College of Natural Science
Department of Mathematics

Graduate Project Report on Existence and approximation of solution of explicit second order nonlinear Neumann problems

Submitted in partial fulfilment of the requirements for the Degree of Master of Science in Mathematics

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Contents

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The undersigned hereby certify that they have read and recommend to the school of graduate studies for acceptance of a project entitled **Existence and Approximation of solution of explicit second order nonlinear Neumann problems** by Simachew Sinshaw in partial fulfillment of the requirements for the degree of master of Science.

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Abstract

In this paper we shall present the basic theory of Neumann boundary value problem (BVP) together with a discussion of some of the powerful methods that are used to solve the Neumann boundary value problem (NBVP). Especialy the paper focuses on the upper and lower solutions of NBVP. .The main objective of this paper is to investigate the existence and approximation of solutions of second order nonlinear Neuman problems.

The paper consists of four main sections: The first section, the introductory part, deals about general introduction and some preliminary considerations. In the second section,the quasilinearization technique, and generalization quasilinearization technique will be defined and discussed;upper and lower solutions of Neumann BVP and elaborate through examples. And then some basic properties of NBVP (theorems on NBVP) are stated and clarified where the proofs of most of these properties are also include. Having familiarized with NBVP in the second section, in the third section,(the core part of the paper), we will come across with Generalized quasilinearization technique .

Introduction

Ordinary and partial differential equation have long played important roles in the history of modelling physical phenomena; they continue to serve as indispensable tools in future investigations. In the theory and applications of ordinary and partial differential equations, one may be interested to find a solution to a differential equation satisfying certain defined conditions. If the conditions are given at only one point of the independent variable, we have initial conditions; where as if the conditions are given at more than one point of the independent variable, they are called boundary conditions (BC). The problem of finding a solution of an n^{th} order differential equation together with n-initial conditions is called an initial value problem(IVB), however if the n-boundary conditions are considered, the problem is called boundary value problem (BVP).
Boundary value problems for nonlinear second order equations are particularly important because of numerous applications in science and technology. In this paper as in most physical applications, boundary conditions are always imposed at end points of an interval.

To study existence and approximation of solutions of some second order nonlinear Neumann problem of the form

$$-y''(x) = f(x, y(x)), x \in [0, 1], \tag{1}$$

$$y'(0) = A,\ y'(0) = B.$$

in the presence of a lower solution α and an upper solution β with $\alpha \geq \beta$ on [0, 1].We use the quasilinearization technique for the existence and approximation of solutions.We show that under suitable conditions the sequence of approximates obtained by the method of quasilinearization converges quadratically to a solution of the original problem.
There is a vast literature dealing with the solvability of nonlinear boundary-value problems with the method of upper and lower solution and the quasilinearization technique in the case where the lower solution α and the upper

solution β are ordered as $\alpha \leq \beta$. The case where the upper and lower solutions are in the reversed order has also received some attention. Existence results for Neumann problems in the presence of lower and upper solutions in the reversed order recently attracted the attention of many researchers. In these papers, the monotone iterative technique is used to show for existence of a solution y such that $\alpha \geq y \geq \beta$.

The purpose of this paper is to study the quasilinearization technique for the solution of the original problem in the case upper and lower solutions are in the reversed order.
We discuss some basic known existence results for a solution of the BVP (1). The key assumption is that the function $f(x,y)-\lambda y$ is non-increasing in y for some λ. We approximate our problem by a sequence of linear problems by the method of quasilinearization and prove that under some suitable conditions there exist monotone sequences of solutions of linear problems converging to a solution of the BVP (1).

Chapter 1

Differential Equations and the method of Upper and Lower solutions

The method of upper and lower solutions is a tool that one uses when trying to prove the existence of a solution to a non-linear differential equation. First we define upper and lower solutions and give a few theorems about their properties. Then in periodic problems we discuss what periodic solutions are and present more theorems which show the relationships between them and upper and lower solutions.

1.1 Strict upper and lower solutions for first order ODE

Let $F : RXR \to R$ be a C^1 function (i.e., F is continuous and continuously differentiable. We consider the DE

$$y'(x) = F(x, y(x)) \tag{1.1}$$

Let J be an interval, open or closed, and $\alpha \in C^1(J, R)$.We say that α is a strict lower solution of (1.1) on J provided

$$\alpha'(x) < F(x, \alpha(x))$$

for all $x \in J$. If $\beta \in C^1(J, R)$. we say that β is a strict upper solution of (1.1) on J provided

$$\beta'(x) > F(x, \beta(x))$$

for all $x \in J$

Theorem 1.1.1. *Let α be a strict lower solution of (1.1) on the interval $[x_0, \infty)$. Let $y_0 > \alpha(x_0)$. Then the solution to (1.1) satisfying $y(x_0) = y_0$, with maximal right interval of existence $[x_0, x_0 + \delta)$; satisfies*

$$y(x) > \alpha(x)$$

on $[x_0, x_0 + \delta)$.

Proof
suppose the conclusion is false, and let $c = \inf_{x \geq x_o} \{x | y(x) \leq \alpha(x)\}$.The set $\{x | y(x) \leq \alpha(x)\}$ is closed since the two functions y and α are continuous, so we have $c \in \{x | y(x) \leq \alpha(x)\}$. Thus $y(c) \leq \alpha(c)$, while $y(x) > \alpha(x)$ for all $x \in [x_0, c)$. Continuity would then force $y(c) = \alpha(c)$. Let $u(x) = y(x) - \alpha(x)$. Then $u(x) > 0$ on $[x_0, c)$ and $u(c) = 0$. Thus $u'(c) \leq 0$. But on $[x_0, x_0 + \delta)$ we have

$$u'(x) = y'(x) - \alpha'(x) > F(x, y(x)) - F(x, \alpha(x))$$

and so at $x = c$ we have

$$u'(c) > F(c, y(c)) - F(c, \alpha(c)) = 0$$

which contradicts $u'(c) \leq 0$. This proves the theorem.

Periodic Problems
Let $F \in C^1(RXR, R)$; and suppose there is a number $T > 0$ such that $F(x + T, t) = F(x, t)$ for all $(x, t) \in RXR$. Consider the DE

$$y' = F(x, y). \tag{1.2}$$

We are interested in the existence of T-periodic solutions of (1.2). A T-periodic solution is a solution $y = y(x)$ satisfying (1.2) for all $x \in R$ such that $y(x + T) = y(x)$ for all x. In short, y is periodic with period T. It is obvious that any T-periodic solution y will satisfy the boundary conditions

$$y(0) = y(T)$$

Theorem 1.1.2. *Let $\alpha(x) \leq \beta(x)$ respectively, strict lower and upper solutions of (1.2) on $[0, T]$. Suppose also that*

$$\alpha(0) \leq \alpha(T)$$

and

$$\beta(0) \geq \beta(T)$$

Then (1.2),(1.3) has a solution $y^*(x)$ satisfying $\alpha(x) \leq y^*(x) \leq \beta(x)$ for $0 \leq x \leq T$. Which has a T-periodic solution.

Proof Let $J = [\alpha(0), \beta(0)]$ and let $t \in J$. Let $y(x;t)$ be the solution to (1.2) with $y(0,t) = t$. By the theorems of the previous section $y(x,t)$ is a solution on $[0,T]$ and satisfies $\alpha(x) \leq y(x,t) \leq \beta(x)$ for $0 \leq x \leq T$. Thus $y(T;t) \in [\alpha(T), \beta(T)] \subset J$. Thus the mapping $t \rightarrowtail y(T;t)$ maps J into itself. Let Φ denote this mapping, so $\Phi(t) := y(T;t)$ and $\Phi(J) \subset J$. Φ is continuous, so it follows that Φ has a fixed point. That is, there is an $t^* \in J$ such that $\Phi(t^*) = t^*$. It now follows that the solution y^* of (1.2) with $y^*(0) = t^*$ satisfies (1.3). This proves the theorem.

1.2 Strict upper and lower solutions for second order ODE

In what follows, we discuss upper and lower solutions for second-order differential equations. (Since we discussed strict solutions in the previous section, we will now contrast that discussion with solutions that are no longer strict.) First let us define second-order differential equation: it is an equation in which a second derivative appears but no higher derivatives than that are found. Another important concept is that of a boundary value problem. A boundary value problem is similar to an initial value problem in that it also consists of a differential equation, but instead of initial conditions, boundary conditions are specified. For ordinary differential equations, the boundary refers to endpoints of an interval, (whereas for partial differential equations, the boundary more generally refers to some curve in the domain of the function of interest). By adapting the definition and theorem of Section 1.1 in [2], we will investigate periodic solutions to the boundary value problem

$$y'' = f(x,y), \tag{1.3}$$
$$y(0) = y(2\Pi),\ y'(0) = y'(2\Pi)$$

where f is a continuous function.

Definition 1.2.1. *A function $\alpha \in C^2((0,2\Pi)) \bigcap\ C^1([0,2\Pi])$ is said to be a lower solution of (1.3) if the following two conditions are met:*
(a) $\alpha''(x) \geq f(x,\alpha(x))$ for all $x \in (0,2\Pi)$;
(b) $\alpha(0) = \alpha(2\Pi),\ \alpha'(0) \geq \alpha'(2\Pi))$
A function $\beta \in C^2((0,2\Pi)) \bigcap\ C^1([0,2\Pi])$ is said to be an upper solution of (1.3) if the following two conditions are met:
(a) $\beta''(x) \leq f(x,\beta(x))$ for all $x \in (0,2\Pi)$;
(b) $\beta(0) = \beta(2\Pi),\ \beta'(0) \leq \beta'(2\Pi))$

Solution of BVP and Green's Function

The solution of the boundary value problem takes the form

$$y(x) = \int_a^b G(x,\xi) f(\xi) d\xi \tag{1.4}$$

where the Greens function is the piecewise defined function

$$G(x,\xi)= \begin{cases} \frac{y_1(\xi)y_2(x)}{w}, & a \leq \xi \leq x; \\ \frac{y_1(x)y_2(\xi)}{w}, & x \leq \xi \leq b. \end{cases}$$

The Green's function satisfies several properties,which we will explore further in the next section. For example, the Green functions satisfies the boundary conditions at x=a and x=b. Thus,

$$G(a,\xi) = \frac{y_1(a)y_2(\xi)}{W} = 0$$

$$G(b,\xi) = \frac{y_1(\xi)y_2(b)}{W} = 0$$

Also, the Greens function is symmetric in its arguments. Interchanging the arguments gives

$$G(\xi,x)= \begin{cases} \frac{y_1(x)y_2(\xi)}{w}, & a \leq x \leq \xi; \\ \frac{y_1(\xi)y_2(x)}{w}, & \xi \leq x \leq b. \end{cases}$$

But a careful look at the original form shows that

$$G(x,\xi) = G(\xi,x) \tag{1.5}$$

1.3 Properties of Green's function

We have noted some properties of Green functions in the last section. In this section we will elaborate on some these properties as a tool for quickly constructing Green functions for boundary value problems. Here is a list of the properties based upon our previous solution.

1. Proportionality

Let $Q = [a,b]X[a,b]$ be a square in the $x\xi$-plane. Partition Q using the line $\xi = x$ in to two triangles Q_1 and Q_2 as follows

$$Q_1 = \{(x,\xi) : a \leq \xi \leq x \leq b\}$$

$$Q_2 = \{(x,\xi) : a \leq x \leq \xi \leq b\}$$

For $x < \xi$ we are on the second branch and $G(x, \xi)$ is proportional to $y_1(x)$. Thus, since $y_1(x)$ is a solution of the homogeneous equation, then so is $G(x, \xi)$. For $x > \xi$ we are on the first branch and $G(x, \xi)$ is proportional to $y_2(x)$. So, once again $G(x, \xi)$ ia solution of the homogeneous problem.
2. Symmetry or Reciprocity

$$G(x, \xi) = G(\xi, x)$$

We had shown this in the last section.
3. Continuity of G at $x = \xi$:

$$G(\xi^+, \xi) = G(\xi^-, \xi)$$

Here we have defined

$$G(\xi^+, \xi) = \lim_{x \downarrow \xi} G(x, \xi), x > \xi,$$

$$G(\xi^-, \xi) = \lim_{x \uparrow \xi} G(x, \xi), x < \xi,$$

Setting $x = \xi$ in both branches,we have

$$\frac{y_1(\xi)y_2(\xi)}{w} = \frac{y_1(\xi)y_2(\xi)}{w}$$

4. Jump Discontinuity of $\frac{\partial G}{\partial x}$ at $x = \xi$

$$\frac{\partial G(\xi^+, \xi)}{\partial x} - \frac{\partial G(\xi^-, \xi)}{\partial x} = 1$$

This case is not as obvious specially for Sturm-Louville problem,

$$\frac{\partial}{\partial x}(p(x)\frac{\partial G(x, \xi)}{\partial x}) + q(x)G(x, \xi) = 0$$

We first compute the derivatives by nothing which branch is involved and then evaluate the derivatives and subtract them. Thus, we have

$$\frac{\partial G(\xi^+, \xi)}{\partial x} - \frac{\partial G(\xi^-, \xi)}{\partial x} = \frac{1}{pw}y_1(\xi)y_2'(\xi) + \frac{1}{pw}y_1'(\xi)y_2(\xi)$$

$$= -\frac{y_1'(\xi)y_2(\xi) - y_1(\xi)y_2'(\xi)}{p(\xi)(y_1(\xi)y_2'(\xi) - y_1'(\xi)y_2(\xi))}$$

$$= \frac{1}{p(\xi)}$$

1.4 Upper and lower solutions and differential inequalities

The defect $P\phi$ of a function ϕ with respect to the explicit first order ordinary differential equation,

$$y' = f(x, y)$$

is the function

$$P\phi = \phi'(x) - f(x, \phi(x))$$

If y is a solution of the explicit ordinary differential equation then

$$Py = 0$$

Thus, the defect tells us how close to the solution is the function ϕ.
Lemma
Let ϕ and ψ be defined and differentiable on the half open interval

$$J_0 = \{x : \xi < x \leq \xi + c; c > 0\}$$

If $\phi(x) < \psi(x)$ on the open interval

$$J_\epsilon = \{x : \xi < x < \xi + \epsilon; \epsilon > 0\}$$

for any $\epsilon > 0$, then one of the following two solutions holds.
i) $\phi < \psi \ on \ J_0$
ii) $\exists x_0 \in J_0$ such that $\phi(x) < \psi(x)$ for $\xi < x < x_0, \ \phi(x_0) = \psi(x_0) \ and \ \phi'(x_0) \geq \psi'(x_0)$.
Proof
Suppose i) dose not hold.
then $\exists x_0 > \xi$ such that $\phi(x_0) = \psi(x_0)$. since, $\phi(x) < \psi(x)$ for $\xi < x < x_0$
We have $\frac{\phi(x_0)-\phi(x_0-h)}{h} > \frac{\psi(x_0)-\psi(x_0-h)}{h} for 0 < h \ll 1$
Taking the limit of the difference quotients as $h \to 0^+$ establishes the second inequality in ii)
Theorem(Comparison)
Let the functions $\phi(x)$ and $\psi(x)$ be differentiable on

$$J_0 = \{x : \xi < x \leq \xi + a; \ a > 0\}$$

If the following conditions hold
i) $\phi(x) < \psi(x)$ for $\xi < x < \xi + \varepsilon; \ \varepsilon > 0$

ii) $p\phi < p\psi$ on J_0
then $\phi < \psi$ on J_0.
Proof
It suffices to show that the second inequality in ii) of the Lemma will not hold.Following the Lemma, suppose

$$\phi(x_0) = \psi(x_0)$$

But then at x_0,

$$\phi'(x_0) = p\phi + f(x_0, \phi(x_0))$$

and

$$\psi'(x_0) = p\psi + f(x_0, \psi(x_0))$$

Since, $p\phi < p\psi$ we obtain

$$\phi'(x_0) < \psi'(x_0)$$

And hence, $\phi'(x_0) \ngeq \psi'(x_0)$ This completes the proof.

Every statement on differential inequality for interval to the right of ξ,

$$J = [\xi, \xi + a], a > 0$$

has a counter part for the interval to the left of ξ,

$$J^- = [\xi - a, \xi], a > 0$$

The following corollary which we shall state without proof is formulated for an interval lying to the left of the point ξ.

Corollary
Let ϕ and ψ be differentiable functions on the interval

$$J_0^- = \{[x : \xi - a \leq x < \xi], a > 0\}$$

If ϕ and ψ satisfy the following inequalities
a) $\phi(x) < \psi(x)\ for\ \xi - \epsilon < x < \xi, \epsilon > 0$ arbitrary
b) $p\phi > p\psi\ on\ J_0^-$ then $\phi < \psi\ on\ J_0^-$
The concept and notions introduced thus for,naturally users to the following theorem on estimates.
Theorem
Let $w(x)$ and $u(x)$ be respectively, the lower and upper solutions of the cauchy problem

$$y' = f(x, y)\ on\ J = [\xi, \xi + a]$$

$$y(\xi) = \eta$$

. If y is the exact solution then we have

$$w(x) \leq y(x) \leq u(x) \; on \; J_0 = (\xi, \xi + a]$$

Representative Examples

The commonest method of determining upper and lower solutions of an ODE is by perturbing the equation. We allow small changes in $f(x, y)$ so as to find two functions $f_1(x, y)$ and $f_2(x, y)$ such that

$$f_1(x, y) < f(x, y), \; f_2(x, y) > f(x, y)$$

and such that

$$w' = f_1(x, w(x)) \; and \; u' = f_2(x, u(x))$$

can be solved explicitly.

For the sake of demonstration, consider Bernolli differential equation with initial data,

$$\begin{cases} y' = x^2 + y^2 \\ y(0) = 1 \end{cases}$$

Since $y^2 < x^2 + y^2$ for any $x \neq 0$, taking $f_1(x, w) = w^2$ and hence, the ODE

$$\begin{cases} w' = w^2 \\ w(0) = 1 \end{cases}$$

The solution of this separable ODE,

$$w(x) = \frac{1}{1 - x}$$

is the lower solution of Bernoulli. Again for $0 \leq x \ll 1$ We have $0 \leq x^2 < 1$

$$x^2 + y^2 < y^2 + 1$$

Setting, $f_2(x, y) = y^2 + 1$ We obtain $f_2(x, u(x)) = u^2 + 1$.

This intern leads to the initial value problem

$$\begin{cases} u' = u^2 + 1 \\ u(0) = 1 \end{cases}$$

The solution of this explicit ODE is then

$$u(x) = \tan(x + \frac{\pi}{4})$$

Which is the upper solution of the original problem, Bernolli differential equation. Consequently, the following estimates is immediate,

$$\frac{1}{1 - x} < y(x) < \tan(x + \frac{\pi}{4}).$$

Chapter 2

Linear Neumann boundary value problems of second order

2.1 Homogeneous BVP

this chapter we need to solve the boundary value problems which are linear by using lower and upper solutions.Indeed,the BVP

$$-y''(x) + My(x) = 0 \; if \; x \in [o, 1]$$

$$y'(0) = 0, \; y'(1) = 0 \tag{2.1}$$

Has only the trivial solution if $M \neq -n^2\pi^2, n \in Z$. The auxiliary equation for (2.1) is$-r^2 + M = 0$.The nature of the roots varies for M negative,zero, and positive. So we will consider these case separately.
Case 1:$M < 0$
Let $M = -\lambda^2$ for $\lambda > 0$.The roots of the auxiliary equation are $r = \pm i\lambda$,so the general solution to (2.1) is

$$y(x) = c_1 \cos \lambda x + c_2 \sin \lambda x$$

$$\Rightarrow y'(x) = -\lambda c_1 \sin \lambda x + \lambda c_2 \cos \lambda x$$

$$\Rightarrow y'(0) = -\lambda c_1 \sin \lambda 0 + \lambda c_2 \cos \lambda 0$$

$$= 0 + \lambda c_2 \cos \lambda 0$$

$$= \lambda c_2 = 0$$

But $\lambda > 0$ so that $c_2 = 0$ and

$$y'(1) = -\lambda c_1 \sin \lambda + \lambda c_2 \cos \lambda = 0$$

$$\lambda c_1 \sin\lambda = \lambda c_2 \cos\lambda$$

$$c_1 \sin\lambda = c_2 \cos\lambda$$

With $c_2 = 0$ then the last equation becomes $c_1 \sin\lambda = 0$.
To obtain a nontrivial solution, we must have $c_1 \neq 0$, so $\sin\lambda = 0$

$$\Rightarrow \sin\lambda = 0 = \sin n\pi, n = 1, 2, 3 \ldots$$

$$\Rightarrow \lambda = n\pi$$

$$\Rightarrow \sqrt{-M} = \lambda = n\pi$$

$$\Rightarrow -M = n^2\pi^2$$

$$\Rightarrow M = -n^2\pi^2$$

Therefore the trivial solution if $M \neq -n^2\pi^2$
case 2:$M > 0$
Let $M = \lambda^2$ for $\lambda > 0$ .The roots of auxiliary equation are $r = \pm\lambda$,so a general solution to (2.1) is $y(x) = c_1 e^{\lambda x} + c_2 e^{-\lambda x}$ .It is more convenient to express the general solution in terms of hyperbolic functions.Thus, we write $y(x)$ in the equivalent form

$$y(x) = c_1 e^{\lambda x} + c_2 e^{-\lambda x}$$

$$y(x) = \frac{c_1}{2}(e^{\lambda x} + e^{\lambda x}) + \frac{c_2}{2}(e^{-\lambda x} + e^{-\lambda x})$$

$$y(x) = \frac{c_1}{2}(e^{\lambda x} + e^{\lambda x}) + \frac{c_2}{2}(e^{-\lambda x} + e^{-\lambda x}) + \frac{c_1}{2}(e^{-\lambda x} - e^{-\lambda x}) + \frac{c_2}{2}(e^{\lambda x} - e^{\lambda x})$$

$$y(x) = \frac{c_1}{2}(e^{\lambda x} + e^{-\lambda x}) + \frac{c_2}{2}(e^{\lambda x} - e^{-\lambda x}) + \frac{c_2}{2}(e^{-\lambda x} + e^{\lambda x}) + \frac{c_2}{2}(e^{-\lambda x} - e^{\lambda x})$$

$$y(x) = c_1 \cosh\lambda x + c_1 \sinh\lambda x + c_2 \cosh\lambda x - c_2 \sinh\lambda x$$

$$y(x) = (c_1 + c_2)\cosh\lambda x + (c_1 - c_2)\sinh\lambda x$$

$$y(x) = \bar{c_1}\cosh\lambda x + \bar{c_2}\sinh\lambda x$$

substituting into the boundary conditions in (2.1) yields

$$y'(x) = \lambda\bar{c_1}\sinh\lambda x + \lambda\bar{c_2}\cosh\lambda x$$

$$y'(0) = \lambda\bar{c_1}\sinh\lambda 0 + \lambda\bar{c_2}\cosh\lambda 0$$

$$y'(0) = 0 + \lambda\bar{c_2}$$

$$y'(0) = \lambda\bar{c_2} = 0$$

But$\lambda > 0$ so that $\bar{c_2} = 0$
And

$$y'(x) = \lambda\bar{c_1}\sinh\lambda x + \lambda\bar{c_2}\cosh\lambda x$$

$$\Rightarrow y'(1) = \lambda\bar{c_1}\sinh\lambda 1 + \lambda\bar{c_2}\cosh\lambda 1 = 0$$

$$\Rightarrow y'(1) = \lambda\bar{c_1}\sinh\lambda + \lambda\bar{c_2}\cosh\lambda = 0$$

$$\lambda\bar{c_1}\sinh\lambda = -\lambda\bar{c_2}\cosh\lambda$$

$$\bar{c_1}\sinh\lambda = -\bar{c_2}\cosh\lambda$$

With $\bar{c_2} = 0$ then requires $\bar{c_1}\sinh\lambda = 0$. Now $\bar{c_1} = 0$ or $\sinh\lambda = 0$.
To obtain a nontrivial solution, we must have $\bar{c_1} \neq 0$, so $\sinh\lambda = 0$ but$\lambda \neq 0$ then $\sinh\lambda \neq 0$ .So that once again the only solution of the BVP is the trivial solution .
Case :3 $M = 0$
Here zero is a double root of the auxiliary equation. So a general solution to (2.1) is

$$r^2 = 0$$

$$r = 0$$

$$y(x) = c_1e^{\lambda x} + xc_2e^{\lambda x}$$

$$\Rightarrow y'(x) = \lambda c_1e^{\lambda x} + c_2e^{\lambda x} + \lambda xc_2e^{\lambda x}$$

$$\Rightarrow y'(x) = \lambda c_1e^{\lambda x} + c_2(e^{\lambda x} + \lambda xe^{\lambda x})$$

$y'(0) = 0 + c_2$ with $\lambda = 0$

$$y'(0) = c_2 = 0$$

And

$$y'(x) = \lambda c_1e^{\lambda x} + c_2(e^{\lambda x} + \lambda e^{\lambda x})$$

$$\Rightarrow y'(1) = \lambda c_1e^{\lambda} + c_2(e^{\lambda} + \lambda e^{\lambda})$$

$y'(1) = 0 + c_2$ with $\lambda = 0$

$$y'(1) = c_2 = 0$$

Therefore c_1 can be any number.So that once again the only solution of the BVP is the the trivial solution.
For $M \neq -n^2\Pi^2$ and any $\sigma \in C[0,1]$, the unique solution of the linear problem

$$-y''(x) + My(x) = \sigma(x)\ if\ x \in [o,1]$$
$$y'(0) = A,\ y'(1) = B \tag{2.2}$$

is given by

$$y(x) = P_\lambda(x) + \int G_\lambda(x,s)\sigma(s)ds$$

Where
$p_\lambda(x)=\begin{cases} \frac{1}{\sqrt{\lambda}\sin\sqrt{\lambda}}(A\cos\sqrt{\lambda}(1-x))-B\cos\sqrt{\lambda}(x), & M=-\lambda,\lambda>0; \\ \frac{1}{\sqrt{\lambda}\sinh\sqrt{\lambda}}(B\cosh\sqrt{\lambda}(x))-A\cosh\sqrt{\lambda}(1-x), & M=\lambda,\lambda>0. \end{cases}$
and(for $M=-\lambda$),
$G_\lambda(x,s)=-\frac{1}{\sqrt{\lambda}\sin\sqrt{\lambda}}\begin{cases} \cos\sqrt{\lambda}(1-s)\cos\sqrt{\lambda}x, & 0\le x\le s\le 1; \\ \cos\sqrt{\lambda}(1-x)\cos\sqrt{\lambda}s, & 0\le s\le x\le 1. \end{cases}$
And (for $M=\lambda$),
$G_\lambda(x,s)=-\frac{1}{\sqrt{\lambda}\sinh\sqrt{\lambda}}\begin{cases} \cosh\sqrt{\lambda}(1-s)\cosh\sqrt{\lambda}x, & 0\le x\le s\le 1; \\ \cosh\sqrt{\lambda}(1-x)\cosh\sqrt{\lambda}s, & 0\le s\le x\le 1. \end{cases}$
is the Greens function of the problem. For $M=-\lambda$,we note that $G_\lambda(x,s)\le 0$ if $0<\sqrt{\lambda}\le\frac{\pi}{2}$. Moreover, for such values of M and λ , we have, $P_\lambda(x)\le 0$ if $A\le 0\le B$, and $P_\lambda(x)\ge 0$ if $A\ge 0\ge B$. Thus we have the following anti-maximum Principle.

2.2 Anti-maximum Principle

Let $0\le M\le\frac{\pi^2}{4}$. If $A\le 0\le B$ and $\sigma(x)\ge 0$, then a solution $y(x)$ of (2.2) is such that $y(x)\le 0$. If $A\ge 0\ge B$ and $\sigma(x)\le 0$, then $y(x)\ge 0$.

Consider the nonlinear Neumann problem

$$\begin{aligned} -y''(x) &= f(x,y(x))\ if\ x\in[0,1] \\ y'(0) &= A,\ y'(1)=B \end{aligned} \tag{2.3}$$

where $f:[0,1]XR\to R$ is continuous and $A,B\in R$. Let us recall the concept of lower and upper solutions.

Definition 2.2.1. *Let $\alpha\in C^2[0,1]$. We say that α is a lower solution (or subsolution) of (2.3), if*

$$-\alpha''(x)\le f(x,\alpha(x))\ if\ x\in[0,1]$$

$$\alpha'(0)\ge A,\ \alpha'(1)\le B$$

Definition 2.2.2. *Let $\beta\in C^2[0,1]$. We say that β is an upper solution (or supersolution) of (2.3), if*

$$-\beta''(x)\ge f(x,\beta(x))\ if\ x\in[o,1]$$

$$\beta'(0)\le A,\ \beta'(1)\ge B$$

2.3 Existence of solution via upper and lower solutions

Theorem 2.3.1. ***(Upper and Lower solutions method).*** *Let* $0 < \lambda \leq \frac{\pi^2}{4}$*. Assume that* α *and* β *are respectively lower and upper solutions of (2.3) such that* $\alpha(x) \geq \beta(x), x \in [0,1]$*. If* $f(x,y) - \lambda y$ *is non-increasing in* y*, then there exists a solution* y *of the boundary value problem (2.3) such that*

$$\alpha(x) \geq y(x) \geq \beta(x), x \in [0,1]$$

.

Proof. Define $p(\alpha(x), y, \beta(x)) = \{\min \alpha(x), \max\{y, \beta(x)\}\}$, then $p(\alpha(x), y, \beta(x))$ satisfies $\beta(x) \leq p(\alpha(x), y, \beta(x)) \leq \alpha(x), y \in R, x \in [0,1]$. Consider the modified boundary value problem

$$\begin{aligned} -y''(x) - \lambda y(x) &= F(x, y(x)), x \in [0,1] \\ y'(0) = A, \ y'(1) &= B \end{aligned} \tag{2.4}$$

Where

$$F(x,y) = f(x, p(\alpha(x), y, \beta(x))) - \lambda p(\alpha(x), y, \beta(x))$$

. This is equivalent to the integral equation

$$y(x) = p_\lambda(x) + \int_0^1 G_\lambda(x,s) F(s, y(s)) ds \tag{2.5}$$

Since $p_\lambda(x)$ and $F(x, y(x))$ are continuous and bounded, this integral equation has a fixed point by the Schauder fixed point theorem. Thus, problem (2.4) has a solution. Moreover,

$$F(x, \alpha(x)) = f(x, \alpha(x)) - \lambda\alpha(x) \geq -\alpha''(x) - \lambda\alpha(x), x \in [0,1]$$

$$F(x, \beta(x)) = f(x, \beta(x)) - \lambda\beta(x) \leq -\beta''(x) - \lambda\beta(x), x \in [0,1]$$

Thus α, β are lower and upper solutions of (2.4). Further, we note that any solution $y(x)$ of (2.4) with the property $\beta(x) \leq y(x) \leq \alpha(x), x \in [0,1]$ is also a solution of (2.3). Now, we show that any solution y of (2.4) does satisfy $\beta(x) \leq y(x) \leq \alpha(x), x \in [0,1]$. For this, set $v(x) = \alpha(x) - y(x)$, then $v'(0) \geq 0, v'(1) \leq 0$. In view of the non-increasing property of the function $f(x,y) - \lambda y$ in y , the definition of lower solution and the fact that $p(\alpha(x), y, \beta(x)) \leq \alpha(x)$, we have

$$-v''(x) - \lambda v = (-\alpha''(x) - \lambda\alpha(x)) - (-y''(x) - \lambda y(x))$$

$$\leq (f(x,\alpha(x)) - \lambda\alpha(x)) - f(x, p(\alpha(x), y, \beta(x))) - \lambda p(\alpha(x), y, \beta(x)) \leq 0$$

By the anti-maximum principle, we obtain $v(x) \geq 0, x \in [0,1]$.
Similarly, $y(x) \geq \beta(x), x \in [0,1]$.

Theorem 2.3.2. *Assume that α and β are lower and upper solutions of the boundary value problem (2.3) respectively. If $f : [0,1]XR \to R$ is continuous and*

$$f(x,\alpha(x)) - \lambda\alpha(x) \leq f(x,\beta(x)) - \lambda\beta(x) \tag{2.6}$$

for some $0 < \lambda \leq \frac{\pi^2}{4}, x \in [0,1]$ then $\alpha(x) \geq \beta(x), x \in [0,1]$.

Proof

Define $m(x) = \alpha(x) - \beta(x), x \in [0,1]$, then $m(x) \in C^2[0,1]$ and $m'(0) \geq 0, m'(1) \leq 0$. In view of (2.5) and the definition of upper and lower solution, we have

$$-m''(x) - \lambda m(x) = (-\alpha''(x) - \lambda\alpha(x)) - (-\beta''(x) - \lambda\beta(x))$$

$$\leq (f(x,\alpha(x)) - \lambda\alpha(x)) - f(x,\beta(x)) - \lambda\beta(x) \leq 0$$

Thus, by anti-maximum principle, $m(x) \geq 0, x \in [0,1]$.

Chapter 3

Nonlinear Neumann BVP of second order

In this section we take a closer look at the explicit second order ODE

$$y'' + f(x, y(x)) = 0; 0 < x < 1$$

Where $f : [0, 1]XR \to R$ is a non-linear function along with Neumann condition on the boundary,

$$y'(0) = A,\ y'(1) = B$$

3.1 Quasilinearization Technique

The method of quasilinearization, being a powerful technique for obtaining approximate solutions of nonlinear differential equations, has recently been studied and extended extensively. The fact that one can obtain a monotone sequence of solutions of linear problems, whose convergence to a solution of the original nonlinear problem is quadratic, is one of the reasons for the popularity of this technique.
For the approximation of solution, we develop the method of quasilinearization. We assume f and its partial derivatives up to second order are continuous. We now approximate our problem by the method of quasilinearization. Lets state the following assumption.
(A1) $\alpha, \beta \in c^2[0, 1]$ are respectively lower and upper solutions of (2.3) such that $\alpha(x) \geq \beta(x), x \in [0, 1] = I$.
(A2) $f(x, y), f_y(x, y), f_{yy}(x, y)$ are continuous on IXR and are such that $0 < f_y(x, y) \leq \frac{\pi^2}{4}$ and $f_{yy}(x, y) \in IX[\min \beta(x), \max \alpha(x)]$.

Theorem 3.1.1. *Under assumptions (A1)-(A2), there exists a monotone sequence $\{W_n\}$ of solutions converging uniformly and quadratically to a solution of the problem (2.3).*

Proof. Taylors theorem and the condition $f_{yy}(x,y) \leq 0$ imply that

$$f(x,y) \leq f(x,t) + f_y(x,t)(y-t) \tag{3.1}$$

for $(x,y),(x,t) \in IX[\min \beta(x), \max \alpha(x)]$. Define

$$F(x,y,t) = f(x,t) + f_y(x,t)(y-t) \tag{3.2}$$

$y,t \in R, x \in I$.Then $F(x,y,t)$ is continuous and satisfies the relations

$$f(x,y) \leq F(x,y,t)$$

$$f(x,y) = F(x,y,y) \tag{3.3}$$

for $(x,y),(x,t) \in IX[\min \beta(x), \max \alpha(x)]$. Let
$\lambda = \max f_y(x,y) : (x,y) \in IX[\min \beta(x), \max \alpha(x)]$,then $0 < \lambda \leq \frac{\pi^2}{4}$.
Now, set $w_0 = \beta$ and consider the linear problem

$$-y''(x) - \lambda y(x) = F(x, p(\alpha(x), y(x), w_0(x)) - \lambda p(\alpha(x), y(x), w_0(x))), x \in I$$

$$y'(0) = A,\ y'(1) = B \tag{3.4}$$

This is equivalent to the integral equation

$$y(x) = P_\lambda(x) + \int_0^1 G_\lambda(x,s)[F(s, p(\alpha(s), y(s), w_0(s))) - \lambda p(\alpha(s), y(s), w_0(s))]ds$$

Since $P_\lambda(x)$ and $F(x, p(\alpha, y, w_0), w_0) - \lambda p(\alpha, y, w_0)$ are continuous and bounded, this integral equation has a fixed point w_1 (say) by the Schauder fixed point theorem .Moreover,

$$F(x, p(\alpha(x), w_0(x), w_0(x)), w_0(x)) - \lambda p(\alpha(x), w_0(x), w_0(x)) = f(x, w_0(x)) - \lambda w_0(x)$$

$$\leq -w_0''(x) - \lambda w_0(x), x \in I$$

And

$$F(x, p(\alpha(x), \alpha(x), w_0(x)), w_0(x)) - \lambda p(\alpha(x), \alpha(x), w_0(x)) \geq f(x, \alpha(x)) - \lambda \alpha(x)$$

$$\geq -\alpha''(x) - \lambda \alpha(x), x \in I$$

This implies that α, w_0 are lower and upper solutions of (3.4). Now, we show that $w_0(x) \leq w_1(x) \leq \alpha(x)$ on I.
For this, set $v(x) = w_1(x) - w_0(x)$, then the boundary conditions imply that

$$v'(0) \geq 0,\ v'(1) \leq 0$$

. Further, in view of the condition $f_y(x, y) \leq \lambda$ for $(x, y) \in IX[\min \beta(x), \max \alpha(x)]$ and (3.2), we have

$$-v''(x) - \lambda v(x) = (-w_1''(x) - \lambda w_1(x) - (-w_0''(x) - \lambda w_0(x)))$$

$$\leq (f_y(x, w_0(x)) - \lambda)(p(\alpha(x), w_1(x), w_0(x)) - w_0(x)) \leq 0$$

Thus, by anti-maximum principle, we obtain $v(x) \geq 0, x \in I$. Similarly, $\alpha(x) \geq w_1(x)$. Thus,

$$w_0(x) \leq w_1(x) \leq \alpha(x), x \in I \tag{3.5}$$

In view of (3.3) and the fact that w_1 is a solution of (3.4) with the property (3.5), we have

$$-w_1''(x) = F(x, w_1(x), w_0(x)) \geq f(x, w_1(x))$$
$$w_1'(0) = A,\ w_1'(1) = B \tag{3.6}$$

which implies that w_1 is an upper solution of (2.3). Now, consider the problem

$$-y''(x) - \lambda y(x) = F(x, p(\alpha(x), y(x), w_1(x)), w_1(x)) - \lambda p(\alpha(x), y(x), w_1(x)), x \in I$$
$$y_1'(0) = A,\ y_1'(1) = B \tag{3.7}$$

Denote by w_2 a solution of (3.7). In order to show that

$$w_1(x) \leq w_2(x) \leq \alpha(x), x \in I \tag{3.8}$$

set $v(x) = w_2(x) - w_1(x)$, then $v'(0) = 0, v'(1) = 0$. Further, in view of (3.2) and the condition $f_y(x, y) \leq \lambda$ for $(t, x) \in IX[\min \beta(x), \max \alpha(x)]$, we obtain

$$-v''(x) - \lambda v(x) \leq F(x, p(\alpha(x), w_2(x), w_1(x)), w_1(x)) - \lambda p(\alpha(x), w_2(x), w_1(x))$$

$$-f(x, w_1(x)) - \lambda w_1(x)$$
$$\leq (f_y(x, w_1(x)) - \lambda)(p(\alpha(x), w_2(x), w_1(x)) - w_1(x)) \leq 0, x \in I$$

Hence $w_2(x) \geq w_1(x)$ follows from the anti-maximum principle.
Similarly, we can show that $\alpha(x) \geq w_2(x)$ on I. Continuing this process, we obtain a monotone sequence $\{w_n\}$ of solutions satisfying

$$W_0(x) \leq w_1(x) \leq w_2(x) \leq \ldots \leq w_n(x) \leq \alpha(x), x \in I \tag{3.9}$$

where, the element w_n of the sequence $\{w_n\}$ that for $x \in I$, satisfies

$$-y''(x) - \lambda y(x) = F(x, p(\alpha(x), y(x), w_{n-1}(x)), w_{n-1}(x)) - \lambda p(\alpha(x), y(x), w_{n-1}(x)), x \in I$$

$$y_1'(0) = A,\ y_1'(1) = B$$

That is,

$$-w_n''(x) = F(x, w_n(x), w_{n-1}(x)), x \in I$$

$$w_n'(0) = A,\ w_n'(1) = B$$

It follows that the convergence of the sequence is uniform. If $y(x)$ is the limit point of the sequence, since F is continuous,we have

$$\lim_{n\to\infty} F(x, w_n(x), w_{n-1}(x)) = F(x, y(x), y(x)) = f(x, y(x))$$

which implies that,y is a solution of the boundary value problem (2.3). Now, we show that the convergence of the sequence is quadratic. For this, set $e_n(x) = y(x) - w_n(x), x \in I, n \in N$, where y is a solution of (2.3). Note that, $e_n(x) \geq 0$ on I and $e_n'(0) = 0, e_n'(1) = 0$.
Let $\rho = \min f_y(x, y) : (x, y) \in IX[\min \beta(x), \max \alpha(x)]$, then $0 < \rho < \frac{\pi^2}{4}$.
Using Taylors theorem and (3.2), we obtain

$$-e_n''(x) = -y''(x) + w_{(n)}''(x) = f(x, y(x)) - F(x, w_n(x), (w_{n-1}(x)))$$

$$= f(x, w_{n-1}(x)) + f_y(x, w_{n-1}(x))(y(x) - w_{n-1}(x)) + \frac{f_{yy}(x), \varepsilon(x)}{2}(y(x) - w_{n-1}(x)) -$$

$$[f(x, w_{n-1}(x)) + f_y(x, w_{n-1}(x))(w_n(x) - w_{n-1}(x))]$$

$$= f_y(x, w_{n-1}(x))e_n(x) + \frac{f_{yy}(x, \varepsilon(x))}{2} e_{n-1}^2(x)$$

$$\geq \rho e_n(x) + \frac{f_{yy}(x, \varepsilon(x))}{2} \|e_{n-1}\|^2, x \in I \tag{3.10}$$

where, $w_{n-1}(x) < \varepsilon(x) < y(x)$. Thus, by comparison of results, the error function e_n satisfies $e_n(x) \leq r(x), x \in I$, where r is the unique solution of the boundary-value problem

$$-r''(x) - \rho r(x) = \frac{f_{yy}(x, \varepsilon(x))}{2} \|e_{n-1}\|^2, x \in I \tag{3.11}$$

and

$$r(x) = \int G(x,s)\frac{f_{yy}(x,\varepsilon(x))}{2}\|e_{n-1}\|^2 ds \leq \delta\|e_{n-1}\|^2$$

$$\delta = \max\{\frac{1}{2}/G_\rho(x,s)f_{yy}(s,y) : (x,y) \in Ix[\min\beta(x), \max\alpha(x)]\}$$

where $\delta = \max\frac{1}{2}/G_\rho(x,s)f_{yy}(s,y) : (x,y) \in Ix[\min\beta(x), \max\alpha(x)]$.
Thus$\|e_n\| \leq \delta\|e_{n-1}\|^2$

3.2 Generalized Quasillinearization Technique

Now we introduce an auxiliary function ϕ to relax the convexity conditions on the function f and hence prove results on the generalized quasilinearization. Let
(B1) $\alpha, \beta \in c^2(I)$are lower and upper solutions of (2.3) respectively, such that $\alpha(x) \geq \beta(x)$ on I.
(B2) $f \in C^2(IXR)$ and is such that $0 < f_y(x,y) \leq \frac{\pi^2}{4}$ for $(x,y) \in IX[\min\beta(x), \max\alpha(x)]$ and$\frac{\partial^2}{\partial y^2}(f(x,y)) + \phi(x,y) \leq 0$ on $IX[\min\beta(x), \max\alpha(x)]$, for some function $\phi \in c^2(IXR)$ satisfies $\phi''(x,y) \leq 0$ on $IX[\min\beta(x), \max\alpha(x)]$.

Theorem 3.2.1. *Under assumptions (B1)-(B2), there exists a monotone sequence $\{w_n\}$ of solutions converging uniformly and quadratically to a solution of the problem (2.3).*

Proof Define $F : IXR \to R$ by

$$F(x,y) = f(x,y) + \phi(x,y) \tag{3.12}$$

. Then, in view of (B2), we have $F(x,y) \in c^2(IXR)$ and

$$F_{yy}(x,y) \leq 0 \; on \; IX[\min\beta(x), \max\alpha(x)] \tag{3.13}$$

, which implies

$$f(x,y) \leq F(x,t) + F_y(x,t)(y-t) - \phi(x,y) \tag{3.14}$$

for $(x,y),(x,t) \in IX[\min\beta(x), \max\alpha(x)]$. Using Taylors theorem on ϕ, we obtain $\phi(x,y) = \phi(x,t) + \phi_y(x,t)(y-t) + \frac{\phi_{yy}}{2!^{\text{‘}}}\|y-t\|^2$ where $y,t \in R, x \in I$ and μ lies between y and t. In view of (B2), we have

$$\phi(x,y) \leq \phi(x,t) + \phi_y(x,t)(y-t) \tag{3.15}$$

for $(x,y),(x,t) \in IX[\min\beta(x), \max\alpha(x)]$ And

$$\phi(x,y) \geq \phi(x,t) + \phi_y(x,t)(x-t) - \frac{M}{2}\|y-t\|^2 \tag{3.16}$$

for $(x, y), (x, t) \in IX[\min \beta(x), \max \alpha(x)]$ Where

$$M = \max\{\phi_{yy}(x, y) : (x, y) \in Ix[\min \beta(x), \max \alpha(x)]\}$$

Using (3.16) in (3.14), we obtain

$$f(x, y) \leq f(x, t) + f_y(x, t)(y - t) + \frac{M}{2}\|y - t\|^2 \tag{3.17}$$

for $(x, y), (x, t) \in IX[\min \beta(x), \max \alpha(x)]$. Define

$$F^\star(x, y, t) = f(x, t) + f_y(x, t)(y - t) + \frac{M}{2}\|y - t\|^2 \tag{3.18}$$

for $x \in I, y, t \in R$. then, $F^\star(x, y, t)$ is continuous and for $(x, y), (x, t) \in IX[\min \beta(x), \max \alpha(x)]$, satisfies the following relations

$$\begin{aligned} f(x, y) &\leq F^\star(x, y, t) \\ f(x, y) &= F^\star(x, y, y) \end{aligned} \tag{3.19}$$

Now, we set $\beta = w_0$ and consider the Neumann problem

$$-y''(x) - \lambda y(x) = F(x, p(\alpha(x), y(x), w_0(x)), w_0(x)) - \lambda p(\alpha(x), y(x), w_0(x)), x \in I$$

$$y'(0) = A, \ y'(1) = B \tag{3.20}$$

where λ and p are the same as defined in Theorem 3.1. Since

$$F^\star(x, p(\alpha, y, w_0), w_0) - \lambda p(\alpha, y, w_0)$$

is continuous and bounded, it follows that the problem (3.20) has a solution. Also, we note that any solution y of (3.20) which satisfies

$$w_0(x) \leq y(x) \leq \alpha(x), x \in I \tag{3.21}$$

, is a solution of

$$-y''(x) = F^\star(x, y(x), w_0(x))$$
$$y'(0) = A, \ y'(1) = B$$

and in view of (3.19),$F^\star(x, y(x), w_0(x)) \geq f(x, y(x))$. It follows that any solution y of (3.20) with the property (3.21) is an upper solution of (2.3).

Now, set $v(x) = \alpha(x) - y(x)$, where y is a solution of (3.20), then $v'(0) \geq 0, v'(1) \leq 0$. Moreover,using (B2) and (3.19), we obtain

$$-v''(x) - \lambda v(x) \leq (-\alpha''(x) - \lambda\alpha(x) - (-y''(x) - \lambda y(x)))$$

$$\leq f(x,\alpha(x)) - [F^{\star}(x, p(\alpha(x), y(x), w_0(x)), w_0(x)) - \lambda p(\alpha(x), y(x), w_0(x))]$$

$$\leq (f_y(x, w_1(x)) - \lambda)(p(\alpha(x), w_2(x), w_1(x)) - w_1(x)), x \in I$$

$$\leq f(x,\alpha(x)) - \alpha(x) - [f(x, p(\alpha(x), y(x), w_0(x))) - \lambda p(\alpha(x), y(x), w_0(x))] \leq 0$$

Hence, by anti-maximum principle,$\alpha(x) \geq y(x), x \in I$. Similarly,$w_0(x) \leq y(x), x \in I$. Continuing this process we obtain a monotone sequence $\{w_n\}$ of solutions satisfying

$$w_0(x) \leq w_1(x) \leq w_2(x) \leq w_3(x) \leq \ldots \leq w_{n-1}(x) \leq w_n(x) \leq \alpha(x), x \in I.$$

The same arguments as in Theorem 3.1.1, shows that the sequence converges to a solution y of the boundary value problem (2.3). Now we show that the convergence of the sequence of solutions is quadratic. For this, we set $e_n(x) = y(x) - w_n(x), x \in I$, where y is a solution of the boundary-value problem (2.3).

Note that,$e_n(x) \geq 0$ on I and,$e'_n(0) = 0, e'_n(1) = 0$. Using Taylors theorem, (3.15) and the fact that $\|w_n - w_{n-1}\| \leq \|e'_{n-1}\|$, we obtain

$$-e''_n(x) = -y''(x) + w''_n(x)$$

$$= (F(x, y(x)) - \phi(x, y(x))) - F^{\star}(x, w_n(x), w_{n-1}(x))$$

$$\Rightarrow F(x, w_{n-1}(x)) + F_y(x, w_{n-1}(x))(y(x) - w_{n-1}(x)) + \frac{F_{yy}(x,\varepsilon(x))}{2}(y(x) - w_{n-1}(x))^2$$

$$-[\phi(x, w_{n-1}(x)) + \phi_y(x, w_{n-1}(x))(y(x) - w_{n-1}(x))]^2$$

$$-[f(x, w_{n-1}(x)) + f_y(x, w_{n-1}(x))(w_n(x) - w_{n-1}(x) + \frac{m}{2}\|w_n - w_{n-1}\|^2)]$$

$$= f_y(x, w_{n-1}(x))e_n(x) + \frac{F_{yy}(x,\varepsilon(x))}{2}e^2_{n-1}(x) - \frac{m}{2}\|w_n - w_{n-1}\|^2$$

$$\geq f_y(x, w_{n-1}(x))e_n(x) - \frac{F_{yy}(x,\varepsilon(x))}{2}e^2_{n-1}(x) + \frac{m}{2}\|w_n - w_{n-1}\|^2$$

$$\geq \rho e_n(x) - Q\|e_{n-1}\|^2, x \in I$$

where, $w_{n-1}(x) \leq \varepsilon(x) \leq y(x)$,

$$Q = \max\{\frac{\mid F_{yy}(x,y) \mid}{2} + \frac{m}{2} : (x,y) \in Ix[\min \beta(x), \max \alpha(x)]\}$$

and ρ is defined as in Theorem 3.1.1 Thus, by comparison of results $e_n(x) \leq r(x), x \in I$, where r is a unique solution of the linear problem

$$-r''(x) - \rho r(x) = -Q\|e_{n-1}\|^2, x \in I$$

$$r'(0) = 0,\ r'(1) = 0,$$

and

$$r(x) = Q \int (G_\rho(x,s) \|e_{n-1}\|^2) ds \leq \sigma \|e_{n-1}\|^2$$

where $\sigma = Q \max\{| G_\rho(x,s) | : (x,s) \in IxI\}$. Thus$\|e_n\| \leq \sigma \|e_{n-1}\|^2 \|e_n\| \leq \sigma \|e_{n-1}\|^2$

Summary

A quasilinearization technique is applied to obtain monotone sequences of lower and upper solutions converging quadratically to the unique solution of a second order ordinary differential equation with nonlinear neumann boundary conditions. In fact, the existence of the unique solution of the boundary value problem is established by relaxing the convexity assumptions on the nonlinear functions present in the differential equation and the boundary conditions.

Appendix

A.1 Theorem (Existence of Fixed points)

Let T be a continuous mapping on a Banach space X. Then the following statements hold true:
1. If there exist $x, y \in X$ such that

$$\lim_{n\to\infty} T^n(x) = y$$

then y is a fixed point for T, i.e T(y)=y. 2.If $T(X)$ is a compact set in X and for each $\in > 0$ there exists $a, x_\epsilon \in X$ such that

$$|| \, Tx_\epsilon - x_\epsilon \, || < \epsilon$$

then T has a fixed point.

Proof Set $y_n = T^n(x), n = 1, 2, \ldots$ If T is a continuous mapping then

$$T(y) = T(\lim_{n\to\infty} y_n) = \lim_{n\to\infty} T(y_n) = \lim_{n\to\infty} y_{n+1} = y,$$

which proves the first statement. Assume that the assumptions of 2 are satisfied. Then for $n = 1, 2, \ldots$ there are $x_n \in X$ such that $|| \, T(x_n) - x_n \, || < \frac{1}{n}$. $T(X)$ is a compact set implies that there exist a convergent subsequence $(T(x_{n_k}))_{k=1}^{\infty}$ of $(T(x_n))_{n=1}^{\infty}$. Call the limit point x. Then x is a fixed point for T since also the sequence $(x_{n_k})_{k=1}^{\infty}$ converges to x according to (1)and T is continuous.

A.2 Theorem (Banach Fixed point Theorem)

Let T be a contraction on a Banach space X.Then T has a unique fixed point.

Proof Fix an arbitrary element $z \in X$ and consider the sequence

$$(T^n(z))_{n=1}^{\infty}$$

Set $Z_n = T^n(z)$ for $n = 1, 2, \ldots$ .We note that

$$\| z_n - z_m \| \leq \| z_n - z_{n-1} \| + \ldots + \| z_{m+1} - z_m \|$$

$$= \| T(z_{n-1}) - T_{z_{n-2}} \| + \ldots + \| T(z_m) - T(z_{m-1}) \| \leq$$
$$\leq C \| z_{n-1} - z_{n-2} \| + \ldots + C \| z_m - z_{m-1} \| \leq \ldots \leq$$
$$\leq (C^{n-1} + C^{n-2} + \ldots + C^{m-1}) \| z_1 - z \| \leq \frac{C^{m-1}}{1-c} \| z_1 - z \|$$

Where we have assumed $n > m \geq 1$. This yields $\| z_1 - z_m \| \to 0$ as $n, m \to \infty$ and hence $(z_n)_{n=1}^{\infty}$ is a Cauchy sequence.Since X is a Banach Space the sequence converges, i.e, there is a $x_0 \in X$ such that $z_n \to x_0$ as $n \to \infty$. Here x_0 is a fixed point for T Since

$$\| T(x_0 - x_0) \| \leq \| T(x_0 - T(z_n)) \| + \| z_{n+1} - x_0 \| \leq C \| x_0 - z_n + \| + \| z_{n+1} - x_0 \|$$

Where the LHS is independent of n and the RHS tends to 0 as $n \to \infty$. The uniqueness follows from the contraction property for T. If $x_0 \neq y_0$ both are fixed points of T then we get

$$\| x_0 - y_0 \| = \| T(x_0 - T(y_0)) \| \leq C \| x_0 - y_0 \| < \| x_0 - y_0 \|$$

Which yields a contradiction.

A.3 Theorem (Brauwer Fixed point Theorem)

Assume that K is a compact convex subset of R^n and that $T : K \to K$ is a continuous mapping. Then T has a fixed point in K.

Proof Assume that $K = \bar{B}(0,1)$ and that has no fixed point.Define the mapping $A : B(\bar{0},1) \to B(\bar{0},1)$ as follows: For every inner point $x \in B(\bar{0},1)$ let $\tilde{x}$ denote the point on the boundary $\partial B(0,1)$ that is the intersection of the ray from $T(x)$ through x and the boundary $\partial B(0,1)$. The ray is always well-defined since T has no fixed point.Now set

$$A(x) = \begin{cases} \tilde{x}, if x \in B(0,1) \\ x, if x \in \partial B(0,1) \end{cases}$$

A.4 Theorem (Schauder Fixed point Theorem)

Assume that K is a convex compact set in a Banach space X and that $T : K \to K$ is a continuous mapping. Then T has a fixed point.

Proof This will be done by approximating the compact se K by compact sets $K_n, n = 1, 2, \ldots$ in finite-dimensional spaces and approximation becomes better and better for larger n. Brouwer's fixed point theorem gives a sequence of points (x_n) that are fixed points for the sequence (T_n),from which a converging subsequence of points (x_{n_k}) can be extracted. The limit element of this sequence will be a fixed point for T. For Every positive integer n we define mappings P_n, called Schauder projections,as follows: The compactness of K implies that there are finitely many elements $x_1, \ldots x_k \in K$ such that

$$K \subset \bigcup_{i=1}^{k} B(x_i, \frac{1}{n}).$$

$$f_i = \max(0, \frac{1}{n} - \| x - x_i \|), i = 1 \ldots k.$$

For every $x \in K$ there exists an i such that$f_i(x) > 0$. This implies that $\sum_{i=1}^{k} f_i(x) > 0$ for all $x \in K$. Now set $K_n = cox_1, \ldots, x_k$ and $P_n(x) = \frac{\sum_{i=1}^{k} f_i(x)x_i}{\sum_{i=1}^{k} f_i(x)}, x \in K$ Finally We define $T_n = P_nT \mid_{k_n}$. We can now apply Brouwer's theorem to every mapping

$$T_n : K_n \to K_n, n = 1, 2, \ldots$$

his yields a sequence of fixed points x_n for T_n. We obtain $P_nT(x_n) = x_n$, and hence we get

$$\| T(x_n - x_n) \| < \frac{1}{n}$$