A NONDEGENERATE THREE-LEVEL LASER WITH THE CAVITY MODES DRIVEN BY COHERENT LIGHT

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To my Parents.

Abstract

In this thesis we study the squeezing and statistical properties of the light produced by a nondegenerate three-level laser, with the cavity modes driven by coherent light. With the aid of the master equation, we obtain stochastic differential equations. Applying the solutions of the resulting differential equations, we calculate the quadrature variance, the squeezing spectrum , the mean and variance of the photon number sum and difference. For a linear gain coefficient of 75 and for a cavity damping constant of 0.8, the maximum intracavity squeezing is found at steady state and at threshold to be 64%.

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Contents

Chapter 1

Introduction

There has been a considerable interest in the analysis of the quantum properties of the squeezed light generated by various quantum optical systems [1-10]. In a squeezed light the fluctuations in one quadrature is below the vacuum level at the expense of enhanced fluctuations in the conjugate quadrature, with the product of the uncertainties in the two quadratures satisfying the uncertainty relation. Squeezed light has potential applications in precision measurements and noiseless communications [11,12]. Squeezed light can be generated by quantum optical processes such as parametric oscillation [3-7], second harmonic generation [1,13,14], and four-wave mixing [13-15].

It has been also found that a three-level laser can generate squeezed light under certain conditions [16-21]. In such a laser, three-level atoms in a cascade configuration are injected at a constant rate into a cavity coupled to a vacuum reservoir via a single-port mirror. When a three-level atom makes a transition from the top to bottom level via the intermediate level, two photons are generated. The two photons are highly correlated and this correlation is responsible for the squeezing of the light generated by a three-level laser.

In this thesis, we consider a nondegenerate three-level laser, with the cavity modes driven by coherent light. We first drive the master equation in the linear and adiabatic approximations. Then using this master equation, we obtain stochastic differential equations. Applying the solutions of the resulting differential equations, we calculate the quadrature variance, the squeezing spectrum, the mean and variance of the photon number sum and difference.

Chapter 2

Stochastic Differential Equations and Their Solutions

The first two sections of this chapter focus on developing the master equation and the stochastic differential equations. In the last section, we apply matrix method to find solutions of the differential equations.

2.1 The Master Equation

In a nondegenerate three-level laser, nondegenerate three-level atoms in a cascade configuration are injected at a constant rate r_a, and removed from the cavity after a certain time τ. We denote the top, intermediate and bottom levels of a three-level atom by $|a\rangle, |b\rangle$, and $|c\rangle$, respectively. In addition, we assume the cavity mode to be at resonance with two transitions $|a\rangle \to |b\rangle$ and $|b\rangle \to |c\rangle$, with direct transition between levels $|a\rangle$ and $|c\rangle$ to be dipole forbidden. The Hamiltonian describing the interaction of a three-level atom with the cavity modes is expressible as

$$\hat{H}_I = ig[|a\rangle\langle b|\hat{a} - \hat{a}^\dagger|b\rangle\langle a| + |b\rangle\langle c|\hat{b} - \hat{b}^\dagger|c\rangle\langle b|] \tag{2.1.1}$$

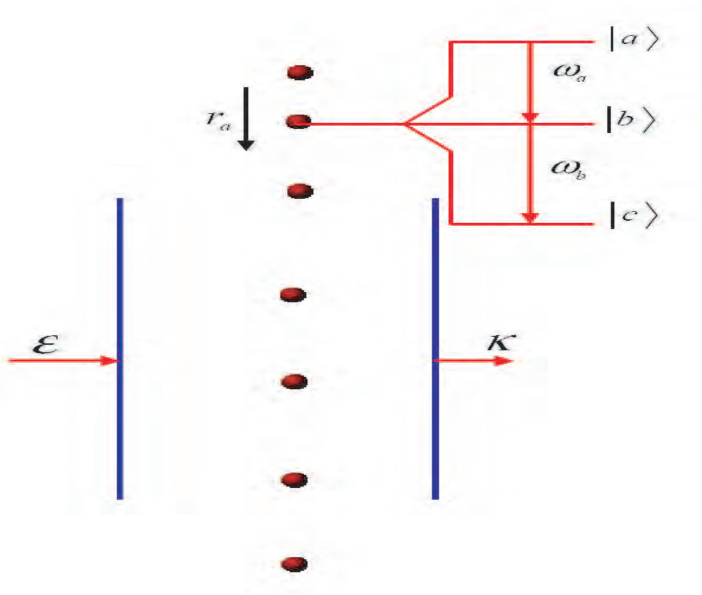


Fig. 2.1 A nondegenerate three-level laser with the cavity modes driven by coherent light.

where $\hat{a}$ and $\hat{b}$ are the annihilation operators for the cavity modes and g is the coupling constant between the three-level atom and the cavity modes. We take the initial state of a single three-level atom to be

$$|\psi_A(0)\rangle = C_a|a\rangle + C_c|c\rangle \tag{2.1.2}$$

and hence the density operator of a single atom is

$$\hat{\rho}_A(0) = \rho_{aa}^{(0)}|a\rangle\langle a| + \rho_{ac}^{(0)}|a\rangle\langle c| + \rho_{ca}^{(0)}|c\rangle\langle a| + \rho_{cc}^{(0)}|c\rangle\langle c| \tag{2.1.3}$$

where

$$\rho_{aa}^{(0)} = C_a^* C_a, \tag{2.1.4}$$

$$\rho_{ac}^{(0)} = C_a C_c^*, \tag{2.1.5}$$

$$\rho_{ca}^{(0)} = C_c C_a^*, \tag{2.1.6}$$

$$\rho_{cc}^{(0)} = C_c^* C_c. \tag{2.1.7}$$

Suppose $\hat{\rho}_{AR}(t, t_j)$ is the density operator for a single atom plus the cavity modes at time t, with the atom injected at time t_j such that $(t - \tau) \leq t_j \leq t$. The density operator for all atoms in the cavity plus the cavity mode at time t can then be written as

$$\hat{\rho}_{AR}(t) = r_a \sum_j \hat{\rho}_{AR}(t, t_j) \triangle t_j, \tag{2.1.8}$$

where $r_a \triangle t_j$ represents the number of atoms injected into the cavity in a time $\triangle t_j$. Now converting the summation into integration in the limit $\triangle t_j \to 0$, we have

$$\hat{\rho}_{AR}(t) = r_a \int_{t-\tau}^{t} \hat{\rho}_{AR}(t, t_j) \triangle t_j \tag{2.1.9}$$

and on differentiating with respect to t, there follows

$$\frac{d}{dt}\hat{\rho}_{AR}(t) = r_a[\hat{\rho}_{AR}(t,t) - \hat{\rho}_{AR}(t, t-\tau)] + r_a \int_{t-\tau}^{t} \frac{\partial}{\partial t}\hat{\rho}_{AR}(t, t')dt'. \tag{2.1.10}$$

We observe that $\hat{\rho}_{AR}(t, t)$ is the density operator for the cavity modes plus an atom injected at time t. This operator can thus be expressed as

$$\hat{\rho}_{AR}(t, t) = \hat{\rho}_A(t)\hat{\rho}(t), \tag{2.1.11}$$

with $\hat{\rho}(t)$ being the density operator for the cavity modes alone. We also note that $\hat{\rho}_{AR}(t, t-\tau)$ represents the density operator for an atom plus the cavity modes at time t, with the atom being removed from the cavity at this time. This operator can also be put in the form

$$\hat{\rho}_{AR}(t, t-\tau) = \hat{\rho}_A(t-\tau)\hat{\rho}(t). \tag{2.1.12}$$

Now in view of Eqs. (2.1.11) and (2.1.12), one can rewrite Eq. (2.1.10) as

$$\frac{d}{dt}\hat{\rho}_{AR}(t) = r_a[\hat{\rho}_A(t) - \hat{\rho}_A(t-\tau)]\hat{\rho}(t) + r_a\int_{t-\tau}^{t}\frac{\partial}{\partial t}\hat{\rho}_{AR}(t,t')dt'. \tag{2.1.13}$$

The density operator evolves in time according to

$$\frac{\partial}{\partial t}\hat{\rho}_{AR}(t,t') = -i[\hat{H}_I, \hat{\rho}_{AR}(t,t')], \tag{2.1.14}$$

so that using this and taking into account Eq. (2.1.9), one can put Eq. (2.1.13) in the form

$$\frac{d}{dt}\hat{\rho}_{AR}(t) = r_a[\hat{\rho}_A(t) - \hat{\rho}_A(t-\tau)]\hat{\rho}(t) - i[\hat{H}_I, \hat{\rho}_{AR}(t)]. \tag{2.1.15}$$

Furthermore, tracing over the atomic variables and taking into account the damping of the cavity modes by a vacuum reservoir [1], we have

$$\begin{aligned}\frac{d\hat{\rho}}{dt} &= -iTr_A[\hat{H}_I, \hat{\rho}_{AR}(t,t')] + \frac{1}{2}\kappa(2\hat{a}\hat{\rho}\hat{a}^\dagger - \hat{\rho}\hat{a}^\dagger\hat{a} - \hat{a}^\dagger\hat{a}\hat{\rho}) \\ &+ \frac{1}{2}\kappa(2\hat{b}\hat{\rho}\hat{b}^\dagger - \hat{\rho}\hat{b}^\dagger\hat{b} - \hat{b}^\dagger\hat{b}\hat{\rho}),\end{aligned} \tag{2.1.16}$$

where we have used the fact that

$$Tr\hat{\rho}_A(t) = Tr\hat{\rho}_A(t-\tau) = 1 \tag{2.1.17}$$

and κ is the cavity damping constant. Employing Eq. (2.1.1), the master equation for the cavity modes, can be put in the form

$$\begin{aligned}\frac{d\hat{\rho}}{dt} &= (\hat{\rho}_{ab}\hat{a}^\dagger - \hat{a}^\dagger\hat{\rho}_{ab} + \hat{\rho}_{bc}\hat{b}^\dagger - \hat{b}^\dagger\hat{\rho}_{bc} + \hat{a}\hat{\rho}_{ba} - \hat{\rho}_{ba}\hat{a} + \hat{b}\hat{\rho}_{cb} - \hat{\rho}_{cb}\hat{b}) \\ &+ \frac{1}{2}\kappa(2\hat{a}\hat{\rho}\hat{a}^\dagger - \hat{\rho}\hat{a}^\dagger\hat{a} - \hat{a}^\dagger\hat{a}\hat{\rho}) + \frac{1}{2}\kappa(2\hat{b}\hat{\rho}\hat{b}^\dagger - \hat{\rho}\hat{b}^\dagger\hat{b} - \hat{b}^\dagger\hat{b}\hat{\rho}),\end{aligned} \tag{2.1.18}$$

in which the matrix element $\hat{\rho}_{\alpha\beta}$ is defined by

$$\hat{\rho}_{\alpha\beta} = \langle\alpha|\hat{\rho}_{AR}|\beta\rangle, \tag{2.1.19}$$

with $\alpha, \beta = a, b, c$. On the other hand, we see from Eq. (2.1.15) that

$$\frac{d\hat{\rho}_{\alpha\beta}}{dt} = [r_a\langle\alpha|\hat{\rho}_A(t)|\beta\rangle - r_a\langle\alpha|\hat{\rho}_A(t-\tau)|\beta\rangle]\hat{\rho}(t) - i[\langle\alpha|\hat{H}_I\hat{\rho}_{AR}|\beta\rangle - \langle\alpha|\hat{\rho}_{AR}\hat{H}_I|\beta\rangle] - \gamma\hat{\rho}_{\alpha\beta}, \tag{2.1.20}$$

where the last term is included to account for the decay of atoms due to spontaneous emission. Here γ, considered to be the same for all the three-levels, is the atomic decay rate. We assume that the atoms are removed from the cavity after they have decayed to a level other than the intermediate or the bottom level [1]. We then see that

$$\langle\alpha|\hat{\rho}_A(t-\tau)|\beta\rangle = 0 \tag{2.1.21}$$

and hence Eq. (2.1.20) reduces to

$$\frac{d\hat{\rho}_{\alpha\beta}}{dt} = r_a\langle\alpha|\hat{\rho}_A(t)|\beta\rangle\hat{\rho}(t) - i[\langle\alpha|\hat{H}_I\hat{\rho}_{AR}|\beta\rangle - \langle\alpha|\hat{\rho}_{AR}\hat{H}_I|\beta\rangle] - \gamma\hat{\rho}_{\alpha\beta}, \tag{2.1.22}$$

Applying this equation and taking into account Eqs. (2.1.1) and (2.1.2), one readily obtains

$$\frac{d\hat{\rho}_{ab}}{dt} = g(\hat{\rho}_{ac}\hat{b}^\dagger + \hat{a}\hat{\rho}_{bb} - \hat{\rho}_{aa}\hat{a}) - \gamma\hat{\rho}_{ab}, \tag{2.1.23}$$

$$\frac{d\hat{\rho}_{bc}}{dt} = g(\hat{b}\hat{\rho}_{cc} - \hat{\rho}_{bb}\hat{b} - \hat{a}^\dagger\hat{\rho}_{ac}) - \gamma\hat{\rho}_{bc}, \tag{2.1.24}$$

$$\frac{d\hat{\rho}_{aa}}{dt} = r_a\rho_{aa}^{(0)}\hat{\rho}(t) + g(\hat{a}\hat{\rho}_{ba} + \hat{\rho}_{ab}\hat{a}^\dagger) - \gamma\hat{\rho}_{aa}, \tag{2.1.25}$$

$$\frac{d\hat{\rho}_{bb}}{dt} = g(\hat{b}\hat{\rho}_{cb} - \hat{\rho}_{ba}\hat{a} - \hat{a}^\dagger\hat{\rho}_{ab}) - \gamma\hat{\rho}_{bb}, \tag{2.1.26}$$

$$\frac{d\hat{\rho}_{ac}}{dt} = r_a\rho_{ac}^{(0)}\hat{\rho}(t) + g(\hat{a}\hat{\rho}_{bc} - \hat{\rho}_{ab}\hat{b}) - \gamma\hat{\rho}_{ac}, \tag{2.1.27}$$

$$\frac{d\hat{\rho}_{cc}}{dt} = r_a\rho_{cc}^{(0)}\hat{\rho}(t) - g(\hat{b}^\dagger\hat{\rho}_{bc} + \hat{\rho}_{cb}\hat{b}) - \gamma\hat{\rho}_{cc}. \tag{2.1.28}$$

We confine ourselves in a linear analysis and this can be achieved by dropping the g-terms in Eqs. (2.1.25) - (2.1.28) and imposing the condition that $\kappa \ll \gamma$ (the good-cavity limit). Under this condition, the cavity mode variables change slowly compared with the atomic variables. Since the atomic variables reach steady state in the relatively short period of γ^{-1}, one can take the time derivatives of such variables to be zero, while keeping the zero-order atomic and cavity mode variables at time t. This procedure may be referred to as the adiabatic approximation scheme. Thus upon dropping the g-terms and applying the adiabatic approximation scheme [1] in Eqs. (2.1.25) - (2.1.28), we get

$$\hat{\rho}_{aa} = \frac{r_a\rho_{aa}^{(0)}}{\gamma}\hat{\rho}(t), \tag{2.1.29}$$

$$\hat{\rho}_{bb} = 0, \tag{2.1.30}$$

$$\hat{\rho}_{ac} = \frac{r_a\rho_{ac}^{(0)}}{\gamma}\hat{\rho}(t), \tag{2.1.31}$$

$$\hat{\rho}_{cc} = \frac{r_a\rho_{cc}^{(0)}}{\gamma}\hat{\rho}(t). \tag{2.1.32}$$

Now combination of Eqs. (2.1.23), (2.1.29), (2.1.30), and (2.1.31) as well as Eqs. (2.1.24), (2.1.30), (2.1.31), and (2.1.32) leads to

$$\frac{d\hat{\rho}_{ab}}{dt} = \frac{gr_a}{\gamma}(\hat{\rho}_{ac}^{(0)}\hat{\rho}\hat{b}^\dagger + \hat{\rho}_{aa}^{(0)}\hat{\rho}\hat{a}) - \gamma\hat{\rho}_{ab}, \tag{2.1.33}$$

$$\frac{d\hat{\rho}_{bc}}{dt} = \frac{gr_a}{\gamma}(\hat{\rho}_{cc}^{(0)}\hat{b}\hat{\rho} - \hat{\rho}_{ac}^{(0)}\hat{a}^\dagger\hat{\rho}) - \gamma\hat{\rho}_{bc}. \tag{2.1.34}$$

Using once more the adiabatic approximation scheme, we easily find

$$\hat{\rho}_{ab} = \frac{gr_a}{\gamma^2}(\hat{\rho}_{ac}^{(0)}\hat{\rho}\hat{b}^\dagger + \hat{\rho}_{aa}^{(0)}\hat{\rho}\hat{a}), \tag{2.1.35}$$

$$\hat{\rho}_{bc} = \frac{gr_a}{\gamma^2}(\hat{\rho}_{cc}^{(0)}\hat{b}\hat{\rho} - \hat{\rho}_{ac}^{(0)}\hat{a}^\dagger\hat{\rho}). \tag{2.1.36}$$

Finally, on account of Eqs. (2.1.35) and (2.1.36), the master equation for the cavity modes turns out to be

$$\begin{aligned}\frac{d\hat{\rho}}{dt} &= Q(2\hat{a}^\dagger\hat{\rho}\hat{a} - \hat{a}\hat{a}^\dagger\hat{\rho} - \hat{\rho}\hat{a}\hat{a}^\dagger) + R(2\hat{b}\hat{\rho}\hat{b}^\dagger - \hat{\rho}\hat{b}^\dagger\hat{b} - \hat{b}^\dagger\hat{b}\hat{\rho}) \\ &+ S(\hat{\rho}\hat{b}^\dagger\hat{a}^\dagger + \hat{b}^\dagger\hat{a}^\dagger\hat{\rho} - 2\hat{a}^\dagger\hat{\rho}\hat{b}^\dagger) + T(\hat{a}\hat{b}\hat{\rho} + \hat{\rho}\hat{a}\hat{b} - 2\hat{b}\hat{\rho}\hat{a}) \\ &+ U(2\hat{a}\hat{\rho}\hat{a}^\dagger - \hat{\rho}\hat{a}^\dagger\hat{a} - \hat{a}^\dagger\hat{a}\hat{\rho}) + U(2\hat{b}\hat{\rho}\hat{b}^\dagger - \hat{\rho}\hat{b}^\dagger\hat{b} - \hat{b}^\dagger\hat{b}\hat{\rho}), \qquad (2.1.37)\end{aligned}$$

where

$$Q = \frac{1}{2}A\rho_{aa}^{(0)}, \qquad (2.1.38)$$

$$R = \frac{1}{2}A\rho_{cc}^{(0)}, \qquad (2.1.39)$$

$$S = \frac{1}{2}A\rho_{ac}^{(0)}, \qquad (2.1.40)$$

$$T = \frac{1}{2}A\rho_{ca}^{(0)}, \qquad (2.1.41)$$

$$U = \frac{1}{2}\kappa, \qquad (2.1.42)$$

$$A = \frac{2g^2 r_a}{\gamma^2} \qquad (2.1.43)$$

with A being the linear gain coefficient. In addition, the interaction of the cavity modes and coherent driving modes, treated classically, can be described by

$$\hat{H}_s = i\varepsilon(\hat{a}^\dagger - \hat{a} + \hat{b}^\dagger - \hat{b}), \qquad (2.1.44)$$

where ε is a constant proportional to the amplitude of the driving mode. The equation of evolution of the density operator associated with this Hamiltonian has the form

$$\frac{d\hat{\rho}}{dt} = -\varepsilon(\hat{a}\hat{\rho} - \hat{\rho}\hat{a} - \hat{a}^\dagger\hat{\rho} + \hat{\rho}\hat{a}^\dagger + \hat{b}\hat{\rho} - \hat{\rho}\hat{b} + \hat{b}^\dagger\hat{\rho} + \hat{\rho}\hat{b}^\dagger). \qquad (2.1.45)$$

On account of Eqs. (2.1.37) and (2.1.45) the master equation for a nondegenerate three-level laser, with the cavity modes driven by coherent light modes and coupled to a two-mode vacuum reservoir can be written as

$$\begin{aligned}\frac{d\hat{\rho}}{dt} &= -\varepsilon(\hat{a}\hat{\rho} - \hat{\rho}\hat{a} - \hat{a}^\dagger\hat{\rho} + \hat{\rho}\hat{a}^\dagger + \hat{b}\hat{\rho} - \hat{\rho}\hat{b} + \hat{b}^\dagger\hat{\rho} + \hat{\rho}\hat{b}^\dagger) \\ &+ Q(2\hat{a}^\dagger\hat{\rho}\hat{a} - \hat{a}\hat{a}^\dagger\hat{\rho} - \hat{\rho}\hat{a}\hat{a}^\dagger) + R(2\hat{b}\hat{\rho}\hat{b}^\dagger - \hat{\rho}\hat{b}^\dagger\hat{b} - \hat{b}^\dagger\hat{b}\hat{\rho}) \\ &+ S(\hat{\rho}\hat{b}^\dagger\hat{a}^\dagger + \hat{b}^\dagger\hat{a}^\dagger\hat{\rho} - 2\hat{a}^\dagger\hat{\rho}\hat{b}^\dagger) + T(\hat{a}\hat{b}\hat{\rho} + \hat{\rho}\hat{a}\hat{b} - 2\hat{b}\hat{\rho}\hat{a}) \\ &+ U(2\hat{a}\hat{\rho}\hat{a}^\dagger - \hat{\rho}\hat{a}^\dagger\hat{a} - \hat{a}^\dagger\hat{a}\hat{\rho}) + U(2\hat{b}\hat{\rho}\hat{b}^\dagger - \hat{\rho}\hat{b}^\dagger\hat{b} - \hat{b}^\dagger\hat{b}\hat{\rho}). \end{aligned} \tag{2.1.46}$$

2.2 The Stochastic Differential Equations

We next proceed to determine the stochastic differential equations for the cavity mode variables. To this end, applying the relation

$$\frac{d}{dt}\langle A\rangle = \frac{d}{dt}Tr(\hat{\rho}\hat{A}) \tag{2.2.1}$$

along with Eq. (2.1.46), one readily finds

$$\frac{d}{dt}\langle\hat{a}\rangle = \varepsilon - (U - Q)\langle\hat{a}\rangle - S\langle\hat{b}^\dagger\rangle, \tag{2.2.2}$$

$$\frac{d}{dt}\langle\hat{a}^2\rangle = 2\varepsilon\langle\hat{a}\rangle - 2(U - Q)\langle\hat{a}^2\rangle - 2S\langle\hat{a}\hat{b}^\dagger\rangle, \tag{2.2.3}$$

$$\frac{d}{dt}\langle\hat{a}^\dagger\hat{a}\rangle = 2Q + \varepsilon(\langle\hat{a}^\dagger\rangle + \langle\hat{a}\rangle) - 2(U - Q)\langle\hat{a}^\dagger\hat{a}\rangle - S\langle\hat{a}^\dagger\hat{b}^\dagger\rangle - T\langle\hat{a}\hat{b}\rangle, \tag{2.2.4}$$

$$\frac{d}{dt}\langle\hat{b}\rangle = \varepsilon - (U + R)\langle\hat{b}\rangle + S\langle\hat{a}^\dagger\rangle, \tag{2.2.5}$$

$$\frac{d}{dt}\langle\hat{b}^2\rangle = 2\varepsilon\langle\hat{b}\rangle - 2(U + R)\langle\hat{b}^2\rangle + 2S\langle\hat{a}^\dagger\hat{b}\rangle, \tag{2.2.6}$$

$$\frac{d}{dt}\langle\hat{b}^\dagger\hat{b}\rangle = \varepsilon(\langle\hat{b}^\dagger\rangle + \langle\hat{b}\rangle) - 2(U + R)\langle\hat{b}^\dagger\hat{b}\rangle + S\langle\hat{a}^\dagger\hat{b}^\dagger\rangle + T\langle\hat{a}\hat{b}\rangle, \tag{2.2.7}$$

$$\frac{d}{dt}\langle\hat{a}\hat{b}\rangle = \varepsilon(\langle\hat{a}^\dagger\rangle + \langle\hat{b}\rangle) - (2U + R - Q)\langle\hat{a}\hat{b}\rangle - S\langle\hat{b}^\dagger\hat{b}\rangle + S\langle\hat{a}^\dagger\hat{a}\rangle + S, \tag{2.2.8}$$

$$\frac{d}{dt}\langle \hat{a}^\dagger \hat{b}\rangle = \varepsilon(\langle \hat{a}^\dagger\rangle + \langle \hat{b}\rangle) - (2U + R - Q)\langle \hat{a}^\dagger \hat{b}\rangle + S\langle \hat{a}^{\dagger 2}\rangle - T\langle \hat{b}^2\rangle. \tag{2.2.9}$$

We see that Eqs. (2.2.2) - (2.2.9) are in normal order. By taking $\hat{a} \to \alpha$, $\hat{a}^\dagger \to \alpha^*$, $\hat{b} \to \beta$, and $\hat{b}^\dagger \to \beta^*$, the corresponding c-number equations are

$$\frac{d}{dt}\langle \alpha\rangle = \varepsilon - (U - Q)\langle \alpha\rangle - S\langle \beta^*\rangle, \tag{2.2.10}$$

$$\frac{d}{dt}\langle \alpha^2\rangle = 2\varepsilon\langle \alpha\rangle - 2(U - Q)\langle \alpha^2\rangle - 2S\langle \alpha\beta^*\rangle, \tag{2.2.11}$$

$$\frac{d}{dt}\langle \alpha^*\alpha\rangle = 2Q + \varepsilon(\langle \alpha^*\rangle + \langle \alpha\rangle) - 2(U - Q)\langle \alpha^*\alpha\rangle - S\langle \alpha^*\beta^*\rangle - T\langle \alpha\beta\rangle, \tag{2.2.12}$$

$$\frac{d}{dt}\langle \beta\rangle = \varepsilon - (U + R)\langle \beta\rangle + S\langle \alpha^*\rangle, \tag{2.2.13}$$

$$\frac{d}{dt}\langle \beta^2\rangle = 2\varepsilon\langle \beta\rangle - 2(U + R)\langle \beta^2\rangle + 2S\langle \alpha^*\beta\rangle, \tag{2.2.14}$$

$$\frac{d}{dt}\langle \beta^*\beta\rangle = \varepsilon(\langle \beta^*\rangle + \langle \beta\rangle) - 2(U + R)\langle \beta^*\beta\rangle + S\langle \alpha^*\beta^*\rangle + T\langle \alpha\beta\rangle, \tag{2.2.15}$$

$$\frac{d}{dt}\langle \alpha\beta\rangle = \varepsilon(\langle \alpha\rangle + \langle \beta\rangle) - (2U + R - Q)\langle \alpha\beta\rangle - S\langle \beta * \beta\rangle + S\langle \alpha^*\alpha\rangle + S, \tag{2.2.16}$$

$$\frac{d}{dt}\langle \alpha^*\beta\rangle = \varepsilon(\langle \alpha^*\rangle + \langle \beta\rangle) - (2U + R - Q)\langle \alpha^*\beta\rangle + S\langle \alpha^{*2}\rangle - T\langle \beta^2\rangle. \tag{2.2.17}$$

On account of Eqs. (2.2.10) and (2.2.13), one can write the stochastic differential equations

$$\frac{d}{dt}\alpha = \varepsilon - (U - Q)\alpha - S\beta^* + f_\alpha(t), \tag{2.2.18}$$

$$\frac{d}{dt}\beta^* = \varepsilon - (U + R)\beta^* + S\alpha + f_\beta^*(t), \tag{2.2.19}$$

where $f_\alpha(t)$ and $f_\beta(t)$ are noise forces the properties of which remain to be determined. If we compare Eq. (2.2.10) and the expectation value of Eq. (2.2.18), we can see that

$$\langle f_\alpha(t)\rangle = 0. \tag{2.2.20}$$

In addition, comparing Eq. (2.2.13) and the expectation value of Eq. (2.2.19), we also see that

$$\langle f_\beta(t)\rangle = 0. \tag{2.2.21}$$

Moreover, using Eqs. (2.2.18) and (2.2.19), one readily finds

$$\frac{d}{dt}\langle \alpha^2\rangle = 2\varepsilon\langle\alpha\rangle - 2(U-Q)\langle\alpha^2\rangle - 2S\langle\alpha\beta^*\rangle + 2\langle\alpha f_\alpha(t)\rangle, \tag{2.2.22}$$

$$\frac{d}{dt}\langle \beta^2\rangle = 2\varepsilon\langle\beta\rangle - 2(U+R)\langle\beta^2\rangle + 2S\langle\alpha^*\beta\rangle + 2\langle\beta f_\beta(t)\rangle, \tag{2.2.23}$$

$$\begin{aligned}\frac{d}{dt}\langle\alpha^*\alpha\rangle &= \varepsilon(\langle\alpha^*\rangle + \langle\alpha\rangle) - 2(U-Q)\langle\alpha^*\alpha\rangle - S\langle\alpha^*\beta^*\rangle - T\langle\alpha\beta\rangle \\ &+ \langle\alpha f_\alpha^*(t)\rangle + \langle\alpha^* f_\alpha(t)\rangle,\end{aligned} \tag{2.2.24}$$

$$\begin{aligned}\frac{d}{dt}\langle\beta^*\beta\rangle &= \varepsilon(\langle\beta^*\rangle + \langle\beta\rangle) - 2(U+R)\langle\beta^*\beta\rangle + S\langle\alpha^*\beta^*\rangle + T\langle\alpha\beta\rangle \\ &+ \langle\beta f_\beta^*(t)\rangle + \langle\beta^* f_\beta(t)\rangle,\end{aligned} \tag{2.2.25}$$

$$\begin{aligned}\frac{d}{dt}\langle\alpha\beta\rangle &= \varepsilon(\langle\alpha\rangle + \langle\beta\rangle) - (2U+R-Q)\langle\alpha\beta\rangle - S\langle\beta*\beta\rangle + S\langle\alpha^*\alpha\rangle \\ &+ \langle\alpha f_\beta(t)\rangle + \langle\beta f_\alpha(t)\rangle,\end{aligned} \tag{2.2.26}$$

$$\begin{aligned}\frac{d}{dt}\langle\alpha^*\beta\rangle &= \varepsilon(\langle\alpha^*\rangle + \langle\beta\rangle) - (2U+R-Q)\langle\alpha^*\beta\rangle + S\langle\alpha^{*2}\rangle - T\langle\beta^2\rangle \\ &+ \langle\alpha^* f_\beta(t)\rangle + \langle\beta f_\alpha^*(t)\rangle.\end{aligned} \tag{2.2.27}$$

Comparing Eqs. (2.2.11), (2.2.12), (2.2.14), (2.2.15), (2.2.16) and (2.2.17) with Eqs. (2.2.22) - (2.2.27), one readily finds

$$\langle\alpha f_\alpha(t)\rangle = 0, \tag{2.2.28}$$

$$\langle\alpha f_\alpha^*(t)\rangle + \langle\alpha^* f_\alpha(t)\rangle = 2Q, \tag{2.2.29}$$

$$\langle \beta f_\beta(t) \rangle = 0, \tag{2.2.30}$$

$$\langle \beta f_\beta^*(t) \rangle + \langle \beta^* f_\beta(t) \rangle = 0, \tag{2.2.31}$$

$$\langle \alpha f_\beta(t) \rangle + \langle \beta f_\alpha(t) \rangle = S, \tag{2.2.32}$$

$$\langle \alpha^* f_\beta(t) \rangle + \langle \beta f_\alpha^*(t) \rangle = 0. \tag{2.2.33}$$

Furthermore, the solution of Eq. (2.2.18) can be written as

$$\alpha(t) = \alpha(0)e^{-(U-Q)t} + \int_0^t e^{-(U-Q)(t-t')}[\varepsilon - T\beta^*(t') + f_\alpha(t')]dt'. \tag{2.2.34}$$

Multiplying both side of Eq. (2.2.34) by $f_\alpha^*(t)$ and taking the expectation value of the resulting expression, we have

$$\begin{aligned} \langle \alpha(t) f_\alpha^*(t) \rangle &= \langle \alpha(0) f_\alpha^*(t) \rangle e^{-(U-Q)t} + \int_0^t e^{-(U-Q)(t-t')}[\varepsilon \langle f_\alpha^*(t) \rangle - T\langle \beta^*(t') f_\alpha^*(t) \rangle \\ &+ \langle f_\alpha(t') f_\alpha^*(t) \rangle]dt'. \end{aligned} \tag{2.2.35}$$

Applying Eq. (2.2.20) and using the fact that the noise force at time t doesn't affect the cavity mode variables at earlier time [1], we get

$$\langle \alpha(t) f_\alpha^*(t) \rangle = \int_0^t e^{-(U-Q)(t-t')} \langle f_\alpha(t') f_\alpha^*(t) \rangle dt'. \tag{2.2.36}$$

Taking the complex conjugate of this equation, we get

$$\langle \alpha^*(t) f_\alpha(t) \rangle = \int_0^t e^{-(U-Q)(t-t')} \langle f_\alpha^*(t') f_\alpha(t) \rangle dt'. \tag{2.2.37}$$

Adding Eqs. (2.2.36) and (2.2.37) and using Eq. (2.2.29), we find

$$\int_0^t e^{-(U-Q)(t-t')}[\langle f_\alpha(t') f_\alpha^*(t) \rangle + \langle f_\alpha^*(t') f_\alpha(t) \rangle]dt' = 2Q. \tag{2.2.38}$$

We assume

$$\langle f_\alpha(t') f_\alpha^*(t) \rangle = \langle f_\alpha^*(t') f_\alpha(t) \rangle. \tag{2.2.39}$$

In view of this assumption, Eq. (2.2.38) turns out to be

$$\int_0^t e^{-(U-Q)(t-t')}\langle f_\alpha(t')f_\alpha^*(t)\rangle dt' = Q. \tag{2.2.40}$$

Based of the resulting expression, we can write [1,15]

$$\langle f_\alpha(t')f_\alpha^*(t)\rangle = 2Q\delta(t-t'). \tag{2.2.41}$$

Furthermore, multiplying both side of Eq. (2.2.34) by $f_\beta(t)$ and taking the expectation value of the result, we have

$$\begin{aligned}\langle\alpha(t)f_\beta(t)\rangle &= \langle\alpha(0)f_\beta(t)\rangle e^{-(U-Q)t} + \int_0^t e^{-(U-Q)(t-t')}[\varepsilon\langle f_\beta(t)\rangle - T\langle\beta^*(t')f_\beta(t)\rangle \\ &+ \langle f_\alpha(t')f_\beta(t)\rangle]dt'. \end{aligned}\tag{2.2.42}$$

Applying Eq. (2.2.21) and using the fact that the noise force at time t doesn't affect the cavity mode variables at earlier time, we find

$$\langle\alpha(t)f_\beta(t)\rangle = \int_0^t e^{-(U-Q)(t-t')}\langle f_\alpha(t')f_\beta(t)\rangle dt'. \tag{2.2.43}$$

The formal solution of the complex conjugate of Eq. (2.2.19) has the form

$$\beta(t) = \beta(0)e^{-(U+R)t} + \int_0^t e^{-(U+R)(t-t')}[\varepsilon + S\alpha^*(t') + f_\beta(t')]dt'. \tag{2.2.44}$$

Multiplying Eq. (2.2.44) by $f_\alpha(t)$ and taking the expectation value of the resulting expression, we get

$$\begin{aligned}\langle\beta(t)f_\alpha(t)\rangle &= \langle\beta(0)f_\alpha(t)\rangle e^{-(U+R)t} + \int_0^t e^{-(U+R)(t-t')}[\varepsilon\langle f_\alpha(t)\rangle - S\langle\alpha^*(t')f_\alpha(t)\rangle \\ &+ \langle f_\beta(t')f_\alpha(t)\rangle]dt'. \end{aligned}\tag{2.2.45}$$

Applying Eq. (2.2.20) and using the fact that the noise force at time t doesn't affect the cavity mode variables at earlier time, we obtain

$$\langle\beta(t)f_\alpha(t)\rangle = \int_0^t e^{-(U+R)(t-t')}\langle f_\beta(t')f_\alpha(t)\rangle dt'. \tag{2.2.46}$$

Adding Eqs. (2.2.43) and (2.2.46) and applying Eq. (2.2.32), we get

$$\int_0^t e^{-(U-Q)(t-t')}\langle f_\alpha(t')f_\beta(t)\rangle dt' + \int_0^t e^{-(U+R)(t-t')}\langle f_\beta(t')f_\alpha(t)\rangle dt' = S. \tag{2.2.47}$$

We assume

$$\langle f_\alpha(t')f_\beta(t)\rangle = \langle f_\beta(t')f_\alpha(t)\rangle. \tag{2.2.48}$$

On account of this assumption, we can see that

$$\int_0^t [e^{-(U-Q)(t-t')} + e^{-(U+R)(t-t')}]\langle f_\beta(t')f_\alpha(t)\rangle dt' = S. \tag{2.2.49}$$

Based on the resulting expression, we can write [1,15]

$$\langle f_\alpha(t')f_\beta(t)\rangle = \langle f_\beta(t')f_\alpha(t)\rangle = S\delta(t-t'). \tag{2.2.50}$$

It can also be established in a similar manner that

$$\langle f_\alpha(t')f_\alpha(t)\rangle = 0, \tag{2.2.51}$$

$$\langle f_\beta(t')f_\beta(t)\rangle = 0, \tag{2.2.52}$$

$$\langle f_\beta(t')f^*_\beta(t)\rangle = 0, \tag{2.2.53}$$

$$\langle f^*_\alpha(t')f_\beta(t)\rangle = \langle f^*_\alpha(t)f_\beta(t')\rangle = 0. \tag{2.2.54}$$

2.3 Solutions of the Stochastic Differential Equations

In the previous section we have found two stochastic differential equations for $\alpha(t)$ and $\beta^*(t)$. In this section we seek to obtain the solutions of these coupled stochastic differential equations using matrix method. On account of Eqs. (2.2.18) and (2.2.19) we can write the matrix equation as

$$\frac{dJ(t)}{dt} = MJ(t) + E(t), \tag{2.3.1}$$

where

$$J(t) = \begin{pmatrix} \alpha(t) \\ \beta^*(t) \end{pmatrix}, \tag{2.3.2}$$

$$M = \begin{pmatrix} Q - U & -S \\ T & -(U + R) \end{pmatrix} \tag{2.3.3}$$

and

$$E(t) = \begin{pmatrix} f_\alpha(t) + \varepsilon \\ f^*_\beta(t) + \varepsilon \end{pmatrix}. \tag{2.3.4}$$

To solve Eq. (2.3.1), we need to find the eigenvalues and eigenvectors of M such that

$$MV' = \lambda V' \tag{2.3.5}$$

where

$$V' = \begin{pmatrix} x \\ y \end{pmatrix}, \tag{2.3.6}$$

with the normalization condition

$$x^2 + y^2 = 1. \tag{2.3.7}$$

Equation (2.3.5) can be rewritten as

$$(M - \lambda I)V' = 0 \tag{2.3.8}$$

where I is an identity operator. Equation (2.3.8) has a non-trivial solution provided that

$$|M - \lambda I| = 0. \tag{2.3.9}$$

This implies that

$$\lambda^2 + (2U + R - Q)\lambda + ((U - Q)(U + R) + TS) = 0. \tag{2.3.10}$$

Solving this quadratic equation for λ, the eigen values of the matrix M to be

$$\lambda_1 = \frac{-(2U + R - Q) + \sqrt{(2U + R - Q)^2 - 4((U - Q)(U + R) + TS)}}{2} \tag{2.3.11}$$

and

$$\lambda_2 = \frac{-(2U + R - Q) - \sqrt{(2U + R - Q)^2 - 4((U - Q)(U + R) + TS)}}{2}. \tag{2.3.12}$$

Using Eqs. (2.3.5) and (2.3.6) along with the normalization condition given by Eq. (2.3.7) , we easily find the corresponding eigen vectors to be

$$V_1' = \frac{1}{\sqrt{S^2 + (Q - U - \lambda_1)^2}} \begin{pmatrix} S \\ Q - U - \lambda_1 \end{pmatrix} \tag{2.3.13}$$

and

$$V_2' = \frac{1}{\sqrt{S^2 + (Q - U - \lambda_2)^2}} \begin{pmatrix} S \\ Q - U - \lambda_2 \end{pmatrix}. \tag{2.3.14}$$

We construct a matrix V consisting of the eigenvectors of M as a column matrix

$$V = (V_1' V_2'). \tag{2.3.15}$$

We then find

$$V = \begin{pmatrix} \frac{S}{\sqrt{S^2+(Q-U-\lambda_1)^2}} & \frac{S}{\sqrt{S^2+(Q-U-\lambda_2)^2}} \\ \frac{Q-U-\lambda_1}{\sqrt{S^2+(Q-U-\lambda_1)^2}} & \frac{Q-U-\lambda_2}{\sqrt{S^2+(Q-U-\lambda_2)^2}} \end{pmatrix}. \tag{2.3.16}$$

The inverse of this matrix has the form

$$V^{-1} = \begin{pmatrix} \frac{-(Q-U-\lambda_2)\sqrt{S^2+(Q-U-\lambda_1)^2}}{S\lambda} & \frac{\sqrt{S^2+(Q-U-\lambda_1)^2}}{\lambda} \\ \frac{(Q-U-\lambda_1)\sqrt{S^2+(Q-U-\lambda_2)^2}}{S\lambda} & -\frac{\sqrt{S^2+(Q-U-\lambda_2)^2}}{\lambda} \end{pmatrix} \tag{2.3.17}$$

where $\lambda = \lambda_2 - \lambda_1$. Applying the identity operator $I = VV^{-1}$ in Eq. (2.3.1), we have

$$\frac{dJ(t)}{dt} = VV^{-1}MVV^{-1}J(t) + E(t) \tag{2.3.18}$$

and multiplying Eq. (2.3.18) on the left by V^{-1}, we get

$$\frac{d}{dt}\Big(V^{-1}J(t)\Big) = DV^{-1}J(t) + V^{-1}E(t) \tag{2.3.19}$$

in which

$$D = V^{-1}MV = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}. \tag{2.3.20}$$

For physically realizable solution λ_1 and λ_2 must have negative values. To this end, the solution of Eq. (2.3.19), can be written as

$$V^{-1}J(t) = e^{Dt}V^{-1}J(0) + \int_0^t e^{D(t-t')}V^{-1}E(t')dt' \tag{2.3.21}$$

from which follows

$$J(t) = Ve^{Dt}V^{-1}J(0) + \int_0^t Ve^{D(t-t')}V^{-1}E(t')dt'. \tag{2.3.22}$$

Since D is diagonal, we have

$$e^{Dt} = \begin{pmatrix} e^{\lambda_1 t} & 0 \\ 0 & e^{\lambda_2 t} \end{pmatrix}. \tag{2.3.23}$$

It is easy to see that

$$Ve^{Dt}V^{-1}J(0) = \begin{pmatrix} \frac{1}{\lambda}(Be^{\lambda_2 t} - Ce^{\lambda_1 t})\alpha(0) & \frac{S}{\lambda}(e^{\lambda_1 t} - e^{\lambda_2 t})\beta^*(0) \\ \frac{BC}{S\lambda}(e^{\lambda_2 t} - e^{\lambda_1 t})\alpha(0) & \frac{1}{\lambda}(Be^{\lambda_1 t} - Ce^{\lambda_2 t})\beta^*(0) \end{pmatrix} \tag{2.3.24}$$

and

$$Ve^{D(t-t')}V^{-1}E(t') = \begin{pmatrix} \frac{1}{\lambda}J_1 & \frac{S}{\lambda}J_2 \\ -\frac{BC}{S\lambda}J_3 & \frac{1}{\lambda}J_4 \end{pmatrix} \tag{2.3.25}$$

where

$$B = Q - U - \lambda_1, \tag{2.3.26}$$

$$C = Q - U - \lambda_2, \tag{2.3.27}$$

$$J_1 = (Be^{\lambda_2(t-t')} - Ce^{\lambda_1(t-t')})(f_\alpha(t') + \varepsilon), \tag{2.3.28}$$

$$J_2 = (e^{\lambda_1(t-t')} - e^{\lambda_2(t-t')})(f_\beta^*(t') + \varepsilon), \tag{2.3.29}$$

$$J_3 = (e^{\lambda_2(t-t')} - e^{\lambda_1(t-t')})(f_\alpha(t') + \varepsilon), \tag{2.3.30}$$

$$J_4 = (Be^{\lambda_1(t-t')} - Ce^{\lambda_2(t-t')})(f_\beta^*(t') + \varepsilon). \tag{2.3.31}$$

Integrating Eq. (2.3.25), we have

$$\int_0^t Ve^{D(t-t')}V^{-1}E(t')dt' = \begin{pmatrix} \frac{1}{\lambda}\int_0^t J_1 dt' & \frac{S}{\lambda}\int_0^t J_2 dt' \\ \frac{BC}{S\lambda}\int_0^t J_3 dt' & \frac{1}{\lambda}\int_0^t J_4 dt' \end{pmatrix}. \tag{2.3.32}$$

Using Eqs. (2.3.24) and (2.3.32) in Eq. (2.3.22), we get

$$\begin{aligned} J(t) &= \begin{pmatrix} \frac{1}{\lambda}(Be^{\lambda_2 t} - Ce^{\lambda_1 t})\alpha(0) & \frac{S}{\lambda}(e^{\lambda_1 t} - e^{\lambda_2 t})\beta^*(0) \\ \frac{BC}{S\lambda}(e^{\lambda_2 t} - e^{\lambda_1 t})\alpha(0) & \frac{1}{\lambda}(Be^{\lambda_1 t} - Ce^{\lambda_2 t})\beta^*(0) \end{pmatrix} \\ &+ \begin{pmatrix} \frac{1}{\lambda}\int_0^t J_1 dt' & \frac{S}{\lambda}\int_0^t J_2 dt' \\ \frac{BC}{S\lambda}\int_0^t J_3 dt' & \frac{1}{\lambda}\int_0^t J_4 dt' \end{pmatrix}. \end{aligned} \tag{2.3.33}$$

With the aid of Eqs. (2.3.2), (2.3.28) - (2.3.31), and (2.3.33), one easily see that

$$\begin{aligned} \alpha(t) &= \frac{1}{\lambda}(Be^{\lambda_2 t} - Ce^{\lambda_1 t})\alpha(0) + \frac{S}{\lambda}(e^{\lambda_1 t} - e^{\lambda_2 t})\beta^*(0) + \frac{\varepsilon(S-C)}{\lambda}\int_0^t e^{\lambda_1(t-t')}dt' \\ &+ \frac{\varepsilon(B-S)}{\lambda}\int_0^t e^{\lambda_2(t-t')}dt' + \frac{1}{\lambda}\int_0^t (Be^{\lambda_2(t-t')} - Ce^{\lambda_1(t-t')})f_\alpha(t')dt' \\ &+ \frac{S}{\lambda}\int_0^t (e^{\lambda_1(t-t')} - e^{\lambda_2(t-t')})f_\beta^*(t')dt' \end{aligned} \tag{2.3.34}$$

and

$$\begin{aligned}
\beta(t) &= \frac{BC}{S\lambda}(e^{\lambda_2 t} - e^{\lambda_1 t})\alpha(0) + \frac{1}{\lambda}(Be^{\lambda_1 t} - Ce^{\lambda_2 t})\beta^*(0) + \frac{\varepsilon B(S-C)}{\lambda S}\int_0^t e^{\lambda_1(t-t')}dt' \\
&+ \frac{\varepsilon C(B-S)}{\lambda S}\int_0^t e^{\lambda_2(t-t')}dt' + \frac{BC}{S\lambda}\int_0^t (e^{\lambda_2(t-t')} - e^{\lambda_1(t-t')})f_\alpha(t')dt' \\
&+ \frac{1}{\lambda}\int_0^t (Be^{\lambda_1(t-t')} - Ce^{\lambda_2(t-t')})f_\beta^*(t')dt' \qquad (2.3.35)
\end{aligned}$$

Carrying out some of the integrations in Eqs. (2.3.34) and (2.3.35), we finally get

$$\begin{aligned}
\alpha(t) &= \frac{\varepsilon(S-C)}{\lambda\lambda_1}(e^{\lambda_1 t} - 1) + \frac{\varepsilon(B-S)}{\lambda\lambda_2}(e^{\lambda_2 t} - 1) + \frac{1}{\lambda}(Be^{\lambda_2 t} - Ce^{\lambda_1 t})\alpha(0) \\
&+ \frac{S}{\lambda}(e^{\lambda_1 t} - e^{\lambda_2 t})\beta^*(0) + F(t), \qquad (2.3.36)
\end{aligned}$$

$$\begin{aligned}
\beta^*(t) &= \frac{\varepsilon B(S-C)}{\lambda\lambda_1 S}(e^{\lambda_1 t} - 1) + \frac{\varepsilon C(B-S)}{\lambda\lambda_2 S}(e^{\lambda_2 t} - 1) + \frac{BC}{S\lambda}(e^{\lambda_2 t} - e^{\lambda_1 t})\alpha(0) \\
&+ \frac{1}{\lambda}(Be^{\lambda_1 t} - Ce^{\lambda_2 t})\beta^*(0) + G(t). \qquad (2.3.37)
\end{aligned}$$

where

$$\begin{aligned}
F(t) &= \frac{1}{\lambda}\int_0^t (Be^{\lambda_2(t-t')} - Ce^{\lambda_1(t-t')})f_\alpha(t')dt' \\
&+ \frac{S}{\lambda}\int_0^t (e^{\lambda_1(t-t')} - e^{\lambda_2(t-t')})f_\beta^*(t')dt', \qquad (2.3.38)
\end{aligned}$$

$$\begin{aligned}
G(t) &= \frac{BC}{S\lambda}\int_0^t (e^{\lambda_2(t-t')} - e^{\lambda_1(t-t')})f_\alpha(t')dt' \\
&+ \frac{1}{\lambda}\int_0^t (Be^{\lambda_1(t-t')} - Ce^{\lambda_2(t-t')})f_\beta^*(t')dt'. \qquad (2.3.39)
\end{aligned}$$

In the next two chapters we apply the cavity mode variables $\alpha(t)$ and $\beta^*(t)$ to calculate all quantities of interest in our analysis of the squeezing and statistical properties of the light generated by the system under consideration.

Chapter 3

Quadrature Fluctuations

In the previous chapter we have found the explicit forms of the two cavity mode variables $\alpha(t)$ and $\beta^*(t)$. Here we apply these results to calculate the quadrature variances and the squeezing spectrum of the two-mode light generated by the optical system under consideration.

3.1 Quadrature Variances

In order to study the squeezing properties of a two-mode light, we introduce the quadrature operators defined by [1]

$$\hat{c}_+ = \frac{1}{\sqrt{2}}(\hat{a}_+ + \hat{b}_+) \tag{3.1.1}$$

$$\hat{c}_- = \frac{1}{\sqrt{2}}(\hat{a}_- + \hat{b}_-), \tag{3.1.2}$$

in which

$$\hat{a}_+ = \hat{a}^\dagger + \hat{a}, \tag{3.1.3}$$

$$\hat{a}_- = i(\hat{a}^\dagger - \hat{a}), \tag{3.1.4}$$

$$\hat{b}_+ = \hat{b}^\dagger + \hat{b}, \tag{3.1.5}$$

$$\hat{b}_{-} = i(\hat{b}^{\dagger} - \hat{b}). \tag{3.1.6}$$

Using Eqs. (3.1.3) - (3.1.6) in Eqs. (3.1.1) and (3.1.2) we have

$$\hat{c}_{+} = \frac{1}{\sqrt{2}}(\hat{a}^{\dagger} + \hat{a} + \hat{b}^{\dagger} + \hat{b}), \tag{3.1.7}$$

$$\hat{c}_{-} = \frac{i}{\sqrt{2}}(\hat{a}^{\dagger} - \hat{a} + \hat{b}^{\dagger} - \hat{b}). \tag{3.1.8}$$

Using the above results, the quadrature variances of the two quadrature operators in the normal order have the form

$$\begin{aligned}\Delta c_{\pm}^{2} &= 1 + \frac{1}{2}\Bigg[[(\langle\hat{a}^{\dagger 2}\rangle + \langle\hat{a}^{2}\rangle + \langle\hat{b}^{\dagger 2}\rangle + \langle\hat{b}^{2}\rangle + 2\langle\hat{a}^{\dagger}\hat{b}^{\dagger}\rangle + 2\langle\hat{a}\hat{b}\rangle) - (\langle\hat{a}^{\dagger}\rangle^{2} + \langle\hat{a}\rangle^{2} \\ &+ \langle\hat{b}^{\dagger}\rangle^{2} + \langle\hat{b}\rangle^{2} + 2\langle\hat{a}^{\dagger}\rangle\langle\hat{b}^{\dagger}\rangle + 2\langle\hat{a}\rangle\langle\hat{b}\rangle)] \pm [(2\langle\hat{a}^{\dagger}\hat{a}\rangle + 2\langle\hat{b}^{\dagger}\hat{b}\rangle + 2\langle\hat{a}^{\dagger}\hat{b}\rangle \\ &+ 2\langle\hat{b}^{\dagger}\hat{a}\rangle) - (2\langle\hat{a}^{\dagger}\rangle\langle\hat{a}\rangle + 2\langle\hat{a}^{\dagger}\rangle\langle\hat{b}\rangle + 2\langle\hat{b}^{\dagger}\rangle\langle\hat{a}\rangle + 2\langle\hat{b}^{\dagger}\rangle\langle\hat{b}\rangle)]\Bigg].\end{aligned} \tag{3.1.9}$$

The c-number function corresponding to Eq. (3.1.9) associated with the normal ordering can be written as

$$\begin{aligned}\Delta c_{\pm}^{2} &= 1 + \frac{1}{2}\Bigg[[(\langle\alpha^{*2}\rangle + \langle\alpha^{2}\rangle + \langle\beta^{*2}\rangle + \langle\beta^{2}\rangle + 2\langle\alpha^{*}\beta^{*}\rangle + 2\langle\alpha\beta\rangle) - (\langle\alpha^{*}\rangle^{2} + \langle\alpha\rangle^{2} \\ &+ \langle\beta^{*}\rangle^{2} + \langle\beta\rangle^{2} + 2\langle\alpha^{*}\rangle\langle\beta^{*}\rangle + 2\langle\alpha\rangle\langle\beta\rangle)] \pm [(2\langle\alpha^{*}\alpha\rangle + 2\langle\beta^{*}\beta\rangle + 2\langle\alpha^{*}\beta\rangle \\ &+ 2\langle\beta^{*}\alpha\rangle) - (2\langle\alpha^{*}\rangle\langle\alpha\rangle + 2\langle\alpha^{*}\rangle\langle\beta\rangle + 2\langle\beta^{*}\rangle\langle\alpha\rangle + 2\langle\beta^{*}\rangle\langle\beta\rangle)]\Bigg].\end{aligned} \tag{3.1.10}$$

To find the quadrature variance of the two-mode light under consideration, we now apply Eqs. (2.3.36) and (2.3.37) to calculate the various expectation values that appeared on Eq. (3.1.10). But we restrict ourselves such that the cavity modes are initially in vacuum state. Mathematically, we can express this initial condition as

$$\langle\alpha(0)\rangle = \langle\alpha(0)^{*}\alpha(0)\rangle = 0, \tag{3.1.11}$$

$$\langle\beta(0)\rangle = \langle\beta(0)^{*}\beta(0)\rangle = 0, \tag{3.1.12}$$

$$\langle\beta(0)\alpha(0)\rangle = \langle\beta^*(0)\alpha(0)\rangle = 0, \tag{3.1.13}$$

Now, making use of these assumptions along with Eqs. (2.2.20), (2.2.21), (2.2.38) and (2.3.39), we can easily see that

$$\langle F(t)\rangle = 0, \tag{3.1.14}$$

$$\langle G(t)\rangle = 0. \tag{3.1.15}$$

Applying Eq. (2.3.36) along with Eqs. (3.1.11) - (3.1.14), we find

$$\begin{aligned}
\langle\alpha^*(t)\alpha(t)\rangle &= \frac{\epsilon^2}{\lambda^2\lambda_1^2}(S-C)(T-C)(e^{\lambda_1 t}-1)^2 + \frac{\epsilon^2}{\lambda^2\lambda_2^2}(B-S)(B-T)(e^{\lambda_2 t}-1)^2 \\
&+ \frac{\epsilon^2}{\lambda^2\lambda_1\lambda_2}[(T-C)(B-S)+(B-T)(S-C)](e^{\lambda_1 t}-1)(e^{\lambda_2 t}-1) \\
&+ \frac{1}{\lambda}(Be^{\lambda_2 t}-Ce^{\lambda_1 t})\langle\alpha^*(0)F(t)\rangle + \frac{T}{\lambda}(e^{\lambda_1 t}-e^{\lambda_2 t})\langle\beta(0)F(t)\rangle \\
&+ \frac{1}{\lambda}(Be^{\lambda_2 t}-Ce^{\lambda_1 t})\langle\alpha(0)F^*(t)\rangle + \frac{S}{\lambda}(e^{\lambda_1 t}-e^{\lambda_2 t})\langle\beta^*(0)F^*(t)\rangle \\
&+ \langle F^*(t)F(t)\rangle.
\end{aligned} \tag{3.1.16}$$

Using the fact that the noise force at time t does not affect the cavity mode variables at earlier time, we note that

$$\langle\alpha^*(0)F(t)\rangle = \langle\alpha^*(0)\rangle\langle F(t)\rangle = 0, \tag{3.1.17}$$

$$\langle\beta(0)F(t)\rangle = \langle\beta(0)\rangle\langle F(t)\rangle = 0. \tag{3.1.18}$$

Applying these results and their complex conjugate, Eq. (3.1.16) takes the form

$$\begin{aligned}
\langle\alpha^*(t)\alpha(t)\rangle &= \frac{\epsilon^2}{\lambda^2}\left[\frac{(S-C)(T-C)}{\lambda_1^2}(e^{\lambda_1 t}-1)^2 + \frac{(B-S)(B-T)}{\lambda_2^2}(e^{\lambda_2 t}-1)^2\right] \\
&+ \frac{\epsilon^2}{\lambda^2}\left[\frac{(T-C)(B-S)+(B-T)(S-C)}{\lambda_1\lambda_2}\right](e^{\lambda_2 t}-1)(e^{\lambda_1 t}-1) \\
&+ \langle F^*(t)F(t)\rangle.
\end{aligned} \tag{3.1.19}$$

On account of Eqs. (2.3.38) and (2.3.39), we have

$$\begin{aligned}\langle F^*(t)F(t)\rangle &= \frac{1}{\lambda^2}\int_0^t (Be^{\lambda_2(t-t')} - Ce^{\lambda_1(t-t')})(Be^{\lambda_2(t-t'')} - Ce^{\lambda_1(t-t'')})\langle f_\alpha^*(t'')f_\alpha(t')\rangle dt'dt'' \\ &+ \frac{S}{\lambda^2}\int_0^t (e^{\lambda_1(t-t')} - e^{\lambda_2(t-t')})(Be^{\lambda_2(t-t'')} - Ce^{\lambda_1(t-t'')})\langle f_\alpha^*(t'')f_\beta^*(t')\rangle dt'dt'' \\ &+ \frac{T}{\lambda^2}\int_0^t (Be^{\lambda_2(t-t')} - Ce^{\lambda_1(t-t')})(e^{\lambda_1(t-t'')} - e^{\lambda_2(t-t'')})\langle f_\beta(t'')f_\alpha(t')\rangle dt'dt'' \\ &+ \frac{ST}{\lambda^2}\int_0^t (e^{\lambda_1(t-t')} - e^{\lambda_2(t-t')})(e^{\lambda_1(t-t'')} - e^{\lambda_2(t-t'')})\langle f_\beta(t'')f_\beta^*(t')\rangle dt'dt''.\end{aligned} \tag{3.1.20}$$

Using the correlation relations given by Eqs. (2.2.41), (2.2.50) and (2.2.53) then carrying out the integrations, we obtain

$$\begin{aligned}\langle F^*(t)F(t)\rangle &= \left(\frac{B^2Q - BST}{\lambda^2\lambda_2}\right)(e^{2\lambda_2 t} - 1) + \left(\frac{C^2Q - CST}{\lambda^2\lambda_1}\right)(e^{2\lambda_1 t} - 1) \\ &- \left(\frac{2(2BCQ - ST(B+C))}{\lambda^2(\lambda_1+\lambda_2)}\right)(e^{(\lambda_1+\lambda_2)t} - 1).\end{aligned} \tag{3.1.21}$$

Substituting Eq. (3.1.21) into Eq. (3.1.19), we have

$$\begin{aligned}\langle \alpha^*(t)\alpha(t)\rangle &= \frac{\epsilon^2}{\lambda^2}\left[\frac{(S-C)(T-C)}{\lambda_1^2}(e^{\lambda_1 t} - 1)^2 + \frac{(B-S)(B-T)}{\lambda_2^2}(e^{\lambda_2 t} - 1)^2\right] \\ &+ \frac{\epsilon^2}{\lambda^2}\left[\frac{(T-C)(B-S) + (B-T)(S-C)}{\lambda_1\lambda_2}\right](e^{\lambda_2 t} - 1)(e^{\lambda_1 t} - 1) \\ &+ \left(\frac{B^2Q - BST}{\lambda^2\lambda_2}\right)(e^{2\lambda_2 t} - 1) + \left(\frac{C^2Q - CST}{\lambda^2\lambda_1}\right)(e^{2\lambda_1 t} - 1) \\ &- \left(\frac{2(2BCQ - ST(B+C))}{\lambda^2(\lambda_1+\lambda_2)}\right)(e^{(\lambda_1+\lambda_2)t} - 1).\end{aligned} \tag{3.1.22}$$

At steady state, Eq. (3.1.22) takes the form

$$\begin{aligned}\langle \alpha^*(t)\alpha(t)\rangle_{ss} &= \frac{2(2BCQ - ST(B+C))}{\lambda^2(\lambda_1+\lambda_2)} - \frac{B^2Q - BST}{\lambda^2\lambda_2} - \frac{C^2Q - CST}{\lambda^2\lambda_1} \\ &+ \frac{\epsilon^2}{\lambda^2}\left[\frac{(S-C)(T-C)}{\lambda_1^2} + \frac{(S-B)(T-B)}{\lambda_2^2}\right] \\ &+ \frac{\epsilon^2}{\lambda^2\lambda_1\lambda_2}[(T-C)(B-S) + (B-T)(S-C)],\end{aligned} \tag{3.1.23}$$

where ss stands for steady state.

Applying Eq. (2.3.37) along with Eqs. (3.1.11), (3.1.12), (3.1.13) and (3.1.15), we have

$$\begin{aligned}
\langle\beta^*(t)\beta(t)\rangle &= \frac{\epsilon^2}{\lambda^2 ST}\left[\frac{B^2(S-C)(T-C)}{\lambda_1^2}(e^{\lambda_1 t}-1)^2+\frac{C^2(B-S)(B-T)}{\lambda_2^2}(e^{\lambda_2 t}-1)^2\right]\\
&+ \frac{BC\epsilon^2}{\lambda^2 ST}\left[\frac{(S-C)(B-T)+(T-C)(B-S)}{\lambda_1\lambda_2}\right](e^{\lambda_2 t}-1)(e^{\lambda_1 t}-1)\\
&+ \frac{BC}{\lambda S}(e^{\lambda_2 t}-e^{\lambda_1 t})\langle\alpha(0)G^*(t)\rangle+\frac{1}{\lambda}(Be^{\lambda_1 t}-Ce^{\lambda_2 t})\langle\beta^*(0)G^*(t)\rangle\\
&+ \frac{BC}{\lambda T}(e^{\lambda_2 t}-e^{\lambda_1 t})\langle\alpha^*(0)G(t)\rangle+\frac{1}{\lambda}(Be^{\lambda_1 t}-Ce^{\lambda_2 t})\langle\beta(0)G(t)\rangle\\
&+ \langle G^*(t)G(t)\rangle. \qquad (3.1.24)
\end{aligned}$$

One can also show in a similar fashion that

$$\begin{aligned}
\langle\beta^*(t)\beta(t)\rangle &= \frac{\epsilon^2}{\lambda^2 ST}\left[\frac{B^2(S-C)(T-C)}{\lambda_1^2}(e^{\lambda_1 t}-1)^2+\frac{C^2(B-S)(B-T)}{\lambda_2^2}(e^{\lambda_2 t}-1)^2\right]\\
&+ \frac{BC\epsilon^2}{\lambda^2 ST}\left[\frac{(S-C)(B-T)+(T-C)(B-S)}{\lambda_1\lambda_2}\right](e^{\lambda_1 t}-1)(e^{\lambda_2 t}-1)\\
&+ \langle G^*(t)G(t)\rangle. \qquad (3.1.25)
\end{aligned}$$

With the aid of Eq. (2.3.39), we can write

$$\begin{aligned}
\langle G^*(t)G(t)\rangle &= \frac{B^2C^2}{\lambda^2 ST}\int_0^t(e^{\lambda_2(t-t')}-e^{\lambda_1(t-t')})(e^{\lambda_2(t-t'')}-e^{\lambda_1(t-t'')})\langle f_\alpha^*(t'')f_\alpha(t')\rangle dt'dt''\\
&+ \frac{BC}{\lambda^2 T}\int_0^t(Be^{\lambda_1(t-t')}-Ce^{\lambda_2(t-t')})(e^{\lambda_2(t-t'')}-e^{\lambda_1(t-t'')})\langle f_\alpha^*(t'')f_\beta^*(t')\rangle dt'dt''\\
&+ \frac{BC}{\lambda^2 S}\int_0^t(e^{\lambda_2(t-t')}-e^{\lambda_1(t-t')})(Be^{\lambda_1(t-t'')}-Ce^{\lambda_2(t-t'')})\langle f_\beta(t'')f_\alpha(t')\rangle dt'dt''\\
&+ \frac{1}{\lambda^2}\int_0^t(Be^{\lambda_1(t-t')}-Ce^{\lambda_2(t-t')})(Be^{\lambda_1(t-t'')}-Ce^{\lambda_2(t-t'')})\langle f_\beta(t'')f_\beta^*(t')\rangle dt'dt''.
\end{aligned}$$

$$(3.1.26)$$

On account of Eqs. (2.2.41), (2.2.50) and (2.2.53), and carrying out the integration

Eq. (3.1.26) turns out to be

$$\begin{aligned}\langle G^*(t)G(t)\rangle &= \frac{QB^2C^2}{\lambda^2 ST}\left[\frac{1}{\lambda_1}(e^{2\lambda_1 t}-1)+\frac{1}{\lambda_2}(e^{2\lambda_2 t}-1)-\frac{4}{\lambda_1+\lambda_2}(e^{(\lambda_1+\lambda_2)t}-1)\right]\\ &+ \frac{BC}{\lambda^2}\left[\frac{2(B+C)}{\lambda_1+\lambda_2}(e^{(\lambda_1+\lambda_2)t}-1)-\frac{B}{\lambda_1}(e^{2\lambda_1 t}-1)-\frac{C}{\lambda_2}(e^{2\lambda_2 t}-1)\right].\end{aligned} \tag{3.1.27}$$

Using Eq. (3.1.27) in Eq. (3.1.25), we get

$$\begin{aligned}\langle \beta^*(t)\beta(t)\rangle &= \frac{\epsilon^2}{\lambda^2 ST}\left[\frac{B^2(S-C)(T-C)}{\lambda_1^2}(e^{\lambda_1 t}-1)^2+\frac{C^2(B-S)(B-T)}{\lambda_2^2}(e^{\lambda_2 t}-1)^2\right]\\ &+ \frac{BC\epsilon^2}{\lambda^2 ST}\left[\frac{(S-C)(B-T)+(T-C)(B-S)}{\lambda_1\lambda_2}\right](e^{\lambda_1 t}-1)(e^{\lambda_2 t}-1)\\ &+ \frac{QB^2C^2}{\lambda^2 ST}\left[\frac{1}{\lambda_1}(e^{2\lambda_1 t}-1)+\frac{1}{\lambda_2}(e^{2\lambda_2 t}-1)-\frac{4}{\lambda_1+\lambda_2}(e^{(\lambda_1+\lambda_2)t}-1)\right]\\ &+ \frac{BC}{\lambda^2}\left[\frac{2(B+C)}{\lambda_1+\lambda_2}(e^{(\lambda_1+\lambda_2)t}-1)-\frac{B}{\lambda_1}(e^{2\lambda_1 t}-1)-\frac{C}{\lambda_2}(e^{2\lambda_2 t}-1)\right].\end{aligned} \tag{3.1.28}$$

Evaluating Eq. (3.1.28) at steady state, we finally get

$$\begin{aligned}\langle \beta^*(t)\beta(t)\rangle_{ss} &= \frac{B^2C}{\lambda^2\lambda_1}+\frac{C^2B}{\lambda^2\lambda_2}-\frac{2BC(B+C)}{\lambda^2(\lambda_1+\lambda_2)}-\frac{QB^2C^2}{(\lambda_1+\lambda_2)\lambda_1\lambda_2 ST}\\ &+ \frac{\epsilon^2}{\lambda^2 ST}\left[\frac{B^2(S-C)(T-C)}{\lambda_1^2}+\frac{C^2(B-S)(B-T)}{\lambda_2^2}\right]\\ &+ \frac{BC\epsilon^2}{\lambda^2\lambda_1\lambda_2 ST}\left[(S-C)(B-T)+(T-C)(B-S)\right].\end{aligned} \tag{3.1.29}$$

Applying Eqs. (2.3.36) and (2.3.37) and taking into account the fact that a noise force at a later time doesn't affect cavity mode variables at earlier time, we find

$$\begin{aligned}\langle \alpha(t)\beta(t)\rangle &= \frac{\epsilon^2}{\lambda^2 T}\left[\frac{B(S-C)(T-C)}{\lambda_1^2}(e^{\lambda_1 t}-1)^2+\frac{C(B-T)(B-S)}{{\lambda_2}^2}(e^{\lambda_2 t}-1)^2\right]\\ &+ \frac{\epsilon^2}{\lambda^2 T}\left[\frac{C(S-C)(B-T)+B(B-S)(T-C)}{\lambda_1\lambda_2}\right](e^{\lambda_1 t}-1)(e^{\lambda_2 t}-1)\\ &+ \langle F(t)G^*(t)\rangle.\end{aligned} \tag{3.1.30}$$

In view of Eqs. (2.3.38) and (2.3.39), we have

$$
\begin{aligned}
\langle F(t)G^*(t)\rangle &= \frac{BC}{\lambda^2 T}\int_0^t (Be^{\lambda_2(t-t')} - Ce^{\lambda_1(t-t')})(e^{\lambda_2(t-t'')} - e^{\lambda_1(t-t'')})\langle f_\alpha^*(t'')f_\alpha(t')\rangle dt'dt'' \\
&+ \frac{BCS}{\lambda^2 T}\int_0^t (e^{\lambda_1(t-t')} - e^{\lambda_2(t-t')})(e^{\lambda_2(t-t'')} - e^{\lambda_1(t-t'')})\langle f_\alpha^*(t'')f_\beta^*(t')\rangle dt'dt'' \\
&+ \frac{1}{\lambda^2}\int_0^t (Be^{\lambda_2(t-t')} - Ce^{\lambda_1(t-t')})(Be^{\lambda_1(t-t'')} - Ce^{\lambda_2(t-t'')})\langle f_\beta(t'')f_\alpha(t')\rangle dt'dt'' \\
&+ \frac{S}{\lambda^2}\int_0^t (e^{\lambda_1(t-t')} - e^{\lambda_2(t-t')})(Be^{\lambda_1(t-t'')} - Ce^{\lambda_2(t-t'')})\langle f_\beta(t'')f_\beta^*(t')\rangle dt'dt''.
\end{aligned}
\tag{3.1.31}
$$

On account of Eqs. (2.2.41), (2.2.50) and (2.2.53), Eq. (3.1.31) reduces to

$$
\begin{aligned}
\langle F(t)G^*(t)\rangle &= \frac{2BCQ}{\lambda^2 T}\left[\frac{B}{2\lambda_2}(e^{2\lambda_2 t} - 1) + \frac{C}{2\lambda_1}(e^{\lambda_1 t} - 1) - \frac{B+C}{\lambda_1+\lambda_2}(e^{(\lambda_1+\lambda_2)t} - 1)\right] \\
&+ \frac{S}{\lambda^2}\left[\frac{(B+C)^2}{\lambda_1+\lambda_2}(e^{(\lambda_1+\lambda_2)t} - 1) - \frac{BC}{\lambda_2}(e^{2\lambda_2 t} - 1) - \frac{BC}{\lambda_1}(e^{2\lambda_2 t} - 1)\right].
\end{aligned}
\tag{3.1.32}
$$

Substituting this result into Eq. (3.1.30), we have

$$
\begin{aligned}
\langle \alpha(t)\beta(t)\rangle &= \frac{\epsilon^2}{\lambda^2 T}\left[\frac{B(S-C)(T-C)}{{\lambda_1}^2}(e^{\lambda_1 t} - 1)^2 + \frac{C(B-T)(B-S)}{{\lambda_2}^2}(e^{\lambda_2 t} - 1)^2\right] \\
&+ \frac{\epsilon^2}{\lambda^2 T}\left[\frac{C(S-C)(B-T) + B(B-S)(T-C)}{\lambda_1\lambda_2}\right](e^{\lambda_1 t} - 1)(e^{\lambda_2 t} - 1) \\
&+ \frac{2BCQ}{\lambda^2 T}\left[\frac{B}{2\lambda_2}(e^{2\lambda_2 t} - 1) + \frac{C}{2\lambda_1}(e^{\lambda_1 t} - 1) - \frac{B+C}{\lambda_1+\lambda_2}(e^{(\lambda_1+\lambda_2)t} - 1)\right] \\
&+ \frac{S}{\lambda^2}\left[\frac{(B+C)^2}{\lambda_1+\lambda_2}(e^{(\lambda_1+\lambda_2)t} - 1) - \frac{BC}{\lambda_2}(e^{2\lambda_2 t} - 1) - \frac{BC}{\lambda_1}(e^{2\lambda_2 t} - 1)\right] \\
&.
\end{aligned}
\tag{3.1.33}
$$

Evaluating Eq. (3.1.33) at steady state, we get

$$\begin{aligned}
\langle\alpha(t)\beta(t)\rangle_{ss} &= \frac{2BCQ}{\lambda^2 T}\left[\frac{B+C}{\lambda_1+\lambda_2}-\frac{B}{2\lambda_2}-\frac{C}{2\lambda_1}\right]+\frac{S}{\lambda^2}\left[\frac{BC}{\lambda_2}+\frac{BC}{\lambda_1}-\frac{(B+C)^2}{\lambda_1+\lambda_2}\right]\\
&+ \frac{\epsilon^2}{\lambda^2 T}\left[\frac{B(S-C)(T-C)}{{\lambda_1}^2}+\frac{C(B-T)(B-S)}{{\lambda_2}^2}\right]\\
&+ \frac{\epsilon^2}{\lambda^2\lambda_1\lambda_2 T}\Big[C(S-C)(B-T)+B(B-S)(T-C)\Big]. \qquad (3.1.34)
\end{aligned}$$

Similar calculation for the remaining expectation values in Eq. (3.1.10) at any time t and at steady state shows that

$$\begin{aligned}
\langle\alpha(t)^2\rangle &= \frac{\epsilon^2}{\lambda^2}\left[\frac{S-C}{\lambda_1}(e^{\lambda_1 t}-1)+\frac{(B-C)}{\lambda_2}(e^{\lambda_2 t}-1)\right]^2, \qquad (3.1.35)\\
\langle\alpha(t)^2\rangle_{ss} &= \frac{\epsilon^2}{\lambda^2}\left[\frac{C-S}{\lambda_1}+\frac{(S-B)}{\lambda_2}\right]^2, \qquad (3.1.36)\\
\langle\beta(t)^2\rangle &= \frac{\epsilon^2}{\lambda^2 T^2}\left[\frac{B(T-C)}{\lambda_1}(e^{\lambda_1 t}-1)+\frac{C(B-T)}{\lambda_2}(e^{\lambda_2 t}-1)\right]^2, \qquad (3.1.37)\\
\langle\beta(t)^2\rangle_{ss} &= \frac{\epsilon^2}{\lambda^2 T^2}\left[\frac{B(C-T)}{\lambda_1}+\frac{C(T-B)}{\lambda_2}\right]^2, \qquad (3.1.38)\\
\langle\alpha^*(t)\beta(t)\rangle &= \frac{\epsilon^2}{\lambda^2 T}\left[B(\frac{T-C}{\lambda_1})^2(e^{\lambda_1 t}-1)^2+C(\frac{B-T}{\lambda_2})^2(e^{\lambda_2 t}-1)^2\right]\\
&+ \frac{(B+C)\epsilon^2}{\lambda^2\lambda_1\lambda_2 T}(B-T)(T-C)(e^{\lambda_1 t}-1)(e^{\lambda_2 t}-1), \qquad (3.1.39)\\
\langle\alpha^*(t)\beta(t)\rangle_{ss} &= \frac{\epsilon^2}{\lambda^2 T}\left[B(\frac{T-C}{\lambda_1})^2+C(\frac{B-T}{\lambda_2})^2\right]\\
&+ \frac{(B+C)\epsilon^2}{\lambda^2\lambda_1\lambda_2 T}(B-T)(T-C). \qquad (3.1.40)
\end{aligned}$$

On account of Eqs. (2.3.36) and (2.3.37) along with (3.1.11) - (3.1.15), we have

$$\langle\alpha(t)\rangle = \frac{\epsilon}{\lambda\lambda_1}(S-C)(e^{\lambda_1 t}-1)+\frac{\epsilon}{\lambda\lambda_2}(B-S)(e^{\lambda_2 t}-1), \qquad (3.1.41)$$

$$\langle\beta(t)\rangle = \frac{B\epsilon}{\lambda\lambda_1 T}(T-C)(e^{\lambda_1 t}-1)+\frac{C\epsilon}{\lambda\lambda_2 T}(B-T)(e^{\lambda_2 t}-1). \qquad (3.1.42)$$

In view of these two equations, we easily see that

$$\langle\alpha(t)\rangle^2 = \frac{\epsilon^2}{\lambda^2}\left[\frac{S-C}{\lambda_1}(e^{\lambda_1 t}-1)+\frac{B-C}{\lambda_2}(e^{\lambda_2 t}-1)\right]^2, \tag{3.1.43}$$

$$\langle\alpha(t)\rangle_{ss}^2 = \frac{\epsilon^2}{\lambda^2}\left[\frac{C-S}{\lambda_1}+\frac{(S-B)}{\lambda_2}\right]^2, \tag{3.1.44}$$

$$\langle\beta(t)\rangle^2 = \frac{\epsilon^2}{\lambda^2 T^2}\left[\frac{B(T-C)}{\lambda_1}(e^{\lambda_1 t}-1)+\frac{C(B-T)}{\lambda_2}(e^{\lambda_2 t}-1)\right]^2, \tag{3.1.45}$$

$$\langle\beta(t)\rangle_{ss}^2 = \frac{\epsilon^2}{\lambda^2 T^2}\left[\frac{B(C-T)}{\lambda_1}+\frac{C(T-B)}{\lambda_2}\right]^2, \tag{3.1.46}$$

$$\begin{aligned}\langle\alpha^*(t)\rangle\langle\beta(t)\rangle &= \frac{B\epsilon^2}{\lambda^2 T}\left[B\left(\frac{T-C}{\lambda_1}\right)^2(e^{\lambda_1 t}-1)^2+C\left(\frac{B-T}{\lambda_2}\right)^2(e^{\lambda_2 t}-1)^2\right]\\ &+ \frac{(B+C)\epsilon^2}{\lambda^2\lambda_1\lambda_2 T}(B-T)(T-C)(e^{\lambda_1 t}-1)(e^{\lambda_2 t}-1),\end{aligned} \tag{3.1.47}$$

$$\begin{aligned}\langle\alpha^*(t)\rangle_{ss}\langle\beta(t)\rangle_{ss} &= \frac{\epsilon^2}{\lambda^2 T}\left[B\left(\frac{T-C}{\lambda_1}\right)^2+C\left(\frac{B-T}{\lambda_2}\right)^2\right]\\ &+ \frac{(B+C)\epsilon^2}{\lambda^2\lambda_1\lambda_2 T}(B-T)(T-C),\end{aligned} \tag{3.1.48}$$

$$\begin{aligned}\langle\alpha^*(t)\rangle\langle\alpha(t)\rangle &= \frac{\epsilon^2}{\lambda^2}\left[\frac{(T-C)(B-S)+(B-T)(S-C)}{\lambda_1\lambda_2}\right](e^{\lambda_2 t}-1)(e^{\lambda_1 t}-1)\\ &+ \frac{\epsilon^2}{\lambda^2\lambda_1^2}(S-C)(T-C)(e^{\lambda_1 t}-1)^2\\ &+ \frac{\epsilon^2}{\lambda^2\lambda_2^2}(B-S)(B-T)(e^{\lambda_2 t}-1)^2\end{aligned} \tag{3.1.49}$$

$$\begin{aligned}\langle\alpha^*(t)\rangle_{ss}\langle\alpha(t)\rangle_{ss} &= \frac{\epsilon^2}{\lambda^2}\left[\frac{(C-T)(C-S)}{\lambda_1^2}+\frac{(S-B)(T-B)}{\lambda_2^2}\right]\\ &+ \frac{\epsilon^2}{\lambda^2}\left[\frac{(C-T)(S-B)+(T-B)(C-S)}{\lambda_1\lambda_2}\right],\end{aligned} \tag{3.1.50}$$

$$\begin{aligned}\langle\beta^*(t)\rangle\langle\beta(t)\rangle &= \frac{BC\epsilon^2}{\lambda^2}\left[\frac{(S-C)(B-T)+(T-C)(B-S)}{\lambda_1\lambda_2 ST}\right](e^{\lambda_1 t}-1)(e^{\lambda_2 t}-1)\\ &+ \frac{B^2\epsilon^2}{\lambda^2\lambda_1^2 ST}(S-C)(T-C)(e^{\lambda_1 t}-1)^2\\ &+ \frac{C^2\epsilon^2}{\lambda^2\lambda_2^2}(B-S)(B-T)(e^{\lambda_2 t}-1)^2,\end{aligned} \tag{3.1.51}$$

$$\langle\beta^*(t)\rangle_{ss}\langle\beta(t)\rangle_{ss} = \frac{\epsilon^2}{\lambda^2 ST}\left[\frac{B^2(C-S)(C-T)}{\lambda_1^2} + \frac{C^2(S-B)(T-B)}{\lambda_2^2}\right] + \frac{\epsilon^2 BC}{\lambda^2}\left[\frac{(T-B)(C-S)+(S-B)(C-T)}{\lambda_1\lambda_2 ST}\right], \quad (3.1.52)$$

$$\langle\alpha(t)\rangle\langle\beta(t)\rangle = \frac{\epsilon^2}{\lambda^2}\left[\frac{C(S-C)(B-T)+B(B-S)(T-C)}{\lambda_1\lambda_2 T}\right](e^{\lambda_1 t}-1)(e^{\lambda_2 t}-1) + \frac{B\epsilon^2}{\lambda^2{\lambda_1}^2 T}(S-C)(T-C)(e^{\lambda_1 t}-1)^2 + \frac{C\epsilon^2}{\lambda^2{\lambda_2}^2 T}(B-T)(B-S)(e^{\lambda_2 t}-1)^2, \quad (3.1.53)$$

$$\langle\alpha(t)\rangle_{ss}\langle\beta(t)\rangle_{ss} = \frac{\epsilon^2}{\lambda^2 T}\left[\frac{B(C-S)(C-T)}{\lambda_1^2} + \frac{C(S-B)(T-B)}{\lambda_2^2}\right] + \frac{\epsilon^2}{\lambda^2}\left[\frac{C(C-S)(T-B)+B(S-B)(C-T)}{\lambda_1\lambda_2 T}\right]. \quad (3.1.54)$$

Comparisons of Eqs. (3.1.35), (3.1.37) and (3.1.39) with Eqs. (3.1.43), (3.1.45) and (3.1.47) shows that

$$\langle\alpha^2(t)\rangle = \langle\alpha(t)\rangle^2, \quad (3.1.55)$$

$$\langle\beta^2(t)\rangle = \langle\beta(t)\rangle^2, \quad (3.1.56)$$

$$\langle\alpha^*(t)\beta(t)\rangle = \langle\alpha^*(t)\rangle\langle\beta(t)\rangle. \quad (3.1.57)$$

In view of these results and their complex conjugates, Eq. (3.1.10) takes the form

$$\Delta c_\pm^2 = 1 + \Big[\langle\alpha^*(t)\alpha(t)\rangle + \langle\beta^*(t)\beta(t)\rangle - \langle\alpha^*(t)\rangle\langle\alpha(t)\rangle - \langle\beta^*(t)\rangle\langle\beta(t)\rangle\Big] \pm \Big[\langle\alpha^*(t)\beta^*(t)\rangle + \langle\alpha(t)\beta(t)\rangle - \langle\alpha^*(t)\rangle\langle\beta^*(t)\rangle - \langle\alpha(t)\rangle\langle\beta(t)\rangle\Big]. \quad (3.1.58)$$

Employing Eqs. (3.1.22), (3.1.28), (3.1.33), (3.1.49), (3.1.51), (3.1.53) and its complex conjugate in Eq. (3.1.58), the quadrature variances become

$$\begin{aligned}
\Delta c_{\pm}^2 &= 1+\frac{QB^2C^2}{\lambda^2 ST}\left[\frac{1}{\lambda_1}(e^{2\lambda_1 t}-1)+\frac{1}{\lambda_2}(e^{2\lambda_2 t}-1)-\frac{4}{\lambda_1+\lambda_2}(e^{(\lambda_1+\lambda_2)t}-1)\right]\\
&+ \frac{BC}{\lambda^2}\left[\frac{2(B+C)}{\lambda_1+\lambda_2}(e^{(\lambda_1+\lambda_2)t}-1)-\frac{B}{\lambda_1}(e^{2\lambda_1 t}-1)-\frac{C}{\lambda_2}(e^{2\lambda_2 t}-1)\right]\\
&+ \left(\frac{B^2Q-BST}{\lambda^2\lambda_2}\right)(e^{2\lambda_2 t}-1)+\left(\frac{C^2Q-CST}{\lambda^2\lambda_1}\right)(e^{2\lambda_1 t}-1)\\
&- \left(\frac{2(2BCQ-ST(B+C))}{\lambda^2(\lambda_1+\lambda_2)}\right)(e^{(\lambda_1+\lambda_2)t}-1)\Bigg]\\
&\pm \left[\frac{2BCQ}{\lambda^2 T}\left[\frac{B}{2\lambda_2}(e^{2\lambda_2 t}-1)+\frac{C}{2\lambda_1}(e^{\lambda_1 t}-1)-\frac{B+C}{\lambda_1+\lambda_2}(e^{(\lambda_1+\lambda_2)t}-1)\right]\right.\\
&+ \frac{S}{\lambda^2}\left[\frac{(B+C)^2}{\lambda_1+\lambda_2}(e^{(\lambda_1+\lambda_2)t}-1)-\frac{BC}{\lambda_2}(e^{2\lambda_2 t}-1)-\frac{BC}{\lambda_1}(e^{2\lambda_2 t}-1)\right]\\
&+ \frac{2BCQ}{\lambda^2 S}\left[\frac{B}{2\lambda_2}(e^{2\lambda_2 t}-1)+\frac{C}{2\lambda_1}(e^{\lambda_1 t}-1)-\frac{B+C}{\lambda_1+\lambda_2}(e^{(\lambda_1+\lambda_2)t}-1)\right]\\
&+ \frac{T}{\lambda^2}\left[\frac{(B+C)^2}{\lambda_1+\lambda_2}(e^{(\lambda_1+\lambda_2)t}-1)-\frac{BC}{\lambda_2}(e^{2\lambda_2 t}-1)-\frac{BC}{\lambda_1}(e^{2\lambda_2 t}-1)\right]. \qquad (3.1.59)
\end{aligned}$$

Evaluating the Eq. (3.1.59) at steady state, we get

$$\begin{aligned}
(\Delta c_{\pm}^2)_{ss} &= 1+\frac{2[2BCQ-(ST+BC)(B+C)]}{\lambda^2(\lambda_1+\lambda_2)}+\frac{B(C^2-BQ+ST)}{\lambda^2\lambda_2}\\
&+ \frac{C(B^2-CQ+ST)}{\lambda^2\lambda_1}\pm\frac{S+T}{\lambda^2 ST}\left[\frac{(B+C)[2BCQ-ST(B+C)]}{\lambda_1+\lambda_2}\right.\\
&+ \left.\frac{BC(ST-CQ)}{\lambda_1}+\frac{BC(ST-BQ)}{\lambda_2}\right]. \qquad (3.1.60)
\end{aligned}$$

To simplify our task, it is more convenient to introduce a new parameter η as

$$\rho_{aa}^{(0)} = \frac{1-\eta}{2}. \qquad (3.1.61)$$

We consider the case in which the atoms are initially in a superposition of the upper and the lower levels. Thus, in view of Eqs. (2.1.2) - (2.1.7) the following conditions hold

$$\rho_{aa}^{(0)}+\rho_{cc}^{(0)} = 1, \qquad (3.1.62)$$

$$|\rho_{ac}^{(0)}|^2 = \rho_{aa}^{(0)}\rho_{cc}^{(0)}. \tag{3.1.63}$$

Using these equations along with Eq. (3.1.61), we get

$$\rho_{cc}^{(0)} = \frac{1+\eta}{2}, \tag{3.1.64}$$

$$\rho_{ac}^{(0)} = \frac{1}{2}\sqrt{1-\eta^2}e^{i\theta}, \tag{3.1.65}$$

$$\rho_{ca}^{(0)} = \frac{1}{2}\sqrt{1-\eta^2}e^{-i\theta} \tag{3.1.66}$$

where θ is the phase angle. Since the value of $\rho_{aa}^{(0)}$ is between 0 and 1 then it is not difficult to see that the value of η lies in the interval $-1 \leq \eta \leq 1$. Using Eqs. (3.1.61), (3.1.64), (3.1.65) and (3.1.66) in Eqs. (2.1.38) - (2.1.41), we have

$$Q = \frac{1}{4}A(1-\eta), \tag{3.1.67}$$

$$R = \frac{1}{4}A(1+\eta), \tag{3.1.68}$$

$$S = \frac{1}{4}A\sqrt{1-\eta^2}e^{i\theta}, \tag{3.1.69}$$

$$T = \frac{1}{4}A\sqrt{1-\eta^2}e^{-i\theta}. \tag{3.1.70}$$

Employing Eqs. (2.1.42) and (3.1.67) - (3.1.70) in Eqs. (2.3.11) and (2.3.12), we obtain

$$\lambda_1 = -\frac{1}{2}\kappa, \tag{3.1.71}$$

$$\lambda_2 = -\frac{1}{2}(\kappa + A\eta), \tag{3.1.72}$$

$$\lambda = \lambda_2 - \lambda_1 = -\frac{1}{2}A\eta. \tag{3.1.73}$$

Furthermore, substituting Eqs. (2.1.42), (3.1.67) and (3.1.71) - (3.1.73) into Eqs. (2.3.26) and (2.3.27), we get

$$B = \frac{1}{4}A(1-\eta), \tag{3.1.74}$$

$$C = \frac{1}{4}A(1+\eta). \tag{3.1.75}$$

Finally, substituting Eqs. (3.1.67) - (3.1.75) into Eq. (3.1.60), the quadrature variances at steady state become [20]

$$\Delta c_\pm^2 = \frac{4(A^2\eta^2 + 3A\eta\kappa + 2\kappa^2) + A(1-\eta)(A + 3A\eta + 4\kappa) + A^2(1-\eta^2)}{4(A^2\eta^2 + 3A\eta\kappa + 2\kappa^2)} \pm \frac{A\cos\theta\sqrt{1-\eta^2}(A + A\eta + 2\kappa)}{2(A^2\eta^2 + 3A\eta\kappa + 2\kappa^2)}. \tag{3.1.76}$$

Equation (3.1.76) is independent of ε which shows that the two mode driving light

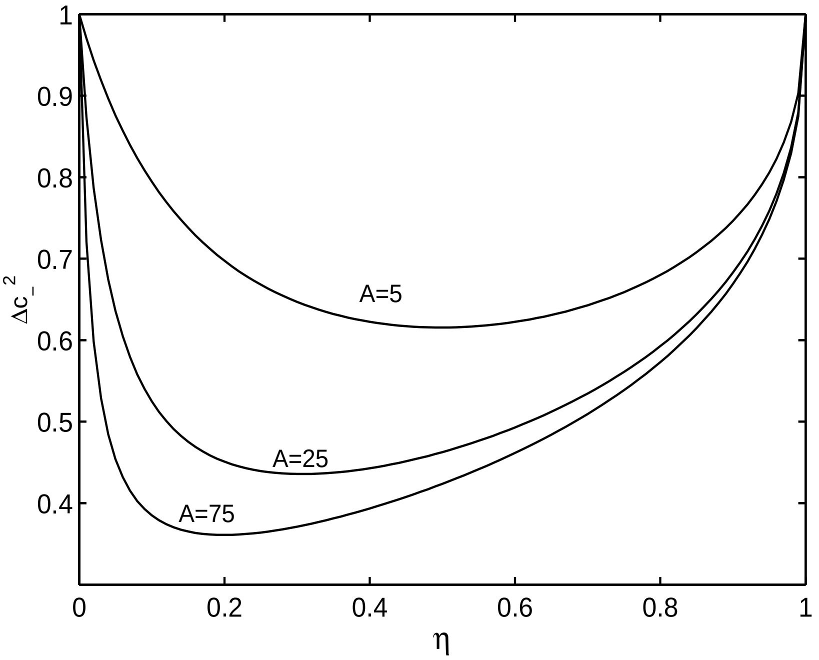


Fig. 3.1 A plot of the variance of the minus quadrature [Eq. (3.1.76)] versus η for $\theta = 0$, $\kappa = 0.8$ and $A = 5, 25, 75$.

has no effect on the squeezing. From Fig. 3.1 we see that squeezing occurs for values of η between 0 and 1. And we also observe that the squeezing increases with the linear gain coefficient. For $A = 75$ and $\kappa = 0.8$ the maximum squeezing is found to be 64%.

3.2 Squeezing Spectrum

We next seek to obtain the squeezing spectrum for the light modes generated by the nondegenerate three-level laser. The squeezing spectrum for a two-mode light is defined as [1,16]

$$S_{\pm}^{out}(\omega) = 2Re\int_0^{\infty}\langle \hat{c}_{\pm}^{out}(t), \hat{c}_{\pm}^{out}(t+\tau)\rangle_{ss} e^{i\omega\tau} d\tau, \tag{3.2.1}$$

where Re denotes the real part. Using the fundamental commutation relation $[\hat{a}(t), \hat{a}^{\dagger}(t')] = [\hat{b}(t), \hat{b}^{\dagger}(t')] = \delta(t-t')$, we have

$$\langle \hat{c}_{\pm}^{out}(t), \hat{c}_{\pm}^{out}(t+\tau)\rangle_{ss} = \delta(\tau) \pm \langle : \hat{c}_{\pm}^{out}(t), \hat{c}_{\pm}^{out}(t+\tau) :\rangle_{ss}, \tag{3.2.2}$$

where :: stands for normal ordering. Using Eq. (3.2.2) in Eq. (3.2.1), we have

$$S_{\pm}^{out}(\omega) = 2Re\int_0^{\infty}\delta(\tau)d\tau \pm 2Re\int_0^{\infty}\langle : \hat{c}_{\pm}^{out}(t), \hat{c}_{\pm}^{out}(t+\tau) :\rangle_{ss} e^{i\omega\tau} d\tau. \tag{3.2.3}$$

Carrying out the first integration in Eq. (3.2.3), we get

$$S_{\pm}^{out}(\omega) = 1 \pm 2Re\int_0^{\infty}\langle : \hat{c}_{\pm}^{out}(t), \hat{c}_{\pm}^{out}(t+\tau) :\rangle_{ss} e^{i\omega\tau} d\tau. \tag{3.2.4}$$

The corresponding c-number function associated with the normal ordering is

$$S_{\pm}^{out}(\omega) = 1 \pm 2Re\int_0^{\infty}\langle \gamma_{\pm}^{out}(t), \gamma_{\pm}^{out}(t+\tau)\rangle_{ss} e^{i\omega\tau} d\tau, \tag{3.2.5}$$

where

$$\gamma_{\pm}^{out}(t) = \frac{1}{\sqrt{2}}(\alpha_{out}^{*}(t) \pm \alpha_{out}(t) + \beta_{out}^{*}(t) \pm \beta_{out}(t)). \tag{3.2.6}$$

For a system coupled to a vacuum reservoir, the output and the intracavity variables are related by [1,16]

$$\gamma_{\pm}^{out}(t) = \sqrt{\kappa}\gamma_{\pm}(t). \tag{3.2.7}$$

In view of this fact, Eq. (3.2.5) takes the form

$$S_{\pm}^{out}(\omega) = 1 \pm 2\kappa Re \int_0^{\infty} \langle \gamma_{\pm}(t), \gamma_{\pm}(t+\tau)\rangle_{ss} e^{i\omega\tau} d\tau, \tag{3.2.8}$$

where

$$\gamma_{\pm}(t) = \frac{1}{\sqrt{2}}(\alpha^*(t) \pm \alpha(t) + \beta^*(t) \pm \beta(t)). \tag{3.2.9}$$

In order to evaluate the two-time correlation function that appears in Eq. (3.2.8), we have to obtain the cavity mode variables α and β^* at time $t+\tau$. Replacing 0 by t and t by $t+\tau$ in Eqs. (2.3.36) and (2.3.37), we get

$$\begin{aligned}
\alpha(t+\tau) &= \frac{\varepsilon}{\lambda\lambda_1}(S-C)(e^{\lambda_1\tau}-1) + \frac{\varepsilon}{\lambda\lambda_2}(B-S)(e^{\lambda_2\tau}-1) \\
&+ \frac{1}{\lambda}(Be^{\lambda_2\tau} - Ce^{\lambda_1\tau})\alpha(t) + \frac{S}{\lambda}(e^{\lambda_1\tau} - e^{\lambda_2\tau})\beta^*(t) \\
&+ F(t+\tau), \qquad (3.2.10)\\
\beta^*(t+\tau) &= \frac{B\varepsilon}{\lambda\lambda_1 S}(S-C)(e^{\lambda_1\tau}-1) + \frac{C\varepsilon}{\lambda\lambda_2 S}(B-S)(e^{\lambda_2\tau}-1) \\
&+ \frac{BC}{\lambda S}(e^{\lambda_2\tau} - e^{\lambda_1\tau})\alpha(t) + \frac{1}{\lambda}(Be^{\lambda_1\tau} - Ce^{\lambda_2\tau})\beta^*(t) \\
&+ G(t+\tau). \qquad (3.2.11)
\end{aligned}$$

For the sake of simplicity, Eqs. (3.2.10) and (3.2.11) can be rewritten as

$$\alpha(t+\tau) = u_1(\tau) + u_2(\tau)\alpha(t) + u_3(\tau)\beta^*(t) + F(t+\tau), \tag{3.2.12}$$

$$\beta^*(t+\tau) = v_1(\tau) + v_2(\tau)\alpha(t) + v_3(\tau)\beta^*(t) + G(t+\tau) \tag{3.2.13}$$

where

$$
\begin{aligned}
u_1(\tau) &= \frac{\varepsilon}{\lambda\lambda_1}(S-C)(e^{\lambda_1\tau}-1)+\frac{\varepsilon}{\lambda\lambda_2}(B-S)(e^{\lambda_2\tau}-1), && (3.2.14)\\
u_2(\tau) &= \frac{1}{\lambda}(Be^{\lambda_2\tau}-Ce^{\lambda_1\tau}), && (3.2.15)\\
u_3(\tau) &= \frac{S}{\lambda}(e^{\lambda_1\tau}-e^{\lambda_2\tau}), && (3.2.16)\\
v_1(\tau) &= \frac{B\varepsilon}{\lambda\lambda_1 S}(S-C)(e^{\lambda_1\tau}-1)+\frac{C\varepsilon}{\lambda\lambda_2 S}(B-S)(e^{\lambda_2\tau}-1), && (3.2.17)\\
v_2(\tau) &= \frac{BC}{\lambda S}(e^{\lambda_2\tau}-e^{\lambda_1\tau}), && (3.2.18)\\
v_3(\tau) &= \frac{1}{\lambda}(Be^{\lambda_1\tau}-Ce^{\lambda_2\tau}). && (3.2.19)
\end{aligned}
$$

Based on Eq. (3.2.9), one can write

$$\langle\gamma_\pm(t+\tau)\rangle=\frac{1}{\sqrt{2}}(\langle\alpha^*(t+\tau)\rangle\pm\langle\alpha(t+\tau)\rangle+\langle\beta^*(t+\tau)\rangle\pm\langle\beta(t+\tau)\rangle), \quad (3.2.20)$$

so that applying the quantum regression theorem [1] and taking into account Eq. (3.2.9), we find

$$
\begin{aligned}
\langle\gamma_\pm(t)\gamma_\pm(t+\tau)\rangle &= \frac{1}{2}\Big[[\langle\alpha^*(t)\alpha^*(t+\tau)\rangle+\langle\beta^*(t)\beta^*(t+\tau)\rangle+\langle\alpha(t)\alpha(t+\tau)\rangle\\
&+ \langle\beta(t)\alpha(t+\tau)\rangle+\langle\alpha^*(t)\beta^*(t+\tau)\rangle+\langle\beta^*(t)\alpha^*(t+\tau)\rangle\\
&+ \langle\alpha(t)\beta(t+\tau)\rangle+\langle\beta(t)\beta(t+\tau)\rangle]\pm[\langle\alpha(t)\alpha^*(t+\tau)\rangle\\
&+ \langle\beta(t)\alpha^*(t+\tau)\rangle+\langle\alpha^*(t)\alpha(t+\tau)\rangle+\langle\beta^*(t)\alpha(t+\tau)\rangle\\
&+ \langle\alpha(t)\beta^*(t+\tau)\rangle+\langle\beta(t)\beta^*(t+\tau)\rangle+\langle\alpha^*(t)\beta(t+\tau)\rangle\\
&+ \langle\beta^*(t)\beta(t+\tau)\rangle\Big]. \quad (3.2.21)
\end{aligned}
$$

Multiplying Eq. (3.2.12) by $\alpha^*(t)$ and taking the expectation value, we find

$$
\begin{aligned}
\langle\alpha^*(t)\alpha(t+\tau)\rangle &= u_1(\tau)\langle\alpha^*(t)\rangle+u_2(\tau)\langle\alpha^*(t)\alpha(t)\rangle+u_3(\tau)\langle\alpha^*(t)\beta^*(t)\rangle\\
&+ \langle\alpha^*(t)F(t+\tau)\rangle. \quad (3.2.22)
\end{aligned}
$$

With the aid of the fact that the noise force at time $t+\tau$ doesn't affect the cavity mode variables at earlier time, we have

$$\langle \alpha^*(t)F(t+\tau)\rangle = 0. \tag{3.2.23}$$

In view of this equation, Eq. (3.2.22) takes the form

$$\begin{aligned}\langle \alpha^*(t)\alpha(t+\tau)\rangle &= u_1(\tau)\langle \alpha^*(t)\rangle + u_2(\tau)\langle \alpha^*(t)\alpha(t)\rangle \\ &+ u_3(\tau)\langle \alpha^*(t)\beta^*(t)\rangle.\end{aligned} \tag{3.2.24}$$

Multiplying the complex conjugate of Eq. (3.2.13) by $\alpha(t)$ and taking the expectation value, we get

$$\begin{aligned}\langle \alpha(t)\beta(t+\tau)\rangle &= v_1^*(\tau)\langle \alpha(t)\rangle + v_2^*(\tau)\langle \alpha(t)\alpha^*(t)\rangle + v_3^*(\tau)\langle \alpha(t)\beta(t)\rangle \\ &+ \langle \alpha(t)G^*(t+\tau)\rangle.\end{aligned} \tag{3.2.25}$$

On account of the fact that the noise force at time $t+\tau$ doesn't affect the cavity mode variables at earlier time, we have

$$\langle \alpha(t)G^*(t+\tau)\rangle = 0. \tag{3.2.26}$$

Using this fact in Eq. (3.2.25), we find

$$\langle \alpha(t)\beta(t+\tau)\rangle = v_1^*(\tau)\langle \alpha(t)\rangle + v_2^*(\tau)\langle \alpha(t)\alpha^*(t)\rangle + v_3^*(\tau)\langle \alpha(t)\beta(t)\rangle. \tag{3.2.27}$$

We can also show in a similar fashion that

$$\langle\beta(t)\alpha^*(t+\tau)\rangle = u_1^*(\tau)\langle\beta(t)\rangle + u_2^*(\tau)\langle\beta(t)\alpha^*(t)\rangle + u_3^*(\tau)\langle\beta^2(t)\rangle, \quad (3.2.28)$$

$$\langle\alpha(t)\alpha(t+\tau)\rangle = u_1(\tau)\langle\alpha(t)\rangle + u_2(\tau)\langle\alpha^2(t)\rangle + u_3(\tau)\langle\alpha(t)\beta^*(t)\rangle, \quad (3.2.29)$$

$$\langle\beta(t)\alpha(t+\tau)\rangle = u_1(\tau)\langle\beta(t)\rangle + u_2(\tau)\langle\beta(t)\alpha(t)\rangle + u_3(\tau)\langle\beta(t)\beta^*(t)\rangle, (3.2.30)$$

$$\langle\alpha(t)\beta^*(t+\tau)\rangle = v_1(\tau)\langle\alpha(t)\rangle + v_2(\tau)\langle\alpha(t)\alpha(t)\rangle + v_3(\tau)\langle\alpha(t)\beta^*(t)\rangle, (3.2.31)$$

$$\langle\beta(t)\beta^*(t+\tau)\rangle = v_1(\tau)\langle\beta(t)\rangle + v_2(\tau)\langle\beta(t)\alpha(t)\rangle + v_3(\tau)\langle\beta(t)\beta^*(t)\rangle, (3.2.32)$$

$$\langle\beta(t)\beta(t+\tau)\rangle = v_1^*(\tau)\langle\beta(t)\rangle + v_2^*(\tau)\langle\beta(t)\alpha^*(t)\rangle + v_3^*(\tau)\langle\beta(t)\beta^*(t)\rangle. (3.2.33)$$

Substituting Eq. (3.2.24) and Eqs. (3.2.27) - (3.2.33) into Eq. (3.2.21), we get

$$\begin{aligned}\langle\gamma_\pm(t)\gamma_\pm(t+\tau)\rangle &= \frac{1}{2}\Big[[u_1^*(\tau) \pm u_1(\tau) + v_1(\tau) \pm v_1^*(\tau)][\langle\alpha^*(t)\rangle \pm \langle\alpha(t)\rangle \\ &+ \langle\beta^*(t)\rangle \pm \langle\beta(t)\rangle] + (u_2^*(\tau) \pm v_2^*(\tau))\langle\alpha^{*2}(t)\rangle \\ &+ (u_2(\tau) \pm v_2(\tau))\langle\alpha^2(t)\rangle + (v_3^*(\tau) \pm u_3^*(\tau))\langle\beta^2(t)\rangle \\ &+ (v_3(\tau) \pm u_3(\tau))\langle\beta^{*2}(t)\rangle \\ &+ [v_2^*(\tau) + v_2(\tau) \pm (u_2^*(\tau) + u_2(\tau))]\langle\alpha^*(t)\alpha(t)\rangle \\ &+ [u_3^*(\tau) + u_3(\tau) \pm (v_3^*(\tau) + v_3(\tau))]\langle\beta^*(t)\beta(t)\rangle \\ &+ [u_2(\tau) + v_3^*(\tau) \pm (v_2(\tau) + u_3^*(\tau))]\langle\alpha(t)\beta(t)\rangle \\ &+ [u_2^*(\tau) + v_3(\tau) \pm (v_2^*(\tau) + u_3(\tau))]\langle\alpha^*(t)\beta^*(t)\rangle \\ &+ [u_3^*(\tau) + v_2^*(\tau) \pm (u_2^*(\tau) + v_3^*(\tau))]\langle\alpha^*(t)\beta(t)\rangle \\ &+ [v_2(\tau) + u_3(\tau) \pm (u_2(\tau) + v_3(\tau))]\langle\alpha(t)\beta^*(t)\rangle\Big]. \quad (3.2.34)\end{aligned}$$

From Eqs. (3.2.9) and (3.2.20), we also easily see that

$$\begin{aligned}
\langle\gamma_\pm(t)\rangle\langle\gamma_\pm(t+\tau)\rangle &= \frac{1}{2}\Big[[u_1^*(\tau)\pm u_1(\tau)+v_1(\tau)\pm v_1^*(\tau)][\langle\alpha^*(t)\rangle\pm\langle\alpha(t)\rangle \\
&+ \langle\beta^*(t)\rangle\pm\langle\beta(t)\rangle]+(u_2^*(\tau)\pm v_2^*(\tau))\langle\alpha^*(t)\rangle^2 \\
&+ (u_2(\tau)\pm v_2(\tau))\langle\alpha(t)\rangle^2+(v_3^*(\tau)\pm u_3^*(\tau))\langle\beta(t)\rangle^2 \\
&+ [v_3(\tau)\pm u_3(\tau)]\langle\beta^*(t)\rangle^2 \\
&+ [v_2^*(\tau)+v_2(\tau)\pm(u_2^*(\tau)+u_2(\tau))]\langle\alpha^*(t)\rangle\langle\alpha(t)\rangle \\
&+ [u_3^*(\tau)+u_3(\tau)\pm(v_3^*(\tau)+v_3(\tau))]\langle\beta^*(t)\rangle\langle\beta(t)\rangle \\
&+ [u_2(\tau)+v_3^*(\tau)\pm(v_2(\tau)+u_3^*(\tau))]\langle\alpha(t)\rangle\langle\beta(t)\rangle \\
&+ [u_2^*(\tau)+v_3(\tau)\pm(v_2^*(\tau)+u_3(\tau))]\langle\alpha^*(t)\rangle\langle\beta^*(t)\rangle \\
&+ [u_3^*(\tau)+v_2^*(\tau)\pm(u_2^*(\tau)+v_3^*(\tau))]\langle\alpha^*(t)\rangle\langle\beta(t)\rangle \\
&+ [v_2(\tau)+u_3(\tau)\pm(u_2(\tau)+v_3(\tau))]\langle\alpha(t)\rangle\langle\beta^*(t)\rangle\Big]. \quad (3.2.35)
\end{aligned}$$

Applying Eqs. (3.2.34) and (3.2.35) along with Eqs. (3.1.55) - (3.1.57) and their complex conjugates, we have

$$\begin{aligned}
\langle\gamma_\pm(t),\gamma_\pm(t+\tau)\rangle &= \frac{1}{2}\Big[[v_2^*(\tau)+v_2(\tau)\pm(u_2^*(\tau)+u_2(\tau))](\langle\alpha^*(t)\alpha(t)\rangle-\langle\alpha^*(t)\rangle\langle\alpha(t)\rangle) \\
&+ [u_3^*(\tau)+u_3(\tau)\pm(v_3^*(\tau)+v_3(\tau))](\langle\beta^*(t)\beta(t)\rangle-\langle\beta^*(t)\rangle\langle\beta(t)\rangle) \\
&+ [u_2(\tau)+v_3^*(\tau)\pm(v_2(\tau)+u_3^*(\tau))](\langle\alpha(t)\beta(t)\rangle-\langle\alpha(t)\rangle\langle\beta(t)\rangle) \\
&+ [u_2^*(\tau)+v_3(\tau)\pm(v_2^*(\tau)+u_3(\tau))](\langle\alpha^*(t)\beta^*(t)\rangle \\
&- \langle\alpha^*(t)\rangle\langle\beta^*(t)\rangle)\Big]. \qquad (3.2.36)
\end{aligned}$$

On account of Eqs. (3.1.19), (3.1.25), (3.1.30), (3.1.49), (3.1.51) and (3.1.53), Eq. (3.2.36) takes the form

$$\begin{aligned}\langle\gamma_{\pm}(t),\gamma_{\pm}(t+\tau)\rangle &= \frac{1}{2}\Big[[v_2^*(\tau)+v_2(\tau)\pm(u_2^*(\tau)+u_2(\tau))]\langle F^*(t)F(t)\rangle \\ &+ [u_3^*(\tau)+u_3(\tau)\pm(v_3^*(\tau)+v_3(\tau))]\langle G^*(t)G(t)\rangle \\ &+ [u_2(\tau)+v_3^*(\tau)\pm(v_2(\tau)+u_3^*(\tau))]\langle F(t)G^*(t)\rangle \\ &+ [u_2^*(\tau)+v_3(\tau)\pm(v_2^*(\tau)+u_3(\tau))]\langle F^*(t)G(t)\rangle\Big]. \quad (3.2.37)\end{aligned}$$

Using Eqs. (3.2.14) - (3.2.19) in Eq. (3.2.37), we have

$$\langle\gamma_{\pm}(t),\gamma_{\pm}(t+\tau)\rangle = f_{\pm}(t)e^{\lambda_2\tau}-g_{\pm}(t)e^{\lambda_1\tau} \quad (3.2.38)$$

where

$$\begin{aligned}f_{\pm}(t) &= \frac{1}{2}\Big[\frac{BC(S+T)\pm 2BST}{\lambda ST}\langle F^*(t)F(t)\rangle \\ &- \frac{(S+T)\pm 2C}{\lambda}\langle G^*(t)G(t)\rangle \\ &+ \frac{(B-C)S\pm(BC-ST)}{\lambda S}\langle F(t)G^*(t)\rangle \\ &+ \frac{(B-C)T\pm(BC-ST)}{\lambda T}\langle F^*(t)G(t)\rangle\Big] \quad (3.2.39)\end{aligned}$$

and

$$\begin{aligned}g_{\pm}(t) &= \frac{1}{2}\Big[\frac{BC(S+T)\pm 2CST}{\lambda ST}\langle F^*(t)F(t)\rangle \\ &- \frac{(S+T)\pm 2B}{\lambda}\langle G^*(t)G(t)\rangle \\ &+ \frac{(C-B)S\pm(BC-ST)}{\lambda S}\langle F(t)G^*(t)\rangle \\ &+ \frac{(C-B)T\pm(BC-ST)}{\lambda T}\langle F^*(t)G(t)\rangle\Big]. \quad (3.2.40)\end{aligned}$$

At steady state, Eq. (3.2.38) can be written as

$$\langle\gamma_{\pm}(t),\gamma_{\pm}(t+\tau)\rangle_{ss} = (f_{\pm}(t))_{ss}e^{\lambda_2\tau}-(g_{\pm}(t))_{ss}e^{\lambda_1\tau}. \quad (3.2.41)$$

Substituting Eq. (3.2.41) into Eq. (3.2.8), we have

$$S_{\pm}^{out}(\omega) = 1 \pm 2\kappa Re\int_0^\infty \left[(f_\pm(t))_{ss}e^{\lambda_2+i\omega\tau} - (g_\pm(t))_{ss}e^{\lambda_1+i\omega\tau}\right]d\tau. \quad (3.2.42)$$

Carrying out the integration, we get

$$S_{\pm}^{out}(\omega) = 1 \pm 2\kappa Re\left[(f_\pm(t))_{ss}\left(\frac{1}{-\lambda_2 - i\omega}\right) - (g_\pm(t))_{ss}\left(\frac{1}{-\lambda_1 - i\omega}\right)\right] \quad (3.2.43)$$

Since $f_\pm(t)$ and $g_\pm(t)$ are real, Eq. (3.2.43) can be written as

$$S_{\pm}^{out}(\omega) = 1 \pm 2\kappa\left[\frac{(g_\pm(t))_{ss}\lambda_1}{\lambda_1^2+\omega^2} - \frac{(f_\pm(t))_{ss}\lambda_2}{\lambda_2^2+\omega^2}\right]. \quad (3.2.44)$$

In view of Eqs. (3.1.21), (3.1.27) and (3.1.32) along with Eqs. (3.1.67) - (3.1.75), Eqs. (3.2.39) and (3.2.40) takes the form

$$(f_\pm(t))_{ss} = \frac{A\sqrt{1-\eta^2}(A\eta^2 - (1-2\eta)\kappa)\cos(\theta)}{2\eta(A^2\eta^2 + 3A\eta\kappa + 2\kappa^2)} \pm \frac{A(1-\eta)(A\eta^2 - (1-\eta)\kappa)}{2\eta(A^2\eta^2 + 3A\eta\kappa + 2\kappa^2)} \quad (3.2.45)$$

and

$$(g_\pm(t))_{ss} = \frac{-A\sqrt{1-\eta^2}\cos(\theta)}{2\eta(A\eta + 2\kappa)} \pm \frac{A(-1+\eta^2)}{2\eta(A\eta+2\kappa)}. \quad (3.2.46)$$

On account of Eqs. (3.1.71), (3.1.72), (3.2.45) and (3.2.46), Eq. (3.2.44) becomes

$$S_{\pm}^{out}(\omega) = 1 + \frac{2A\kappa(1-\eta)(\kappa(A + A\eta + \kappa) + 4\omega^2)}{(\kappa^2 + 4\omega^2)((A\eta+\kappa)^2 + 4\omega^2)} \pm \frac{2A\kappa\sqrt{1-\eta^2}\cos(\theta)(\kappa(A+\kappa) + 4\omega^2)}{(\kappa^2+4\omega^2)((A\eta+\kappa)^2+4\omega^2)}. \quad (3.2.47)$$

For $\omega = 0$ and for $\theta = 0$, Eq. (3.2.47) turns out to be

$$S_{\pm}^{out}(0) = 1 + \frac{2A((1-\eta)(A + A\eta + \kappa) \pm \sqrt{1-\eta^2}(A+\kappa))}{(A\eta+\kappa)^2}. \quad (3.2.48)$$

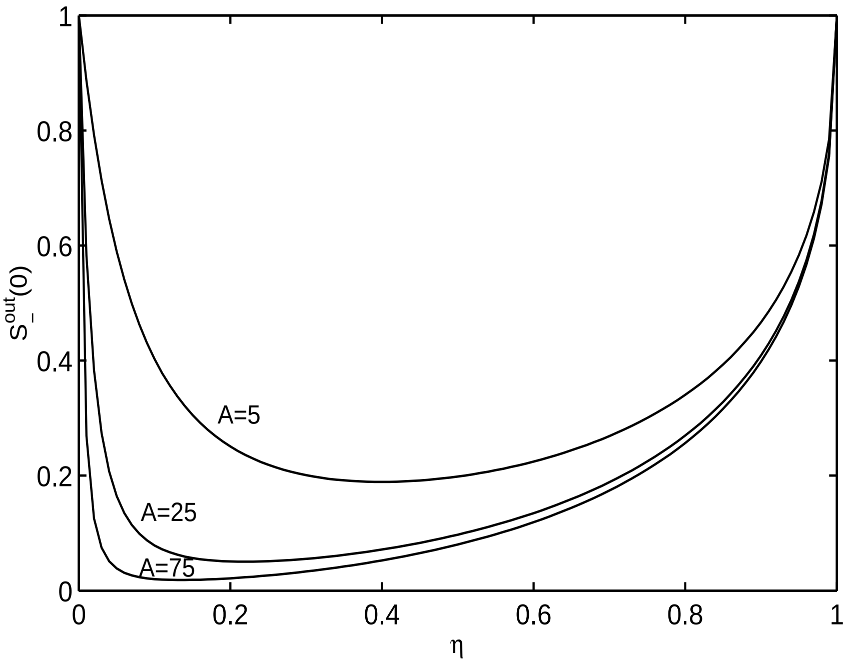


Fig. 3.2 A plot of the squeezing spectrum [Eq. (3.2.48)] versus η for $k = 0.8$ and $A = 5, 25, 75$.

Equation (3.2.48) is independent of ε which shows that the two mode driving light has no effect on the squeezing. From Fig. 3.2 we observe that the squeezing of the cavity modes occurs for values of η between 0 and 1. In addition we see that the squeezing increases with the linear gain coefficient.

Chapter 4

Photon Statistics

In this chapter, we seek to calculate the mean of the photon number, the mean of the photon number sum and difference, and the variances of the photon number sum and difference of the two mode-light (modes a and b).

4.1 Mean Photon Number

In this section we want to calculate the mean photon number for the two cavity modes. The photon number operators for the two modes are defined by

$$\hat{n}_a(t) = \hat{a}^\dagger(t)\hat{a}(t), \tag{4.1.1}$$

$$\hat{n}_b(t) = \hat{b}^\dagger(t)\hat{b}(t). \tag{4.1.2}$$

where $\hat{n}_a(\hat{n}_b)$ is the photon number operator for mode $a(b)$. Equations (4.1.1) and (4.1.2) are already in the normal order. Thus, the mean photon numbers are expressible in terms of c-number variables associated with the normal ordering as

$$\bar{n}_a(t) = \langle \alpha^*(t)\alpha(t)\rangle, \tag{4.1.3}$$

$$\bar{n}_b(t) = \langle \beta^*(t)\beta(t)\rangle. \tag{4.1.4}$$

On account of Eqs. (3.1.23) and (3.1.29), Eqs. (4.1.3) and (4.1.4) has at steady state the form

$$\begin{aligned}
\bar{n}_a &= \frac{\varepsilon^2}{\lambda^2}\left[\frac{(S-C)(T-C)}{\lambda_1^2}+\frac{(S-B)(T-B)}{\lambda_2^2}\right] \\
&+ \frac{\varepsilon^2}{\lambda^2}\left[\frac{(T-C)(B-S)+(B-T)(S-C)}{\lambda_1\lambda_2}\right] \\
&+ \frac{2(2BCQ-ST(B+C))}{\lambda^2(\lambda_1+\lambda_2)}-\frac{B^2Q-BST}{\lambda^2\lambda_2}-\frac{C^2Q-CST}{\lambda^2\lambda_1}, \qquad (4.1.5)\\
\bar{n}_b &= \frac{\varepsilon^2}{\lambda^2 ST}\left[\frac{B^2(S-C)(T-C)}{\lambda_1^2}+\frac{C^2(S-B)(T-B)}{\lambda_2^2}\right] \\
&+ \frac{\varepsilon^2}{\lambda^2 ST}\left[\frac{BC[(T-C)(B-S)+(B-T)(S-C)]}{\lambda_1\lambda_2}\right] \\
&+ \frac{B^2C}{\lambda^2\lambda_1}+\frac{C^2B}{\lambda^2\lambda_2}-\frac{2BC(B+C)}{\lambda^2(\lambda_1+\lambda_2)}-\frac{QB^2C^2}{(\lambda_1+\lambda_2)\lambda_1\lambda_2 ST}. \qquad (4.1.6)
\end{aligned}$$

With the aid of Eqs. (3.1.67) - (3.1.75), Eqs. (4.1.5) and (4.1.6) become

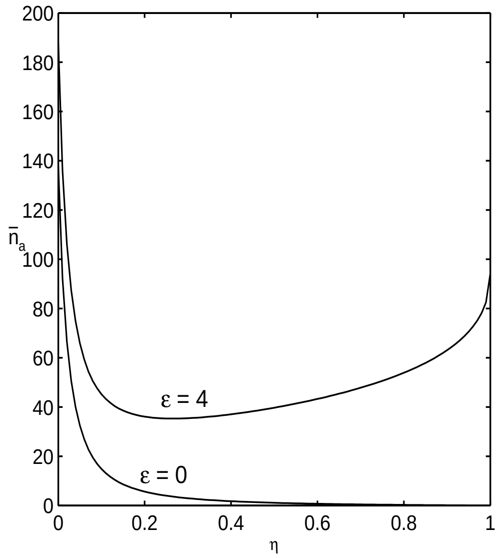
$$\begin{aligned}
\bar{n}_a &= \frac{4\varepsilon^2}{(A\eta+2\kappa)\kappa}+2\varepsilon^2\left[\frac{A(A+A\eta+2\kappa)(1-\sqrt{1-\eta^2}\cos\theta)}{(A\eta+2\kappa)^2\kappa^2}\right] \\
&+ \frac{A(1-\eta)(A+3A\eta+4\kappa)}{4(A\eta+\kappa)(A\eta+2\kappa)}, \qquad (4.1.7)\\
\bar{n}_b &= \frac{4\varepsilon^2}{(A\eta+2\kappa)\kappa}+2\varepsilon^2\left[\frac{A(A-A\eta-2\kappa)(1-\sqrt{1-\eta^2}\cos\theta)}{(A\eta+2\kappa)^2\kappa^2}\right] \\
&+ \frac{A^2(1-\eta^2)}{4(A\eta+\kappa)(A\eta+2\kappa)}. \qquad (4.1.8)
\end{aligned}$$

For the case in which there is no driving radiation ($\varepsilon = 0$), the mean photon numbers reduce to

$$\bar{n}_a = \frac{A(1-\eta)(A+3A\eta+4\kappa)}{4(A\eta+\kappa)(A\eta+2\kappa)}, \qquad (4.1.9)$$

$$\bar{n}_b = \frac{A^2(1-\eta^2)}{4(A\eta+\kappa)(A\eta+2\kappa)}. \qquad (4.1.10)$$

From Fig. 4.1 and Fig. 4.2 we see that the mean of the photon numbers with the driving light ($\varepsilon \neq 0$) is greater than with out the driving light ($\varepsilon = 0$). In other



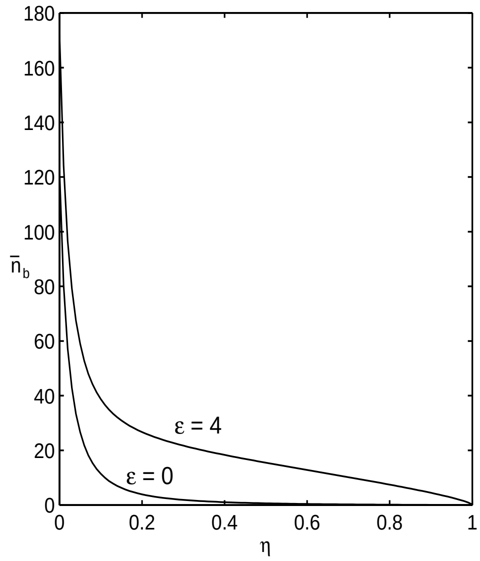


Fig. 4.1 Left: Plots of $\bar{n}_a$ [Eqs. (4.1.7) and (4.1.9)] versus η, for $A = 25$, $\kappa = 0.8$, and $\varepsilon = 0, 4$.

Fig. 4.2 Right: Plots of $\bar{n}_b$ [Eqs. (4.1.8) and (4.1.10)] versus η, for $A = 25$, $\kappa = 0.8$, and $\varepsilon = 0, 4$.

words, the driving light increases the mean of the photon numbers. We also observe that the mean photon number of mode a is greater than that of mode b.

We are also intersted in the mean of the photon number sum and difference of the two-mode light. We define the photon number sum $\hat{n}_+$ and difference $\hat{n}_-$ by

$$\hat{n}_\pm(t) = \hat{n}_a(t) \pm \hat{n}_b(t). \tag{4.1.11}$$

Upon taking the expectation value of Eq. (4.1.11) and applying Eqs. (4.1.7) and (4.1.8), the mean of the photon number sum and difference are found at steady state to be

$$\begin{aligned} \bar{n}_+ &= \frac{8\varepsilon^2}{(A\eta + 2\kappa)\kappa} + \frac{4\varepsilon^2 A^2 (1 - \sqrt{1-\eta^2}\cos\theta)}{(A\eta + 2\kappa)^2\kappa^2} + \frac{A(1-\eta)(A + 3A\eta + 4\kappa)}{4(A\eta + \kappa)(A\eta + 2\kappa)} \\ &+ \frac{A^2(1-\eta^2)}{4(A\eta + \kappa)(A\eta + 2\kappa)}, \end{aligned} \tag{4.1.12}$$

$$\bar{n}_{-} = \frac{4\varepsilon^2 A(A\eta+\kappa)(1-\sqrt{1-\eta^2}\cos\theta)}{(A\eta+2\kappa)^2\kappa^2} + \frac{A(1-\eta)(A+3A\eta+4\kappa)}{4(A\eta+\kappa)(A\eta+2\kappa)} - \frac{A^2(1-\eta^2)}{4(A\eta+\kappa)(A\eta+2\kappa)}. \quad (4.1.13)$$

For the case in which there is no driving radiation ($\varepsilon = 0$), we find

$$\bar{n}_{\pm} = \frac{A(1-\eta)(A+3A\eta+4\kappa)}{4(A\eta+\kappa)(A\eta+2\kappa)} \pm \frac{A^2(1-\eta^2)}{4(A\eta+\kappa)(A\eta+2\kappa)}. \quad (4.1.14)$$

From Fig. 4.3 and Fig. 4.4 we see that the mean of the photon number sum and

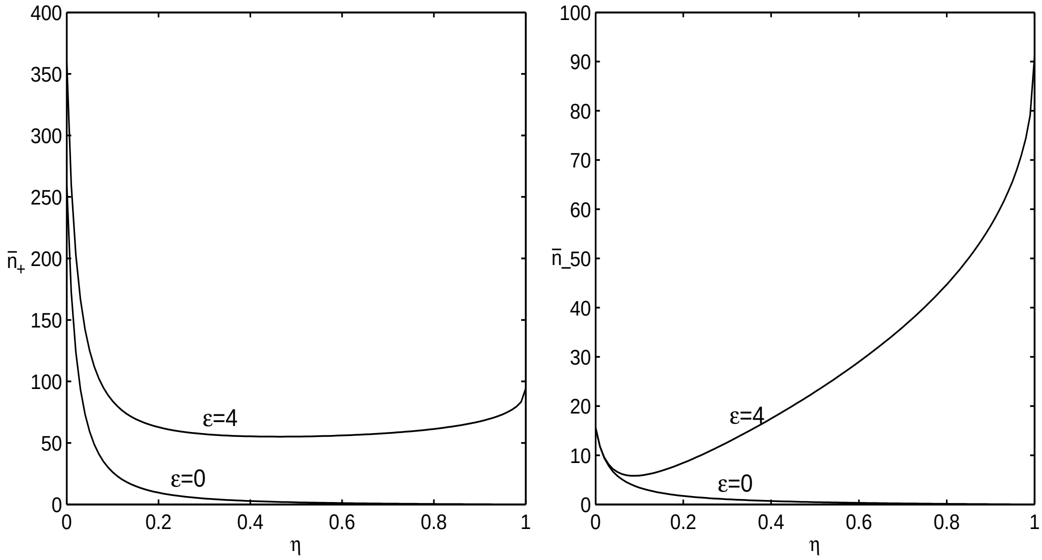


Fig. 4.3 Left: Plots of $\bar{n}_+$ [Eqs. (4.1.12) and (4.1.14)] versus η, for $A = 25$, $\kappa = 0.8$, and $\varepsilon = 0, 4$.

Fig. 4.4 Right: Plots of $\bar{n}_-$ [Eqs. (4.1.13) and (4.1.14)] versus η, for $A = 25$, $\kappa = 0.8$, and $\varepsilon = 0, 4$.

difference are larger in the presence of driving light than in its absence.

4.2 Variances of the Photon Number Sum and Difference

The variances of the photon number sum and difference are given by

$$\Delta n_{\pm}^2(t) = \langle \hat{n}_{\pm}^2(t)\rangle - \langle \hat{n}_{\pm}(t)\rangle^2. \tag{4.2.1}$$

In view of Eq. (4.1.11), Eq. (4.2.1) takes the form

$$\begin{aligned}\Delta n_{\pm}^2(t) &= \langle \hat{n}_a^2(t)\rangle - \langle \hat{n}_a(t)\rangle^2 + \langle \hat{n}_b^2(t)\rangle - \langle \hat{n}_b(t)\rangle^2 \\ &\pm 2[\langle \hat{n}_a(t)\hat{n}_b(t)\rangle - \langle \hat{n}_a(t)\rangle\langle \hat{n}_b(t)\rangle].\end{aligned} \tag{4.2.2}$$

On account of Eqs. (4.1.1) and (4.1.2) and putting the operators in the normal order, we get

$$\begin{aligned}\Delta n_{\pm}^2(t) &= \langle \hat{a}^{\dagger 2}(t)\hat{a}^2(t)\rangle - \langle \hat{a}^{\dagger}(t)\hat{a}(t)\rangle^2 + \langle \hat{a}^{\dagger}(t)\hat{a}(t)\rangle \\ &+ \langle \hat{b}^{\dagger 2}(t)\hat{b}^2(t)\rangle - \langle \hat{b}^{\dagger}(t)\hat{b}(t)\rangle^2 + \langle \hat{b}^{\dagger}(t)\hat{b}(t)\rangle \\ &\pm 2[\langle \hat{a}^{\dagger}(t)\hat{b}^{\dagger}(t)\hat{a}(t)\hat{b}(t)\rangle - \langle \hat{a}^{\dagger}(t)\hat{a}(t)\rangle\langle \hat{b}^{\dagger}(t)\hat{b}(t)\rangle].\end{aligned} \tag{4.2.3}$$

The variances can be expressed in terms of the corresponding c-number variables associated with the normal ordering as

$$\begin{aligned}\Delta n_{\pm}^2(t) &= \langle \alpha^{*2}(t)\alpha^2(t)\rangle - \langle \alpha^*(t)\alpha(t)\rangle^2 + \langle \alpha^*(t)\alpha(t)\rangle \\ &+ \langle \beta^{*2}(t)\beta^2(t)\rangle - \langle \beta^*(t)\beta(t)\rangle^2 + \langle \beta^*(t)\beta(t)\rangle \\ &\pm 2[\langle \alpha^*(t)\beta^*(t)\alpha(t)\beta(t)\rangle - \langle \alpha^*(t)\alpha(t)\rangle\langle \beta^*(t)\beta(t)\rangle].\end{aligned} \tag{4.2.4}$$

With the aid of Eqs. (2.3.36) and (2.3.37), the cavity mode variables at time t can be written as

$$\alpha(t) = u_1(t) + u_2(t)\alpha(0) + u_3(0)\beta^*(0) + F(t), \tag{4.2.5}$$

$$\beta^*(t) = v_1(t) + v_2(t)\alpha(0) + v_3(t)\beta^*(0) + G(t). \tag{4.2.6}$$

where

$$u_1(t) = \frac{\varepsilon}{\lambda\lambda_1}(S-C)(e^{\lambda_1 t}-1) + \frac{\varepsilon}{\lambda\lambda_2}(B-S)(e^{\lambda_2 t}-1), \tag{4.2.7}$$

$$u_2(t) = \frac{1}{\lambda}(Be^{\lambda_2 t} - Ce^{\lambda_1 t}), \tag{4.2.8}$$

$$u_3(t) = \frac{S}{\lambda}(e^{\lambda_1 t} - e^{\lambda_2 t}), \tag{4.2.9}$$

$$v_1(t) = \frac{B\varepsilon}{\lambda\lambda_1 S}(S-C)(e^{\lambda_1 t}-1) + \frac{C\varepsilon}{\lambda\lambda_2 S}(B-S)(e^{\lambda_2 t}-1), \tag{4.2.10}$$

$$v_2(t) = \frac{BC}{\lambda S}(e^{\lambda_2 t} - e^{\lambda_1 t}), \tag{4.2.11}$$

$$v_3(t) = \frac{1}{\lambda}(Be^{\lambda_1 t} - Ce^{\lambda_2 t}). \tag{4.2.12}$$

Taking into account the noise force at time t doesn't affect the cavity mode variables at earlier time and with the aid of Eqs. (3.1.14) and (3.1.15), we have

$$\begin{aligned}\langle \alpha^{*2}(t)\alpha(t)^2\rangle &= u_1^{*2}u_1^2 + u_1^{*2}\langle F^2\rangle + u_1^2\langle F^{*2}\rangle + 4u_1^*u_1\langle F^*F\rangle \\ &+ 2u_1^*\langle F^*F^2\rangle + 2u_1\langle FF^{*2}\rangle + \langle F^{*2}F^2\rangle,\end{aligned} \tag{4.2.13}$$

$$\begin{aligned}\langle \beta^{*2}(t)\beta^2(t))\rangle &= v_1^{*2}v_1^2 + v_1^{*2}\langle G^2\rangle + v_1^2\langle G^{*2}\rangle + 4v_1^*v_1\langle G^*G\rangle \\ &+ 2v_1^*\langle G^*G^2\rangle + 2v_1\langle GG^{*2}\rangle + \langle G^{*2}G^2\rangle,\end{aligned} \tag{4.2.14}$$

$$\begin{aligned}\langle \alpha^*(t)\alpha(t)\beta^*(t)\beta(t)\rangle &= u_1^*u_1v_1^*v_1 + u_1^*u_1\langle G^*G\rangle + u_1^*v_1\langle G^*F\rangle \\ &+ u_1^*v_1^*\langle FG\rangle + u_1^*\langle FG^*G\rangle + u_1v_1\langle F^*G^*\rangle \\ &+ u_1v_1^*\langle F^*G\rangle + u_1\langle F^*G^*G\rangle + v_1v_1^*\langle F^*F\rangle \\ &+ v_1\langle F^*FG^*\rangle + v_1^*\langle F^*FG\rangle + \langle F^*FG^*G\rangle,\end{aligned} \tag{4.2.15}$$

$$\langle(\alpha^*(t)\alpha(t))\rangle^2 = u_1^{*2}(t)u_1^2 + 2u_1^*u_1\langle F^*F\rangle + \langle F^*F\rangle^2, \tag{4.2.16}$$

$$\langle(\beta^*(t)\beta(t))\rangle^2 = (v_1^*v_1)^2 + 2v_1^*v_1\langle G^*G\rangle + \langle G^*G\rangle^2, \tag{4.2.17}$$

$$\begin{aligned}
\langle \alpha^*(t)\alpha(t)\rangle &= u_1^* u_1 + \langle F^* F\rangle, &(4.2.18)\\
\langle \beta^*(t)\beta(t)\rangle &= v_1^* v_1 + \langle G^* G\rangle, &(4.2.19)\\
\langle \alpha^*(t)\alpha(t)\rangle\langle \beta^*(t)\beta(t)\rangle &= u_1^* u_1 v_1^* v_1 + u_1^* u_1 \langle G^* G\rangle + v_1^* v_1 \langle F^* F\rangle \\
&+ \langle F^* F\rangle\langle G^* G\rangle. &(4.2.20)
\end{aligned}$$

Substituting Eqs. (4.2.13) - (4.2.20) into Eq. (4.2.4), we get

$$\begin{aligned}
\Delta n_{\pm}^2(t) &= \Big[u_1^* u_1 + \langle F^* F\rangle + u_1^{*2}\langle F^2\rangle + u_1(t)^2\langle F^{*2}\rangle + 2u_1^*(t)u_1\langle F^* F\rangle \\
&+ 2u_1\langle F F^{*2}\rangle + 2u_1^*\langle F^* F^2\rangle + [\langle F^{*2}F^2\rangle - \langle F^* F\rangle^2] + v_1^* v_1 \\
&+ \langle G^* G\rangle + v_1^{*2}\langle G^2\rangle + v_1^2\langle G^{*2}\rangle + 2v_1^* v_1\langle G^* G\rangle + 2v_1^*\langle G^* G^2\rangle \\
&+ 2v_1\langle G G^{*2}\rangle + [\langle G^{*2}G^2\rangle - \langle G^* G\rangle^2]\Big] \\
&\pm 2\Big[u_1^* v_1\langle G^* F\rangle + u_1^* v_1^*\langle F G\rangle + u_1^*\langle F G^* G\rangle + u_1 v_1\langle F^* G^*\rangle \\
&+ u_1 v_1^*\langle F^* G\rangle + u_1\langle F^* G^* G\rangle + v_1\langle F^* F G^*\rangle + v_1^*\langle F^* F G\rangle \\
&+ [\langle F^* F G^* G\rangle - \langle F^* F\rangle\langle G^* G\rangle]\Big]. &(4.2.21)
\end{aligned}$$

With the aid of Eqs. (2.3.38) and (2.3.39) along with Eqs. (2.2.41) and (2.2.50) - (2.2.54), one can easily show that[1]

$$\langle F^2(t)\rangle = 0, \qquad (4.2.22)$$

$$\langle G^2(t)\rangle = 0, \qquad (4.2.23)$$

$$\langle F(t)G(t)\rangle = 0. \qquad (4.2.24)$$

[1]See the procedure we used to calculate $\langle F^* F\rangle$ in chapter three.

Applying these results in Eq. (4.2.21), we have

$$\begin{aligned}
\Delta n_{\pm}^2(t) &= \Big[u_1^* u_1 + \langle F^* F\rangle + 2u_1^* u_1 \langle F^* F\rangle + 2u_1 \langle F F^{*2}\rangle \\
&+ 2u_1^* \langle F^* F^2\rangle + [\langle F^{*2} F^2\rangle - \langle F^* F\rangle^2] + v_1^* v_1 \\
&+ \langle G^* G\rangle + 2v_1^* v_1 \langle G^* G\rangle + 2v_1^* \langle G^* G^2\rangle + 2v_1 \langle G G^{*2}\rangle \\
&+ [\langle G^{*2} G^2\rangle - \langle G^* G\rangle^2]\Big] \pm 2\Big[u_1^* v_1 \langle G^* F\rangle + u_1^* \langle F G^* G\rangle \\
&+ u_1 v_1^* \langle F^* G\rangle + u_1 \langle F^* G^* G\rangle + v_1 \langle F^* F G^*\rangle + v_1^* \langle F^* F G\rangle \\
&+ [\langle F^* F G^* G\rangle - \langle F^* F\rangle \langle G^* G\rangle]\Big]. \qquad (4.2.25)
\end{aligned}$$

Next we seek to evaluate the expectation values that appear in Eq. (4.2.25). Equations (2.3.38) and (2.3.39) can be rewritten as

$$F(t) = F_1(t) + F_2(t), \qquad (4.2.26)$$

$$G(t) = G_1(t) + G_2(t), \qquad (4.2.27)$$

where

$$F_1(t) = \int_0^\infty \left(\frac{S}{\lambda} f_\beta^*(t') - \frac{C}{\lambda} f_\alpha(t')\right) e^{\lambda_1 (t-t')}, \qquad (4.2.28)$$

$$F_2(t) = \int_0^\infty \left(\frac{B}{\lambda} f_\alpha(t') - \frac{S}{\lambda} f_\beta^*(t')\right) e^{\lambda_2 (t-t')}, \qquad (4.2.29)$$

$$G_1(t) = \int_0^\infty \left(\frac{B}{\lambda} f_\beta^*(t') - \frac{BC}{S\lambda} f_\alpha(t')\right) e^{\lambda_1 (t-t')}, \qquad (4.2.30)$$

$$G_2(t) = \int_0^\infty \left(\frac{BC}{S\lambda} f_\alpha(t') - \frac{C}{\lambda} f_\beta^*(t')\right) e^{\lambda_2 (t-t')}. \qquad (4.2.31)$$

On account of Eqs. (4.2.26), one can write

$$\begin{aligned}
\langle F^* F^2\rangle &= \langle F_1^* F_1^2\rangle + \langle F_1^* F_2^2\rangle + 2\langle F_1^* F_1 F_2\rangle \\
&+ \langle F_2^* F_1^2\rangle + \langle F_2^* F_2^2\rangle + 2\langle F_2^* F_1 F_2\rangle. \qquad (4.2.32)
\end{aligned}$$

Applying Eqs. (4.2.28) and (4.2.29) along with Eqs. (2.2.20) and (2.2.21), one can verify that

$$\langle F_1(t)\rangle = \langle F_2(t)\rangle = 0. \tag{4.2.33}$$

We now want to determine weather F_1 and F_2 are Gaussian variables or not. To this end, differentiating both sides of Eq. (4.2.28) with respect to t, we have

$$\frac{d}{dt}F_1(t) = \frac{S}{\lambda}f^*_\beta(t) - \frac{C}{\lambda}f_\alpha(t) + \lambda_1 F_1(t). \tag{4.2.34}$$

Upon taking the expectation value of Eq. (4.2.34) and taking into account Eqs. (2.2.20) and (2.2.21), we find

$$\frac{d}{dt}\langle F_1(t)\rangle = \lambda_1\langle F_1(t)\rangle. \tag{4.2.35}$$

It can also be shown in a similar manner that

$$\frac{d}{dt}\langle F_2(t)\rangle = \lambda_2\langle F_2(t)\rangle. \tag{4.2.36}$$

Equations (4.2.35) and (4.2.36) show that F_1 and F_2 are Gaussian variables. One can also verify that G_1 and G_2 are Gaussian variables. Since F_1 and F_2 are Gaussian variables with zero mean [1], we see that

$$\langle F_1^* F_1^2\rangle = \langle F_1^* F_2^2\rangle = \langle F_1^* F_1 F_2\rangle = \langle F_2^* F_1^2\rangle = \langle F_2^* F_2^2\rangle = \langle F_2^* F_1 F_2\rangle = 0. \tag{4.2.37}$$

In view of this result, Eq. (4.2.32) turns out to be

$$\langle F^*(t)F(t)^2\rangle = 0. \tag{4.2.38}$$

Similar calculations show that

$$\langle G^* G^2\rangle = \langle F G^* G\rangle = \langle F^* F G^*\rangle = 0. \tag{4.2.39}$$

Now, with the aid of Eqs. (4.2.38) and (4.2.39), Eq. (4.2.25) takes the form

$$\begin{aligned}\Delta n_{\pm}^2(t) &= \Big[u_1^* u_1 + (2u_1^* u_1 + 1)\langle F^* F\rangle + [\langle F^{*2} F^2\rangle - \langle F^* F\rangle^2] + v_1^* v_1 \\ &+ (2v_1^* v_1 + 1)\langle G^* G\rangle + [\langle G^{*2} G^2\rangle - \langle G^* G\rangle^2]\Big] \pm 2\Big[u_1^* v_1 \langle G^* F\rangle \\ &+ u_1 v_1^* \langle F^* G\rangle + [\langle F^* F G^* G\rangle - \langle F^* F\rangle\langle G^* G\rangle]\Big]. \qquad (4.2.40)\end{aligned}$$

On account of Eqs. (4.2.26) and (4.2.27), we can write

$$\begin{aligned}\langle F^{*2} F^2\rangle &= \langle F_1^{*2} F_1^2\rangle + 2\langle F_1^{*2} F_1 F_2\rangle + 2\langle F_1^2 F_1^* F_2^*\rangle + 4\langle F_1^* F_1 F_2^* F_2\rangle + \langle F_1^{*2} F_2^2\rangle \\ &+ \langle F_2^2 F_2^* F_1^*\rangle + \langle F_2^{*2} F_1^2\rangle + 2\langle F_2^{*2} F_2 F_1\rangle + \langle F_2^{*2} F_2^2\rangle, \qquad (4.2.41)\end{aligned}$$

$$\begin{aligned}\langle G^{*2} G^2\rangle &= \langle G_1^{*2} G_1^2\rangle + 2\langle G_1^{*2} G_1 G_2\rangle + 2\langle G_1^2 G_1^* G_2^*\rangle + 4\langle G_1^* G_1 G_2^* G_2\rangle + \langle G_1^{*2} G_2^2\rangle \\ &+ \langle G_2^2 G_2^* G_1^*\rangle + \langle G_2^{*2} G_1^2\rangle + 2\langle G_2^{*2} G_2 G_1\rangle + \langle G_2^{*2} G_2^2\rangle, \qquad (4.2.42)\end{aligned}$$

$$\begin{aligned}\langle F^* F G^* G\rangle &= \langle F_1^* F_1 G_1^* G_1\rangle + \langle F_1^* F_1 G_1^* G_2\rangle + \langle F_1^* F_1 G_2^* G_1\rangle + \langle F_1^* F_1 G_2^* G_2\rangle \\ &+ \langle F_1^* F_2 G_1^* G_1\rangle + \langle F_1^* F_2 G_1^* G_2\rangle + \langle F_1^* F_2 G_2^* G_1\rangle + \langle F_1^* F_2 G_2^* G_2\rangle \\ &+ \langle F_2^* F_1 G_1^* G_1\rangle + \langle F_2^* F_1 G_1^* G_2\rangle + \langle F_2^* F_1 G_2^* G_1\rangle + \langle F_2^* F_1 G_2^* G_2\rangle \\ &+ \langle F_2^* F_2 G_1^* G_1\rangle + \langle F_2^* F_1 G_1^* G_2\rangle + \langle F_2^* F_1 G_2^* G_1\rangle \\ &+ \langle F_2^* F_1 G_2^* G_2\rangle. \qquad (4.2.43)\end{aligned}$$

Since F_1, F_2, G_1 and G_2 are Gaussian variables with zero mean, Eqs. (4.2.41), (4.2.42) and (4.2.43) can be rewritten as

$$\begin{aligned}\langle F^{*2} F^2\rangle &= \langle F_1^{*2}\rangle\langle F_1^2\rangle + 2\langle F_1^* F_1\rangle^2 + 2\langle F_1^{*2}\rangle\langle F_1 F_2\rangle + 4\langle F_1^* F_1\rangle\langle F_1^* F_2\rangle + 2\langle F_1^2\rangle\langle F_1^* F_2^*\rangle \\ &+ 4\langle F_1 F_1^*\rangle\langle F_1 F_2^*\rangle + 4\langle F_1^* F_1\rangle\langle F_2^* F_2\rangle + 4\langle F_1^* F_2^*\rangle\langle F_1 F_2\rangle + 4\langle F_1^* F_2\rangle\langle F_1 F_2^*\rangle \\ &+ \langle F_1^{*2}\rangle\langle F_2^2\rangle + 2\langle F_1^* F_2\rangle^2 + \langle F_2^{*2}\rangle\langle F_1^2\rangle + 2\langle F_2^* F_1\rangle^2 + 2\langle F_2^{*2}\rangle\langle F_2 F_1\rangle \\ &+ 4\langle F_2^* F_2\rangle\langle F_2^* F_1\rangle + \langle F_2^{*2}\rangle\langle F_2^2\rangle + 2\langle F_2^* F_2\rangle^2, \qquad (4.2.44)\end{aligned}$$

$$\begin{aligned}\langle G^{*2}G^2\rangle &= \langle G_1^{*2}\rangle\langle G_1^2\rangle + 2\langle G_1^*G_1\rangle^2 + 2\langle G_1^{*2}\rangle\langle G_1G_2\rangle + 4\langle G_1^*G_1\rangle\langle G_1^*G_2\rangle + 2\langle G_1^2\rangle\langle G_1^*G_2^*\rangle \\ &+ 4\langle G_1G_1^*\rangle\langle G_1G_2^*\rangle + 4\langle G_1^*G_1\rangle\langle G_2^*G_2\rangle + 4\langle G_1^*G_2^*\rangle\langle G_1G_2\rangle + 4\langle G_1^*G_2\rangle\langle G_1G_2^*\rangle \\ &+ \langle G_1^{*2}\rangle\langle G_2^2\rangle + 2\langle G_1^*G_2\rangle^2 + \langle G_2^{*2}\rangle\langle G_1^2\rangle + 2\langle G_2^*G_1\rangle^2 + 2\langle G_2^{*2}\rangle\langle G_2G_1\rangle \\ &+ 4\langle G_2^*G_2\rangle\langle G_2^*G_1\rangle + \langle G_2^{*2}\rangle\langle G_2^2\rangle + 2\langle G_2^*G_2\rangle^2, \end{aligned} \tag{4.2.45}$$

$$\begin{aligned}\langle F^*FG^*G\rangle &= \langle F_1^*F_1\rangle\langle G_1^*G_1\rangle + \langle F_1^*G_1^*\rangle\langle F_1G_1\rangle + \langle F_1^*G_1\rangle\langle F_1G_1^*\rangle + \langle F_1^*F_1\rangle\langle G_1^*G_2\rangle \\ &+ \langle F_1^*G_1^*\rangle\langle F_1G_2\rangle + \langle F_1^*G_2\rangle\langle F_1G_1^*\rangle + \langle F_1^*F_1\rangle\langle G_2^*G_1\rangle + \langle F_1^*G_2^*\rangle\langle F_1G_1\rangle \\ &+ \langle F_1^*G_1\rangle\langle F_1G_2^*\rangle + \langle F_1^*F_1\rangle\langle G_2^*G_2\rangle + \langle F_1^*G_2^*\rangle\langle F_1G_2\rangle + \langle F_1^*G_2^*\rangle\langle F_1G_2\rangle \\ &+ \langle F_1^*F_1\rangle\langle G_2^*G_2\rangle + \langle F_1^*G_2^*\rangle\langle F_1G_2\rangle + \langle F_1^*G_2\rangle\langle F_1G_2^*\rangle + \langle F_1^*F_2\rangle\langle G_1^*G_2\rangle \\ &+ \langle F_1^*G_1^*\rangle\langle F_2G_2\rangle + \langle F_1^*G_2\rangle\langle F_2G_1^*\rangle + \langle F_1^*F_2\rangle\langle G_2^*G_1\rangle + \langle F_1^*G_2^*\rangle\langle F_2G_1\rangle \\ &+ \langle F_1^*G_1\rangle\langle F_2G_2^*\rangle + \langle F_1^*F_2\rangle\langle G_2^*G_2\rangle + \langle F_1^*G_2^*\rangle\langle F_2G_2\rangle + \langle F_1^*G_2\rangle\langle F_2G_2^*\rangle \\ &+ \langle F_2^*F_1\rangle\langle G_1^*G_1\rangle + \langle F_2^*G_1^*\rangle\langle F_1G_1\rangle + \langle F_2^*G_1\rangle\langle F_1G_1^*\rangle + \langle F_2^*F_1\rangle\langle G_1^*G_2\rangle \\ &+ \langle F_2^*G_1^*\rangle\langle F_1G_2\rangle + \langle F_2^*G_2\rangle\langle F_1G_1^*\rangle + \langle F_2^*F_1\rangle\langle G_2^*G_1\rangle + \langle F_2^*G_2^*\rangle\langle F_1G_1\rangle \\ &+ \langle F_2^*G_1\rangle\langle F_1G_2^*\rangle + \langle F_2^*F_1\rangle\langle G_2^*G_2\rangle + \langle F_2^*G_2^*\rangle\langle F_1G_2\rangle + \langle F_2^*G_2\rangle\langle F_1G_2^*\rangle \\ &+ \langle F_2^*F_2\rangle\langle G_1^*G_1\rangle + \langle F_2^*G_1^*\rangle\langle F_2G_1\rangle + \langle F_2^*G_1\rangle\langle F_2G_1^*\rangle + \langle F_2^*F_1\rangle\langle G_1^*G_2\rangle \\ &+ \langle F_2^*G_1^*\rangle\langle F_1G_2\rangle + \langle F_2^*G_2\rangle\langle F_1G_1^*\rangle + \langle F_2^*F_1\rangle\langle G_2^*G_1\rangle + \langle F_2^*G_2^*\rangle\langle F_1G_1\rangle \\ &+ \langle F_2^*G_1\rangle\langle F_1G_2^*\rangle + \langle F_2^*F_1\rangle\langle G_2^*G_2\rangle + \langle F_2^*G_2^*\rangle\langle F_1G_2\rangle \\ &+ \langle F_2^*G_2\rangle\langle F_1G_2^*\rangle. \end{aligned} \tag{4.2.46}$$

Applying Eqs. (4.2.28) - (4.2.31) along with Eqs. (2.2.41) and (2.2.50) - (2.2.54), it can easily be shown that

$$\langle F_1^2\rangle = \langle F_1F_2\rangle = \langle F_2^2\rangle = 0, \tag{4.2.47}$$

$$\langle G_1^2\rangle = \langle G_1G_2\rangle = \langle G_2^2\rangle = 0, \tag{4.2.48}$$

$$\langle F_1 G_1 \rangle = \langle F_1 G_2 \rangle = \langle F_2 G_1 \rangle = \langle F_2 G_2 \rangle = 0. \tag{4.2.49}$$

Using Eqs. (4.2.47) - (4.2.49) and their complex conjugate in Eqs. (4.2.44) - (4.2.46) and making some rearrangement, we have

$$\langle F^{*2} F^2 \rangle = 2\langle F^* F \rangle^2, \tag{4.2.50}$$

$$\langle G^{*2} G^2 \rangle = 2\langle G^* G \rangle^2, \tag{4.2.51}$$

$$\langle F^* F G^* G \rangle = \langle F^* F \rangle \langle G^* G \rangle + \langle F^* G \rangle \langle G^* F \rangle. \tag{4.2.52}$$

Using these results in Eq. (4.2.40), we obtain

$$\begin{aligned} \Delta n_{\pm}^2(t) &= \Big[(u_1^*(t)u_1(t)) + (v_1^*(t)v_1(t)) + (2(u_1^*(t)u_1(t)) + 1)\langle F^*(t)F(t)\rangle \\ &+ (2(v_1^*(t)v_1(t)) + 1)\langle G^*(t)G(t)\rangle + \langle F^*(t)F(t)\rangle^2 + \langle G^*(t)G(t)\rangle^2\Big] \\ &\pm 2\Big[(u_1(t)v_1^*(t))\langle F^*(t)G(t)\rangle + (u_1^*(t)v_1(t))\langle G^*(t)F(t)\rangle \\ &+ \langle F^*(t)G(t)\rangle\langle G^*(t)F(t)\rangle\Big]. \end{aligned} \tag{4.2.53}$$

In view of Eqs. (3.1.21), (3.1.27), (3.1.32), (4.2.7) and (4.2.10) and their complex conjugate, the various expressions in Eq. (4.2.53) at steady state take the form

$$(u_1^*(t)u_1(t))_{ss} = \frac{\varepsilon^2(B\lambda_1 - C\lambda_1 + S\lambda)(B\lambda_1 - C\lambda_2 + T\lambda)}{\lambda^2\lambda_1^2\lambda_2^2}, \tag{4.2.54}$$

$$(v_1^*(t)v_1(t))_{ss} = \frac{\varepsilon^2(C\lambda_1(S-B) + B\lambda_2(C-S))(C\lambda_1(T-B) + B\lambda_2(C-T))}{\lambda^2\lambda_1^2\lambda_2^2 ST}, \tag{4.2.55}$$

$$(u_1(t)v_1^*(t))_{ss} = \frac{\varepsilon^2(B\lambda_1 - C\lambda_1 + S\lambda)(-B(C\lambda + T\lambda_2) - CT\lambda_1)}{\lambda^2\lambda_1^2\lambda_2^2 T}, \tag{4.2.56}$$

$$\langle F^*(t)F(t)\rangle_{ss} = \frac{2(2BCQ - ST(B+C))}{\lambda^2(\lambda_1+\lambda_2)} - \frac{B^2Q - BST}{\lambda^2\lambda_2} - \frac{C^2Q - CST}{\lambda^2\lambda_1}, \tag{4.2.57}$$

$$\langle G^*(t)G(t)\rangle_{ss} = \frac{B^2C}{\lambda^2\lambda_1} + \frac{C^2B}{\lambda^2\lambda_2} - \frac{2BC(B+C)}{\lambda^2(\lambda_1+\lambda_2)} - \frac{QB^2C^2}{(\lambda_1+\lambda_2)\lambda_1\lambda_2 ST}, \tag{4.2.58}$$

$$\langle F(t)G^*(t)\rangle_{ss} = \frac{2BCQ}{\lambda^2 T}\left[\frac{B+C}{\lambda_1+\lambda_2} - \frac{B}{2\lambda_2} - \frac{C}{2\lambda_1}\right] + \frac{S}{\lambda^2}\left[\frac{BC}{\lambda_2} + \frac{BC}{\lambda_1} - \frac{(B+C)^2}{\lambda_1+\lambda_2}\right]. \tag{4.2.59}$$

Applying Eqs. (3.1.67) - (3.1.75) in Eqs. (4.2.54) - (4.2.59) and substituting the resulting expressions and their complex conjugate into Eq. (4.2.53), the variances of the photon number sum and difference become

$$\begin{aligned}
(\Delta n^2_{\pm})_{ss} \;=\;& \Bigg[\frac{\varepsilon^2[A^2(5-\eta^2) + (A+A\eta+2\kappa)^2 + 8\kappa(\kappa+A\eta)]}{\kappa^2(A\eta+\kappa)^2} \\
&- \frac{2\varepsilon^2 A(3A+A\eta+2\kappa)\sqrt{1-\eta^2}\cos\theta}{\kappa^2(A\eta+\kappa)^2} \\
&+ \Bigg(1 + \frac{2\varepsilon^2[A(1-\eta^2) + (A+A\eta+2\kappa)^2]}{\kappa^2(A\eta+\kappa)^2} \\
&- \frac{4\varepsilon^2 A\sqrt{1-\eta^2}(A+A\eta+2\kappa)\cos\theta}{\kappa^2(A\eta+\kappa)^2}\Bigg)\Bigg(\frac{A(1-\eta)(A+3A\eta+4\kappa)}{4(A\eta+\kappa)(A\eta+2\kappa)}\Bigg) \\
&+ \frac{A^2(1-\eta^2)[8\varepsilon^2(A^2+2A\eta\kappa+2\kappa^2 - A^2(\sqrt{1-\eta^2})\cos\theta) + \kappa^2(A\eta+\kappa)^2]}{4\kappa^2(A\eta+\kappa)^3(A\eta+2\kappa)} \\
&+ \frac{A^2(1-\eta)^2(A+3A\eta+4\kappa)^2 + A^4(1-\eta^2)^2}{16(A\eta+\kappa)^2(A\eta+2\kappa)^2}\Bigg] \\
&\pm\; 2\Bigg[\frac{\varepsilon^2 A\sqrt{1-\eta^2}(A+A\eta+2\kappa)[(2\kappa(A\eta+\kappa) - A^2(1-\eta^2))\cos\theta + A^2\sqrt{1-\eta^2}]}{\kappa^2(A\eta+\kappa)^3(A\eta+2\kappa)} \\
&+ \frac{A^2(1-\eta^2)(A+A\eta+2\kappa)^2}{16(A\eta+\kappa)^2(A\eta+2\kappa)^2}\Bigg].
\end{aligned} \tag{4.2.60}$$

From Eq. (4.2.60) we see that the variance of the mean photon number sum and difference increases with increasing of the amplitude of the driving light. For the case in which there is no driving light ($\varepsilon = 0$), Eq. (4.2.60) becomes

$$\begin{aligned}
(\Delta n^2_{\pm})_{ss} \;=\;& \frac{A(1-\eta)(A+3A\eta+4\kappa) + A^2(1-\eta^2)}{4(A\eta+\kappa)(A\eta+2\kappa)} \\
&+ \frac{A^2(1-\eta)^2(A+3A\eta+4\kappa)^2 + A^4(1-\eta^2)^2}{16(A\eta+\kappa)^2(A\eta+2\kappa)^2}\Bigg] \\
&\pm \frac{A^2(1-\eta^2)(A+A\eta+2\kappa)^2}{8(A\eta+\kappa)^2(A\eta+2\kappa)^2}.
\end{aligned} \tag{4.2.61}$$

From Fig. 4.5 we observe that the variance of the photon number sum is greater

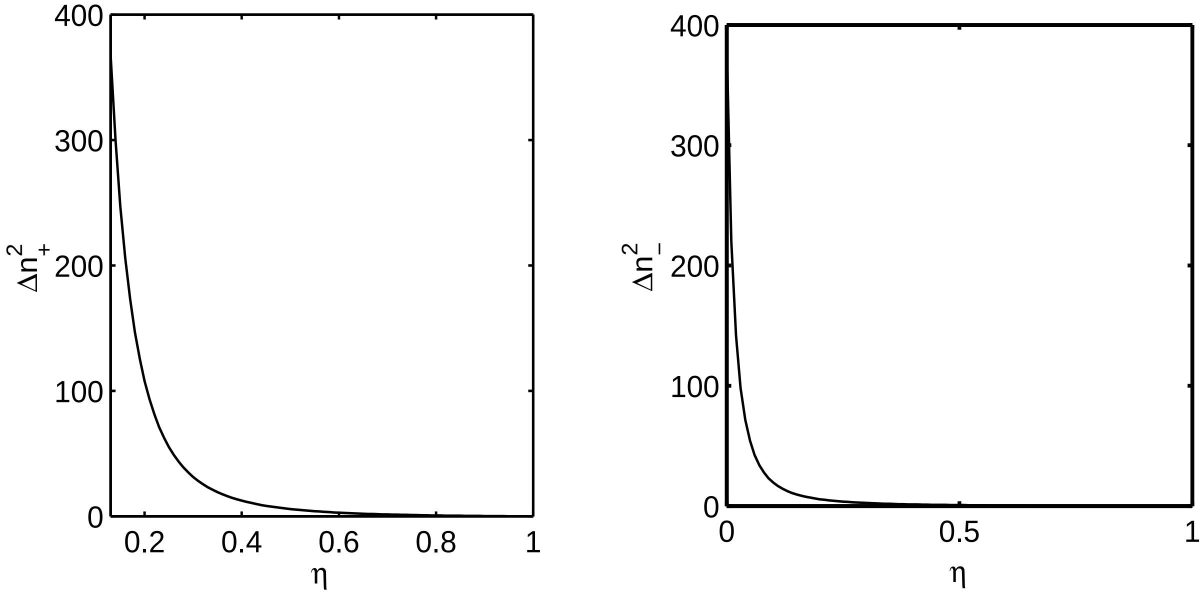


Fig. 4.5 Plots of the variance of the photon number sum and difference [Eq. (4.2.61)] versus η for $\varepsilon = 0$, $\kappa = 0.8$ and $A = 25$.

than that of the photon number difference.

Chapter 5

Conclusion

In this thesis we have considered a nondegenerate three-level laser, with the cavity modes driven by coherent light. First we have derived the master equation in the linear and adiabatic approximations. Then using this master equation, we have obtained stochastic differential equations. Applying the solutions of the resulting differential equations, we have calculated the quadrature variance and the squeezing spectrum. In addition, using the same solutions we have determined the mean and the variance of the photon number sum and difference.

We have found for a linear gain coefficient of 75 and for a cavity damping constant of 0.8, the maximum intracavity squeezing at steady state and at threshold to be 64%. We have also seen that the two-mode driving light has no effect on the squeezing of the cavity modes.

Unlike the squeezing , the driving light affects the mean photon numbers and the variances of the photon number sum and difference. We have also found that increasing the amplitude of the driving light increases the mean photon numbers and the variances of the photon number sum and difference. Furthermore, we have seen that the mean photon number of mode a is greater than that of mode b.

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DECLARATION

I hereby declare that this thesis is my original work and has not been presented for a degree in any university and that all sources of material used for the thesis have been duly acknowledged.

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