ADDIS ABEBA UNIVERSITY
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DIFFERENTIAL EQUATIONS

SEMINAR REPORT ON:

POSITIVE SOLUTION FOR HIGHER ORDER NONLINEAR EIGENVALUE PROBLEM

(PREPARED IN THE PARTIAL FULFILLMENT OF MSc DEGREE IN MATHEMATICS).

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Table of contents **page**

Preface:

In this paper I tried to characterize the value of λ (eigenvalue) for which a higher order boundary value problem will have a positive solution.

In the process of characterization I used many aspects of analysis. In other words this work shows the application of analysis to differential equations.

The paper is generally divided in to three different parts. The first part starts with second order nonlinear differential equations. i.e.

$$\left.\begin{aligned} &u'' + \lambda a(t) f(u) = 0 \qquad t \in (0,1) \\ &u(0) = u(1) = 0 \end{aligned}\right\} \quad (1)$$

In this part the value of λ will be characterized so that the BVP (1) will have a positive solution.

The second part of this paper is the generalization of the first part to n^{th} order. i.e.

$$\left.\begin{aligned} &u^{(n)} + \lambda H(t,u) = \lambda K(t,u), \qquad n \geq 2,\ t \in (0,1) \\ &u^{(i)}(0) = u^{(p)}(1) = 0, \qquad 0 \leq i \leq n-2 \\ &\textit{where } \ 0 \leq p \leq n-1, \textit{but fixed and } \lambda > 0 \end{aligned}\right\} \quad (2)$$

The BVP (2) is the most general form of (1) above and λ will be characterized so that the BVP (2) will have a positive solution.

Finally, considering the BVP (2) a special case $\lambda = 1$ will be studied. In this section sufficient condition will be given so that the BVP (2) will have a positive solution for $\lambda = 1$.

Introduction

In real world not all values are meaningful to our work. Therefore to overcome these kinds of problems we should limit ourselves to the values that fit to the real world application. Characterizing eigenvalues of the differential equations to get only positive solutions is one way of solving such problems.

Differential equations describe many areas of the real world. To mention some; speed of change of population size, flow rates, transportation laws conservation laws are expressed (modeled) using differential equations. In some of these applications only positive solutions are meaningful. So this work is used in such areas of applications.

In the entire paper two fixed point theorems (Krasnosel'skii and shauder) are stated and used as a tool for the characterization and existence of λ and positive solutions respectively. Also Banach space is constructed from the set of continuous functions defined on $[0,1]$ with supremum norm. And to ensure the existence of positive solution a cone is defined on the Banach space. Further certain properties of Green's function is established which used later.

Finally, sublinearity and superlinearity of conditions on a function is discussed to ensure the existence of a positive solution for $\lambda = 1$.

1. Preliminaries

Definitions

1.1 A Euclidean vector space E is said to be complete if every Cauchy sequence in E converges to an element of E.

1.2 A Banach space is a normed space which is complete as a metric space.

1.3. A set P is convex if and only if

$$for\ x, y \in p\ and\ t \in [0,1]\ implies\ tx + (1-t)y \in p$$

1.4. An operator T is convex if and only if D(T)(domain of T) is convex and

$$for\ all\ x, y \in D(T)\ with\ x < y\ and\ for\ all\ t \in [0,1]$$
$$T(tx + (1-t)y) \le tTx + (1-t)Ty$$

and T is concave if and only if –T is convex.

1.5. Let B be a Banach space over R (real).A nonempty closed convex set $p \subset B$ is said to be a cone provided the following are satisfied;

$$(a)\ If\ y \in P\ and\ \alpha \ge 0,\ then\ \alpha y \in p$$
$$(b)\ if\ y \in p\ and\ -y \in p,\ then\ y = 0$$

1.6. Let X be a normed space. A set $S \subseteq X$ is called sequentially compact if every sequence $\{x_n\}\ in\ S$ has a subsequence converging to an element in X.

1.7. A map $T : X \to Y$ between two metric spaces is said to be

i. Compact if $T(B)$ is sequentially compact in Y a bounded subset B of X.

ii. T is completely continuous if it is compact on each bounded subset in its domain.

The following facts are used later.

Let $B = \{y : y \in c[0,1]\}$: the set of continuous real valued functions defined on $[0,1]$.

B is a vector space over R (real).

Since continuous real valued function defined on closed and bounded interval is bounded we can define supremum norm on B.

Let $\|y\| = \sup_{t\in[0,1]} |y(t)|$ this indeed defines a norm for if,

For $y \in B$

$\|y\| = 0 \Leftrightarrow \sup_{t\in[0,1]} |y(t)| = 0$ since $0 \le |y(t)| \le \sup_{t\in[0,1]} |y(t)| = 0 \Leftrightarrow y(t) = 0$, $\forall t \in [0,1]$

$\|\alpha y\| = \sup_{t\in[0,1]} |\alpha y(t)| = |\alpha| \sup_{t\in[0,1]} |y(t)| = |\alpha| \|y\|$

For $y_1, y_2 \in B$ and $t \in [0,1]$

$$\begin{aligned}
&|y_1(t) + y_2(t)| \le |y_1(t)| + |y_2(t)| \le \sup|y_1(t)| + \sup|y_2(t)| = \|y_1\| + \|y_2\| \\
&\Rightarrow \sup|y_1(t) + y_2(t)| \le \|y_1\| + \|y_2\| \\
&\Rightarrow \|y_1 + y_2\| \le \|y_1\| + \|y_2\|
\end{aligned}$$

Now we want to show B is complete with respect to the norm defined above. For this let $\{y_n\}$ be a Cauchy sequence in B.

Fact 1. A Cauchy sequence has a convergent subsequence.

Let $\{y_{n_k}\}$ be a subsequence of $\{y_n\}$ which converges to y .i.e.

$y_{n_k} \to y$ as $n_k \to \infty$. If we show $y \in B$ and $y_n \to y$, then B will be Banach space with the norm defined above.

Fact 2. If $\{f_n\}$ is a sequence of continuous functions on a set F and $f_n \to f$ uniformly on F then f is continuous on F.

From fact 2 since $\{y_{n_k}\}$ is a sequence of continuous functions on $[0,1]$ we find y is continuous on $[0,1]$. This implies $y \in B$.

Next we will show $y_n \to y$. For this, since $y_{n_k} \to y$ as $n_k \to \infty$ we have;

for $\varepsilon > 0, \exists N_1 \ni \|y_{n_k} - y\| < \varepsilon/2 \quad , \forall n_k \ge N_1$ *

and since $\{y_n\}$ is a Cauchy sequence we have;

if $\varepsilon > 0 \ \exists N_2 > 0 \ \ni \ \left\|y_m - y_{n_k}\right\| < \varepsilon/2 \quad , \forall m, n_k \geq N_2$ $\quad * *$

now let $M = \max\{N_1, N_2\}$

From * and * * we find that,

$$\left\|y_m - y\right\| = \left\|y_m - y_{n_k} + y_{n_k} - y\right\| \leq \left\|y_m - y_{n_k}\right\| + \left\|y_{n_k} - y\right\|$$
$$< \varepsilon/2 + \varepsilon/2 = \varepsilon$$

Letting $m \to \infty$ $\left\|y_m - y\right\| < \varepsilon \quad \forall m \geq M$. Therefore the sequence $\{y_n\}$ converges to y .This implies that the space B is Banach space.

Theorem 1. (Krasnosel'skii fixed point theorem)

Let B be a Banach space, and let $C(\subset B)$ be a cone. Assume Ω_1, Ω_2 be open subsets of B with $0 \in \Omega_1, \overline{\Omega_1} \subset \Omega_2$ and let

$$S : C \cap \left(\overline{\Omega_2} \setminus \Omega_1\right) \to C$$

Be completely continuous operator such that, either

(a) $\|Sy\| \leq \|y\|, y \in C \cap \partial\Omega_1, and \|Sy\| \geq \|y\|, y \in C \cap \partial\Omega_2$

OR

(b) $\|Sy\| \geq \|y\|, y \in C \cap \partial\Omega_1, and \|Sy\| \leq \|y\|, y \in C \cap \partial\Omega_2$

Then, S has a fixed point in $C \cap \left(\overline{\Omega_2} \setminus \Omega_1\right)$, where $\partial\Omega_1$ is boundary of Ω_1.

2. Second order boundary value problem

In the first part of this work we saw positive solutions for second order nonlinear eigenvalue problem.

i.e. For the boundary value problem below:

$$u'' + \lambda a(t) f(u) = 0 \qquad t \in (0,1) \qquad 1.1$$

$$u(0) = u(1) = 0 \qquad 1.2$$

Where

(a) $f : [0,\infty) \to [0,\infty)$ *is continuous*

(b) $a : [0,1] \to [0,\infty)$ is continuous and does not vanish identically on any subinterval ,and

(c) $f_0 = \lim_{x \to 0^+} \frac{f(x)}{x}$ *and* $f_\infty = \lim_{x \to \infty} \frac{f(x)}{x}$ *exist*

Now the Green's function for

$$-y'' = 0$$

$$y(0) = y(1) = 0$$

Will be

$$G(t,s) = \begin{cases} t(1-s), & 0 \le t \le s \le 1 \\ s(1-t), & 0 \le s \le t \le 1 \end{cases} \qquad 1.3$$

From which

$$G(t,s) > 0 \quad on \ (0,1)\times(0,1) \qquad 1.4$$

$$G(t,s) \le G(s,s) = s(1-s), \qquad 0 \le t,\ s \le 1 \qquad 1.5$$

And

$$G(t,s) \ge \frac{1}{4} G(s,s) = \frac{1}{4} s(1-s),\ t \in \left[\frac{1}{4}, \frac{3}{4}\right],\ s \in [0,1] \qquad 1.6$$

Now let's characterize λ so that 1.1, 1.2 will has a positive solution.

N ote that $u(t)$ is a solution of 1.1, 1.2 if and only if

$$u(t)=\lambda\int_0^1 G(t,s)a(s)f(u(s))ds, \quad t\in[0,1] \qquad (*)$$

That is, (*) can be written as

$$\begin{aligned} u(t) &= \lambda\left[\int_0^t G(t,s)a(s)f(u(s))ds+\int_t^1 G(t,s)a(s)f(u(s))ds\right] \\ &= \lambda\left[\int_0^t s(1-t)a(s)f(u(s))ds-\int_1^t t(1-s)a(s)f(u(s))ds\right] \\ &= \lambda\left[(1-t)\int_0^t sa(s)f(u(s))ds-t\int_1^t (1-s)a(s)f(u(s))ds\right] \qquad ......(**) \end{aligned}$$

Therefore

$$\begin{aligned} u'(t) &= \lambda\left[-1\int_0^t sa(s)f(u(s))ds+(1-t)ta(t)f(u)-\int_1^t (1-s)a(t)f(u(s))ds-(1-t)ta(t)f(u)\right] \\ &= \lambda\left[-1\int_0^t sa(s)f(u(s))ds-\int_1^t (1-s)a(t)f(u(s))ds\right] \end{aligned}$$

Thus

$$\begin{aligned} u''(t) &= \lambda\left[-ta(t)f(u)-(1-t)a(t)f(u)\right] \\ &= -\lambda a(t)f(u) \end{aligned}$$

this implies

$$u''+\lambda a(t)f(u)=0$$

And it is clear that from (**)

$$u(0)=u(1)=0$$

Therefore $u(t)$ given in (*) is the solution of the BVP 1.1, 1.2.

Now let $B(Banach)=C[0,1]$ with norm $\|x\|=\sup_{0\le t\le 1}|x(t)|$

And let P be a cone defined by

$$P=\left\{x\in B : x(t)\ge 0\, on\,[0,1]\ and\ \min_{1/4\le t\le 3/4} x(t)\ge \frac{1}{4}\|x\|\right\}$$

Once again let $\tau\in[0,1]$ be defined by

$$\int_{1/4}^{3/4} G(\tau,s)a(s)\,ds = \max_{0\le t\le 1}\int_{1/4}^{3/4} G(t,s)a(s)\,ds \qquad 1.7$$

Theorem 1.1. Assume that conditions (a), (b) and (c) hold. Then, for each λ satisfying

$$\frac{4}{\left(\int_{1/4}^{3/4} G(\tau,s)a(s)\,ds\right)f_\infty} < \lambda < \frac{1}{\left(\int_0^1 s(1-s)a(s)\,ds\right)f_0} \qquad (1.8)$$

There exist at least one solution of 1.1, 1.2 in P.

Proof: Let λ be given as in (1.8). Now let $\varepsilon > 0$ be chosen such that

$$\frac{4}{\left(\int_{1/4}^{3/4} G(\tau,s)a(s)\,ds\right)(f_\infty-\varepsilon)} \le \lambda \le \frac{1}{\left(\int_0^1 s(1-s)a(s)\,ds\right)(f_0+\varepsilon)} \qquad (***)$$

Now let's define an integral operator $T: P \to B$ by

$$Tu(t) = \lambda\int_0^1 G(t,s)a(s)f(u(s))\,ds, \qquad u\in P. \qquad (1.9)$$

Now we need to find a fixed point of T in the cone P, and that fixed point is the solution of the BVP (1.1), (1.2) and it is positive as it can be seen from (1.9).

From (1.4), for $u\in P$, $Tu(t)\ge 0$ on $[0,1]$.

Also, for $u \in P$, (1.5) implies:

$$\begin{aligned} Tu(t) &= \lambda\int_0^1 G(t,s)a(s)f(u(s))\,ds \\ &\le \lambda\int_0^1 s(1-s)a(s)f(u(s))\,ds \end{aligned}$$

So that

$$\|Tu\| \le \lambda\int_0^1 s(1-s)a(s)f(u(s))\,ds \qquad (1.10)$$

And from (1.6)

$$\min_{1/4\le t\le 3/4} Tu(t) = \min_{1/4\le t\le 3/4} \lambda \int_0^1 G(t,s)a(s)f(u(s))ds$$
$$\ge \frac{\lambda}{4}\int_0^1 s(1-s)a(s)f(u(s))ds$$
$$\ge \frac{1}{4}\|Tu\|$$

This shows that, $T: P \to P$ .therefore T is completely continuous operator.

Now consider f_0, by assumption (c) and definition of limit

$\exists \delta_1 > 0$ *such that* $f(x) \le (f_0 + \varepsilon)$ *for* $0 < x \le \delta_1$

-Now if $u \in P$ with $\|u\| = \delta_1$ is chosen then (1.10) implies that

$$\|Tu\| \le \lambda \int_0^1 s(1-s)a(s)f(u(s))ds$$
$$\le \lambda \int_0^1 s(1-s)a(s)(f_0+\varepsilon)u(s)ds$$
$$\le \lambda \int_0^1 s(1-s)a(s)ds(f_0+\varepsilon)\|u\|$$
$$(***) \Rightarrow \qquad \le \|u\|$$

Therefore

$$\|Tu\| \le \|u\|$$

Now to apply theorem 1(krasnoselskii) set

$$\Omega_1 = \{x \in B : \|x\| < \delta_1\}$$

then

$$\partial\Omega_1 = \{x \in B : \|x\| = \delta_1\}$$

Therefore, $\|Tu\| \le \|u\|$ for $u \in P \cap \partial\Omega_1$ (1.11)

Next consider f_∞, using assumption (c) and definition of limit

$$\exists \bar{\delta}_2 > 0 \ni f(x) \ge (f_\infty - \varepsilon)x, \quad \forall x \ge \bar{\delta}_2$$

and let $\delta_2 = \max\{2\delta_1, 4\bar{\delta}_2\}$ *with* $\Omega_2 = \{x \in B : \|x\| < \delta_2\}$

Since $\delta_2 \ge 2\delta_1$, $\overline{\Omega}_1 \subset \Omega_2$

Now if $u \in P$, with $\delta_2 = \|u\|$, then $\min_{1/4 \le t \le 3/4} u(t) \ge \frac{1}{4}\|u\| = \frac{1}{4}\delta_2 \ge \frac{1}{4} \times 4\bar{\delta}_2 = \bar{\delta}_2$

Now

$$Tu(\tau) = \lambda \int_0^1 G(\tau,s)a(s)f(u(s))ds$$
$$\ge \lambda \int_{1/4}^{3/4} G(\tau,s)a(s)(f_\infty - \varepsilon)u(s)ds$$
$$\ge \lambda \int_{1/4}^{3/4} G(\tau,s)a(s)ds(f_\infty - \varepsilon)\frac{\|u\|}{4}, \quad u \in P$$

(**) $\Rightarrow$ $\ge \|u\|$

Thus $\|Tu\| \ge \|u\| \quad for \quad u \in P \cap \partial\Omega_2$ (1.12)

Where $\partial\Omega_2 = \{x \in B : \|x\| = \delta_2\}$

Therefore (1.11) and (1.12) satisfy theorem 1 of (i) and hence T has a fixed point $u(t) \in P \cap (\bar{\Omega}_2 \setminus \Omega_1)$. Thus (1.9) implies that $u(t)$ is positive and solution of the BVP 1.1, 1.2. Therefore λ is an eigenvalue for the BVP 1.1, 1.2.

Theorem 1.2: Suppose assumptions (a), (b) and (c) hold. Then for each λ satisfying

$$\frac{4}{\left(\int_{1/4}^{3/4} G(\tau,s)a(s)ds\right)f_0} < \lambda < \frac{1}{\left(\int_0^1 s(1-s)a(s)ds\right)f_\infty}$$

Then there exists at least one solution of the BVP 1.1, 1.2 in P

Proof: choose

$\varepsilon > 0 \ni$

$$\frac{4}{\left(\int_{1/4}^{3/4} G(\tau,s)a(s)ds\right)(f_0 - \varepsilon)} \le \lambda \le \frac{1}{\left(\int_0^1 s(1-s)a(s)ds\right)(f_\infty + \varepsilon)} \quad (1.13)$$

Let T be the completely continuous operator defined in the proof of theorem 1.1

Now consider $f_0 = \lim_{x \to 0} \frac{f(x)}{x}$, by definition of limit we have,

$\exists \delta_1 > 0 \ni f(x) \geq (f_0 - \varepsilon), \quad 0 < x \leq \delta_1$. And set for $u \in P, \; \|u\| = \delta_1$

Now

$$\begin{aligned} Tu(\tau) &= \lambda \int_0^1 G(\tau,s)a(s)f(u(s))ds \\ &\geq \lambda \int_{1/4}^{3/4} G(\tau,s)a(s)(f_0 - \varepsilon)u(s)ds \\ &\geq \lambda \int_{1/4}^{3/4} G(\tau,s)a(s)ds(f_0 - \varepsilon)\frac{\|u\|}{4}, \quad u \in P \\ (1.13) \Rightarrow \quad &\geq \|u\| \end{aligned}$$

Then if

$$\Omega_1 = \{x \in B : \|x\| < \delta_1\}$$

then $\|Tu\| \geq \|u\| \quad for \quad u \in P \cap \partial\Omega_1$ ....(1.14)

Finally consider f_∞, as $f_\infty = \lim_{x \to \infty} \frac{f(x)}{x}$, by definition of limit we have

$\exists \overline{\delta}_2 > 0 \; such \; that \; f(x) \leq (f_\infty + \varepsilon)x \quad for \; \forall x \geq \overline{\delta}_2$ ....(1.15)

Now two cases will be considered

i. if f is bounded

ii. if f is unbounded

(i). If f is bounded

Let $N > 0 \ni f(x) \leq N, \quad 0 < x < \infty$

and let $\delta_2 = \max\left\{2\delta_1, N\lambda \int_0^1 s(1-s)a(s)ds\right\}$

then for $u \in P \quad with \quad \|u\| = \delta_2$

$$\begin{aligned} Tu(t) &= \lambda \int_0^1 G(t,s)a(s)f(u(s))ds \\ boundedness\; of\; f\; and\; (1.5) \Rightarrow \quad &\leq \lambda \int_0^1 s(1-s)a(s)dsN \\ &\leq \delta_2 = \|u\| \end{aligned}$$

Therefore

$$\|Tu\| \le \|u\|$$

If we set $\Omega_2 = \{x \in B : \|x\| < \delta_2\}$ *then* $\overline{\Omega_1} \subset \Omega_2$

therefore $\|Tu\| \le \|u\|$ *for* $u \in P \cap \partial\Omega_2$

(ii). If f is unbounded

Now let $\delta_2 > \max\{2\delta_1, \overline{\delta}_2\} \ni f(x) \le f(\delta_2),\ 0 < x \le \delta_2$

choose $u \in P$ *with* $\|u\| = \delta_2$

then

$$Tu(t) = \lambda \int_0^1 G(t,s) a(s) f(u(s)) ds$$

$$(1.5) \Rightarrow \quad \le \lambda \int_0^1 s(1-s) a(s) f(u(s)) ds$$

$$\text{unboundedness of } f \Rightarrow \quad \le \lambda \int_0^1 s(1-s) a(s) f(\delta_2) ds$$

$$(1.15) \Rightarrow \quad \le \lambda \int_0^1 s(1-s) a(s) ds (f_\infty + \varepsilon) \delta_2$$

$$(1.13) \Rightarrow \quad \le \delta_2 = \|u\|$$

Therefore $\|Tu\| \le \|u\|$

If $\Omega_2 = \{x \in B : \|x\| < \delta_2\}$, then $\|Tu\| \le \|u\|$ *for* $u \in P \cap \partial\Omega_2$

In both cases theorem 1 of (ii) is satisfied.

Therefore T has a fixed point $u(t) \in P \cap (\overline{\Omega_2} \setminus \Omega_1)$.Thus (1.9) implies that $u(t)$ is positive and solution of the BVP 1.1, 1.2. This concludes that λ is an eigenvalue for the BVP 1.1, 1.2.

Next let's generalize the problem to n^{th} order boundary value problem.

3. Eigenvalue Characterization for (n, p) boundary value problems.

Consider the (n, p) boundary value problem:

$$u^{(n)} + \lambda H(t,u) = \lambda K(t,u)\,, \qquad n \geq 2,\ t \in (0,1) \qquad (1)$$

$$u^{(i)}(0) = u^{(p)}(1) = 0\,, \qquad 0 \leq i \leq n-2 \qquad (2)$$

where $0 \leq p \leq n-1, \textit{but fixed and } \lambda > 0$

Objective: the value of λ will be characterized so that the boundary value problem has a positive solution.

A positive solution y of (1), (2) we mean is that $y \in C^{n}(0,1)$ and y satisfies (1) and fulfils (2).

If, for a particular λ , the boundary value problem $(1),(2)$ has a positive solution, then λ is called an eigenvalue and the corresponding function y is called eigenfunction for $(1)\,and\,(2)$.

Now let E be the set consisting of such values of λ such that $(1),(2)$ has a positive solution.

$E = \{\lambda > 0 : (1),(2)\, \textit{has a positive solution}\}$

Assumptions: Throughout this paper the following assumptions will be considered.

There exist continuous functions

$f:[0,\infty)\to(0,\infty)$ *and*
$k,k_1,h,h_1:(0,1)\to R$, *such that*,

(H-1) f is nondecreasing

(H-2) for $u\in[0,\infty)$,

$$h(t)\frac{H(t,u)}{f(u)}\le h_1(t),\ k(t)\le\frac{K(t,u)}{f(u)}\le k_1(t);$$

(H-3) $h(t)-k_1(t)$ is nonnegative and is not identically zero on any subinterval of (0, 1)

(H-4) $\int_0^1(1-t)^{n-p-1}\left[h_1(t)-k(t)\right]dt<\infty$

To obtain the solution for the BVP it requires a mapping whose kernel $G(t,s)$ is the Green's function of the BVP

$$-y^{(n)}=0.$$
$$y^{(p)}(1)=y^{(i)}(0)=0,\ \ 0\le i\le n-2$$

Where $0\le p\le n-1$ but fixed, then the Green's function will become;

$$G(t,s)=\frac{1}{(n-1)!}\begin{cases} t^{n-1}(1-s)^{n-p-1}-(t-s)^{n-1}, & 0\le s\le t\le 1\\ t^{n-1}(1-s)^{n-p-1}, & 0\le t\le s\le 1\end{cases}$$

And

$$\frac{\partial^i}{\partial t^i}G(t,s)\ge 0,\qquad 0\le i\le p,\qquad (t,s)\in[0,1]\times[0,1]$$

Lemma 1. For $(t,s)\in[0,1]\times[0,1]$,

$$G(t,s)\le\frac{1}{(n-1)!}(1-s)^{n-p-1}$$

. Proof. This is a direct consequence of the Green's function given above

Lemma 2. *For* $(t,s)\in\left[\frac{1}{4},\frac{3}{4}\right]\times[0,1]$, *we have*

$$G(t,s)\geq\left(\frac{1}{4}\right)^{n-1}\frac{1}{(n-1)!}(1-s)^{n-p-1}\phi(s)$$

where $0\leq\phi(s)\leq 1$ is given by

$$\phi(s)=\begin{cases}1-(1-s)^{p}, & s\leq t\\ 1, & t\leq s\end{cases}$$

Proof:

For $0\leq s\leq t$, *from the Green's function given above*

$$\begin{aligned}(n-1)!G(t,s)&\geq t^{n-1}(1-s)^{n-p-1}-(t-ts)^{n-1}\\ &=t^{n-1}(1-s)^{n-p-1}\left[1-(1-s)^{p}\right]\\ &\geq\left(\frac{1}{4}\right)^{n-1}(1-s)^{n-p-1}\phi(s)\end{aligned}$$

For $t\leq s\leq 1$, the inequality is direct.

Notations: the following notations are frequently used in the build up of the result.

Let $v(t)=h_1(t)-k(t)$ *and* $u(t)=h(t)-k_1(t)$ (3)

For a nonnegative y *on* $[0,1]$, *denote*

$$\alpha=\frac{1}{(n-1)!}\int_0^1(1-s)^{n-p-1}v(s)f(y(s))ds \qquad (4)$$

$$\beta=\frac{1}{(n-1)!}\int_0^1(1-s)^{n-p-1}\phi(s)u(s)f(y(s))ds \qquad (5)$$

from (H-2) and (H-3) it is clear that $\alpha\geq\beta>0$

Further let's define

$$\gamma = \left(\frac{1}{4}\right)^{n-1} \frac{\beta}{\alpha} \tag{6}$$

And note that $0 < \gamma < 1$

Let the Banach space be constructed by the set of continuous functions defined on the closed interval $[0,1]$. i.e.

$$B = \{y \mid y \in c[0,1]\} \ \textit{be equiped with norm}$$

$$\|y\| = \sup_{t\in[0,1]} |y(t)|, \textit{and let}$$

$$C = \left\{ y \in B \mid y(t) \textit{ is nonnegative on } [0,1] : \min_{t\in\left[\frac{1}{4},\frac{3}{4}\right]} y(t) \geq \gamma \|y\| \right\}$$

It is clear that C is a cone in B. Further, let

$$C_M = \{y \in C \mid \|y\| \leq M\}$$

Where C_M is bounded, closed, convex, subset of C.

And let's define the operator

$$S : C \to B \ \textit{by}$$

$$Sy(t) = \int_0^1 G(t,s)\left[H(s,y) - K(s,y)\right] ds, \ t \in [0,1] \tag{2.1}$$

To obtain a positive solution of the BVP a fixed point of the operator λS in the cone C is needed.i.e $y(t)$ is a solution of the BVP (1), (2) if and only if

$$y(t) = \lambda \int_0^1 G(t,s)\left[H(s,y) - K(s,y)\right] ds, \ t \in [0,1]$$

To obtain the desired result we find that the operator S satisfy the following;

$$Uy(t) \le Sy(t) \le Vy(t), \qquad t \in [0,1]$$

$$\text{where } \; Uy(t) = \int_0^1 G(t,s)u(s)f(y(s))ds$$

$$\text{and } \; Vy(t) = \int_0^1 G(t,s)v(s)f(y(s))ds$$

To verify this , from (H-2)

$$H(t,u) \le h_1(t)f(u) \qquad (2.2)$$

$$\text{and } \; K(t,u) \ge k(t)f(u) \qquad (2.3)$$

from (2.2) *and* (2.3) *we find that,*

$$H(t,u) - K(t,u) \le [h_1(t) - k(t)]f(u)$$
$$= v(t)f(u).$$

therefore,

$$\int_0^1 G(t,s)[H(t,u) - K(t,u)]ds \le \int_0^1 G(t,s)v(s)f(y(s))ds \qquad (2.4)$$

again from $(H-2)$

$$h(t)f(u) \le H(t,u) \qquad (2.5)$$

$$k_1(t)f(u) \ge K(t,u) \qquad (2.6)$$

from (2.5) *and*.(2.6) *we get,*

$$H(t,u) - K(t,u) \ge [h(t) - k_1(t)]f(u)$$
$$= u(t)f(u)$$

therefore

$$\int_0^1 G(t,s)[H(t,u) - K(t,u)]ds \ge \int_0^1 G(t,s)[h(t) - k_1(t)]f(y(s))ds$$
$$= \int_0^1 G(t,s)u(s)f(y(s))ds \qquad (2.7)$$

from (2.4) *and* (2.7) *we have,*

$$Uy(t) \le Sy(t) \le Vy(t), \qquad t \in [0,1] \qquad (2.8)$$

$$\text{where } \; Uy(t) = \int_0^1 G(t,s)u(s)f(y(s))ds$$

$$\text{and } \; Vy(t) = \int_0^1 G(t,s)v(s)f(y(s))ds$$

Theorem 3.1: (Schauder fixed point theorem)

Let M be a nonempty, closed, bounded, convex subset of a B- space, and Suppose $T:M\to M$ is a compact operator .Then T has a fixed point.

Now let's show the operator S defined in (2.1) is compact on the cone C .Let's consider the case $u(t)$ is unbounded in a deleted right neighborhood of 0 and also in a deleted left neighborhood of 1.Since $v(t)\geq u(t)$, $v(t)$ is also unbounded near 0 and 1.

For $m\in\{1,2,3,\ldots\}$, define $u_m,v_m:[0,1]\to R$ by

$$u_m(t)=\begin{cases} u\left(\frac{1}{m+1}\right) , & 0\leq t\leq\frac{1}{m+1} \\ u(t), & \frac{1}{m+1}\leq t\leq\frac{m}{m+1} \\ u\left(\frac{m}{m+1}\right), & \frac{m}{m+1}\leq t\leq\frac{1}{m+1} \end{cases} \tag{2.9}$$

$$v_m(t)=\begin{cases} v\left(\frac{1}{m+1}\right), & 0\leq t\leq\frac{1}{m+1} \\ v(t), & \left(\frac{m}{m+1}\right)\frac{1}{m+1}\leq t\leq\frac{m}{m+1} \\ v\left(\frac{m}{m+1}\right), & \frac{m}{m+1}\leq t\leq\frac{1}{m+1} \end{cases} \tag{2.10}$$

and the operators $U_m,V_m:C\to B$ *by*

$$U_my(t)=\int_0^1 G(t,s)u_m(s)f(y(s))ds \tag{2.11}$$

$$V_my(t)=\int_0^1 G(t,s)v_m(s)f(y(s))ds \tag{2.12}$$

From the definitions (2.9) and (2.10) for each m both $U_m \; and \, V_m$ are compact operators on C.

Next, to show $V \; and \; U$ are compact operators on C, let $M > 0 \; and \; y \in C_M$

$$
\begin{aligned}
\left|v_m y(t) - Vy(t)\right| &= \int_0^1 G(t,s)\left|v_m(s) - v(s)\right| f(y(s))\, d \\
&= \int_0^{\frac{1}{m+1}} G(t,s)\left|v_m(s) - v(s)\right| f(y(s))\, ds \\
&\quad + \int_{\frac{m}{m+1}}^1 G(t,s)\left|v_m(s) - v(s)\right| f(y(s))\, ds \\
&\quad + \int_{\frac{1}{m+1}}^{\frac{m}{m+1}} G(t,s)\left|v_m(s) - v(s)\right| f(y(s))\, ds
\end{aligned}
$$

$$
lemma\,1\, and\, (H-1) \Rightarrow \quad \le \frac{f(M)}{(n-1)!}\left[\int_0^{\frac{1}{m+1}} (1-s)^{n-p-1}\left| v\left(\frac{1}{m+1}\right) - v(s)\right| ds + \int_{\frac{m}{m+1}}^1 (1-s)^{n-p-1}\left| v\left(\frac{1}{m+1}\right) - v(s)\right| ds \right]
$$

Assumption (H-4) shows the integrablity of $(1-t)^{n-p-1} v(t)$. Therefore taking the limit as $m \to \infty$, V_m converges uniformly to $V \; on \; C_M$ .compactness of V_m implies V is compact on C. Similarly U_m converges uniformly to $U \, on \, C_M$ and U is compact on C .therefore (2.8) shows that the operator S is compact on C.

Now it remains to show that S maps the bounded set C_M in to C_M , then by using theorem 3.1 S will have a fixed point and indeed this fixed point is positive (observe (2.1)). And λ will be characterized so that the BVP (1), (2) will have a positive solution. For this the following theorem will be proved.

Theorem 3.2. There exists a $c > 0$ such that the interval $(0, c] \subseteq E$.

Proof: let $M > 0$ be given. Define

$$c = M\left[\frac{f(M)}{(n-1)!}\int_0^1 (1-s)^{n-p-1} v(s)ds\right]^{-1} \tag{2.13}$$

Let $\lambda \in (0, c]$. We shall prove that $(\lambda S)(C_M) \subseteq C_M$, that is λS maps bounded sets in to bounded sets. For this, let $y \in C_M$.

Now first let's show $\lambda Sy \in C$ .From (2.8) and (H-3) we have,

$$(\lambda Sy)(t) \geq \lambda \int_0^1 G(t,s)u(s)f(y(s))ds \geq 0, \quad t[0,1] \tag{2.14}$$

Further, it follows from (2.8) and lemma 1 that

$$\begin{aligned} Sy(t) &\leq \int_0^1 G(t,s)v(s)f(y(s))ds \\ &\leq \frac{1}{(n-1)!}\int_0^1 (1-s)^{n-p-1} v(s)f(y(s))ds = \alpha, \qquad t \in [0,1]. \end{aligned}$$

Thus

$$\|Sy\| \leq \alpha \tag{2.15}$$

Now, on using (2.8), lemma 2 and (2.15), for $t \in \left[\frac{1}{4}, \frac{3}{4}\right]$ we have

$$\begin{aligned} (\lambda Sy)(t) &\geq \lambda \int_0^1 G(t,s)u(s)f(y(s))ds \\ &\geq \lambda \left(\frac{1}{4}\right)^{n-1} \frac{1}{(n-1)!}\int_0^1 (1-s)^{n-p-1} \phi(s)u(s)f(y(s))ds \\ &= \lambda \left(\frac{1}{4}\right)^{n-1} \beta \\ &\geq \lambda \left(\frac{1}{4}\right)^{n-1} \beta \frac{\|Sy\|}{\alpha} = \lambda\gamma\|Sy\| = \gamma\|\lambda Sy\|. \end{aligned}$$

Therefore

$$\min_{t\in[1/4,3/4]}(\lambda Sy)(t)\geq\gamma\|\lambda Sy\| \qquad \text{.....................}(2.16)$$

And (2.14) and (2.16) imply $\lambda Sy\in C$.

Next we want to show that $\lambda Sy\in C_M$. i.e., $\|\lambda Sy\|\leq M$.

So for this, using (2.8), lemma 1 and (2.13) successively, we find that

$$(\lambda Sy)(t)\leq\lambda\int_0^1 G(t,s)v(s)f(y(s))ds$$
$$\leq\frac{\lambda}{(n-1)!}\int_0^1(1-s)^{n-p-1}v(s)f(M)ds\leq M,\;\; t\in[0,1]$$

This implies

$$\|\lambda Sy\|\leq M \qquad .$$

Hence $(\lambda S)(C_M)\subseteq C_M$. This shows that for any bounded subset C_M of the cone C $(\lambda S)(C_M)$ also lies in C_M. Therefore any sequence $\{z_n\}$ taken from $(\lambda S)(C_M)$ is bounded and since C_M is closed subset of the Banach space B C_M is also a Banach space. This implies that $\{z_n\}$ has a convergent subsequence in C_M . Therefore $(\lambda S)(C_M)$ is sequentially compact subset of C_M. Hence λS is compact operator which maps each bounded set C_M into itself. So from shauder fixed point theorem we get that λS has fixed point in C_M and this fixed point is indeed a positive solution of the BVP (1), (2). This shows that λ is an eigenvalue of the BVP. i.e $\lambda\in E$. Since λ is arbitrary element of $(0,c]$, $(0,c]\subseteq E$.

Therefore

for $$\lambda\in\left(0\,,\;M\left[\frac{f(M)}{(n-1)!}\int_0^1(1-s)^{n-p-1}v(s)ds\right]^{-1}\right]$$

the BVP (1), (2) a positive solutions.

To get more specific results further conditions must be fulfilled. The following theorems will provide such conditions.

Theorem 3.3: suppose that

$$\lambda_0 \in E \textit{ then, for each } 0 \le \lambda \le \lambda_0,\ \lambda \in E$$

Thus E is an interval.

Theorem 3.4:

Let λ *be an eigenvalue of the BVP* $(1),(2)$ *and* $y \in C$ *be a corresponding eigenfunction.* *If* $y^{n-1}(0) = q$ *for some* $q > 0$, *then* λ *satisfies*

$$g(v)q\left[f\left(\frac{q}{(n-1)!}\right)\right]^{-1} \le \lambda \le g(u)q\left[f(0)\right]^{-1}, \tag{2.17}$$

where

$$g(z) = \left[\int_0^1 (1-s)^{n-p-1} z(s)\,ds\right]^{-1} \tag{2.18}$$

Proof: For $m \in \{1,2,\ldots\}$, *we define* $f_m = f * \psi_m$, *where* ψ_m Is a standard mollifier (10, 19) such that f_m is Lipschitz and converges uniformly to f.

For a fixed m, *let* λ_m *be an eigenvalue and* y_m, *with* $y_m^{n-1}(0) = q$, *be a corresponding eigenfunction of the BVP*

$$y_m^{n} + \lambda_m H_m(t, y_m) = \lambda_m K_m(t, y_m), t \in [0,1], \tag{2.19}$$

$$\left.\begin{aligned} y_m^{i}(0) &= 0, \quad 0 \le i \le n-2 \\ y_m^{p}(1) &= 0, \end{aligned}\right\} \tag{2.20}$$

where H_m *and* K_m *converge uniformly to* H *and* K *respectively, and from* (3.5) *and* (3.6) *we have,*

$$u_m(t) \le \frac{H_m(t,z) - K_m(t,z)}{f_m(z)} \le v_m(z) \qquad (2.21)$$

clearly y_m *is the unique solution of the IVP*

$$\left.\begin{aligned} y_m^i(0) = 0, \;\; 0 \le i \le n-2 \\ y_m^{n-1}(1) = q \end{aligned}\right\} \qquad (2.22)$$

Since

$$\begin{aligned} y_m^n(t) &= \lambda_m\left[K_m(t,y_m) - H_m(t,y_m)\right] \\ &\le -\lambda_m u_m(t) f_m(y_m(t)) \le 0, \end{aligned}$$

we have y_m^{n-1} *is nonincreasing and hence*

$$y_m^{n-1}(t) \le y_m^{n-1}(0) = q, \quad t \in [0,1] \qquad (2.23)$$

Noting that

$$y_m^i(t) = \int_0^t y_m^{i+1}(s)ds, \; 0 \le i \le n-2, \; t \in [0,1] \qquad (2.24)$$

$$(2.24) \Rightarrow y_m^{n-2}(t) = \int_0^t y_m^{n-1}(s)ds \le \int_0^t q ds = qt \;, t \in [0,1] \qquad (2.25)$$

applying (20) *and continuing Integrating, we will have*

$$y_m(t) \le q\frac{t^{n-1}}{(n-1)!} \le \frac{q}{(n-1)!}, \; t \in [0,1] \qquad (2.26)$$

Now, from (2.19),(2.21),and(2.26) we get for $t \in [0,1]$,

$$\lambda_m u_m(t) f_m(0) \le -y_m^{(n)}(t) \le \lambda_m v_m(t) f_m\left(\frac{q}{(n-1)!}\right) \qquad (2.27)$$

Integrating (2.27) from 0 to t provides

$$\theta_1(t) \le y_m^{(n-1)}(t) \le \theta_2(t), \; t \in [0,1] \qquad (2.28)$$

Where

$$\theta_1(t) = q - \lambda_m f_m\left(\frac{q}{(n-1)!}\right)\int_0^t v_m(s)ds$$

and

$$\theta_2(t) = q - \lambda_m f_m(0)\int_0^t u_m(s)ds$$

Continuing integration process , we get for $0 \le p \le n-1$,

$$\theta_3(t) \le y_m^{(p)}(t) \le \theta_4(t),\ t \in [0,1] \qquad (2.29)$$

Where

$$\theta_3(t) = \frac{q}{(n-p-1)!}t^{n-p-1} - \lambda_m f_m\left(\frac{q}{(n-1)!}\right)\int_0^t \frac{(t-s)^{n-p-1}}{(n-p-1)!}v_m(s)ds$$

and

$$\theta_4(t) = \frac{q}{(n-p-1)!}t^{n-p-1} - \lambda_m f_m(0)\int_0^t \frac{(t-s)^{n-p-1}}{(n-p-1)!}u_m(s)ds$$

In order to have $y_m^p(1) = 0$ (see (2.20)), from (2.29) it is nessary that

$\theta_3(1) \le 0$ *and* $\theta_4(1) \ge 0$, Or equivalently,

$$\lambda_m \ge g(v_m)q\left[f_m\left(\frac{q}{(n-1)!}\right)\right]^{-1} \qquad (2.30)$$

and

$$\lambda_m \le g(u_m)q\left[f_m(0)\right]^{-1} \qquad (2.31)$$

From (2.30) *and* (2.31) we get

$$g(v_m)q\left[f_m\left(\frac{q}{(n-1)!}\right)\right]^{-1} \le \lambda_m \le g(u_m)q\left[f_m(0)\right]^{-1} \qquad (2.32)$$

It follows from (2.28) that $\left\{y_m^{(n-1)}\right\}_{m=1}^{\infty}$ is a uniformly bounded sequence on $[0,1]$.Using the initial conditions (2.22) and repeated integration; we find that

$\left\{y_m^{(i)}\right\}_{m=1}^{\infty}, 0 \le i \le n-1$ is a uniformly bounded sequence. Thus there exists a subsequence, which can be relabeled as $\{y_m\}_{m=1}^{\infty}$, that converges uniformly (in fact, in $C^{(n-1)}$ norm) to some y on $[0,1]$. We note that each $y_m(t)$ can be expressed as

$$y_m(t) = \lambda_m \int_0^1 G(t,s)\left[H_m(s,y_m) - K_m(s,y_m)\right]ds \quad t \in [0,1] \tag{2.33}$$

Since $\{\lambda_m\}_{m=1}^{\infty}$ is a bounded sequence (from(2.32)), there is a subsequence, which can be relabeled as $\{\lambda_m\}_{m=1}^{\infty}$, that converges to some λ. Letting $m \to \infty$ in (2.33) yields

$$y(t) = \lambda \int_0^1 G(t,s)\left[H(s,y) - K(s,y)\right]ds, \quad t \in [0,1].$$

This means that y is an eigenfunction of the BVP corresponding to the eigenvalue λ. Further, $y^{(n-1)}(0) = q$, and (2.17) follows from (2.32) immediately.

Theorem 3.5:

Let λ be an eigen value of the BVP and $y \in C$ be a corresponding eigenfunction. Further, let

$\eta = \|y\|$ *and* $\rho = \max_{t\in[0,1]} \left|y^{n-2}(t)\right|$ *Then*

$$\lambda \geq \frac{\eta}{f(\eta)}(n-1)!\left[\int_0^1 (1-s)^{n-p-1} v(s)\,ds\right]^{-1} \tag{2.34}$$

and

$$\lambda \leq \frac{\eta}{f(\gamma\eta)}\left[\int_{\frac{3}{4}}^{\frac{1}{4}} G\left(\frac{1}{2},s\right)u(s)\,ds\right]^{-1} \tag{2.35}$$

also there exists a $c > 0$ such that

$$\lambda \leq \frac{\rho}{f(c\rho)}\frac{1}{(n-2)!}\left[\int_{\frac{1}{4}}^{\frac{1}{2}} G\left(\frac{1}{2},s\right)u(s)\,ds\right]^{-1} \tag{2.36}$$

Pr*oof: First we shall prove (2.34). For this, let $t_0 \in [0,1]$ be such that*

$$\eta = \|y\| = y(t_0)$$

Then, applying (2.8) and lemma 1 we find

$$\eta = y(t_0) = (\lambda S y)(t_0) \leq \lambda \int_0^1 G(t_0,s) v(s) f(y(s))\,ds$$

$$\leq \tfrac{\lambda}{(n-1)!}\int_0^1 (1-s)^{n-p-1} v(s) f(y(s))\,ds$$

$$\leq \tfrac{\lambda}{(n-1)!} f(\eta)\int_0^1 (1-s)^{n-p-1} v(s)\,ds$$

from which (2.34) follows.

Next, applying (2.8) *and the fact that* $\min_{t\in[1/4,3/4]} y(t) \geq \eta\gamma$, *we get*

$$\eta \geq y\left(\frac{1}{2}\right) \geq \lambda \int_0^1 G\left(\frac{1}{2},s\right)v(s) f(y(s))ds$$

$$\geq \lambda \int_{\frac{1}{4}}^{\frac{3}{4}} G\left(\frac{1}{2},s\right)v(s) f(y(s))ds$$

$$\geq \lambda f(\gamma\eta) \int_{\frac{1}{4}}^{\frac{3}{4}} G\left(\frac{1}{2},s\right)v(s)ds$$

which gives (2.35)

Finally, to prove (2.36) *we see the relation bellow*

$$y^{(i)}(t) = \int_0^t y^{(i+1)}(s)ds, \quad 0 \leq i \leq n-3, t \in [0,1] \tag{2.37}$$

and the nonnegativity of y *implies that* $y^{(n-2)}$ *is nonnegative on* $[0,1]$.

It is also observed that $y^{(n)}$ *is nonpositive and hence* $y^{(n-2)}$ *is concave on* $[0,1]$.

Thus, there exists a unique $t \in [0,1]$ *such that* $\rho = \max_{t\in[0,1]} y^{(n-2)}(t) = y^{(n-2)}(t_1)$.

Now two case will be considerd.

Case 1. $y^{(n-2)}(1) = 0$

Here, $y^{(n-2)}(1) = y^{(n-2)}(0) = 0$. *Thus, it follows from concavity of* $y^{(n-2)}$ *that*

$$y^{(n-2)}(t) \geq \begin{cases} \dfrac{\rho}{t_1}t, & t \in [0,t_1] \\ \dfrac{\rho}{1-t_1}(1-t), & t \in [t_1,1] \end{cases}$$

$$\geq \rho t(1-t), t \in [0,1] \tag{2.38}$$

applying (2.37) *and* (2.38), *we get*

$$y^{(n-3)}(t) = \int_0^t y^{(n-2)}(s)ds \geq \int_0^t \rho s(1-s)ds = \rho\left(\frac{t^2}{2},\frac{t^3}{3}\right), t \in [0,1]$$

continuing the Integration process, we find

$y(t) \geq \rho\psi(t), t \in [0,1]$

where $\psi(t) = \dfrac{t^{n-1}}{(n-1)!} - 2\dfrac{t^n}{n!}$

observe that $\psi'(t) = \dfrac{t^{n-2}}{(n-2)!}\left(1 - \dfrac{2t}{n-1}\right)$

is nonnegative for $t \in I \equiv \left[0, \dfrac{n-1}{2}\right]$*.Hence in particular*

$\psi(t)$ *is non decreasIng for* $t \in \left[\dfrac{1}{4}, \dfrac{1}{2}\right] \subseteq I$*.It follows from* (2.38) *that*

$$y(t) \geq c\rho, \ t \in \left[\frac{1}{4}, \frac{1}{2}\right] \tag{2.39}$$

where
$$c = \psi\left(\frac{1}{4}\right) = \frac{1}{4^{n-1}(n-1)!} - \frac{2}{4^n n!} > 0 \tag{2.40}$$

Now relation (2.37) provides

$$y^{(n-3)}(t) = \int_0^t y^{(n-2)}(s)ds \leq \int_0^t \rho ds = \rho t \quad t \in [0,1].$$

Using the above inequality and (2.37), we will have

$$y(t) \leq \rho\frac{t^{n-2}}{(n-2)!} \leq \frac{\rho}{(n-2)!}, \quad t \in [0,1] \tag{2.41}$$

In view of (2.41), (2.8) and (2.39), we find

$$\begin{aligned}\frac{\rho}{(n-2)!} \geq y\left(\frac{1}{2}\right) &\geq \lambda\int_0^1 G\left(\frac{1}{2}, s\right)u(s)f(y(s))ds \\ &\geq \lambda\int_{\frac{1}{4}}^{\frac{1}{2}} G\left(\frac{1}{2}, s\right)u(s)f(y(s))ds \\ &\geq \lambda f(c\rho)\int_{\frac{1}{4}}^{\frac{1}{2}} G\left(\frac{1}{2}, s\right)u(s)ds\end{aligned}$$

From which (2.36) follows.

Case 2. $y^{(n-2)}(1) > 0$

In this case, $y^{(n-2)}(0) = 0, y^{(n-2)}(1) \neq 0$ *hence by the concavity of* $y^{(n-2)}$,
we have $y^{(n-2)}(t) \geq y^{(n-2)}(1)t \geq y^{(n-2)}(1)t(1-t),\ t \in [0,1]$ (2.42)

Using similar technique to that of case1, it follows from (2.42) and successive integrations that

$$y(t) \geq y^{(n-2)}(1)\psi(t), \qquad t \in [0,1] \tag{2.43}$$

This leads to (2.39), where

$$c = \frac{y^{(n-2)}(1)}{\rho}\left[\frac{1}{4^{n-1}(n-1)!} - \frac{2}{4^n n!}\right] > 0 \tag{2.44}$$

The rest is similar to case 1.

This completes the proof of the theorem.

Theorem 3.6:

Let

$$F_B = \left\{ f : \frac{u}{f(u)} \text{ is bounded for } u \in [0,\infty) \right\},$$

$$F_0 = \left\{ f : \lim_{u\to\infty} \frac{u}{f(u)} = 0 \right\}, \quad F_\infty = \left\{ f : \lim_{u\to\infty} \frac{u}{f(u)} = \infty \right\}$$

(a) *If* $f \in F_B$, *then* $E = (0,c)$ *or* $(0,c]$ *for some* $c \in (0,\infty)$.

(b) *If* $f \in F_0$, *then* $E = (0,c]$ *for some* $c \in (0,\infty)$.

(c) *If* $f \in F_\infty$, *then* $E = (0,\infty)$.

Proof: (a) this is immediate from (2.35) and (2.36)

(b) since $F_0 \subseteq F_B$, it follows from case (a) that

$$E = (0,c) \text{ or } (o,c] \text{ for some } c \in (0,\infty).$$

In particular, $c = \sup E$

Let $\{\lambda_m\}_{m=1}^{\infty}$ be a monotonically increasing sequence in E which converges to c, and let $\{y_m\}_{m=1}^{\infty}$ in C be a corresponding sequence of eigenfunctions. Further, let $\eta_m = \|y_m\|$. Then, (2.35) implies that no subsequence of $\{\eta_m\}_{m=1}^{\infty}$ can diverge to infinity.

Thus, there exists $M > 0$ *such that* $\eta_m \le M$ *for all m.* So y_m is uniformly bounded.Hence, there is a subsequence of $\{y_m\}_{m=1}^{\infty}$, relablled as the original sequence , which converges uniformly to some $y \in C$. Noting that $\lambda_m S y_m = y_m$,

We have

$$cSy_m = \frac{c}{\lambda_m} y_m. \qquad (2.45)$$

Since $\{cSy_m\}_{m=1}^{\infty}$ is relatively compact, y_m converges to y and λ_m converges to c, letting $m \to \infty$ in (2.45) gives $cSy = y$, that is, $c \in E$ .this completes the proof for (b).

(c) Follows from the fact that E is an interval and (2.34).

Example.1. Consider the boundary value problem

$$y^{(4)} + \lambda \frac{1}{\left(5+2t^3-t^4\right)^r}\left(12y+5\right)^r = 0 \qquad t \in (0,1)$$

$$y(0) = y'(0) = y''(0) = y^{(p)}(1) = 0$$

Where $0 \le p \le 3$ but fixed, $\lambda > 0$ $and\, r \ge 0$

Solution:

Now if we take $f(y) = (12y+5)^r$, then

$$\frac{H(t,y)}{f(y)} = \frac{1}{\left(5+2t^3-t^4\right)^r} \quad and \quad \frac{K(t,y)}{f(y)} = 0$$

Hence, we may take

$$h_1(t) = \frac{2}{\left(5+2t^3-t^4\right)^r}, \qquad h(t) = \left(\frac{1}{2\left(5+2t^3-t^4\right)^r}\right)$$

And $k_1(t) = k(t) = 0$

$\Rightarrow$ All the hypotheses (i.e. (H-1)-(H-4))are satisfied.

Now regarding r three cases will be considered

Case 1: For $0 \le r < 1$

$f \in F_\infty$ by theorem 3.6 (c) the $E = (0,\infty)$

For example when $p = \lambda = 2$ and $r = 0$ the BVP has a positive solution given by

$$y(t) = t^3 \frac{(2-t)}{12}$$

Case 2: For $r = 1$

Since $f \in F_B$ by theorem 3.6(a) the set E is an open or a half closed interval.

Further, if $p = 2, E$ contains the interval$(0,2]$.

Case 3: For $r > 1$

Since $f_0 \in F_0$, by theorem 3.6(b) the set E is a half closed interval.

Once again if $p=2$, then $(0,2] \subseteq E$.

Example 2: Consider the BVP

$$y'' + \lambda \frac{\sin \pi t}{8 + 5 \sin \pi t}(5y+8)^r = 0, \quad t \in (0,1)$$
$$y(0) = y^{(p)}(1) = 0$$

Where $0 \le p \le 1$ but fixed $\lambda > 0$ *and* $r \ge 0$,

If we choose $f(y) = (5y+8)^r$,then we can have

$$h_1(t) = \frac{3\sin \pi t}{(8+5\sin \pi t)^r}, \; h(t) = \frac{\sin \pi t}{4(8+5\sin \pi t)^r}$$

$$\Rightarrow k(t) = k_1(t) = 0$$

And all the hypotheses are satisfied ((H-1)-(H-4)).

And we note that if $p=0$, $\lambda = \pi^2$ and $r=0$, and then the BVP will have a positive solution given by

$$y(t) = \sin \pi.$$

4. Special case: When $\lambda = 1$

Definition: A function f is said to be sublinear if

$$f_0 = \lim_{u \to 0^+} \frac{f(u)}{u} = \infty \quad \text{and} \quad f_\infty = \lim_{u \to \infty} \frac{f(u)}{u} = 0.$$

and superlinear if $f_0 = 0$ and $f_\infty = \infty$.

Theorem 4.1: suppose that f is either superlinear or sublinear. Then the boundary value problem has a positive solution.

Proof: to obtain a positive solution of the BVP, we shall seek a fixed point of the operator S (defined earlier) in the cone C .we have seen that S is compact on the cone C .further, we observe from the proof of theorem (2.2) that S maps C in to itself. Also, the standard arguments yield that S is completely continuous.

Case 1: suppose that f is superlinear.Since $f_0 = 0$, we may choose $\in, \delta > 0$ such that

$$f(u) \leq \in u, \qquad 0 < u \leq \delta \qquad (3.1)$$

And

$$\tfrac{\in}{(n-1)!} \int_0^1 (1-s)^{n-p-1} v(s) ds \leq 1 \qquad (3.2)$$

Let $y \in C$ such that $\|y\| = \delta$. Then, applying (2.8), (3.1), lemma 1 and (3.2) successively, we find for $t \in [0,1]$,

$$\begin{aligned} Sy(t) &\leq \int_0^1 G(t,s) v(s) f(y(s)) ds \\ &\leq \in \int_0^1 G(t,s) v(s) y(s) ds \\ &\leq \frac{\in}{(n-1)!} \int_0^1 (1-s)^{n-p-1} v(s) y(s) ds \end{aligned}$$

$$\leq \frac{\in}{(n-1)!}\int_0^1 (1-s)^{n-p-1} v(s)\|y\| ds \leq \|y\|$$

Hence

$$\|Sy\| \leq \|y\| \qquad (3.3)$$

If we set $\Omega_1 = \{y \in B : \|y\| < \delta\}$, then (3.3) holds for $y \in C \cap \partial\Omega_1$.

Next, since $f_\infty = \infty$, we may choose $M, N > 0$ such that

$$f(u) \geq Mu, \quad u \geq N \qquad (3.4)$$

and

$$M\gamma \int_{\frac{1}{4}}^{\frac{3}{4}} G\left(\frac{1}{2}, s\right) u(s)\, ds \geq 1 \qquad (3.5)$$

Let $y \in C$ be such that $\|y\| = N_1 \equiv \max\left\{2\delta, \frac{N}{\gamma}\right\}$. Thus for $t \in \left[\frac{1}{4}, \frac{3}{4}\right]$,

$$y(t) \geq \gamma\|y\| \geq \gamma\frac{N}{\gamma} = N,$$

This in view of (3.4) leads to

$$f(y(t)) \geq My(t), \quad t \in \left[\frac{1}{4}, \frac{3}{4}\right] \qquad (3.6)$$

Using (2.8), (3.6) and (3.5), we find

$$\begin{aligned} Sy\left(\frac{1}{2}\right) &\geq \int_0^1 G\left(\frac{1}{2}, s\right) u(s) f(y(s))\, ds \\ &\geq \int_{\frac{1}{4}}^{\frac{3}{4}} G\left(\frac{1}{2}, s\right) u(s) f(y(s))\, ds \\ &\geq M\int_{\frac{1}{4}}^{\frac{3}{4}} G\left(\frac{1}{2}, s\right) u(s) y(s)\, ds \\ &\geq M\int_{\frac{1}{4}}^{\frac{3}{4}} G\left(\frac{1}{2}, s\right) u(s)\gamma\|y\|\, ds \geq \|y\| \end{aligned}$$

Therefore

$$\|Sy\| \geq \|y\| \qquad (3.9)$$

If we set $\Omega_2 = \{y \in B : \|y\| < N_1\}$, then (3.9) holds for $y \in C \cap \partial\Omega_2$.

In view of (3.3) and (3.9), it follows from **theorem1** that S has a fixed point $y \in C \cap (\overline{\Omega_2} \setminus \Omega_1)$, such that $\delta \leq \|y\| \leq N_1$. This y is a positive solution of the BVP (1), (2).

Case 2: suppose that f is sublinear. Since $f_0 = \infty$, then by definition of limits there exist $L, \xi > 0$ such that

$$f(u) \geq Lu, \qquad 0 < u \leq \xi \qquad (3.10)$$

And

$$L\gamma \int_0^1 G\left(\frac{1}{2}, s\right) u(s) ds \geq 1 \qquad (3.11)$$

Let $y \in C$ be such that $\|y\| = \xi$. On using (2.8),(3.10) and (3.11) successively, we get

$$\begin{aligned} Sy\left(\frac{1}{2}\right) &\geq \int_0^1 G\left(\frac{1}{2}, s\right) u(s) f(y(s)) ds \\ &\geq L \int_0^1 G\left(\frac{1}{2}, s\right) u(s) y(s) ds \\ &\geq L \int_{\frac{1}{4}}^{\frac{3}{4}} G\left(\frac{1}{2}, s\right) u(s) y(s) ds \\ &\geq L \int_{\frac{1}{4}}^{\frac{3}{4}} G\left(\frac{1}{2}, s\right) u(s) \gamma \|y\| ds \geq \|y\| \end{aligned}$$

From which (3.9) follows immediately. If we set $\Omega_1 = \{y \in B : \|y\| < \xi\}$, then (3.9) holds for $y \in C \cap \partial\Omega_1$.

Next, in view of $f_\infty = 0$, we may choose $J, \theta > 0$ such that

$$f(u) \leq \theta u, \quad u \geq J \qquad (3.12)$$

and $\frac{\theta}{(n-1)!}\int_0^1 (1-s)^{n-p-1} v(s)\,ds \le 1$ (3.13)

Let $J_1 = \max\{2\xi, J\}$.

Since f is nondecreasing

$$\Rightarrow \quad f(u) \le f(J_1) \text{ for } 0 < u \le J_1$$

In view of (3.12), this implies that

$$f(u) \le \theta J_1, \qquad 0 < u \le J_1 \tag{3.14}$$

Let $y \in C$ be such that $\|y\| = J_1$. Then it follows from (3.14) that

$$f(y(t)) \le \theta J_1, \quad t \in [0,1] \tag{3.15}$$

On using (2.8), (3.15), Lemma1 and (3.13) successively, we get for $t \in [0,1]$ that

$$\begin{aligned} Sy(t) &\le \int_0^1 G(t,s)v(s)f(y(s))\,ds \\ &\le \theta J_1 \int_0^1 G(t,s)v(s)\,ds \\ &\le \frac{\theta J_1}{(n-1)!}\int_0^1 (1-s)^{n-p-1} v(s)\,ds \\ &\le J_1 = \|y\| \end{aligned}$$

From which (3.3) follows immediately.

If we set $\Omega_2 = \{y \in B : \|y\| < J_1\}$, then (3.3) holds for $y \in C \cap \partial\Omega_2$.

Now we have obtained (3.9) and (3.3), it follows from **theorem 1** that S has a fixed point $y \in C \cap (\overline{\Omega}_2 \setminus \Omega_1)$, such that $\xi \le \|y\| \le J_1$. This y is a positive solution of the BVP (1), (2).

Example1: consider the boundary value problem

$$y^3 + \frac{\pi^3 \sin \pi t}{(5-4\cos \pi t)^r}(4y+1)^r = 0, \quad t \in (0,1)$$
$$y(0) = y'(0) = y^{(p)}(1) = 0$$

Where $0 \le p \le 2$ but fixed and $0 \le r < 1$.

Taking $f(y) = (4y+1)^r$, then $f_0 = \lim_{y \to 0^+} \frac{f(y)}{y} = \lim_{y \to 0^+} \frac{(4y+1)^r}{y} = \infty$

and $f_\infty = \lim_{y \to \infty} \frac{f(y)}{y} = \lim_{y \to \infty} \frac{(4y+1)^r}{y} = 0$, which implies f is sublinear. We find that

$$\frac{H(t,y)}{f(y)} = \frac{\pi^3 \sin \pi t}{(5-4\cos \pi t)^r} \quad and \quad \frac{K(t,y)}{f(y)} = 0$$

Hence we may choose

$$h_1(t) = \frac{\pi^4 \sin \pi t}{(5-4\cos \pi t)^r}, \quad h(t) = \frac{\pi^2 \sin \pi t}{(5-4\cos \pi t)^r}$$

And $k(t) = k_1(t) = 0$.

All the conditions of theorem 4.1 are fulfilled and therefore the boundary value problem has a positive solution. We note that when $p = 1$ and $r = 0$ the BVP will reduce to:

$$y^3 + \pi^3 \sin \pi t = 0 \quad t \in (0,1)$$
$$y(0) = y'(0) = y'(1) = 0.$$

And this BVP has a unique positive solution $y(t) = 1 - \cos \pi t$.

Example 2: consider the boundary value problem

$$y'' + \frac{1}{\left(3 + 2t + -t^2\right)^r}(2y+3)^r = 0, \qquad t \in (0,1)$$

$$y(0) = y^{(p)}(1) = 0$$

Where $0 \le p \le 1$, but fixed. And $0 \le r < 1$.

Choosing $f(y) = (2y+3)^r$ (which is sublinear), we may take

$$h_1(t) = \frac{5}{\left(3+2t-t^2\right)^r}, \quad h(t) = \frac{1}{3\left(3+2t-t^2\right)^r}$$

And $k(t) = k_1(t) = 0$.

Once again, all the conditions of theorem 4.1 are satisfied and so the boundary value problem has a positive solution. Indeed, when $p = 1$, one such solution is given by

$$y(t) = \frac{t(2-t)}{2}.$$

REFERENCE

1. Recent trends in differential equations, volume 1, world scientific series in applicable analysis. (1992).
2. Eberhard Zeidler, nonlinear functional analysis and its applications-1, fixed-point theorems, (1986).
3. Angus E. Taylor and David C.Wiley and publishing Co. (1980).
4. Richard R. Goldberg, methods of real analysis, (1964).
5. H.L.Royden, Real analysis, second edition.