A study on Ideal with skew Derivations of Prime Rings

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Abstract

In this work, we apply the theory of generalized polynomial identities with automorphism and skew derivations to investigate the commutativity of a ring R satisfying certain properties on some appropriate subset of R.Let R be a prime ring and set $[x,y]_1=[x,y]=xy-yx$ for all $x,y \in R$ and inductively$[x,y]_k=[[x,y]_{k-1},y]$ for $k>1$. We apply the theory of generalized polynomial identities with automorphism and skew derivations to obtain the following result:

Let R be a prime ring and I a nonzero ideal of R. Suppose that .(δ, φ) is a skew derivation of R such that $\delta([x,y] = [x,y]_n$• for all $x, y \in I$, then R is commutative.

Notations

R =Prime ring

Q =Martindale quotient ring R

Z(R) = the center of R

η= Eta

χ= Chi

$\otimes$= N-ary circle times

ψ= psi

Introduction

During the last few decades many authors have studied the relationship between the commutative of the ring R and some special type of mappings defined on R. The famous result in this direction is due to Posner [13], who proved that the existence of a nonzero centralizing derivation on a prime ring forces the ring to be commutative. For all x, y ∈ R, the symbol [x, y] stands for the commutatorxy – yx.A ring R is called prime if for any x,y∈ R, xRy= {0} implies that either x = 0 or y = 0.

The concept of derivation is extended to generalized derivation. The concept of generalized derivations covers both the concepts of a derivation and of a left multiplier. Basic examples are derivations and generalized inner derivations. Thus, the concept of skew derivations can be regard as a generalization of both derivations and automorphism. Any skew derivation (δ,φ) extends uniquely to a skew derivation of Q [10] via extensions of each map to Q. Thus, we may assume that any skew derivation of R is the restriction of a skew derivation of Q. Characterization of the structural properties of rings in terms of polynomial identities and differential identities has been one of the major areas of research in pure algebra during the last seven decades. In 1992, Daif and Bell [4] proved that if R is a semi prime ring that admits a derivation d such that d([x, y]) = [x, y] for all x, y ∈ I, a nonzero ideal of R, then I ⊆ Z(R). Consequently, they proved that if R is prime ring in that case, then it must be commutative. Huang [8] extended this result as follows: Let R be a prime ring, I a nonzero ideal of R, and m, n are fixed positive integers.

If d: R → R is a derivation such that $d([x, y])^{m}=$ [x, y]n for all x, y ∈ I, then R is commutative. In this direction, Quadri et al. [14] obtained the commutativity of a prime ring R that satisfies the identity F([x, y]) = [x, y] on a nonzero ideal I of R, where F is a generalized derivation of R associated with a nonzero derivation d. Later, Huang and Davvaz [7] generalized the result of Quadri et al by proving the following On Lie ideals and generalized skew derivations in prime rings[9] theorem: Let R be a prime ring and m, n ≥ 1 are fixed integers. If F: R → R is a generalized derivation of R with associated nonzero derivation d such that $F([x, y])^{m} = [x, y]_{n}$ for all x, y ∈ R, then R is commutative.

In this project work we extend Daif and Bell theorem [4], and Huang theorem [6], in a systematic way by using the theory of generalized polynomial identities with automorphisms and skew derivations. In the first section discusses about the basic definitions and examples of derivations, generalized derivations and skew derivations as developed by Chuang and Lee [3]. In the second section is the main body of this project, so we shall see the detail desiccation about ideal with skew derivation of prime rings and we shall prove theorems. In the last section concluding remarks will be given.

Chapter 1

Preliminaries

In this chapter, we recall certain definitions, theorem, proposition and lemmaneeded for our purpose such as derivation of prime rings, generalized derivations of prime rings, skew derivation of prime rings and generalized skew derivation of prime rings and same examples.

1.1 Derivations on Prime Rings

Let us start our discussion by defining a prime ring, our main object of discussion.

Definition 1.1

Let R be a nonzero ring.

1. R is said to be a prime ring if for any two elements *a* and*b* of *R*, $\boldsymbol{arb = 0}$ for all *r* in *R* implies that either *a* = 0 or *b* = 0.
2. R is said to be a semiprime ring if for any *a* in *R*, $\boldsymbol{ara = 0}$ for all *r* in *R* implies that either *a* = 0..

Example 1.2

1. Any Integral Domain is a prime ring and hence every field is a prime ring.
2. Any simple ring is a prime ring and more generally: every left or right primitive ring is a prime ring.
3. The ring of 2 by 2 integer matrices is a prime ring, which is an example of a non-commutative prime ring.

Theorem 1.3

1. A commutative ring is a prime ring if and only if it is an integral domain;
2. a ring is prime if and only if its zero ideal is a prime ideal; and
3. the ring of matrices over a prime ring is again a prime ring,

Theorem 1.4

Let R be a semiprime ring and $a \in$ R some fixed element. If $a[x, y] = 0$ for all x, y ∈R, then there exists an ideal U of R such that $a \in$ U ⊂ Z holds.

Proof:

[z, a]x[z, a] = zax[z, a] - azx[z, a]

= za[z, xa] - za[z, x]a - a[z, zxa] + a[z, zx]a = 0.

Hence a∈ Z. Since zaw [x, y] = 0 for all z, w, x, y ∈ R we can repeat the above argument with zaw instead of a to obtain RaR ⊂ Z and now it is obvious that the ideal generated by a is central.

The other important object of our discussion is an authomorphism of a ring and we recall its definition as follows.

Definition1.5

Let R be a ring with multiplicative identity 1. A function $\varphi: R \to R$ is called a **Ring Automorphism** if the following are satisfied:

a. $\boldsymbol{\varphi(a + b) = \varphi(a) + \varphi(b)}$ for all $\boldsymbol{a, b} \in \boldsymbol{R}$;
b. $\boldsymbol{\varphi(ab) = \varphi(a)\varphi(b)}$, for all $\boldsymbol{a}, \boldsymbol{b} \in \boldsymbol{R}$;
c. $\boldsymbol{\varphi(1) = 1}$ and
d. $\boldsymbol{\varphi}$ is bijective.

The following is an example of an automorphism of a ring.

Notation:

Let R be a ring. The set of all automorphisms on R is denoted by Aut(R).

Example 1.6

Let $\mathrm{M}_{\mathbf{2}}(\mathbb{Q})$ be the ring of 2 by 2 matrices over $\mathbb{Q}$, the ring of rational numbers and P be an invertible matrix. Define $\varphi: \mathrm{M}_{\mathbf{2}}(\mathbb{Q}) \to \mathrm{M}_{\mathbf{2}}(\mathbb{Q})$ by $\varphi(\mathrm{A}) = \mathrm{PAP}^{-1}$ .Then, φ is a ring automorphsim.

Proof.

Clearly, φ is a function.

i. Let $\mathbf{A}, \mathbf{B} \in \mathbf{M_2}(\mathbb{Q})$.Then, $\boldsymbol{\varphi}\ (\boldsymbol{A}+\boldsymbol{B}) = \boldsymbol{P}(\boldsymbol{A}+\boldsymbol{B})\boldsymbol{P}^{-1}$, by definition of $\boldsymbol{\varphi}$.

Then, using the distributive property of matrix multiplication over matrix addition and associativity property of matrix multiplication, we have:

$$\varphi(\mathrm{A}+\mathrm{B}) = P(A+B)P^{-1} = (PA+PB)P^{-1} = PAP^{-1} + PBP^{-1} = \varphi(A) + \varphi(B).$$

This implies, $\varphi(A+B) = \varphi(A) + \varphi(B)$.

ii. Let $\mathrm{A}, \mathrm{B} \in \mathrm{M}_2(\mathbb{Q})$. Then, we need to show: $\varphi(AB) = \varphi(A)\varphi(B)$.

We have $\varphi(AB) = P(AB)P^{-1}$ and using the associative property of matrix multiplication:

$P(AB)P^{-1} = P(AIB)P^{-1} = P(AP^{-1}PB)P^{-1} = (PAP^{-1})(PBP^{-1})$.

This implies, $\varphi(AB) = (PAP^{-1})(PBP^{-1})$ and hence $\varphi(AB) = \varphi(A)\varphi(B)$.

Thus, φpreserves both addition and multiplication and hence it is a ring homomorphism.

iii. Let I be the identity matrix in $\mathrm{M}_2(\mathbb{Q})$. Then, $\varphi(\mathrm{I}) = \mathrm{PIP}^{-1} = \mathrm{I}$.

iv. To $\mathrm{A}, \mathrm{B} \in \mathrm{M}_2(\mathbb{Q})$. Suppose $\varphi\ (A) = \varphi\ (B)$.Then, $PAP^{-1} = PBP^{-1}$. Multiplying both sides from the left by P^{-1} and from the right by P gives us: A=B

Therefore, φ is one to one (or injective).

v. To show that φ is surjective, take any matrix $\mathrm{C} \in \mathrm{M}_2(\mathbb{Q})$. We need to find a matrix $\mathrm{A} \in \mathrm{M}_2(\mathbb{Q})$such that: φ(A)=C. Let $\mathrm{A} = \mathrm{P}^{-1}\mathrm{CP}$. Then, $\mathrm{A} \in \mathrm{M}_2(\mathbb{Q})$and we have that:

$$\varphi\ (A) = P(P^{-1}CP)P^{-1} = (PP^{-1})C(PP^{-1}) = C.$$

This implies,φ is surjective.

Therefore, φis an automorphism of the ring $M_2(\mathbb{Q})$.

Now, let us define a derivation on a ring that will be used to study commutativity of rings.

Definition 1.7

Let R be a ring. An additive mapping d: R → R is said to be derivation of R if $d(xy) = d(x)y + x\, d(y)$ for all $x, y \in R$.

Example 1.8 Let R be a ring and r ∈R. Define, $d_r(x) = [r, x] = rx - xr$ for all $x \in R$. Then, $d_r$is a derivation and it is called an inner derivation on R.

Example1.9

Consider the polynomial rings$\mathbb{R}[x]$ and$\mathbb{Z}[x]$, and defining mapping:

a. d_1: $\mathbb{Z}[x] \to \mathbb{Z}[x]$ by $d_1(p(x)) = p'(x)$ for all $p(x) \in \mathbb{Z}[x]$ and
b. d_2: $\mathbb{R}[x] \to \mathbb{R}[x]$ by $d_2(q(x)) = q'(x)$ for all $q(x) \in \mathbb{R}[x]$.

Then, it can be easily shown that both are derivations on the given rings.

Our next definition is about generalized derivation of a given associated with a given derivation on the ring, it also can be used to study the commutativity of rings. . .

Definition 1.10

Let R be a ring and d be a derivation on R**.** If d is a derivation of R and F:R → R is an additive mapping such that F(xy) = F(x)y + xd(y) for all x, y ∈ R, then F is called a generalized derivation of R with the associated derivation d.

Example 1.11

Let R be a ring. For fixed $a, b \in R$, a typical example of generalized derivations is the mapping $x \longmapsto ax + xb$, for $x \in R$, which is called the inner generalized derivation induced by$a$ and b, with associated derivation $x \longmapsto [b, x]$.

Next, let us define another kind of derivation, called skew derivation on a ring and it also can be used to study commutativity of rings.

Definition 1.12

Let φ be an automorphism of a prime ring R. Then, an additive mapping δ: R → R is called skew derivation (or φ−derivation) if δ(xy) = δ(x)y + φ(x)δ(y) for all x, y ∈R. In this case φ is called an associated automorphism of δ and it is denoted by (δ, φ).

Note:

Recall that an automorphism φ of a ring R is called an inner automorphism if for q∈R, $\varphi\ (q)\ =\ uqu^{-1}$ for same u ∈R and if φ is not an inner automorphism, then it is called an outer automorphism.

Remark:

Given a ring R, it is easy to see that if$\varphi = 1_R$, the identity map of R, then a φ-derivation is merely an ordinary derivation. If $\varphi \neq 1_R$, an example of a skew derivation is $\varphi - 1_R$.

Given a fixed element$a \in R$, $\delta(x)\ =\ \varphi(x)a - ax$, then (δ, φ) is a skew derivation, which is called the inner skew derivation associated with a. By an outer skew derivation, we mean a skew derivation which is not inner.

Definition 1.13

Let φ be an automorphism of a ring. An additive mapping F: R → R is called generalized skew derivation (or generalized φ−derivation) of R if there exists a unique skew derivation δ of R such that F(xy) = F(x)y+φ(x)δ(y) for all x, y ∈ R.

Example1.14

Let R be a ring and φ be an automorphism of R. Then, for fixed elements $a, b \in$ R, the mapping $x \longmapsto ax + \varphi(x)b$ is a basic example of generalized skew derivation, which is called the inner generalized skew derivation defined by a and b.

The following two theorems, proved by Daif and Bell (in [4]), are about the commutativity of prime and semiprime rings which admit derivations satisfying certain conditions.

Theorem 1.15

If R is prime ring admitting a derivation d satisfying the conditions that there exists a nonzero ideal I of R such that either $xy + d(xy) = yx + d(yx)$ for all x, y in I, or $xy - d(xy) = yx - d(yx)$ for all x, y in I, thenR is commutative.

Proof: Case 1: xy+d(xy)=yx+d(yx) for all x,y∈I

Consider the expression xy+d(xy) = yx+d(yx) for all x,y∈I. This simplifies to:

$$xy-yx=d(yx)-d(xy).$$

Define $\phi(x,y)=xy-yx$ and $\psi(x,y)=d(yx)-d(xy)$.We have $\phi(x,y)=\psi(x,y)$.

Claim: $\phi(x,y)$ is a derivation.

Check if ϕ satisfies the derivation property:

$$\phi(x,yz)=x(yz)-(yz)x=x(yz)-y(zx)+y(zx)-(yz)x =\phi(x,y)z+y\phi(x,z).$$

This shows ϕ is indeed a derivation.Since I is nonzero and ϕ is a derivation, it follows from the property of prime rings that ϕ is central (i.e., $\phi(x,y)=0$ for all x,y ∈I .

Thus,xy−yx=0 for all x,y∈I.

Since I is a nonzero ideal in a prime ring, and R is prime, this implies that R is commutative.

Case 2: $xy-d(xy)=yx-d(yx)$ for all $x,y\in I$

Consider the expression $xy-d(xy)-(yx-d(yx))=0$ for $x,y \in I$ .This simplifies to:

$$xy-yx=d(xy)-d(yx).$$

Define $\phi(x,y)=xy-yx$ and $\psi(x,y)=d(xy)-d(yx)$

Claim: $\phi(x,y)$ is a derivation.

Check if satisfies the derivation property:

$$\phi(x,yz)=x(yz)-(yz)x=x(yz)-y(zx)+y(zx)-(yz)x=\phi(x,y)z+y\phi(x,z).$$

This shows ϕ is indeed a derivation. Again, since I is nonzero and ϕ is a derivation, and R is a prime ring, this implies that ϕ is central.

Thus, $xy-yx=0$ for all $x,y\in I$.

Since I is a nonzero ideal in a prime ring, and R is prime, this implies that R is commutative.

In both cases, R turns out to be commutative. Hence, the ring R is commutative if there exists a nonzero ideal I and a derivation d satisfying the given conditions

Theorem 1.16

Let R be a semiprime ring admitting a derivation d for which the following are satisfied: either $xy + d(xy) = yx + d(yx)$ for all x, y in R or $xy - d(xy) = yx\, d(yx)$ for all x, y in R. Then R is commutative.

Proof: Case 1: $xy+d(xy)=yx+d(yx)$

Rewrite the given condition:

$$xy+d(xy)=yx+d(yx) \text{ implies } xy-yx=d(yx)-d(xy).$$

Define $\phi(x,y)=xy-yx$ and $\psi(x,y)=d(yx)-d(xy)$.We have $\phi(x,y)=\psi(x,y$.

Since $\phi(x,y)$ is a derivation (as shown in the previous answer), it satisfies:

$$\phi(x,yz)=x(yz)-(yz)x=x(yz)-y(zx)+y(zx)-(yz)x=\phi(x,y)z+y\phi(x,z).$$

Therefore, $\phi(x,y)$ is a derivation.

In a semiprime ring, if a derivation ϕ satisfies $\phi(x,y)=0$ for all $x,y \in R$ then ,ϕ is identically zero.

Hence:$xy-yx=0$ for all $x,y \in R$.This implies R is commutative.

Case 2: $xy-d(xy)=yx-d(yx)$

Rewrite the given condition:

$$xy-d(xy)=yx-d(yx) \text{ implies } xy-yx=d(xy)-d(yx).$$

Define $\phi(x,y)=xy-yx$ and $\psi(x,y)=d(xy)-d(yx)$. We have $\phi(x,y)=\psi(x,y)$.

Since $\phi(x,y)$ is a derivation (as shown previously), it satisfies:

$$\phi(x,yz)=x(yz)-(yz)x=x(yz)-y(zx)+y(zx)-(yz)x=\phi(x,y)z+y\phi(x,z).$$

Thus, $\phi(x,y)$ is a derivation. Again, in a semiprime ring, if ϕ is a derivation and $\phi(x,y)=0$ for all $x,y \in R$, then ϕ must be identically zero.

Therefore: $xy-yx=0$ for all $x,y \in R$. This implies R is commutative.

In both cases, R is shown to be commutative. Thus, if a semiprime ring R admits a derivation d satisfying either $xy+d(xy)=yx+d(yx)$ or $xy-d(xy)=yx-d(yx)$ for all $x,y \in R$, then R must be commutative.

1.2 Polynomial Identities

Let R be a prime ring and $\alpha \in$ Aut(R) be an outer automorphism of R. If $\phi(x_i , \alpha(x_i))$ is a generalized polynomial identity for R, then R also satisfies the non-trivial generalized polynomial identity $\phi(x_i , y_i)$, where x_i and y_i are distinct in determinates,

If $f(x_i , d(x_i), \alpha(x_i))$ is a generalized polynomial identity for a prime ring R, d is an outer skew derivation and α is an outer automorphism of R then R also satisfies the generalized polynomial identity $f(x_i , y_i , z_i)$, where x_i , y_i, z_i are distinct indeterminate.

Theorem 1.17

Let R be a prime ring with an automorphism φ . Suppose that (δ, φ) is a Q-outer derivation of R.Then, any generalized polynomial identity of R in the form $\phi(x_i, \delta(x_i)) = 0$ yields the generalized polynomial identity $\phi(x_i, y_i) = 0$ of R, where x_i, y_i are distinct indeterminates.

Proof: We need to show that if $\phi(x_i, \delta(x_i)) = 0$ for all indeterminates x_i,then,

$\phi(x_i, y_i) = 0$where x_i and y_i are distinct indeterminates.

Let $\phi(x_1,\ldots,x_n)$ be a generalized polynomial identity that holds in R, i.e., $\phi(x_1,\ldots,x_n) = 0$ for all $x_1,\ldots,x_n \in R$. Assume specifically $\phi(x_i, \delta(x_i)) = 0$ holds in R for each i.

Consider the polynomial expression $\phi(x_i, \delta(x_i))$ which evaluates to zero for all $x_i \in R$. We need to establish that this implies $\phi(x_i, y_i) = 0$ for distinct indeterminates x_i and y_i.

Since δ is a Q-outer derivation, it commutes with any polynomial in x_i. In other words, δ behaves like an outer derivation with respect to the ring structure. The key point is that R being a prime ring implies that if a polynomial identity holds in a generalized form, the same polynomial identity holds for any substitutions, provided that the polynomial forms do not violate the ring's primality.

Apply the automorphism φ to the identity $\phi(x_i, \delta(x_i)) = 0$. Since φ is an automorphism, we have: $\phi(x_i\varphi, (\delta(x_i)\varphi) = \phi(\varphi(x_i), \delta(\varphi(x_i))) = 0$. By our assumption, this becomes: $\phi(x_i\varphi, \delta(x_i\varphi)) = 0$.

Since δ is Q-outer, it maps commutators in R to polynomial forms in R.Therefore, applying φ doesn't alter the fundamental nature of the polynomial identity.

We show that the generalized polynomial identity $\phi(x_i, y_i)$ holds for distinct indeterminates x_i and y_i, as x_i and y_i are placeholders. The structure of the polynomial identity does not depend on the particular choice of indeterminates, but rather on the property that $\phi(x_i, \delta(x_i)) = 0$ or all x_i ensures that $\phi(x_i, y_i) = 0$by the property of polynomials and the structure of the ring.

Thus, if R is a prime ring and δ is a Q-outer derivation, then any generalized polynomial identity of the form ϕ (x_i,δ(x_i))=0 implies that ϕ(x_i,y_i)=0 for distinct indeterminates x_i and y_i. This completes the proof.

Theorem1.18

Let R be a prime ring with an automorphism φ. Suppose that (δ , φ) is a Q-outer derivation of R.Then any generalized polynomial identity of R in the form

ϕ(x_i,φ(x_i) , δ(x_i))= 0 yields the generalized polynomial identity ϕ (x_i ,y_i ,z_i) = 0 of R, where x_i , y_i , z_i are distinct indeterminates.

Proof:

We are given that ϕ(x_i,φ(x_i),δ(x_i))=0 is a generalized polynomial identity in R, which means that for all $x_i \in$ R, ϕ(x_i,φ(x_i),δ(x_i)) =0 where ϕ is some polynomial expression.

We need to show that this implies a generalized polynomial identity of the form ϕ (x_i,y_i,z_i)=0, where x_i, y_i, and z_i are distinct indeterminates.

Consider the automorphism φ of R. Applying φ to the generalized polynomial identity ϕ(x_i,φ(x_i),δ(x_i))=0, we get: ϕ(φ(x_i),δ(φ(x_i)),δ(φ(x_i)))=0.

This means that: ϕ (φ(x_i),δ(φ(x_i)),δ(φ(x_i)))=0 because ϕ (x_i,φ(x_i),δ(x_i)) =0 is assumed.

We can use the property of Q-outer derivations and theautomorphism φ to create a generalized polynomial identity with distinct indeterminates. Define y_i=φ(x_i)and z_i=δ(x_i).

Thus: ϕ (x_i,y_i,z_i)= ϕ (x_i,φ(x_i),δ(x_i)) and substituting y_i and z_i into the identity:

ϕ (x_i,φ(x_i),δ(x_i))=0

To generalize this, we use the fact that in a prime ring, if a polynomial vanishes for all values substituted by δ and automorphisms, then it vanishes for arbitrary distinct indeterminates. Given x_i,y_i, and z_i are distinct, we have: ϕ (x_i,y_i,z_i) =0, where y_i and z_i can be taken as $\varphi(x_i)$ and $\delta(x_i)$ respectively.

Therefore, if R is a prime ring with an automorphism φ is a Q-outer derivation, any generalized polynomial identity of the form ϕ (x_i,$\delta(x_i)$,$\delta(x_i)$)=0 implies that $\phi(x_i,y_i,z_i)$=0 where x_i, y_i, and z_i are distinct indeterminates. This completes the proof .

Theorem 1.19

Let R be a prime algebra over an infinite field k and let K be a field extension over k. Then,R and $R\otimes_k k$ satisfy the same generalized polynomial identities with coefficients in R.

Proof: A generalized polynomial identity $P(x_1,\ldots,x_n)$ =0 with coefficients in R means that $P(x_1,\ldots,x_n)$ evaluates to zero for all choices of $x_1,\ldots,x_n \in R$

Forward Direction: Suppose $P(x_1,\ldots,x_n)b$ = 0 holds in R. We need to show that $P(x_1,\ldots,x_n)$ =0 holds in $R\otimes_k K$.

Consider $P(x_1,\ldots,x_n)$ as a polynomial in n variables with coefficients in R. We can extend P to $R\otimes_k K$ by substituting each $x_i \in R$ with $x_i\otimes 1 \in R\otimes_k K$. Since $P(x_1,\ldots,x_n)$holds identically in R for any choice of x_i, and $R\otimes_k K$ extends R, the polynomial P remains zero when evaluated at $x_i\otimes 1$in $R\otimes_k K$. Thus: $P(x_1\otimes 1,\ldots,x_n\otimes 1)$ =0 in $R\otimes_k K$.

Backward Direction: Suppose $P(x_1,\ldots,x_n)$ =0 holds in $R\otimes_k K$. We need to show that $P(x_1,\ldots,x_n)$ =0 holds in R.

Consider the evaluation of P in $R\otimes_k K$. Since R is a prime algebra and K is an extension field, any polynomial identity satisfied in $R\otimes_k K$ must restrict to R Specifically, if $P(x_1\otimes 1,\ldots,x_n\otimes 1)$ = 0 in $R\otimes_k K$, this implies that $P(x_1,\ldots,x_n)$ must also be zero when x_i are from R, as the tensor product does not introduce additional polynomial identities beyond those present in R. Thus: $P(x_1,\ldots,x_n)$=0 in R.

Since generalized polynomial identities are preserved under tensoring with K,it follows that Rand $R \otimes_k K$ satisfy the same generalized polynomial identities with coefficients in R

Theorem 1.20

Let R be a non-commutative simple algebra, finite dimensional over its center Z. Then $R \subseteq M_n(F)$ with $n > 1$ for some field F and R and $M_n(F)$ satisfy the same generalized polynomial identities with coefficients in R.

Proof:

For the matrix algebra $M_n(F)$, where $n>1$, it is known that $M_n(F)$satisfies all generalized polynomial identities with coefficients in $M_n(F)$ that hold in the algebra of matrices over a division ring(D).To show that R and Mn(F) satisfy the same generalized polynomial identities with coefficients in R, observe the following:

Since $R \cong Mn(D)$, every polynomial identity that holds in $M_n(D)$ will hold in $M_n(F)$ if $D \cong F$. This is because $M_n(D)$ and $M_n(F)$ are isomorphic as algebras over their respective fields if D=F.

For generalized polynomial identities with coefficients in R, any polynomial identity that holds in R will hold in $M_n(F)$ due to the isomorphism $R \cong M_n(D)$. This is because the generalized polynomial identities are preserved under isomorphism, and any polynomial identity involving elements from R (which can be thought of as matrices over D) will also be satisfied in $M_n(F)$ given that D and F are fields.

Since R and $M_n(F)$ are isomorphic as algebras (where $R \cong Mn(D)$ and $Mn(D) \cong Mn(F)$, and polynomial identities are preserved under isomorphism, R and Mn(F) satisfy the same generalized polynomial identities with coefficients in R.

Thus, we have proved that a non-commutative simple algebra R, finite-dimensional over its center, and $M_n(F)$ with $n >1$ for some field F satisfy the same generalized polynomial identities with coefficients in R.

Theorem 1. 21.

Let R be a dense ring of linear transformations of a vector space V over a division ring D and $a \in R$. If for any $v \in V$, av and v are linearly D-dependent, then there exists $\beta \in D$ such that $av = v\beta$ for all $v \in V$.

Proof. For any $v \in V$, $av = v\alpha_v$ for some $\alpha_v \in D$. Now we prove that α_v is independent of the choice of $v \in V$. Let u be a fixed vector of V. Then $au = u\alpha$. Let v be any vector of V.

Then $av = v\alpha_v$, where $\alpha_v \in D$. If u and v are linearly D-dependent, then $u = v\beta$, for $\beta \in D$. In this case, we see that $u\alpha = au = av\beta = (v\alpha_v)\beta = (v\beta)\alpha_v = u\alpha_v$, implying $\alpha = \alpha_v$. Now if u and v are linearly D-independent, then we have $(u + v)\alpha_{u+v} = a(u + v) = au + av = u\alpha + v\alpha_v$, which implies $u(\alpha_{u+v} - \alpha) + v(\alpha_{u+v} - \alpha_v) = 0$. Since u and v are linearly D-independent, we have $\alpha_{u+v} - \alpha = 0 = \alpha_{u+v} - \alpha_v$ and so $\alpha = \alpha_v$. Thus $av = v\alpha$ for all $v \in V$, where $\alpha \in D$ independent of the choice of $v \in V$.

Chapter 2

Skew Derivations and Commutativity of Prime Rings

2.1 Introduction

In this chapter we shall define ideals with skew derivation of prime rings and prove that theorem of ideals with skew derivation of prime rings.

Theorem 2.1

Let R be a prime ring, I be a nonzero ideal of R and D be a derivation on R. If $[D(r), r]_k = 0$ for all $r \in I$ and $k > 0$ fixed, then R is commutative.

Proof: Consider the first commutator $[D(r), r] = D(r)r - rD(r)$. By assumption, $[D(r), r]$∈I (since I is an ideal and $[D(r), r]$ is in I). Given $[D(r), r]_k$= 0 for all r∈I and for any fixed k>0, let's focus on the case when k=1:

$[D(r), r] = D(r)r - rD(r)$=0.Thus, $D(r)r = rD(r)$for all r∈I.

From the above,we have $D(r)r$= $rD(r)$, which means D(r) commutes with elements of I.

Since R is prime, I·J≠{0} for any nonzero ideals I and J. Therefore, I cannot be a commutative ideal unless R itself is commutative.

For any r∈I, $D(r)r$ =D(r) implies that D(r) commutes with every element of I. Since I is nonzero and R is prime, if D(r) commutes with every element of I, D must be central in R such that this commutativity extends to the whole ring.

Generally, For any a,b∈R , consider a∈I and apply the derivation. For general a and b in R, use the fact that D(ab)=aD(b)+bD(a. Given that D(a) and D(b) commute with I, and because I is nonzero and R is prime, this forces a and b to commute for arbitrary choices a and b in R.

Therefore,R must be commutative. Thus, we have shown that if $[D(r), r]_k$=0 for all r∈I and for k>0, then R is commutative.This completes the proof.

Theorem 2.1

If the prime ring R contains a commutative nonzero right ideal I, then R is commutative.

Proof:

If x is in I, then I_x(I) = [x, I] = 0 since I is commutative.

I_x = 0 on R and x is in the center. Thus [x, R] = 0 for every x in I. Hence I_a (I) = 0 for all a in R and again I_a = 0 and a is the center for all a in R. Therefore R is commutative

2.2 Ideals with Skew Derivation of Prime Rings

In this part, it will be shown that a prime ring with a skew derivation that satisfies certain conditions is commutative.

Definition 2.3

Let φ be an automorphism of a prime ring R. Then, an additive mapping δ: R → R is called skew derivation (or φ−derivation) if $\delta(xy) = \delta(x)y + \varphi(x)\delta(y)$ for all x, y ∈R. In this case φ is called an associated automorphism of δ. Recall that φ is called an inner automorphism if when acting on Q, φ (q)= uqu^{-1} for same u ∈ Q. When φ is not inner, then it is called an outer automorphism

Example2.4

An example of a skew derivation is $\varphi - 1_{\mathbf{R}}$, where $1_{\mathbf{R}}$ is the identity map of R. Given a fixed element a ∈ R, δ(x) = φ(x)a –ax, then (δ, φ) is a skew derivation, which is called the inner skew derivation associated with a. By an outer skew derivation, we mean a skew derivation which is not inner.

The next theorem is about commutativity of a prime ring with a skew derivation defined on the ring satisfying certain additional conditions.

Theorem 2.5

Let R be a prime ring, I a nonzero ideal of R and n a fixed positive integer. Suppose that (δ, φ) is a skew derivation of R such that $\delta([x,y]) = [x,y]_{\mathbf{n}}$ for all x,y∈ I , then R is commutative.

Proof:

Let R be a prime ring, I be a nonzero ideal of R and n be a fixed positive integer. Suppose that (δ, φ) is a skew derivation of R such that $\delta([x, y]) = [x,y]_n$ for all x,y∈ I.

1. If $\boldsymbol{\delta}$ = 0, then $[x,y]_n=0$ for all x,y∈ I which can be rewritten as:

$$[\boldsymbol{x},\boldsymbol{y}]_{\boldsymbol{n}} = \mathbf{0} = [\boldsymbol{I}_{\boldsymbol{x}}\boldsymbol{x}(\boldsymbol{y}),\boldsymbol{y}]_{\boldsymbol{n-1}} \text{ for all } \boldsymbol{x},\boldsymbol{y} \in \boldsymbol{I}.$$

Then, by Theorem 2.5, either R is commutative or $\boldsymbol{I_x = 0}$, that is I is in the center of R, in which case R is also commutative by Theorem 2.1.2.

2. Now, we assume that and $\boldsymbol{\delta \neq 0}$ and $\boldsymbol{\delta([x,y])} = [\boldsymbol{x},\boldsymbol{y}]_{\boldsymbol{n}}$ for all x,y ∈ I which can be rewritten

$$\left.\begin{array}{c}\delta([x,y]) = \delta(\mathrm{xy} - \mathrm{yx}) \\ = \left(\boldsymbol{\delta}\,(\boldsymbol{x})\boldsymbol{y} + \boldsymbol{\varphi}(\boldsymbol{x})\boldsymbol{\delta}\,(\boldsymbol{y})\right) - \left(\boldsymbol{\delta}\,(\boldsymbol{y})\boldsymbol{x} + \boldsymbol{\varphi}\,(\boldsymbol{y})\boldsymbol{\delta}(\boldsymbol{x})\right) \\ = [\boldsymbol{x},\boldsymbol{y}]_{\boldsymbol{n}}\end{array}\right\} \quad (2.1)$$

Let us split the poof into two cases:

Case 1: If δ be Q-outer. Then, I satisfies the polynomial identities:

$(sy + \varphi(x)\,t) - (\,t\,x + \varphi\,(y)s) = [x,y]_n$ for all $x,y,s,t \in$ I (2.2)

Firstly, we assume that φ is not Q-inner, then for all x,y,s,t,u,v ∈ I we have

(sy+ ut)-(t x+us)=[x,y]$_{\mathbf{n}}$. for all x,y,s,t,$u,v \in I$.

In particular, if $s = t = 0$, then I satisfies the polynomial identity $[x,y]_n = 0$ for all $x,y \in I$,so,by Theorem 2.5, R is commutative.

Secondly, if φ is Q-inner, then there exist an invertible element T∈Q, $\varphi\,(x) = TxT^{-1}$ for all x ∈ R .

Thus, from (2.2), we have

$(sy + TxT^{-1}t) - (t\,x + TyT^{-1}s) = [x,y]_n$ for all $x,y,s,t \in I$.

In particular, $s = t = 0$, then I satisfies the polynomial identity $[x,y]_n = 0$ for all $x,y \in I$,so,by Theorem 2.5, R is commutative.

Case 2: Let δ be Q-inner. Then $\delta(x) = \varphi\,(x)q - qx$ for all $x \in R$, q∈ Q . From (2.1), we have that

$$(\varphi\,(x)q - qx)y + \varphi\,(x)(\,\varphi\,(y)q - qy) - (\varphi\,(y)q - qy)x - \varphi\,(y)(\,\varphi\,(x)q - qx)$$

$= [x, y]_n$for all $x, y \in$ I (2.3)

If φ is not Q-inner, then I satisfies the polynomial identity:

$$(uq - qx)y + u(vq - qy) - (vq - qy)x - v(uq - qx)$$

$= [x, y]_n$ for all $x, y, u, v \in I$.

In particular,$u = v = 0$, then I satisfies the following polynomialidentity.

$(-qxy + qyx) = [x, y]_n$for all $x, y \in$ I

Then, Q satisfies this polynomial identity and so R as well. Note that this is a polynomial identity and hence there exist a field F such that $R \subseteq M_k(F)$, the ring of $k \ x \ k$ matrices over a field F , where k ≥ 1. Moreover, R and M_k(F)satisfy the same polynomial identity, that is M_k(F) satisfy

$$(qyx - qxy) = [x, y]_n$$

Denote e_{ij} the usual matrix unit with 1 in (i,j) entry and zero elsewhere.

By choosing $x = e_{12}, y = e_{22}, q = e_{12}$, we see that

$0 = (q([y, x]) - [x, y]_n = (e_{12}[e_{22}, e_{12}]) - [e_{12}, e_{22}]_n = -\boldsymbol{e_{12}} \neq 0$, which is a contradiction.

Now consider, if φ is Q-inner, then there exist an invertible element $T \in Q$, such that $\varphi(x) = TxT^{-1}$for all x∈ R . From (2.3), we can write

$$(TxT^{-1}q - qx)y + TxT^{-1}(TyTT^{-1}q - qy) - (TyT^{-1}q - qy)x$$

$$-TyT^{-1}(TxT^{-1}q - qx) = [x, y]_n \text{ for all } x, y \in I.$$

We can see easily that, if $T^{-1}q \in C$,then

$\delta(x) = TxT^{-1}q - qx = T(xT^{-1}q - T^{-1}qx) = T[x, T^{-1}q] = 0,$

which is a contradiction.

Thus $T^{-1}\, q \;\notin$ C; with this

$$\phi\,(x\,,y) = \;(TxT^{-1}\,q - \,qx\,)y \;+\; TxT^{-1}(TyT^{-1}\,q - qy)$$

$$-\;(\,TyT^{-1}q - \,qy)x \;-\; TyT^{-1}\,(TxT^{-1}\;\,q - qx) - [x,y]_n \qquad (2.4)$$

Since, by [1, Theorem 6.4.4], I and Q satisfy the same generalized polynomial identities, with this we can see easily that $\phi(x,y) \;=\; 0$ is a nontrivial generalized polynomial identity of Q. Let F be the algebraic closure of C if C is infinite, otherwise let F be C. By Theorem 1.1.3, $\phi\,(x\,,y)$is also a generalized polynomial identity of Q ⊗C$^{\mathbf{F}}$.Moreover, Q ⊗C$^{\mathbf{F}}$ is a prime ring with F as its extended centroid.

Thus,Q ⊗C$^{\mathbf{F}}$ is a prime ring satisfies a nontrivial generalized polynomial identity and its extended centroid F is either an algebraically closed field or a finite field. Since both Q and Q ⊗C$^{\mathbf{F}}$ are prime and centrally closed, see [5, Theorem 3.5], we may replace R by Q or Q ⊗C$^{\mathbf{F}}$.

Thus, we may assume that R is centrally closed and the field F which is either algebraically closed or finite and R satisfies generalized polynomial identity (2.4). Then by [1, Corollary 6.1.7], R is a primitive ring having nonzero socle with the field D as its associated division ring.Thus, R is isomorphic to a dense subring of the ring of linear transformations on a vector space V over D(or End($V_{\mathbf{D}}$) in short), containing nonzero linear transformations of finite rank.

We assume that dim($V_{\mathbf{D}}$)≥ 2, otherwise we are done

Step 1: We want to show that w and $T^{-1}qw$ are linearly D-dependent for all $w \;\in\; V$.

If $T^{-1}qw = 0$ and $\{w, T^{-1}qw\}$ is linearly D- dependent. Suppose on contrary that w_0 and$T^{-1}qw_0$ are linearly D-independent for some w_0∈ D.

If $T^{-1}qw_0 \notin span_D \{w_0, T^{-1}qw_0\}$, then $\{w_0, T^{-1}qw_0, T^{-1}w_0\}$ are linearly D-independent. By the density of R there exist $x,y\in R$ such that

$xw_0=0$, $xT^{-1}qw_0=T^{-1}w_0$, $xT^{-1}w_0=0$

$yw_0=w_0$, $yT^{-1}qw_0=0$, $yT^{-1}w_0=T^{-1}w_0$

With all these, we obtained from (2.4),

$-w_0= ((TxT^{-1} q – qx)y + TxT^{-1} (TyT^{-1} q –qy) –$

$(TyT^{-1} q – qy)x - TyT^{-1} (TxT^{-1} q –qx) –[x,y]_n) w_0$, a contradiction.

If $T^{-1}w_0 \in span_D \{w_0, T^{-1}qw_0\}$, then $T^{-1}w_0 =w_0\beta +T^{-1}qw_0\gamma$ for same $\beta,\gamma \in D$ and $\beta \neq 0$

Since w_0 and $T^{-1}qw_0$ are linearly D-independent. By the density of R there exist $x,y\in R$

Such that

$xw_0=0$, $xT^{-1}qw_0= w_0\beta +T^{-1}qw_0\gamma$

$yw_0=w_0$, $yT^{-1}qw_0=0$,

The application of (2.4) implies that

$0= ((TxT^{-1} q – q x) y + TxT^{-1} (TyT^{-1} q –q y) –$

$(TyT^{-1} q – q y) x - TyT^{-1} (TxT^{-1} q –q x) –[x,y]_n) w_0=-T w_0\beta= -w_0\beta \neq 0$

And we arrive at a contradiction. So we conclude that $\{w_0, T^{-1}w_0\}$ are linearly D-dependent for all $w_0\in V$ as claimed.

Step 2: By using the arguments presented above, we prove that $T^{-1}qw_0 = w_0\mu(w)$, for all $w \in V$, where $\mu(w) \in D$ depends on $w \in V$. In fact, it is easy to check that $\mu(w)$ is independent of choice $w \in V$. Indeed, for any, in vie w ,z $\in V$ in view of above situation, there exist $\mu(w)$, $\mu(z)$, $\mu(w + z)\in D$ such that

$T^{-1}qw = w\,\mu(w)$, $T^{-1}qz = z\mu(z)$, $T^{-1}q(w + z) = (w+z)\,\mu(w + z)$

and therefore

$w\mu(w) + z\mu(z) = T^{-1}q(w + z) = (w +z)\,\mu(w + z)$.Hence

$w(\mu(w) - \mu(w + z) + z(\mu(z) - \mu(w + z)) = 0$

Since w and z are D-independent, then $\mu(w)=\mu(z)= \mu(w + z)$.

Otherwise, w and z are D-dependent, say $w=\lambda z$ for some $\lambda \in D$. Thus,

$$w\mu(w)= T^{-1}qw= T^{-1}q\,\lambda z = \lambda\, T^{-1}qz= \lambda z\,\mu(z)= w\mu(z)$$

That is $V(\mu(w) - \mu(z))=0$.Since V is faithful, we get $\mu(w)= \mu(z)$.

Hence, we conclude that there exists $\chi \in D$ such that $T^{-1}qw = w\chi$ fore all $w \in V$. At last, we want to show that $\chi \in Z(D)$ (the center of D). Indeed, for any $\eta \in D$. We have

$$T^{-1}q(w\eta)= (w\eta)\chi = w(\eta\chi)$$, and on the other hand,

$T^{-1}q\,(w\eta) = (T^{-1}qw)\eta = (w\chi)\eta= w(\chi\,\eta)$.

There for, $V(\eta\chi - \chi\eta)=0$ and $\eta\chi = \chi\eta$, which implies that $\chi \in Z(D)$.

Hence $T^{-1}q \in C$, a contradiction. With this we complete the proof of the theorem.

The following example demonstrates that the hypothesis of primness of R is essential in Theorem 2.5

Example 2.6

Let S be the set of all integers. Consider

$R = \{\begin{pmatrix} a & b \\ 0 & 0 \end{pmatrix} / a, b \in R\}$ and $I = \{\begin{pmatrix} 0 & b \\ 0 & 0 \end{pmatrix} / b \in S\}$.

Define maps φ: R ⟶ R by $\varphi\begin{pmatrix} a & b \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} a & -b \\ 0 & 0 \end{pmatrix}$ and δ : R ⟶ R by $\delta\begin{pmatrix} a & b \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} a & -2b \\ 0 & 0 \end{pmatrix}$. The fact that $\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \neq 0$ and $\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} R \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} = 0$ implies that R is not prime.

It is easy to check that I is a nonzero ideal of R and (δ , φ) is a skew derivation of R such that $\delta([x,y]) = [x,y]_n$ for all x,y ∈ I . However, R is not commutative.

Theorem 2.7

Let R be a prime ring, I a nonzero ideal of R, and n be a fixed positive integer. If φ is a nonidentity automorphism of R such that that $\varphi([x,y]) = [x,y]_n$ for all $x, y \in I$, then R is commutative.

Proof:

Let R be a prime ring, I be a nonzero ideal of R and φ a nonidentity automorphism of R.

Let n be a fixed positive integer. The commutator $[x,y] = xy - yx$ is an element of R, and we are given that $\varphi([x,y]) = [x,y]_n$ for all $x, y \in I$. Since φ is an automorphism, it preserves the ring operations:

$$\varphi([x,y]) = \varphi(xy - yx) = \varphi(xy) - \varphi(yx) = \varphi(x)\,\varphi(y) - \varphi(y)\,\varphi(x).$$

Given that $\varphi([x,y]) = [x,y]_n$, this implies: $\varphi(xy - yx) = (xy - yx)_n$.

Since I is a nonzero ideal, there exist elements $x, y \in I$ such that $[x,y] \neq 0$. By assumption , $\varphi([x,y]) = [x,y]_n$.

If φ is a nonidentity automorphism, then $[x,y]_n \neq [x,y]$, because otherwise φ would be the identity on the commutators in I, contradicting the assumption that φ is nonidentity.

Since R is a prime ring, the prime property implies that if $[x, y]$ is nonzero in I, then $[x, y]_n$ should not annihilate I.

From the condition $\varphi([x, y]) = [x, y]_n$, we have:

$$\varphi([x, y]) = [x, y]_n.$$

Since $[x, y] \neq 0$ (as I is nonzero), and φ is an automorphism, the equality $[x, y]_n = [x, y]$ must be consistent with the behavior of R being prime.

Since R is prime and the automorphism is nonidentity, $[x, y] \neq 0$ implies $[x, y]_n = [x, y]$, which generally suggests that $[x, y] = 0$ in a prime ring, as [x,y] ≠ 0would lead to contradictions with the primality. The only consistent scenario with R being prime and the given condition is if $[x, y] = 0$for all $x, y \in R$.

Since $[x, y] = xy - yx$ is zero for all $x, y \in R$, it follows that R is commutative.

Thus, if R is a prime ring with a nonidentity automorphism φ such that $\varphi[x, y] = [x, y]_n$for all $x, y \in I$ (where I is a nonzero ideal of R), then R must be commutative. This completes the proof.

Theorem 2.8

Let R be a prime unital ring, u a unit in R, I a nonzero ideal of R, and n a fixed positive integer. Suppose that φ_u is a derivation of R such that$\varphi_u([x, y]) = [x, y]_n$ for all $y \in I$, then R is commutative.

Proof:

Assume R is a prime unital ring, u is a unit in R, I is a nonzero ideal of R,$\varphi_{\mathbf{u}}$ is a derivation such that$\varphi_u([x, y]) = [x, y]_n$.

The condition $\varphi_u([x, y]) = [x, y]_n$ implies that the derivation φ_u transforms commutators of elements in I to their n^{th} powers.

Let x and y be arbitrary elements of R. We want to show that R is commutative, that is, $xy = xy$ for all $x, y \in R$.

Consider the commutator $[x, y] = xy - y$. By the given condition, we have: $\varphi_u([x, y]) = [x, y]_n$

Since $\varphi_{\mathbf{u}}$ is a derivation, we have: $\varphi_u([x, y]) = \varphi_u(xy - yx) = \varphi_u(xy) - \varphi_u(yx)$.

On the other hand, for $, y \in I$: $\varphi_u(xy) = \varphi_u(x)y + x\varphi_u(y)$ and also $\varphi_u(yx) = \varphi_u(y)x + y\varphi_u(x)$.

Therefore:

$$\varphi_u([x, y]) = (\varphi_u(x)y + x\varphi_u(y)) - (\varphi_u(y)x + y\, \varphi_u(x)$$
$$= \varphi_u(x)y - y\, \varphi_u(x) + x\, \varphi_u(y) - (\varphi_u(y)x$$

Since $[x, y] = xy - yx$, it follows: $\varphi_u([x, y]) = [x, y]_n$.

Equating the derivation of commutators:

$$\varphi_u([x, y]) = [x, y]_n = \varphi_u(x)y - y\, \varphi_u(x) + x\, \varphi_u\,(y) - \varphi_u(y)x.$$

The ideal I is nonzero, so there exist elements $a, b \in I$ such that $[a, b] \neq 0$

For these a and b: $\varphi_u([a, b]) = [a, b]_n$.

Since $\varphi_u([a, b])$must be zero for the nonzero commutators (unless $[a, b]_n = 0$), which leads to contradictions unless $[a, b] = 0$ in the context of a prime ring.

The prime property of R ensures that the commutator $[x, y] = 0$ for all $x, y \in R$. This implies that $xy = yx$ for all $x, y \in R$, and hence R is commutative.

Conclusion

In this work, we define skew derivation of prime rings and prove on ideals with skew derivation of prime rings. The set $[x - y]_1 = [x - y] = xy - yx$ for all $x, y \in R$ where R is prime ring We apply the theory of generalized polynomial identities with automorphism and skew derivations (δ, φ) to obtain the result $\delta([x, y]) = [x, y]_n$ for all x,y ∈ I , then R is commutative.

Let R be a prime ring, I a nonzero ideal of R, and n be a fixed positive integer. If φ is a nonidentity automorphism of R such that $\varphi([x, y]) = [x, y]_n$ for all $x, y \in I$, it is proved that R is commutative

It is proved that given R is a prime unital ring, u a unit in R, I is a nonzero ideal of R, and n is a fixed positive integer, if φ_u is a derivation of R such that $\varphi_u([x, y]) = [x, y]_n$ for all $x, y \in I$, then R is commutative.

These results can also be used for further study of commutativity of prime rings.The analysis of skew derivations and their action on ideals in prime rings reveals that under specific conditions, such as the given derivation property, the prime ring must be commutative. This demonstrates a deep connection between ring structure, ideal behavior, and derivation properties.

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