INTRODUCTION

The primary purpose of differential equation is to serve as a tool for the study of change in physical world as nothing is permanent except change. Theory of stability in linear system is a fundamental component for the next concept and stability in nonlinear system focuses on techniques to determine the equilibrium solution or stability of constant and to describe the behavior of nonlinear equation near to its equilibrium point .The first step when studying a nonlinear system is to linearize the system. Nonlinear differential equation may have several fixed points, all, none or some of which may be stable.

Continuous dynamical systems that involve differential equations mostly contain parameters. It can happen that a slight variation in a parameter can have significant impact on the solution. The main questions of interest in this paper are:

- How to compute stability boundaries of equilibrium and limit cycles in the parameter space?
- How to predict qualitative changes in system's behavior (bifurcations) occurring at these equilibrium points?

Bifurcation theory is the mathematical study of changes in the qualitative or topological structure of a given family, such as the integral curves of a family of vector fields, and the solutions of a family of differential equations. Most commonly applied to the mathematical study of dynamical systems, a bifurcation occurs when a small smooth change made to the parameter values (the bifurcation parameters) of a system causes a sudden 'qualitative' or topological change in its behavior. Bifurcations occur in both continuous systems (described by ODEs, or PDEs), and discrete systems (described by maps).The name bifurcation was first introduced by Hernia Poincare in 1885.

A wide variety of systems are modeled by ordinary differential equations depending upon second and higher order derivatives but there is an easy device that we will reduce any higher order ordinary differential equation to an equivalent first order system. The upshot is that for all practical purpose we need to analyze first order system.

CHAPTER 1: FIRST ORDER SYSTEMS

[1]A first order system of ordinary differential equation has the general form

$$\begin{cases} \dfrac{dx_1}{dt} = f_1\,(t, x_1, x_2, \dots, x_n) \\[2ex] \dfrac{dx_2}{dt} = f_2\,(t, x_1, x_2, \dots, x_n) \\ \quad\vdots \\ \dfrac{dx_n}{dt} = f_n\,(t, x_1, x_2, \dots, x_n) \end{cases}$$

where $x_1, x_2, \dots, x_n$ are scalar functions of an independent variable t or we can write it in a compact form as x (t) = f(t, x).

where $x = (x_1, x_2, \dots, x_n)^T$ and $f(t, x) = f_1(t, x_1, x_2 \dots, x_n), \dots, f_n(t, x_1, x_2 \dots, x_n)$ are a vector function of (n+1) variables and we assume that it is continuously differentiable. By a solution to the differential equation we mean a vector valued function x (t) that is defined and continuously differentiable on an interval [a,b]. Moreover, it satisfies the differential equation on its interval of definitions. That is as t varies each solution x (t) serves to parameterize a curve $C \subseteq R^n$, also known as a trajectory or orbit of the system. In this context the word trajectory, orbit and space curve are used to refer to solution of a differential equation.

1.1 Linear systems

Linear systems of equations constitute an important class of systems. This is because linear systems often arise in applications because a thorough knowledge of linear systems required for the study of nonlinear systems.

The ordinary differential equation $\dot{x}(t)$= f (t, x) is called linear if f has of the form f(t,x)=Ax +B(t) where A is n x n matrix and B(t) is a continuous vector valued function. If A is constant matrix and B (t) = 0, we call it to be a linear homogenous system with constant coefficients. The general solution of the homogeneous system is a linear combination of the basis of the vector space. That is x (t) = $c_1v_1 + c_2v_2 + \cdots + c_nv_n$ where $c_1, c_{2,\ldots,} c_n$ are arbitrary constants.

Key points

- Linear systems satisfy the properties of superposition and homogeneity. Any system that does not satisfy these properties is nonlinear.
- Linear systems have one equilibrium point at the origin. Nonlinear systems may have many equilibrium points.
- Stability needs to be precisely defined for nonlinear systems.

1.2 Stability of solutions of Linear System

[2] **Definition**: Stability means that the trajectories do not change too much under small perturbations. When a body is in stable equilibrium, even after disturbed, it will come to its original state and when it is in unstable equilibrium, once disturbed, it gets permanently disturbed and cannot come to its original state.

The system of differential equation $\dot{x}$(t) = Ax has a solution x (t) = 0. This solution is represented by the origin of the coordinates in the solution space and the origin is called an equilibrium point. Since $\dot{x}$= 0 when x = 0 If the determinant of the matrix A, det (A) ≠ 0, then the origin is the only equilibrium point.

Definition: Equilibrium point is a point on a dynamical system at which there is no dynamics.

The phase portrait for homogeneous system

$$\begin{cases} \dot{x} = a\,x + by \\ \dot{y} = cx + dy \end{cases} \quad , ad - bc \neq 0,\ a, b, c, d \in \mathbb{R}.$$

The characteristic polynomial $\lambda^2 - (a+d)\lambda +$ (ad-bc) =0

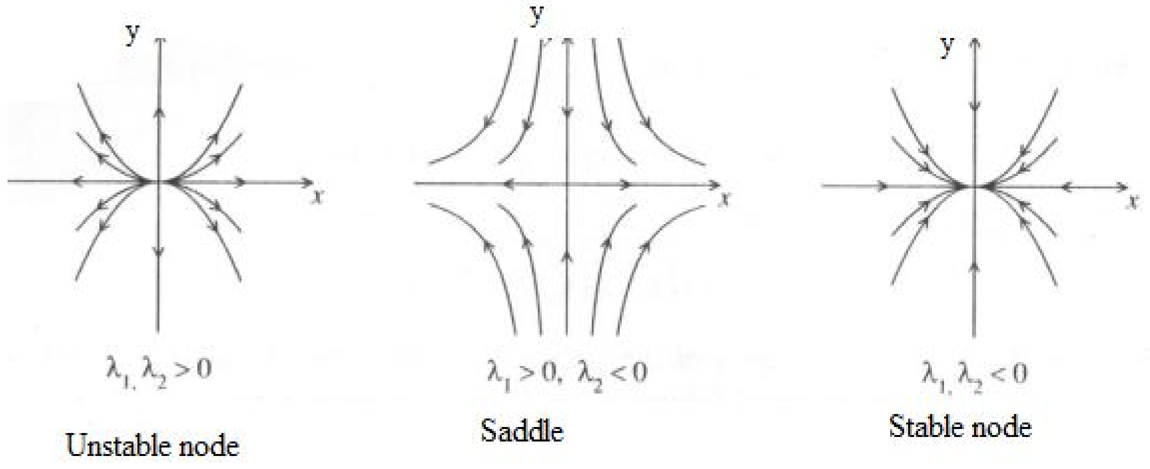
Root of characteristic equation	Equilibrium point
1. Real, different, negative	stable node
2. Real, different, positive	un Stable node
3. Real, opposite sign	saddle (unstable)
4. Complex, negative real part	stable spiral
5. Complex, positive real part	un stable spiral
6. Complex, zero real part	center (neutrally stable)
7. Real, equal, negative	stable (improper node)
8. Real, equal, positive	unstable (improper node)

(See figure 1.1 and 1.2)

Definition: The phase portrait of a linear system is stable if the real parts of all roots of the characteristic equations are negative and unstable if the real part of all roots of the characteristic equations of at least one root is positive. That is the origin is said to be a stable equilibrium point for the system $\dot{x}(t) = Ax$ if $x(t) \to 0$ as $t \to \infty$ for all initial points or complex numbers with real part negative.

$x(t) = e^{\alpha t}(p \cos \beta t + q \sin \beta t)$, where $p = c_1 + c_2$ and $q = i(c_1 - c_2)$

If $\alpha < 0$, $\lim_{t \to \infty} x(t) = 0$ and $\alpha > 0$ $\lim_{t \to \infty} x(t) = \infty$, a saddle point is always an unstable steady state



a b c

Figure 1.1 Saddle, stable and unstable node

If the eigenvalues of a matrix A are complex, i.e $\lambda = \alpha \pm \beta$,

The general solution is $x(t) = e^{\alpha t}\left(p \cos \beta t + q \sin \beta t\right)$

If $\alpha < 0$, then the $\lim_{t\to\infty} x(t) = 0$ hence x (t) =0 is a stable focus

If $\alpha > 0$ then the $\lim_{t\to\infty} x(t) = \infty$, hence x (t) =0 is an unstable

If $\alpha = 0$ then $x(t) = x(t) = p\cos(\beta t) + q\sin(\beta t)$, where $p = c_1 - c_2$ and $q = i\,(c_1 - c_2)$ hence the orbits are ellipses. Consider $x(t) = e^{\alpha t}\left(p \cos \beta t + q \sin \beta t\right)$

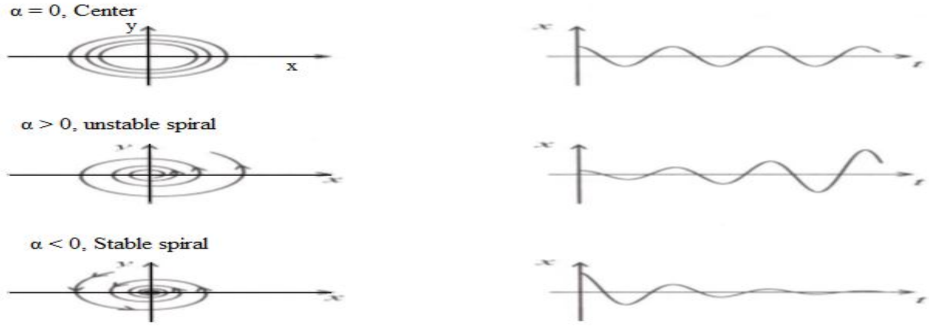


Figure 1.2 center, stable spiral and unstable spiral

Repeated root case: Suppose the characteristic polynomial $\lambda^2-(a+d)\lambda+$ (ad-bc) has a repeated root $\lambda\neq 0$. This occurs if $(a + d)^2 - 4(ad - bc) = 0$ in which the root is $\lambda = \frac{a+d}{2}$. If $b=c=0$and $a=d$ the repeated root $\lambda=a$ all orbits associated with the general solution

$x(t) = c_1e^{\lambda t}+c_2te^{\lambda t}$, then the orbit tend to origin as $t\to\infty$ if $\lambda<0$ and tend away from origin as $t\to\infty$ if $\lambda>0$.The phase plane is called a stable star point and unstable star point respectively.

Proposition: Let$\delta=\det A$ and $\gamma=$ trace A and consider the linear system $\dot{x}=Ax$ if $\delta>0$ and $\gamma^2-4\delta\geq 0$ then the linear system has a node at the origin; it is stable if $\gamma<0$ and unstable if $\gamma>0$.

Consider homogenous linear second order differential equation $4\ddot{x}-4\dot{x}+x=0$.

The characteristic equation is $4\lambda^2-4\lambda+1=0$

The characteristic root is $\lambda=\frac{1}{2}$ (Repeated root)

The fundamental system of solution is $\left\{e^{t/2},te^{t/2}\right\}$.

The general solution is $x(t) = c_1e^{t/2}+c_2te^{t/2}$ which tends to ∞ as $t\to\infty$

(Since the exponential terms tend to 0 faster than polynomials as $t\to\infty$)

❖ Consider $\begin{cases}\dot{x}=-x+y\\ y'=-y\end{cases}$

The characteristic polynomial is $(\lambda+1)^2$ with repeated eigenvalue $\lambda=-1$

The general solution $x(t)=(c_1+c_2t)e^{-t}$. For $c_2=0$ we have the solution pairs

$x(t)=c_1e^{-t}$and $Y(t)=0$ whose orbits are half line lying on x-axis. These orbits approach the origin as $t\to\infty$. If we choose $c_2\neq 0$ we find that the ratio $\frac{y}{x}=\frac{c_2}{c_1+tc_2}$ tends to zero as $t\to\infty$.

This means that all other orbits approach the origin tangentially to x-axis. The resulting portrait is a stable improper node.

1.3 Nonlinear system

Any function that does not satisfy superposition and homogeneity is nonlinear. We saw that any linear system $\dot{x}$=Ax has a unique solution through each point x_0 in the plane space $\mathbb{R}^n$ .The solution is given by $x(t) = x_0 e^{At}$ and is defined for all $f \in \mathbb{R}$.In this section we begin our study on nonlinear systems of differential equation $\dot{x} = f(x)$ where f: E × $\mathbb{R}^n$ → $\mathbb{R}^n$ and E is open subset of $\mathbb{R}^n$.

We show that under certain conditions on the function f the non linear system $\dot{x} = f(x)$ has a unique solution through each point $x_0 \in$ E defined on maximal interval of existence $(\alpha, \beta) \subset \mathbb{R}^n$ .We establish the Hartman-Grobman theorem and the stable manifold theorem which show that topologically the local behavior of the nonlinear system $\dot{x}$=f(x) near an equilibrium point x_0 where $f(x_0)$ is typically determined by the behavior of the linear system $\dot{x}$=Ax near the origin when the matrix A=D $f(x_0)$ the derivative of f at x_0.

1.4 The fundamental existence –uniqueness theorem

[4] We establish the fundamental existence –uniqueness theorem for a nonlinear system of ordinary differential equation $\dot{x} = f(x)$ under the hypothesis that $f \in C^1$ (E) where E is open subset of $\in IR^n$.

THEOREM: Let E be open subset of $\in IR^n$ containing x_0 and assume that $f \in C^1$ (E) then there exists an $a > 0$ such that the initial value problem $\dot{x} = f(x)$

$$x(0) = x_0$$

has a unique solution x(t) on the interval $[-a, a]$.

PROOF: Since $f \in C^1$ (E), and there is an ϵ_neighborhood N $\epsilon(x_0) \subset E$ and a constant $k > 0$ such that for all x , $y \in N\epsilon(x_0)$, $|f(x) - f(y)| \leq k|x - y|$.

Let $b = \frac{\epsilon}{2}$ then the continuous function f(x) is bounded on the compact set $N_0 = \{x \in \mathbb{R} : |x - x_0| \leq b\}$

Let M = $\max_{x \in N_0} |f(x)|$. Let the successive approximation $u_k(t)$ be defined $u_0(t) = x_0$, $u_{k+1}(t) = x_0 + \int_0^t f(u_k(s))ds$ for k=0,1,2…then assuming that there exists an $a > 0$ such that) $u_k(t)$ is defined and continuous on $[-a, a]$ and satisfies $\max_{[-a,a]}|u_k(t) - x| \leq b$ it follows that $f(u_k(t))$ is defined and continuous on $[-a, a]$ and satisfies $|u_{k+1}(t) + x_0| \leq \int_0^t |f(k(s))| ds \leq$ Ma for all $t \in [-a, a]$.Thus choosing , $0 < a < b/M$ it follow by induction that $u_k(t)$ is defined and continuous and satisfies

$\max_{[-a,a]}|u_k(t) - x_0| \leq b$ for all $t \in [-a, a]$.

Next since for all $t \in [-a, a]$ and k=, 1, 2, 3… $u_k(t) \in N0$ b for all $t \in [-a, a]$ it follows from Lipchitz condition satisfied by f that for all $t \in [-a, a]$

$$|u_2(t) - u_1(t)| \leq \int_0^t | f(u_2(s)) - f(u_1(s))| \, ds$$

$$\leq k \int_0^t |(u_2(s)) - (u_1(s))| \, ds$$

$$\leq k \quad \max_{[-a,a]}|(u_1(s)) - x_0|$$

$$\leq kab$$

And assuming that $\max_{[-a,a]}|u_j(t) - u_{j-1}(t)| \leq (ka)^{j-1} b$ for some integer $j \geq 2$

It follows that for all $t \in [-a, a], |u_{j+1}(t) - u_j(t)| \leq \int_0^t |f(u_j(s)) - f(u_{j-1}(s))| \, ds$

$$\leq k \int_0^t |(u_j(s))) - (u_{j-1}(s)| \, ds$$

$$\leq ka \max_{[-a,a]} |u_j(t) - u_{j-1}(t)|$$

$$\leq (ka)^j b$$

Thus it follows by induction that $\max_{[-a,a]}|u_j(t) - u_{j-1}(t)| \leq (ka)^{j-1} b$ holds for j=2, 3,…

Setting α = ka and choosing, $0 < a < {}^{b}/_{k}$ we see that

For $m \geq k \geq N$ and $t \in [-a, a]$

$$|u_m(t) - u_k(t)| \leq \sum_{j=k}^{m-1} |u_{j+1}(t) - u_j(t)|$$

$$\leq \sum_{j=N}^{\infty} |u_{j+1}(t) - u_j(t)|$$

$$\leq \sum_{j=N}^{\infty} \alpha^J b = {\frac{\alpha}{1-\alpha^{b}}}^{N}$$

Thus the last quantity approaches zero as $N \to \infty$. Therefore for all $\varepsilon > 0$ there exist N such that

m.k $> N \Rightarrow \|u_m - u_k\| = \max_{[-a,a]}|u_m(t) - u_k(t)| < \varepsilon$

That is $\{u_k\}$ is a Cauchy sequence of continuous function in C ($[-a, a]$)

$u_k(t)$ Converges to a continuous function u (t) uniformly for all $t \in [-a, a]$ as $k \to \infty$ Taking both sides

$\lim_{k\to\infty} u_{k+1}(t) = \lim_{k\to\infty} x_0 + \int_0^t f(u_k(s)ds \Rightarrow$ u (t) = $\lim_{k\to\infty} u_k(t)$

satisfies the integral equation u(t)= $x_0 + \int_0^t f(u_k(s))ds$ for all t $\in[-a, a]$ we have used the fact that the integral and the limit can be interchanged since the limit is u(t)= $\lim_{k\to\infty} u_k(t)$ is uniform for all t $\in[-a, a]$ since u(t) is continuous ,f(u(t)) is continuous and differentiable and

$\dot{u}(t)$= f(u(t)) for all t $\in[-a, a]$ furthermore $u(0)=x_0$ and from

$\max_{[-a,a]}|u_k(t) - x_0|$ Thus u (t) is the solution of the initial value problem) $\dot{x} = f(x)$

x (0) = x_0 on$[-a, a]$.

It remains to show that it is the only solution.

Let u (t) and v (t) be two solutions of the initial value problem $\dot{x}$=f(x)

x (0) =x_0 on$[-a, a]$. Then the

Continuous function $|u(t) - v(t)|$ achieves its maximum at some point $t_1 \in[-a, a]$

It follows $||u-v||= \max_{[-a,a]}|u(t) - v(t)|$

$$=\left|\int_0^t f(u(s)) - f(v(s))ds\right|$$

$$\le \int_0^{t_1}|f(u(s)) - f(v(s))|ds$$

$$\le k\int_0^{t_1}|u(s) - v(s)|ds$$

$$\le ka \max_{[-a,a]}|u(t) - v(t)|$$

$$\le ka \; ||u-v||$$

But ka $<$1 this last inequality can only be satisfied if $||u-v||$=0 Thus (t) =v (t) on$[-\alpha, a]$.

1.5 LINEARIZATION TECHNIQUES

Linearization is the process of replacing the nonlinear system model by its linear counterpart in a small region about its equilibrium point. A good place to start analyzing the non linear system $\dot{x}$ = f(x) is to determine the equilibrium point of $\dot{x}$ = f(x) and to describe the behavior of the system near its equilibrium point.

Definition: A point $x_o \in IR^n$ is called an equilibrium point or critical point of $\dot{x}$= f(x) if $\dot{x}$ = f (x_0) =0.

Definition: (Hyperbolic point): The equilibrium is said to be hyperbolic if all eigenvalues of the jacobian matrix have non-zero real parts.

Definition ;(Non-Hyperbolic point): If at least one eigenvalue of the Jacobian matrix is zero or has a zero real part, then the equilibrium is said to be non-hyperbolic. The linear system $\dot{x}$ = Ax with matrix A = D f (x_o) is called linearization of f(x) at x_o.

Consider the following function on a single variable; The Taylor series expansion of f(x) about the equilibrium point x_0 is given by f(x)≅ f(x_0) + $\frac{D^2 f(x_0)(x-x_0)^2}{2}$ +…

If x_0 = 0 is an equilibrium point of $\dot{x}$ = f(x) then f (0) =0 and the

Taylor's theorem f(x) ≅ D f (x_o) x+ $\frac{D^2 f(x_0)(x-x_0)^2}{2}$ + …

It follows that the linear function D f (x_o) is a good first approximation to the non linear function f(x) near x_o = 0 and it is reasonable to expect that the behavior of nonlinear system $\dot{x}$= f(x) near the point x_o = 0 will be approximated by the behavior of its linearization at x_o =0.

Thus if f is a differentiable function, the derivative Df(x_o) is given by the n x n Jacobean matrix

$$D\,f(x) = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} & \cdots & \frac{\partial f_1}{\partial x_n} \\ \vdots & & \ddots & \vdots \\ \frac{\partial f_m}{\partial x_1} & \frac{\partial f_m}{\partial x_2} & \cdots & \frac{\partial f_m}{\partial x_n} \end{bmatrix}$$

THEOREM: (Hartman-Grobman theorem)

If x_0 is a hyperbolic equilibrium point of the n^{th} order autonomous system $\dot{x}$ = f(x) then there is a homeomorphism from $\mathbb{R}^n$ to $\mathbb{R}^n$ defined in neighborhood of the maps trajectories of a nonlinear system to trajectories of the local linear system . In other words the linrarized system has the same phase portrait as the original system in sufficiently small neighborhood of the hyperbolic fixed point of the nonlinear system. In dynamical systems, only the solutions of linear systems may be found explicitly. The problem is that in general real life problems may only be modeled by nonlinear systems.

The main idea is to approximate a nonlinear system by a linear one (around the equilibrium point). Of course, we do hope that the behavior of the solutions of the linear system will be the same as the nonlinear one. But this is not always true

NOTE: Let E be an open sub set of $\mathcal{R}^2$ containing the origin and let f $\in \mathbf{c}^2$(E). suppose that the origin is a hyperbolic critical point of $\dot{x}$ = f(x) then the origin is a stable or unstable node for the nonlinear system $\dot{x}$ = f(x) if and only if it is stable or unstable for the linear system $\dot{x} = Ax$ with A=Df (x_0) and the origin is stable or unstable focus for the nonlinear system $\dot{x}$ = f(x) if and only if it is stable or unstable for the linear system $\dot{x} = Ax$ with A=D f (x_0).

The origin is a proper node for a nonlinear system if and only if it is a proper node for linear system $\dot{x} = Ax$ with A=D f (x_0).

NOTE Let E be an open sub set of $\mathcal{R}^2$ containing the origin and f be analytic in E with f (0) =0 suppose that the origin is a center for the linear system $\dot{x} = Ax$ with A=D f(x_0) then the origin is either center or focus for the nonlinear system $\dot{x}$ = f(x) .

Remember that in nonlinear systems, stability, about an equilibrium point:

- Is dependent on initial conditions.
- Local vs. global stability is important.
- Possibility of limit cycles.

A center for a linear system can be either remains a center or change to stable or unstable focus with the addition of nonlinear terms $\dot{x} = Ax$ with A=D f (x_0).

Consider the nonlinear differential equation $\ddot{x} + x - 0.25x^2 = 0$

Evaluate the equilibrium point and determine its nature.

Solution: Let $\dot{x}$=y

$$\dot{y} = \ddot{x} = 0.25x^2 - x$$

$$\dot{x} = f(x) = \begin{bmatrix} y \\ 0.25x^2 - x \end{bmatrix}$$

The equilibrium point is $x' = f(x) = 0$

$\Rightarrow$ y = 0 and $0.25x^2 - x = 0$

$\Rightarrow$ x = 0 or x = 4

Therefore (0, 0) and (4, 0) are equilibrium points.

D f (0, 0) = $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ = A, The eigenvalues λ_1, λ_2 = $\pm$ i

The equilibrium point (0, 0) is nonhyperbolic.

The nature of the nonlinear equation near to (0, 0) is either center or focus.

The nature of the nonlinear equation near to (4, 0) is saddle since λ_1, λ_2=1,-1

The equilibrium point (4, 0) is hyperbolic

Consider the Parameter dependent nonlinear differential equation f(x) = $\ddot{x}$-$\varepsilon(1-x^2)\dot{x}+x$

Linearize the equation

Let $\dot{x}$ = y

$\dot{y}$ = $\ddot{x}$ = ε(1-x^2) x'-x

$$\dot{x} = \begin{pmatrix} x \\ y \end{pmatrix}' = \begin{pmatrix} y \\ \varepsilon(1-x^2)x'-x \end{pmatrix} = \begin{pmatrix} y \\ \varepsilon(1-x^2)y-x \end{pmatrix}$$

The equilibrium point y = 0 and ε (1-x^2) y-x=0 y= 0 and x = 0

Therefore (0, 0) is the only equilibrium point.

D f (x_o) = $\begin{pmatrix} 0 & 1 \\ -2x\varepsilon y-1 & \varepsilon(1-x^2 \end{pmatrix}$, D f (0, 0) = $\begin{pmatrix} 0 & 1 \\ -1 & \varepsilon \end{pmatrix}$ = A

The linear approximation x' = Ax = $\begin{bmatrix} 0 & 1 \\ -1 & \varepsilon \end{bmatrix}$x

The characteristics equation, $\lambda^2 - \varepsilon\lambda + 1 = 0$

The corresponding eigenvalues λ_1, λ_2 = $\dfrac{\varepsilon \pm \sqrt{\varepsilon^2 - 4}}{2}$

Case 1: If $\varepsilon < -2$, both $\lambda_{1,2}$ are negative and real therefore the equilibrium point is stable node

Case2: If ε = -2 then the eigenvalue is -1. Thus the equilibrium point is stable improper node

Case 3 If -2 < ε < 0, then the eigenvalues are complex with negative real part.

Thus the equilibrium point is a stable spiral/ focus.

Case 4: If 0 < ε < 2, λ_1, λ_2 are complex with positive real part.

Therefore the origin is unstable focus.

Case 5: if ε = 0, then, λ_1, λ_2 purely imaginary the equilibrium point is center .(nonhyperbolic)

Case 6: If $\varepsilon > 2$, then $\lambda_{1,2}\ \varepsilon$ IR and positive the equilibrium point is unstable mode.

Case 7: If $\varepsilon =$ 2 then the eigenvalue is 1.Thus the equilibrium point is unstable improper node.

1.6 Center manifold

[3] We presented the Hartman –Grobman theorem, which shows that in a neighborhood of a hyperbolic critical point $x_0 \in E$ the nonlinear system $\dot{x}=f(x)$ is topological conjugate to the linear system $\dot{x} = Ax$ with A=D f(x_0) the neighborhood of the origin .In the section we present the local center manifold theorem, the qualitative behavior in a neighborhood of a non hyperbolic critical x_0 of the nonlinear system $\dot{x}=f(x)$ with x $\in \mathcal{R}^n$ is determined by its behavior on the center manifold near x_0. Since the center manifold is generally of the smaller dimension than the system $\dot{x}=f(x)$. This simplifies the problem of determining the stability and qualitative behavior of the flow near a non hyperbolic critical point of $\dot{x}$=f(x).

CENTER MANIFOLD THEOREM: Consider the system of ODEs

$$\begin{cases} \dot{x} = Ax + G_1(x,y,\lambda) \\ \dot{y} = By + G_2(x,y,\lambda) \end{cases}$$

where x $\in \mathcal{R}^m$, y$\in \mathcal{R}^{n-m}$ $(0 < m < n)$, A and B are m×m (n - m) × (n-m) matrices respectively $G_i(x,y,\lambda)$ (i=1, 2) are continuously on λ and C^r(r>1) on (x, y) $\in \mathcal{R}^m \times \mathcal{R}^{n-m}$ and $G_i\ (x, y, \lambda$i=1, 2) = O($|x| - |y|$) $\forall \lambda \in \mathcal{R}^1$.

THEOREM: Suppose that all eigenvalues of A have non negative (non positive) real part and all the eigenvalues of B have negative (resp. positive) real parts then the system

$$\begin{cases} \dot{x} = Ax + G_1(x,y,\lambda) \\ \dot{y} = By + G_2(x,y,\lambda) \end{cases}$$

with the condition $G_i\ (x, y, \lambda\ ) = O\ (|x| - |y|)$ $\forall \lambda \in \mathcal{R}$

There exists C^r function called the **center manifold function h (. , λ): D→ $\mathcal{R}^{n-m}$,D⊏ $\mathcal{R}^m$** a neighborhood of x=0 such that h(x, λ) is continuous on λ and

1) h (0, λ)=0, $\dot{h}$=(0, λ)=0

2)The set $M_\lambda=\{(x,y): x \in D \sqsubset R^n\ \ , y = h(x,y)\}$ called the center manifold is a local invariant of $\dot{x}$ *and* $\dot{y}$

3) If M_λ is positively invariant (resp. negatively invariant) namely Z(t , ψ) $\in M_\lambda$ (resp. Z(-t, ψ) $\in M_\lambda$) $\forall t > 0$ then is an attracting set of $\dot{x}$ and $\dot{y}$ (resp. a repelling set) that is there a neighborhood U $\sqsubset R^n$ of M_λ as $\psi \in$U.

We have $\lim_{t\to\infty} dist(z(t,\psi),M_\lambda)$is =0 (resp. $\lim_{t\to\infty} dist(z(-t,\psi),M_\lambda)$ =0) where

Z (t, ψ) = (x (t,ψ), M_λ) is the solution of $\begin{cases} \dot{x} = \mathbf{Ax} + G_1(x,y,\lambda) \\ \dot{y} = \mathrm{By} + G_2(x,y,\lambda) \end{cases}$ with the IVP Z(0, ψ)

Property 1 means that the center manifold $M_\lambda \subseteq R^n$ is tangent to the eigen space R^m of A at Z (0, ψ) = ψ.

Consider the system

$$\begin{cases} \dot{x} = Ax + G_1(x,y,z,\lambda) \\ \dot{y} = By + G_2(x,y,z,\lambda) \\ \dot{z} = Cz + G_3(x,y,z,\lambda) \end{cases}$$

where x $\in \mathcal{R}^m$, y$\in \mathcal{R}^k$ z $\in \mathcal{R}^n$

$G_i(x,y,z,\lambda)$ = O (|x|, |y|, |z|) i=1, 2, 3 then we have

Let the eigenvalues of A have positive real part the eigenvalues of B negative real part the eigenvalues of C have zero real parts then the system

$$\begin{cases} \dot{x} = Ax + G_1(x,y,z,\lambda) \\ \dot{y} = By + G_2(x,y,z,\lambda) \\ \dot{z} = Cz + G_3(x,y,z,\lambda) \end{cases}$$

has three locally invariant manifolds M^u, M^s, M^c which are tangent to the eigen space of A, B, C respectively

T_w=OM^u = $\mathcal{R}$

T_w= OM^S = $\mathcal{R}^K$

T_w = OM^C = $\mathcal{R}^n$ where M^u is the unstable manifold M^S is the stable manifold and M^C is the center manifold.

Consider the linearized system $x' = \begin{bmatrix} -3 & 0 & -2 \\ -4 & -1 & -4 \\ 3 & 1 & 3 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ where x = (x, y, z)^T

Find the equation of center manifold at the origin.

$$A = \begin{bmatrix} -3 & 0 & -2 \\ -4 & -1 & -4 \\ 3 & 1 & 3 \end{bmatrix}$$

The characteristic equation is $\lambda^3 + \lambda^2 + \lambda + 1 = 0$

The eigenvalues of A are $\lambda_1, \lambda_2, \lambda_3 =$ -1, i and –i respectively

The eigen vector corresponding to eigenvalue $\lambda_1 = -1$ is $V_1 = \begin{pmatrix} -1 \\ -1 \\ 1 \end{pmatrix}$

The eigen vector corresponding to eigenvalue $\lambda_2 = i$ is $V_2 = \begin{pmatrix} 3+i \\ -6+2i \\ 5 \end{pmatrix}$

The eigen vector corresponding to eigenvalue $\lambda_3 = -i$ is $V_3 = \begin{pmatrix} -3-i \\ -6-2i \\ 5 \end{pmatrix}$

Therefore $x(t) = \alpha \begin{pmatrix} -1 \\ -1 \\ 1 \end{pmatrix} e^{-t} + \beta \begin{pmatrix} -3+i \\ -6+2i \\ 5 \end{pmatrix} e^{it} + \gamma \begin{pmatrix} -3-i \\ -6-2i \\ 5 \end{pmatrix} e^{-it}$

α is a real constant and β & γ are complex constants.

The system has stable manifold, put $\beta = \gamma = 0$ and $\alpha = 1$

$x(t) = \begin{pmatrix} -1 \\ -1 \\ 1 \end{pmatrix} e^{-t} = (-e^{-t}, -e^{-t}, -e^{-t})^T$

The center manifold is, put $\beta = \gamma = 1$, $\alpha = 0$

Then $x(t) = \begin{pmatrix} -3-i \\ -6-2i \\ 5 \end{pmatrix} e^{-it} + \gamma \begin{pmatrix} -3+i \\ -6+2i \\ 5 \end{pmatrix} e^{it}$

Which defines the plane y – 2x =0

Depending on the initial values as t $\to \infty$ solutions approach a periodic solution of the center which lie in the plane y – 2x =0

CHAPTER 2: BIFURCATION ANALYSIS OF DYNAMICAL SYSTEMS

[3] Differential equation that arises in applications often contains unspecified numerical constants called parameters or coefficients. The graph of $\dot{x}$=f(x) and consequently the phase line portrait of an autonomous system $\dot{x}$=f(x) depend on the values assigned to the parameter appearing in f(x). In applications it is often important to understand how changes in parameter values alter the phase line portrait. Parameter may change, for example, from naturally occurring events or from deliberate manipulations by humans.

Bifurcation theory is the study of how changes in parameters alter the phase line portrait and the asymptotic dynamics of the equation. Bifurcation is a qualitative change of dynamics.

Stability**:** a system is called stable when its state does not change over time, e.g. d(state variables)/dt = 0. **B**ifurcation: Stability may change as a parameter is changed causing emergence or disappearance of new stable points "bifurcation". This happens when dynamical system changes suddenly as a parameter is varied, hence it is said to have undergone a bifurcation. [5]

Bifurcation theory is the mathematical study of changes in the qualitative or topological structure of a given family of vector field, and the solution of a family of differential equations.
An equilibrium point at which no bifurcation occurs is called hyperbolic fixed point. The name "bifurcation" was first introduced by Hernia Poincare in 1885 in the first paper in mathematics.
Consider the Vander pal's equation $\ddot{x}$- $\varepsilon(1\text{-}x^2)$ $\dot{x}$ + x = 0

The equilibrium point is (0, 0) , The eigen values are $\lambda_1, \lambda_2 = \dfrac{\varepsilon \pm \sqrt{\varepsilon^2 - 4}}{2}$

$\varepsilon \le -2$	$-2 < \varepsilon < 0$	ε =0	$0 < \varepsilon < 2$	$\varepsilon < 2$
$\lambda_1 < \lambda_2 < 0$	Complex with $\alpha > 0$	Complex with $\alpha = 0$, nonhyperbolic	Complex with $\alpha > 0$	$0 < \lambda_1 < \lambda_2$
Stable node	Stable focus	Center , focus Or node	Unstable focus	Unstable node

Here 0 is a critical value. Critical values are demarcation points that bring change in qualitative behavior very small change in parameter can produce large change in qualitative behavior of solution.

This is called bifurcation theory

Types of Bifurcation

1. Local Bifurcation
2. Global Bifurcation

2.1.1 Local Bifurcation

(8) Definition A local bifurcations is a bifurcation that can be analyzed purely in terms of a change in the linearization around single invariant set or attractor**.**
This deals changes in the local stability properties of equilibrate, periodic orbits or other invariant sets as parameters cross through critical points. Bifurcation is a general change on the local or global phase picture.

Locally, when the real part, of all eigenvalues are different from zero the phase picture is locally equivalent to its linear part by Hartman's theorem. It might change when some real, part becomes zero. The Hartman-Grobman Theorem shows that hyperbolic equilibrium are structurally stable. The converse is also true. Non-hyperbolic equilibria are not structurally stable
Local bifurcation is the bifurcation that involves equilibrium.

A local bifurcation occurs when a parameter change causes the stability of an equilibrium (or fixed point) to change. In continuous systems, this corresponds to the real part of an eigenvalue of an equilibrium passing through zero. In both continuous and discrete systems cases, the equilibrium is non-hyperbolic at the bifurcation point. The topological changes in the phase portrait of the system can be confined to arbitrarily small neighborhoods of the bifurcating fixed points by moving the bifurcation parameter close to the bifurcation point (hence 'local')

These bifurcations can be classified in terms of the eigenvalues of the Jacobean matrix associated with the parameter dependent equilibrium. More technically, consider the continuous dynamical system described by the ODE $\dot{x} = f(x, \lambda) \quad f : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}^n.$

A local bifurcation occurs at (x_0, λ_0), if the Jacobian matrix D $f(x_0, \lambda_0)$ has an eigenvalue with zero real part. If the eigenvalue is equal to zero, the bifurcation is a steady state bifurcation, but if the eigenvalue is non-zero but purely imaginary, this is a Hopf bifurcation.

Types of local bifurcations

Local bifurcation includes

a. Saddle node bifurcation
b. Transcritical bifurcation
c. Pitch fork bifurcation
d. Period doubling bifurcation
e. Hopf bifurcation

2.1.1) Saddle Node Bifurcation

It is a local bifurcation in which two fixed points or (equilibrium) of a dynamical system collide and annihilate each other. One of the equilibrium point is stable (the node) and the other is unstable (saddle) (see fig 2.1)

The normal form of a saddle node bifurcation is $\dot{u} = \mu - u^2$

Here u is the state variable and μ is the bifurcation parameter.

The equilibrium point is $\dot{u} = 0 \Rightarrow u = \pm\sqrt{\mu}$

Case1 For $\mu > 0$ there are two equilibrium points at $u = \pm\sqrt{\mu}$

$$D\,f(u, \mu) = -2u, \quad D\,f(\pm\sqrt{\mu}, \mu) = \pm 2\sqrt{\mu}$$

The critical point at $u=\sqrt{\mu}\ is\ stable$ while the critical point at $u= -\sqrt{\mu}$ is an unstable.

Case2 For $\mu=0$ there is only one critical point at u=0 and it is a nonhyperbolic since

D f (0,0)=0. Therefore $\mu=0$ is a bifurcation value.

Case3: For $\mu < 0$ there is no critical point. (see fig 2.1)

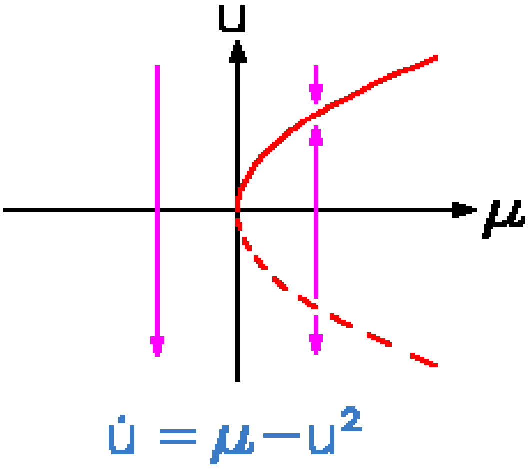


Figure 2.1 Saddle Node Bifurcations

Note: The solid lines depict stable behavior and the dashed lines depict unstable behavior.

Consider the following system of equation

$$\begin{cases} \dot{x} = \mu - x^2 \\ \dot{y} = -y \end{cases}$$

The critical points are found by solving the equations $\dot{x} = \dot{y} = 0$

$\Longrightarrow$ -y = 0 or $\mu - x^2 = 0$

$\Rightarrow$ $x = \pm\sqrt{\mu}$

Case1: When $\mu < 0$, there is no critical point in the plane

Case 2: When $\mu = 0$, there is one critical point at the origin and it is nonhyperbolic.

The solution curve may be found by solving the differential equation $\frac{dy}{dx} = \frac{\dot{y}}{\dot{x}} = \frac{y}{x^2}$

The solution is to be $y = ke^{-1/x}$ where k is constant.

In this case it is stable for x > 0 and unstable for x < 0. .(see fig 2.2)

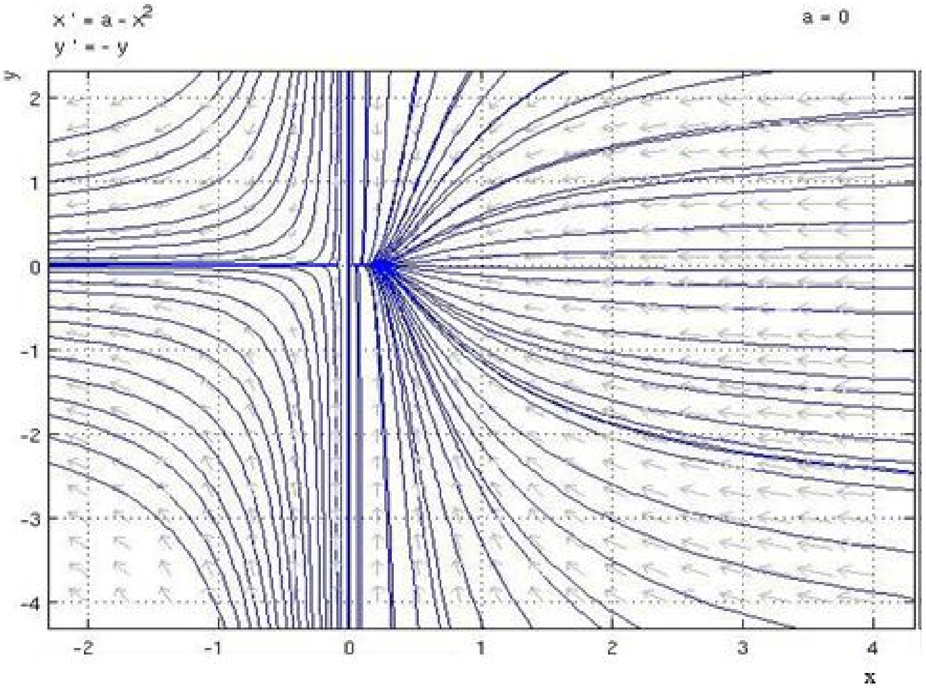


Figure 2.2: Saddle

Case 3: When $\mu > 0$, there are two critical points at ($\sqrt{\mu}$, 0) and (-$\sqrt{\mu}$, 0) that is one saddle point and one node.

The Jacobian matrix A = $\begin{bmatrix} -2x & 0 \\ 0 & -1 \end{bmatrix}$

$$D\,f\,(\mu\,,0) = \begin{bmatrix} -2\sqrt{\mu} & 0 \\ 0 & -1 \end{bmatrix}$$

The eigenvalues and eigen vectors are given by λ_1= -2$\sqrt{\mu}$, $(1,0)^T$ and λ_2= -1, $(1,0)^T$ respectively .

For μ=9 the equilibrium point on coordinate axis will be (-3, 0) and (3, 0) (see fig 2.3)

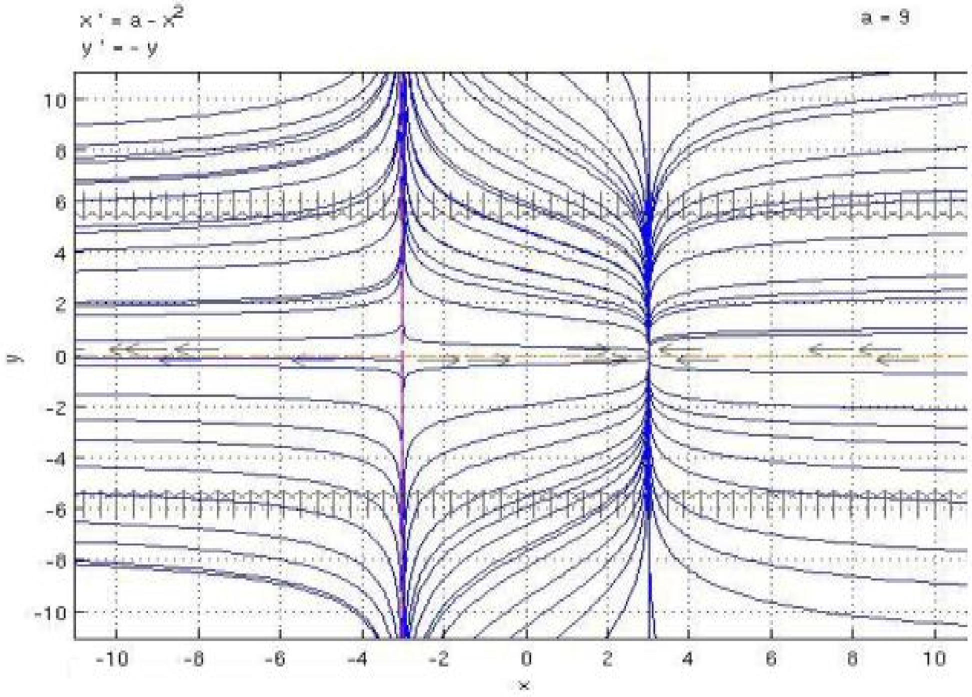


Figure 2.3 Saddle nodes with Two Equilibrium Points

When $\mu < 0$ there are no critical points and as μ passes through zero the qualitative behavior changes and two critical points bifurcate from the origin. As μ increases, the critical points move farther and farther apart.

2.1.2 Transcritical Bifurcation

[4] A transcritical bifurcation is a particular kind of local bifurcation, meaning that it is characterized by an equilibrium having an eigenvalue whose real part passes through zero.

A transcritical bifurcation is one in which a fixed point exists for all values of a parameter and is never destroyed. However, such a fixed point interchanges its stability with another fixed point

The normal form of a trancritical bifurcation is $\frac{du}{dt} = \mu u - u^2$

The equilibrium points are at u=0 and u=μ

Case1: For $\mu = 0$ there is only one critical point at u=0 and it is nonhyperbolic.

Since D f (0, 0) = 0, $\mu = 0$ is a bifurcation value.

Case2: For $\mu > 0$, D f (u, μ) = μ - 2u= - μ

Therefore the critical point at u = μ is stable

Case3: For $\mu < 0$, $Df(u, \mu) = \mu - 2u = \mu > 0 \Rightarrow$ the critical point at $u = \mu$ is unstable. As the parameter varied or the stable equilibrium become unstable and the unstable became stable

(See fig 2.4)

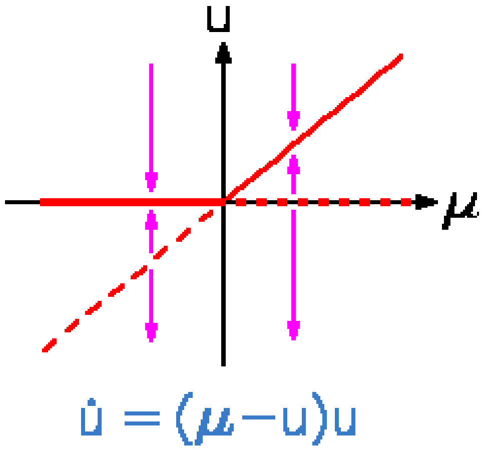


Figure 2.4: A transcritical bifurcation

Note: The solid lines depict stable behavior and the dashed lines depict unstable behavior.

Consider the following system of equations.

$$\begin{cases} \dfrac{dx}{dt} = \mu x - x^2 \\ \dfrac{dy}{dt} = -y \end{cases}$$

The critical points are found by solving the equation $\dfrac{dx}{dt} = 0 = \dfrac{dy}{dt}$. There are either one or two critical points depending up on the value of the parameter μ .

Case 1: When $\mu = 0$ there is one non hyperbolic critical point at the origin the solution curve satisfy the differential equation $\dfrac{dy}{dx} = \dfrac{y}{x^2}$

$$\frac{dx}{dt} = \mu x - x^2$$

$$\frac{dy}{dt} = -y$$

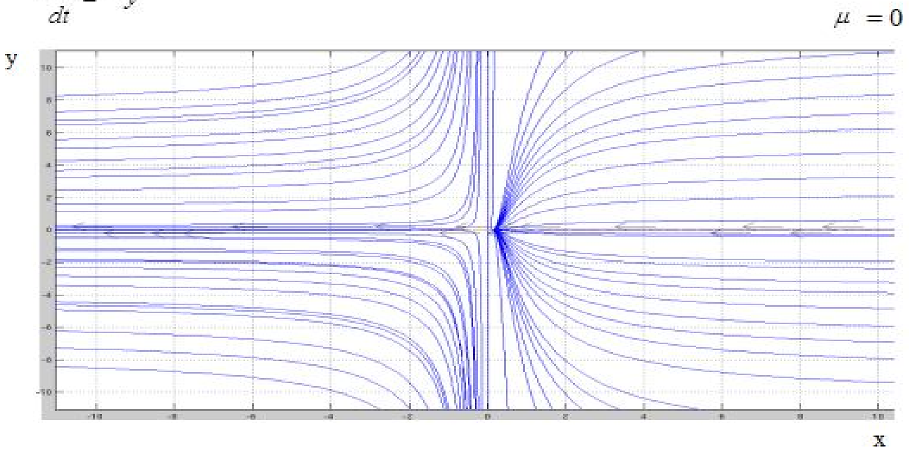


Figure 2.5: One equilibrium point

Case 2 When μ < 0 there are two critical points, one at (0, 0) and the other at (μ, 0) the origin is a stable node and (−3, 0) is a saddle point.(see fig 2.6)

$$\frac{dx}{dt} = \mu x - x^2$$

$$\frac{dy}{dt} = -y$$

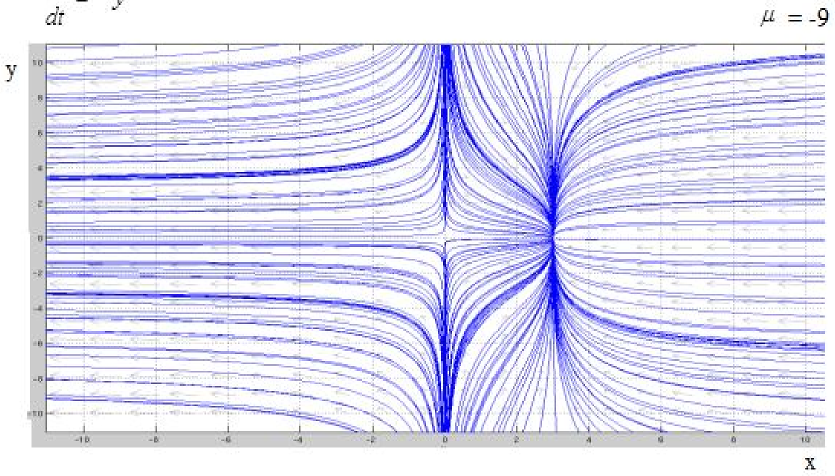


Figure 2.6: Two equilibrium point

***Case* 3**: when μ > 0 there are two critical points, one at (0, 0) and the other at (μ, 0) the origin is now a saddle point and (3, 0) is a stable node **(see fig 2.7)**

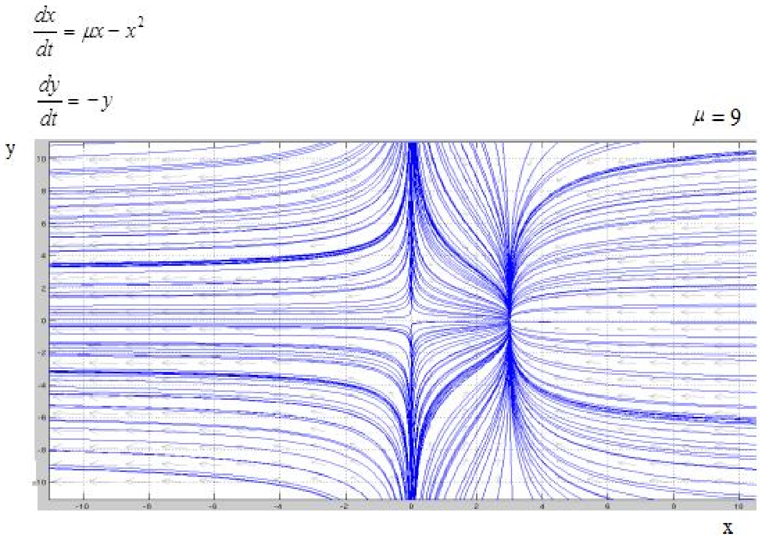


Figure 2.7: Two equilibrium point

2.1.3 Pitch fork bifurcation

- Suited for problems that have symmetry.
- There are two stable equilibrium which are separated by an unstable equilibrium.
- Pitch fork bifurcation have two types: super and subcritical pitch fork bifurcation.

A) Supper Critical :

The normal form of supercritical bifurcation is $\frac{dx}{dt} = rx - x^3$

For r < 0, there is one stable equilibrium at x = 0 (see fig 2.8)

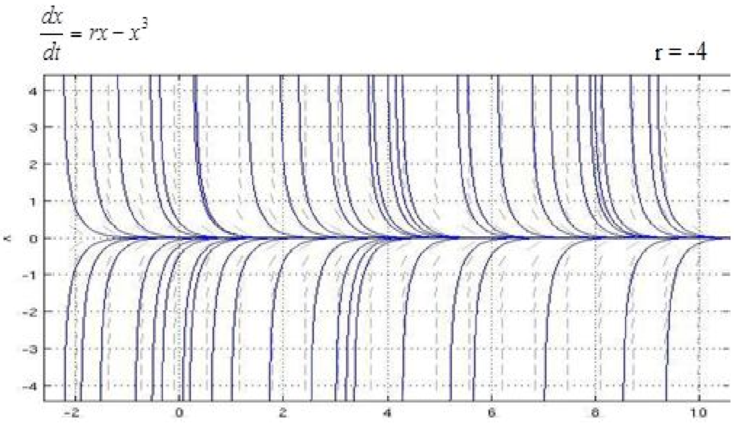


Figure 2.8 one equilibrium point

For r>0, there is an unstable equilibrium at x=0 and two stable equilibrium at x = $\pm\sqrt{r}$ (see fig 2.9)

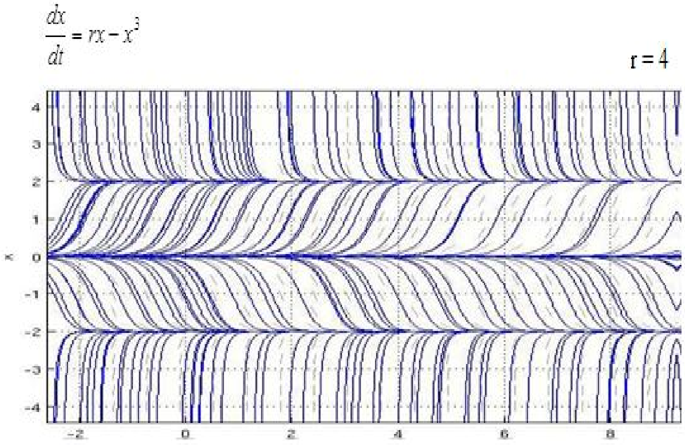


Figure 2.9 Three equilibrium points

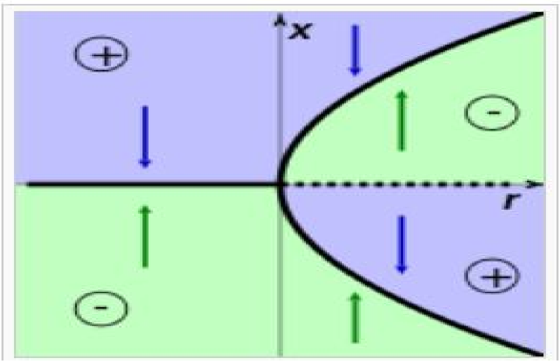


Figure 2.10: Supper critical

Note: The solid lines depict stable behavior and the dashed lines depict unstable behavior

B) Sub Critical: The normal form for the subcritical case is $\frac{dx}{dt} = rx + x^3$

In this case, the equilibrium point is at x=0 or x = $\pm\sqrt{-r}$

For r < 0, the equilibrium at x =0 is stable and

There are two unstable equilibriums at x =$\pm\sqrt{-r}$.

If $r > 0$, the equilibrium at $x = 0$ is unstable.

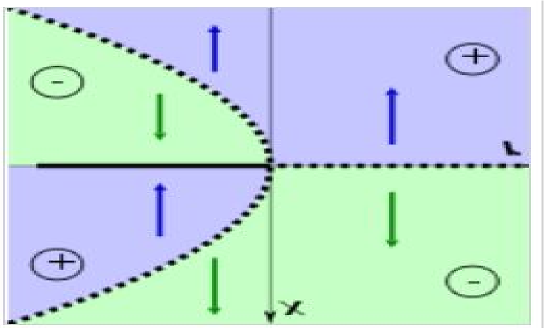


Figure 2.11: Sub critical

Note: The solid lines depict stable behavior and the dashed lines depict unstable behavior

2.2 Global Bifurcation

[8] **Definition**: A local bifurcations is a bifurcation that cannot be analyzed purely in terms of a change in the linearization around single invariant set or attractor**.** This deal with larger invariant sets of the system collides with each other, or with equilibrates of the system. They cannot be detected purely by a stability analysis of the equation.

Global bifurcations occur when 'larger' invariant sets, such as periodic orbits, collide with equilibria. This causes changes in the topology of the trajectories in the phase space which cannot be confined to a small neighborhood, as is the case with local bifurcations. In fact, the changes in topology extend out to an arbitrarily large distance (hence 'global').

Examples of global bifurcations include:

2.2.1 Homoclinic: bifurcation in which a limit cycle collides with a saddle point.

2.2.2 Heteroclinic bifurcation in which a limit cycle collides with two or more saddle points.

2.2.3 Infinite-period bifurcation in which a stable node and saddle point simultaneously occur on a limit cycle.

2.2.4 Blue sky catastrophe in which a limit cycle collides with a nonhyperbolic cycle.

CHAPTER 3: ECONOMIC DEVELOPMENT

3.1 INTRODUCTION

[7] Our starting point is the so-called Solow-Swan model named after Robert (Bob) Solow and Trevor Swan, or simply the Solow model, named after the more famous of the two economists. These economists published two path breaking articles in the same year, 1956 (Solow, 1956; Swan, 1956) introducing the Solow model. Bob Solow later developed many implications and applications of this model and was awarded the Nobel Prize in economics for his contributions. This model has shaped the way we approach not only economic growth but also the entire field of macroeconomics.

The Solow model is remarkable in its simplicity. Looking at it today, one may fail to appreciate how much of an intellectual breakthrough it was. Before the advent of the Solow growth model, the most common approach to economic growth built on the model developed by Roy Harrod and Evsey Domar (Harrod, 1939; Domar, 1946). The Harrod-Domar model emphasized potential dysfunctional aspects of economic growth, for example, how economic growth could go hand-in-hand with increasing unemployment on this model. The Solow model demonstrated why the Harrod-Domar model was not an attractive place to start. At the center of the Solow growth model, distinguishing it from the Harrod-Domar model is the neoclassical aggregate production function. This function not only enables the Solow model to make contact with microeconomics, but it also serves as a bridge between the model and the data.

An important feature of the Solow model is that it is a simple and abstract representation of a complex economy. At first, it may appear too simple or too abstract. After all, to do justice to the process of growth or macroeconomic equilibrium, we have to consider households and individuals with different tastes, abilities, incomes, and roles in society; various sectors; and multiple social interactions.

Production in the model takes place 3 inputs.

Physical capital ….. ……………..k(t)
Labor force…..…… …………….L(t)
Technology/ knowledge …. …….A(t)

The aggregate production function for the unique final good is written as Y(t)= F(K(t), L(t), A(t)) where Y(t) is the total amount of production of the final good at time t or the flow of output produced at a time t.

Physical capital: is the capital stock or the durable physical inputs such as machines, buildings. It is a rival.

Labor force: is total employment or the input associated with human body. Employment can be measured in different ways. For example, we may want to think of as corresponding to hours of employment or to number of employees. The capital stock K(t) corresponds to the quantity of “machines” (or more specifically, equipment and structures) used in production, and it is typically measured in terms of the value of the machines. There are also multiple ways of thinking of capital (and equally many ways of specifying how capital comes into existence). Since the objective here is to start with a simple workable model, I make the rather sharp simplifying assumption that capital is the same as the final good of the economy. However, instead of being consumed, capital is used in the production process of more goods. To take a concrete example, think of the final good as “corn.” Corn can be used both for consumption and as an input, as seed, for the production of more corn tomorrow. Capital then corresponds to the amount of corn used as seed for further production

Technology: Technology has no natural unit, and A(t) is simply a shifter of the production function . For mathematical convenience, I often represent A(t) in terms of a constant or number, but it is useful to bear in mind that, at the end of the day, it is a representation of a more abstract concept.

A major assumption of the Solow growth model (and of the neoclassical growth model) is that technology is free: it is publicly available as a nonexcludable, nonrival good. Public goods are nonrival (theycan be used by many people simultaneously) and also nonexcludable (it is technologically or legally impossible to prevent people from using such goods. Technology is a good candidate for a nonexcludable, nonrival good; once the society has some knowledge useful for increasing the efficiency of production, this knowledge can be used by any firm without impinging on the use of it by others. Moreover, it is typically difficult to prevent firms from using this knowledge (at least once it is in the public domain and is not protected by patents). For example, once the society knows how to make wheels, everybody can use that knowledge to make wheels without diminishing the ability of others to do the same (thus making the knowledge to produce wheels nonrival). Moreover, unless somebody has a well-enforced patent on wheels, anybody can decide to produce wheels (thus making the knowhow to produce wheels nonexcludable). The implication of the assumptions that technology is nonrival and nonexcludable is that A(t) is freely available to all potential firms in the economy and firms do not have to pay for making use of this technology. Departing from models in which technology is freely available is a major step.

Technology can improve over time: The same amount of capital and labor yields a larger quantity of output in 2000 than 1800. Because the technology employed in 2000 is superior.

Technology can also differ from country to country: The same amount of capital and labor yields a larger quantity of output in Japan than Ethiopia since the technology available in Japan is better.

In a closed economy no public spending, all output is devoted to consumption or gross investment. So $Y(t) = c(t) + I(t)$ where $c(t)$ is the amount to be consumed.

$$\Rightarrow Y(t) - c(t) = I(t)$$

$\Rightarrow s(t) = Y(t) - c(t)$ that is the amount saved equals the amount invested.

In other words, the saving rate of a closed economy represents the fraction of GDP that an economy devotes to investment.

Let s (.) be the fraction of output that is saved so 1-s (.) = c (t), fraction of output that is consumed.

The net increase in the stock of physical capital at a point in time equals gross investment less depreciation. That is

$$K'(t) = I(t) - \delta K$$

$$K'(t) = sF(K(t), L(t), T(t)) - \delta K \quad \text{Where} \quad K'(t) = \frac{\partial K(t)}{\partial t} \quad \text{and} \quad 0 \le s \le 1$$

$\dot{K}$ (t) determines the dynamics of K for given technology and labor.

3.2 The Neoclassical Model of slow -Swan

[5] The process of economic growth depends on the shape of production function. Now we consider the neoclassical production function. We say the production function is neoclassical if

a) Constant returns to scale**:** When output increased by the same proportion as of input increased. Or if we multiply capital and labor by a constant λ we get λ amount of output. That is $F(\lambda k, \lambda L) = \lambda F(K, L)$ for $\lambda > 0$, this property is called homogeneity of degree one in capital and labor.

b) Assumption (Continuity, Differentiability, Positive and Diminishing Marginal Products, and Constant Returns to Scale):

The production functio*n* $F: R_+^3 \rightarrow R_+$ *is* twice differentiable in K and L, and satisfies

$$F_K(K,L,A) \equiv \frac{\partial F(K,L,A)}{\partial K} > 0, \qquad F_L(K,L,A) \equiv \frac{\partial F(K,L,A)}{\partial L} > 0,$$

$$F_{KK}(K,L,A) \equiv \frac{\partial^2 F(K,L,A)}{\partial K^2} < 0, \qquad F_{LL}(K,L,A) \equiv \frac{\partial^2 F(K,L,A)}{\partial L^2} < 0. \quad \text{Or}$$

The production function F is concave

All of the components of assumptions are important. First, the notation F: $R_+^3 \to R_+$ implies that the production function takes nonnegative arguments (i.e., $K,L \in R_+$) and maps to nonnegative levels of output ($Y \in R_+$). It is natural that the level of capital and the level of employment should be positive. The second important aspect of assumption is that F is a continuous function in its arguments and is also differentiable. There are many interesting production functions that are not differentiable, and some interesting ones that are not even continuous. But working with differentiable functions makes it possible to use differential calculus, and the loss of some generality is a small price to pay for this convenience. The assumption also specifies that marginal products are positive (so that the level of production increases with the amount of inputs); this restriction also rules out some potential production functions and can be relaxed without much complication. More importantly, the assumption also requires that the marginal products of both capital and labor are diminishing, that is, $F_{KK}<0$ and $F_{LL}<0$, so that more capital, holding everything else constant, increases output by less and less. And the same applies to labor.

c) The Inada conditions:

The third defining characteristic of the neoclassical production function is the marginal product of capital or labor approaches infinity as capital or labor goes zero and vice versa. F satisfies the Inada conditions:

$$\lim_{L\to 0}\frac{dF}{dL} = \lim_{K\to 0}\frac{dF}{dK} = \infty \quad , \qquad \lim_{L\to\infty}\frac{dF}{dL} = \lim_{K\to\infty}\frac{dF}{dK} = 0$$

d) Essentiality: An input is essential if a strictly positive amount is needed to produce positive amount of output. When zero units of input is used for either capital or labor then nothing is produced i.e F (0, L,) = 0 = F (K, 0) for all L and K

Per Capita Variables:

When we say that a country is rich or poor, we tend to think in terms of output per person. Consider the basic Solow growth model in continuous time, then there exists a unique steady- Satisfies per capita output is given by $y = f(k)$, and per capita consumption is given by $c = (1 - s) f$. We can construct a model in per capital terms and study the behavior of the per caption quantities of GDP, consumption and capital.

$Y = F(K(t), L(t))$ (divide both sides by L),$\Rightarrow \frac{Y}{L} = F(\frac{K}{L}, 1)$ where $k = \frac{K}{L}$capital per worker and $\frac{Y}{L} = y = f(k)$ is out per worker.

Production per person is determined by the amount of physical capital each person has access to very large economies, such as china can have less output or income per person than small economies such as Netherlands.

(The Cobb-Douglas Production Function) : Let us consider the most common example of production function used in macroeconomics, the Cobb-Douglas production function. The Cobb-Douglas production function can be written as

$$Y(t) = AK^{\alpha}(t)L^{1-\alpha}(t), \quad 0 < \alpha < 1.$$

$$y = Ak^{\alpha}$$

Exercise: Verify that Cobb-Douglas satisfies the conditions of a neoclassical production function.

3.3 The fundamental Equation of Solow- Swan Model

[7] We now analyze the dynamic behavior of the economy described by the neoclassical production function. The resulting growth model is called a Solow swan model. The change in the capital stock overtime is given by $K' = I(t) - \delta K$ (Divide both sides by L)

$$\frac{K(t)'}{L} = sF\left(\frac{K(t)}{L}, \frac{L(t)}{L}\right)$$

$$\frac{K(t)'}{L} = s\,F\left(\frac{K}{L}, 1\right) - \delta\,\frac{K}{L}$$

$$\frac{K(t)'}{L} = s\,f(k) - \delta\,k \qquad \text{where } f(k) = F\left(\frac{K}{L}, 1\right)$$

Take the derivative of K with respect to L

$$k = \frac{K}{L} \Rightarrow k' = \left(\frac{K}{L}\right)' = \frac{K'}{L} - \frac{K}{L}.\frac{L'}{L} \qquad \text{Where } \frac{L'}{L} = n$$

$$k' = \frac{K'}{L} - n\,\frac{K}{L}$$

Now substitute the expression for $\frac{\dot{K}}{L} \Rightarrow \frac{\dot{K}}{L} = s\,F(k,1) - \delta\,K$

$$\dot{k} = \frac{\dot{K}}{L} - n\,\frac{K}{L} \qquad \text{where } k = \frac{K}{L}$$

$$k = s\,f(k) - n\,\frac{K}{L} - \delta k \qquad \text{where } f(k)=Ak^{\alpha} \text{ is the Cobb-Douglas Production Function}$$

$$\dot{k} = sAk^{\alpha} - nk - \delta k$$

$\dot{k}=sAk^{\alpha} - (n+\delta)\,k$ is the fundamental differential equation of a slow- swan model.

This nonlinear differential equation depends on only on k. 3.3The Steady State

We know have the necessary tools to analyze the behavior of the model overtime by considering the steady state dynamics.

3.4: The steady state

Definition: A steady state is a situation in which the different quantities grow at constant rates. In slow-swan model the steady state corresponds to $\dot{k}= 0$ is the intersection of $(n+\delta)k$ and s f(k). Hence in the neoclassical model, the per capita quantities k,y, and c don't grow in the steady state. A proportional upward shift of the production function or an increase in sf(k) curve upward and leads to an increase k. An increase in n and δ movies the (n+ δ) k line up wards and leads to a decrease in k, k = $(\frac{n+\delta}{AS})^{\frac{1}{\alpha-1}}$ is the equilibrium point of the model on the production.

The equilibrium conditions are consistent with a steady state, that is, a situation in which the various quantities grow at constant (possibly zero) rates. The steady-state growth rates of k and c must be zero.

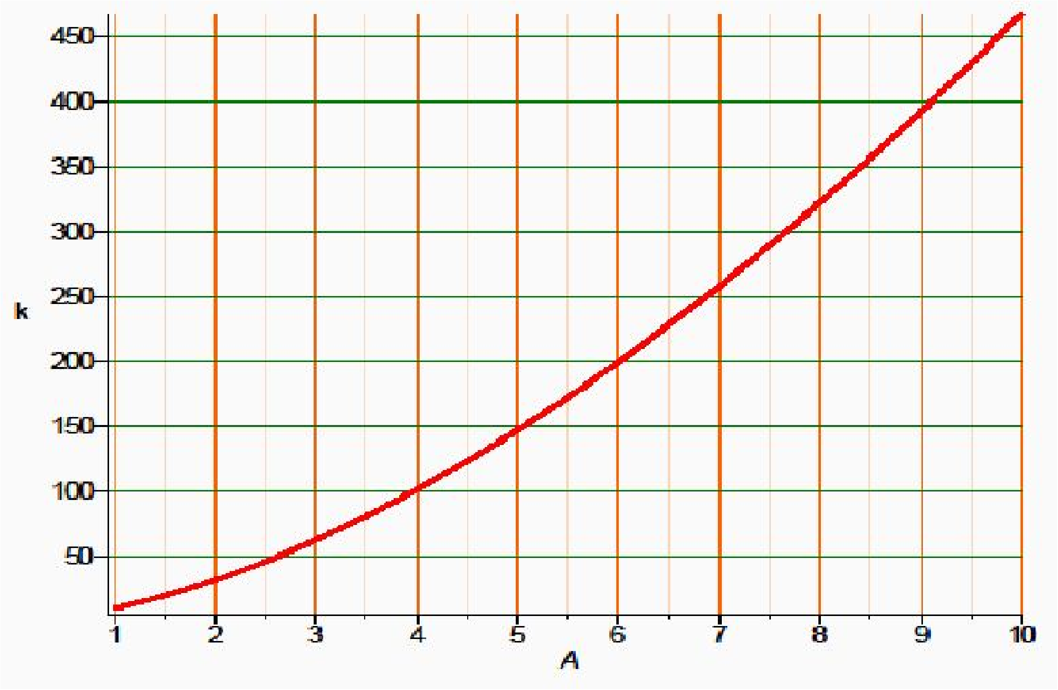


Figure 3.1 the advancement of technology on production

k = $(\frac{n+\delta}{AS})^{\frac{1}{\alpha-1}}$ where δ-depreciation rate,

n- Population growth rate,

A-technology and

s- Saving rate

Note that, as we saw graphically for a more general production function *f (k)*, k rises with the saving rate *s* and the level of technology *A*, and falls with the rate of population growth n and the depreciation rate δ. Therefore countries with higher saving rates and better technologies will have higher **Capital-labor** ratios and will be richer. Those with greater (technological) depreciation will tend to have lower capital-labor ratios and will be poorer.

We have shown that output per capita and capital per capita grow at rate of technological progress and without technology we do not have growth in per capita variables y, k and c. i.e. technological progress is the source of sustained (long - run) growth in k, y and c.

Increase in technology would raise both marginal productivities of capital and labor and thus the rental capital and the real wage.

Questions to Think

1. Does an increase in the savings rate increase the long run level of consumption per capita?

2. What happen to the technology that it is not assumed as a linear constant like capital and labor?

Summery

The mathematical formulation of a numerous physical problems results in a nonlinear ordinary differential equation. In general nonlinear ordinary differential equation cannot be solved analytically even when possible it is not the easiest way to determine the qualitative behavior of the system. Thus the equilibrium solutions which correspond to configuration, in which the physical system does not move, only occur in every day situation if they are stable.

An unstable equilibrium will not appear in practice, since slight perturbation of the system or if physical surroundings will immediately dislodge the system faraway from equilibrium

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