Approximating Common Solutions of Variational Inequality, Equilibrium and Fixed Point Problems

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Declaration

I, Tesfalem Hadush Meche, with student ID number $GSR/2793/05$, hereby declare that this thesis is my own work and that it has not been previously submitted for assessment or completion of any post graduate qualification to another university or for another qualification.

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Abstract

Many of the most important problems arising in nonlinear analysis reduce to solving a given equation, which in turn may be reduced to finding the fixed points of a certain mapping or solutions of variational inequality and equilibrium problems. Because of the relation between the fixed point problem, variational inequality and equilibrium problems, finding common solutions of these problems is an important field of research.

In this thesis, we introduce and study an iterative algorithm which converges strongly to a common element of the set of fixed points of a more general class of Lipschitz hemicontractive-type multi-valued mappings and the set of solutions of variational inequality problem in real Hilbert spaces. In addition, we have obtained strong convergence theorems of an iterative process for finding a common solution of the fixed point problem for Lipschitz hemicontractive-type multi-valued mapping and the generalized equilibrium problem in the framework of real Hilbert spaces. We also extend this result to a finite family of generalized equilibrium problems. Furthermore, a viscosity-type approximation method is introduced for approximating a common element of the set of fixed points of a nonexpansive multi-valued mapping, the sets of solutions of a split equilibrium and a variational inequality problems.

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Contents

Chapter 1

Introduction

1.1 Background of the Study

Fixed point theory is the search for a combination of conditions on a set X and a mapping $T : X \longrightarrow X$ which, in turn, assures that T leaves at least one point x in X fixed, that is, $x = Tx$ for some $x \in X$. For a set-valued (or multi-valued) mapping $T : X \longrightarrow 2^X$, a fixed point is a point in X that belongs to its image under T. It has been viewed that many of the most important nonlinear problems in applied mathematics reduce to solving a given equation which in turn may be reduced to finding the fixed points of a certain mapping. Because of its important linkages with applied mathematics, such as optimization, variational analysis and differential equations, fixed point theory has long been considered as a fundamental part of nonlinear functional analysis.

The existence of fixed points of a mapping T depends on the geometry of the underlying space and the nature of T. Existence theorems are concerned with launching sufficient conditions under which the mapping will have fixed points, but do not point out how to find them. The study of existence of fixed points for single-valued nonlinear mappings in Banach spaces was discussed by several mathematicians. For instance, in 1912, Brouwer [5] proved the existence of fixed points for continuous functions defined on compact and convex subsets of Euclidean space, $\mathbb{R}^n$. In 1922, Banach [3] proved the existence and uniqueness of fixed point of single-valued contraction self-mapping of complete metric space, called *Banach Contraction*

Mapping Principle. In general, a single-valued nonexpansive self-mapping of complete metric space need not have a fixed point. For example, the mapping $T : \mathbb{R} \to \mathbb{R}$ defined by $Tx = x+1$ is nonexpansive but has no fixed point. However, some existence results of fixed points have been obtained by Browder [6] and Kirk [26] for nonexpansive mapping in Hilbert spaces. Moreover, existence of fixed points for single-valued mappings have been extended and generalized to multi-valued mappings using Hausdorff metric in several ways. For instance, Nadler [35] in 1969 generalized the classical principle (or Banach Contraction Mapping Principle) to the case of multi-valued contraction mapping and established that a multi-valued contraction mapping possesses a fixed point in a complete metric space. Dozo [28] proved that a multi-valued nonexpansive mapping has a fixed point in Banach spaces satisfying Opial's condition.

In the direction of finding fixed points of mappings (when they exists or assuming existence), many researchers concentrate on providing iterative algorithms (or schemes) for approximation of such fixed points. Banach Contraction Mapping Principle is one of the fundamental result which asserts that the sequence of Picard iteration converges to the unique fixed point of a contraction mapping in a complete metric space. Subsequently, many iterative algorithms, such as Mann iteration, Halpern Iteration, Krasnoselskii iteration, Ishikawa iteration, are introduced. Since then, several improvements and generalizations had appeared for approximation of fixed points of both single-valued and multi-valued mappings; see, for example, [11, 14, 24, 30, 41, 43, 44, 45] and the references cited therein. Approximation of fixed points is related to approximation of solutions of a variational inequality and equilibrium problems.

Variational inequality problem is the problem of searching for a point

x in a subset C of a Hilbert space H such that $\langle Ax, y-x\rangle \geq 0$, for all $y \in C$, where $A : C \longrightarrow H$ is a given mapping. Variational inequality problem was first introduced by Stampacchia [46] in 1964. Since then, it has been extended and generalized in several directions, using new and powerful methods, to study a wide class of problems in a unified and general framework. Existence and approximation of solutions are important aspects of study of variational inequalities. The variatinal inequality problem is a general problem formulation that covers many problems, such as optimization problems, complementarity problems, fixed point problems etc. This alternative equivalent formulation has played a major role in finding solutions of variational inequality problems using iterative algorithms. In recent years, much attention has been given to developing efficient iterative algorithms for treating solution problems of variational inequalities; see, for example, [2, 16, 22, 56, 57, 60, 61, 62] and the references therein. Constructing and studying an iterative algorithm that converges to a common solution of a fixed point problem for multi-valued mapping and a variational inequality problem is one of our focus in this thesis.

Equilibrium problem is the problem of finding a point $x \in C$ such that $F(x, y) \geq 0$ for all $y \in C$, where $F : C \times C \longrightarrow \mathbb{R}$ is a bifunction. Equilibrium problems which were introduced by Blum and Oettli [4] in 1994, have had a great impact and influence in pure and applied sciences. It has been shown that numerous problems in physics, optimization and economics reduce to finding a solution of equilibrium problem. The equilibrium problem includes and unifies a wide class of problems, such as variational inequality problems, complementarity problems, Nash equilibrium problems, optimization problems, fixed point problems. One of the most important and interesting topics in the theory of equilibrium is to develop algorithms for solving equilibrium problems and their generalizations (see, [17] and the references cited therein). Since the equilibrium problems have very close relations with both the fixed point and the variational inequality problems, finding a common solution of these problems in Banach space has drawn many people's attention (see, for example, [15, 16, 49, 52] and the reference therein). Equilibrium problems have been extended and generalized in several directions. An important and useful generalization of equilibrium problem is the generalized equilibrium problem. Generalized equilibrium problems were introduced and studied by Takahashi and Takahashi [50] in 2008. It includes equilibrium problems, fixed point problems and variational inequality problems as special cases. Since then, many authors devoted their efforts in finding a solution of generalized equilibrium problem and which is also a fixed point of some nonlinear mappings in Hilbert spaces; see, for example, [8, 9, 19, 21, 33, 50] and the references therein. In this thesis, we also emphasis in finding a solution of generalized equilibrium problems which is also a fixed point of a multi-valued mapping by using iterative algorithms in a real Hilbert space.

On the other hand, split equilibrium problem, which seems more general, is another class of equilibrium problem which was introduced by He [20] in 2012. This problem includes two equilibrium problems in which one is enabled to find a solution of one problem which is an image of a solution of the other under a continuous linear mapping. Split equilibrium problem includes split variational inequality problem and split fixed point problem as special cases and this has been widely studied. For more details concerning this, the reader may refer to [10, 20, 25] and the references cited therein.

In this thesis, some iterative algorithms which converge strongly to a common element of the set of solutions of fixed point problem for multi-valued mappings, and the set of solutions of the classical variational inequality problem; and the set of common solutions of a finite family of generalized equilibrium problems; and the set of solutions of a split equilibrium problem are introduced and studied in the framework of real Hilbert spaces.

The rest of this thesis is organized as follows. In Section 1.2, on the preliminaries of our work, some basic definitions and facts which will be frequently used in this thesis are recalled. The concepts of the classical variational inequality, equilibrium and split equilibrium problems are also discussed. In Chapter 2, we give a detailed review of materials which are related to our work. In particular, Section 2.1 deals with the literature review related to the existence and approximation of fixed points of nonlinear mappings in metric spaces. It also includes the literature on approximation methods for a common solution of variational inequality and fixed point problems. In Section 2.2, we present literature review of approximation methods for a common solution of generalized equilibrium and fixed point problems in Hilbert spaces. The literature review related to iterative algorithms for finding a common element of the sets of solutions of split equilibrium, variational inequality and fixed point problems are stated in Section 2.3. Section 2.4 is devoted to the statement of problems of this work. The main aim and objectives of our work are stated in Section 2.5. In addition, the significance, scope and methodology of this study are part of this chapter. In Chapter 3, we provide some lammas which are needed in the proof of our main results. In Chapter 4, we state our main results and prove strong convergence theorems. Specifically, in Section 4.1, we introduce an iterative algorithm and establish its strong convergence to a common solution of a variational inequality and a fixed point problems. In Section 4.2, we introduce another iterative algorithm and prove convergence results for approximating a common solution of generalized equilibrium problem and fixed point problem for a Lipschitz hemicontractive-type mapping. In Section 4.3, we present our results on approximation of a common solution of a finite family of generalized equilibrium problems and fixed point problem for hemicontractive-type multi-valued mapping. Our results on approximation of common solution of split equilibrium problem, variational inequality problem and fixed point problem for nonexpansive multi-valued mapping are presented in Section 4.4. Finally in Chapter 5, the discussion, conclusion and recommendations are presented. At the end of this thesis, bibliography of references are given.

1.2 Preliminaries

In this section, we shall present some important definitions and concepts, such as the metric projection, Hausdorff metric, nonlinear mappings, the notions of variational inequality and equilibrium problems, which will play crucial role in the subsequent chapters.

1.2.1 Basic Definitions

We begin this subsection with the definition of best approximation (or metric projection) which is very important in what follows.

Definition 1.2.1. *Let C be a nonempty subset of a real normed space X and let $x \in X$. An element $y_0 \in C$ is said to be a best approximation of x if*

$$\|x - y_0\| = d(x, C) = \inf\{\|x - y\| : y \in C\}.$$

The set(possibly empty) of all best approximations of x in C is denoted by $P_C x = \{y \in C : \|x - y\| = d(x, C)\}$. If $P_C x$ is nonempty for each $x \in X$, then the subset C is said to be *proximinal set.*

Definition 1.2.2. *Let C be a nonempty subset of a normed space X. The mapping $P_C : X \longrightarrow 2^C$ given by*

$$P_C x = \{y \in C : \|x - y\| = d(x, C)\}$$

is called the metric projection from X onto C.

Lemma 1.2.3. *[12] Let C be a nonempty, closed and convex subset of a Hilbert space H. Then, for every $x \in H$, there exists a unique element $z \in C$ such that $||x - z|| = d(x, C)$. That is, the metric projection P_C from H onto C is nonempty and single-valued.*

Definition 1.2.4. *A Banach space X is said to be uniformly convex, if for any $\epsilon \in (0, 2]$, there exists $\delta = \delta(\epsilon) > 0$ such that for $x, y \in X$ with $\|x\| \leq 1, \|y\| \leq 1$ and $\|x - y\| \geq \epsilon$, then $\|\frac{x+y}{2}\| \leq 1 - \delta$. This says that if x and y are in the closed unit ball $B_X = \{x \in X : \|x\| \leq 1\}$ with $\|x - y\| \geq \epsilon > 0$, the midpoint of x and y lies inside the unit ball B_X at a distance of at least δ from the unit sphere $S_X = \{x \in X : \|x\| = 1\}$.*

It is known that Hilbert spaces, L_p and l_p, for $1 < p < \infty$, are uniformly convex Banach spaces (see, *for example*, [12]).

The following definition is due to Opial [36].

Definition 1.2.5. *A Banach space X is said to satisfy Opial's condition if for each x in X and each sequence $\{x_n\}$ that is weakly convergent to x, the inequality*

$$\liminf_{n\to\infty} \|x_n - x\| < \liminf_{n\to\infty} \|x_n - y\|$$

holds for every $y \in X$ with $y \neq x$.

Note that every Hilbert space H satisfies the Opial's condition (see, [36]).

1.2.2 Single-Valued Nonlinear Mappings

Definition 1.2.6. *Let (X, d) be a metric space. Let C be a nonempty subset of X. Then, a mapping $T : C \longrightarrow X$ is called $L-$Lipschitzian if there exists $L \geq 0$ such that*

$$d(Tx, Ty) \leq Ld(x, y), \; \forall \; x, y \in C.$$

If $L < 1$, then T is called contraction (or contractive mapping) and if $L \leq 1$, it is called nonexpansive.

We observe that every contraction (or contractive mapping) is non-expansive and every nonexpansive mapping is Lipschitzian mapping. Note that the inclusions are proper as can be seen from the following example.

Example 1.2.7. *i) The identity mapping defined on a normed linear space X is nonexpansive but it is not contraction.*

ii) The mapping $T : [0, 1] \longrightarrow \mathbb{R}$ defined by $Tx = x^4 + x + 1$ is Lipschitzian mapping with Lipschitz constant 5 but not nonexpansive.

Definition 1.2.8. *Let (X, d) be a metric space. A point $x \in X$ is called a fixed point of a mapping $T : X \longrightarrow X$ if $x = Tx$. We denote the set of fixed points of T by $F(T)$, that is,*

$$F(T) = \{x \in X : x = Tx\}.$$

Example 1.2.9. *i) Let $X = \mathbb{R}$ and $T : X \longrightarrow X$ be given by $Tx = x^3 + x + 1$. Then, $x = -1$ is a fixed point of T (and is the unique fixed point of T), that is, $F(T) = \{-1\}$.*

ii) Let $X = \mathbb{R}$ and $T : X \longrightarrow X$ be given by $Tx = x^2 + x + 1$. Then, T has no fixed points, that is, $F(T) = \emptyset$.

iii) Let $X = \mathbb{R}$ and $T : X \longrightarrow X$ be given by $Tx = x^2 + 4x + 2$. Then, $x = -1$ and $x = -2$ are fixed points of T, that is, $F(T) = \{-1, -2\}$ and hence fixed points need not be unique.

As the above examples illustrate, fixed points may not exist, and if they exist, they may not be unique.

Definition 1.2.10. *Let C be a nonempty subset of a normed linear space X. A mapping $T : C \longrightarrow X$ with $F(T) \neq \emptyset$ is said to be quasi-nonexpansive if*

$$\|Tx - p\| \leq \|x - p\|, \text{ for all } x \in C,\ p \in F(T).$$

The following example shows that the class of quasi-nonexpansive mappings includes properly the class of nonexpansive mappings with nonempty set of fixed points.

Example 1.2.11. Let $X = \mathbb{R}$ and $C = [0, 1]$. Define a mapping $T : C \longrightarrow X$ by

$$Tx = \begin{cases} x^2 \sin(1/x), & \text{if } x \neq 0, \\ 0, & \text{if } x = 0. \end{cases}$$

Then, zero is the only fixed point of T and for $x \in C$,

$$|Tx - 0| = |x^2 \sin(1/x)| \leq |x^2| \leq |x| = |x - 0|,\ \forall x \in C.$$

Hence, T is quasi-nonexpansive. However, for $x = \frac{2}{\pi}$ and $y = \frac{1}{\pi}$, we have that

$$|Tx - Ty| = |\frac{4}{\pi^2} \sin\frac{\pi}{2} - \frac{1}{\pi^2} \sin\pi| = \frac{4}{\pi^2} > \frac{1}{\pi} = |x - y|.$$

Thus, T is not nonexpansive.

Definition 1.2.12. *Let C be a nonempty subset of a real Hilbert space H. Then, a mapping $T : C \longrightarrow H$ is said to be pseudocontractive if it satisfies the following inequality*

$$\langle Tx - Ty, x - y\rangle \leq ||x - y||^2, \ \forall x, y \in C.$$

Equivalently, T is called pseudocontractive if

$$\|Tx - Ty\|^2 \leq \|x - y\|^2 + \|(I - T)x - (I - T)y\|^2, \tag{1.1}$$

holds for all $x, y \in C$, where I is the identity mapping on C.

Definition 1.2.13. *Let C be a nonempty subset of a real Hilbert space H. A mapping $T : C \longrightarrow H$ is called $k-$strictly pseudocontractive if there exists $k \in [0, 1)$ such that*

$$\|Tx - Ty\|^2 \leq \|x - y\|^2 + k\|(I - T)x - (I - T)y\|^2, \tag{1.2}$$

for all $x, y \in C$.

We note that, if $k \leq 1$ in (1.2), then T is pseudocontractive but if $k = 0$ in (1.2), then T is nonexpansive. Thus, the class of $k-$strictly pseudocontractive mappings is a subclass of the class of Lipschitz pseudocontractive mappings and a superclass of the class of nonexpansive mappings (see [7]).

The following lemma shows that if a mapping T is a continuous pseudocontractive mapping, then the set of fixed points of T, $F(T)$, is closed and convex.

Lemma 1.2.14. *[64] Let C be a nonempty, closed and convex subset of a real Hilbert space H and $T : C \longrightarrow C$ be a continuous pseudocontractive mapping. Then, $F(T)$ is a closed and convex subset of C.*

Note that since every nonexpansive and $k-$strictly pseudocontractive mappings are continuous pseudocontractive, then Lemma 1.2.14 also holds for nonexpansive and $k-$strictly pseudocontractive.

In Definition 1.2.12 and 1.2.13, if $F(T)$ is nonempty, then inequality (1.1) and (1.2) reduce to the following relations,

$$\|Tx - p\|^2 \leq \|x - p\|^2 + \|x - Tx\|^2, \ \forall x \in C, \ p \in F(T)$$

and

$$\|Tx - p\|^2 \leq \|x - p\|^2 + k\|x - Tx\|^2, \ \forall x \in C, \ p \in F(T),$$

respectively. This leads to our next definition.

Definition 1.2.15. *Let C be a nonempty subset of a real Hilbert space H. A mapping $T : C \longrightarrow H$ is said to be*

i. hemicontractive if $F(T)$ is nonempty and

$$\|Tx - p\|^2 \leq \|x - p\|^2 + \|x - Tx\|^2,$$

holds for all $x \in C$, $p \in F(T)$;

ii. demicontractive if $F(T)$ is nonempty and there exists $k \in [0, 1)$ such that

$$\|Tx - p\|^2 \leq \|x - p\|^2 + k\|x - Tx\|^2,$$

holds for all $x \in C$, $p \in F(T)$.

It is easy to verify that every pseudocontractive mapping T with $F(T) \neq \emptyset$ is hemicontractive, while every $k-$strictly pseudocontractive mapping T with $F(T) \neq \emptyset$ is demicontractive. Note that the inclusions are strict as can be seen from the following example:

Example 1.2.16. *Let $X = \mathbb{R}$ and $C = [0,1]$. Let $T : C \longrightarrow X$ be defined by*

$$Tx = \begin{cases} x^2 \sin(1/x), & \text{if } x \neq 0, \\ 0, & \text{if } x = 0. \end{cases}$$

Then, zero is the only fixed point of T and for all $x \in C$, we have

$$\begin{aligned} |Tx - 0|^2 &= |x^2 \sin(1/x)|^2 \leq |x^4| \\ &\leq |x|^2 \leq |x-0|^2 + k|x - Tx|^2, \ \forall\, k \in [0,1). \end{aligned}$$

Hence, T is a demicontractive mapping and so it is hemicontractive mapping (since $k < 1$). However, if we take $x = \frac{2}{\pi}$ and $y = \frac{1}{\pi}$, then we get that

$$|Tx - Ty|^2 = |\frac{4}{\pi^2} \sin\frac{\pi}{2} - \frac{1}{\pi^2} \sin\pi|^2 = \frac{16}{\pi^4}.$$

And

$$\begin{aligned} |x-y|^2 + |x - y - (Tx - Ty)|^2 &= \frac{1}{\pi^2} + \left(\frac{\pi - 4}{\pi^2}\right)^2 \\ &= \frac{2\pi^2 - 8\pi + 16}{\pi^4} \\ &< \frac{16}{\pi^4} = |Tx - Ty|^2, \end{aligned}$$

which shows that T is not pseudocontractive mapping. Hence, T is not $k-$ strictly pseudocontractive mapping.

Remark 1.2.17. *Note that the class of quasi-nonexpansive mappings is included in the class of demicontractive mappings which in turn is included in the class of hemicontractive mappings.*

We now define another class of nonlinear mappings in Hilbert spaces.

Definition 1.2.18. *Let C be a nonempty subset of a real Hilbert space H. Then, a mapping $A : C \longrightarrow H$ is called*

i) $\alpha-$inverse strongly monotone if there exists a positive real number α such that

$$\langle Ax - Ay, x - y\rangle \geq \alpha\|Ax - Ay\|^2, \ \forall\, x, y \in C.$$

ii) monotone if

$$\langle Ax - Ay, x - y\rangle \geq 0,\ \forall\, x, y \in C.$$

It is clear that if $A : C \longrightarrow H$ is $\alpha-$inverse strongly monotone mapping, then A is $\frac{1}{\alpha}-$Lipschitz continuous. Observe from Definition 1.2.18 that the class of monotone mappings properly includes the class of $\alpha-$inverse strongly monotone mappings. To see this, consider the following example.

Example 1.2.19. *Let $H = \mathbb{R}$ and $C = (0, 1]$. Let $A : C \longrightarrow H$ given by $Ax = -\frac{1}{x}$. Then, for every $x, y \in C$, we have*

$$\langle x - y, Ax - Ay\rangle = \langle x - y, -\frac{1}{x} + \frac{1}{y}\rangle = \frac{1}{xy}||x - y||^2 \geq 0,$$

which shows that A is monotone, however, since

$$|Ax - Ay| = |\frac{-1}{x} + \frac{1}{y}| = \frac{1}{xy}|x - y|,$$

and $\frac{1}{xy} \to \infty$ as $xy \to 0$. Then A is not Lipschitzian and so it is not $\alpha-$inverse strongly monotone mapping.

We note that the class of pseudocontractive mappings, apart from being a generalization of the nonexpansive mappings, are closely connected with the class of monotone mappings as shown in the lemma below.

Lemma 1.2.20. *Let C be a nonempty subset of a real Hilbert space H. A mapping $A : C \longrightarrow H$ is monotone if and only if $T := I - A$ is a pseudocontractive mapping.*

Proof. Let $x, y \in C$ be arbitrary. Then, A is monotone if

$$\begin{aligned} 0 &\leq \langle Ax - Ay, x - y\rangle = \langle Ax - Ay - (x - y) + x - y, x - y\rangle \\ &= \langle Ax - Ay - (x - y), x - y\rangle + \langle x - y, x - y\rangle \end{aligned}$$

$$
\begin{aligned}
&= -\langle x - Ax - (y - Ay), x - y\rangle + \|x - y\|^2 \\
&= -\langle Tx - Ty, x - y\rangle + \|x - y\|^2,
\end{aligned}
$$

which implies that $\langle Tx - Ty, x - y\rangle \leq \|x - y\|^2$ and hence T is pseudocontractive. One may also easily show that the other side holds. This completes the proof. $\square$

Observe that the zeros of a monotone mapping A are the fixed points of pseudocontractive mapping $T := I - A$.

The following lemma (see [51]) shows that the relation between nonexpansive and $\alpha-$inverse strongly monotone mapping:

Lemma 1.2.21. *Let C be a nonempty subset of a real Hilbert space H. Let $T : C \longrightarrow C$ be a nonexpansive mapping, then $A := I - T$ is $\frac{1}{2}-$inverse strongly monotone mapping.*

Proof. Suppose T is nonexpansive. Then, for $x, y \in C$, we have that

$$
\begin{aligned}
\|Ax - Ay\|^2 &= \|x - y - (Tx - Ty)\|^2 \\
&= \langle x - y - (Tx - Ty), x - y - (Tx - Ty)\rangle \\
&= \|x - y\|^2 - 2\langle x - y, Tx - Ty\rangle + \|Tx - Ty\|^2 \\
&\leq \|x - y\|^2 - 2\langle x - y, Tx - Ty\rangle + \|x - y\|^2 \\
&= 2\|x - y\|^2 - 2\langle x - y, Tx - Ty\rangle \\
&= 2\langle x - y, x - y - (Tx - Ty)\rangle \\
&= 2\langle x - y, Ax - Ay\rangle.
\end{aligned}
$$

Thus, $\langle x - y, Ax - Ay\rangle \geq \frac{1}{2}\|Ax - Ay\|^2$ for all $x, y \in C$ and hence, we obtain the desired result. $\square$

Definition 1.2.22. *Let C be a nonempty subset of a real Hilbert space H and $T : C \longrightarrow H$ be a mapping. The mapping $(I - T)$ is said to be demiclosed at zero if for any sequence $\{x_n\}$ in C such that*

$x_n \rightharpoonup x \in C$ and $x_n - Tx_n \to 0$ as $n \to \infty$, we have that $Tx = x$, where "$\rightharpoonup$" denote weak convergence.

Concerning the demiclosedness principle of some class of nonlinear mappings, we have the following lemma which is proved in [64].

Lemma 1.2.23. *Let C be a nonempty, closed and convex subset of a real Hilbert space H and $T : C \longrightarrow C$ be a continuous pseudocontractive mapping. Then, $(I - T)$ is demiclosed at zero.*

Since every nonexpansive and $k-$strictly pseudocontractive mappings are continuous pseudocontractive, then Lemma 1.2.23 also holds for nonexpansive and $k-$strictly pseudocontractive mappings.

1.2.3 Hausdorff Metric

Let (X, d) be a metric space. Let $CB(X)$ denote the collection of all nonempty closed and bounded subsets of X and $K(X)$ denote the collection of all nonempty and compact subsets of X . It is clear that $K(X)$ is included in $CB(X)$. For $x \in X$ and $A, B \in CB(X)$, define

(i) $d(x, A) = \inf\{d(x, y) : y \in A\}$

(ii) $\delta(A, B) = \sup\{d(x, B) : x \in A\}$

(iii) $D(A, B) = \max\{\delta(A, B), \delta(B, A)\}$

Proposition 1.2.24. *[1] Let (X, d) be a metric space. Then, D is a metric on $CB(X)$. The metric D on $CB(X)$ is called the Hausdorff metric.*

Note that if (X, d) is a complete metric space, then $(CB(X), D)$ is also complete metric space.

Example 1.2.25. Let $X = \mathbb{R}$ with the usual metric $d(x, y) = |x - y|$, $A = [1, 4]$ and $B = [6, 10]$. Then, we observe that for each $x \in A$, the closest point in B that gives the smallest distance will always be $y = 6$. That is, $d(x, B) = \inf\{d(x, y) : y \in B\} = d(x, 6) = |x - 6|$. Therefore, $\delta(A, B) = \sup\{d(x, B) : x \in A\} = \sup\{d(x, 6) : x \in A\}$. The point $x = 1$ in A maximizes this distance. Hence,

$$\delta(A, B) = d(1, 6) = |6 - 1| = 5.$$

Similarly, we find that $\delta(B, A) = \sup\{d(y, 4) : y \in B\}$, since the point $x = 4$ will give the smallest distance to any point in B. The point $y = 10$ in B maximizes this distance, so we have

$$\delta(B, A) = d(4, 10) = |10 - 4| = 6.$$

Thus,

$$D(A, B) = \max\{\delta(A, B), \delta(B, A)\} = 6.$$

The following lemma follows from the definition of Hausdorff metric. For the sake of completeness, we include the proof.

Lemma 1.2.26. *Let (X, d) be a metric space. Let $A, B \in CB(X)$ and $a \in A$. Then for $\epsilon > 0$, there must exist a point $b \in B$ such that*

$$d(a, b) \leq D(A, B) + \epsilon.$$

Proof. Let $a \in A$ and $\epsilon > 0$ be given. Then, by the definition of $d(a, B)$, $d(a, B) = \inf\{d(a, b) : b \in B\}$, there exists $b \in B$ such that

$$d(a, b) \leq d(a, B) + \epsilon.$$

Thus, from the definition of D, we conclude that

$$d(a, b) \leq D(A, B) + \epsilon,$$

for some $b \in B$. This completes the proof. $\square$

1.2.4 Multi-Valued Mappings

Definition 1.2.27. *Let C be a nonempty subset of a metric space (X, d). Then, a multi-valued mapping $T : C \longrightarrow CB(C)$ is said to be $L-$Lipschizian if there exists $L \geq 0$ such that for all $x, y \in C$,*

$$D(Tx, Ty) \leq Ld(x, y).$$

If $L < 1$, then T is called contraction and if $L \leq 1$, we say that the multi-valued mapping T is nonexpansive.

An element $x \in C$ is called a *fixed point* of a multi-valued mapping $T : C \longrightarrow CB(C)$ if $x \in Tx$. The set of fixed points of T is denoted by $F(T)$, that is, $F(T) = \{x \in C : x \in Tx\}$.

Definition 1.2.28. *Let C be a nonempty subset of a real Hilbert space H. A multi-valued mapping $T : C \longrightarrow CB(C)$ with $F(T)$ nonempty is called quasi-nonexpansive if for all $x \in C$ and $p \in F(T)$,*

$$D(Tx, Tp) \leq \|x - p\|.$$

Note that every nonexpansive multi-valued mapping with nonempty set of fixed points is quasi-nonexpansive. But the converse may not hold (see, *for example*, [53]).

Definition 1.2.29. *Let C be a nonempty subset of a real Hilbert space H. A multi-valued mapping $T : C \longrightarrow CB(C)$ is said to be $k-$strictly pseudocontractive in the sense of Chidume et al.[14] if there exists a constant $k \in [0, 1)$ such that*

$$D^2(Tx, Ty) \leq \|x - y\|^2 + k\|(x - u) - (y - v)\|^2, \tag{1.3}$$

for all $x, y \in C$, $u \in Tx, v \in Ty$, where $D^2(Tx, Ty) = (D(Tx, Ty))^2$. If $k \leq 1$ in (1.3), then T is said to be pseudocontractive mapping.

We remark from the definitions that the class of multi-valued k-strictly pseudocontractive mappings contains properly the class of multi-valued nonexpansive mappings and the class of multi-valued pseudocontractive mappings contains properly the class of multi-valued $k-$strictly pseudocontractive mappings (see, [53]).

Definition 1.2.30. *Let C be a nonempty subset of a real Hilbert space H. A multi-valued mapping $T : C \longrightarrow CB(C)$ is said to be hemicontractive-type in the sense of Woldeamanuel et al.[53], if $F(T) \neq \emptyset$ and for all $p \in F(T)$, $x \in C$, we have*

$$D^2(Tx, Tp) \leq \|x - p\|^2 + \|x - u\|^2, \ \forall \ u \in Tx.$$

And T is said to be demicontractive-type if $F(T) \neq \emptyset$ and theere exists $k \in [0, 1)$ such that for all $p \in F(T)$, $x \in C$, we have

$$D^2(Tx, Tp) \leq \|x - p\|^2 + k\|x - u\|^2, \ \forall \ u \in Tx.$$

We observe that every pseudocontractive multi-valued mapping T with $F(T) \neq \emptyset$ and $T(p) = \{p\}, \forall p \in F(T)$ is hemicontractive-type mapping, while every $k-$strictly pseudocontractive multi-valued mapping T with $F(T) \neq \emptyset$ and $T(p) = \{p\}, \forall p \in F(T)$ is demicontractive-type mapping. It is also easy to see that every quasi-nonexpansive mapping is demicontractive-type mapping and every demicontractive-type mapping is hemicontractive-type mapping. As the following example shows, the inclusions are strict (see also [53]).

Example 1.2.31. *Let $T : \mathbb{R} \longrightarrow CB(\mathbb{R})$ be given by*

$$Tx = [-3x, -2x], \ \textit{if } x \geq 0 \textit{ and } Tx = [-2x, -3x], \ \textit{if } x < 0.$$

Then, zero is the only fixed point of T, i.e., $F(T) = \{0\}$ and for all $x \in \mathbb{R} \setminus \{0\}$, we have

$$D(Tx, T0) = \max\left\{\sup_{a \in Tx} d(a, T0), \sup_{b \in T0} d(b, Tx)\right\}$$

$$= \max\left\{\sup_{a\in Tx}|a|, d(0,Tx)\right\}$$
$$= \max\{3|x|, 2|x|)\} = 3|x| > |x-0|, \quad (1.4)$$

which shows that T is not a quasi-nonexpansive mapping. And for every $x \in \mathbb{R}$,

$$d(x,Tx) = \inf_{a\in Tx}|a-x| = |x+2x| = 3|x|.$$

Thus, from (1.4), we have

$$\begin{aligned} D^2(Tx,T0) &= 9|x|^2 = |x-0|^2 + 8|x|^2 \\ &\le |x-0|^2 + 9|x|^2 = |x-0|^2 + (d(x,Tx))^2 \\ &\le |x-0|^2 + |x-u|^2, \text{ for all } u \in Tx. \end{aligned}$$

This shows that T is a hemicontractive-type mapping. However, it is not a pseudocontractive mapping. Indeed, for $x = 1,\ y = 2,\ u = -3 \in Tx$ and $v = -4 \in Ty$, we have

$$\begin{aligned} D^2(Tx,Ty) &= 9 > 1+4 = |x-y|^2 + |x-u-(y-v)|^2 \\ &> |x-y|^2 + k|x-u-(y-v)|^2,\ \forall\, k \in [0,1). \end{aligned}$$

It is also easy to see that T is a demicontractive-type mapping with $k \in [\frac{8}{9}, 1)$ but not $k-$strictly pseudocontractive mapping.

Definition 1.2.32. *Let H be a real Hilbert space. Let 2^H denote the set of all subsets of H except empty set. A multi-valued mapping $A : H \longrightarrow 2^H$ is called monotone if for every $x, y \in H$,*

$$\langle x-y, u-v\rangle \ge 0,\ \forall\, u \in Ax \text{ and } v \in Ay \text{ holds .}$$

Such a mapping A is called *maximal* if its graph $G(A)$, $G(A) = \{(x,y) : y \in Ax\}$, is not properly contained in the graph of any other monotone mapping.

Lemma 1.2.33. *Let H be a real Hilbert space. A multi-valued monotone mapping $A : H \longrightarrow 2^H$ is maximal if and only if, for $(x,u) \in H \times H$, $\langle x-y, u-v\rangle \geq 0$ for every $(y,v) \in G(A)$ implies $u \in Ax$.*

Proof. Suppose A is maximal monotone and for $(x,u) \in H \times H$, we have $\langle x-y, u-v\rangle \geq 0$ for every $(y,v) \in G(A)$. We will show that $u \in Ax$. Now, suppose $u \notin Ax$. Then the mapping B defined, for every $y \in H$, by

$$By = \begin{cases} Ay, & \textit{if}\, y \neq x, \\ Ay \cup \{u\}, & \textit{if}\, y = x, \end{cases}$$

is a monotone mapping that contains the graph of A, which is a contradiction to the assumption that A is maximal. Hence, $u \in Ax$. Conversely, let $(x,u) \in H \times H$ be arbitrary such that $\langle x-y, u-v\rangle \geq 0$, for all $(y,v) \in G(A)$ implies $u \in Ax$. We show that A is maximal. Let $B : H \longrightarrow 2^H$ be another monotone mapping such that $G(A) \subset G(B)$. Let $(z,w) \in G(B)$, then since B is monotone, we have that

$$\langle z-y, w-v\rangle \geq 0, \text{ for all } (y,v) \in G(A) \subset G(B).$$

This implies that $w \in Az$, that is, $(z,w) \in G(A)$. Thus, $G(A) = G(B)$ and hence A is maximal. This completes the proof. □

Definition 1.2.34. *Let C be a nonempty, closed and convex subset of a real Hilbert space H. The normal cone to C at $v \in C$, denoted by N_Cv, is given by*

$$N_Cv = \{w \in H : \langle v-u, w\rangle \geq 0, \forall u \in C\}.$$

Note that the mapping $v \longmapsto N_Cv$ is monotone. To see this, let $w \in N_Cv$ and $z \in N_Cu$, then from the definition we have $\langle v-u, w\rangle \geq 0$ and $\langle u-v, z\rangle \geq 0$. Thus, by adding, we get that $\langle v-u, w-z\rangle \geq 0$, for all $w \in N_Cv$ and $z \in N_Cu$. And so, it is monotone.

Definition 1.2.35. *Let C be a nonempty subset of a real Hilbert space H and $T : C \longrightarrow CB(C)$ be a multi-valued mapping. Then, $(I-T)$ is said to be demiclosed at zero if for any sequence $\{x_n\}$ in C such that $x_n \rightharpoonup x \in C$ and $\lim_{n\to\infty} d(x_n, Tx_n) = 0$, we have that $x \in Tx$.*

1.2.5 Variational Inequality Problem

In this subsection, we consider the classical variational inequality which was initially introduced by Stampacchia [46] in 1964.

Definition 1.2.36. *Let C be a nonempty, closed and convex subset of a real Hilbert space H and $A : C \longrightarrow H$ be a nonlinear mapping. The classical variational inequality problem is to find an element $u \in C$ such that*

$$\langle v - u, Au \rangle \geq 0, \quad \textit{for all } v \in C. \tag{1.5}$$

The set of solutions of the classical variational inequality problem is denoted by $VI(C, A)$. That is,

$$VI(C, A) = \{u \in C : \langle v - u, Au \rangle \geq 0, \ \forall v \in C\}.$$

It is easy to see from the definitions that $u \in VI(C, A)$ if and only if $-Au \in N_C u$. An equivalent expression of this is to write

$$0 \in Au + N_C u$$

such a relation is called a generalized inclusion.

Lemma 1.2.37. *Let C be a nonempty, closed and convex subset of a real Hilbert space H and $T : C \longrightarrow C$ be any nonlinear mapping. Define a mapping $A : C \longrightarrow H$ by $Ax = x - Tx$, then $VI(C, A) = F(T)$.*

Proof. If a point $p \in C$ is a fixed point of T, then it is clear that $p \in VI(C, A)$ because $Ap = 0$. Conversely, let $p \in VI(C, A)$, then $\langle Ap, y - p\rangle \geq 0$, for all $y \in C$. In particular, since $Tp \in C$, taking $y = Tp$, we obtain that $\langle p - Tp, Tp - p\rangle \geq 0$, which gives that $Tp = p$. Thus, $VI(C, A) = F(T)$. This completes the proof. □

We observe from Lemma 1.2.37 that the set of solutions of the classical variational inequality problem is equivalent to the set of fixed points of the mapping T. Recall that the fixed point problem is to find a point $x \in C$ such that $x = Tx$, where $T : C \longrightarrow C$ is a given mapping. In this case fixed point problem is a special case of variational inequality problem.

Remark 1.2.38. *Let C be a nonempty, closed and convex subset of a real Hilbert space H. Let A be a nonlinear mapping of C into H. If $C = H$, then $VI(C, A) = A^{-1}(0)$.*

Proof. Let $z \in A^{-1}0$, then it follows directly from the definition that $z \in VI(C, A)$. Conversely, let $z \in VI(C, A)$, then we have that $\langle Az, y - z\rangle \geq 0$, for all $y \in C = H$. Thus, taking $y = z - Az$, we get $0 \leq \langle Az, -Az\rangle = -\|Az\|^2$, which gives that $Az = 0$, i.e., $z \in A^{-1}(0)$. Hence, $VI(C, A) = A^{-1}(0)$. This completes the proof. □

Thus, system of equations can be considered as a variational inequality problem.

Lemma 1.2.39. *[34] Let A be a nonlinear mapping from a nonempty, closed and convex subset C of a real Hilbert space H into H. Then, in the context of variational inequality problem, we have that*

$$u \in VI(C, A) \text{ if and only if } u = P_C(u - \lambda Au),\ \forall \lambda > 0, \tag{1.6}$$

where P_C is the metric projection from H onto C.

We note from Lemma 1.2.39 that solving the variational inequality problem, is equivalent to finding a fixed point of the mapping $P_C(I - \lambda A)$. Thus, fixed point methods can be implemented in finding solutions of variational inequality problems.

The following lemma is due to Zegeye [58].

Lemma 1.2.40. *Let C be a nonempty, closed and convex subset of a real Hilbert space H and A be a continuous monotone mapping from C into H. Then, $VI(C, A)$ is closed and convex subset of C.*

1.2.6 Equilibrium Problems

We begin this subsection with the definition of equilibrium problem which was introduced by Blum and Oettli [4] in 1994.

Definition 1.2.41. *Let C be a nonempty, closed and convex subset of a real Hilbert space H and $F : C \times C \longrightarrow \mathbb{R}$ be a bifunction. The equilibrium problem is the problem of finding a point $z \in C$ such that*

$$F(z, y) \geq 0, \text{ for all } y \in C. \tag{1.7}$$

The set of solutions of problem (1.7) is denoted by $EP(F)$. That is,

$$EP(F) = \{z \in C : F(z, y) \geq 0, \ \forall y \in C\}.$$

Let C and H be as in Definition 1.2.41. Given a mapping $A : C \longrightarrow H$, if we define $F(x, y) = \langle Ax, y - x\rangle$, for all $x, y \in C$, then $z \in EP(F)$ if and only if $\langle Az, y - z\rangle \geq 0$, for all $y \in C$ and hence $z \in VI(C, A)$. In addition, let $T : C \longrightarrow C$ be a mapping. If $F(x, y) = \langle x - Tx, y - x\rangle$, for all $x, y \in C$, then the equilibrium problem is equivalent to the fixed point problem for the mapping T. This shows that equilibrium problem contains classical variational inequality problem and fixed point problem as a special case.

Let C be a nonempty, closed and convex subset of a real Hilbert space H and let $g : C \longrightarrow \mathbb{R}$ be a given mapping. The *minimization problem* is to find an element $x \in C$ such that

$$g(x) \leq g(y), \text{ for all } y \in C.$$

If we define $F(x, y) = g(y) - g(x)$ for all $x, y \in C$, then the equilibrium problem determined by F is equivalent to the minimization problem relative to g. Hence, equilibrium problem includes minimization problem as a special case.

For solving an equilibrium problem, we shall make use of the following assumption.

Assumption 1.2.42. *Let C be a nonempty, closed and convex subset of a real Hilbert space H. In the sequel, let F be a bifunction from $C \times C$ into $\mathbb{R}$ satisfying the following conditions:*

(A1) $F(x, x) = 0, \forall\, x \in C$;

(A2) F *is monotone, i.e.,* $F(x, y) + F(y, x) \leq 0, \forall\, x, y \in C$;

(A3) $\lim_{t \downarrow 0} F(tz + (1-t)x, y) \leq F(x, y), \forall\, x, y, z \in C$;

(A4) for each $x \in C$, $y \mapsto F(x, y)$ *is convex and lower semicontinuous.*

Definition 1.2.43. *Let C be a nonempty, closed and convex subset of a real Hilbert space H. Let $F : C \times C \longrightarrow \mathbb{R}$ be a bifunction and $A : C \longrightarrow H$ be a nonlinear mapping. The generalized equilibrium problem is to find $z \in C$ such that*

$$F(z, y) + \langle Az, y - z \rangle \geq 0, \ \forall\, y \in C. \tag{1.8}$$

In this thesis, the set of solutions of problem (1.8) is denoted by $EP(F, A)$, that is,

$$EP(F, A) = \{z \in C : F(z, y) + \langle Az, y - z \rangle \geq 0, \ \forall\, y \in C\}.$$

In the case of $A \equiv 0$, the problem (1.8) reduces to the equilibrium problem (1.7) and hence $EP(F,A)$ is given by $EP(F)$. In the case of $F \equiv 0$, the problem (1.8) reduces to the classical variational inequality problem and hence $EP(F,A)$ is given by $VI(C,A)$.
Therefore, generalized equilibrium problem contains classical variational inequality and equilibrium problems as special cases.

Clearly, if $z \in VI(C,A) \cap EP(F)$, then $z \in EP(F,A)$. However, the converse may not be hold as we can see from the following example.

Example 1.2.44. *Let $H = \mathbb{R}$ and $C = [0,1]$. Define $F : C \times C \to \mathbb{R}$ by $F(x,y) = x + y$ and $A : C \to H$ by $Ax = \frac{x}{2}$. Then, for $x = 1$, we have that*

$$\begin{aligned} F(x,y) + \langle Ax, y - x \rangle &= (1 + y) + \frac{1}{2}(y - 1) \\ &= \frac{1}{2}(3y + 1) \geq 0, \end{aligned}$$

for all $y \in C$, which implies that $1 \in EP(F,A)$. But, since

$$\langle A(1), y - 1 \rangle = \frac{1}{2}(y - 1) \leq 0,$$

for all $y \in C$, then $1 \notin VI(C,A)$ and hence $1 \notin EP(F) \cap VI(C,A)$.

Let F be a bifunction from $C \times C$ into $\mathbb{R}$ satisfying Assumption 1.2.42 and $A : C \longrightarrow H$ be a continuous monotone mapping. Define $G : C \times C \longrightarrow \mathbb{R}$ by

$$G(x,y) = F(x,y) + \langle Ax, y - x \rangle,$$

then it is easy to see that the bifunction G satisfies Assumption 1.2.42. Thus, the generalized equilibrium problem (1.8) is equivalent to the following equilibrium problem:

$$\text{find } z \in C \text{ such that } G(z,y) \geq 0, \text{ for all } y \in C.$$

1.2.7 Split Equilibrium Problem

Let H_1 and H_2 be real Hilbert spaces and let C and Q be nonempty, closed and convex subsets of H_1 and H_2, respectively. Let $F_1 : C \times C \longrightarrow \mathbb{R}$ and $F_2 : Q \times Q \longrightarrow \mathbb{R}$ be bifunctions. Let $B : H_1 \longrightarrow H_2$ be a bounded linear mapping, then the *split equilibrium problem (SEP)* is the problem of finding a point $z \in C$ such that

$$F_1(z, x) \geq 0, \text{ for all } x \in C \ , \tag{1.9}$$

and such that

$$y^* = Bz \in Q \text{ solves } F_2(y^*, y) \geq 0, \text{ for all } y \in Q \ . \tag{1.10}$$

The set of solutions of SEP (1.9) and (1.10) is denoted by Ω. That is,

$$\Omega = \{z \in C : z \in EP(F_1) \text{ and } Bz \in EP(F_2)\}.$$

If we consider separately, (1.9) is the equilibrium problem given in (1.7). From (1.9) and (1.10), it is easy to see that the SEP contains a pair of equilibrium problems. If $H_1 = H_2$, $B = I$, $Q = C$ and $F_2 \equiv 0$, then the split equilibrium problem reduces to the equilibrium problem (1.7). Thus, the split equilibrium problem is quite general. Given mappings $A_1 : C \longrightarrow H_1$ and $A_2 : Q \longrightarrow H_2$. If we take $F_1(x, y) = \langle A_1 x, y - x\rangle$ and $F_2(u, v) = \langle A_2 u, v - u\rangle$, then the split equilibrium problem (SEP) becomes split variational inequality problem (SVIP), that is, to find a point $z \in C$ such that

$$z \in VI(C, A_1) \text{ and such that } Bz \in VI(Q, A_2),$$

which was studied by Censor et al. [10]. So, split equilibrium problem includes split variational inequality as special case.

Chapter 2

Literature Review

2.1 Fixed Point and Variational Inequality Problems

In this section, we shall discuss some developments on iterative algorithms as captured in the existing literature for approximation of fixed points of some nonlinear single-valued and multi-valued mappings. We also explore some results on problem of finding a common element of set of fixed points of nonlinear mappings and set of solutions of the classical variational inequality problems which have been considered by some authors in Hilbert spaces.

One of the well-known and widely applied result in fixed point theory is the Banach Contraction Mapping Principle. In metric space setting it can be briefly stated as follows:

Theorem 2.1.1. *(Banach Contraction Mapping Principle) Let (X, d) be a complete metric space and $T : X \longrightarrow X$ be a contraction mapping. Then, T has a unique fixed point in X and for arbitrary point $x_0 \in X$, the Picard iterative process defined by*

$$x_{n+1} = Tx_n = T^{n+1}x_0, \ \forall \ n \geq 0, \tag{2.1}$$

converges strongly to the unique fixed point of T.

If we omit the assumption that the domain is complete, then the Banach Contraction Mapping Principle fails. For example, the mapping $T : (0, 1) \longrightarrow (0, 1)$ given by $Tx = \frac{x}{2}$ is a contraction with no fixed point. Unlike in the case of the Banach Contraction Mapping

Principle, the fixed point of a nonexpansive mapping may not exist. For instance, the mapping $T : [0, \infty) \longrightarrow [0, \infty)$ given by $Tx = x + 2$ is nonexpansive which has no fixed point. For this reason, to ensure the existence of fixed point of a nonexpansive mapping T we must assume additional conditions on T and/or on the underlying space. In this direction, Browder [6] showed that a nonexpansive mapping defined on a bounded, closed and convex subset of a uniformly convex Banach space has a fixed point. However, it may not be unique even when it exists as we can see easily that an identity mapping is nonexpansive with many fixed points.

We now turn to the approximation methods of fixed points (if they exist) of a nonexpansive mapping. The sequence of Picard iterates (2.1) in general does not converge to a fixed point (if it exists) of a nonexpansive mapping as we can see from the following example.

Example 2.1.2. *Let $T : [-1, 1] \longrightarrow [-1, 1]$ be given by $Tx = -x$. Then, T is a nonexpansive mapping with $F(T) = \{0\}$. But, for any $x_0 \neq 0$, the Picard sequence, $(-1)^n x_0$, does not converge to 0.*

However, in 1955, Krasnoselskii [27] proved that if T is a nonexpansive self-mapping defined on a convex and compact subset C of a uniformly convex Banach space X, then the iterative sequence defined from any $x_0 \in C$ by

$$x_{n+1} = \frac{1}{2}x_n + \frac{1}{2}Tx_n \tag{2.2}$$

converges to a fixed point of T. A further generalization of (2.2) due to Shaefer [42] for the approximation of fixed points of nonexpansive mapping T, starting from an initial point x_0, is the following:

$$x_{n+1} = (1 - \lambda)x_n + \lambda Tx_n, \text{ for } \lambda \in (0, 1),$$

which is now known as the Krasnoselskii iterative algorithm in light of [27] and has been studied extensively by various researchers (see, *e.g.,* [14]). Another widely used iterative algorithms for approximating fixed points of nonexpansive mapping are the Mann and the Halpern iterative algorithms.

Mann iterative algorithm, initially introduced by Mann [30] for approximation of fixed point of a nonexpansive mapping T, is defined recursively starting from arbitrary point x_0 in the domain of T by

$$x_{n+1} = (1 - \alpha_n)x_n + \alpha_n T x_n, \ \forall n \geq 0,$$

where $\{\alpha_n\}$ is a sequence of numbers in $(0, 1)$ satisfying some conditions. Since then, it has been studied by many authors for approximating fixed points of several mappings in Banach spaces, (see, *for example*, [37, 38, 41, 51, 59]). It is, however, known that the sequence $\{x_n\}$ generated by the Mann iterative algorithm converges weakly to a fixed point of a mapping T.

Another iterative algorithm introduced for approximation of fixed point of a mapping T defined on nonempty convex subset of an appropriate Banach space is Halpern iterative algorithm. It was first studied by Halpern [18], is defined by the following recursive formula: x_0, u in the domain of T,

$$x_{n+1} = \alpha_n u + (1 - \alpha_n)T x_n, \ \forall \ n \geq 0,$$

where $\{\alpha_n\}$ is a sequence of numbers in $(0, 1)$. Since then, Halpern iteration has been considered by several authors (see, [8, 13, 25, 50, 53] and references therein). In contrast to Mann algorithm, Halpern algorithm converges strongly to a fixed point (if it exists) of T provided that the control sequence $\{\alpha_n\}$ satisfies some appropriate conditions.

As an extension to Halpern iterative algorithm, Moudafi [32] established the following iterative algorithm for approximation of fixed point of a mapping T called Viscosity algorithm: x_0 in domain of T,

$$x_{n+1} = \alpha_n f(x_n) + (1 - \alpha_n)T x_n, \ \forall n \geq 0,$$

where f is a contraction mapping defined from domain of T into itself, and α_n is a sequence in $(0,1)$ satisfying certain restrictions. Many authors have dedicated to the study of the convergence of this algorithm to fixed points of some nonlinear mappings (see, *for example*, [32, 39, 49, 52, 55, 58]).

In 1974, Ishikawa [24] introduced the following iterative algorithm for approximating a fixed point of Lipschitz pseudocontractive self-mapping T of a nonempty, convex and compact subset C of a Hilbert space H:

$$\begin{cases} x_0 \in C, \\ y_n = (1-\beta_n)x_n + \beta_n T x_n, \\ x_{n+1} = (1-\alpha_n)x_n + \alpha_n T y_n, \end{cases} \tag{2.3}$$

for all $n \geq 0$, where $\{\alpha_n\}$ and $\{\beta_n\}$ are sequences of positive numbers satisfying the following conditions:

$$(i)\ 0 \leq \alpha_n \leq \beta_n < 1;\ (ii) \lim_{n\to\infty} \beta_n = 0;\ (iii) \sum_{n=0}^{\infty} \alpha_n \beta_n = \infty.$$

The iterative algorithm (2.3) with the parameters $\{\alpha_n\}$ and $\{\alpha_n\}$ satisfying (i), (ii) and (iii) is called the Ishikawa iterative algorithm. This iterative algorithm has been modified in different directions by many mathematicans for approximation of fixed points. In its original form, the Ishikawa iterative algorithm does not include the Mann iterative algorithm, due to the assumption (i). In fact if $\beta_n = 0$, then $\alpha_n = 0$ as well. For this reason, to obtain Ishikawa-type iterative algorithm which should include Mann iterative algorithm as a special case, some authors, have modified the condition (i) to a weaker condition of the form $0 \leq \alpha_n,\ \beta_n < 1$.

We now discuss the existence and approximation methods of fixed points of some multi-valued mappings. In 1969, Nadler [35] proved the existence of fixed points for multi-valued contractive mapping in a complete metric space. More precisely, Nadler [35] proved the following theorem:

Theorem 2.1.3. *[35] Let (X, d) be a complete metric space and $T : X \longrightarrow CB(X)$ be a multi-valued contractive mapping. Then, T has a fixed point in X.*

However, unlike to the single-valued contractive mapping, the fixed point (when it exists) of a multi-valued contractive is not necessarily unique as the following example shows:

Example 2.1.4. *Let $C = [0, 1]$ and $T : C \longrightarrow CB(C)$ be defined by $Tx = [\frac{x}{2}, 1]$. Then, for $x, y \in C$ with $x \leq y$, we have that $Ty = [\frac{y}{2}, 1] \subset [\frac{x}{2}, 1] = Tx$ and*

$$\begin{aligned} D(Tx, Ty) &= \max\left\{\sup_{a\in Tx} d(a, Ty), \sup_{b\in Ty} d(b, Tx)\right\} \\ &= \max\left\{\sup_{a\in Tx} |a - \frac{y}{2}|, 0\right\} = \frac{1}{2}|x - y|. \end{aligned}$$

By symmetry, if $x > y$, $D(Tx, Ty) = \frac{1}{2}|x - y|$. Hence, T is a multi-valued contractive mapping. Moreover, since $\frac{x}{2} \leq x \leq 1$ for all $x \in C$, we have that $F(T) = [0, 1]$. That is, T has many fixed points.

Fixed point theorems for single-valued nonexpansive mappings have been extended to multi-valued mappings. The first result in this direction was established by Markin [31] in Hilbert spaces. In 1973, Dozo [28] extended the result of Markin [31] to Banach spaces satisfying the Opial's condition (see, [36]). In fact, Dozo [28] proved the following theorem.

Theorem 2.1.5. *Let X be a Banach space which satisfies Opial's condition and C be a nonempty, convex and weakly compact subset of X. If $T : C \longrightarrow K(C)$ is a compact-valued nonexpansive mapping, then there exists $x_0 \in C$ such that $x_0 \in Tx_0$.*

Many research efforts have been devoted to developing iterative algorithms for approximating fixed points (if they exist) for multi-valued mappings.

Mann-type and Ishikawa-type iterative algorithms for multi-valued mappings were first considered by Sastry and Babu [41] in 2005. Since then, many authors considered an iterative algorithms for multi-valued mappings (see, *for example*, [8, 14, 43, 44, 53] and references therein). Moreover, Sastry and Babu [41] introduced the following definitions of Mann-type and Ishikawa-type iterative algorithms for multi-valued mappings.

Let C be a nonempty convex subset of a real Hilbert space H and $P(C)$ denote the collection of all nonempty bounded and proximinal subsets of C. Let $T : C \longrightarrow P(C)$ be a multi-valued mapping with $p \in Tp$.

(a) The sequence of Mann-type iterates is defined by

$$\begin{cases} x_0 \in C, \\ x_{n+1} = \alpha_n x_n + (1-\alpha_n) y_n, \end{cases} \tag{2.4}$$

for all $n \geq 0$, where $y_n \in Tx_n$ is such that $\|y_n - p\| = d(p, Tx_n)$, and $\{\alpha_n\}$ is a sequence of numbers such that $0 \leq \alpha_n < 1$ and $\sum_{n=0}^{\infty} \alpha_n = \infty$.

(b) The sequence of Ishikawa-type iterates is defined by

$$\begin{cases} x_0 \in C, \\ y_n = (1-\beta_n) x_n + \beta_n z_n, \\ x_{n+1} = (1-\alpha_n) x_n + \alpha_n u_n, \end{cases} \tag{2.5}$$

for all $n \geq 0$, where $z_n \in Tx_n$ is such that $\|z_n - p\| = d(p, Tx_n)$, $u_n \in Ty_n$ is such that $\|u_n - p\| = d(p, Ty_n)$, and $\{\alpha_n\}$ and $\{\beta_n\}$ sequences of positive numbers satisfying the conditions:

$$(i)\ 0 \leq \alpha_n, \beta_n < 1;\ (ii) \lim_{n\to\infty} \beta_n = 0;\ (iii) \sum_{n=0}^{\infty} \alpha_n \beta_n = \infty.$$

They proved strong convergence of the above iterative algorithms to fixed points of nonexpansive mapping T whose domain is a nonempty, compact and convex subset of a Hilbert space H. Panyanak [37] extended the results of Sastry and Babu [41] to uniformly convex Banach spaces. In fact, Panyanak [37] proved the following theorem:

Theorem 2.1.6. *Let C be a nonempty, compact and convex subset of a uniformly convex Banach space X. Suppose that a nonexpansive mapping $T : C \longrightarrow P(C)$ has a fixed point p. Then, the Mann-type and Ishikawa-type iterative algorithms defined by (2.4) and (2.5) converges strongly to fixed points of T.*

In 2013, Chidume et al.[14] studied a Krasnoselskii-type iterative algorithm and also proved that the algorithm converges strongly to a fixed point of multi-valued $k-$strictly pseudocontractive mapping in a real Hilbert space under appropriate conditions. In 2015, Woldeamanuel et al.[53] considered the problem of finding a common point of the set of fixed points of a finite family of Lipschitz hemicontractive-type multi-valued mappings T_i with Lipschitz constants L_i, for $i = 1, 2, ...N$ $(N \in \mathbb{N})$ defined from a nonempty, closed and convex subset C of a real Hilbert space into itself, and they introduced the following iterative algorithm from arbitrary $x_1, w \in C$:

$$\begin{cases} y_n = (1-\beta_n)x_n + \beta_n u_n, \; u_n \in T_n x_n \\ z_n = (1-\gamma_n)x_n + \gamma_n w_n, \; w_n \in T_n y_n \\ x_{n+1} = \alpha_n w + (1-\alpha_n) z_n, \; \forall n \geq 1, \end{cases} \tag{2.6}$$

where $T_n := T_{n(modN)+1}$, and $\{\alpha_n\}, \{\beta_n\}, \{\gamma_n\} \subset (0,1)$ satisfy the following conditions:

i) $0 \leq \alpha_n \leq c < 1$ such that $\lim_{n\to\infty} \alpha_n = 0$ and $\sum_{n=1}^{\infty} \alpha_n = \infty$,

ii) $0 < a \leq \gamma_n \leq \beta_n \leq b < \frac{1}{\sqrt{4L^2+1}+1}$, for $L := \max\{L_i : i = 1, 2, ..., N\}$.

Then, they proved that the sequence $\{x_n\}$ generated by (2.6) converges strongly to some point p in $\bigcap_{i=1}^{N} F(T_i)$ nearest to w.

Recently, finding a common element of the set of fixed points of nonlinear mappings and the set of solutions of a variational inequality problem has been considered by many authors (see, *for example*, [2, 16, 23, 34, 51, 59] and references therein).

In 2003, Takahashi and Toyoda [51] considered an iterative algorithm for finding a common element of the set of solutions of a variational inequality problem for an $\alpha-$inverse strongly monotone mapping and the set of fixed points of a nonexpansive mapping and then they established the following weak convergence theorem:

Theorem 2.1.7. *Let C be a nonempty, closed and convex subset of a real Hilbert space H. Let A be an $\alpha-$inverse strongly monotone mapping from C into H, and T be a nonexpansive mapping from C into itself such that $F(T) \cap VI(C, A)$ is nonempty. Let $\{x_n\}$ be a sequence generated by*

$$\begin{cases} x_0 \in C, \\ x_{n+1} = \alpha_n x_n + (1 - \alpha_n) T P_C (x_n - \gamma_n A x_n), \end{cases}$$

for all $n \geq 0$, where $\{\gamma_n\} \subset [a, b]$ for some $a, b \in (0, 2\alpha)$ and $\{\alpha_n\} \subset [c, d]$ for some $c, d \in (0, 1)$. Then, $\{x_n\}$ converges weakly to $p \in F(T) \cap VI(C, A)$, where $p = \lim_{n\to\infty} P_{F(T)\cap VI(C,A)} x_n$.

Subsequently, Iiduka and Takahashi [22], introduced an iterative algorithm for finding a common element of the set of fixed points of a nonexpansive nonself mapping and the set of solutions of a variational inequality problem for an $\alpha-$inverse strongly monotone mapping in a real Hilbert space. Then, they obtained strong convergence theorem which improves the results in [51]. More precisely, they proved the following theorem:

Theorem 2.1.8. *Let C be a nonempty, closed and convex subset of a real Hilbert space H. Let A be an $\alpha-$inverse strongly monotone mapping from C into H, and T be a nonexpansive nonself-mapping from C into H such that $F(T)\cap VI(C,A)$ is nonempty. Let $\{x_n\}$ be given by*

$$\begin{cases} x_1 = x \in C, \\ x_{n+1} = P_C(\alpha_n u + (1-\alpha_n)TP_C(x_n - \lambda_n A x_n)), \end{cases}$$

for all $n \geq 1$, where $\{\lambda_n\}$ is a sequence in $[0, 2\alpha]$ and $\{\alpha_n\}$ a sequence in $[0,1)$ satisfying the following conditions:

i) $0 < a \leq \lambda_n \leq b < 2\alpha,\ \lim_{n\to\infty} \alpha_n = 0,\ \sum_{n=1}^{\infty} \alpha_n = \infty;$

ii) $\sum_{n=1}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty$ *and* $\sum_{n=1}^{\infty} |\lambda_{n+1} - \lambda_n| < \infty.$

Then, $\{x_n\}$ converges strongly to $p = P_{F(T)\cap VI(C,A)}u$.

Furthermore, in 2006, Nadezhkina and Takahashi [34] introduced an iterative algorithm for finding a common element of the set of fixed points of a nonexpansive self mapping and the set of solutions of a variational inequality problem for monotone and Lipschitz continuous mapping, and then they obtained the following weak convergence theorem.

Theorem 2.1.9. *Let C be a nonempty, closed and convex subset of a real Hilbert space H. Let A be a monotone and $L-$Lipschitz continuous mapping from C into H, and T be a nonexpansive mapping from C into itself such that $F(T)\cap VI(C,A)$ is nonempty. Let $\{x_n\}$ and $\{y_n\}$ be sequences generated from arbitrary $x_0 \in C$ by*

$$\begin{cases} y_n = P_C(x_n - \gamma_n A x_n), \\ x_{n+1} = \alpha_n x_n + (1-\alpha_n)TP_C(x_n - \gamma_n A y_n), \end{cases}$$

for every $n \geq 0$*, where* $\{\gamma_n\} \subset [a,b]$ *for some* $a,b \in (0,\frac{1}{L})$ *and* $\{\alpha_n\} \subset [c,d]$ *for some* $c,d \in (0,1)$*. Then, the sequences* $\{x_n\}$ *and* $\{y_n\}$ *converge weakly to the same point* $p \in F(T) \cap VI(C,A)$*, where* $p = \lim_{n\to\infty} P_{F(T)\cap VI(C,A)}(x_n)$*.*

Recently, Alghamdi et al. [2] extended the results of Nadezhkina and Takahashi [34] from nonexpansive mapping to a continuous pseudocontractive mapping. Moreover, they constructed a *new iterative algorithm* for finding a common element of the set of fixed points of continuous pseudocontractive mapping and the set of solutions of a variational inequality problem for Lipschitz monotone mapping. They proved the following strong convergence theorem in the setting of Hilbert spaces:

Theorem 2.1.10. *Let* C *be a nonempty, closed and convex subset of a real Hilbert space* H*. Let* $T : C \longrightarrow C$ *be a continuous pseudocontractive mapping and* $A : C \longrightarrow H$ *be a Lipschitzian monotone mapping with Lipschitz constant* L*. Assume that* $\mathcal{F} = F(T) \cap VI(C,A)$ *is nonempty. Let* $\{x_n\}$ *be a sequence generated from an arbitrary* $x_0, u \in C$ *by*

$$\begin{cases} z_n = P_C[x_n - \gamma_n A x_n], \\ x_{n+1} = \alpha_n u + (1-\alpha_n)\left(a_n x_n + b_n w_n + c_n P_C[x_n - \gamma_n A z_n]\right), \end{cases} \tag{2.7}$$

for all $n \geq 0$*, where* P_C *is a metric projection from* H *onto* C*,* $\langle y - w_n, Tw_n\rangle - \frac{1}{r_n}\langle y - w_n, (1+r_n)w_n - x\rangle \leq 0, \forall\, y \in C$*,* $\gamma_n \subset [a,b]$ *for some* $a,b \in (0,\frac{1}{L})$ *and* $\{a_n\},\{b_n\},\{c_n\} \subset (e,f)$ *and* $\{\alpha_n\} \subset (0,c)$ *for some* $c,e,f \in (0,1)$ *satisfying the following conditions: (i)* $a_n+b_n+c_n = 1$*; (ii)* $\lim_{n\to\infty} \alpha_n = 0$*,* $\sum_{n=0}^{\infty} \alpha_n = \infty$*. Then,* $\{x_n\}$ *converges strongly to the point* p *of* $\mathcal{F}$ *nearest to* u*.*

We remark that the results of Alghamdi et al.[2] improve, unify and extend the corresponding results that have been proved previously for approximation of a common solution of fixed point problem for some single-valued mappings and classical variational inequality problems. In particular, all the theorems proved in [2] hold for Lipschitz pseudocontractive mapping because every Lipschitz mapping is continuous.

2.2 Common Solutions of Fixed Point and Generalized Equilibrium Problems

In 1994, Blum and Oettli [4] have shown that several classes of problems, such as variational inequality, optimization, complementary and Nash equilibrium problems can be formulated as equilibrium problems. Then, many techniques and algorithms have been developed for the existence and approximation of a solution of equilibrium problems (see, *for example*, [17, 47] and some references cited therein).

In this section, we review some strong convergence theorems for finding a common element of the set of solutions of equilibrium problems and the set of fixed points of some nonlinear mappings.

In 2007, Takahashi and Takahashi [49] introduced an iterative algorithm for finding a common element of the set of solutions of equilibrium problem and the set of fixed points of a single-valued nonexpansive mapping in a real Hilbert space. In particular, they proved the following theorem:

Theorem 2.2.1. *Let C be a nonempty, closed and convex subset of a real Hilbert space H and $F : C \times C \longrightarrow \mathbb{R}$ be a bifunction satisfying Assumption 1.2.42. Let $f : H \longrightarrow H$ be a contraction mapping and $T : C \longrightarrow H$ be a nonexpansive mapping such that $F(T) \cap EP(F)$*

is nonempty. Let $\{x_n\}$ *be a sequence generated from* $x_1 \in H$ *by*

$$\begin{cases} F(z_n, y) + \frac{1}{r_n}\langle y - z_n, z_n - x_n\rangle \geq 0, \ \forall\, y \in C, \\ x_{n+1} = \alpha_n f(x_n) + (1-\alpha_n)Tz_n, \end{cases} \tag{2.8}$$

for all $n \geq 1$, *where* $\{\alpha_n\} \subset [0,1]$ *and* $\{r_n\} \subset (0,\infty)$ *satisfying:*

(i) $\lim\limits_{n\to\infty} \alpha_n = 0, \sum\limits_{n=1}^{\infty} \alpha_n = \infty$ *and* $\sum\limits_{n=1}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty$;

(ii) $\liminf\limits_{n\to\infty} r_n > 0$ *and* $\sum\limits_{n=1}^{\infty} |r_{n+1} - r_n| < \infty$.

Then, $\{x_n\}$ *and* $\{z_n\}$ *converge strongly to* $p = P_{F(T)\cap EP(F)}f(p)$.

In addition, Wang et al.[52] studied an iterative algorithm by viscosity approximation method for obtaining a common solution of an equilibrium problem, a variational inequality problem and a fixed point problem for a nonexpansive mapping in Hilbert spaces. Also, The authors obtained a strong convergence theorem which improves and extends the results of Takahashi and Takahashi [49]. In fact, they proved the following theorem:

Theorem 2.2.2. *Let* C *be a nonempty, closed and convex subset of a real Hilbert space* H. *Let* $F : C \times C \longrightarrow \mathbb{R}$ *be a bifunction satisfying Assumption 1.2.42 and* f *be a contraction mapping from* C *into itself. Let* A *be an* $\alpha-$*inverse strongly monotone mapping from* C *into* H *and* T *be a nonexpansive mapping from* C *into itself such that* $F(T) \cap EP(F) \cap VI(C, A) \neq \emptyset$. *Suppose* $x_1 \in C$ *and* $\{x_n\}, \{z_n\}$ *are sequences generated by*

$$\begin{cases} F(z_n, y) + \frac{1}{r_n}\langle y - z_n, z_n - x_n\rangle \geq 0, \ \forall\, y \in C, \\ x_{n+1} = \alpha_n f(x_n) + \beta_n x_n + \gamma_n(\mu T z_n + (1-\mu)P_C(z_n - \lambda_n A z_n), \end{cases}$$

for every $n \geq 1$, *where* $\mu \in [0,1]$, $\{r_n\} \subset (0,\infty)$, $\{\lambda_n\} \subset [a,b]$ *with* $0 < a < b < 2\alpha$ *and* $\{\alpha_n\}, \{\beta_n\}$ *and* $\{\gamma_n\}$ *are sequences in* $[0,1]$ *and* *satisfy* $\alpha_n + \beta_n + \gamma_n = 1$ *for every* $n \geq 1$. *If* $\{\alpha_n\}, \{\beta_n\}, \{\lambda_n\}$ *and* $\{r_n\}$ *are chosen so that*

$$i) \quad \lim_{n\to\infty} \alpha_n = 0,\ \sum_{n=1}^{\infty} \alpha_n = \infty,\ 0 < \liminf_{n\to\infty} \beta_n < \limsup_{n\to\infty} \beta_n < 1,$$
$$ii) \quad \liminf_{n\to\infty} r_n > 0,\ \lim_{n\to\infty} |r_{n+1} - r_n| = 0,\ \lim_{n\to\infty} |\lambda_{n+1} - \lambda_n| = 0.$$

Then, $\{x_n\}$ *and* $\{z_n\}$ *converge strongly to the same point* $p = P_{F(T)\cap EP(F)\cap VI(C,A)} f(p)$.

On the other hand, in 2008, Moudafi [33] introduced an iterative algorithm for finding a common element of the set of solutions of a generalized equilibrium problem and the set of fixed points of a nonexpansive single-valued mapping in a Hilbert space setting; and obtained the following weak convergence theorem:

Theorem 2.2.3. *Let* C *be a nonempty, closed and convex subset of a real Hilbert space* H *and* $F : C \times C \longrightarrow \mathbb{R}$ *be a bifunction satisfying Assumption 1.2.42. Let* A *be an* $\alpha-$*inverse strongly monotone mapping from* C *into* H *and* T *be a nonexpansive mapping from* C *into itself such that* $F(T) \cap EP(F, A)$ *is nonempty. Let* $x_0 \in C$ *and* $\{z_n\}$, $\{x_n\}$ *be sequences generated by*

$$\begin{cases} F(z_n, y) + \langle Ax_n, y - z_n\rangle + \frac{1}{r_n}\langle y - z_n, z_n - x_n\rangle \geq 0,\ \forall\, y \in C, \\ x_{n+1} = \alpha_n x_n + (1 - \alpha_n)Tz_n,\ \forall n \geq 0, \end{cases}$$

where $\{r_n\} \subset [a, b]$ *for some* $a, b \in (0, 2\alpha)$ *and* $\{\alpha_n\} \subset [c, d]$ *for some* $c, d \in (0, 1)$. *Then,* $\{x_n\}$ *converges weakly to* $p \in F(T) \cap EP(F, A)$, *where* $p = \lim_{n\to\infty} P_{F(T)\cap EP(F,A)}(x_n)$.

In [50], Takahashi and Takahashi proposed an iterative algorithm and proved that the algorithm converges strongly to a common element of the set of fixed points of a nonexpansive single-valued mapping and the set of solutions of a generalized equilibrium problem for

$\alpha-$inverse strongly monotone mapping. They proved the following theorem:

Theorem 2.2.4. *Let C be a nonempty, closed and convex subset of a real Hilbert space H and $F : C\times C \longrightarrow \mathbb{R}$ be a bifunction satisfying Assumption 1.2.42. Let A be an $\alpha-$inverse strongly monotone mapping from C into H and $T : C \longrightarrow C$ be a nonexpansive mapping such that $F(T)\cap EP(F, A)$ is nonempty. Let $u, x_1 \in C$ and $\{x_n\}$ be a sequence generated by*

$$\begin{cases} F(z_n, y) + \langle Ax_n, y - z_n\rangle + \frac{1}{r_n}\langle y - z_n, z_n - x_n\rangle \geq 0, \ \forall\, y \in C, \\ y_n = \alpha_n u + (1 - \alpha_n)z_n, \\ x_{n+1} = \beta_n x_n + (1 - \beta_n)Ty_n, \ \forall\, n \geq 1, \end{cases}$$

where $\{\alpha_n\} \subset [0, 1], \{\beta_n\} \subset [0, 1]$ and $\{r_n\} \subset [0, 2\alpha]$ satisfying

$$\begin{aligned} &i) \quad 0 < a \leq r_n \leq b < 2\alpha, \ 0 < c \leq \beta_n \leq d < 1, \\ &ii) \quad \lim_{n\to\infty} |r_n - r_{n+1}| = 0, \ \lim_{n\to\infty} \alpha_n = 0 \ \text{and} \ \sum_{n=1}^{\infty} \alpha_n = \infty. \end{aligned}$$

Then, $\{x_n\}$ converges strongly to $p = P_{F(T)\cap EP(F,A)}(u)$.

Furthermore, Ceng et al.[9] introduced an iterative algorithm for finding a common element of the set of solutions of a generalized equilibrium problem (1.8) and the set of fixed points of a continuous pseudocontractive mapping in a Hilbert space settings. In fact, they proved the following strong convergence theorem which improves and extends the results of Moudafi [33], Takahashi and Tekahashi [50].

Theorem 2.2.5. *Let C be a nonempty, closed and convex subset of a real Hilbert space H and $F : C \times C \longrightarrow \mathbb{R}$ be a bifunction satisfying Assumption 1.2.42. Let $A : C \longrightarrow H$ be an $\alpha-$inverse strongly monotone mapping, $f : C \longrightarrow C$ be a contraction mapping, $S : C \longrightarrow C$ be a nonexpansive mapping, and $T : C \longrightarrow C$ be a* *continuous pseudocontractive mapping such that* $F(T) \cap EP(F, A)$ *is nonempty. For* $x_0 \in C$*, let* $\{x_n\}$ *be defined by*

$$\begin{cases} F(z_{n-1}, y) + \langle Ax_{n-1}, y - z_{n-1}\rangle \\ +\frac{1}{r_n}\langle y - z_{n-1}, z_{n-1} - x_{n-1}\rangle \geq 0, \ \forall\, y \in C, \\ y_n = \beta_n f(x_{n-1}) + \gamma_n S z_{n-1} + (1 - \beta_n - \gamma_n)x_{n-1}, \\ x_n = \alpha_n y_n + (1 - \alpha_n)Tx_n, \ \forall\, n \in \mathbb{N}, \end{cases} \tag{2.9}$$

where $\{\alpha_n\}, \{\beta_n\}, \{\gamma_n\} \subset (0, 1]$ *and* $\{r_n\} \subset (0, 2\alpha]$ *satisfying:*

(i) $\lim_{n\to\infty} \alpha_n = 0,\ \sum_{n=1}^{\infty} \beta_n = \infty;$ (ii) $\lim_{n\to\infty} (\frac{\gamma_n}{\beta_n}) = 0,\ \beta_n + \gamma_n \leq 1,\ \forall n \in \mathbb{N}.$

Then, the sequence $\{x_n\}$ *generated by (2.9) converges strongly to* $p = P_{F(T)}f(p)$*. Assume additionally that* $\lim_{n\to\infty} r_n = r > 0$ *and that the existence of* $\lim_{n\to\infty} ||x_n - z_n||$ *implies* $\lim_{n\to\infty} ||x_n - z_n|| = 0$*, then* $p = P_{F(T)\cap EP(F,A)}f(p)$*.*

We remark that Theorem 2.2.5 is more general than Theorem 2.2.4 in the sense that the class of nonexpansive mappings is extended to the class of pseudocontractive mappings.

More recently, Buangern et al. [8] introduced and studied a modified iterative algorithm for approximating a common element of the set of solutions of generalized equilibrium problem and the set of fixed points of nonexpansive multi-valued mapping. Then, they established a strong convergence theorem of the sequence generated by their iteration process in a real Hilbert space which extends the results in [33] and [50]. More precisely, they proved the following theorem:

Theorem 2.2.6. *Let* C *be a nonempty, convex and weakly compact subset of a real Hilbert space* H *and* $F : C \times C \longrightarrow \mathbb{R}$ *be a bifunction satisfying Assumption 1.2.42. Let* A *be an* $\alpha-$*inverse strongly monotone mapping from* C *into* H *and* $T : C \longrightarrow K(C)$ *be a nonexpansive* *multi-valued mapping such that* $F(T) \cap EP(F, A)$ *is nonempty. Let* $u, x_1 \in C$ *and* $\{x_n\}$ *be a sequence generated by*

$$\begin{cases} F(z_n, y) + \langle Ax_n, y - z_n \rangle + \frac{1}{r_n}\langle y - z_n, z_n - x_n \rangle \geq 0, \ \forall\, y \in C, \\ y_n = \alpha_n u + (1 - \alpha_n) z_n, \\ x_{n+1} = \beta_n x_n + (1 - \beta_n) v_n, \ \forall\, n \geq 1, \end{cases}$$

where $v_n \in Ty_n$ *such that* $\|v_n - v_{n+1}\| \leq D(Ty_n, Ty_{n+1})$*,* $\{\alpha_n\}, \{\beta_n\} \subset [0, 1]$ *and* $\{r_n\} \subset [0, 2\alpha]$ *satisfying the following conditions:*

i) $0 < a \leq r_n \leq b < 2\alpha, \ 0 < c \leq \beta_n \leq d < 1$*;*

ii) $\lim_{n\to\infty}(r_n - r_{n+1}) = 0, \ \lim_{n\to\infty} \alpha_n = 0$ *and* $\sum_{n=1}^{\infty} \alpha_n = \infty$.

If $Tp = \{p\}$*, for all* $p \in F(T)$ *, then the sequence* $\{x_n\}$ *converges strongly to* $q = P_{F(T)\cap EP(F,A)}(u)$*.*

We note that Theorem 2.2.6 is more general than Theorems 2.2.3 and 2.2.4 in the sense that the class of single-valued nonexpansive mappings is extended to the class of multi-valued nonexpansive mappings. However, we see that the restriction that C is weakly compact imposed in Theorem 2.2.6 is strong. In addition, in Theorem 2.2.6 the mapping T is assumed to have compact values which is also a strong condition.

On the other hand, in 2012, Zhang [63] considered the problem of finding a common element of the set of common solutions of a finite family of equilibrium problems and the set of fixed points of $k-$strictly pseudocontractive single-valued mapping; and obtained a weak convergence theorem in the setting of Hilbert spaces. Although the result of Zhang [63] extends the corresponding results of [47] and [49] to a finite family of equilibrium problem, it is restricted to equilibrium problems and fixed points of $k-$strictly pseudocontractive single-valued mapping which is a subclass of Lipschitz hemicontractive-type mapping.

2.3 Solutions of Split Equilibrium, Variational Inequality and Fixed Point problems

In 2012, He [20] introduced a new class of equilibrium problems, called split equilibrium problem, which enable us to find a solution of one equilibrium problem such that its image under a given bounded linear mapping is a solution of another equilibrium problem (possibly in different spaces). He [20] also constructed some iterative algorithms for solution of such problems in a Hilbert space settings and obtained some strong and weak convergence theorems. In particular, the following theorems were proved in [20].

Theorem 2.3.1. *Let C and Q be a nonempty, closed and convex subset of real Hilbert spaces H_1 and H_2, respectively. Let $I := \{1, 2, \cdots, N\}$ denote a finite index set and for any $i \in I$, let $F_i : C \times C \longrightarrow \mathbb{R}$ be a bifunction with $\bigcap_{i=1}^{N} EP(F_i) \neq \emptyset$. Let $A : H_1 \longrightarrow H_2$ be a bounded linear mapping with adjoint B and $G : Q \times Q \longrightarrow \mathbb{R}$ be a bifunction with $EP(G) \neq \emptyset$. Let $\{x_n\}$ and $\{z_n^i\}(i \in I)$ be sequences generated from $x_1 \in C$ by*

$$\begin{cases} F_i(z_n^i, y) + \frac{1}{r_n}\langle y - z_n^i, z_n^i - x_n\rangle \geq 0, \ y \in C, i \in I, \\ \tau_n = \frac{1}{N}\left(z_n^1 + \cdots + z_n^N\right), \\ G(w_n, z) + \frac{1}{r_n}\langle z - w_n, w_n - A\tau_n\rangle \geq 0, \ z \in Q, \\ x_{n+1} = P_C(\tau_n + \mu B(w_n - A\tau_n)), \ \forall\, n \in \mathbb{N}, \end{cases} \tag{2.10}$$

where $\{r_n\} \subset (0, \infty)$ with $\liminf_{n\to\infty} r_n > 0$, P_C is a metric projection from H_1 onto C and $\mu \in (0, \frac{1}{||B||^2})$ is a constant. Suppose that $\Omega =$

$\{p \in \bigcap_{i=1}^{N} EP(F_i) : Ap \in EP(G)\} \neq \emptyset$, *then the sequences* $\{x_n\}$ *and* $\{z_n^i\}$ *generated by (2.10) converge weakly to a point* $p^* \in \Omega$, *while* $\{w_n\}$*converges weakly to* $Ap^* \in EP(G)$.

Theorem 2.3.2. *Let* C *and* Q *be a nonempty, closed and convex subset of a real Hilbert spaces* H_1 *and* H_2, *respectively. Let* $I := \{1, 2, \cdots, N\}$ *denotes a finite index set and for any* $i \in I$, *let* $F_i : C \times C \longrightarrow \mathbb{R}$ *be a bifunction with* $\bigcap_{i=1}^{N} EP(F_i) \neq \emptyset$. *Let* $A : H_1 \longrightarrow H_2$ *be a bounded linear mapping with adjoint* B *and* $G : Q \times Q \longrightarrow \mathbb{R}$ *be a bifunction with* $EP(G) \neq \emptyset$. *Let* $C_1 = C$ *and* $\{x_n\}$ *and* $\{z_n^i\}(i \in I)$ *be sequences generated from* $x_1 \in C$ *by*

$$\begin{cases} F_i(z_n^i, y) + \frac{1}{r_n}\langle y - z_n^i, z_n^i - x_n\rangle \geq 0,\ y \in C, i \in I \\ \tau_n = \frac{1}{N}\left(z_n^1 + \cdots + z_n^N\right), \\ G(w_n, z) + \frac{1}{r_n}\langle z - w_n, w_n - A\tau_n\rangle \geq 0,\ z \in Q, \\ y_n = P_C(\tau_n + \mu B(w_n - A\tau_n)), \\ C_{n+1} = \{v \in C_n : ||y_n - v|| \leq ||\tau_n - v|| \leq ||x_n - v||, \\ x_{n+1} = P_{C_{n+1}}(x_1), \forall n \in \mathbb{N}, \end{cases}$$

where $\{r_n\} \subset (0, \infty)$ *with* $\liminf_{n\to\infty} r_n > 0$, P_C *is a metric projection from* H_1 *onto* C *and* $\mu \in (0, \frac{1}{||B||^2})$ *is a constant. Suppose that* $\Omega = \{p \in \bigcap_{i=1}^{N} EP(F_i) : Ap \in EP(G)\} \neq \emptyset$, *then the sequences* $\{x_n\}$ *and* $\{z_n^i\}(i \in I)$ *converge strongly to a point* $p^* \in \Omega$, *while* $\{w_n\}$*converges strongly to* $Ap^* \in EP(G)$.

Recently, Kazmi and Rizvi [25] studied the problem of finding a common element of the set of solutions of a split equilibrium problem, the set of solutions of a variational inequality problem and the set of fixed points of a nonexpansive single-valued mapping. Then, they proved the following strong convergence theorem in a Hilbert space settings.

Theorem 2.3.3. *Let H_1 and H_2 be real Hilbert spaces. Let C and Q be nonempty, convex and closed subsets of H_1 and H_2, respectively. Let $A : C \longrightarrow H_1$ be an $\alpha-$inverse strongly monotone mapping and $B : H_1 \longrightarrow H_2$ be a bounded linear mapping. Assume that $F_1 : C \times C \longrightarrow \mathbb{R}$ and $F_2 : Q \times Q \longrightarrow \mathbb{R}$ be bifunctions satisfying Assumption 1.2.42 and F_2 is upper semi-continuous in the first argument. Let $T : C \longrightarrow C$ be nonexpansive mapping such that $\Theta = F(T) \cap \Omega \cap VI(C, A) \neq \emptyset$. For a given $x_0, u \in C$, let the iterative sequences $\{z_n\}$, $\{x_n\}$ and $\{y_n\}$ be generated by*

$$\begin{cases} z_n = T_{r_n}^{F_1}(x_n + \gamma B^*(T_{r_n}^{F_2} - I)Bx_n, \\ y_n = P_C(z_n - \lambda_n A z_n), \\ x_{n+1} = \alpha_n u + \beta_n x_n + \gamma_n T y_n, \ \forall\, n \geq 0, \end{cases}$$

where $\{r_n\} \subset (0, \infty)$, $\{\lambda_n\} \subset (0, 2\alpha)$ and $\gamma \in (0, \frac{1}{L})$, $L = ||BB^*||$ and B^* is the adjoint of B and the sequences $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$ are in $(0, 1)$ satisfying the following conditions:

$$\begin{aligned}
&(i) \quad \alpha_n + \beta_n + \gamma_n = 1, \ (ii) \lim_{n\to\infty} \alpha_n = 0, \sum_{n=0}^{\infty} \alpha_n = \infty, \\
&(iii) \quad 0 < \liminf_{n\to\infty} \beta_n \leq \limsup_{n\to\infty} \beta_n < 1, \\
&(iv) \quad \liminf_{n\to\infty} r_n > 0, \sum_{n=1}^{\infty} |r_{n+1} - r_n| < \infty, \\
&(v) \quad \lim_{n\to\infty} \left(\frac{\gamma_{n+1}}{1 - \beta_{n+1}} - \frac{\gamma_n}{1 - \beta_n} \right) = 0, \\
&(vi) \quad 0 < \liminf_{n\to\infty} \lambda_n \leq \limsup_{n\to\infty} \lambda_n < 1, \ \text{and} \ \lim_{n\to\infty} |\lambda_{n+1} - \lambda_n| = 0.
\end{aligned}$$

Then, the sequence $\{x_n\}$ converges strongly to $p \in \Theta$, where $p = P_\Theta u$.

We observe that Theorem 2.3.3 improves and generalizes the results of He [20]. However, the upper semi-continuity assumption imposed on F_2 is strong. Furthermore, we note that the conclusion holds if A is restricted to be $\alpha-$inverse strongly monotone mapping.

2.4 Statement of the Problem

Based on literature in this thesis, we consider the following problems:

1. *Can we obtain an iterative algorithm which converges strongly to a common element of the set of solutions of classical variational inequality problem and the set of fixed points of a Lipschitz hemicontractive-type mapping*?

2. *Can an iterative algorithm be introduced which converges strongly to a common solution of a generalized equilibrium problem and a fixed point problem for more general class of Lipschitz hemicontractive-type mappings without the requirement that the domain is weakly compact*?

3. *Is it possible to construct an iterative algorithm which converges strongly to a common solution of a finite family of generalized equilibrium problems and a fixed point problem for Lipschitz hemicontractive-type mapping*?

4. *Can we find an iterative process which converges strongly to a common solution of fixed point problem for a nonexpansive multi-valued mapping, a variational inequality and a split equilibrium problems without the assumption that F_2 is upper semi-continuous*?

2.5 Aim and Objectives

In this thesis, our main aim is to find a common solution of a classical variational inequality, equilibrium and fixed point problems in the setting of Hilbert spaces. In particular, the main objectives of this thesis are:

1. To construct an iterative algorithm which yields the conclusion of Alghamdi et al.[2] for the more general class of Lipschitz hemicontractive-type mappings.
2. To prove a strong convergence theorem for approximating a fixed point of Lipschitz hemicontractive-type mapping which is also a solution of generalized equilibrium problem.
3. To obtain the conclusion of Buangern et al. [8] for hemicontractive-type and continuous monotone mappings.
4. To introduce an iterative algorithm for finding a common solution of a finite family of generalized equilibrium problems and a fixed point problem for Lipschitz hemicontractive-type mapping.
5. To construct an iterative algorithm for approximating a common solution of a fixed point problem for nonexpansive multi-valued mapping, a variational inequality and a split equilibrium problems.
6. To establish strong convergence theorem for a common solution of a fixed point problem for nonexpansive multi-valued mapping, a variational inequality and a split equilibrium problems.

2.6 Significance of the study

Fixed points of multi-valued mappings, solutions of variational inequality and equilibrium problems are of great interest in many applications such as, optimization problems, differential equations, economics, game theory etc. Thus, this thesis will enhance the study of approximation of common solutions of variational inequality, equilibrium and fixed point problems. In addition, the thesis will serve as reference for readers that are interested in finding solutions of these problems and applications.

2.7 Scope of the Study

This thesis is limited to the study of an iterative algorithms for approximating a common element of the set of fixed points of Lipschitz hemicontractive-type multi-valued mapping, the set of solutions of classical variational inequality problem for continuous monotone mapping and the set of solutions of equilibrium problems in real Hilbert spaces.

2.8 Methodology

In this thesis, we critically review existing literature on approximation of fixed points, solutions of variational inequality and equilibrium problems. Then, we study distinct iterative algorithms and prove strong convergence theorems for finding common solutions of fixed point problem for multi-valued mappings, variational inequality and equilibrium problems in the framework of real Hilbert spaces.

Chapter 3

Theory of Methods

3.1 Some Auxiliary Mathematical Tools

In this section, we shall state some facts and results from the existing literature which are used in the proof of our main results. We proceed as follows:

Lemma 3.1.1. *[1] Let H be a real Hilbert space. Then the following hold:*

1. $\|x-y\|^2 = \|x\|^2 + \|y\|^2 - 2\langle x, y\rangle, \ \forall \ x, y \in H;$
2. $\|x+y\|^2 \leq \|x\|^2 + 2\langle y, x+y\rangle, \ \forall \ x, y \in H.$

Lemma 3.1.2. *[59] Let H be a real Hilbert space. Let $\alpha_i \in [0,1]$, $i = 0, 1, 2, ..., n$, be such that $\alpha_0 + \alpha_1 + ... + \alpha_n = 1$, then for $x_i \in H$, $i = 0, 1, 2, ..., n$,*

$$\|\alpha_0 x_0 + \alpha_1 x_1 + ... + \alpha_n x_n\|^2 = \sum_{i=0}^{n} \alpha_i \|x_i\|^2 - \sum_{0 \leq i,j \leq n} \alpha_i \alpha_j \|x_i - x_j\|^2.$$

Lemma 3.1.3. *[48] Let C be a nonempty, closed and convex subset of a real Hilbert space H. Then, the metric projection P_C is characterized by the following properties: for $x \in H$,*

i. $z = P_C x \in C$ *if and only if* $\langle x - z, z - y\rangle \geq 0, \ \forall \ y \in C,$ *and*

ii. $\|y - P_C x\|^2 \leq \|x - y\|^2 - \|x - P_C x\|^2, \ \forall \ x \in H, \ \forall \ y \in C.$

Lemma 3.1.4. *[51] Let C be a nonempty subset of a real Hilbert space and $A : C \longrightarrow H$ be an $\alpha-$inverse strongly monotone mapping. Then, $I - \lambda A$ is a nonexpansive mapping, for all $0 < \lambda \leq 2\alpha$.*

Lemma 3.1.5. *[54] Let $\{a_n\}$ be a sequence of nonnegative real numbers satisfying the following relation:*

$$a_{n+1} \leq (1-\alpha_n)a_n + \alpha_n\delta_n, \text{ for } n \geq n_0,$$

where $\{\alpha_n\} \subset (0,1)$ and $\{\delta_n\} \subset \mathbb{R}$ satisfy the following conditions:

$$\lim_{n\to\infty} \alpha_n = 0, \ \sum_{n=1}^{\infty} \alpha_n = \infty, \ \text{ and } \limsup_{n\to\infty} \delta_n \leq 0.$$

Then, $\lim\limits_{n\to\infty} a_n = 0$.

Lemma 3.1.6. *[29] Let $\{a_n\}$ be a sequence of real numbers such that there exist a subsequence $\{n_i\}$ of $\{n\}$ such that $a_{n_i} < a_{n_i+1}$, for all $i \in \mathbb{N}$. Then there exists a nondecreasing sequence $\{m_k\} \subset \mathbb{N}$ such that $m_k \to \infty$ and the following properties are satisfied by all (sufficiently large) numbers $k \in \mathbb{N}$:*

$$a_{m_k} \leq a_{m_k+1} \text{ and } a_k \leq a_{m_k+1}.$$

In fact, $m_k = \max\{j \leq k : a_j \leq a_{j+1}\}$.

Lemma 3.1.7. *[25] Let C be a nonempty, closed and convex subset of a real Hilbert space H. Let $T : C \longrightarrow H$ be a nonexpansive mapping with $F(T)$ nonempty. Then, for every $x \in C$ and $y \in F(T)$, we have*

$$\langle x - Tx, y - Tx\rangle \leq \frac{1}{2}||Tx - x||^2.$$

Lemma 3.1.8. *[2, 40] Let A be monotone and $L-$Lipschitz mapping from a nonempty, closed and convex subset C of H into H. Let B be a mapping defined by*

$$Bv = \begin{cases} Av + N_C v, & \text{if } v \in C, \\ \emptyset, & \text{if } v \notin C. \end{cases}$$

Then, B is a maximal monotone mapping and $0 \in Bv$ if and only if $v \in VI(C, A)$.

Lemma 3.1.9. *[4] Let C be a nonempty, closed and convex subset of a real Hilbert space H and F be a bifunction from $C \times C$ into $\mathbb{R}$ satisfying Assumption 1.2.42. Then, for any $r > 0$ and $x \in H$, there exists $z \in C$ such that*

$$F(z,y) + \frac{1}{r}\langle y - z, z - x\rangle \geq 0, \ \forall \ y \in C.$$

Lemma 3.1.10. *[17] Let C be a nonempty, closed and convex subset of a real Hilbert space H and F be a bifunction from $C \times C$ into $\mathbb{R}$ satisfying Assumption 1.2.42. For $r > 0$ and $x \in H$, define a mapping $T_r^F : H \longrightarrow C$ as follows:*

$$T_r^F x = \{z \in C : F(z,y) + \frac{1}{r}\langle y - z, z - x\rangle \geq 0, \forall \ y \in C\},$$

for all $x \in H$. Then, the following hold:

(1) T_r^F is single valued;

(2) T_r^F is firmly nonexpansive, i.e.,
$\|T_r^F x - T_r^F y\|^2 \leq \langle T_r^F x - T_r^F y, x - y\rangle, \ \forall \ x, y \in H;$

(3) $F(T_r^F) = EP(F)$;

(4) $EP(F)$ is closed and convex.

The following lemma follows directly from Lemma 2.3 of [50].

Lemma 3.1.11. *For $r > 0$, let $T_r^F : H \longrightarrow C$ be defined as in Lemma 3.1.10. Then, for any $x \in H$ and for any $r_1, r_2 > 0$,*

$$||T_{r_1}^F x - T_{r_2}^F x|| \leq \frac{|r_1 - r_2|}{r_1}\left(||T_{r_1}^F x|| + ||x||\right).$$

Lemma 3.1.12. *[16] Let C be a nonempty, closed and convex subset of a real Hilbert space H. Let $T : C \longrightarrow CB(C)$ be a nonexpansive multi-valued mapping with $F(T) \neq \emptyset$ and $T(p) = \{p\}$, for all $p \in F(T)$. Then, $F(T)$ is closed and convex subset of C.*

Chapter 4

Main Results

4.1 Approximation of a Common Solution of Variational Inequality Problem and Fixed Point Problem

In this section, we give a seemingly new iterative algorithm and prove strong convergence theorems for classical variational inequality problem and fixed point problem for Lipschitz hemicontractive-type mapping in the framework of Hilbert spaces.

Let C be a nonempty, closed and convex subset of a real Hilbert space H. Let $A : C \longrightarrow H$ be a monotone mapping and $T : C \longrightarrow CB(C)$ be a hemicontractive-type multi-valued mapping. Let $\{\gamma_n\}$ be a positive real sequence, and $\{\lambda_n\}, \{a_n\}, \{b_n\}, \{c_n\}$ and $\{\alpha_n\}$ be real sequences in $(0, 1)$ with $a_n + b_n + c_n = 1$, for all $n \geq 0$. For any $x_0 \in C$, we find $v_0 \in Tx_0$ and $y_0 \in C$ such that

$$y_0 = (1 - \lambda_0)x_0 + \lambda_0 v_0,$$

and by Lemma 1.2.3, the metric projection P_C from H onto C is nonempty for every point in H. So, we can find $z_0, u_0 \in C$ such that

$$\begin{cases} z_0 = P_C(x_0 - \gamma_0 A x_0), \\ u_0 = P_C(x_0 - \gamma_0 A z_0). \end{cases}$$

Observe that as a consequence of Lemma 1.2.26, there exists $w_0 \in Ty_0$ such that $\|v_0 - w_0\| \leq 2D(Tx_0, Ty_0)$. Now, for fixed $u \in C$, let

$$x_1 = \alpha_0 u + (1 - \alpha_0)(a_0 x_0 + b_0 w_0 + c_0 u_0).$$

Then, since $x_1 \in C$, we can find $v_1 \in Tx_1$ and $z_1, u_1, y_1 \in C$ such that

$$\begin{cases} z_1 = P_C(x_1 - \gamma_1 A x_1), \\ u_1 = P_C(x_1 - \gamma_1 A z_1), \\ y_1 = (1-\lambda_1)x_1 + \lambda_1 v_1. \end{cases}$$

Again, by Lemma 1.2.26, there exists $w_1 \in Ty_1$ such that $\|v_1 - w_1\| \leq 2D(Tx_1, Ty_1)$. Let

$$x_2 = \alpha_1 u + (1-\alpha_1)(a_1 x_1 + b_1 w_1 + c_1 u_1).$$

Inductively, we generate the sequence $\{x_n\} \subset C$ as follows:

$$\begin{cases} z_n = P_C(x_n - \gamma_n A x_n), \\ u_n = P_C(x_n - \gamma_n A z_n), \\ y_n = (1-\lambda_n)x_n + \lambda_n v_n, \\ x_{n+1} = \alpha_n u + (1-\alpha_n)(a_n x_n + b_n w_n + c_n u_n), \ \forall\, n \geq 0, \end{cases} \tag{4.1}$$

where $v_n \in Tx_n$ and $w_n \in Ty_n$ such that $\|v_n - w_n\| \leq 2D(Tx_n, Ty_n)$.

Now, we are in a position to state and prove our strong convergence theorems.

Theorem 4.1.1. *Let C be a nonempty, closed and convex subset of a real Hilbert space H. Let $A : C \longrightarrow H$ be a $d-$Lipschitz monotone mapping. Let $T : C \longrightarrow CB(C)$ be a Lipschitz hemicontractive-type mapping with Lipschitz constant L. Assume that $\mathcal{F} = F(T) \cap VI(C, A)$ is nonempty and $Tp = \{p\}$ for all $p \in \mathcal{F}$. Let $\{x_n\}$ be a sequence generated by (4.1), where $\{\gamma_n\} \subset (0, \frac{1}{d})$ and $\{a_n\}, \{b_n\}, \{c_n\}$, $\{\alpha_n\} \subset (0,1)$ satisfy the following conditions:*
(i) $a_n + b_n + c_n = 1$; (ii) $b_n + c_n \leq \lambda_n \leq \lambda < \frac{1}{\sqrt{1+4L^2}+1}$, $\forall n \geq 0$. Then, the sequence $\{x_n\}$ is bounded.

Proof. Let $p \in \mathcal{F}$. Then, from Lemma 3.1.3 (ii) and (4.1), we have

$$\begin{aligned} \|u_n - p\|^2 &= \|P_C(x_n - \gamma_n A z_n) - p\|^2 \\ &\leq \|x_n - \gamma_n A z_n - p\|^2 - \|x_n - \gamma_n A z_n - u_n\|^2 \end{aligned}$$

$$\begin{aligned}
&= \langle x_n - \gamma_n A z_n - p, x_n - \gamma_n A z_n - p\rangle \\
&\quad - \langle x_n - \gamma_n A z_n - u_n, x_n - \gamma_n A z_n - u_n\rangle \\
&= \|x_n - p\|^2 - \|x_n - u_n\|^2 + 2\gamma_n\langle A z_n, p - u_n\rangle \\
&= \|x_n - p\|^2 - \|x_n - u_n\|^2 \\
&\quad + 2\gamma_n\langle A z_n - Ap + Ap, p - z_n + z_n - u_n\rangle \\
&= \|x_n - p\|^2 - \|x_n - u_n\|^2 + 2\gamma_n\Big(\langle A z_n - Ap, p - z_n\rangle \\
&\quad + \langle Ap, p - z_n\rangle + \langle A z_n, z_n - u_n\rangle\Big).
\end{aligned}$$

Since A is monotone mapping and $p \in VI(C, A)$, we get that

$$\begin{aligned}
\|u_n - p\|^2 &\leq \|x_n - p\|^2 - \langle x_n - u_n, x_n - u_n\rangle + 2\gamma_n\langle A z_n, z_n - u_n\rangle \\
&= \|x_n - p\|^2 - \|x_n - z_n\|^2 - 2\langle x_n - z_n, z_n - u_n\rangle \\
&\quad - \|z_n - u_n\|^2 + 2\gamma_n\langle A z_n, z_n - u_n\rangle \\
&= \|x_n - p\|^2 - \|x_n - z_n\|^2 - \|z_n - u_n\|^2 \\
&\quad + 2\langle x_n - \gamma_n A z_n - z_n, u_n - z_n\rangle. \qquad (4.2)
\end{aligned}$$

And from Lemma 3.1.3 (i), (4.1), Cauchy Schwartz inequality and the fact that A is $d-$Lipschitz mapping, we get that

$$\begin{aligned}
\langle x_n - \gamma_n A z_n - z_n, u_n - z_n\rangle &= \langle x_n - \gamma_n A x_n - z_n, u_n - z_n\rangle \\
&\quad + \langle \gamma_n A x_n - \gamma_n A z_n, u_n - z_n\rangle \\
&\leq \langle \gamma_n A x_n - \gamma_n A z_n, u_n - z_n\rangle \\
&\leq \gamma_n d\|x_n - z_n\|.\|u_n - z_n\|. \qquad (4.3)
\end{aligned}$$

Thus, from (4.2) and (4.3), we obtain that

$$\begin{aligned}
\|u_n - p\|^2 &\leq \|x_n - p\|^2 - \|x_n - z_n\|^2 - \|z_n - u_n\|^2 \\
&\quad + 2\gamma_n d\|x_n - z_n\|.\|u_n - z_n\| \\
&\leq \|x_n - p\|^2 - \|x_n - z_n\|^2 - \|z_n - u_n\|^2 \\
&\quad + \gamma_n^2 d^2\|x_n - z_n\|^2 + \|z_n - u_n\|^2 \\
&= \|x_n - p\|^2 + (\gamma_n^2 d^2 - 1)\|x_n - z_n\|^2. \qquad (4.4)
\end{aligned}$$

In addition, from (4.1), Lemma 3.1.2, the definition of hemicontractive-type mapping and the fact that $v_n \in Tx_n$, we have that

$$\begin{aligned}
\|y_n - p\|^2 &= \|(1-\lambda_n)(x_n - p) + \lambda_n(v_n - p)\|^2 \\
&= (1-\lambda_n)\|x_n - p\|^2 + \lambda_n\|v_n - p\|^2 \\
&\quad - \lambda_n(1-\lambda_n)\|x_n - v_n\|^2 \\
&= (1-\lambda_n)\|x_n - p\|^2 + \lambda_n \left(d(v_n, Tp)\right)^2 \\
&\quad - \lambda_n(1-\lambda_n)\|x_n - v_n\|^2 \\
&\leq (1-\lambda_n)\|x_n - p\|^2 + \lambda_n D^2(Tx_n, Tp) \\
&\quad - \lambda_n(1-\lambda_n)\|x_n - v_n\|^2 \\
&\leq (1-\lambda_n)\|x_n - p\|^2 + \lambda_n\Big(\|x_n - p\|^2 + \|x_n - v_n\|^2\Big) \\
&\quad - \lambda_n(1-\lambda_n)\|x_n - v_n\|^2 \\
&= \|x_n - p\|^2 + \lambda_n^2\|x_n - v_n\|^2. \qquad (4.5)
\end{aligned}$$

Thus, from Lemma 3.1.2, (4.4), the fact that $w_n \in Ty_n$ and the definition of hemicontractive-type mapping, we have the following:

$$\begin{aligned}
\|x_{n+1} - p\|^2 &= \|\alpha_n u + (1-\alpha_n)(a_n x_n + b_n w_n + c_n u_n) - p\|^2 \\
&\leq \alpha_n\|u - p\|^2 + (1-\alpha_n) \\
&\quad \times \|a_n(x_n - p) + b_n(w_n - p) + c_n(u_n - p)\|^2 \\
&\leq \alpha_n\|u - p\|^2 + (1-\alpha_n)\Big(a_n\|x_n - p\|^2 + b_n\|w_n - p\|^2 \\
&\quad + c_n\|u_n - p\|^2\Big) - (1-\alpha_n)a_n b_n\|w_n - x_n\|^2 \\
&\leq \alpha_n\|u - p\|^2 + (1-\alpha_n)a_n\|x_n - p\|^2 + (1-\alpha_n)b_n \\
&\quad \times D^2(Ty_n, Tp) + (1-\alpha_n)c_n\|u_n - p\|^2 \\
&\quad -(1-\alpha_n)a_n b_n\|w_n - x_n\|^2 \\
&\leq \alpha_n\|u - p\|^2 + (1-\alpha_n)a_n\|x_n - p\|^2 + (1-\alpha_n)b_n \\
&\quad \times \Big(\|y_n - p\|^2 + \|y_n - w_n\|^2\Big) + (1-\alpha_n)c_n \\
&\quad \times \Big(\|x_n - p\|^2 + (\gamma_n^2 d^2 - 1)\|x_n - z_n\|^2\Big) \\
&\quad - (1-\alpha_n)a_n b_n\|w_n - x_n\|^2.
\end{aligned}$$

Then, using (4.5) and condition (i), we have that

$$\begin{aligned}
\|x_{n+1}-p\|^2 &\leq \alpha_n\|u-p\|^2+(1-\alpha_n)(a_n+c_n)\|x_n-p\|^2 \\
&\quad +(1-\alpha_n)b_n\Big(\|x_n-p\|^2+\lambda_n^2\|x_n-v_n\|^2\Big) \\
&\quad +(1-\alpha_n)b_n\|y_n-w_n\|^2+(1-\alpha_n)c_n \\
&\quad \times(\gamma_n^2d^2-1)\|x_n-z_n\|^2-(1-\alpha_n)a_nb_n\|w_n-x_n\|^2 \\
&= \alpha_n\|u-p\|^2+(1-\alpha_n)\|x_n-p\|^2+(1-\alpha_n)b_n\lambda_n^2 \\
&\quad \times\|x_n-v_n\|^2+(1-\alpha_n)b_n\|y_n-w_n\|^2 \\
&\quad +(1-\alpha_n)c_n(\gamma_n^2d^2-1)\|x_n-z_n\|^2 \\
&\quad -(1-\alpha_n)a_nb_n\|w_n-x_n\|^2. \qquad (4.6)
\end{aligned}$$

Furthermore, since T is $L-$Lipschitz mapping, from (4.1), Lemma 3.1.2 and the assumption that $||v_n-w_n||\leq 2D(Tx_n,Ty_n)$, we get that

$$\begin{aligned}
\|y_n-w_n\|^2 &= \|(1-\lambda_n)(x_n-w_n)+\lambda_n(v_n-w_n)\|^2 \\
&= (1-\lambda_n)\|x_n-w_n\|^2+\lambda_n\|v_n-w_n\|^2 \\
&\quad -\lambda_n(1-\lambda_n)\|x_n-v_n\|^2 \\
&\leq (1-\lambda_n)\|x_n-w_n\|^2+4\lambda_nD^2(Tx_n,Ty_n) \\
&\quad -\lambda_n(1-\lambda_n)\|x_n-v_n\|^2 \\
&\leq (1-\lambda_n)\|x_n-w_n\|^2+4\lambda_nL^2\|x_n-y_n\|^2 \\
&\quad -\lambda_n(1-\lambda_n)\|x_n-v_n\|^2.
\end{aligned}$$

Thus, since $||x_n-y_n||=\lambda_n||x_n-v_n||$, we obtain that

$$\begin{aligned}
\|y_n-w_n\|^2 &\leq (1-\lambda_n)\|x_n-w_n\|^2+4\lambda_n^3L^2\|x_n-v_n\|^2 \\
&\quad -\lambda_n(1-\lambda_n)\|x_n-v_n\|^2 \\
&= (1-\lambda_n)\|x_n-w_n\|^2 \\
&\quad +\lambda_n(4L^2\lambda_n^2+\lambda_n-1)\|x_n-v_n\|^2. \qquad (4.7)
\end{aligned}$$

Now, substituting (4.7) into (4.6), we obtain that

$$\|x_{n+1}-p\|^2 \leq \alpha_n\|u-p\|^2+(1-\alpha_n)\|x_n-p\|^2$$

$$
\begin{aligned}
& + (1-\alpha_n) b_n \lambda_n^2 \|x_n - v_n\|^2 + (1-\alpha_n) b_n \Big((1-\lambda_n) \\
& \times \|x_n - w_n\|^2 - \lambda_n (1 - 4L^2\lambda_n^2 - \lambda_n) \|x_n - v_n\|^2 \Big) \\
& - (1-\alpha_n) a_n b_n \|w_n - x_n\|^2 \\
& + (1-\alpha_n) c_n (\gamma_n^2 d^2 - 1) \|x_n - z_n\|^2 \\
= \; & \alpha_n \|u - p\|^2 + (1-\alpha_n) \|x_n - p\|^2 \\
& + \Big((1-\alpha_n) b_n \lambda_n^2 - (1-\alpha_n) b_n \lambda_n (1 - 4L^2\lambda_n^2 - \lambda_n) \Big) \\
& \times \|x_n - v_n\|^2 + \Big((1-\alpha_n) b_n (1-\lambda_n) \\
& - (1-\alpha_n) a_n b_n \Big) \|x_n - w_n\|^2 \\
& + (1-\alpha_n) c_n (\gamma_n^2 d^2 - 1) \|x_n - z_n\|^2,
\end{aligned}
$$

which gives

$$
\begin{aligned}
\|x_{n+1} - p\|^2 \; \le \; & \alpha_n \|u - p\|^2 + (1-\alpha_n) \|x_n - p\|^2 \\
& - (1-\alpha_n) b_n \lambda_n \Big(1 - 4L^2\lambda_n^2 - 2\lambda_n \Big) \|x_n - v_n\|^2 \\
& + (1-\alpha_n) b_n \Big(1 - a_n - \lambda_n \Big) \|x_n - w_n\|^2 \\
& + (1-\alpha_n) c_n (\gamma_n^2 d^2 - 1) \|x_n - z_n\|^2 \\
= \; & \alpha_n \|u - p\|^2 + (1-\alpha_n) \|x_n - p\|^2 \\
& - (1-\alpha_n) b_n \lambda_n \big(1 - 4L^2\lambda_n^2 - 2\lambda_n \big) \|x_n - v_n\|^2 \\
& + (1-\alpha_n) b_n \Big(b_n + c_n - \lambda_n \Big) \|x_n - w_n\|^2 \\
& + (1-\alpha_n) c_n (\gamma_n^2 d^2 - 1) \|x_n - z_n\|^2. \qquad (4.8)
\end{aligned}
$$

Now, since from the hypothesis, we have $\gamma_n < \frac{1}{d}$ and

$$1 - 4L^2\lambda_n^2 - 2\lambda_n \ge 1 - 4L^2\lambda^2 - 2\lambda > 0 \text{ and } b_n + c_n - \lambda_n \le 0, \quad (4.9)$$

for all $n \ge 0$, then inequality (4.8) implies that

$$
\begin{aligned}
\|x_{n+1} - p\|^2 \; \le \; & \alpha_n \|u - p\|^2 + (1-\alpha_n) \|x_n - p\|^2 \\
\le \; & \alpha_n \max\{\|u - p\|^2, \|x_n - p\|^2\} \\
& + (1-\alpha_n) \max\{\|u - p\|^2, \|x_n - p\|^2\} \\
= \; & \max\{\|u - p\|^2, \|x_n - p\|^2\}.
\end{aligned}
$$

Thus, by induction, we have that

$$\|x_{n+1} - p\|^2 \leq \max\{\|u - p\|^2, \|x_0 - p\|^2\}, \ \forall n \geq 0,$$

which implies that $\{x_n\}$ is bounded. This completes the proof. $\square$

Theorem 4.1.2. *Let C be a nonempty, closed and convex subset of a real Hilbert space H. Let $A : C \longrightarrow H$ be a $d-$Lipschitz monotone mapping and $T : C \longrightarrow CB(C)$ be a Lipschitz hemicontractive-type mapping with Lipschitz constant L. Assume that $\mathcal{F} = F(T) \cap VI(C, A)$ is nonempty, closed and convex, $(I - T)$ is demiclosed at zero and $Tp = \{p\}$, for all $p \in \mathcal{F}$. Let $\{x_n\}$ be a sequence given by (4.1), where $\{\gamma_n\} \subset [a, b]$ for some $a, b \in (0, \frac{1}{d}), \{a_n\}, \{b_n\}, \{c_n\} \subset [e, f]$, and $\{\alpha_n\} \subset (0, c)$ for some $c, e, f \in (0, 1)$ satisfying the following conditions:*
(i) $a_n + b_n + c_n = 1$; (ii) $\lim\limits_{n\to\infty} \alpha_n = 0, \sum\limits_{n=0}^{\infty} \alpha_n = \infty$;
(iii) $b_n + c_n \leq \lambda_n \leq \lambda < \frac{1}{\sqrt{1+4L^2}+1}, \ \forall n \geq 0$.
Then, the sequence $\{x_n\}$ converges strongly to the point $q = P_{\mathcal{F}}(u)$.

Proof. Clearly, from Theorem 4.1.1 the sequence $\{x_n\}$, hence $\{y_n\}$, $\{u_n\}$, $\{z_n\}$ are bounded. Let $q = P_{\mathcal{F}}(u)$. Then, using (4.1), Lemma 3.1.1 (2), the fact that $q \in \mathcal{F}$ and following the methods used to get (4.8), we obtain that

$$\begin{aligned}
\|x_{n+1} - q\|^2 &= \|\alpha_n u + (1 - \alpha_n)\big(a_n x_n + b_n w_n + c_n u_n\big) - q\|^2 \\
&\leq (1 - \alpha_n)\|a_n x_n + b_n w_n + c_n u_n - q\|^2 \\
&\quad + 2\alpha_n\langle u - q, x_{n+1} - q\rangle \\
&\leq (1 - \alpha_n)a_n\|x_n - q\|^2 + (1 - \alpha_n)b_n\|w_n - q\|^2 \\
&\quad + (1 - \alpha_n)c_n\|u_n - q\|^2 - (1 - \alpha_n)a_n b_n \\
&\quad \times \|w_n - x_n\|^2 - (1 - \alpha_n)a_n c_n\|u_n - x_n\|^2 \\
&\quad + 2\alpha_n\langle u - q, x_{n+1} - q\rangle
\end{aligned}$$

$$
\begin{aligned}
&\leq (1-\alpha_n)\|x_n-q\|^2-(1-\alpha_n)b_n\lambda_n \\
&\quad \times \big(1-4L^2\lambda_n^2-2\lambda_n\big)\|x_n-v_n\|^2+(1-\alpha_n)b_n \\
&\quad \times \big(b_n+c_n-\lambda_n\big)\|x_n-w_n\|^2-(1-\alpha_n)a_nc_n \\
&\quad \times \|u_n-x_n\|^2+(1-\alpha_n)c_n(\gamma_n^2d^2-1)\|x_n-z_n\|^2 \\
&\quad +2\alpha_n\langle u-q, x_{n+1}-q\rangle. \qquad (4.10)
\end{aligned}
$$

This together with (4.9) implies that

$$
\begin{aligned}
\|x_{n+1}-q\|^2 &\leq (1-\alpha_n)\|x_n-q\|^2 \\
&\quad +2\alpha_n\langle u-q, x_{n+1}-q\rangle. \qquad (4.11)
\end{aligned}
$$

Now, we consider two cases:

Case 1. Suppose that there exists $n_0 \in \mathbb{N}$ such that $\{\|x_n-q\|\}$ is decreasing for all $n \geq n_0$. Then, we get that $\{\|x_n-q\|\}$ is convergent. Thus, from (4.10) and (4.9), we have that

$$
\begin{aligned}
(1-\alpha_n)b_n\lambda_n(1-4L^2\lambda_n^2-2\lambda_n)\|x_n-v_n\|^2 &\leq (1-\alpha_n)\|x_n-q\|^2 \\
&\quad -\|x_{n+1}-q\|^2 \\
&\quad +2\alpha_n\langle u-q, x_{n+1}-q\rangle.
\end{aligned}
$$

Hence, from (4.9), the fact that $\alpha_n \to 0$ as $n \to \infty$, the assumptions of $\{b_n\}$ and $\{\lambda_n\}$, we have that

$$||x_n-v_n|| \to 0 \text{ as } n \to \infty, \qquad (4.12)$$

and from (4.10), (4.9), the assumptions of $\{a_n\}$ and $\{c_n\}$, and the fact that $\alpha_n \to 0$ as $n \to \infty$, we also have that

$$||u_n-x_n|| \to 0, \quad ||z_n-x_n|| \to 0 \text{ as } n \to \infty. \qquad (4.13)$$

Moreover, from (4.1) and (4.12), we obtain that

$$
\begin{aligned}
\|y_n-x_n\| &= \|(1-\lambda_n)x_n+\lambda_nv_n-x_n\| \\
&= \lambda_n\|x_n-v_n\| \to 0 \text{ as } n \to \infty, \qquad (4.14)
\end{aligned}
$$

and hence Lipschitz continuity of T, (4.12), (4.14) and the assumption that $||v_n - w_n|| \leq 2D(Tx_n, Ty_n)$ imply that

$$\begin{aligned} \|w_n - x_n\| &\leq \|w_n - v_n\| + \|v_n - x_n\| \\ &\leq 2D(Tx_n, Ty_n) + \|v_n - x_n\| \\ &\leq 2L\|x_n - y_n\| + \|v_n - x_n\| \to 0 \qquad (4.15) \end{aligned}$$

as $n \to \infty$. In addition, from the fact that $\alpha_n \to 0$ as $n \to \infty$, (4.13) and (4.15) we have that

$$\begin{aligned} \|x_{n+1} - x_n\| &= \|\alpha_n(u - x_n) + (1 - \alpha_n) \\ &\quad \times \Big(b_n(w_n - x_n) + c_n(u_n - x_n)\Big)\| \\ &\leq \alpha_n\|u - x_n\| + (1 - \alpha_n)b_n\|w_n - x_n\| \\ &\quad + (1 - \alpha_n)c_n\|u_n - x_n\| \to 0 \qquad (4.16) \end{aligned}$$

as $n \to \infty$, and from (4.12) we get

$$d(x_n, Tx_n) \leq ||x_n - v_n|| \to 0 \text{ as } n \to \infty. \qquad (4.17)$$

Furthermore, from Theorem 4.1.1, we have that $\{x_{n+1}\}$ is a bounded sequence in H. Thus, since every bounded sequence in Hilbert space has a weakly convergent subsequence, without loss of generality, we can choose a subsequence $\{x_{n_i+1}\}$ of $\{x_{n+1}\}$ and a point, say $z \in H$, such that $x_{n_i+1} \rightharpoonup z$ as $i \to \infty$ and

$$\limsup_{n\to\infty}\langle u - q, x_{n+1} - q\rangle = \lim_{i\to\infty}\langle u - q, x_{n_i+1} - q\rangle.$$

Then, from (4.16), we have $x_{n_i} \rightharpoonup z$ as $i \to \infty$ and hence from (4.13), we get that $u_{n_i} \rightharpoonup z$ and $z_{n_i} \rightharpoonup z$ as $i \to \infty$. Thus, from (4.17) and the demiclosedness of $(I - T)$ at zero, we obtain that

$$z \in F(T).$$

Next, we show that $z \in VI(C, A)$. But, since A is $d-$Lipschitz continuous and from (4.13), we have $z_n - u_n = z_n - x_n + x_n - u_n \to 0$

as $n \to \infty$, we get that

$$||Az_n - Au_n|| \leq d||z_n - u_n|| \to 0 \text{ as } n \to \infty.$$

Let

$$Bv = \begin{cases} Av + N_C v, & \text{if} \quad v \in C, \\ \emptyset, & \text{if} \quad v \notin C. \end{cases} \tag{4.18}$$

Then, by Lemma 3.1.8, B is a maximal monotone mapping and $0 \in Bv$ if and only if $v \in VI(C, A)$. Let $(v, w) \in G(B)$, where $G(B)$ denote the graph of B, then we have $w \in Bv = Av + N_C v$ and hence $w - Av \in N_C v$. Thus, from the definition of $N_C v$, we get that

$$\langle v - y, w - Av \rangle \geq 0, \ \forall y \in C.$$

On the other hand, since $u_n = P_C(x_n - \gamma_n Az_n)$ and $v \in C$, then by Lemma 3.1.3 (i), we have that

$$\langle x_n - \gamma_n Az_n - u_n, u_n - v \rangle \geq 0,$$

and hence

$$\left\langle v - u_n, \frac{(u_n - x_n)}{\gamma_n} + Az_n \right\rangle \geq 0. \tag{4.19}$$

Therefore, since A is monotone mapping, from $w - Av \in N_C v$, (4.19) and $u_{n_i} \in C$, we obtain

$$\begin{aligned}
\langle v - u_{n_i}, w \rangle &\geq \langle v - u_{n_i}, Av \rangle \\
&\geq \langle v - u_{n_i}, Av \rangle - \left\langle v - u_{n_i}, \frac{(u_{n_i} - x_{n_i})}{\gamma_{n_i}} + Az_{n_i} \right\rangle \\
&= \left\langle v - u_{n_i}, Av + Au_{n_i} - Au_{n_i} \right\rangle \\
&\quad - \left\langle v - u_{n_i}, \frac{(u_{n_i} - x_{n_i})}{\gamma_{n_i}} \right\rangle - \langle v - u_{n_i}, Az_{n_i} \rangle \\
&= \langle v - u_{n_i}, Av - Au_{n_i} \rangle - \left\langle v - u_{n_i}, \frac{(u_{n_i} - x_{n_i})}{\gamma_{n_i}} \right\rangle \\
&\quad + \langle v - u_{n_i}, Au_{n_i} - Az_{n_i} \rangle
\end{aligned}$$

$$\geq \langle v - u_{n_i}, Au_{n_i} - Az_{n_i}\rangle - \Big\langle v - u_{n_i}, \frac{(u_{n_i} - x_{n_i})}{\gamma_{n_i}}\Big\rangle.$$

This implies that as $i \to \infty$, we obtain that $\langle v - z, w\rangle \geq 0$. Then, since B is maximal monotone, by Lemma 1.2.33, we get that $0 \in Bz$ and hence $z \in VI(C, A)$. Therefore,

$$z \in \mathcal{F} = F(T) \cap VI(C, A).$$

Thus, since $q = P_{\mathcal{F}}(u)$ and $x_{n_i+1} \rightharpoonup z$, then using Lemma 3.1.3 (i), we obtain that

$$\begin{aligned}\limsup_{n\to\infty}\langle u - q, x_{n+1} - q\rangle &= \lim_{i\to\infty}\langle u - q, x_{n_i+1} - q\rangle. \\ &= \langle u - q, z - q\rangle \leq 0. \qquad (4.20)\end{aligned}$$

Hence, it follows from (4.11), (4.20), condition (ii) and Lemma 3.1.5 that

$$\|x_n - q\| \to 0 \text{ as } n \to \infty\ .$$

Consequently, $x_n \to q = P_{\mathcal{F}}(u)$.

Case 2. Suppose that there exists a subsequence $\{n_j\}$ of $\{n\}$ such that

$$\|x_{n_j} - q\| < \|x_{n_j+1} - q\|,$$

for all $j \in \mathbb{N}$. Then, by Lemma 3.1.6, there exist a nondecreasing sequence $\{m_k\} \subset \mathbb{N}$ such that $m_k \to \infty$, and

$$\|x_{m_k} - q\| \leq \|x_{m_k+1} - q\| \text{ and } \|x_k - q\| \leq \|x_{m_k+1} - q\|, \qquad (4.21)$$

for all $k \in \mathbb{N}$. Thus, replacing n by m_k, from (4.21), (4.10), (4.9) and the fact that $\alpha_n \to 0$, we get that

$$\begin{gathered}||x_{m_k} - v_{m_k}|| \to 0, ||u_{m_k} - x_{m_k}|| \to 0, \\ \text{and } ||z_{m_k} - x_{m_k}|| \to 0 \text{ as } k \to \infty\ .\end{gathered}$$

Hence, following the same method as in Case 1, we obtain that

$$\limsup_{k\to\infty}\langle u-q, x_{m_k+1}-q\rangle \le 0. \tag{4.22}$$

Now, from (4.11), we have that

$$\begin{aligned}\|x_{m_k+1}-q\|^2 &\le (1-\alpha_{m_k})\|x_{m_k}-q\|^2\\ &\quad + 2\alpha_{m_k}\langle u-q, x_{m_k+1}-q\rangle, \end{aligned}\tag{4.23}$$

and hence (4.21) and (4.23) imply that

$$\begin{aligned}\alpha_{m_k}\|x_{m_k}-q\|^2 &\le \|x_{m_k}-q\|^2-\|x_{m_k+1}-q\|^2\\ &\quad + 2\alpha_{m_k}\langle u-q, x_{m_k+1}-q\rangle\\ &\le 2\alpha_{m_k}\langle u-q, x_{m_k+1}-q\rangle.\end{aligned}$$

Hence, the fact that $\alpha_{m_k}>0$ imply that

$$\|x_{m_k}-q\|^2 \le 2\langle u-q, x_{m_k+1}-q\rangle.$$

Thus, using (4.22), we get that $\|x_{m_k}-q\|\to 0$ as $k\to\infty$. This together with (4.23) and (4.22) implies that $\|x_{m_k+1}-q\|\to 0$ as $k\to\infty$. Since $\|x_k-q\|\le\|x_{m_k+1}-q\|$, for all $k\in\mathbb{N}$, we get that $x_k\to q$ as $k\to\infty$. Therefore, from the above two cases, we can conclude that $\{x_n\}$ converges strongly to $q=P_{\mathcal{F}}(u)$. This completes the proof. □

If, in Theorem 4.1.2, we assume that T is a single-valued Lipschitz hemicontractive mapping, then we obtain the following corollary.

Corollary 4.1.3. *Let C be a nonempty, closed and convex subset of a real Hilbert space H. Let $A: C\longrightarrow H$ be a $d-$Lipschitz monotone mapping. Let $T: C\longrightarrow C$ be a Lipschitz hemicontractive mapping with Lipschitz constant L. Assume that $\mathcal{F}=F(T)\cap VI(C,A)$ is nonempty, closed and convex and $(I-T)$ is demiclosed at zero. Let*

$\{x_n\}$ be a sequence generated from an arbitrary $x_0, u \in C$ by

$$\begin{cases} z_n = P_C(x_n - \gamma_n A x_n), \\ u_n = P_C(x_n - \gamma_n A z_n), \\ y_n = (1-\lambda_n)x_n + \lambda_n T x_n, \\ x_{n+1} = \alpha_n u + (1-\alpha_n)(a_n x_n + b_n T y_n + c_n u_n), \end{cases}$$

for all $n \geq 0$, where $\{\gamma_n\} \subset [a,b]$ for some $a, b \in (0, \frac{1}{d})$, $\{a_n\}$, $\{b_n\}$, $\{c_n\} \subset [e,f]$, and $\{\alpha_n\} \subset (0,c)$ for some $c, e, f \in (0,1)$ satisfying the following conditions:
(i) $a_n + b_n + c_n = 1$; (ii) $\lim_{n\to\infty} \alpha_n = 0$, $\sum_{n=0}^{\infty} \alpha_n = \infty$;
(iii) $b_n + c_n \leq \lambda_n \leq \lambda < \frac{1}{\sqrt{1+L^2}+1}$.
Then, the sequence $\{x_n\}$ converges strongly to the point $q = P_{\mathcal{F}}(u)$.

If, in Theorem 4.1.2, we assume that $A \equiv 0$ so that $VI(C,A) = C$, then we obtain the following corollary:

Corollary 4.1.4. *Let C be a nonempty, closed and convex subset of a real Hilbert space H. Let $T : C \longrightarrow CB(C)$ be a Lipschitz hemicontractive-type mapping with Lipschitz constant L. Assume that $F(T)$ is nonempty, closed and convex, $(I - T)$ is demiclosed at zero and $Tp = \{p\}$, for all $p \in F(T)$. Let $\{x_n\}$ be a sequence generated from an arbitrary $x_0, u \in C$ by*

$$\begin{cases} y_n = (1-\lambda_n)x_n + \lambda_n v_n, \\ x_{n+1} = \alpha_n u + (1-\alpha_n)((1-a_n)x_n + a_n w_n), \end{cases}$$

for all $n \geq 0$, where $v_n \in Tx_n$ and $w_n \in Ty_n$ such that $\|v_n - w_n\| \leq 2D(Tx_n, Ty_n)$ and $\{a_n\} \subset [e,f]$, and $\{\alpha_n\} \subset (0,c)$ for some $c, e, f \in (0,1)$ satisfying the following conditions:
(i) $\lim_{n\to\infty} \alpha_n = 0$, $\sum \alpha_n = \infty$; (ii) $a_n \leq \lambda_n \leq \lambda < \frac{1}{\sqrt{1+4L^2}+1}$. Then, the sequence $\{x_n\}$ converges strongly to the point $q = P_{\mathcal{F}}(u)$.

If, in Theorem 4.1.2, we assume that $C = H$, then we have $VI(C, A) = A^{-1}(0)$ and $P_H = I$, identity mapping on H. Hence, we have the following corollary:

Corollary 4.1.5. *Let H be a real Hilbert space. Let $T : H \longrightarrow CB(H)$ be a Lipschitz hemicontractive-type mapping with Lipschitz constant L. Let $A : H \longrightarrow H$ be a $d-$Lipschitz monotone mapping. Assume that $\mathcal{F} = F(T) \cap A^{-1}(0)$ is nonempty, closed and convex, $(I - T)$ is demiclosed at zero and $Tp = \{p\}$, for all $p \in \mathcal{F}$. Let $\{x_n\}$ be a sequence generated from an arbitrary $x_0, u \in C$ by*

$$\begin{cases} z_n = x_n - \gamma_n A x_n, \\ u_n = x_n - \gamma_n A z_n, \\ y_n = (1 - \lambda_n) x_n + \lambda_n v_n, \\ x_{n+1} = \alpha_n u + (1 - \alpha_n)(a_n x_n + b_n w_n + c_n u_n), \end{cases}$$

for all $n \geq 0$, where $v_n \in Tx_n$ and $w_n \in Ty_n$ such that $\|v_n - w_n\| \leq 2D(Tx_n, Ty_n)$ and $\gamma_n \subset [a, b]$ for some $a, b \in (0, \frac{1}{d})$, $\{a_n\}, \{b_n\}, \{c_n\} \subset [e, f]$, and $\{\alpha_n\} \subset (0, c)$ for some $c, e, f \in (0, 1)$ satisfying the following conditions:(i) $a_n + b_n + c_n = 1$; (ii) $\lim_{n\to\infty} \alpha_n = 0$, $\sum \alpha_n = \infty$; (iii) $b_n + c_n \leq \lambda_n \leq \lambda < \frac{1}{\sqrt{1+4L^2}+1}$. Then, the sequence $\{x_n\}$ converges strongly to the point $q = P_{\mathcal{F}}(u)$.

Remark 4.1.6. *We note that, since every quasi-nonexpansive and demicontractive mappings are hemicontractive, the results obtained in this section for hemicontractive (single and multi-valued) mapping also hold for quasi-nonexpansive and demicontractive mappings provided that the indicated conditions are satisfied.*

Remark 4.1.7. *Again we remark that, since every Lipschitz pseudocontractive multi-valued mapping T with $F(T) \neq \emptyset$ and $T(p) = \{p\}, \forall p \in F(T)$ is a Lipschitz hemicontractive-type mapping, the results given in this section also hold for Lipschitz pseudocontractive multi-valued mapping T with $F(T) \neq \emptyset$ and $T(p) = \{p\}, \forall p \in F(T)$.*

Hence, it is also true for nonexpansive and $k-$strictly pseudocontractive multi-valued mappings provided that the specified assumptions are satisfied because every nonexpansive and $k-$strictly pseudocontractive mappings are Lipschitz pseudocontractive.

4.2 Approximation of a Common Solution of Generalized Equilibrium Problem and Fixed Point Problem

In this section, we furnish an iterative algorithm and prove strong convergence theorems for a generalized equilibrium problem and a fixed point problem for Lipschitz hemicontractive-type mapping.

Let C be a nonempty, closed and convex subset of a real Hilbert space H. Let $A : C \longrightarrow H$ be a continuous monotone mapping and $F : C \times C \longrightarrow \mathbb{R}$ be a bifunction satisfying Assumption 1.2.42. Let $T : C \longrightarrow CB(C)$ be a hemicontractive-type multi-valued mapping. Let $\{r_n\}$ be a positive sequence of real numbers and $\{\lambda_n\}, \{\beta_n\}, \{\gamma_n\}$ and $\{\alpha_n\}$ be sequences in $(0, 1)$ such that $\alpha_n + \beta_n + \gamma_n = 1$, for all $n \geq 1$. Let $x_1, u \in C$ be arbitrary. Then, using Lemma 3.1.9, we can find $z_1 \in C$ such that

$$F(z_1, y) + \langle Az_1, y - z_1\rangle + \frac{1}{r_1}\langle y - z_1, z_1 - x_1\rangle \geq 0,\ \forall\, y \in C.$$

And for $u_1 \in Tz_1$, we find $y_1 \in C$ such that

$$y_1 = (1 - \lambda_1)z_1 + \lambda_1 u_1.$$

By Lemma 1.2.26, there exists $v_1 \in Ty_1$ such that $\|u_1 - v_1\| \leq 2D(Tz_1, Ty_1)$ and then we calculate $x_2 \in C$ by

$$x_2 = \alpha_1 u + \beta_1 v_1 + \gamma_1 z_1.$$

Inductively, we construct the sequence $\{x_n\} \subset C$ by the following:

$$\begin{cases} F(z_n, y) + \langle Az_n, y - z_n \rangle \\ +\frac{1}{r_n}\langle y - z_n, z_n - x_n \rangle \geq 0, \ \forall \, y \in C, \\ y_n = (1 - \lambda_n)z_n + \lambda_n u_n, \\ x_{n+1} = \alpha_n u + \beta_n v_n + \gamma_n z_n, \end{cases} \tag{4.24}$$

for all $n \geq 1$, where $u_n \in Tz_n$ and $v_n \in Ty_n$ are such that $\|u_n - v_n\| \leq 2D(Tz_n, Ty_n)$.

We now present our results concerning the iterative sequence generated by (4.24), assuming mild conditions in our parameters. We proceed as follows:

Theorem 4.2.1. *Let C be a nonempty, closed and convex subset of a real Hilbert space H. Let $A : C \longrightarrow H$ be a continuous monotone mapping and $F : C \times C \longrightarrow \mathbb{R}$ be a bifunction satisfying Assumption 1.2.42. Let $T : C \longrightarrow CB(C)$ be a Lipschitz hemicontractive-type mapping with Lipschitz constant L such that $\Gamma = F(T) \cap EP(F, A)$ is nonempty and $Tp = \{p\}$ for all $p \in \Gamma$. Let $\{x_n\}$ be a sequence generated by (4.24), where $\{\beta_n\}, \{\gamma_n\}, \{\alpha_n\}$ and $\{\lambda_n\}$ satisfy the following conditions:*
(i) $\alpha_n + \beta_n + \gamma_n = 1$; (ii) $\alpha_n + \beta_n \leq \lambda_n \leq \lambda < \frac{1}{\sqrt{1+4L^2}+1}$, for all $n \geq 1$. Then, the sequence $\{x_n\}$ is bounded.

Proof. Define $G : C \times C \longrightarrow \mathbb{R}$ by $G(x, y) = F(x, y) + \langle Ax, y - x \rangle$. Then, G satisfies Assumption 1.2.42. Let $p \in \Gamma$, then, $p \in EP(F, A)$ and $p \in EP(F, A)$ is equivalent to $G(p, y) \geq 0$, for all $y \in C$. Thus, from (4.24) and using Lemma 3.1.10, z_n can be rewritten as $z_n = T^G_{r_n} x_n$ and hence, we have $p = T^G_{r_n} p$. So, by Lemma 3.1.10, which assures that $T^G_{r_n}$ is nonexpansive, we obtain that

$$\|z_n - p\| = \|T^G_{r_n} x_n - T^G_{r_n} p\| \leq \|x_n - p\|. \tag{4.25}$$

Since T is hemicontractive-type mapping and $u_n \in Tz_n$, then from (4.24), (4.25) and Lemma 3.1.2, we get

$$\begin{aligned}
\|y_n - p\|^2 &= \|(1-\lambda_n)z_n + \lambda_n u_n - p\|^2 \\
&= (1-\lambda_n)\|z_n - p\|^2 + \lambda_n\|u_n - p\|^2 \\
&\quad - \lambda_n(1-\lambda_n)\|z_n - u_n\|^2 \\
&\leq (1-\lambda_n)\|z_n - p\|^2 + \lambda_n D^2(Tz_n, Tp) \\
&\quad - \lambda_n(1-\lambda_n)\|z_n - u_n\|^2 \\
&\leq (1-\lambda_n)\|z_n - p\|^2 + \lambda_n\Big(\|z_n - p\|^2 + \|z_n - u_n\|^2\Big) \\
&\quad - \lambda_n(1-\lambda_n)\|z_n - u_n\|^2 \\
&= \|z_n - p\|^2 + \lambda_n\|z_n - u_n\|^2 - \lambda_n(1-\lambda_n)\|z_n - u_n\|^2 \\
&\leq \|x_n - p\|^2 + \lambda_n^2\|z_n - u_n\|^2. \qquad (4.26)
\end{aligned}$$

On the other hand, since T is hemicontractive-type mapping and $v_n \in Ty_n$, we obtain from (4.24) and (4.26) that

$$\begin{aligned}
\|v_n - p\|^2 &= d^2(v_n, Tp) \leq D^2(Ty_n, Tp) \\
&\leq \|y_n - p\|^2 + \|y_n - v_n\|^2 \\
&\leq \|x_n - p\|^2 + \lambda_n^2\|z_n - u_n\|^2 + \|y_n - v_n\|^2. \qquad (4.27)
\end{aligned}$$

It follows from (4.24) that

$$\|z_n - y_n\|^2 = \lambda_n^2\|z_n - u_n\|^2. \qquad (4.28)$$

Since T is $L-$Lipschitzian mapping and $\|u_n - v_n\| \leq 2D(Tz_n, Ty_n)$, then using (4.28) and Lemma 3.1.2, we get that

$$\begin{aligned}
\|y_n - v_n\|^2 &= \|(1-\lambda_n)(z_n - v_n) + \lambda_n(u_n - v_n)\|^2 \\
&= (1-\lambda_n)\|z_n - v_n\|^2 + \lambda_n\|u_n - v_n\|^2 \\
&\quad - \lambda_n(1-\lambda_n)\|z_n - u_n\|^2 \\
&\leq (1-\lambda_n)\|z_n - v_n\|^2 + 4\lambda_n D^2(Tz_n, Ty_n) \\
&\quad - \lambda_n(1-\lambda_n)\|z_n - u_n\|^2
\end{aligned}$$

$$
\begin{aligned}
&\leq (1-\lambda_n)\|z_n - v_n\|^2 + 4\lambda_n L^2\|z_n - y_n\|^2 \\
&\quad - \lambda_n(1-\lambda_n)\|z_n - u_n\|^2 \\
&= (1-\lambda_n)\|z_n - v_n\|^2 + 4\lambda_n^3 L^2\|z_n - u_n\|^2 \\
&\quad - \lambda_n(1-\lambda_n)\|z_n - u_n\|^2 \\
&= (1-\lambda_n)\|z_n - v_n\|^2 \\
&\quad + \lambda_n(4L^2\lambda_n^2 + \lambda_n - 1)\|z_n - u_n\|^2. \qquad (4.29)
\end{aligned}
$$

Hence, substituting (4.29) into (4.27), we obtain that

$$
\begin{aligned}
\|v_n - p\|^2 &\leq \|x_n - p\|^2 + \lambda_n^2\|z_n - u_n\|^2 + (1-\lambda_n)\|z_n - v_n\|^2 \\
&\quad + \lambda_n(4L^2\lambda_n^2 + \lambda_n - 1)\|z_n - u_n\|^2 \\
&= \|x_n - p\|^2 + (1-\lambda_n)\|z_n - v_n\|^2 \\
&\quad + \lambda_n(4L^2\lambda_n^2 + 2\lambda_n - 1)\|z_n - u_n\|^2. \qquad (4.30)
\end{aligned}
$$

Thus, from (4.25), (4.30) and Lemma 3.1.2, we have that

$$
\begin{aligned}
\|x_{n+1} - p\|^2 &= \|\alpha_n u + \beta_n v_n + \gamma_n z_n - p\|^2 \\
&\leq \alpha_n\|u - p\|^2 + \beta_n\|v_n - p\|^2 + \gamma_n\|z_n - p\|^2 \\
&\quad - \beta_n\gamma_n\|z_n - v_n\|^2 \\
&\leq \alpha_n\|u - p\|^2 + \beta_n\Big(\|x_n - p\|^2 + (1-\lambda_n)\|z_n - v_n\|^2 \\
&\quad + \lambda_n(4L^2\lambda_n^2 + 2\lambda_n - 1)\|z_n - u_n\|^2\Big) \\
&\quad + \gamma_n\|x_n - p\|^2 - \beta_n\gamma_n\|z_n - v_n\|^2 \\
&= \alpha_n\|u - p\|^2 + (\beta_n + \gamma_n)\|x_n - p\|^2 \\
&\quad + \beta_n(1 - \gamma_n - \lambda_n)\|z_n - v_n\|^2 \\
&\quad - \beta_n\lambda_n(1 - 4L^2\lambda_n^2 - 2\lambda_n)\|z_n - u_n\|^2.
\end{aligned}
$$

Then, by condition (i), we have that

$$
\begin{aligned}
\|x_{n+1} - p\|^2 &\leq \alpha_n\|u - p\|^2 + (1-\alpha_n)\|x_n - p\|^2 \\
&\quad - \beta_n\lambda_n(1 - 4L^2\lambda_n^2 - 2\lambda_n)\|z_n - u_n\|^2 \\
&\quad + \beta_n(\alpha_n + \beta_n - \lambda_n)\|z_n - v_n\|^2, \qquad (4.31)
\end{aligned}
$$

and from condition (ii), we get

$$1 - 4L^2\lambda_n^2 - 2\lambda_n \geq 1 - 4L^2\lambda^2 - 2\lambda > 0 \text{ and}$$
$$\alpha_n + \beta_n - \lambda_n \leq 0, \tag{4.32}$$

for all $n \geq 1$. Therefore, from (4.31) and (4.32), we infer that

$$\begin{aligned} \|x_{n+1} - p\|^2 &\leq \alpha_n \|u - p\|^2 + (1 - \alpha_n)\|x_n - p\|^2 \\ &\leq \max\{\|u - p\|^2, \|x_n - p\|^2\}. \end{aligned}$$

Thus, by induction, we have that

$$\|x_n - p\|^2 \leq \max\{\|u - p\|^2, \|x_1 - p\|^2\},$$

which implies that $\{x_n\}$ is bounded. This completes the proof. $\square$

Theorem 4.2.2. *Let C be a nonempty, closed and convex subset of a real Hilbert space H. Let $A : C \longrightarrow H$ be a continuous monotone mapping and $F : C \times C \longrightarrow \mathbb{R}$ be a bifunction satisfying Assumption 1.2.42. Let $T : C \longrightarrow CB(C)$ be a Lipschitz hemicontractive-type mapping with Lipschitz constant L such that $\Gamma = F(T) \cap EP(F, A)$ is nonempty, closed and convex. Assume that $(I - T)$ is demiclosed at zero and $Tp = \{p\}$ for all $p \in \Gamma$. Let $\{x_n\}$ be a sequence generated by (4.24) , where $\{r_n\} \subset (0, \infty)$ such that $\lim_{n\to\infty} r_n = r$, for some $0 < r < \infty$, $\{\beta_n\}, \{\gamma_n\} \subset [a, b]$, $\{\alpha_n\} \subset (0, c)$ for some $a, b, c \in (0, 1)$ and $\{\lambda_n\}$ satisfying the following conditions:*
(i) $\alpha_n + \beta_n + \gamma_n = 1$; (ii) $\lim_{n\to\infty} \alpha_n = 0$, $\sum_{n=1}^{\infty} \alpha_n = \infty$;
(iii) $\alpha_n + \beta_n \leq \lambda_n \leq \lambda < \frac{1}{\sqrt{1+4L^2}+1}$. Then, the sequence $\{x_n\}$ converges strongly to the point $q = P_\Gamma(u)$.

Proof. Since Γ is nonempty, closed and convex subset of C, we get that P_Γ is well defined. Clearly, from Theorem 4.2.1 the sequence $\{x_n\}$, hence $\{y_n\}$ and $\{z_n\}$ are bounded. Now, Let $q = P_\Gamma(u)$. Then,

$q \in \Gamma$. Thus, using (4.24), Lemma 3.1.2 and Lemma 3.1.1 (2), we obtain

$$\begin{aligned} \|x_{n+1} - q\|^2 &= \|\alpha_n u + \beta_n v_n + \gamma_n z_n - q\|^2 \\ &\leq \|\beta_n(v_n - q) + \gamma_n(z_n - q)\|^2 + 2\alpha_n\langle u - q, x_{n+1} - q\rangle \\ &\leq \beta_n\|v_n - q\|^2 + \gamma_n\|z_n - q\|^2 - \beta_n\gamma_n\|z_n - v_n\|^2 \\ &\quad + 2\alpha_n\langle u - q, x_{n+1} - q\rangle. \end{aligned} \tag{4.33}$$

Since, by Lemma 3.1.10, $T^G_{r_n}$ is firmly nonexpansive and $q = T^G_{r_n}q$, from Lemma 3.1.1 (1) , we have that

$$\begin{aligned} \|z_n - q\|^2 &= \|T^G_{r_n}x_n - T^G_{r_n}q\|^2 \\ &\leq \langle z_n - q, x_n - q\rangle \\ &= \frac{1}{2}\Big(\|z_n - q\|^2 + \|x_n - q\|^2 - \|x_n - z_n\|^2\Big), \end{aligned}$$

which gives that

$$\|z_n - q\|^2 \leq \|x_n - q\|^2 - \|x_n - z_n\|^2. \tag{4.34}$$

Thus, since $q \in \Gamma$, substituting (4.30) and (4.34) into (4.33) and using condition (i), we obtain that

$$\begin{aligned} \|x_{n+1} - q\|^2 &\leq \beta_n\Big(\|x_n - q\|^2 + \lambda_n\left(4L^2\lambda_n^2 + 2\lambda_n - 1\right) \\ &\quad \times \|z_n - u_n\|^2 + (1 - \lambda_n)\|z_n - v_n\|^2\Big) \\ &\quad + \gamma_n(\|x_n - q\|^2 - \|x_n - z_n\|^2) \\ &\quad - \beta_n\gamma_n\|z_n - v_n\|^2 + 2\alpha_n\langle u - q, x_{n+1} - q\rangle \\ &= (\beta_n + \gamma_n)\|x_n - q\|^2 - \beta_n\lambda_n(1 - 4L^2\lambda_n^2 - 2\lambda_n) \\ &\quad \times \|z_n - u_n\|^2 + \beta_n(1 - \gamma_n - \lambda_n)\|z_n - v_n\|^2 \\ &\quad - \gamma_n\|z_n - x_n\|^2 + 2\alpha_n\langle u - q, x_{n+1} - q\rangle \\ &= (1 - \alpha_n)\|x_n - q\|^2 - \beta_n\lambda_n(1 - 4L^2\lambda_n^2 - 2\lambda_n) \\ &\quad \times \|z_n - u_n\|^2 + \beta_n(\alpha_n + \beta_n - \lambda_n)\|z_n - v_n\|^2 \\ &\quad - \gamma_n\|z_n - x_n\|^2 + 2\alpha_n\langle u - q, x_{n+1} - q\rangle. \end{aligned} \tag{4.35}$$

It follows from (4.32) and (4.35) that

$$\begin{aligned} \|x_{n+1}-q\|^2 &\leq (1-\alpha_n)\|x_n-q\|^2 \\ &\quad +2\alpha_n\langle u-q, x_{n+1}-q\rangle. \end{aligned} \tag{4.36}$$

Next, we consider two cases:

Case 1. Suppose that there exists $n_0 \in \mathbb{N}$ such that $\{\|x_n - q\|\}$ is decreasing for all $n \geq n_0$. Then, we get that $\{\|x_n - q\|\}$ is convergent. Thus, from (4.35) and (4.32), we have that

$$\begin{aligned} \beta_n\lambda_n\Big(1-4L^2\lambda_n^2-2\lambda_n\Big)\|z_n-u_n\|^2 &\leq (1-\alpha_n)\|x_n-q\|^2 \\ &\quad -\|x_{n+1}-q\|^2 \\ &\quad +2\alpha_n\langle u-q, x_{n+1}-q\rangle. \end{aligned}$$

Hence, since $\{x_n\}$ is bounded, from (4.32), the fact that $\alpha_n \to 0$ as $n \to \infty$ and the assumptions of $\{\beta_n\}$ and $\{\lambda_n\}$, we have that

$$\|z_n - u_n\| \to 0 \text{ as } n \to \infty, \tag{4.37}$$

which implies that

$$d(z_n, Tz_n) \leq \|z_n - u_n\| \to 0 \text{ as } n \to \infty. \tag{4.38}$$

In addition, from (4.32) and (4.35), we find that

$$\begin{aligned} \gamma_n\|z_n-x_n\|^2 &\leq (1-\alpha_n)\|x_n-q\|^2-\|x_{n+1}-q\|^2 \\ &\quad +2\alpha_n\langle u-q, x_{n+1}-q\rangle. \end{aligned}$$

Thus, since $\alpha_n \to 0$ as $n \to \infty$, it follows from the assumption of $\{\gamma_n\}$ and the boundedness of $\{x_n\}$ that

$$\lim_{n\to\infty}\|z_n - x_n\| = 0. \tag{4.39}$$

Next, using the Lipschitz continuity of T, triangle inequality, (4.37) and the assumption that $||u_n - v_n|| \leq 2D(Tz_n, Ty_n)$, we get that

$$\|z_n - v_n\| \leq \|z_n - u_n\| + \|u_n - v_n\|$$

$$
\begin{aligned}
&\leq \|z_n - u_n\| + 2D(Tz_n, Ty_n) \\
&\leq \|z_n - u_n\| + 2L\|z_n - y_n\| \\
&= \|z_n - u_n\| + 2L\lambda_n\|z_n - u_n\| \to 0 \text{ as } n \to \infty. \qquad (4.40)
\end{aligned}
$$

Therefore, from (4.39), (4.40) and the fact that $\alpha_n \to 0$ as $n \to \infty$, we obtain that

$$
\begin{aligned}
\|x_{n+1} - x_n\| &\leq \|x_{n+1} - z_n\| + \|z_n - x_n\| \\
&= \|\alpha_n(u - z_n) + \beta_n(v_n - z_n)\| + \|z_n - x_n\| \\
&\leq \alpha_n\|u - z_n\| + \beta_n\|v_n - z_n\| + \|z_n - x_n\| \to 0 \qquad (4.41)
\end{aligned}
$$

as $n \to \infty$.

Now, we claim that $\limsup_{n\to\infty}\langle u - q, x_{n+1} - q\rangle \leq 0$. Since $\{x_{n+1}\}$ is a bounded sequence in a real Hilbert space H, there exists a subsequence $\{x_{n_i+1}\}$ of $\{x_{n+1}\}$ and an element in H, say z, such that

$$
x_{n_i+1} \rightharpoonup z \text{ and } \limsup_{n\to\infty}\langle u - q, x_{n+1} - q\rangle = \lim_{i\to\infty}\langle u - q, x_{n_i+1} - q\rangle.
$$

Since, C is weakly closed, we have $z \in C$ and from (4.41) it follows that $x_{n_i} \rightharpoonup z$ as $i \to \infty$. From (4.39), we obtain that $z_{n_i} \rightharpoonup z$ as $i \to \infty$ and hence using the assumption that $(I - T)$ is demiclosed at zero and (4.38), we conclude that

$$
z \in F(T).
$$

In addition, from (4.39) and the fact that $z_{n_i} = T^G_{r_{n_i}}x_{n_i}$, we obtain that

$$
\|x_{n_i} - T^G_{r_{n_i}}x_{n_i}\| = \|x_{n_i} - z_{n_i}\| \to 0 \text{ as } i \to \infty.
$$

Thus, we obtain from Lemma 3.1.11 that

$$
\begin{aligned}
||x_{n_i} - T^G_r x_{n_i}|| &\leq ||x_{n_i} - T^G_{r_{n_i}}x_{n_i}|| + ||T^G_{r_{n_i}}x_{n_i} - T^G_r x_{n_i}|| \\
&\leq ||x_{n_i} - T^G_{r_{n_i}}x_{n_i}|| + \frac{|r_{n_i} - r|}{r}\left(||T^G_r x_{n_i}|| + ||x_{n_i}||\right),
\end{aligned}
$$

which implies that $||x_{n_i} - T_r^G x_{n_i}|| \to 0$ as $i \to \infty$.

Since T_r^G is nonexpansive, we get $(I - T_r^G)$ is demiclosed at zero and hence, since $x_{n_i} \rightharpoonup z$ as $i \to \infty$, we obtain that $z = T_r^G z$, that is, $F(z, y) + \langle Az, y - z\rangle \geq 0$, for all $y \in C$. Hence, we have

$$z \in EP(F, A).$$

Therefore, $z \in \Gamma$. Hence, since $q = P_\Gamma(u)$ and $x_{n_i} \rightharpoonup z$, from Lemma 3.1.3 (i), we get that

$$\begin{aligned}\limsup_{n\to\infty}\langle u - q, x_{n+1} - q\rangle &= \lim_{i\to\infty}\langle u - q, x_{n_i+1} - q\rangle \\ &= \langle u - q, z - q\rangle \leq 0. \end{aligned} \tag{4.42}$$

Thus, from (4.36), (4.42) and Lemma 3.1.5, we get that

$$\|x_n - q\| \to 0 \text{ as } n \to \infty \ .$$

Hence, $x_n \to q = P_\Gamma(u)$.

Case 2. Suppose that there exists a subsequence $\{n_j\}$ of $\{n\}$ such that

$$\|x_{n_j} - q\| < \|x_{n_j+1} - q\|,$$

for all $j \in \mathbb{N}$. Then, by Lemma 3.1.6, there exists a nondecreasing sequence $\{m_k\} \subset \mathbb{N}$ such that $m_k \to \infty$, and

$$\|x_{m_k} - q\| \leq \|x_{m_k+1} - q\| \text{ and } \|x_k - q\| \leq \|x_{m_k+1} - q\|, \tag{4.43}$$

for all $k \in \mathbb{N}$. Thus, replacing n by m_k, from (4.35), (4.32), (4.43) and the fact that $\alpha_n \to 0$ as $n \to \infty$, we get that

$$||z_{m_k} - u_{m_k}|| \to 0, ||z_{m_k} - x_{m_k}|| \to 0 \text{ as } k \to \infty \ .$$

Hence, by the same argument as in Case 1, we obtain that

$$\limsup_{k\to\infty}\langle u - q, x_{m_k+1} - q\rangle \leq 0. \tag{4.44}$$

But, from (4.36), we have that

$$\begin{aligned}\|x_{m_k+1} - q\|^2 &\leq (1 - \alpha_{m_k})\|x_{m_k} - q\|^2 \\ &\quad +2\alpha_{m_k}\langle u - q, x_{m_k+1} - q\rangle, \qquad (4.45)\end{aligned}$$

and hence (4.43) and (4.45) imply that

$$\begin{aligned}\alpha_{m_k}\|x_{m_k} - q\|^2 &\leq \|x_{m_k} - q\|^2 - \|x_{m_k+1} - q\|^2 \\ &\quad + 2\alpha_{m_k}\langle u - q, x_{m_k+1} - q\rangle \\ &\leq 2\alpha_{m_k}\langle u - q, x_{m_k+1} - q\rangle.\end{aligned}$$

Hence, the fact that $\alpha_{m_k} > 0$ implies that

$$\|x_{m_k} - q\|^2 \leq 2\langle u - q, x_{m_k+1} - q\rangle.$$

Thus, using (4.44), we get that $\|x_{m_k} - q\| \to 0$ as $k \to \infty$. This together with (4.45) and (4.44) implies that $\|x_{m_k+1} - q\| \to 0$ as $k \to \infty$. Since $\|x_k - q\| \leq \|x_{m_k+1} - q\|$, for all $k \in \mathbb{N}$, we get that $x_k \to q$ as $k \to \infty$. Consequently, from both cases above, we deduce that $\{x_n\}$ converges strongly to $q = P_\Gamma(u)$. This completes the proof. $\square$

If, in Theorem 4.2.2, we assume that $A \equiv 0$, then A is a continuous monotone mapping and $EP(F, A) = EP(F)$. Hence, we obtain the following corollary:

Corollary 4.2.3. *Let C be a nonempty, closed and convex subset of a real Hilbert space H. Let $F : C \times C \longrightarrow \mathbb{R}$ be a bifunction satisfying Assumption 1.2.42. Let $T : C \longrightarrow CB(C)$ be a Lipschitz hemicontractive-type mapping with Lipschitz constant L such that $\Gamma = F(T) \cap EP(F)$ is nonempty, closed and convex. Assume that $(I - T)$ is demiclosed at zero and $Tp = \{p\}$, for all $p \in \Gamma$. Let*

$x_1, u \in C$ be arbitrary and $\{x_n\}$ be a sequence generated by

$$\begin{cases} F(z_n, y) + \frac{1}{r_n}\langle y - z_n, z_n - x_n\rangle \geq 0, \ \forall\, y \in C, \\ y_n = (1-\lambda_n)z_n + \lambda_n u_n, \\ x_{n+1} = \alpha_n u + \beta_n v_n + \gamma_n z_n, \end{cases}$$

for all $n \geq 1$, where $u_n \in Tz_n$ and $v_n \in Ty_n$ such that $\|u_n - v_n\| \leq 2D(Tz_n, Ty_n)$, and $\{r_n\} \subset (0, \infty)$ such that $\lim_{n\to\infty} r_n = r$ for some $0 < r < \infty$, $\{\beta_n\}, \{\gamma_n\} \subset [a, b]$, $\{\alpha_n\} \subset (0, c)$ for some $a, b, c \in (0, 1)$ and $\{\lambda_n\}$ satisfy the following conditions:
(i) $\alpha_n + \beta_n + \gamma_n = 1$; (ii) $\lim_{n\to\infty} \alpha_n = 0$, $\sum_{n=1}^{\infty} \alpha_n = \infty$;
(iii) $\alpha_n + \beta_n \leq \lambda_n \leq \lambda < \frac{1}{\sqrt{1+4L^2}+1}$. Then, the sequence $\{x_n\}$ converges strongly to the point $q = P_\Gamma(u)$.

If, in Theorem 4.2.2, we assume that $F(x, y) = 0$, for all $x, y \in C$ so that $EP(F, A) = VI(C, A)$, then we obtain the following corollary:

Corollary 4.2.4. *Let C be a nonempty, closed and convex subset of a real Hilbert space H. Let $A : C \longrightarrow H$ be a continuous monotone mapping and $T : C \longrightarrow CB(C)$ be a Lipschitz hemicontractive-type mapping with Lipschitz constant L such that $\Gamma = F(T) \cap VI(C, A)$ is nonempty, closed and convex. Assume that $(I - T)$ is demiclosed at zero and $Tp = \{p\}$, for all $p \in \Gamma$. Let $x_1, u \in C$ be arbitrary and $\{x_n\}$ be a sequence generated by*

$$\begin{cases} \langle Az_n, y - z_n\rangle + \frac{1}{r_n}\langle y - z_n, z_n - x_n\rangle \geq 0, \ \forall\, y \in C, \\ y_n = (1-\lambda_n)z_n + \lambda_n u_n, \\ x_{n+1} = \alpha_n u + \beta_n v_n + \gamma_n z_n, \end{cases}$$

for all $n \geq 1$, where $u_n \in Tz_n$ and $v_n \in Ty_n$ such that $\|u_n - v_n\| \leq 2D(Tz_n, Ty_n)$, and $\{r_n\} \subset (0, \infty)$ such that $\lim_{n\to\infty} r_n = r$ for some $0 < r < \infty$, $\{\beta_n\}, \{\gamma_n\} \subset [a, b]$, $\{\alpha_n\} \subset (0, c)$ for some $a, b, c \in (0, 1)$ and $\{\lambda_n\}$ satisfy the following conditions:

(i) $\alpha_n + \beta_n + \gamma_n = 1$; *(ii)* $\lim_{n\to\infty} \alpha_n = 0$, $\sum_{n=1}^{\infty} \alpha_n = \infty$;
(iii) $\alpha_n + \beta_n \leq \lambda_n \leq \lambda < \frac{1}{\sqrt{1+4L^2}+1}$. *Then, the sequence* $\{x_n\}$ *converges strongly to the point* $q = P_\Gamma(u)$.

If, in Theorem 4.2.2, we assume that T is a single-valued hemicontractive mapping from C into itself, then we obtain the following corollary:

Corollary 4.2.5. *Let C be a nonempty, closed and convex subset of a real Hilbert space H. Let $A : C \longrightarrow H$ be a continuous monotone mapping and $F : C \times C \longrightarrow \mathbb{R}$ be a bifunction satisfying Assumption 1.2.42. Let $T : C \longrightarrow C$ be a Lipschitz hemicontractive mapping with Lipschitz constant L such that $\Gamma = F(T) \cap EP(F, A)$ is nonempty, closed and convex. Assume that $(I - T)$ is demiclosed at zero. Let $x_1, u \in C$ be arbitrary and $\{x_n\}$ be a sequence generated by*

$$\begin{cases} F(z_n, y) + \langle Az_n, y - z_n\rangle + \frac{1}{r_n}\langle y - z_n, z_n - x_n\rangle \geq 0, \ \forall\, y \in C, \\ y_n = (1 - \lambda_n)z_n + \lambda_n T z_n, \\ x_{n+1} = \alpha_n u + \beta_n T y_n + \gamma_n z_n, \end{cases}$$

for all $n \geq 1$, where $\{r_n\} \subset (0, \infty)$ such that $\lim_{n\to\infty} r_n = r$ for some $0 < r < \infty$, $\{\beta_n\}, \{\gamma_n\} \subset [a, b]$, $\{\alpha_n\} \subset (0, c)$ for some $a, b, c \in (0, 1)$ and $\{\lambda_n\}$ satisfy the following conditions:
(i) $\alpha_n + \beta_n + \gamma_n = 1$; *(ii)* $\lim_{n\to\infty} \alpha_n = 0$, $\sum_{n=1}^{\infty} \alpha_n = \infty$;
(iii) $\alpha_n+\beta_n \leq \lambda_n \leq \lambda < \frac{1}{\sqrt{1+L^2}+1}$. *Then, the sequence* $\{x_n\}$ *converges strongly to the point* $q = P_\Gamma(u)$.

If, in Corollary 4.2.5, we assume that $T = I$, then we obtain the following corollary on generalized equilibrium problem.

Corollary 4.2.6. *Let C be a nonempty, closed and convex subset of a real Hilbert space H. Let $A : C \longrightarrow H$ be a continuous monotone* *mapping and $F : C \times C \longrightarrow \mathbb{R}$ be a bifunction satisfying Assumption 1.2.42 such that $EP(F, A)$ is nonempty. Let $x_1, u \in C$ be arbitrary and $\{x_n\}$ be a sequence generated by*

$$\begin{cases} F(z_n, y) + \langle Az_n, y - z_n\rangle + \frac{1}{r_n}\langle y - z_n, z_n - x_n\rangle \geq 0, \ \forall\, y \in C, \\ x_{n+1} = \alpha_n u + (1 - \alpha_n)z_n, \end{cases}$$

for all $n \geq 1$, where $\{r_n\} \subset (0, \infty)$ such that $\lim_{n\to\infty} r_n = r$, for some $0 < r < \infty$ and $\{\alpha_n\} \subset (0, c)$ for some $c \in (0, 1)$ such that $\lim_{n\to\infty} \alpha_n = 0$ and $\sum_{n=1}^{\infty} \alpha_n = \infty$. Then, the sequence $\{x_n\}$ converges strongly to the point $q = P_{EP(F,A)}(u)$.

Remark 4.2.7. *We remark that, since every quasi-nonexpansive and demicontractive mappings are hemicontractive, the results obtained in this section for hemicontractive mapping also hold for quasi-nonexpansive and demicontractive mappings provided that the indicated conditions are satisfied.*

Remark 4.2.8. *Observe that, since every Lipschitz pseudocontractive multi-valued mapping T with $F(T) \neq \emptyset$ and $T(p) = \{p\}, \forall\, p \in F(T)$ is Lipschitz hemicontractive-type, the results given in this section also hold for Lipschitz pseudocontractive multi-valued mapping T with $F(T) \neq \emptyset$ and $T(p) = \{p\}, \forall\, p \in F(T)$. Hence, it is also true for nonexpansive and $k-$strictly pseudocontractive multi-valued mappings provided that the specified assumptions are satisfied because every nonexpansive and $k-$strictly pseudocontractive mappings are Lipschitz pseudocontractive.*

4.3 Approximation of a Common Solution of a Finite Family of Generalized Equilibrium Problems and Fixed Point Problem

In this section, we prove the strong convergence of an iterative algorithm to a common element of the set of fixed points of Lipschitz hemicontractive-type multi-valued mapping and the set of common solutions of a finite family of generalized equilibrium problems in real Hilbert spaces.

Let C be a nonempty, closed and convex subset of a real Hilbert space H. Let $A_m : C \longrightarrow H$ and $F_m : C \times C \longrightarrow \mathbb{R}$, $m = 1, 2, ..., N$, be a finite family of continuous monotone mappings and bifunctions satisfying Assumption 1.2.42, respectively. Let $T : C \longrightarrow CB(C)$ be a hemicontractive-type multi-valued mapping. Let $\{r_{m,n}\} \subset (0, \infty)$, $\{a_n\}, \{b_n\}, \{c_n\}, \{e_n\} \subset (0, 1)$ such that $b_n+c_n+e_n = 1$, and $\{d_{m,n}\} \subset (0, 1]$ such that $\sum_{m=1}^{N} d_{m,n} = 1$ for all $n \geq 1$ and $m \in \{1, 2, \ldots, N\}$. Let $x_1, u \in C$ be arbitrary. Then, for each $m = 1, 2, \ldots, N$, using Lemma 3.1.9, we can find $z_{m,1} \in C$ such that

$$F(z_{m,1}, y) + \langle Az_{m,1}, y - z_{m,1}\rangle + \frac{1}{r_{m,1}}\langle y - z_{m,1}, z_{m,1} - x_1\rangle \geq 0, \ \forall \ y \in C,$$

we then calculate $w_1 \in C$ by $w_1 = \sum_{m=1}^{N} d_{m,1} z_{m,1}$. And for $v_1 \in Tw_1$, we find $y_1 \in C$ such that

$$y_1 = a_1 v_1 + (1 - a_1)w_1.$$

By Lemma 1.2.26, there exists $u_1 \in Ty_1$ such that $\|v_1 - u_1\| \leq 2D(Tw_1, Ty_1)$ and then we calculate $x_2 \in C$ by

$$x_2 = b_1 u + c_1 u_1 + e_1 w_1.$$

Inductively, we generate a sequence $\{x_n\} \subset C$ as follows:

$$\begin{cases} F_m(z_{m,n}, y) + \langle A_m z_{m,n}, y - z_{m,n}\rangle \\ +\frac{1}{r_{m,n}}\langle y - z_{m,n}, z_{m,n} - x_n\rangle \geq 0, \ m = 1, \ldots, N, \ \forall\, y \in C, \\ w_n = \sum_{m=1}^{N} d_{m,n} z_{m,n}, \\ y_n = a_n v_n + (1 - a_n) w_n, \\ x_{n+1} = b_n u + c_n u_n + e_n w_n, \end{cases} \tag{4.46}$$

for all $n \geq 1$, where $v_n \in Tw_n$ and $u_n \in Ty_n$ are such that $\|v_n - u_n\| \leq 2D(Tw_n, Ty_n)$.

Next, we state and prove our convergence results of the sequence generated by (4.46).

Theorem 4.3.1. *Let C be a nonempty, closed and convex subset of a real Hilbert space H. Let $A_m : C \longrightarrow H$ and $F_m : C \times C \longrightarrow \mathbb{R}$, $m = 1, 2, ..., N$, be a finite family of continuous monotone mappings and bifunctions satisfying Assumption 1.2.42, respectively. Let $T : C \longrightarrow CB(C)$ be a Lipschitz hemicontractive-type mapping with Lipschitz constant L. Assume that $\Theta = \bigcap_{m=1}^{N} EP(F_m, A_m) \bigcap F(T)$ is nonempty and $Tp = \{p\}$ for all $p \in \Theta$. Let $\{x_n\}$ be a sequence generated by (4.46), where $b_n + c_n \leq a_n \leq d < \frac{1}{\sqrt{1+4L^2}+1}$, $\forall n \geq 1$. Then, the sequence $\{x_n\}$ is bounded.*

Proof. Let $p \in \Theta$. Then, we have $Tp = \{p\}$ and $F_m(p, y) + \langle A_m p, y - p\rangle \geq 0$, for all $m = 1, 2, ..., N$ and $y \in C$. For each $m = 1, 2, ..., N$, define $G_m : C \times C \longrightarrow \mathbb{R}$ by $G_m(x, y) := F_m(x, y) + \langle A_m x, y - x\rangle$. Then, G_m is a finite family of bifunctions satisfying Assumption 1.2.42 and $p \in EP(F_m, A_m)$ is equivalent to $G_m(p, y) \geq 0$, for all $m = 1, 2, ..., N$ and $y \in C$. Hence, using the definition , $z_{m,n}$ can be rewritten as $z_{m,n} = T^{G_m}_{r_{m,n}} x_n := \{G_m(z_{m,n}, y) + \frac{1}{r_{m,n}}\langle y - z_{m,n}, z_{m,n} - x_n\rangle \geq 0, \ \forall\ y \in C\}$, $m = 1, 2, ..., N$, hence, by Lemma 3.1.10, we obtain $p = T_{r_{m,n}}^{G_m} p$, for all $m = 1, 2, ..., N$. In view of Lemma 3.1.10, $T_{r_{m,n}}^{G_m}$ is nonexpansive, for all $m = 1, 2, ..., N$. Thus, we have that

$$\|z_{m,n} - p\| = \|T_{r_{m,n}}^{G_m} x_n - T_{r_{m,n}}^{G_m} p\| \leq \|x_n - p\|. \tag{4.47}$$

Then, from (4.46) and (4.47), we have the following:

$$\begin{aligned} \|w_n - p\| &= \|\sum_{m=1}^{N} d_{m,n} z_{m,n} - p\| = \|\sum_{m=1}^{N} d_{m,n} z_{m,n} - \sum_{m=1}^{N} d_{m,n} p\| \\ &\leq \sum_{m=1}^{N} d_{m,n} \|x_n - p\| = \|x_n - p\|. \end{aligned} \tag{4.48}$$

And using Lemma 3.1.2, the fact that T is hemicontractive-type mapping and $v_n \in Tw_n$, we find that

$$\begin{aligned} \|y_n - p\|^2 &= \|a_n(v_n - p) + (1 - a_n)(w_n - p)\|^2 \\ &= a_n\|v_n - p\|^2 + (1 - a_n)\|w_n - p\|^2 \\ &\quad - a_n(1 - a_n)\|w_n - v_n\|^2 \\ &\leq a_n D^2(Tw_n, Tp) + (1 - a_n)\|w_n - p\|^2 \\ &\quad - a_n(1 - a_n)\|w_n - v_n\|^2 \\ &\leq a_n\big(\|w_n - p\|^2 + \|w_n - v_n\|^2\big) + (1 - a_n)\|w_n - p\|^2 \\ &\quad - a_n(1 - a_n)\|w_n - v_n\|^2 \\ &= \|w_n - p\|^2 + a_n^2\|w_n - v_n\|^2. \end{aligned}$$

This combined with (4.48) gives that

$$\|y_n - p\|^2 \leq \|x_n - p\|^2 + a_n^2\|w_n - v_n\|^2. \tag{4.49}$$

In addition, since T is hemicontractive-type mapping and $u_n \in Ty_n$, from (4.49), we get that

$$\begin{aligned} \|u_n - p\|^2 &= d^2(u_n, Tp) \leq D^2(Ty_n, Tp) \\ &\leq \|y_n - p\|^2 + \|y_n - u_n\|^2 \\ &\leq \|x_n - p\|^2 + a_n^2\|w_n - v_n\|^2 + \|y_n - u_n\|^2. \end{aligned} \tag{4.50}$$

It follows from (4.46) that

$$\begin{aligned}\|w_n - y_n\|^2 &= \|w_n - (a_n v_n + (1-a_n)w_n)\|^2 \\ &= a_n^2\|w_n - v_n\|^2 \end{aligned} \tag{4.51}$$

Thus, since $\|v_n - u_n\| \leq 2D(Tw_n, Ty_n)$ and T is a $L-$Lipschitzian mapping , from Lemma 3.1.2 and (4.51), we get that

$$\begin{aligned}\|y_n - u_n\|^2 &= \|a_n v_n + (1-a_n)w_n - u_n\|^2 \\ &= a_n\|v_n - u_n\|^2 + (1-a_n)\|w_n - u_n\|^2 \\ &\quad - a_n(1-a_n)\|w_n - v_n\|^2 \\ &\leq (1-a_n)\|w_n - u_n\|^2 + 4a_n D^2(Tw_n, Ty_n) \\ &\quad - a_n(1-a_n)\|w_n - v_n\|^2 \\ &\leq (1-a_n)\|w_n - u_n\|^2 + 4a_n L^2\|w_n - y_n\|^2 \\ &\quad - a_n(1-a_n)\|w_n - v_n\|^2 \\ &= (1-a_n)\|w_n - u_n\|^2 + 4a_n^3 L^2\|w_n - v_n\|^2 \\ &\quad - a_n(1-a_n)\|w_n - v_n\|^2 \\ &= (1-a_n)\|w_n - u_n\|^2 \\ &\quad + a_n(4L^2a_n^2 + a_n - 1)\|w_n - v_n\|^2. \end{aligned} \tag{4.52}$$

Hence, substituting (4.52) into (4.50), we get that

$$\begin{aligned}\|u_n - p\|^2 &\leq \|x_n - p\|^2 + a_n^2\|w_n - v_n\|^2 + (1-a_n)\|w_n - u_n\|^2 \\ &\quad + a_n(4L^2a_n^2 + a_n - 1)\|w_n - v_n\|^2 \\ &= \|x_n - p\|^2 + (1-a_n)\|w_n - u_n\|^2 \\ &\quad + a_n(4L^2a_n^2 + 2a_n - 1)\|w_n - v_n\|^2. \end{aligned} \tag{4.53}$$

Then, from Lemma 3.1.2, (4.48), (4.46) and (4.53), we obtain that

$$\begin{aligned}\|x_{n+1} - p\|^2 &= \|b_n u + c_n u_n + e_n w_n - p\|^2 \\ &\leq b_n\|u - p\|^2 + c_n\|u_n - p\|^2 + e_n\|w_n - p\|^2 \\ &\quad - c_n e_n\|w_n - u_n\|^2 \end{aligned}$$

$$\begin{aligned}
&\leq b_n\|u-p\|^2 + c_n\Big(\|x_n-p\|^2 + (1-a_n)\|w_n-u_n\|^2 \\
&\quad + a_n(4L^2a_n^2+2a_n-1)\|w_n-v_n\|^2\Big) \\
&\quad + e_n\|w_n-p\|^2 - c_ne_n\|w_n-u_n\|^2 \\
&\leq b_n\|u-p\|^2 + c_n\|x_n-p\|^2 + e_n\|x_n-p\|^2 \\
&\quad + c_n(1-e_n-a_n)\|w_n-u_n\|^2 \\
&\quad - c_na_n(1-4L^2a_n^2-2a_n)\|w_n-v_n\|^2 \\
&= b_n\|u-p\|^2 + (1-b_n)\|x_n-p\|^2 + c_n(1-e_n-a_n) \\
&\quad \times \|w_n-u_n\|^2 - c_na_n(1-4L^2a_n^2-2a_n)\|w_n-v_n\|^2 \\
&\leq b_n\|u-p\|^2 + (1-b_n)\|x_n-p\|^2 \\
&\quad - c_na_n(1-4L^2a_n^2-2a_n)\|w_n-v_n\|^2 \\
&\quad + c_n(b_n+c_n-a_n)\|w_n-u_n\|^2. \qquad (4.54)
\end{aligned}$$

Moreover, from the hypothesis, we have that

$$\begin{aligned}
1-4L^2a_n^2-2a_n &\geq 1-4L^2d^2-2d>0 \text{ and} \\
& b_n+c_n-a_n \leq 0, \qquad (4.55)
\end{aligned}$$

for all $n \geq 1$. Thus, from (4.54) and (4.55), we conclude that

$$\begin{aligned}
\|x_{n+1}-p\|^2 &\leq b_n\|u-p\|^2 + (1-b_n)\|x_n-p\|^2 \\
&\leq \max\{\|u-p\|^2, \|x_n-p\|^2\}.
\end{aligned}$$

Hence, by induction, the sequence $\{x_n\}$ is bounded. This completes the proof. □

Theorem 4.3.2. *Let C be a nonempty, closed and convex subset of a real Hilbert space H. Let $A_m : C \longrightarrow H$ and $F_m : C \times C \longrightarrow \mathbb{R}$, for $m = 1, 2, ..., N$, be a finite family of continuous monotone mappings and bifunctions satisfying Assumption 1.2.42, respectively. Let $T : C \longrightarrow CB(C)$ be a Lipschitz hemicontractive-type mapping with Lipschitz constant L. Assume that $\Theta = \bigcap_{m=1}^{N} EP(F_m, A_m) \bigcap F(T)$*

is nonempty, closed and convex, $(I - T)$ *is demiclosed at zero and* $Tp = \{p\}$ *for all* $p \in \Theta$*. Let* $\{x_n\}$ *be a sequence given by (4.46), where* $\{r_{n,m}\} \subset (0, \infty)$ *such that* $\lim_{n\to\infty} r_{m,n} = r_m$ *for some* $0 < r_m < \infty$ *and for each* $m = 1, 2, \ldots, N$*, and let* $\{a_n\}$*,* $\{b_n\}$*,* $\{c_n\}$*,* $\{e_n\}$ *and* $\{d_{m,n}\}$ *satisfy the following conditions:*
i) $0 < a \le c_n, e_n \le b < 1$ *and* $0 < c \le d_{m,n} \le 1$*,* $\forall m \in \{1, 2, ..., N\}$*;*
ii) $\lim_{n\to\infty} b_n = 0$*,* $\sum_{n=1}^{\infty} b_n = \infty$*; iii)* $b_n + c_n \le a_n \le d < \frac{1}{\sqrt{1+4L^2}+1}$*. Then, the sequence* $\{x_n\}$ *converges strongly to* $q = P_\Theta(u)$*.*

Proof. Since Θ is nonempty, closed and convex subset of H, from Lemma 1.2.3, we see that P_Θ is well defined. Clearly, from Theorem 4.3.1 the sequence $\{x_n\}$, hence $\{z_{m,n}\}, \{w_n\}$ and $\{y_n\}$ are bounded. Now, let $q = P_\Theta(u)$. Then, $q \in \Theta$. Thus, using the fact that $T^{G_m}_{r_{m,n}}$ is firmly nonexpansive and $T^{G_m}_{r_{m,n}} q = q$, for each $m \in \{1, 2, ..., N\}$, and Lemma 3.1.1 (1), we find that

$$\begin{aligned}
\|z_{m,n} - q\|^2 &= \|T^{G_m}_{r_{m,n}} x_n - T^{G_m}_{r_{m,n}} q\|^2 \\
&\le \langle z_{m,n} - q, x_n - q\rangle \\
&= \frac{1}{2}\Big(\|z_{m,n} - q\|^2 + \|x_n - q\|^2 - \|z_{m,n} - x_n\|^2\Big),
\end{aligned}$$

which implies that

$$\|z_{m,n} - q\|^2 \le \|x_n - q\|^2 - \|z_{m,n} - x_n\|^2.$$

This combined with Lemma 3.1.2 gives that

$$\begin{aligned}
||w_n - q||^2 &\le \sum_{m=1}^{N} d_{m,n}||z_{m,n} - q||^2 \\
&\le ||x_n - q||^2 - \sum_{m=1}^{N} d_{m,n}||z_{m,n} - x_n||^2. \qquad (4.56)
\end{aligned}$$

On the other hand, from (4.46), Lemma 3.1.2 and Lemma 3.1.1 (2), we have

$$\|x_{n+1} - q\|^2 = \|b_n u + c_n u_n + e_n w_n - q\|^2$$

$$
\begin{aligned}
&\le \|c_n(u_n - q) + e_n(w_n - q)\|^2 + 2b_n\langle u - q, x_{n+1} - q\rangle \\
&\le c_n\|u_n - q\|^2 + e_n\|w_n - q\|^2 - c_n e_n\|w_n - u_n\|^2 \\
&\quad + 2b_n\langle u - q, x_{n+1} - q\rangle. \qquad (4.57)
\end{aligned}
$$

Thus, since $q \in \Theta$, from (4.53), (4.56) and (4.57), we obtain that

$$
\begin{aligned}
\|x_{n+1} - q\|^2 &\le c_n\Big(\|x_n - q\|^2 + a_n(4L^2a_n^2 + 2a_n - 1)\|w_n - v_n\|^2 \\
&\quad + (1 - a_n)\|w_n - u_n\|^2\Big) + e_n\big(\|x_n - q\|^2 \\
&\quad - \sum_{m=1}^{N} d_{m,n}\|z_{m,n} - x_n\|^2\big) - c_n e_n\|w_n - u_n\|^2 \\
&\quad + 2b_n\langle u - q, x_{n+1} - q\rangle \\
&= (1 - b_n)\|x_n - q\|^2 - c_n a_n\big(1 - 4L^2a_n^2 - 2a_n\big) \\
&\quad \times \|w_n - v_n\|^2 + c_n(b_n + c_n - a_n)\|w_n - u_n\|^2 \\
&\quad - e_n\sum_{m=1}^{N} d_{m,n}\|z_{m,n} - x_n\|^2 \\
&\quad + 2b_n\langle u - q, x_{n+1} - q\rangle. \qquad (4.58)
\end{aligned}
$$

Now, we consider the following two cases:

Case 1. Suppose that there exists $n_0 \in \mathbb{N}$ such that $\{\|x_n - q\|\}$ is decreasing sequence for all $n \ge n_0$. Then, the boundedness of $\{\|x_n - q\|\}$ implies that $\{\|x_n - q\|\}$ is convergent. From (4.55) and (4.58), it follows that

$$
\begin{aligned}
c_n a_n\left(1 - 4L^2a_n^2 - 2a_n\right)\|w_n - v_n\|^2 &\le (1 - b_n)\|x_n - q\|^2 \\
&\quad - \|x_{n+1} - q\|^2 \\
&\quad + 2b_n\langle u - q, x_{n+1} - q\rangle.
\end{aligned}
$$

Thus, from (4.55), the assumptions of $\{c_n\}$ and $\{a_n\}$, and the fact that $b_n \to 0$ as $n \to \infty$, we obtain that

$$
\|w_n - v_n\| \to 0 \text{ as } n \to \infty. \qquad (4.59)
$$

This implies that

$$\lim_{n\to\infty} d(w_n, Tw_n) = 0. \tag{4.60}$$

And from (4.55) and (4.58), we see that

$$\begin{aligned} e_n d_{m,n}\|z_{m,n} - x_n\|^2 \;&\le\; (1-b_n)\|x_n - q\|^2 - \|x_{n+1} - q\|^2 \\ &\quad + 2b_n\langle u - q, x_{n+1} - q\rangle. \end{aligned}$$

This together with the conditions (i) and (ii) implies that

$$\lim_{n\to\infty} \|z_{m,n} - x_n\| = 0. \tag{4.61}$$

In addition, from the $L-$Lipschitz condition of T, the assumption that $||v_n - u_n|| \le 2D(Tw_n, Ty_n)$, (4.51) and (4.59), we get that

$$\begin{aligned} \|w_n - u_n\| &\le \|w_n - v_n\| + \|v_n - u_n\| \\ &\le \|w_n - v_n\| + 2D(Tw_n, Ty_n) \\ &\le \|w_n - v_n\| + 2L\|w_n - y_n\| \\ &= \|w_n - v_n\| + 2La_n\|w_n - v_n\| \to 0 \text{ as } n \to \infty. \end{aligned} \tag{4.62}$$

Furthermore, from (4.46) and the triangle inequality, we get that

$$\begin{aligned} \|x_{n+1} - x_n\| &\le \|x_{n+1} - w_n\| + \|w_n - x_n\| \\ &= \|b_n(u - w_n) + c_n(u_n - w_n)\| + \|\sum_{m=1}^{N} d_{m,n} z_{m,n} - x_n\| \\ &\le b_n\|u - w_n\| + c_n\|u_n - w_n\| + \sum_{m=1}^{N} d_{m,n}\|z_{m,n} - x_n\|. \end{aligned}$$

Thus, since $b_n \to 0$ as $n \to \infty$, from (4.61), (4.62), we obtain that

$$\lim_{n\to\infty} ||x_{n+1} - x_n|| = 0. \tag{4.63}$$

It follows from (4.55) and (4.58) that

$$\begin{aligned} \|x_{n+1} - q\|^2 \;&\le\; (1-b_n)\|x_n - q\|^2 \\ &\quad + 2b_n\langle u - q, x_{n+1} - q\rangle. \end{aligned} \tag{4.64}$$

Next, we show that $\limsup_{n\to\infty}\langle u-q, x_{n+1}-q\rangle \leq 0$. Since $\{x_{n+1}\}$ is a bounded sequence in a real Hilbert space H, without loss of generality, we can choose a subsequence $\{x_{n_i+1}\}$ of $\{x_{n+1}\}$ and a point in H, say z, such that

$$x_{n_i+1} \rightharpoonup z \text{ and } \limsup_{n\to\infty}\langle u-q, x_{n+1}-q\rangle = \lim_{i\to\infty}\langle u-q, x_{n_i+1}-q\rangle.$$

Since, C is weakly closed, we have $z \in C$ and from (4.63) it follows that $x_{n_i} \rightharpoonup z$ as $i \to \infty$. Using (4.61), we see that $||w_{n_i} - x_{n_i}|| \to 0$ as $i \to \infty$ and so $w_{n_i} \rightharpoonup z$ as $i \to \infty$. Thus, demiclosedness of $(I-T)$ at zero and (4.60) imply that

$$z \in F(T).$$

On the other hand, from Lemma 3.1.11 and (4.61), we get that

$$\begin{aligned}
||x_{n_i} - T_{r_m}^{G_m} x_{n_i}|| &\leq ||x_{n_i} - T_{r_{m,n_i}}^{G_m} x_{n_i}|| + ||T_{r_{m,n_i}}^{G_m} x_{n_i} - T_{r_m}^{G_m} x_{n_i}|| \\
&\leq ||x_{n_i} - T_{r_{m,n_i}}^{G_m} x_{n_i}|| \\
&\quad + \frac{|r_m - r_{m,n_i}|}{r_m}\left(||T_{r_m}^{G_m} x_{n_i}|| + ||x_{n_i}||\right),
\end{aligned}$$

which implies that $||x_{n_i} - T_{r_m}^{G_m} x_{n_i}|| \to 0$ as $i \to \infty$, for each $m = 1, 2, \ldots, N$. Using the fact that $(I - T_{r_m}^{G_m})$ is demiclosed at zero for each $m = 1, 2, \ldots, N$, we obtain that $z = T_{r_m}^{G_m} z$, and thus $z \in EP(F_m, A_m)$ for all $m = 1, 2, ..., N$. That is,

$$z \in \bigcap_{m=1}^{N} EP(F_m, A_m).$$

Therefore, $z \in \Theta$. Hence, since $q = P_\Theta(u)$ and $x_{n_i} \rightharpoonup z$, from the property of a metric projection, Lemma 3.1.3 (i), we have that

$$\begin{aligned}
\limsup_{n\to\infty}\langle u-q, x_{n+1}-q\rangle &= \lim_{i\to\infty}\langle u-q, x_{n_i+1}-q\rangle \\
&= \langle u-q, z-q\rangle \leq 0. \qquad (4.65)
\end{aligned}$$

Thus, from (4.64), (4.65), condition (ii) and Lemma 3.1.5, we conclude that

$$\|x_n - q\| \to 0 \text{ as } n \to \infty.$$

That is, $x_n \to q = P_\Theta(u)$.

Case 2. Suppose that there exists a subsequence $\{n_j\}$ of $\{n\}$ such that

$$\|x_{n_j} - q\| < \|x_{n_j+1} - q\|,$$

for all $j \in \mathbb{N}$. Then, by Lemma 3.1.6, there exist a nondecreasing sequence $\{n_k\} \subset \mathbb{N}$ such that $n_k \to \infty$, and

$$\|x_{n_k} - q\| \leq \|x_{n_k+1} - q\| \text{ and } \|x_k - q\| \leq \|x_{n_k+1} - q\|, \quad (4.66)$$

for all $k \in \mathbb{N}$. Thus, replacing n by n_k, and using (4.66), (4.58), (4.55) and the fact that $b_n \to 0$, we find that

$$||w_{n_k} - v_{n_k}|| \to 0, ||z_{m,n_k} - x_{n_k}|| \to 0 \text{ as } k \to \infty \ .$$

Hence, since $q = P_\Theta(u)$, following the same technique as in Case 1, we obtain that

$$\limsup_{k\to\infty} \langle u - q, x_{n_k+1} - q\rangle \leq 0. \quad (4.67)$$

Now, from (4.64), we get that

$$\|x_{n_k+1} - q\|^2 \leq (1 - b_{n_k})\|x_{n_k} - q\|^2 + 2b_{n_k}\langle u - q, x_{n_k+1} - q\rangle, (4.68)$$

and hence (4.66) and (4.68) imply that

$$\begin{aligned} b_{n_k}\|x_{n_k} - q\|^2 &\leq \|x_{n_k} - q\|^2 - \|x_{n_k+1} - q\|^2 \\ &\quad + 2b_{n_k}\langle u - q, x_{n_k+1} - q\rangle \\ &\leq 2b_{n_k}\langle u - q, x_{n_k+1} - q\rangle. \end{aligned}$$

Then, from the fact that $b_{n_k} > 0$, we have that

$$\|x_{n_k} - q\|^2 \leq 2\langle u - q, x_{n_k+1} - q\rangle.$$

It follows from (4.67) that

$$\|x_{n_k} - q\| \to 0 \text{ as } k \to \infty.$$

This together with (4.68) and the fact that $b_{n_k} \to 0$ imply that $\|x_{n_k+1} - q\| \to 0$ as $k \to \infty$. Since $\|x_k - q\| \leq \|x_{n_k+1} - q\|$, for all $k \in \mathbb{N}$, we obtain that $x_k \to q$ as $k \to \infty$. Therefore, we conclude that the sequence $\{x_n\}$ generated by (4.46) converges strongly to the point $q = P_\Theta(u)$. This completes the proof. $\square$

If, in Theorem 4.3.2, we assume that $A_m \equiv 0$, for all $m = 1, 2, ..., N$, then $EP(F_m, A_m) = EP(F_m)$ for each $m = 1, 2, ..., N$. Thus, we obtain the following corollary on a finite family of equilibrium problems and fixed point problem for a Lipschitz hemicontractive-type mapping:

Corollary 4.3.3. *Let C be a nonempty, closed and convex subset of a real Hilbert space H. Let $F_m : C \times C \longrightarrow \mathbb{R}$, for $m = 1, 2, ..., N$, be a finite family of bifunctions satisfying Assumption 1.2.42. Let $T : C \longrightarrow CB(C)$ be a Lipschitz hemicontractive-type mapping with Lipschitz constant L. Assume that $\Theta = \bigcap_{m=1}^{N} EP(F_m) \bigcap F(T)$ is nonempty, closed and convex, $(I - T)$ is demiclosed at zero and $Tp = \{p\}$ for all $p \in \Theta$. Let $\{r_{m,n}\} \subset (0, \infty)$ such that $\lim_{n\to\infty} r_{m,n} = r_m$ for some $0 < r_m < \infty$ and for each $m = 1, 2, \ldots, N$, and let $\{a_n\}$, $\{b_n\}$, $\{c_n\}$, $\{e_n\}$ and $\{d_{m,n}\}$ be real number sequences in $(0, 1)$. Let $\{x_n\}$ be a sequence generated from an arbitrary $x_1, u \in C$ by*

$$\begin{cases} F_m(z_{m,n}, y) + \frac{1}{r_{m,n}}\langle y - z_{m,n}, z_{m,n} - x_n\rangle \geq 0, \\ m = 1, 2, \ldots, N, \ \forall\, y \in C, \\ w_n = \displaystyle\sum_{m=1}^{N} d_{m,n} z_{m,n}, \\ y_n = a_n v_n + (1 - a_n) w_n, \\ x_{n+1} = b_n u + c_n u_n + e_n w_n, \end{cases}$$

for all $n \geq 1$, where $v_n \in Tw_n$, $u_n \in Ty_n$ such that $\|v_n - u_n\| \leq 2D(Tw_n, Ty_n)$, $\{a_n\}$, $\{b_n\}$, $\{c_n\}$, $\{e_n\}$ and $\{d_{m,n}\}$ satisfy the follow-

ing conditions:
i) $b_n + c_n + e_n = 1$ *and* $0 < a \leq c_n, e_n \leq b < 1$*; ii)* $\lim_{n\to\infty} b_n = 0$, $\sum_{n=1}^{\infty} b_n = \infty$*; iii)* $0 < c \leq d_{n,m} \leq 1$ *and* $\sum_{m=1}^{N} d_{n,m} = 1$*;*
iv) $b_n + c_n \leq a_n \leq d < \frac{1}{\sqrt{1+4L^2}+1}$*. Then, the sequence* $\{x_n\}$ *converges strongly to* $q = P_\Theta(u)$*.*

If, in Theorem 4.3.2, we assume that $F_m \equiv 0$, for all $m = 1, 2, ..., N$, then $EP(F_m, A_m) = EP(C, A_m)$ for each $m = 1, 2, ..., N$. Hence, we get the following corollary:

Corollary 4.3.4. *Let* C *be a nonempty, closed and convex subset of a real Hilbert space* H*. Let* $A_m : C \longrightarrow H$*, for* $m = 1, 2, ..., N$*, be a finite family of continuous monotone mappings. Let* $T : C \longrightarrow CB(C)$ *be a Lipschitz hemicontractive-type mapping with Lipschitz constant* L*. Assume that* $\Theta = \bigcap_{m=1}^{N} VI(C, A_m) \bigcap F(T)$ *is nonempty, closed and convex,* $(I - T)$ *is demiclosed at zero and* $Tp = \{p\}$ *for all* $p \in \Theta$*. Let* $\{r_{m,n}\} \subset (0, \infty)$ *such that* $\lim_{n\to\infty} r_{m,n} = r_m$ *for some* $0 < r_m < \infty$ *and for each* $m = 1, 2, \ldots, N$*, and let* $\{a_n\}$*,* $\{b_n\}$*,* $\{c_n\}$*,* $\{e_n\}$ *and* $\{d_{n,m}\}$ *be real number sequences in* $(0, 1)$*. Let* $\{x_n\}$ *be a sequence generated from an arbitrary* $x_1, u \in C$ *by*

$$\begin{cases} \langle A_m z_{m,n}, y - z_{m,n}\rangle + \frac{1}{r_{m,n}}\langle y - z_{m,n}, z_{m,n} - x_n\rangle \geq 0, \\ m = 1, 2, \ldots, N, \ \forall\, y \in C, \\ w_n = \sum_{m=1}^{N} d_{m,n} z_{m,n}, \\ y_n = a_n v_n + (1 - a_n) w_n, \\ x_{n+1} = b_n u + c_n u_n + e_n w_n, \end{cases}$$

for all $n \geq 1$*, where* $v_n \in Tw_n$*,* $u_n \in Ty_n$ *such that* $\|v_n - u_n\| \leq 2D(Tw_n, Ty_n)$*,* $\{a_n\}$*,* $\{b_n\}$*,* $\{c_n\}$*,* $\{e_n\}$ *and* $\{d_{m,n}\}$ *satisfy the following conditions:*

i) $b_n + c_n + e_n = 1$ and $0 < a \leq c_n, e_n \leq b < 1$; ii) $\lim_{n\to\infty} b_n = 0$, $\sum_{n=1}^{\infty} b_n = \infty$; iii) $0 < c \leq d_{n,m} \leq 1$ and $\sum_{m=1}^{N} d_{n,m} = 1$;
iv) $b_n + c_n \leq a_n \leq d < \frac{1}{\sqrt{1+4L^2}+1}$. Then, the sequence $\{x_n\}$ converges strongly to $q = P_\Theta(u)$.

If, in Theorem 4.3.2, we assume that T is a single-valued hemicontractive mapping from C into itself, then we obtain the following corollary:

Corollary 4.3.5. *Let C be a nonempty, closed and convex subset of a real Hilbert space H. Let $T : C \longrightarrow C$ be a Lipschitz hemicontractive mapping with Lipschitz constant L. Let $A_m : C \longrightarrow H$ and $F_m : C \times C \longrightarrow \mathbb{R}$, for $m = 1, 2, ..., N$, be a finite family of continuous monotone mappings and bifunctions satisfying Assumption 1.2.42, respectively. Assume that $\Theta = \bigcap_{m=1}^{N} EP(F_m, A_m) \bigcap F(T)$ is nonempty, closed and convex, and $(I - T)$ is demiclosed at zero. Let $\{r_{m,n}\} \subset (0, \infty)$ such that $\lim_{n\to\infty} r_{m,n} = r_m$ for some $0 < r_m < \infty$ and for each $m = 1, 2, \ldots, N$, and let $\{a_n\}$, $\{b_n\}$, $\{c_n\}$, $\{e_n\}$ and $\{d_{m,n}\}$ be real number sequences in $(0, 1)$. Let $\{x_n\}$ be a sequence generated from an arbitrary $x_1, u \in C$ by*

$$\begin{cases} F_m(z_{m,n}, y) + \langle A_m z_{m,n}, y - z_{m,n}\rangle \\ +\frac{1}{r_{m,n}}\langle y - z_{m,n}, z_{m,n} - x_n\rangle \geq 0,\ m = 1, 2, \ldots, N,\ \forall\, y \in C, \\ w_n = \sum_{m=1}^{N} d_{m,n} z_{m,n}, \\ y_n = a_n T w_n + (1 - a_n) w_n, \\ x_{n+1} = b_n u + c_n T y_n + e_n w_n, \end{cases}$$

for all $n \geq 1$, where $\{a_n\}$, $\{b_n\}$, $\{c_n\}$, $\{e_n\}$ and $\{d_{m,n}\}$ satisfy the following conditions: i) $b_n + c_n + e_n = 1$ and $0 < a \leq c_n, e_n \leq b < 1$; ii) $\lim_{n\to\infty} b_n = 0$, $\sum_{n=1}^{\infty} b_n = \infty$; iii) $0 < c \leq d_{n,m} \leq 1$ and $\sum_{m=1}^{N} d_{n,m} = 1$;

iv) $b_n + c_n \leq a_n \leq d < \frac{1}{\sqrt{1+L^2}+1}$. *Then, the sequence* $\{x_n\}$ *converges strongly to* $q = P_\Theta(u)$.

If, in Corollary 4.3.5, we assume that $T = I$, then we obtain the following corollary:

Corollary 4.3.6. *Let* C *be a nonempty, closed and convex subset of a real Hilbert space* H. *Let* $A_m : C \longrightarrow H$ *and* $F_m : C \times C \longrightarrow \mathbb{R}$, *for* $m = 1, 2, ..., N$, *be a finite family of continuous monotone mappings and bifunctions satisfying Assumption 1.2.42, respectively. Assume that* $\Theta = \bigcap_{m=1}^{N} EP(F_m, A_m)$ *is nonempty. Let* $\{r_{m,n}\} \subset (0, \infty)$ *such that* $\lim_{n\to\infty} r_{m,n} = r_m$ *for some* $0 < r_m < \infty$ *and for each* $m = 1, 2, \ldots, N$, *and let* $\{b_n\}$ *and* $\{d_{m,n}\}$ *be real number sequences in* $(0, 1)$. *Let* $\{x_n\}$ *be a sequence generated from an arbitrary* $x_1, u \in C$ *by*

$$\begin{cases} F_m(z_{m,n}, y) + \langle A_m z_{m,n}, y - z_{m,n}\rangle \\ +\frac{1}{r_{m,n}}\langle y - z_{m,n}, z_{m,n} - x_n\rangle \geq 0,\ m = 1, 2, \ldots, N,\ \forall\, y \in C, \\ w_n = \displaystyle\sum_{m=1}^{N} d_{m,n} z_{m,n}, \\ x_{n+1} = b_n u + (1 - b_n) w_n, \end{cases}$$

for all $n \geq 1$, *where* $\{b_n\}$ *and* $\{d_{m,n}\}$ *satisfy the following conditions: i)* $\lim_{n\to\infty} b_n = 0$, $\sum_{n=1}^{\infty} b_n = \infty$; *ii)* $0 < c \leq d_{n,m} \leq 1$ *and* $\sum_{m=1}^{N} d_{n,m} = 1$. *Then, the sequence* $\{x_n\}$ *converges strongly to* $q = P_\Theta(u)$.

If, in Corollary 4.3.6, we assume that $F_m \equiv 0$, for all $m = 1, 2, ..., N$, then we obtain the following corollary:

Corollary 4.3.7. *Let* C *be a nonempty, closed and convex subset of a real Hilbert space* H. *Let* $A_m : C \longrightarrow H$, *for* $m = 1, 2, ..., N$, *be a finite family of continuous monotone mappings. Assume that*

$\Theta = \bigcap_{m=1}^{N} VI(C, A_m)$ *is nonempty. Let* $\{r_{m,n}\} \subset (0, \infty)$ *such that* $\lim_{n\to\infty} r_{m,n} = r_m$ *for some* $0 < r_m < \infty$ *and for each* $m = 1, 2, \dots, N$, *and let* $\{b_n\}$ *and* $\{d_{m,n}\}$ *be real number sequences in* $(0, 1)$*. Let* $\{x_n\}$ *be a sequence generated from an arbitrary* $x_1, u \in C$ *by*

$$\begin{cases} \langle A_m z_{m,n}, y - z_{m,n}\rangle + \frac{1}{r_{m,n}}\langle y - z_{m,n}, z_{m,n} - x_n\rangle \geq 0, \\ m = 1, 2, \dots, N, \quad \forall\, y \in C, \\ w_n = \displaystyle\sum_{m=1}^{N} d_{m,n} z_{m,n}, \\ x_{n+1} = b_n u + (1 - b_n) w_n, \end{cases}$$

for all $n \geq 1$*, where* $\{b_n\}$ *and* $\{d_{m,n}\}$ *satisfy the following conditions: i)* $\lim_{n\to\infty} b_n = 0$, $\sum_{n=1}^{\infty} b_n = \infty$*; ii)* $0 < c \leq d_{n,m} \leq 1$ *and* $\sum_{m=1}^{N} d_{n,m} = 1$*. Then, the sequence* $\{x_n\}$ *converges strongly to* $q = P_\Theta(u)$*.*

If, in Corollary 4.3.6, we assume that $A_m \equiv 0$, for all $m = 1, 2, ..., N$, then we obtain the following corollary on a finite family of equilibrium problems:

Corollary 4.3.8. *Let* C *be a nonempty, closed and convex subset of a real Hilbert space* H*. Let* $F_m : C \times C \longrightarrow \mathbb{R}$*, for* $m = 1, 2, ..., N$*, be a finite family of bifunctions satisfying Assumption 1.2.42. Assume that* $\Theta = \bigcap_{m=1}^{N} EP(F_m)$ *is nonempty. Let* $\{r_{m,n}\} \subset (0, \infty)$ *such that* $\lim_{n\to\infty} r_{m,n} = r_m$ *for some* $0 < r_m < \infty$ *and for each* $m = 1, 2, \dots, N$, *and let* $\{b_n\}$ *and* $\{d_{m,n}\}$ *be real number sequences in* $(0, 1)$*. Let* $\{x_n\}$ *be a sequence generated from an arbitrary* $x_1, u \in C$ *by*

$$\begin{cases} F_m(z_{m,n}, y) + \frac{1}{r_{m,n}}\langle y - z_{m,n}, z_{m,n} - x_n\rangle \geq 0, \\ m = 1, 2, \dots, N, \quad \forall\, y \in C, \\ w_n = \displaystyle\sum_{m=1}^{N} d_{m,n} z_{m,n}, \\ x_{n+1} = b_n u + (1 - b_n) w_n, \end{cases}$$

for all $n \geq 1$, *where* $\{b_n\}$ *and* $\{d_{m,n}\}$ *satisfy the following conditions: i)* $\lim_{n\to\infty} b_n = 0$, $\sum_{n=1}^{\infty} b_n = \infty$; *ii)* $0 < c \leq d_{n,m} \leq 1$ *and* $\sum_{m=1}^{N} d_{n,m} = 1$. *Then, the sequence* $\{x_n\}$ *converges strongly to* $q = P_{\Theta}(u)$.

Remark 4.3.9. *We note that, since every quasi-nonexpansive and demicontractive mappings are hemicontractive, the results obtained in this section for hemicontractive (single and multi-valued) mapping also hold for quasi-nonexpansive and demicontractive mappings provided that the indicated conditions are satisfied.*

Remark 4.3.10. *Observe that, since every Lipschitz pseudocontractive multi-valued mapping* T *with* $F(T) \neq \emptyset$ *and* $T(p) = \{p\}, \forall\ p \in F(T)$ *is Lipschitz hemicontractive-type, the results given in this section also hold for Lipschitz pseudocontractive multi-valued mapping* T *with* $F(T) \neq \emptyset$ *and* $T(p) = \{p\}, \forall\ p \in F(T)$. *Hence, it is also true for nonexpansive and* $k-$*strictly pseudocontractive multi-valued mappings provided that the specified assumptions are satisfied because every nonexpansive and* $k-$*strictly pseudocontractive mappings are Lipschitz pseudocontractive.*

4.4 Approximation of a Common Solution of Split Equilibrium, Variational Inequality and Fixed Point Problems

In this section, we give an iterative algorithm and prove strong convergence theorems for a split equilibrium problem, a variational inequality problem and a fixed point problem for a nonexpansive multi-valued mapping.

Let H_1 and H_2 be real Hilbert spaces. Let C and Q be nonempty, closed and convex subsets of H_1 and H_2, respectively.

Let $A : C \longrightarrow H_1$ be a monotone mapping and $B : H_1 \longrightarrow H_2$ be a bounded linear mapping. Let $F_1 : C\times C \longrightarrow \mathbb{R}$ and $F_2 : Q\times Q \longrightarrow \mathbb{R}$ be bifunctions satisfying Assumption 1.2.42. Let $f : C \longrightarrow C$ be a $\rho-$contraction mapping and $T : C \longrightarrow CB(C)$ be a nonexpansive multi-valued mapping. Let $s, r, \lambda > 0$ be fixed real numbers, $\{\gamma_n\}$ be a sequence of positive real numbers, and $\{a_n\}, \{b_n\} \subset (0, 1)$, for all $n \geq 0$. For arbitrary $x_0 \in C$ (using Lemmas 3.1.9 and 3.1.10) we find $z_0 \in C$ such that

$$z_0 = T_s^{F_1}(I - \lambda B^*(I - T_r^{F_2})B)x_0,$$

and since by Lemma 1.2.3, the metric projection P_C from H_1 onto C is nonempty and unique for every point in H_1, we find $u_0, y_0 \in C$ such that

$$\begin{cases} u_0 = P_C(z_0 - \gamma_0 A z_0), \\ y_0 = P_C(z_0 - \gamma_0 A u_0). \end{cases}$$

Choose $v_0 \in Ty_0$ and then we compute $x_1 \in C$ by

$$x_1 = a_0 f(x_0) + (1 - a_0)(b_0 x_0 + (1 - b_0)v_0).$$

Then, we can find $z_1, u_1, y_1 \in C$ such that

$$\begin{cases} z_1 = T_s^{F_1}(I - \lambda B^*(I - T_r^{F_2})B)x_1, \\ u_1 = P_C(z_1 - \gamma_1 A z_1), \\ y_1 = P_C(z_1 - \gamma_1 A u_1). \end{cases}$$

Choose $v_1 \in Ty_1$ and then we compute $x_2 \in C$ by

$$x_2 = a_1 f(x_1) + (1 - a_1)(b_1 x_1 + (1 - b_1)v_1).$$

Continuing in this way, we construct inductively a sequence $\{x_n\} \subset C$ by

$$\begin{cases} z_n = T_s^{F_1}(I - \lambda B^*(I - T_r^{F_2})B)x_n, \\ u_n = P_C[z_n - \gamma_n A z_n], \\ y_n = P_C[z_n - \gamma_n A u_n], \\ x_{n+1} = a_n f(x_n) + (1 - a_n)(b_n x_n + (1 - b_n)v_n), \ \forall\, n \geq 0, \end{cases} \tag{4.69}$$

where $v_n \in Ty_n$, P_C is a metric projection from H_1 onto C, and $T_s^{F_1}$ and $T_r^{F_2}$ are defined as in Lemma 3.1.10.

In this section, we shall need the following lemma.

Lemma 4.4.1. *Let C be a nonempty, closed and convex subset of a real Hilbert space H and let $CBC(C)$ denote the collection of all nonempty, closed, bounded and convex subsets of C. Let $T : C \longrightarrow CBC(C)$ be a nonexpansive multi-valued mapping. Then, $(I - T)$ is demiclosed at zero.*

Proof. Let $\{x_n\} \subset C$ be such that $x_n \rightharpoonup x$ and suppose $d(x_n, Tx_n) \to 0$ as $n \to \infty$. We want to show that $0 \in (I - T)x$, that is, $x \in Tx$. Now, since Tx_n is closed and convex subset of H, then by Lemma 1.2.3, there exists an element $y_n \in Tx_n$ such that

$$||x_n - y_n|| = d(x_n, Tx_n).$$

Thus, by assumption, we get that

$$||x_n - y_n|| \to 0 \text{ as } n \to \infty. \tag{4.70}$$

Again since Tx is a closed and convex subset of H, there exists a point $p^* \in Tx$ such that

$$||y_n - p^*|| = d(y_n, Tx) \leq D(Tx_n, Tx). \tag{4.71}$$

Define a function $h : H \longrightarrow [0, \infty)$ by $h(y) := \limsup_{n\to\infty} ||x_n - y||$. Moreover, since

$$||x_n - y||^2 = \langle x_n - y, x_n - y\rangle = ||x_n - x||^2 + 2\langle x_n - x, x - y\rangle + ||x - y||^2,$$

and $x_n \rightharpoonup x$, we obtain that

$$\begin{aligned} h(y) &= \limsup_{n\to\infty} ||x_n - y||^2 = \limsup_{n\to\infty} ||x_n - x||^2 + ||x - y||^2 \\ &= h(x) + ||x - y||^2, \ \forall\, y \in H. \end{aligned}$$

Hence, we have that

$$h(p^*) = h(x) + ||x - p^*||^2. \tag{4.72}$$

On the other way, from the Cauchy Schwartz inequality, we have

$$\begin{aligned}
||x_n - p^*||^2 &= ||x_n - y_n + y_n - p^*||^2 \\
&= \langle x_n - y_n + y_n - p^*, x_n - y_n + y_n - p^* \rangle \\
&= ||x_n - y_n||^2 + 2\langle x_n - y_n, y_n - p^* \rangle + ||y_n - p^*||^2 \\
&\leq ||x_n - y_n||^2 + 2||x_n - y_n||.||y_n - p^*|| + ||y_n - p^*||^2.
\end{aligned}$$

Thus, from (4.70), we get that

$$\limsup_{n\to\infty} ||x_n - p^*||^2 \leq \limsup_{n\to\infty} ||y_n - p^*||^2. \tag{4.73}$$

Furthermore, using (4.71), (4.73) and the fact that T is nonexpansive mapping, we obtain that

$$\begin{aligned}
h(p^*) &= \limsup_{n\to\infty} ||x_n - p^*||^2 \\
&\leq \limsup_{n\to\infty} ||y_n - p^*||^2 \\
&\leq \limsup_{n\to\infty} D^2(Tx_n, Tx) \\
&\leq \limsup_{n\to\infty} ||x_n - x||^2 \\
&= h(x).
\end{aligned} \tag{4.74}$$

Therefore, from (4.72) and (4.74), we infer that

$$||x - p^*||^2 \leq 0,$$

which implies that $x = p^* \in Tx$. Hence, we conclude that $(I - T)$ is demiclosed at zero. This completes the proof. $\square$

Now, we are in a position to state and prove our main theorem.

Theorem 4.4.2. *Let H_1 and H_2 be real Hilbert spaces. Let C and Q be nonempty, closed and convex subsets of H_1 and H_2, respectively. Let $A : C \longrightarrow H_1$ be a $L-$Lipschitz monotone mapping and $B : H_1 \longrightarrow H_2$ be a bounded linear mapping. Let $F_1 : C \times C \longrightarrow \mathbb{R}$ and $F_2 : Q \times Q \longrightarrow \mathbb{R}$ be bifunctions satisfying Assumption 1.2.42. Let $f : C \longrightarrow C$ be a $\rho-$contraction mapping with $0 \leq \rho < \frac{1}{2}$ and $T : C \longrightarrow CBC(C)$ be a nonexpansive multi-valued mapping. Assume that $Tp = \{p\}$, for all $p \in F(T)$ and $\mathcal{F} = F(T) \cap \Omega \cap VI(C, A)$ is nonempty. Let $\{x_n\}$ be a sequence generated by (4.69), where $\{\gamma_n\} \subset [a, b]$ for some $a, b \in (0, \frac{1}{L})$ and $\lambda \in (0, \frac{1}{d})$, for $d = \|B^*B\|$, where B^* is the adjoint of B, and $\{b_n\} \subset [e, f]$ for some $e, f \in (0, 1)$ and $\{a_n\} \subset (0, 1)$ satisfying $\lim_{n\to\infty} a_n = 0$ and $\sum_{n=0}^{\infty} a_n = \infty$. Then, the sequence $\{x_n\}$ converges strongly to the point $q = P_{\mathcal{F}} f(q)$.*

Proof. First, we show that $B^*(I - T_r^{F_2})B$ is a $\frac{1}{2d}-$inverse strongly monotone mapping. Since Lemma 3.1.10 implies that $T_r^{F_2}$ is nonexpansive mapping, then by Lemma 1.2.21, we have $I - T_r^{F_2}$ is $\frac{1}{2}-$inverse strongly monotone mapping. Thus, for all $x, y \in H_1$, from the Cauchy Schwartz inequality, we obtain that

$$\begin{aligned}
&\|B^*(I - T_r^{F_2})Bx - B^*(I - T_r^{F_2})By\|^2 \\
&\quad = \big\langle B^*((I - T_r^{F_2})Bx - (I - T_r^{F_2})By), B^*((I - T_r^{F_2})Bx \\
&\qquad -(I - T_r^{F_2})By)\big\rangle \\
&\quad = \big\langle (I - T_r^{F_2})Bx - (I - T_r^{F_2})By, BB^*((I - T_r^{F_2})Bx \\
&\qquad -(I - T_r^{F_2})By)\big\rangle \\
&\quad \leq d\|(I - T_r^{F_2})Bx - (I - T_r^{F_2})By\|^2 \\
&\quad \leq 2d\langle Bx - By, (I - T_r^{F_2})Bx - (I - T_r^{F_2})By\rangle \\
&\quad = 2d\langle x - y, B^*(I - T_r^{F_2})Bx - B^*(I - T_r^{F_2})By\rangle,
\end{aligned}$$

which implies that $B^*(I - T_r^{F_2})B$ is a $\frac{1}{2d}-$inverse strongly monotone mapping. Thus, from Lemma 3.1.4 and the assumption that $\lambda \in$

$(0, \frac{1}{d})$, we get $I - \lambda B^*(I - T_r^{F_2})B$ is nonexpansive mapping and hence from the nonexpansiveness of $T_s^{F_1}$, we obtain

$$\begin{aligned}&\|T_s^{F_1}(I - \lambda B^*(I - T_r^{F_2})B)x - T_s^{F_1}(I - \lambda B^*(I - T_r^{F_2})B)y\|\\&\leq \|x - y\|. \qquad (4.75)\end{aligned}$$

Since from Lemmas 3.1.12, 3.1.10 and 1.2.40, we get $\mathcal{F}$ is a closed and convex subset of C and f is $\rho-$contraction, we have $P_{\mathcal{F}}f$ is a mapping from C into C and

$$\begin{aligned}\|P_{\mathcal{F}}f(x) - P_{\mathcal{F}}f(y)\| &\leq \|f(x) - f(y)\|\\&\leq \rho\|x - y\|,\end{aligned}$$

for all $x, y \in C$. Hence, $P_{\mathcal{F}}f$ is $\rho-$contraction mapping. Therefore, by Banach Contraction Mapping Principle, there exists a unique point $q \in \mathcal{F} \subset C$ such that $q = P_{\mathcal{F}}f(q)$.

Next, we show that $\{x_n\}$ is bounded. Let $p \in \mathcal{F}$. Then, we have $p \in \Omega$ and hence by Lemma 3.1.10, we get $p = T_s^{F_1}p$ and $Bp = T_r^{F_2}Bp$, which implies that $T_s^{F_1}(I - \lambda B^*(I - T_r^{F_2})B)p = p$. Thus, using (4.69) and (4.75), we get that

$$\begin{aligned}\|z_n - p\| &= \|T_s^{F_1}(I - \lambda B^*(I - T_r^{F_2})B)x_n - p\|\\&\leq \|x_n - p\|. \qquad (4.76)\end{aligned}$$

From Lemma 3.1.3 (ii) and (4.69), we have that

$$\begin{aligned}||y_n - p||^2 &= ||P_C(z_n - \gamma_n Au_n) - p||^2\\&\leq ||z_n - \gamma_n Au_n - p||^2 - ||z_n - \gamma_n Au_n - y_n||^2\\&= \langle z_n - \gamma_n Au_n - p, z_n - \gamma_n Au_n - p\rangle\\&\quad -\langle z_n - \gamma_n Au_n - y_n, z_n - \gamma_n Au_n - y_n\rangle\\&= ||z_n - p||^2 - ||z_n - y_n||^2 + 2\gamma_n\langle Au_n, p - y_n\rangle\\&= ||z_n - p||^2 - ||z_n - y_n||^2 + 2\gamma_n\big(\langle Au_n - Ap, p - u_n\rangle\\&\quad +\langle Ap, p - u_n\rangle + \langle Au_n, u_n - y_n\rangle\big)\end{aligned}$$

$$\begin{aligned}
&\leq ||z_n - p||^2 - ||z_n - y_n||^2 + 2\gamma_n\langle Au_n, u_n - y_n\rangle \\
&= ||z_n - p||^2 - ||z_n - u_n||^2 - 2\langle z_n - u_n, u_n - y_n\rangle \\
&\quad -||u_n - y_n||^2 + 2\gamma_n\langle Au_n, u_n - y_n\rangle \\
&= ||z_n - p||^2 - ||z_n - u_n||^2 - ||u_n - y_n||^2 \\
&\quad +2\langle z_n - \gamma_n Au_n - u_n, y_n - u_n\rangle, \qquad (4.77)
\end{aligned}$$

and since A is $L-$Lipschitz mapping, from Lemma 3.1.3 (i) and (4.69), we obtain

$$\begin{aligned}
\langle z_n - \gamma_n Au_n - u_n, y_n - u_n\rangle &= \langle z_n - \gamma_n Az_n - u_n, y_n - u_n\rangle \\
&\quad +\langle \gamma_n Az_n - \gamma_n Au_n, y_n - u_n\rangle \\
&\leq \langle \gamma_n Az_n - \gamma_n Au_n, y_n - u_n\rangle \\
&\leq \gamma_n L||z_n - u_n||.||y_n - u_n||. \qquad (4.78)
\end{aligned}$$

Thus, substituting (4.78) into (4.77), we get

$$\begin{aligned}
||y_n - p||^2 &\leq ||z_n - p||^2 - ||z_n - u_n||^2 - ||u_n - y_n||^2 \\
&\quad +2\gamma_n L||z_n - u_n||.||y_n - u_n|| \\
&\leq ||z_n - p||^2 - ||z_n - u_n||^2 - ||u_n - y_n||^2 \\
&\quad +\gamma_n L(||z_n - u_n||^2 + ||u_n - y_n||^2) \\
&\leq ||z_n - p||^2 + (\gamma_n L - 1)||z_n - u_n||^2 \\
&\quad +(\gamma_n L - 1)||u_n - y_n||^2. \qquad (4.79)
\end{aligned}$$

Since T is nonexpansive and $\gamma_n L - 1 < 0$, it follows from (4.76), (4.79) and the assumption that $Tp = \{p\}$, for all $p \in F(T)$ that

$$\begin{aligned}
\|v_n - p\| &= d(v_n, Tp) \leq D(Ty_n, Tp) \\
&\leq \|y_n - p\| \\
&\leq ||z_n - p|| \\
&\leq ||x_n - p||. \qquad (4.80)
\end{aligned}$$

Thus, from (4.69) and (4.80), we obtain that

$$\|x_{n+1} - p\| = \|a_n f(x_n) + (1 - a_n)(b_n x_n + (1 - b_n)v_n) - p\|$$

$$
\begin{aligned}
&\leq a_n\|f(x_n)-p\|+(1-a_n)\big(b_n\|x_n-p\| \\
&\quad +(1-b_n)\|v_n-p\|\big) \\
&\leq a_n\|f(x_n)-f(p)\|+a_n\|f(p)-p\| \\
&\quad +(1-a_n)\left(b_n\|x_n-p\|+(1-b_n)\|x_n-p\|\right) \\
&\leq a_n\rho\|x_n-p\|+a_n\|f(p)-p\|+(1-a_n)\|x_n-p\| \\
&= (1-a_n(1-\rho))\|x_n-p\|+a_n(\frac{1-\rho}{1-\rho})\|f(p)-p\| \\
&\leq \max\{\|x_n-p\|,\frac{1}{1-\rho}\|f(p)-p\|\}.
\end{aligned}
$$

By induction, we have that

$$\|x_{n+1}-p\|\leq \max\{\|x_0-p\|,\frac{1}{1-\rho}\|f(p)-p\|\}.$$

Hence, $\{x_n\}$ is bounded and so are $\{f(x_n)\}$, $\{y_n\}$, $\{z_n\}$ and $\{v_n\}$.

On the other hand, since $T_s^{F_1}$ is nonexpansive, we have that
$\|z_n-p\|^2$

$$
\begin{aligned}
&= \|T_s^{F_1}(I-\lambda B^*(I-T_r^{F_2})B)x_n-T_s^{F_1}(I-\lambda B^*(I-T_r^{F_2})B)p\|^2 \\
&\leq \|(I-\lambda B^*(I-T_r^{F_2})B)x_n-(I-\lambda B^*(I-T_r^{F_2})B)p\|^2 \\
&= \|(x_n-p)-\lambda(B^*(I-T_r^{F_2})Bx_n-B^*(I-T_r^{F_2})Bp)\|^2 \\
&= \|x_n-p\|^2-2\lambda\langle x_n-p,B^*(I-T_r^{F_2})Bx_n-B^*(I-T_r^{F_2})Bp\rangle \\
&\quad +\lambda^2\|B^*(I-T_r^{F_2})Bx_n-B^*(I-T_r^{F_2})Bp\|^2.
\end{aligned}
$$

Hence, the $\frac{1}{2d}-$inverse strong monotonicity of $B^*(I-T_r^{F_2})B$ gives that

$$
\begin{aligned}
\|z_n-p\|^2 &\leq \|x_n-p\|^2-\frac{\lambda}{d}\|B^*(I-T_r^{F_2})Bx_n-B^*(I-T_r^{F_2})Bp\|^2 \\
&\quad +\lambda^2\|B^*(I-T_r^{F_2})Bx_n-B^*(I-T_r^{F_2})Bp\|^2 \\
&= \|x_n-p\|^2+\lambda(\lambda-\frac{1}{d})\|B^*(I-T_r^{F_2})Bx_n\|^2. \qquad (4.81)
\end{aligned}
$$

Now, let $q=P_{\mathcal{F}}f(q)$. Then, $q\in\mathcal{F}$. Consequently, using Lemma 3.1.2, (4.80), (4.79) and Lemma 3.1.1 (2), we obtain

$$||x_{n+1}-q||^2 = ||a_n(f(x_n)+(1-a_n)\big[b_nx_n+(1-b_n)v_n\big]-q||^2$$

$$
\begin{aligned}
&\leq (1-a_n)||b_n x_n + (1-b_n)v_n - q||^2 \\
&\quad +2a_n\langle f(x_n) - q, x_{n+1} - q\rangle \\
&\leq (1-a_n)b_n||x_n - q||^2 + (1-a_n)(1-b_n)||v_n - q||^2 \\
&\quad -(1-a_n)b_n(1-b_n)||v_n - x_n||^2 \\
&\quad +2a_n\langle f(x_n) - q, x_{n+1} - q\rangle \\
&\leq (1-a_n)b_n||x_n - q||^2 + (1-a_n)(1-b_n)||y_n - q||^2 \\
&\quad -(1-a_n)b_n(1-b_n)||v_n - x_n||^2 \\
&\quad +2a_n\langle f(x_n) - q, x_{n+1} - q\rangle \\
&\leq (1-a_n)b_n||x_n - q||^2 + (1-a_n)(1-b_n) \\
&\quad \Big[||z_n - q||^2 + (\gamma_n L - 1)||z_n - u_n||^2 + (\gamma_n L - 1) \\
&\quad \times||u_n - y_n||^2\Big] - (1-a_n)b_n(1-b_n)||v_n - x_n||^2 \\
&\quad +2a_n\langle f(x_n) - q, x_{n+1} - q\rangle. \qquad (4.82)
\end{aligned}
$$

Thus, substituting (4.81) into (4.82), we find that

$$
\begin{aligned}
||x_{n+1} - q||^2 &\leq (1-a_n)b_n||x_n - q||^2 + (1-a_n)(1-b_n) \\
&\quad \Big[\|x_n - q\|^2 + \lambda(\lambda - \frac{1}{d})\|B^*(I - T_r^{F_2})Bx_n\|^2\Big] \\
&\quad +(1-a_n)(1-b_n)(\gamma_n L - 1)||z_n - u_n||^2 \\
&\quad +(1-a_n)(1-b_n)(\gamma_n L - 1)||u_n - y_n||^2 \\
&\quad -(1-a_n)b_n(1-b_n)||v_n - x_n||^2 \\
&\quad +2a_n\langle f(x_n) - q, x_{n+1} - q\rangle \\
&= (1-a_n)||x_n - q||^2 + (1-a_n)(1-b_n)\lambda(\lambda - \frac{1}{d}) \\
&\quad \times\|B^*(I - T_r^{F_2})Bx_n\|^2 + (1-a_n)(1-b_n) \\
&\quad \times(\gamma_n L - 1)||z_n - u_n||^2 + (1-a_n)(1-b_n) \\
&\quad \times(\gamma_n L - 1)||u_n - y_n||^2 - (1-a_n)b_n(1-b_n) \\
&\quad \times||v_n - x_n||^2 + 2a_n\langle f(x_n) - q, x_{n+1} - q\rangle. \qquad (4.83)
\end{aligned}
$$

From (4.83) and the fact that $\lambda < \frac{1}{d}$ and $\gamma_n L < 1$, we get

$$||x_{n+1} - q||^2 \leq (1 - a_n)||x_n - q||^2 + 2a_n\langle f(x_n) - q, x_{n+1} - q\rangle. \quad (4.84)$$

But

$$\begin{aligned} \langle f(x_n) - q, x_{n+1} - q\rangle &= \langle f(x_n) - q, x_n - q\rangle \\ &\quad + \langle f(x_n) - q, x_{n+1} - x_n\rangle \\ &\leq \langle f(x_n) - f(q), x_n - q\rangle \\ &\quad + \langle f(q) - q, x_n - q\rangle \\ &\quad + ||x_{n+1} - x_n||.||f(x_n) - q|| \\ &\leq \rho||x_n - q||^2 + \langle f(q) - q, x_n - q\rangle \\ &\quad + ||x_{n+1} - x_n||.||f(x_n) - q||. \quad (4.85) \end{aligned}$$

Thus, substituting (4.85) in (4.84), we obtain that

$$\begin{aligned} ||x_{n+1} - q||^2 &\leq (1 - a_n(1 - 2\rho)||x_n - q||^2 + 2a_n\langle f(q) - q, x_n - q\rangle \\ &\quad + 2a_n||x_{n+1} - x_n||.||f(x_n) - q||. \quad (4.86) \end{aligned}$$

Next, we consider two cases.

Case 1. Suppose that there exists $n_0 \in \mathbb{N}$ such that $\{||x_n - q||\}$ is decreasing for all $n \geq n_0$. Then, we get that $\{||x_n - q||)\}$ is convergent. Thus, from (4.83), the fact that $\gamma_n \leq b < \frac{1}{L}$, for all $n \geq 0$, $\lambda < \frac{1}{d}$ and $a_n \to 0$ as $n \to \infty$, we have that

$$||z_n - u_n|| \to 0, \ ||u_n - y_n|| \to 0 \text{ and } ||v_n - x_n|| \to 0 \quad (4.87)$$

as $n \to \infty$. Moreover, since

$$x_{n+1} - x_n = a_n(f(x_n) - x_n) + (1 - a_n)(1 - b_n)(v_n - x_n),$$

from (4.87) and the fact that $a_n \to 0$ as $n \to \infty$, we get that

$$\lim_{n\to\infty} ||x_{n+1} - x_n|| = 0. \quad (4.88)$$

Again from (4.83), the fact that $\lambda < \frac{1}{d}$, $\gamma_n L < 1$ and $a_n \to 0$ as $n \to \infty$, we obtain that

$$||B^*(I - T_r^{F_2})Bx_n|| \to 0 \text{ as } n \to \infty \ , \tag{4.89}$$

which implies that

$$||x_n - (x_n - \lambda B^*(I - T_r^{F_2})Bx_n)|| \to 0 \text{ as } n \to \infty. \tag{4.90}$$

Since $T_s^{F_1}$ is firmly nonexpansive and $q \in \mathcal{F}$, from the nonexpansiveness of $(I - \lambda B^*(I - T_r^{F_2})B)$, (4.69) and Lemma 3.1.1 (1), we have

$$\begin{aligned}
\|z_n - q\|^2 &= \|T_s^{F_1}(I - \lambda B^*(I - T_r^{F_2})B)x_n - T_s^{F_1}q\|^2 \\
&\leq \langle z_n - q, (I - \lambda B^*(I - T_r^{F_2})B)x_n - q\rangle \\
&= \frac{1}{2}\Big(\|z_n - q\|^2 + \|(I - \lambda B^*(I - T_r^{F_2})B)x_n - q\|^2 \\
&\quad - \|z_n - (I - \lambda B^*(I - T_r^{F_2})B)x_n\|^2\Big) \\
&\leq \frac{1}{2}\Big(\|z_n - q\|^2 + \|x_n - q\|^2 - \|z_n - x_n\|^2 \\
&\quad - 2\lambda\langle z_n - x_n, B^*(I - T_r^{F_2})Bx_n\rangle \\
&\quad -\lambda^2\|B^*(I - T_r^{F_2})Bx_n\|^2\Big),
\end{aligned}$$

which implies that

$$\begin{aligned}
\|z_n - q\|^2 \leq\ & \|x_n - q\|^2 - \|z_n - x_n\|^2 \\
& +2\lambda\langle x_n - z_n, B^*(I - T_r^{F_2})Bx_n\rangle.
\end{aligned} \tag{4.91}$$

Thus, from Lemma 3.1.2, (4.80) and (4.91), we get that

$$\begin{aligned}
\|x_{n+1} - q\|^2 &\leq a_n\|f(x_n) - q\|^2 + (1 - a_n)b_n\|x_n - q\|^2 \\
&\quad +(1 - a_n)(1 - b_n)\|v_n - q\|^2 \\
&\leq a_n\|f(x_n) - q\|^2 + (1 - a_n)b_n\|x_n - q\|^2 \\
&\quad +(1 - a_n)(1 - b_n)\|z_n - q\|^2 \\
&\leq a_n\|f(x_n) - q\|^2 + b_n\|x_n - q\|^2
\end{aligned}$$

$$\begin{aligned}
&+(1-b_n)\|x_n-q\|^2-(1-b_n)\|z_n-x_n\|^2\\
&+2(1-b_n)\lambda\|x_n-z_n\|.\|B^*(I-T_r^{F_2})Bx_n\|\\
\leq\ & a_n\|f(x_n)-q\|^2+\|x_n-q\|^2-(1-b_n)\\
&\times\|z_n-x_n\|^2+2(1-b_n)\lambda\|x_n-z_n\|\\
&\times\|B^*(I-T_r^{F_2})Bx_n\|. \qquad (4.92)
\end{aligned}$$

Therefore, since $\{f(x_n)\}$, $\{x_n\}$ and $\{z_n\}$ are bounded and $a_n \to 0$ as $n \to \infty$, from (4.92) and (4.89), we have

$$||z_n - x_n|| \to 0 \text{ as } n \to \infty. \qquad (4.93)$$

Thus, since

$$v_n - y_n = v_n - x_n + x_n - z_n + z_n - u_n + u_n - y_n,$$

from (4.93)and (4.87), we obtain that

$$||v_n - y_n|| \to 0 \text{ as } n \to \infty.$$

This and the fact that $v_n \in Ty_n$ imply that

$$d(y_n, Ty_n) \leq ||v_n - y_n|| \to 0 \text{ as } n \to \infty. \qquad (4.94)$$

Furthermore, since $\{x_n\}$ is a bounded sequence in Hilbert space H, without loss of generality, we can choose a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ and a point z in H such that $x_{n_i} \rightharpoonup z$ and

$$\limsup_{n\to\infty}\langle f(q) - q, x_n - q\rangle = \lim_{i\to\infty}\langle f(q) - q, x_{n_i} - q\rangle.$$

Thus, (4.93) implies that $z_{n_i} \rightharpoonup z$, then from (4.87), we get that $u_{n_i} \rightharpoonup z$ and $y_{n_i} \rightharpoonup z$. Hence, since Lemma 4.4.1 implies that $(I-T)$ is demiclosed at zero, from (4.94), we get

$$z \in F(T).$$

Since $(I - \lambda B^*(I - T_r^{F_2})B)$ is nonexpansive, from (4.90) and demiclosedness princeple for nonexpansive, we get that

$$z = (I - \lambda B^*(I - T_r^{F_2})B)z,$$

which implies that $B^*(I - T_r^{F_2})Bz = 0$. Thus, using the fact that $Bp = T_r^{F_2}Bp, \forall p \in \mathcal{F}$ and Lemma 3.1.7, we obtain that $Bz = T_r^{F_2}Bz$ hence $Bz \in EP(F_2)$. In addition, from (4.93), we have

$$\lim_{i\to\infty} \|x_{n_i} - T_s^{F_1}(I - \lambda B^*(I - T_r^{F_2})B)x_{n_i}\| = \lim_{i\to\infty} \|x_{n_i} - z_{n_i}\| = 0.$$

Since $T_s^{F_1}(I-\lambda B^*(I-T_r^{F_2})B)$ is nonexpansive, from the demiclosedness of a nonexpansive mapping, we obtain that $z = T_s^{F_1}(I-\lambda B^*(I-T_r^{F_2})B)z$. This with the fact that $Bz = T_r^{F_2}Bz$ gives that $z = T_s^{F_1}z$ and hence $z \in EP(F_1)$. Therefore, $z \in \Omega$.

Next, we show that $z \in VI(C, A)$. But, since A is Lipschitz continuous, from (4.87), we have

$$||Ay_{n_i} - Au_{n_i}|| \to 0 \text{ as } i \to \infty.$$

Let

$$Bv = \begin{cases} Av + N_Cv, & \text{if } v \in C, \\ \emptyset, & \text{if } v \notin C. \end{cases} \tag{4.95}$$

Then, by Lemma 3.1.8, B is a maximal monotone and $0 \in Bv$ if and only if $v \in VI(C, A)$. Let $(v, w) \in G(B)$. Then, we have $w \in Bv = Av + N_Cv$ and hence $w - Av \in N_Cv$. Thus, we get $\langle v - u, w - Av\rangle \geq 0$, for all $u \in C$. On the other hand, since $y_{n_i} = P_C(z_{n_i} - \gamma_{n_i}Au_{n_i})$ and $v \in C$, we have

$$\langle z_{n_i} - \gamma_{n_i}Au_{n_i} - y_{n_i}, y_{n_i} - v\rangle \geq 0,$$

and hence

$$\langle v - y_{n_i}, \frac{(y_{n_i} - z_{n_i})}{\gamma_{n_i}} + Au_{n_i}\rangle \geq 0.$$

Therefore, from $w - Av \in N_Cv$, $y_{n_i} \in C$ and the monotonicity of A, we get

$$\begin{aligned} \langle v - y_{n_i}, w\rangle &\geq \langle v - y_{n_i}, Av\rangle \\ &\geq \langle v - y_{n_i}, Av\rangle - \langle v - y_{n_i}, \frac{(y_{n_i} - z_{n_i})}{\gamma_{n_i}} + Au_{n_i}\rangle \end{aligned}$$

$$\begin{aligned} &= \langle v - y_{n_i}, Av - Ay_{n_i}\rangle + \langle v - y_{n_i}, Ay_{n_i} - Au_{n_i}\rangle \\ &\quad -\langle v - y_{n_i}, \frac{(y_{n_i} - z_{n_i})}{\gamma_{n_i}}\rangle \\ &\geq \langle v - y_{n_i}, Ay_{n_i} - Au_{n_i}\rangle - \langle v - y_{n_i}, \frac{(y_{n_i} - z_{n_i})}{\gamma_{n_i}}\rangle. \end{aligned}$$

This implies that $\langle v - z, w\rangle \geq 0$, as $i \to \infty$. Then, by Lemma 1.2.33, maximality of B gives that $0 \in Bz$ and hence $z \in VI(C, A)$. Therefore,

$$z \in \mathcal{F} = F(T) \cap \Omega \cap VI(C, A).$$

Thus, since $q = P_{\mathcal{F}} f(q)$ and $x_{n_i} \rightharpoonup z$, from the property of metric projection P_C, Lemma 3.1.3 (i), we have

$$\begin{aligned} \limsup_{n\to\infty} \langle f(q) - q, x_n - q\rangle &= \lim_{i\to\infty} \langle f(q) - q, x_{n_i} - q\rangle \\ &= \langle f(q) - q, z - q\rangle \leq 0. \end{aligned} \tag{4.96}$$

Now, since $\rho \in [0, \frac{1}{2})$ and $\{a_n\} \subset (0, 1)$, it follows that $\{a_n(1-2\rho)\} \subset (0, 1)$. So, from the assumptions of $\{a_n\}$, we obtain that

$$\lim_{n\to\infty} a_n(1 - 2\rho) = 0 \text{ and } \sum_{n=1}^{\infty} a_n(1 - 2\rho) = \infty.$$

Hence, from (4.86), (4.88), (4.96), Lemma 3.1.5 and the boundedness of $\{f(x_n)\}$, we conclude that

$$||x_n - q|| \to 0 \text{ as } n \to \infty.$$

Consequently, $x_n \to q = P_{\mathcal{F}} f(q)$.

Case 2. Suppose that there exists a subsequence $\{n_j\}$ of $\{n\}$ such that

$$||x_{n_j} - q|| < ||x_{n_j+1} - q||,$$

for all $j \in \mathbb{N}$. Then, by Lemma 3.1.6, there exist a nondecreasing sequence $\{m_k\} \subset \mathbb{N}$ such that $m_k \to \infty$, and

$$||x_{m_k} - q|| \leq ||x_{m_k+1} - q|| \text{ and } ||x_k - q|| \leq ||x_{m_k+1} - q||, \tag{4.97}$$

for all $k \in \mathbb{N}$. Now, replacing n by m_k from (4.97), (4.83), (4.92) and the fact that $a_n \to 0$ as $n \to \infty$, we get that

$$\begin{aligned} &||u_{m_k} - z_{m_k}|| \to 0, ||v_{m_k} - x_{m_k}|| \to 0, ||y_{m_k} - u_{m_k}|| \to 0, \\ &\text{and } ||z_{m_k} - x_{m_k}|| \to 0 \text{ as } k \to \infty. \end{aligned}$$

Thus, following the same process as in Case 1, we obtain

$$\limsup_{k\to\infty} \langle f(q) - q, x_{m_k} - q\rangle \leq 0. \tag{4.98}$$

Now, from (4.86) we have that

$$\begin{aligned} ||x_{m_k+1} - q||^2 &\leq (1 - a_{m_k}(1-2\rho))||x_{m_k} - q||^2 \\ &\quad +2a_{m_k}\langle f(q) - q, x_{m_k} - q\rangle, \\ &\quad +2a_{m_k}||x_{m_k+1} - x_{m_k}||.||f(x_{m_k}) - q||. \end{aligned} \tag{4.99}$$

And hence (4.97) and (4.99) imply that

$$\begin{aligned} a_{m_k}(1-2\rho)||x_{m_k} - q||^2 &\leq ||x_{m_k} - q||^2 - ||x_{m_k+1} - q||^2 \\ &\quad +2a_{m_k}\langle f(q) - q, x_{m_k} - q\rangle \\ &\quad +2a_{m_k}||x_{m_k+1} - x_{m_k}||.||f(x_{m_k}) - q|| \\ &\leq 2a_{m_k}\langle f(q) - q, x_{m_k} - q\rangle \\ &\quad +2a_{m_k}||x_{m_k+1} - x_{m_k}||.||f(x_{m_k}) - q||. \end{aligned}$$

But, the fact that $a_{m_k} > 0$ implies that

$$\begin{aligned} (1-2\rho)||x_{m_k} - q||^2 &\leq 2\langle f(q) - q, x_{m_k} - q\rangle \\ &\quad +2||x_{m_k+1} - x_{m_k}||.||f(x_{m_k}) - q||. \end{aligned}$$

Thus, using (4.98) and (4.88), we get that $||x_{m_k} - q|| \to 0$ as $k \to \infty$. This together with (4.99), (4.98) and (4.88) imply that $||x_{m_k+1} - q|| \to 0$ as $k \to \infty$. Hence, $||x_k - q|| \leq ||x_{m_k+1} - q||$, for all $k \in \mathbb{N}$ gives that $x_k \to q$ as $k \to \infty$. Therefore, from the above two cases, we can conclude that $\{x_n\}$ converges strongly to a point $q = P_{\mathcal{F}} f(q)$. This completes the proof. $\square$

If, in Theorem 4.4.2, we assume that $f(x) = u \in C$ for all $x \in C$ and T is a single-valued nonexpansive, then we obtain the following result.

Corollary 4.4.3. *Let H_1 and H_2 be real Hilbert spaces. Let C and Q be nonempty, closed and convex subsets of H_1 and H_2, respectively. Let $A : C \longrightarrow H_1$ be a $L-$Lipschitz monotone mapping and $B : H_1 \longrightarrow H_2$ be a bounded linear mapping. Let $F_1 : C \times C \longrightarrow \mathbb{R}$ and $F_2 : Q \times Q \longrightarrow \mathbb{R}$ be bifunctions satisfying Assumption 1.2.42. Let $T : C \longrightarrow C$ be a nonexpansive mapping. Assume that $\mathcal{F} = F(T) \cap \Omega \cap VI(C, A) \neq \emptyset$. Let $\{x_n\}$ be a sequence generated from an arbitrary $x_0 \in C$ by*

$$\begin{cases} z_n = T_s^{F_1}(I - \lambda B^*(I - T_r^{F_2})B)x_n, \\ u_n = P_C[z_n - \gamma_n A z_n], \\ y_n = P_C[z_n - \gamma_n A u_n], \\ x_{n+1} = a_n u + (1 - a_n)(b_n x_n + (1 - b_n)T y_n), \ \forall\, n \geq 0, \end{cases}$$

where P_C is a metric projection from H_1 onto C, $s, r > 0$, $\{\gamma_n\} \subset [a, b]$ for some $a, b \in (0, \frac{1}{L})$ and $\lambda \in (0, \frac{1}{d})$, for $d = \|B^*B\|$, where B^* is the adjoint of B, and $\{b_n\} \subset [e, f]$ for some $e, f \in (0, 1)$ and $\{a_n\} \subset (0, 1)$ satisfying $\lim_{n\to\infty} a_n = 0$ and $\sum_{n=0}^{\infty} a_n = \infty$. Then, the sequence $\{x_n\}$ converges strongly to the point $q = P_{\mathcal{F}}(u)$.

If, in Theorem 4.4.2, we assume that $A \equiv 0$ so that $VI(C, A) = C$, then we obtain the following corollary on a split equilibrium problem and a fixed point problem for nonexpansive multi-valued mapping:

Corollary 4.4.4. *Let H_1 and H_2 be real Hilbert spaces. Let C and Q be nonempty, closed and convex subsets of H_1 and H_2, respectively. Let $B : H_1 \longrightarrow H_2$ be a bounded linear mapping. Let $F_1 : C \times C \longrightarrow \mathbb{R}$ and $F_2 : Q \times Q \longrightarrow \mathbb{R}$ be bifunctions satisfying Assumption 1.2.42. Let $f : C \longrightarrow C$ be a $\rho-$contraction mapping with $0 \leq \rho < \frac{1}{2}$ and*

$T : C \longrightarrow CBC(C)$ be nonexpansive multi-valued mapping. Assume that $Tp = \{p\}$, for all $p \in F(T)$ and $\mathcal{F} = F(T) \cap \Omega \neq \emptyset$. Let $\{x_n\}$ be a sequence generated from an arbitrary $x_0 \in C$ by

$$\begin{cases} z_n = T_s^{F_1}(I - \lambda B^*(I - T_r^{F_2})B)x_n, \\ x_{n+1} = a_n f(x_n) + (1 - a_n)(b_n x_n + (1 - b_n)v_n), \ \forall \ n \geq 0, \end{cases}$$

where $v_n \in Tz_n$, $s, r > 0$, $\lambda \in (0, \frac{1}{d})$, for $d = \|B^*B\|$, where B^* is the adjoint of B, and $\{b_n\} \subset [e, f]$ for some $e, f \in (0,1)$ and $\{a_n\} \subset (0,1)$ satisfying $\lim\limits_{n\to\infty} a_n = 0$ and $\sum\limits_{n=0}^{\infty} a_n = \infty$. Then, the sequence $\{x_n\}$ converges strongly to the point $q = P_{\mathcal{F}} f(q)$.

If, in Theorem 4.4.2, we assume that $A \equiv 0$ and T is an identity mapping on C, then $F(T) \cap VI(C, A) = C$. Thus, we obtain the following corollary:

Corollary 4.4.5. *Let H_1 and H_2 be real Hilbert spaces. Let C and Q be nonempty, closed and convex subsets of H_1 and H_2, respectively. Let $B : H_1 \longrightarrow H_2$ be a bounded linear mapping. Let $F_1 : C \times C \longrightarrow \mathbb{R}$ and $F_2 : Q \times Q \longrightarrow \mathbb{R}$ be bifunctions satisfying Assumption 1.2.42. Assume that $\Omega \neq \emptyset$ and let $f : C \longrightarrow C$ be a $\rho-$contraction with $0 \leq \rho < \frac{1}{2}$. Let $\{x_n\}$ be a sequence generated from an arbitrary $x_0 \in C$ by*

$$\begin{cases} z_n = T_s^{F_1}(I - \lambda B^*(I - T_r^{F_2})B)x_n, \\ x_{n+1} = a_n f(x_n) + (1 - a_n)(b_n x_n + (1 - b_n)z_n), \ \forall \ n \geq 0, \end{cases}$$

where $s, r > 0$, $\lambda \in (0, \frac{1}{d})$, for $d = \|B^*B\|$, where B^* is the adjoint of B, and $\{b_n\} \subset [e, f]$ for some $e, f \in (0,1)$ and $\{a_n\} \subset (0,1)$ satisfying $\lim\limits_{n\to\infty} a_n = 0$ and $\sum\limits_{n=0}^{\infty} a_n = \infty$. Then, the sequence $\{x_n\}$ converges strongly to the point $q = P_\Omega f(q)$.

If, in Theorem 4.4.2, we assume that $H_1 = H_2$, $Q = C$, $B = I$ and $F_2 \equiv 0$, then we obtain the following corollary:

Corollary 4.4.6. *Let C be a nonempty, closed and convex subset of a real Hilbert space H_1. Let $A : C \longrightarrow H_1$ be a $L-$Lipschitz monotone mapping and $F_1 : C \times C \longrightarrow \mathbb{R}$ be a bifunction satisfying Assumption 1.2.42. Let $f : C \longrightarrow C$ be a $\rho-$contraction mapping with $0 \leq \rho < \frac{1}{2}$ and $T : C \longrightarrow CBC(C)$ be a nonexpansive multi-valued mapping. Assume that $Tp = \{p\}$, for all $p \in F(T)$ and $\mathcal{F} = F(T) \cap EP(F_1) \cap VI(C, A) \neq \emptyset$. Let $\{x_n\}$ be a sequence generated from an arbitrary $x_0 \in C$ by*

$$\begin{cases} z_n = T_s^{F_1} x_n, \\ u_n = P_C[z_n - \gamma_n A z_n], \\ y_n = P_C[z_n - \gamma_n A u_n], \\ x_{n+1} = a_n f(x_n) + (1 - a_n)(b_n x_n + (1 - b_n) v_n), \ \forall \ n \geq 0, \end{cases}$$

where $v_n \in Ty_n$, P_C is a metric projection from H_1 onto C, $s > 0$, $\{\gamma_n\} \subset [a, b]$ for some $a, b \in (0, \frac{1}{L})$, $\{b_n\} \subset [e, f]$ for some $e, f \in (0, 1)$ and $\{a_n\} \subset (0, 1)$ satisfying $\lim_{n\to\infty} a_n = 0$ and $\sum_{n=0}^{\infty} a_n = \infty$. Then, the sequence $\{x_n\}$ converges strongly to the point $q = P_{\mathcal{F}} f(q)$.

Chapter 5

Discussion, Conclusion and Recommendation

5.1 Discussion

The results obtained in this thesis extend, improve and unify several recent results in the existing literature on approximation of fixed points of nonlinear mappings, solutions of classical variational inequality and equilibrium problems. In particular, Theorem 4.1.2 extends the results of Alghamdi et al.[2] from the class of single-valued mappings to the class of multi-valued mappings. Moreover, our Theorem 4.1.2 provides strong convergence to a common element of the set of solutions of a variational inequality problem and the set of fixed points of a Lipschitz hemicontractive-type multi-valued mapping which is more general than pseudocontractive and nonexpansive mappings.
The results presented in section 4.1 have been published in the journal *"Creative Mathematics and Informatics, Volume 25, No.2, (2016), 183-196."*

Moreover, Theorem 4.2.2 extends and unifies the results of Takahashi and Takahashi [49, 50], Wang et al.[52], Ceng et al.[9] in the sense that the class of $\alpha-$inverse strongly monotone mappings is extended to the class of continuous monotone mappings. Theorem 4.2.2 further improves and extends Theorem 2.2.6 of [8] in the following way:

i. The result obtained for class of nonexpansive multi-valued mappings is extended to more general class of Lipschitz hemicontractive-type multi-valued mappings.

ii. The result obtained for $\alpha-$inverse strongly monotone mapping is extended to continuous monotone mapping.

iii. In our results, T is assumed to have closed and bounded values instead of compact values and also the domain of the mapping involved is assumed to be closed and convex instead of convex and weakly compact.

The results presented in section 4.2 have been published in the journal *"International Journal of Advanced Mathematical Sciences, Volume 5, issue 1, (2017), 20-26."*

Furthermore, Theorem 4.3.2 extends and generalizes the results of Razani and Yazdi [39], Hao [19], Wang et al.[52], Ceng et al.[9], Huang and Ma [21] to a finite family of generalized equilibrium problems and a Lipschitz hemicontractive-type mapping which contains the class of pseudocontractive and $k-$strictly pseudocontractive mappings. In addition, Our Theorem 4.3.2 extends and improves Theorem 3.7 of Zhang [63] in the sense that the results obtained for a finite family of equilibrium problems is extended to a finite family of generalized equilibrium problems.
The results presented in section 4.3 have been accepted in the journal *"SINET: Ethiopian Journal of Science, (2017)."*

Besides, Theorem 4.4.2 is new and more general than the result of Kazmi and Rizvi [25] (see Theorem 2.3.3 on page 45 of this thesis) in the sense that:

1. The result obtained in [25] is extended from the class of non-expansive single-valued mappings to more general class of non-expansive multi-valued mappings; and from class of $\alpha-$inverse

strongly monotone mappings to more general class of Lipschitz monotone mappings.

2. Theorem 4.4.2 exhibits a viscosity-type iterative algorithm which is more general than the iterative algorithm considered in Theorem 2.3.3.

3. The assumption that F_2 is upper semi-continuous imposed in Theorem 2.3.3 is dispensed with.

5.2 Conclusion

In conclusion, we wish to state that the main aim of this research is met; and all objectives that enhanced the establishment of the aim achieved. All the results obtained in this thesis are of independent of interest.

5.3 Recommendation

We recommend that further work can be carried out on the approximation of a common solution of a fixed point problem for class of multi-valued mappings more general than Lipschitz hemicontractive-type and generalized equilibrium problems. In connection with results obtained in this thesis, which are done in the setting of real Hilbert spaces, it will be of interest if answers can be provided for the following questions:

- Are the results obtained in this thesis extendable to Banach spaces more general than Hilbert spaces? If so, what modifications are needed on our theorems?

- Can the results obtained in this thesis for a classical variational inequality problem be extended to the case of a multi-valued variational inequality problem?
- Is it possible to approximate common element of the set of fixed points of more general multi-valued mappings than nonexpansive mappings, the set of solutions of variational inequality problem and the set of solutions of split equilibrium problem?
- Can some conditions, like demiclosedness and others, imposed in our main theorems be removed or dispensed with?

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