma_{i-1} \subseteq a\partial_i\Gamma_{i-1}$$

But $\partial_i \Gamma_{i-1} \subseteq \Gamma_i$ , Hence $[a, \partial_i]\Gamma_{i-1} = 0$ ,which contradicts the induction hypothesis , we may reach a similar contradiction by assuming that $\beta_i \neq 0$ , Thus the lemma is proved.

**Theorem 3.4.2** If M is a finitely generated non-zero left $A_n$- module then d(M)≥ n

**Proof** Choose a set of generators for M and let $\Gamma$ be the good filtration obtained by giving each of these generators degree zero, Then $\Gamma_o \neq 0$ , let $h(t) = h(t, \Gamma, M)$ be the corresponding Hilbert polynomial by the lemma 3.4.1 , $B_i$ can be embedded in $\mathrm{Hom}_K(\Gamma_i , \Gamma_{2i})$ .

In particular

$$\dim B_i \leq \dim (\mathrm{Hom}_K(\Gamma_i , \Gamma_{2i}) .$$

But, $\mathrm{Hom}_K(\Gamma_i , \Gamma_{2i})$ has dimension $(\dim \Gamma_i).(\dim \Gamma_{2i})$, Thus, assuming that i >>0 we get that dim $B_i \leq h(i)\, h(2i)$

On the other hand. $\dim B_i \leq \binom{i+2n}{2n}$ is a polynomial in i of degree 2n . Hence as a polynomial in i, $h(i)\, h(2i)$ must have degree ≥ 2n , But the degree of $h(i)\, h(2i)$ is 2d(M).Thus d(M) ≥ n as required.

The above proved inequality is called the Bernstein inequality.

### 3. 5.Holonomic Modules

**Definition**:- A finitely generated left $A_n$-module is holonomic if it is zero,or if it has dimension n.

Example :- We already know that the $A_n$ – module ( $\mathbb{C}[X]$ =( $\mathbb{C}[x_1, x_2, \ldots, x_n]$ has dimension n. Hence it is holonomic $A_n$- module.

**Remark** :- Not all $A_n$-modules are holonomic .

For example $A_n$ it self is not a holonomic module because it has dimension 2n.

**Remark :-** Holonomic module has finite dimension .It follows that every submodule is finitely generated .Therefore all holonomic modules are Noetherian

**Proposition 3.5.1** Let n be positive integer. Then sub modules and quotients of holonomic $A_n$ – modules are holonomic.

Proof let M be a left $A_n$ – module and N a submodule of M. from theorem 3. .1 we have $d(N) \leq d(M)$. Since $d(M) = n$ and using Bernstein's inequality . We deduce that $d(N) = d(M/N)$ are also equal to n. Thus N and M/N are holonomic

**Theorem 3.5.2** Holonomic Modules are artinian

**Proof** Let M be a holonomic left $A_n$- module, suppose that M has a descending chain of submodules

$$M = N_o \supseteq N_1 \supseteq .....N_r$$

By theorem 3.3.1 and proposition 3.5.1 it follows that
$m(N_i)=m(N_{i+1}) + m(N_i/N_{i+1})$. Putting these together , we get that

$$m(M) = \sum_{0}^{r-1} m(N_i / N_{i+1}) + m(N_r) \geq r \ .$$

Hence M cannot have a descending chain of more than r sub modules. In particular M cannot have an infinite descending chain hence the theorem.

Remark: - Not all $A_n$-modules are artinian,

For example, $A_n$ is not artinian as a module over itself. Since

$A_n x_n \supseteq A_n x_n^2 \supseteq A_n x_n^3 \supseteq \ldots\ldots$ is an infinite descending chain of submodules.

For the moment we return to the general situation. Let M be a left non-zero module over a ring R. Assume that M is both artinian and noetherian. Suppose we have constructed a chain $N_o \subseteq N_1 \subseteq \ldots \subseteq N_k$ of submodules of M such that $N_i/N_{i-1}$ is irreducible, since M is artinian there exists a submodule $N_{k+1}$ of M such that $N_{k+1}/N_k$ is minimal in the set of non zero submodules of $N/N_k$. Since $N_{k+1}/N_k$ is minimal in the set of non zero submodules of $M/N_k$, then it must be irreducible. Thus in the chain $N_o \subseteq ,\ldots N_k \subseteq N_{k+1}$, the quotient of two adjacent terms is irreducible. This process cannot be continued forever since M is also noetherian. Thus we get a chain

$$0 = N_o \subseteq \cdots \subseteq N_r = M$$

of submodules of M such that $N_i/N_{i-1}$ is irreducible for i=1,2,…,r. such a chain is called a composition series of M. It follows from Jordan Holder Theorem that any two composition series of M have the same number of submodules. The number r is called the length of M.

Finally we know that a holonomic module is both noetherian and Artinian. It must have a composition series. Hence its length is well defined.

Thus we conclude that **length** of modules over the **Weyl Algebra** is well defined.

**SCHOLIUM. 3.5.3** The length of holonomic module cannot exceed its multiplicity.

**Proof** Suppose M be a holonomic module of length k.

We want to show that $m(M) \geq k$

Now $l(M) = k \Rightarrow$ There exists a chain

$0 = M_0 \subseteq M_1 \subseteq \ldots \subseteq M_k = M$, of submodules of M such that $M_i/M_{i-1}$ is simple for each i=1, . . . , k.

By Theorem 3. 3.1 and proposition 3.5 .1, we have that

$m(M_i) = m(M_{i-1}) + m(M_i/M_{i-1})$, putting these together , we get that m (M)

$= \sum_{1}^{k} m(M_i / M_{i-1}) + m(M_o) \geq k$, as required.

**Corollary 3.5.4** A holonomic $A_n$ – modules of multiplicity 1 is irreducible.

Proof Let M be a left holonomic module of multiplicity 1 and suppose that it has a non-zero submodule N. By proposition 3.5.1, We have N and M/N are holonomic,

Again by Theorem 3.1 we have m (M) = m (N) + m (M/N) since m (M) =1 and $N \neq 0$, we have that M/N =0

Thus M=N and M must be irreducible.

Example 1 ) We know that $\mathbb{C}[x]$ is a holonomic $A_1$ – module which is simple, and m ($\mathbb{C}[x]$) =1

2) The holonomic $A_2$ – module $\mathbb{C}[x,y]$ has multiplicity 1. It follows that $l(\mathbb{C}[x,y]) \leq 1$ , since $\mathbb{C}[x,y]$ is irreducible its only sub modules are 0 and itself. Thus $0 \subset (\mathbb{C}[x,y])$ is a composition series for $\mathbb{C}[x,y]$ which implies $l(\mathbb{C}[x,y]) = 1$

**Lemma 3.5.5** Let M be a left $A_n$ – module with a filtration $\Gamma$ with respect to the Bernstein filtration of $A_n$. Suppose that there exist constants. $c_1$, $c_2$ such that for j>>0

$\dim \Gamma_j \leq c_1 j^n/n! + c_2 (j+1)^{n-1}$

Then M is a holonomic $A_n$ – module whose multiplicity cannot exceed $c_1$. In particular M is finitely generated.

**Proof** The proof breaks up naturally in to two parts.

**Part 1**. Every finitely generated sub module of M is holonomic of multiplicity $\leq c_1$,

Let N be a finitely generated sub module of M. Thus N admits a good filtration say $F = \{\Gamma_i \cap N\}_{i \geq 0}$, it follows that there exists

a positive integer r such that $F_j \subseteq \Gamma_{j+r} \cap N$.

In particular, $\dim F_j \leq \dim \Gamma_{j+r}$ Let $h(t)$ be the Hilbert polynomial of N for the filtration $F$, using the polynomial bound of the hypothesis, we conclude that, for very large j,

$h(j) \leq c_1 (j+ r)^n/n! + c_2 (j +r)^{n-1}$

In particular the degree of $h(t)$ cannot exceed n. Hence $d(N) \leq n$ Form the Bernstein inequality we conclude that $d(N) = n$ equals the degree of $h(t)$, Going back to the polynomial in equality above we obtain that $m(N) \leq c_1$

**Part 2**. M is finitely generated.

Consider the ascending chain $N_1 \subseteq N_2 \subseteq \ldots \subseteq N_r$ of finitely generated submodules of M. Each of these is holonomic. By theorem 3.1

$m(N_i) = m(N_{i-1}) + m(N_i/N_{i-1})$ for $i = 1, 2, \ldots r$. Adding them up one has

$$\sum_{2}^{r} m(N_i / N_{i-1}) + m(N_1) = m(N_r) \leq c_1 .$$

In particular, all ascending chains of finitely generated submodules of M have less than $c_1$ steps, Thus M must be finitely generated.

We may now apply part 1 to M itself, and conclude that it is holonomic of multiplicity $\leq c_1$.

Suppose that a polynomial p is chosen is K[X]. Let $K[X, p^{-1}]$ denote the set of rational functions of the form $f/p^r$, where f is a polynomial in K[X]. Recall that rational functions are preserved by partial differentiation and by multiplication by a polynomial. In other words. $K[X, p^{-1}]$ is a left $A_n$-submodule of K(X).

**Theorem 3.5.6** The $A_n$-module $K[X, p^{-1}]$ is holonomic and its multiplicity is $\leq (\deg(p) + 1)^n$.

**Proof** Put $M= K[X, p^{-1}]$, Suppose that p has degrees m.

Set $\Gamma_k = \{ f/p^k : \deg(f) \leq (m+1)\, k\}$

we first check that $\Gamma$ is a filtration for M.

Let $f/p^k \in M$ and assume that deg (f) = s, then

$f/p^k = f.p^s/p^{s+k}$, but $fp^s$ has degree s (m+1), which is $\leq (m+1)(s+k)$

Hence $f/p^k \in \Gamma_{s+k}$, it follows that $M=\bigcup_{k\geq 0}\Gamma_k$

Next suppose that $f/p^k \in \Gamma_k$, equivalently. Deg (f) $\leq (m+1)k$, multiplication by $x_i$ increases the degree of f by 1. Thus

$x_i(f/p^k) = x_i fp/p^{k+1} \in \Gamma_{k+1}$.

Differentiating $f/p^k$ with respect to $x_i$. We get

$\partial_i(f/p^k) = (p\partial_i(f) - kf\partial_i(p))/p^{k+1}$

$\deg(p\partial_i(f) - kf\partial_i(p)) \leq (m+1)k + (m-1)$

$\partial_i(f/p^k) \in \Gamma_{k+1}$

Thus $B_i$. $\Gamma_k \subseteq \Gamma_{k+1}$, since $B_i = B_i^{-1}$, we also have that $B_i\ \Gamma_k \subseteq \Gamma_{k+i}$ finally, the dimension of $\Gamma_k$ cannot exceed the dimension of the vector space of polynomials of degree $\leq (m+1)k$, hence each $\Gamma_k$ is finite dimensional.

Therefore $\Gamma$ is a filtration o M.

And also we have

$$\dim \Gamma_k \leq \binom{(m+1)k+n}{n}$$

Since the two terms of highest degree in k of this binomial number are

$(m+1)^n k^n/n!$ and $(m+1)^{n-1} (n+1)(k+1)^{n-1}/n!$

It follows that

$$\dim \Gamma_k \leq (m+1)^n k^n/n! + (m+1)^{n-1} (n+1) (k+1)^{n-1}/n!$$

For very large values of k.

By lemma 3.5.5, we must have that M is holonomic $A_n$- module of multiplicity $\leq (m+1)^n$,

Example Consider the left $A_2$ – module $\mathbb{C}[x,y]_x$ ,

Set $\Gamma_t = \{f/x^t : \deg(f) \leq 2t\}$ then $\Gamma$ become good filtration of $\mathbb{C}[x,y]_x$ since the dimension of $\Gamma_t$ can not exceeds the dimension of the vector space of polynomials of degree $\leq 2t$, it is suffice to count the non- negative solutions of the equation $\alpha + \beta \leq 2t$, where $\alpha$ and $\beta$ are the degree of a monomials in x and y respectively.

Hence there are $\binom{(2t+2)}{2}$ such solutions

Thus $\dim \Gamma_t \leq \binom{(2t+2)}{2} = (2t+2)(2t+1)/2! = \chi(t,\Gamma,\mathbb{C}[x,y]_x)$

$\Rightarrow \dim \Gamma_t \leq (4t^2+6t+2)/2!$

$\Rightarrow$ By Bernstein inequality we have $d(\mathbb{C}[x,y]_x)=2$ and by lemma 3.5.5 , we get $m(\mathbb{C}[x,y]_x) \leq 4$ , which is the same as the answer which is found by using the formula given in Theorem 3. 5.6.

i.e $m(\mathbb{C}[x,y]_x) \leq (\deg(x)+1)^2 = 4$

Moreover, by SCHOLIUM 3.5.3 we have $l(\mathbb{C}[x,y]_x) \leq 4$

**Proposition 3.5.7** Let $p \in K[x_1,\ldots,x_n]$ and s be a new variable, then $K(s)[X, p^{-1}]p^s$ is a holonomic $A_n(K(s))$ – module, where K (s) is a field of rational function in s.

Proof As we have proved that $K[x,p^{-1}]$ is a module over the ring $A_n(K)$ , it can be easy to show that $K(s)[X, P^{-1}]p^s$ is a module over the ring $A_n(K(s))$, so here we need to show only it is holonomic.

For this put $M = K(s)[X,p^{-1}]p^s$, let $\deg(p) = m$ and

Set $\Gamma_k = \{q \cdot p^s/p^k : \deg(q) \leq (m+1)k\}$

First we need to check that $\Gamma$ is a filtration of M.

Let $q \cdot p^s/p^k \in M$ and assume that $\deg(q) = r$ , Then

$q.p^s/p^k = q.p^r.p^s/p^{r+k}$ , But $\deg(q.p^r) = r(m+1) \leq (m+1)(r+k)$

Hence $q.p^s/p^k \in \Gamma_{k+r}$

$\Rightarrow M = \bigcup_{k\geq 0} \Gamma_k$

Next suppose that $q.p^s/p^k \in \Gamma_k$ , Equivalently deg(q) $\leq$ (m+1)k

Now $x_i (q.p^s/p^k) = x_i q.p^{s+1}/p^{k+1}$. Since deg($x_i$q.p) $\leq$ (m+1) (k+ 1)

We have $x_i (q.p^s/p^k) \in \Gamma_{k+1}$, also

$\partial_i (q.p^s/p^k) = [p\partial_i (q) + (s-k)\, q\partial_i (p)]p^s/p^{k+1}$

Since deg $(p\partial_i (q) + (s-k)\, q\partial_i (p)) \leq (m+1)k + (m-1) < (m+1)k + (m+1)$

$= (m+1)(k+1)$

We have that $\partial_i (q.p^s/p^k) \in \Gamma_{k+1}$, But as $\partial_i$ ,$x_i \in B_1$ and $q.p^s/p^k \in \Gamma_k$ we have

$B_1 . \Gamma_k \subseteq \Gamma_{k+1}$

Since $B_1 B_1 = B_2$ and $B_1 B_1 = B_1^2$ We have $B_2 = B_1^2$ .

In general $B_i = B_1^i$ , so we have that $B_i . \Gamma_k \subseteq \Gamma_{k+i}$ Finally dim($\Gamma_k$) can not exceed the dimension of the vector space of polynomial of degree $\leq$ (m+1)k.

Hence each $\Gamma_k$ is finite dimensional.

There fore $\Gamma$ is a filtration of M.

Let $h(t, \Gamma, M)$ be the Hilbert polynomial of $\Gamma$ . Then for k>>0

$$\dim \Gamma_k \leq h((m+1), \Gamma, M) \leq \binom{(m+1)k+n}{n}.$$

Since the two terms of highest degree in k of this binomial number

are $(m+1)^n k^n /n! + (n+1)nk^{n+1}/2(n)!$

it follows that

dim $\Gamma_k \leq (m+1)^n k^n/n! + (m+n)^{n-1} (n+1)n (k+1)^{n-1}/n!$ , for very large value of k.

In particular the degree of $\chi(t)$ is n.

Thus d(M) =n

Therefore M= K(s) $[X , p^{-1}]p^s$ is holonomic $A_n$ (K(s)) – module.

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