ON THE METHOD OF GREEN'S FUNCTIONS

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A thesis submitted to the department of mathematics, Addis Ababa University in partial fulfillment for the requirements of the degree of master of science in mathematics.

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Declaration

This thesis submitted to the Department of Mathematics, College of Natural and Computational Sciences, Addis Ababa University in partial fulfillment of the degree of Master of Science in Mathematics. As a result, I declare that this thesis has been written by me and no part has been submitted to any other institution and location in order to receive a degree or academic qualification.

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Abstract

In this thesis, the construction of Green's function for IVP and BVP is discussed. We established continuity and symmetric properties of Green's function for ODEs, especially for Sturm-Liouville problems. We demonstrated the method of Green's function in conjunction with the Fourier transform to solve PDEs.

Introduction

The method of Green's function is a technique widely used to solve a non-homogeneous differential equation (ODE and PDE). The method essentially consists in finding an integral operator which produces a solution satisfying a set of prescribed conditions (IC and/or BC). As such, a Green's function is an integral kernel that can be employed to solve differential equations such as ordinary differential equations with initial or boundary value conditions, as well as inhomogeneous partial differential equations (PDE) with boundary conditions. In the study of linear PDEs, Green's function arises in connection with fundamental solutions.

The method of Green's function finds a number of applications in several areas of science such as physics, especially in the study of electricity and magnetism. George Green $(1793 - 1841)$ in his 1823 investigation of electric potential in a bounded domain considered Poisson's equation

$$\Delta u = f$$

and introduced the function for which afterwards Bernhard Riemann coined the nomenclature Green's function. The solution of the above equation is basically,

$$u = \Delta^{-1} f$$

which might be written as

$$u = \int G(x, y) f(y) dy$$

The integral kernel, i.e the function $G(x, y)$ is what we call Green's function.

This thesis is organized as follows, in chapter 1 we visit some mathematical preliminaries that we use in the sequel and in chapter 2 we study Green's function for ODEs. Finally in chapter 3 we investigate Greens function for PDEs.

Chapter 1

Preliminaries

1.1 Heaviside Function and Dirac Delta Function

Definition 1.1.1. *Heaviside function $H(x - x_0)$ is defined as*

$$H(x - x_0) = \begin{cases} 1, & x > x_0 \\ 0, & x < x_0. \end{cases}$$

Let $G(x)$ be a continuous function everywhere except at the point $x = x_0$, at that point it has a jump discontinuity such that

$$G(x) = \begin{cases} G_1(x), & x > x_0 \\ G_2(x), & x < x_0. \end{cases}$$

Then $G(x)$ can be written as

$$G(x) = G_1(x)H(x - x_0) + G_2(x)H(x_0 - x).$$

Define a pulse of unit length starting at x_i as

$$v(x_i) = \begin{cases} 1, & (x_i, \triangle x + x_i) \\ 0, & \textit{otherwise.} \end{cases}$$

Then any function $g(x)$ can be written as

$$g(x) \cong \sum_{i=1}^{N} g(x_i)v(x_i)) \text{ on an interval } (a, b)$$

This is a piecewise constant approximation of $g(x)$. As $\triangle x \to 0$ the number of points $N \to \infty$. Hence usig the limit, the infinite series becomes the integral. Therefore

$$g(x) = \int_a^b g(x_i)\delta(x - x_i)dx$$

where $\delta(x - x_i)$ is the limit of unit pulse $v(x_i)$ divided by $\triangle x$.
The properties of Dirac delta function $\delta(x - \zeta)$, which has several significants are :

$$\delta(x - \zeta) = 0, \ for \ x \neq \zeta$$

$$\int_a^b \delta(x - \zeta)dx = \begin{cases} 0, a, b < \zeta \ or \ \zeta < a, b, \\ 1, a \leq \zeta \leq b \end{cases}$$

and

$$\int_{-\infty}^{\infty} \delta(x - \zeta)g(x)dx = g(\zeta)$$

where $g(x)$ is a sufficiently smooth function, which is called the sifting property or the reproducing property of the delta function. when $g(x) = 1$, then the sifting property becomes

$$\int_{-\infty}^{\infty} \delta(x - \zeta)dx = 1$$

The Green's function $G(x, \zeta)$ is related with Dirac delta function as

$$LG(x, \zeta) = \delta(x - \zeta).$$

where L is linear differential operator.

1.2 Fourier Transform and Inverse of Fourier Transform

Definition 1.2.1. *The Fourier transform of a function f denoted by $\hat{f}(x)$ is given by*

$$\hat{f}(x) = \int f(w)e^{-iw.x}dw.$$

The inverse Fourier transform of a function f, denoted by $f(w)$ is defined as

$$f(w) = \frac{1}{2\pi} \int f(x)e^{iw.x}dx.$$

We can extend the theory of R_1 to n dimensional space. That is for $n = 3$, we have

$$\hat{f}(x, y, z) = \int \int \int f(w_1, w_2, w_3)e^{-i(w_1,w_2,w_3).(x,y,z)}dw_1dw_2dw_3.$$

and

$$f(w_1, w_2, w_3) = \frac{1}{(2\pi)^3} \int \int \int f(x, y, z)e^{i(w_1,w_2,w_3).(x,y,z)}dxdydz.$$

1.3 Leibnitz's Rule

For any continuous function $G(x, \zeta)$ whose derivative $G_x(x, \zeta)$ is piecewise continuous, we have

$$\frac{d}{dx} \int_{f_1(x)}^{f_2(x)} G(x, \zeta)d\zeta = G(x, f_2(x))f_2\prime(x) - G(x, f_1(x))f_1\prime(x) + \int_{f_1(x)}^{f_2(x)} G_x(x, \zeta)d\zeta$$

where $f_1(x)$ and $f_2(x)$ are differentiable function.

1.4 The Method of Eigenfunction Expansions

Consider nonhomogeneous heat equation

$$u_t - ku_{xx} = Q(x,t), x \in (0,L) \tag{1.1}$$

subject to

$$u(0,t) = u(L,t) = 0 \ and \ u(x,0) = f(x) \tag{1.2}$$

The solution of the homogeneous PDE leads to the eigenfunctions

$$\phi_n(x) = \sin \frac{n\pi}{L} x, n = 1,2,3,... \ and \ eigenvalues \ \lambda_n = (\frac{n\pi}{L})^2, n = 1,2,3,...$$

The eigenfunctions depend on the boundary conditions and the PDE. Having the eigenfunctions, we expand the source term

$$Q(x,t) = \sum_{n=1}^{\infty} q_n(t)\phi_n(x), \ where \ q_n(t) = \frac{\int_0^L Q(x,t)\phi_n(x)dx}{\int_0^L \phi_n^2(x)dx}$$

Let

$$u(x,t) = \sum_{n=1}^{\infty} u_n(t)\phi_n(x). \tag{1.3}$$

Then

$$f(x) = u(x,0) = \sum_{n=1}^{\infty} u_n(0)\phi_n(x).$$

Since $f(x)$ is known, we have

$$u_n(0) = \frac{\int_0^L f(x)\phi_n(x)dx}{\int_0^L \phi_n^2(x)dx} \tag{1.4}$$

Differentiate $u(x,t)$ in terms of t once and x twice and then substitute in the given PDE, we get

$$\sum_{n=1}^{\infty} u_n\prime(t)\phi_n(x) = \sum_{n=1}^{\infty}(-k\lambda_n)u_n(t)\phi_n(x) + \sum_{n=1}^{\infty} q_n(t)\phi_n(x).$$

This implies that

$$u_n\prime(t) + k\lambda_n)u_n(t) = q_n(t).$$

Using the method of variation of parameters, we have

$$u_n(t) = u_n(0)e^{-\lambda_n kt} + \int_0^t q_n(\tau)e^{-\lambda_n k(t-\tau)d\tau} \tag{1.5}$$

Thus, by substitute equations (1.5) and (1.4) in an equation (1.3) we summarize the solution of BVP (1.1) subject to (1.2).

Chapter 2

Green's Function for Ordinary Differential Equations

2.1 Linear Boundary Value Problems

Definition 2.1.1. *A differential equation subject to boundary conditions is called boundary value problem . The second order linear differential equation*

$$q_1(x)y'' + q_2(x)y' + q_3(x)y = f(x) \tag{2.1}$$

on the interval $I = [a, b]$ *subject to boundary conditions*

$$\begin{cases} l_1(y) = a_0 y(a) + a_1 y^{'}(a) + b_0 y(b) + b_1 y^{'}(b) = A \\ l_2(y) = c_0 y(a) + c_1 y^{'}(a) + d_0 y(b) + d_1 y^{'}(b) = B \end{cases} \tag{2.2}$$

where a_i, b_i, c_i, d_i, $A \neq 0$ *and* $B \neq 0$, $i = 0, 1$ *are arbitrary constants, is called nonhomogeneous boundary value problem. An associated homogeneous differential equation of* (2.1) *subject to boundary conditions*

$$\begin{cases} l_1(y) = a_0 y(a) + a_1 y^{'}(a) + b_0 y(b) + b_1 y^{'}(b) = 0 \\ l_2(y) = c_0 y(a) + c_1 y^{'}(a) + d_0 y(b) + d_1 y^{'}(b) = 0 \end{cases} \tag{2.3}$$

is called homogeneous boundary value problem.

Theorem 2.1.1. *Let* $y_1(x)$ *and* $y_2(x)$ *be any two linearly independent solutions of the associated homogeneous of DE* (2.1)*. Then the homogeneous boundary value problem*

$$q_1(x)y^{''} + q_2(x)y^{'} + q_3(x)y = 0$$

subject to boundary conditions $l_1(y) = 0, l_2(y) = 0$ *has only the trivial solution if and only if*

$$\Delta = \begin{vmatrix} l_1(y_1) & l_1(y_2) \\ l_2(y_1) & l_2(y_2) \end{vmatrix} \neq 0$$

Proof. Since $y_1(x)$ and $y_2(x)$ are linearly independent solutions of

$$q_1(x)y^{''} + q_2(x)y^{'} + q_3(x)y = 0,$$

its general solution can be written as

$$y(x) = \alpha y_1(x) + \beta y_2(x)$$

Hence

$$y(x) = \alpha y_1(x) + \beta y_2(x)$$

is the solution of

$$q_1(x)y^{''} + q_2(x)y^{'} + q_3(x)y = 0$$

subject to $l_1(y) = 0, l_2(y) = 0$ if and only if

$$l_1(\alpha y_1(x) + \beta y_2(x)) = \alpha l_1(y_1(x)) + \beta l_1(y_2(x)) = 0$$

$$l_2(\alpha y_1(x) + \beta y_2(x)) = \alpha l_2(y_1(x)) + \beta l_2(y_2(x)) = 0$$

Thus, this system has trivial solution if and only if

$$\begin{vmatrix} l_1(y_1) & l_1(y_2) \\ l_2(y_1) & l_2(y_2) \end{vmatrix} \neq 0$$

□

Theorem 2.1.2. *The nonhomogeneous boundary value problem*

$$q_1(x)y'' + q_2(x)y' + q_3(x)y = f(x)$$

subject to boundary conditions

$$l_1(y) = A$$

$$l_2(y) = B$$

has a unique solution if and only if the homogeneous boundary value problem

$$q_1(x)y'' + q_2(x)y' + q_3(x)y = 0$$

subject to boundary conditions

$$l_1(y) = 0$$

$$l_2(y) = 0$$

has only the trivial solution.

Proof. Let $y_1(x)$ and $y_2(x)$ be linearly independent solutions of $q_1(x)y'' + q_2(x)y' + q_3(x)y = 0$ and $w(x)$ be a particular solution of $q_1(x)y'' + q_2(x)y' + q_3(x)y = f(x)$. Then $y(x) = \alpha y_1(x) + \beta y_2(x) + w(x)$ is the general solution of $q_1(x)y'' + q_2(x)y' + q_3(x)y = f(x)$.

However, $y(x)$ is a solution of problem $q_1(x)y'' + q_2(x)y' + q_3(x)y = f(x)$ with $l_1(y) = A, l_2(y) = B$ if and only if

$$l_1(y) = l_1(\alpha y_1 + \beta y_2 + w(x)) = \alpha l_1(y_1) + \beta l_1(y_2) + l_1(w) = A$$

$$l_2(y) = l_2(\alpha y_1 + \beta y_2 + w(x)) = \alpha l_2(y_1) + \beta l_2(y_2) + l_2(w) = B$$

This system has a unique solution if and only if

$$\begin{vmatrix} l_1(y_1) & l_1(y_2) \\ l_2(y_1) & l_2(y_2) \end{vmatrix} \neq 0$$

Applying theorem (2.1.1), $\triangle \neq 0$ implies that the problem

$$q_1(x)y'' + q_2(x)y' + q_3(x)y = 0$$

subject to boundary conditions $l_1(y) = 0, l_2(y) = 0$ has only trivial solution. □

Corollary 2.1.1. *The boundary value problem $y'' = f(x)$ subject to boundary conditions*

$$l_1(y) = a_0 y(a) + a_1 y'(a) = 0$$

$$l_2(y) = d_0 y(b) + d_1 y'(b) = 0$$

has a unique solution if and only if $\Delta = a_0 d_0 (b - a) + a_0 d_1 - a_1 d_0 \neq 0$

Proof. Associated homogeneous of DE $y'' = f(x)$ is $y'' = 0$. The general solution of the given homogeneous equation can be written as $y(x) = \beta_1 + \beta_2 x$, where β_1 and β_2 are arbitrary constants.
$\Rightarrow y_1(x) = 1$ and $y_2(x) = x$. And

$$\begin{cases} l_1(y_1(x)) = a_0 y_1(a) + a_1 y_1'(a) = a_0 \\ l_1(y_2(x)) = a_0 y_2(a) + a_1 y_2'(a) = a_0 a + a_1 \\ l_2(y_1(x)) = d_0 y_1(b) + d_1 y_1'(b) = d_0 \\ l_2(y_2(x)) = d_0 y_2(b) + d_1 y_2'(b) = b d_0 + d_1 \end{cases}$$

$$\Delta = \begin{vmatrix} a_0 & a_0 a + a_1 \\ d_0 & b d_0 + d_1 \end{vmatrix} = (a_0)(b d_0 + d_1) - (d_0)(a_0 a + a_1) = a_0 d_0 (b - a) + a_0 d_1 - a_1 d_0$$

Since $W(y_1, y_2) \neq 0$, $y_1$and y_2 are linearly independent solutions of the given DE. Applying theorem (2.1.1), $y'' = 0$ subject to boundary conditions

$$l_1(y) = a_0 y(a) + a_1 y'(a) = 0$$

$$l_2(y) = d_0 y(b) + d_1 y'(b) = 0$$

has trivial solution if and only if $a_0 d_0 (b - a) + a_0 d_1 - a_1 d_0 \neq 0$.

Therefore, by theorem (2.1.2), the boundary value problem $y^{''} = f(x)$ subject to boundary conditions

$$l_1(y) = a_0 y(a) + a_1 y^{'}(a) = 0$$

$$l_2(y) = d_0 y(b) + d_1 y^{'}(b) = 0$$

has a unique solution if and only if $\Delta = a_0 d_0 (b-a) + a_0 d_1 - a_1 d_0 \neq 0$ □

Theorem 2.1.3. *Let $y_1(x)$ and $y_2(x)$ be the solutions of the boundary value problems*

$$q_1(x)y'' + q_2(x)y' + q_3(x)y = 0$$

subject to

$$l_1(y) = A$$

$$l_2(y) = B$$

and

$$q_1(x)y'' + q_2(x)y' + q_3(x)y = f(x)$$

subject to

$$l_1(y) = 0$$

$$l_2(y) = 0.$$

respectively. Then $y_1(x) + y_2(x)$ is a solution of the boundary value problem

$$q_1(x)y'' + q_2(x)y' + q_3(x)y = f(x)$$

subject to

$$l_1(y) = A$$

$$l_2(y) = B$$

Proof. Since $y_1(x)$ is the solution of

$$q_1(x)y'' + q_2(x)y' + q_3(x)y = 0$$

subject to

$$l_1(y) = A$$

$$l_2(y) = B$$

we have

$$q_1(x)y_1'' + q_2(x)y_1' + q_3(x)y_1 = 0$$

subject to

$$l_1(y_1) = a_0 y_1(a) + a_1 y_1^{'}(a) + b_0 y_1(b) + b_1 y_1^{'}(b) = A$$

$$l_2(y_1) = c_0 y_1(a) + c_1 y_1^{'}(a) + d_0 y_1(b) + d_1 y_1^{'}(b) = B$$

and since $y_2(x)$ is the solution of

$$q_1(x)y'' + q_2(x)y' + q_3(x)y = f(x)$$

subject to

$$l_1(y) = 0$$

$$l_2(y) = 0$$

we have

$$q_1(x)y_2'' + q_2(x)y_2' + q_3(x)y_2 = f(x)$$

subject to

$$l_1(y_2) = a_0 y_2(a) + a_1 y_2^{'}(a) + b_0 y_2(b) + b_1 y_2^{'}(b) = 0$$

$$l_2(y_2) = c_0 y_2(a) + c_1 y_2^{'}(a) + d_0 y_2(b) + d_1 y_2^{'}(b) = 0$$

Now suppose $y(x) = y_1(x) + y_2(x)$. Then

$$q_1(x)y'' + q_2(x)y' + q_3(x)y = [\underbrace{q_1(x)y_1'' + q_2(x)y_1' + q_3(x)y_1}] + [\overbrace{q_1(x)y_2'' + q_2(x)y_2' + q_3(x)y_2}]$$

This implies that

$$q_1(x)y'' + q_2(x)y' + q_3(x)y = f(x)$$

From boundary conditions, we get

$$l_1(y) = l_1(y_1 + y_2) = a_0(y_1 + y_2)(a) + a_1(y_1^{'} + y_2^{'})(a) + b_0(y_1 + y_2)(b) + b_1(y_1^{'} + y_2^{'})(b)$$

$$\Rightarrow l_1(y) = A$$

And

$$l_2(y) = l_2(y_1 + y_2) = c_0(y_1 + y_2)(a) + c_1(y_1^{'} + y_2^{'})(a) + d_0(y_1 + y_2)(b) + d_1(y_1^{'} + y_2^{'})(b)$$

$$\Rightarrow l_2(y) = B$$

Therefore, $y(x) = y_1(x) + y_2(x)$ is a solution of the boundary value problem

$$q_1(x)y'' + q_2(x)y' + q_3(x)y = f(x)$$

subject to

$$l_1(y) = A$$

$$l_2(y) = B$$

□

2.2 Green's Function for Second Order Linear ODEs

Definition 2.2.1. *A Green's function for boundary value problem*

$$q_1(x)y'' + q_2(x)y' + q_3(x)y = 0$$

subject to

$$l_1(y) = 0, l_2(y) = 0$$

is a function $G(x,\zeta)$ for $(x,\zeta) \in [a,b] \times [a,b]$ such that $G(x,\zeta)$ satisfy the following properties:

1. *$G(x,\zeta)$ is continuous on $[a,b] \times [a,b]$.*
2. *$\frac{\partial G(x,\zeta)}{\partial x}$ is continuous in each of the triangles $a \leq x \leq \zeta \leq b$ and $a \leq \zeta \leq x \leq b$. Moreover, $\frac{\partial G(\zeta^+,\zeta)}{\partial x} - \frac{\partial G(\zeta^-,\zeta)}{\partial x} = \frac{1}{q_1(\zeta)}$.*
3. *For any $\zeta \in [a,b]$, $G(x,\zeta)$ is a solution of associated homogeneous of equation (2.1) in each of the intervals $[a,\zeta)$ and $(\zeta,b]$.*
4. *For any $\zeta \in [a,b]$, $G(x,\zeta)$ satisfies the boundary condition $l_1(y) = 0$ and $l_2(y) = 0$.*

Theorem 2.2.1. *Let the associated homogeneous boundary value problem*

$$q_1(x)y'' + q_2(x)y' + q_3(x)y = 0$$

subject to

$$l_1(y) = 0$$

$$l_2(y) = 0$$

has only the trivial solution. Then

1. *there exist a unique Green's function $G(x,\zeta)$ of the associated homogeneous of boundary value problem*

   $$q_1(x)y'' + q_2(x)y' + q_3(x)y = f(x)$$

   with

   $$l_1(y) = 0, l_2(y) = 0$$

2. $$y(x) = \int_a^b G(x,\zeta)f(\zeta)d\zeta = \int_a^x G(x,\zeta)f(\zeta)d\zeta + \int_x^b G(x,\zeta)f(\zeta)d\zeta$$

   is the unique solution of the nonhomogeneous boundary value problem

   $$q_1(x)y'' + q_2(x)y' + q_3(x)y = f(x)$$

   subject to

   $$l_1(y) = 0, l_2(y) = 0$$

Proof. 1. Let $y_1(x)$ and $y_2(x)$ be two linearly independent solutions of the associated homogeneous of the equation (2.1). From property (3) there exist a functions $\lambda_1(\zeta), \lambda_2(\zeta), \mu_1(\zeta)$ and $\mu_2(\zeta)$ such that

$$G(x,\zeta) = \begin{cases} y_1(x)\lambda_1(\zeta) + y_2(x)\lambda_2(\zeta), & a \leq x \leq \zeta \\ y_1(x)\mu_1(\zeta) + y_2(x)\mu_2(\zeta), & \zeta \leq x \leq b. \end{cases}$$

From property (1) and (2), we get

$$y_1(\zeta)\lambda_1(\zeta) + y_2(\zeta)\lambda_2(\zeta) = y_1(\zeta)\mu_1(\zeta) + y_2(\zeta)\mu_2(\zeta)$$

$$y_1\prime(\zeta)\mu_1(\zeta) + y_2\prime(\zeta)\mu_2(\zeta) - [y_1\prime(\zeta)\lambda_1(\zeta) + y_2\prime(\zeta)\lambda_2(\zeta)] = \frac{1}{q_1(\zeta)}$$

Now suppose $v_1(\zeta) = \mu_1(\zeta) - \lambda_1(\zeta)$ and $v_2(\zeta) = \mu_2(\zeta) - \lambda_2(\zeta)$. Then the above two equations becomes

$$y_1(\zeta)v_1(\zeta) + y_2(\zeta)v_2(\zeta) = 0$$

$$y_1\prime(\zeta)v_1(\zeta) + y_2\prime(\zeta)v_2(\zeta)] = \frac{1}{q_1(\zeta)}$$

The Wronskian $W(y_1, y_2) \neq 0$ because of y_1 and y_2 are the linearly independent solutions for all $\zeta \in [a,b]$. Hence, the above equations uniquely determine $v_1(\zeta)$ and $v_1(\zeta)$. Since $v_1(\zeta)+\lambda_1(\zeta) = \mu_1(\zeta)$ and $v_2(\zeta) + \lambda_2(\zeta) = \mu_2(\zeta)$, the Green's function $G(x,\zeta)$ can be written as

$$G(x,\zeta) = \begin{cases} y_1(x)\lambda_1(\zeta) + y_2(x)\lambda_2(\zeta), & a \leq x \leq \zeta \\ y_1(x)\lambda_1(\zeta) + y_2(x)\lambda_2(\zeta) + y_1(x)v_1(\zeta) + y_2(x)v_2(\zeta), & \zeta \leq x \leq b. \end{cases}$$

Finally, from the property (4), we get

$$l_1(y_1)\lambda_1(\zeta) + l_1(y_2)\lambda_2(\zeta) = -b_0(y_1(b)v_1(\zeta) + y_2(b)v_2(\zeta)) - b_1(y_1\prime(b)v_1(\zeta) + y_2\prime(b)v_2(\zeta))$$

$$l_2(y_1)\lambda_1(\zeta) + l_2(y_2)\lambda_2(\zeta) = -d_0(y_1(b)v_1(\zeta) + y_2(b)v_2(\zeta)) - d_1(y_1\prime(b)v_1(\zeta) + y_2\prime(b)v_2(\zeta))$$

Thus, the above linear system of equation uniquely determines $\lambda_1(\zeta)$ and $\lambda_2(\zeta)$ since the associated homogeneous of equation (2.1) subject to

$$l_1(y) = 0$$

$$l_2(y) = 0$$

has only the trivial solution. From the above construction we have the existence of the Green's function $G(x,\zeta)$ and since the associated homogeneous of an equation (2.1) subject to

$$l_1(y) = 0$$

$$l_2(y) = 0$$

has only the trivial solution the constructed Green's function $G(x,\zeta)$ is unique.

2.

$$y(x)=\int_a^b G(x,\zeta)f(\zeta)d\zeta=\int_a^x G(x,\zeta)f(\zeta)d\zeta+\int_x^b G(x,\zeta)f(\zeta)d\zeta$$

since $G(x,\zeta)$ is differentiable in each of the intervals, we get

$$\begin{aligned}y\prime(x)&=G(x,x)f(x)+\int_a^x G_x(x,\zeta)f(\zeta)d\zeta-G(x,x)f(x)+\int_x^b G_x(x,\zeta)f(\zeta)d\zeta\\&=\int_a^x G_x(x,\zeta)f(\zeta)d\zeta+\int_x^b G_x(x,\zeta)f(\zeta)d\zeta\\&=\int_a^b G_x(x,\zeta)f(\zeta)d\zeta\end{aligned}$$

Again since $\frac{\partial G(x,\zeta)}{\partial x}$ is continuous function on an interval (x,ζ) in the triangles $a\le\zeta\le x\le b$ and $a\le x\le\zeta\le b$, for any point (s,s) on the diagonal of the square(that is $x=t$) we have the relations

$$\frac{\partial G}{\partial x}(s,s^-)=\frac{\partial G}{\partial x}(s^+,s)$$
$$\frac{\partial G}{\partial x}(s,s^+)=\frac{\partial G}{\partial x}(s^-,s)$$

$$y\prime\prime(x)=G_x(x,x^-)f(x)+\int_a^x G_{xx}(x,\zeta)f(\zeta)d\zeta-G_x(x,x^+)f(x)+\int_x^b G_{xx}(x,\zeta)f(\zeta)d\zeta$$

using the above relations, we obtain

$$y\prime\prime(x)=[G_x(x^+,x)-G_x(x^-,x)]f(x)+\int_a^b G_{xx}(x,\zeta)f(\zeta)d\zeta$$

Because of $G_x(x^+,x)-G_x(x^-,x)=\frac{1}{q_1(x)}$

$$y\prime\prime(x)=\frac{f(x)}{q_1(x)}+\int_a^b G_{xx}(x,\zeta)f(\zeta)d\zeta$$

Therefore,

$$q_1(x)y\prime\prime(x)+q_2(x)y\prime(x)+q_3(x)y=f(x)+\int_a^b[q_1(x)G_{xx}(x,\zeta)+q_2(x)G_x(x,\zeta)+q_3(x)G(x,\zeta)]$$

$$q_1(x)y\prime\prime(x)+q_2(x)y\prime(x)+q_3(x)y=f(x)$$

From boundary conditions, we have

$$y(a)=\int_a^b G(a,\zeta)f(\zeta)d\zeta\ \textit{and}\ y\prime(a)=\int_a^b G_x(a,\zeta)f(\zeta)d\zeta,$$

and

$$y(b)=\int_a^b G(b,\zeta)f(\zeta)d\zeta\ \textit{and}\ y\prime(b)=\int_a^b G_x(b,\zeta)f(\zeta)d\zeta$$

This implies that

$$l_1(y)=\int_a^b l_1(G(x,\zeta))f(\zeta)d\zeta=0$$

$$l_2(y) = \int_a^b l_l(G(x,\zeta))f(\zeta)d\zeta = 0$$

Therefore, the proof is completed. □

Example 2.2.1. *Find the Green's function $G(x,\zeta)$ for the periodic BVP*

$$y\prime\prime + k^2 y = 0, k > 0$$

subject to

$$y(0) = y(\omega)$$

$$y\prime(0) = y\prime(\omega), \omega > 0$$

Solution 1.

$$y_1(x) = coskx \ and \ y_2(x) = sinkx$$

are linearly independent solutions of

$$y\prime\prime + k^2 y = 0, k > 0$$

however,

$$y\prime\prime + k^2 y = 0, k > 0$$

subject to

$$y(0) = y(\omega)$$

$$y\prime(0) = y\prime(\omega), \omega > 0$$

has trivial solution if and only if

$$\triangle = 4k\sin^2\frac{kw}{2} \neq 0, w \in (0, \frac{2\pi}{k}).$$

From continuity and jump discontinuity property, we have

$$v_1(\zeta)\cos k\zeta + v_2(\zeta)\sin k\zeta = 0$$

$$-kv_1(\zeta)\sin k\zeta + kv_2(\zeta)\sin k\zeta = 1$$

implies that

$$v_1(\zeta) = -\frac{1}{k}\sin k\zeta$$

$$v_2(\zeta) = \frac{1}{k}\cos k\zeta$$

and from boundary conditions

$$\lambda_1(\zeta)(1-\cos kw) - \lambda_1(\zeta)\sin kw = \frac{\sin k(w-\zeta)}{k}$$

$$\lambda_1(\zeta)\sin kw + \lambda_1(\zeta)(1-\cos kw) = \frac{\cos k(w-\zeta)}{k}$$

This implies

$$\lambda_1(\zeta) = \frac{1}{k\sin\frac{kw}{2}}\cos k(\zeta - \frac{w}{2})$$

$$\lambda_2(\zeta) = \frac{1}{k\sin\frac{kw}{2}}\sin k(\zeta - \frac{w}{2})$$

Therefore, the required Green's function is

$$G(x,\zeta) = \begin{cases} \frac{1}{k\sin\frac{kw}{2}}\cos k(x-\zeta+\frac{w}{2}), & 0 \le x \le \zeta \\ \frac{1}{k\sin\frac{kw}{2}}\cos k(\zeta-x+\frac{w}{2}), & \zeta \le x \le w. \end{cases}$$

Formula for Finding Green's Function

Consider a linear second order differential operator

$$Ly(x) = q_1(x)y\prime\prime + q_2(x)y\prime + q_3(x)y = f(x), \ on\ I = [a,b]$$

where $q_1(x), q_2(x), q_3(x)$ are continuous on an interval $I = [a,b]$, $q_1(x) \neq 0$ and f is bounded on $[a,b]$. Now we need to construct the Green's function for boundary value problem

$$Ly(x) = q_1(x)y\prime\prime + q_2(x)y\prime + q_3(x)y = f(x),$$

take the fact that whenever $x \neq \zeta, LG(x,\zeta) = 0$. Suppose that $y_1(x)$ and $y_2(x)$ are two linearly independent solutions of the homogeneous DE

$$Ly(x) = q_1(x)y\prime\prime + q_2(x)y\prime + q_3(x)y = 0,$$

on an interval [a, b]. For both $x > \zeta$ and $x < \zeta$ we can express Green's function $G(x,\zeta)$ in terms of solutions of the problem

$$Ly(x) = q_1(x)y\prime\prime + q_2(x)y\prime + q_3(x)y = 0.$$

Define $y_1(a) = 0$ and $y_2(b) = 0$. i.e each of $y_1(x)$ and $y_2(x)$ obeys one of the homogeneous boundary conditions. On an interval $[a,\zeta)$ the Green's function obeys $LG(x,\zeta) = 0$ and $G(a,\zeta) = 0$. However, any homogeneous solution to $Ly = 0$ obeying $y(a) = 0$ must be direct proportional to $y_1(x)$ and we have

$$G(x,\zeta) = A(\zeta)y_1(x), \ for\ x \in [a,\zeta)$$

where $A(\zeta)$ is a constant of proportionality which is independent of x. Again, on an interval $(\zeta, b]$ the Green's function obeys $LG(x,\zeta) = 0$ and $G(b,\zeta) = 0$. However, any homogeneous solution to $Ly = 0$ obeying $y(b) = 0$ must be direct proportional to $y_2(x)$ and we have

$$G(x,\zeta) = B(\zeta)y_2(x), \ for\ x \in (\zeta,b].$$

where $B(\zeta)$ is a constant of proportionality which is independent of x. Our construction gives us families of Green's function for all $x \in [a,b] - \zeta$ in terms of the functions $A(\zeta)$ and $B(\zeta)$.

Hence we must determine how these two solutions are to be joined together at $x = \zeta$. Suppose that $G(x, \zeta)$ was discontinuous at $x = \zeta$, with the discontinuity modelled by step function. Then $\frac{\partial}{\partial x} G(x, \zeta) \propto \delta(x - \zeta)$, and $\frac{\partial^2}{\partial x^2} G(x, \zeta) \propto \delta\prime(x - \zeta)$. But the problem

$$LG(x, \zeta) = \delta(x - \zeta)$$

shows that $LG(x, \zeta)$ not involves generalized functions above $\delta(x - \zeta)$, and in particular it can not contains the derivatives of δ- function. Therefore, $G(x, \zeta)$ is continuous on an interval $[a, b]$, and in particular $x = \zeta$. Now, from problem

$$LG(x, \zeta) = \delta(x - \zeta)$$

we get

$$\int_{\zeta-\epsilon}^{\zeta+\epsilon} [q_1(x) \frac{\partial^2}{\partial x^2} G(x, \zeta) + q_2(x) \frac{\partial}{\partial x} G(x, \zeta) + q_3(x) G(x, \zeta)] dx = \int_{\zeta-\epsilon}^{\zeta+\epsilon} \delta(x - \zeta) dx$$

$$\Rightarrow \int_{\zeta-\epsilon}^{\zeta+\epsilon} [q_1(x) \frac{\partial^2}{\partial x^2} G(x, \zeta) + q_2(x) \frac{\partial}{\partial x} G(x, \zeta) + q_3(x) G(x, \zeta)] dx = 1$$

From our assumption all coefficient functions i.e $q_1(x), q_2(x), q_3(x)$ are bounded and since $G(x, \zeta)$ is continuous, the $\int_{\zeta-\epsilon}^{\zeta+\epsilon} q_3(x) G(x, \zeta) dx$ is zero as we make the integration region infinitesimally thin. Also, since $G(x, \zeta)$ is continuous, $\frac{\partial}{\partial x} G(x, \zeta)$ must be bounded. So the middle term can not contribute as the integration region shrinks to zero. Finally, we have

$$\lim_{\epsilon \to 0} \int_{\zeta-\epsilon}^{\zeta+\epsilon} q_1(x) \frac{\partial^2 G(x, \zeta)}{\partial x^2} dx = 1$$

Using integration by part, we get

$$\frac{\partial G(x, \zeta)}{\partial x}|_{x=\zeta^+} - \frac{\partial G(x, \zeta)}{\partial x}|_{x=\zeta^-} = \frac{1}{q_1(\zeta)}$$

And since $G(x, \zeta)$ is continuous we have

$$\frac{\partial G(x, \zeta)}{\partial x}|_{x=\zeta^-} = \frac{\partial G(x, \zeta)}{\partial x}|_{x=\zeta^+}$$

we get the system of linear equation

$$A(\zeta) y_1(\zeta) = B(\zeta) y_2(\zeta)$$

$$A(\zeta) y_1\prime(\zeta) - B(\zeta) y_2\prime(\zeta) = \frac{1}{q_1(\zeta)}$$

By applying Cramer's rule, we get

$$A(\zeta) = \frac{y_2(\zeta)}{q_1(\zeta) W(\zeta)}$$

$$B(\zeta) = \frac{y_1(\zeta)}{q_1(\zeta)W(\zeta)}$$

where

$$W(x) = y_1 y_2\prime - y_2 y_1\prime$$

is Wronskian of y_1 and y_2. Therefore, we got the solution $G(x,\zeta)$ of $LG = \delta(x-\zeta)$ satisfying $G(a,\zeta) = G(b,\zeta) = 0$ which is given by

$$G(x,\zeta) = \begin{cases} \frac{y_1(x)y_2(\zeta)}{q_1(\zeta)W(\zeta)}, & a \leq x \leq \zeta \\ \frac{y_2(x)y_1(\zeta)}{q_1(\zeta)W(\zeta)}, & \zeta \leq x \leq b \end{cases}$$

To verify the uniqueness of the Green's function $G(x,\zeta)$, suppose that there is two Green's function $G_1(x,\zeta)$ and $G_2(x,\zeta)$

$$\int_a^b [G_1(x,\zeta) - G_2(x,\zeta)]f(\zeta)d\zeta = \int_a^b G_1(x,\zeta)f(\zeta)d\zeta - \int_a^b G_2(x,\zeta)f(\zeta)d\zeta$$

$$\int_a^b [G_1(x,\zeta) - G_2(x,\zeta)]f(\zeta)d\zeta = f(x) - f(x)$$

$$\Rightarrow \int_a^b [G_1(x,\zeta) - G_2(x,\zeta)]f(\zeta)d\zeta = 0$$

$$\Rightarrow G_1(x,\zeta) - G_2(x,\zeta) = 0, f(\zeta) \neq 0$$

$$\Rightarrow G_1(x,\zeta) = G_2(x,\zeta)$$

Define Green's function $G(x,\zeta)$ of a linear operator L to be the unique solution to the problem

$$LG(x,\zeta) = \delta(x-\zeta)$$

which satisfies homogeneous boundary conditions $G(a,\zeta) = 0$ and $G(b,\zeta) = 0$.
Given the solution of DE

$$LG(x,\zeta) = \delta(x-\zeta)$$

$G(x,\zeta)$, we can solve the more general problem

$$Ly(x) = f(x),$$

with

$$l_1(y) = 0 \ and \ l_2(y) = 0$$

for arbitrary forcing term $f(x)$ such that

$$y(x) = \int_a^b G(x,\zeta)f(\zeta)d\zeta$$

To verify $y(x)$ is the solution of the given DE, compute

$$LG(x,\zeta) = \delta(x-\zeta)$$

by multiplying both sides by $f(\zeta)$ and integrate in terms of $d\zeta$, we have

$$\int_a^b f(\zeta)LG(x,\zeta)d\zeta = \int_a^b f(\zeta)\delta(x-\zeta)d\zeta$$

Since the Green's function is the only function that depends on x, we can interchange differential operator L and integral sign, hence we get

$$L[\int_a^b f(\zeta)G(x,\zeta)d\zeta] = \int_a^b f(\zeta)\delta(x-\zeta)d\zeta$$

$$L[y(x)] = f(x)$$

Hence, The solution of nonhomogeneous equation $Ly = f(x)$ is written as

$$\int_a^b G(x,\zeta)f(\zeta)d\zeta$$

2.3 Green's Functions for Initial Value Problems

The initial value problem is also solved by using Green's function. Suppose that the independent variable is x and the function $y(x) : [x,\infty) \rightarrow R$ that satisfies the differential equation

$$Ly = f(x)$$

subject to the initial condition $y(x_0) = 0$ and $y\prime(x_0) = 0$. Then we want to find $G(x,\zeta)$ such that $LG(x,\zeta) = \delta(x-\zeta)$ for each ζ, the Green's function $G(x,\zeta)$ will solve the homogeneous equation $LG(x,\zeta) = 0$ whenever $x \neq \zeta$. On an interval $x_0 \leq x < \zeta$ construct $G(x,\zeta)$ as a general solution of the homogeneous equation, so that

$$G(x,\zeta) = Ay_1(x) + By_2(x)$$

where $y_1(x)$ and $y_2(x)$ are linearly independent homogeneous solutions. Now by applying initial condition, we get

$$Ay_1(x_0) + By_2(x_0) = 0$$

$$Ay_1\prime(x_0) + By_2\prime(x_0) = 0$$

Since $y_1(x)$ and $y_2(x)$ are linearly independent, the Wronskian $W(y_1,y_2)$ is different from zero. Therefore, the only way to impose both initial conditions is to set $A = B = 0$.

$$\Rightarrow G(x,\zeta) = 0,\ for\ x \in [a,\zeta)$$

For $x > \zeta$, construct the Green's function as a general solution of the homogeneous equation $Ly = 0$, so that

$$G(x,\zeta) = Cy_1(x) + Dy_2(x)$$

Applying the continuity condition and jump discontinuity at $x = \zeta$. Since $G(x,\zeta) = 0$ for $x < \zeta$ we get

$$Cy_1(\zeta) + Dy_2(\zeta) = 0$$

$$Cy_1\prime(\zeta) + Dy_2\prime(\zeta) = \frac{1}{q_1(\zeta)}$$

By applying Cramer's rule, we obtain $C = \frac{-y_2(\zeta)}{q_1(\zeta)W(\zeta)}$ and $D = \frac{y_1(\zeta)}{q_1(\zeta)W(\zeta)}$.
Hence

$$G(x,\zeta) = \begin{cases} 0, & x < \zeta \\ \frac{-y_2(\zeta)}{q_1(\zeta)W(\zeta)}y_1(x) + \frac{y_1(\zeta)}{q_1(\zeta)W(\zeta)}y_2(x), & x > \zeta. \end{cases}$$

By using the Green's function we solve the nonhomogeneous problem

$$Ly = f(x)$$

as

$$y(x) = \int_{x_0}^{\infty} G(x,\zeta)f(\zeta)d\zeta$$
$$= \int_{x_0}^{x} G(x,\zeta)f(\zeta)d\zeta$$

because of the fact that $G(x,\zeta)$ vanishes for $\zeta > x$. This equation show that the solution satisfies a causality condition *i.e* the value of y at independent variable x depends only on the behaviour of the forcing term for earlier times $\zeta \in [x_0, x]$.

Theorem 2.3.1. *The first order nonhomogeneous differential equations*

$$y\prime + p(x)y = f(x)$$

subject to initial value condition

$$y(a) = 0, \ \ for\ x > a$$

has the solution

$$y(x) = \int_{a}^{x} G(x,\zeta)f(\zeta)d\zeta$$

where $G(x,\zeta)$ satisfies the equation $LG(x,\zeta) = \delta(x-\zeta)$ for $x > a$, $G(a,\zeta) = 0$ and the Green's function $G(x,\zeta) = H(x-\zeta)e^{\int_a^x P(x)dx}$

Proof. Given an equation

$$LG(x,\zeta) = \delta(x-\zeta)$$

multiplying both sides by $f(\zeta)$ and integrate in terms of $d\zeta$, we have

$$\int_{a}^{\infty} LG(x,\zeta)f(\zeta)d\zeta = \int_{a}^{\infty} \delta(x-\zeta)f(\zeta)d\zeta$$

Since the Green's function $G(x,\zeta)$ is the only function depend on x, we can interchange linear operator L and the integral sign. That is

$$L[\int_a^\infty G(x,\zeta)f(\zeta)d\zeta] = \int_a^\infty \delta(x-\zeta)f(\zeta)d\zeta$$

$$\Rightarrow Ly(x) = f(x)$$

The integral also satisfies the initial condition

$$y(x) = \int_a^x G(x,\zeta)f(\zeta)d\zeta$$

$$y(a) = \int_a^a G(a,\zeta)f(\zeta)d\zeta = 0$$

To determine the Green's function $G(x,\zeta)$, for $x \neq \zeta$ we use a homogeneous solution of a given differential equation. However, at $x = \zeta$ we expect some singular behaviour $\partial_x G(x,\zeta)$ a dirac delta function type singularity. That is $\partial_x G(x,\zeta)$ will have a jump discontinuity at $x = \zeta$. We integrate the given differential equation on the interval (a,ζ) to determine this jump.

$$\partial_x G(x,\zeta) + p(x)G(x,\zeta) = \delta(x-\zeta)$$

$$G(\zeta^+,\zeta) - G(\zeta^-,\zeta) + \int_{\zeta^-}^{\zeta^+} p(x)G(x,\zeta) = 1$$

$$G(\zeta^+,\zeta) - G(\zeta^-,\zeta) = 1$$

The homogeneous solution of the differential equation is $y_h = e^{-\int p(x)dx}$. Since the Green's function satisfies the homogeneous equation for $x \neq \zeta$, it will be a constant times this homogeneous solution for $x < \zeta$ and $x > \zeta$

$$G(x,\zeta) = \begin{cases} c_1 e^{-\int p(x)dx}, & a < x < \zeta \\ c_2 e^{-\int p(x)dx}, & x > \zeta. \end{cases}$$

In order to satisfies the initial condition $G(a,\zeta) = 0$, the Green's function must vanish on an interval (a,ζ)

$$\Rightarrow G(\zeta^+,\zeta) = 1$$

This determines the constant in the homogeneous solution $c_1 = 0$ and $c_2 = 1$. Therefore,

$$G(x,\zeta) = \begin{cases} 0, & a < x < \zeta \\ e^{-\int p(x)dx}, & x > \zeta. \end{cases}$$

$$G(x,\zeta) = H(x-\zeta)e^{-\int p(x)dx}$$

□

2.4 Higher Order Linear Ordinary Differential Operators

In section (2.2) we have constructed Green's function for linear ordinary differential equations of order two. So, we can extend the technique (or definition) in section (2.2) to higher order linear order ordinary differential equations. The following instance illustrates the appropriate adjustments for the case of higher order ODEs when $n = 3$.

Example 2.4.1. *Take third order ODE*

$$y''' = f(x)$$

subject to boundary condition

$$y(0) = y'(0) = y''(1) = 0$$

To construct Green's function $G(x, \zeta), 0 \leq x \leq 1, 0 \leq \zeta \leq 1$, for the given boundary value problem:
Applying integration by part, we have

$$\int_0^1 Gy'''dx = (Gy'' - G_x y' + G_{xx} y)|_0^1 - \int_0^1 G_{xxx} y dx$$

To obtain the solution of the above problem as

$$y(\zeta) = \int_0^1 G(x, \zeta) f(x) dx$$

put

$$(Gy'' - G_x y' + G_{xx} y)|_0^1 = 0, -G_{xxx} = \delta(x - \zeta)$$

and G, G_x are continuous $0 \leq x \leq 1, 0 \leq \zeta \leq 1$. From boundary condition

$$-G(0, \zeta) y''(0) - G_x(1, \zeta) y'(1) + G_{xx}(1, \zeta) y(1) = 0$$

Hence, take $G(0, \zeta) = G_x(1, \zeta) = G_{xx}(1, \zeta) = 0$, and therefore $(Gy'' - G_x y' + G_{xx} y)|_0^1 = 0$ satisfied. Thus, the Green's function $G(x, \zeta)$

$$G(x, \zeta) = \begin{cases} G_1(x, \zeta), & x < \zeta \\ G_2(x, \zeta), & x > \zeta. \end{cases}$$

has the following properties:

1. *$\frac{\partial^3}{\partial x^3} G_1 = 0$ for $x < \zeta$ and $\frac{\partial^3}{\partial x^3} G_2 = 0$ for $x > \zeta$ and also G, G_x are continuous at $x = \zeta$*
2. *$G_1(0, \zeta) = \frac{\partial}{\partial x} G_2(1, \zeta) = \frac{\partial^2}{\partial x^2} G_2(1, \zeta) = 0$*
3. *The second derivative of G with respect to x has a jump discontinuity of magnitude -1 at $x = \zeta$*

From

$$G(x,\zeta) = \begin{cases} G_1(x,\zeta), & x < \zeta \\ G_2(x,\zeta), & x > \zeta. \end{cases}$$

we have

$$G(x,\zeta) = \begin{cases} \alpha_1 + \alpha_2 x + \alpha_3 x^2, & x < \zeta \\ \beta_1 + \beta_2 x + \beta_3 x^2, & x > \zeta. \end{cases}$$

From properties (1) *and* (3), *we have*

$$(\beta_1 - \alpha_1) + (\beta_2 - \alpha_2)\zeta + (\beta_3 - \alpha_3)\zeta^2 = 0$$

$$(\beta_2 - \alpha_2) + 2(\beta_3 - \alpha_3)\zeta = 0$$

$$2(\beta_3 - \alpha_3) = -1$$

Solving these system of linear equation for $\beta_j - \alpha_j, j = 1, 2, 3,$ *we obtain*

$$\beta_3 - \alpha_3 = -\frac{1}{2}, \beta_2 - \alpha_2 = \zeta \text{ and } \beta_1 - \alpha_1 = \frac{-\zeta^2}{2}$$

From property (2),

$$G_1(0,\zeta) = 0 \text{ implies that } \alpha_1 = 0, \text{and thus } \beta_1 = \frac{-\zeta^2}{2};$$

$$\frac{\partial}{\partial x} G_2(1,\zeta) = 0 \text{ and } \frac{\partial^2}{\partial x^2} G_2(1,\zeta) = 0 \text{ imply } \beta_2 + 2\beta_3 = 0, \text{and} \beta_3 = 0 \text{ respectively.}$$

Hence

$$\beta_2 = 0, \alpha_2 = -\zeta \text{ and } \alpha_3 = \frac{1}{2}$$

Thus,

$$G(x,\zeta) = \begin{cases} -\zeta x + \frac{1}{2}x^2, & x < \zeta \\ -\frac{1}{2}\zeta^2, & x > \zeta. \end{cases}$$

Hence the solution of the above boundary value problem becomes

$$y(\zeta) = \int_0^1 f(x) G(x,\zeta) dx$$

$$= \int_0^\zeta f(x) G_1(x,\zeta) dx + \int_\zeta^1 f(x) G_2(x,\zeta) dx$$

$$= \int_0^\zeta f(x)(-\zeta x + \frac{1}{2}x^2) dx + \int_\zeta^1 f(x)(-\frac{1}{2}\zeta^2) dx$$

Generally consider n^{th} order ordinary differential equation

$$q_n(x) y^n(x) + q_{n-1}(x) y^{n-1}(x) + ... + q_1(x) y'(x) + q_0(x) y(x) = f(x)$$

with $n > 2$, $q_n(x), q_{n-1}(x), ..., q_0(x)$ are continuous on an interval $[0,1]$ and $q_n(x) \neq 0$ with

n boundary conditions. Then to construct the Green's function $G(x, \zeta)$ for the given linear boundary value problem: Using integration by part

$$\int_0^1 GLydx = \int_0^1 G(q_n(x)y^n(x) + q_{n-1}(x)y^{n-1}(x) + ... + q_1(x)y^{'}(x) + q_0(x)y(x))dx$$

gives

$$\int_0^1 GLydx = p(y, G)|_0^1 + \int_0^1 yL^*Gdx$$

where L^*G is adjoint differential operator of G with respect to x, That is

$$L^*G = (-1)^n \frac{d^n}{x^n}(q_1G) + (-1)^{n-1}\frac{d^{n-1}}{x^{n-1}}(q_2G) + ... + (-1)\frac{d}{dx}(q_{n-1}G) + q_nG$$

and

$$p(y, G) = y[q_{n-1}G - \frac{d}{dx}(q_{n-2}G) + ... + (-1)^{n-1}\frac{d^{n-1}}{x^{n-1}}(q_1G)] +$$

$$y\prime[q_{n-2}G - \frac{d}{dx}(q_{n-3}G) + ... + (-1)^{n-2}\frac{d^{n-2}}{x^{n-2}}(q_1G)] \ + ... + y^{(n-1)}(q_1G)$$

By using $p(u, G)|_0^1 = 0$, where $p(u, G)|_0^1$ is a function of u and G and the given boundary conditions on y we can find the boundary condition on G, $B(G)$.
Thus, the Green's function

$$G(x, \zeta) = \begin{cases} G_1(x, \zeta), & x < \zeta \\ G_2(x, \zeta), & x > \zeta, \end{cases}$$

has the following properties:

1. $L^*G_1 = 0$ and $L^*G_2 = 0$, and $G, \frac{\partial}{\partial x}G, ..., \frac{\partial^{n-2}}{\partial x^{n-2}}G$ are continuous on $0 \leq x \leq 1, 0 \leq \zeta \leq 1$ in particular, the derivatives of G_1 and G_2 join up smoothly along $x = \zeta$

2. G satisfies the boundary condition $B(G)$

3. $\frac{\partial^{n-1}}{\partial x^{n-1}}G$ has jump discontinuity at $x = \zeta$ with magnitude $\frac{(-1)^n}{q_1(\zeta)}$.

Furthermore, the general solution of the nonhomogeneous differential equation

$$Ly(x) = f(x)$$

satisfying the homogeneous boundary conditions $y(a) = 0$ and $y(b) = 0$ is given by

$$y(x) = \int_a^b G(x, \zeta)f(\zeta)d\zeta$$

where $G(x, \zeta)$ satisfies

$$LG(x, \zeta) = \delta(x - \zeta)$$

subject to boundary conditions $G(a, \zeta) = G(b, \zeta) = 0$, and both $G(x, \zeta)$ and its first $n - 2$ derivatives are continuous at $x = \zeta$. Moreover

$$\frac{\partial^{n-1}G(x, \zeta)}{\partial x^{n-1}}|_{x=\zeta^+} - \frac{\partial^{n-1}G(x, \zeta)}{\partial x^{n-1}}|_{x=\zeta^-} = \frac{1}{q_n(\zeta)}$$

To verify, given an equation

$$LG(x,\zeta) = \delta(x-\zeta)$$

Multiply both sides by $f(\zeta)$ and integrate both sides in terms of $d\zeta$, we obtain

$$\int_a^b LG(x,\zeta)f(\zeta) = \int_a^b \delta(x-\zeta)f(\zeta)$$

Since Green's function $G(x,\zeta)f(\zeta)$ is the only function depend on x, we can interchange the integral sign and differential operator L.

$$L[\int_a^b G(x,\zeta)f(\zeta)] = \int_a^b \delta(x-\zeta)f(\zeta)$$

by applying property of δ-function, we get

$$L[y(x)] = f(x)$$

Since the boundary conditions

$$G(a,\zeta) = G(b,\zeta) = 0$$

we obtain

$$y(a) = \int_a^b G(a,\zeta)f(\zeta)d\zeta = 0$$

and

$$y(b) = \int_a^b G(b,\zeta)f(\zeta)d\zeta = 0$$

2.5 Sturm-Liouville Problems

Consider the Sturm-Liouville Problem

$$L_\lambda y(x) = (p(x)y\prime)\prime + g(x)y + \lambda y = 0, a < x < b$$

subject to boundary conditions

$$l_1(y) = 0$$

$$l_2(y) = 0$$

where

$$l_i(y) = a_{i1}y(a) + a_{i2}y\prime(a) + b_{i1}y(b) + b_{i2}y\prime(b)$$

and $p(x) \in C^1(a,b)$, $p(x) > 0, x \in [a,b]$. A Green's function for Sturm-Liouville Problem

$$L_\lambda y(x) = (p(x)y\prime)\prime + g(x)y + \lambda y = 0, a < x < b$$

is a function $G(x,\zeta,\lambda)$ for $(x,\zeta) \in [a,b] \times [a,b]$ such that the following properties are holds:

1. $G(x,\zeta,\lambda)$is continuous on an interval $[a,b] \times [a,b]$
2. $\frac{\partial G(x,\zeta,\lambda)}{\partial x}$ is continuous on an interval $[a,\zeta) \times (\zeta,b]$.

Moreover, $\frac{\partial G(\zeta^+,\zeta,\lambda)}{\partial x} - \frac{\partial G(\zeta^-,\zeta,\lambda)}{\partial x} = \frac{1}{p(\zeta)}$

3. For all $\zeta \in [a,b], G(x,\zeta,\lambda)$ solves $L_\lambda G(x,\zeta,\lambda) = 0, x \neq \zeta$

4. For all $\zeta \in [a,b], l_i(G(x,\zeta,\lambda)) = 0$

Theorem 2.5.1. *Suppose λ is not an eigenvalue of Sturm-Liouville Problem*

$$L_\lambda y(x) = (p(x)y\prime)\prime + g(x)y + \lambda y = 0, a < x < b$$

subject to boundary condition

$$l_1(y) = 0$$

$$l_2(y) = 0.$$

Then BVP_λ has a unique Green's function $G(x,\zeta,\lambda)$ and it's symmetric.

Proof. We provided the proof by constructing Green's function.
1. consider

$$L_\lambda y(x) = (p(x)y\prime)\prime + g(x)y + \lambda y = 0, a < x < b$$

subject to boundary condition

$$l_1(y) = a_1 y(a) + a_2 y\prime(a) = 0$$

$$l_2(y) = b_1 y(b) + b_2 y\prime(b) = 0.$$

Select u_i such that $L_\lambda(u_i) = 0$ and $l_i(u_i) = 0$. The solutions u_1 and u_2 must be linearly independent. so suppose $z = c_1 u_1 + c_2 u_2$. Then $l_1(z) = c_2 l_1(u_2)$ and $l_2(z) = c_1 l_2(u_1)$. If $l_1(u_2) = 0$, then u_2 becomes an eigenfunction, but λ is not an eigenvalue. This implies that $c_2 = 0$. In the same fashion $c_1 = 0$. Take the Green's function $G(x,\zeta,\lambda)$ as

$$G(x,\zeta,\lambda) = \begin{cases} Au_1, & a \leq x \leq \zeta \\ Bu_2, & \zeta \leq x \leq b. \end{cases}$$

From continuity condition and jump discontinuity, we have

$$Au_1(\zeta) = Bu_2(\zeta) \text{ and } Bu_2\prime(\zeta) - Au_1\prime(\zeta) = \frac{1}{p(\zeta)}$$

by using substitution, we get

$$A = \frac{u_2(\zeta)}{p(\zeta)W(u_1,u_2)(\zeta)}$$

$$B = \frac{u_1(\zeta)}{p(\zeta)W(u_1,u_2)(\zeta)}$$

Hence,

$$G(x,\zeta,\lambda) = \begin{cases} \frac{u_2(\zeta)}{p(\zeta)W(u_1,u_2)(\zeta)} u_1(x), & a \leq x \leq \zeta \\ \frac{u_1(\zeta)}{p(\zeta)W(u_1,u_2)(\zeta)} u_2(x), & \zeta \leq x \leq b. \end{cases}$$

To verify $G(x,\zeta,\lambda)$ is symmetric, we apply

$$W(u_1,u_2)(\zeta) = W(x_0)e^{-\int_{x_0}^{\zeta}\frac{p\prime(x)}{p(x)}dx} = W(x_0)\frac{p(x_0)}{p(\zeta)}$$

which is known as Abel's formula. This implies that

$$W(u_1,u_2)(\zeta)p(\zeta) = W(x_0)p(x_0)$$

Hence

$$W(u_1,u_2)(\zeta)p(\zeta) = constant$$

Therefore

$$G(x,\zeta,\lambda) = G(\zeta,x,\lambda)$$

□

Example 2.5.1. *Construct the Green's function for*

$$y\prime\prime = 0, 0 < x < 1$$

subject to

$$y(0) = 0, y(1) = 0.$$

$L_\lambda = L_0$ and $\lambda = 0$ is not an eigenvalue. Take $u_1(x) = x$ and $u_2(x) = x - 1$.

$$W(u_1,u_2) = 1$$

and since $p(x) = 1, p(\zeta) = 1$.
Therefore, the required Green's function is

$$G(x,\zeta,\lambda) = \begin{cases} x(\zeta-1), & 0 \le x \le \zeta \\ \zeta(x-1), & \zeta \le x \le 1. \end{cases}$$

2. Consider the special case of Initial Value Problems

$$l_1(u) = u(a)$$

$$l_2(u) = u\prime(a)$$

Assume $u_1(x)$ and $u_2(x)$ are linearly independent solution of $L_\lambda = 0$. we seek the Green's function

$$G(x,\zeta,\lambda) = \begin{cases} 0, & a \le x \le \zeta \\ Au_1(x) + Bu_2(x), & \zeta \le x. \end{cases}$$

From continuity and jump condition, we have

$$Au_1(\zeta) + Bu_2(\zeta) = 0$$

$$Au_1\prime(\zeta) + Bu_2\prime(\zeta) = \frac{1}{p(\zeta)}$$

using Cramer's rule, we got

$$A = \frac{-u_2(\zeta)}{p(\zeta)W(u_1(\zeta), u_2(\zeta))}$$

$$B = \frac{u_1(\zeta)}{p(\zeta)W(u_1(\zeta), u_2(\zeta))}$$

Hence,

$$G(x, \zeta, \lambda) = \begin{cases} 0, & a \leq x \leq \zeta \\ \frac{-u_2(\zeta)}{p(\zeta)W(u_1(\zeta), u_2(\zeta))} u_1(x) + \frac{u_1(\zeta)}{p(\zeta)W(u_1(\zeta), u_2(\zeta))} u_2(x), & \zeta \leq x. \end{cases}$$

Let $u_\zeta(x)$ be the solution of

$$L_\lambda(u_\zeta(x)) = 0, u_\zeta(\zeta) = 0 \ and \ u_\zeta\prime(\zeta) = \frac{1}{p(\zeta)}$$

Thus, the Green's function for the initial value problem satisfies

$$G(x, \zeta) = H(x - \zeta)u_\lambda(x)$$

where $H(x - \zeta)$ is Heaviside function. Such a Green's function is often referred to as the causal fundamental solution. For more general boundary conditions, we might seek Green's function $G(x, \zeta)$ as

$$G(x, \zeta) = H(x - \zeta)u_\zeta + Au_1(x) + Bu_2(x)$$

where $u_1(x)$ and $u_2(x)$ are linearly independent solutions of $L_\lambda = 0$.

Example 2.5.2. *Find the Green's function for*

$$u\prime\prime = 0, 0 < x < 1$$

subject to

$$u(0) + u(1) = 0 \ and \ u\prime(0) + u\prime(1) = 0$$

From above theorem, we seek

$$G(x, \zeta) = H(x - \zeta)u_\zeta(x) + Au_1(x) + Bu_2(x) = E(x, \zeta) + Ax + B$$

where

$$u_\zeta\prime\prime = 0, x > \zeta > 0$$

$$u_\zeta(\zeta^+) = 0, u_\zeta(\zeta^+) = 1.$$

Then

$$E(x, \zeta) = \begin{cases} 0, & 0 \leq x \leq \zeta \\ x - \zeta, & x \leq \zeta \leq 1. \end{cases}$$

and

$$l_1(G) = (E(0,\zeta) + B) + (E(0,\zeta) + A + B) = 2B + A + (1+\zeta) = 0$$

$$l_2(G) = (0 + A) + (1 + A) = 2A + 1 = 0$$

This implies that

$$A = \frac{-1}{2} \ and \ B = \frac{-1}{4} + \frac{\zeta}{2}$$

Therefore,

$$G(x,\zeta) = \begin{cases} \frac{1}{2}x - \frac{1}{4} + \frac{\zeta}{2}, & 0 \le x \le \zeta \\ (x-\zeta) - \frac{1}{4} + \frac{\zeta}{2} - \frac{x}{2}, & x < \zeta \le 1. \end{cases}$$

Take the nonhomogeneous boundary value problem

$$L_\lambda y(x) = (p(x)y\prime)\prime + g(x)y + \lambda y = f(x), a < x < b$$

subject to boundary condition

$$l_1(y) = 0$$

$$l_2(y) = 0$$

where

$$l_i(y) = a_{i1}y(a) + a_{i2}y\prime(a) + b_{i1}y(b) + b_{i2}y\prime(b)$$

and $p(x) \in C^1(a,b)$, $p(x) > 0, x \in [a,b]$. Given any two functions u and v, we have Lagrange identity

$$vL_\lambda(u) - uL_\lambda(v) = \frac{dP(u,v)}{dx}$$

where $P(u,v) = p(x)(u\prime v - v\prime u)$. And by integrating both sides, we got

$$\int_a^b [vL_\lambda(u) - uL_\lambda(v)] = P(u,v)|_{x=a}^{x=b}$$

which is called Green's formula. Let $G(x,\zeta)$ be Green's function for the homogeneous boundary value problem. From Lagrange's identity, for $x \neq \zeta$,we have

$$GL_\lambda(y) - yL_\lambda(G) = \frac{d}{dx}(p(x)(y\prime G - G\prime y))$$

implies that

$$\int_a^{\zeta^-} GL_\lambda(y)dx = p(x)(y\prime G - G\prime y)|_a^{\zeta^-}$$

and

$$\int_{\zeta^+}^b GL_\lambda(y)dx = p(x)(y\prime G - G\prime y)|_{\zeta^-}^b$$

Therefore,

$$\int_a^b GL_\lambda(y)dx = p(x)(y\prime G - G\prime y)|_a^b - p(x)(y\prime G - G\prime y)|_{\zeta^-}^{\zeta^+}$$

Suppose l_1 and l_2 are boundary conditions with the property if u, v satisfy $l_1(u) = 0$ and $l_2(v) = 0$, then

$$p(x)(y\prime G - G\prime y)|_a^b = 0$$

Let us refers to such boundary conditions as regular boundary conditions. Assume the boundary conditions are regular and $l_1(y) = 0$ and $l_2(y) = 0$. Then

$$\int_a^b GL_\lambda(y)dx = -p(x)(y\prime G - G\prime y)|_{\zeta^-}^{\zeta^+}$$

$$= p(x)(\frac{\partial G(\zeta^+, \zeta)}{\partial x} - \frac{\partial G(\zeta^+, \zeta)}{\partial x} y(\zeta)$$

$$= y(\zeta)$$

Theorem 2.5.2. *Let y_1 and y_2 (different from zero) be solutions of problem*

$$L(y) = 0,$$

subject to boundary condition

$$l_1(y_1) = 0$$

$$l_2(y_2) = 0$$

and λ be not an eigenvalue of BVP_λ subject to boundary condition

$$l_1(y) = a_1 y(a) + a_2 y\prime(a)$$

$$l_2(y) = b_1 y(b) + b_2 y\prime(b).$$

Then

$$y(x) = \frac{\gamma_2}{l_2(y_1)} y_1(x) + \frac{\gamma_1}{l_1(y_2)} y_2(x) + \int_a^b G(x, \zeta) f(\zeta) d\zeta$$

is the unique solution of nonhomogeneous boundary value problem

$$L_\lambda y(x) = f(x), a < x < b$$

subject to boundary condition

$$l_1(y) = \gamma_1$$

$$l_2(y) = \gamma_2$$

Proof. Since y_1 and y_2 are different from zero, we have $l_1(y_2) \neq 0$ and $l_2(y_1) \neq 0$. And also since $l_1(G) = l_2(G) = 0$, we obtain

$$l_1(y) = \frac{\gamma_1}{l_1(y_2)} l_1(y_2) = \gamma_1$$

and

$$l_2(y) = \frac{\gamma_2}{l_2(y_1)} l_2(y_1) = \gamma_2$$

To verify the nonhomogeneous differential equation is satisfied, consider

$$u(x) = \int_a^b G(x,\zeta)f(\zeta)d\zeta$$

$$= \int_a^x G(x,\zeta)f(\zeta)d\zeta + \int_x^b G(x,\zeta)f(\zeta)d\zeta$$

$$\Rightarrow u\prime(x) = \int_a^{x^-} \frac{\partial G(x,\zeta)}{\partial x} f(\zeta)d\zeta + G(\zeta^-,\zeta) + \int_{x^+}^b \frac{\partial G(x,\zeta)}{\partial x} f(\zeta)d\zeta - G(\zeta^+,\zeta)$$

$$= \int_a^{x^-} \frac{\partial G(x,\zeta)}{\partial x} f(\zeta)d\zeta + \int_{x^+}^b \frac{\partial G(x,\zeta)}{\partial x} f(\zeta)d\zeta$$

Differentiating $u(x)$ twice, we get

$$u\prime\prime(x) = \int_a^{x^-} G_{xx}(x,\zeta)f(\zeta)d\zeta + G_x(\zeta^-,\zeta)f(\zeta^-) + \int_{x^+}^b G_{xx}(x,\zeta)f(\zeta)d\zeta - G_x(\zeta^+,\zeta)f(\zeta^+)$$

Note that

$$\frac{\partial G(x,x^-)}{\partial x} = \frac{\partial G(x^+,x)}{\partial x}$$

$$\frac{\partial G(x^-,x)}{\partial x} = \frac{\partial G(x,x^+)}{\partial x}$$

To verify it

$$\frac{\partial G(x,x^-)}{\partial x} = \lim_{\epsilon\to 0} \frac{\partial G(x,x-\epsilon)}{\partial x}$$

$$= \lim_{\epsilon\to 0}\lim_{h\to 0} \frac{G(x+h,x-\epsilon) - G(x,x-\epsilon)}{h}$$

The partial exist since the open region $x > \zeta$ away from the diagonal $x = \zeta$ where only one sided derivatives exist. Moreover, since G is smooth for every $x > \zeta$, we can interchange the order of limits, we get

$$\frac{\partial G(x,x^-)}{\partial x} = \lim_{\epsilon\to 0^+}\lim_{h\to 0^+} \frac{G(x+h,x-\epsilon) - G(x,x-\epsilon)}{h}$$

$$= \lim_{h\to 0^+}\lim_{\epsilon\to 0^+} \frac{G(x+h,x-\epsilon) - G(x,x-\epsilon)}{h}$$

$$= \lim_{h\to 0^+} \frac{G(x+h,x) - G(x,x)}{h}$$

$$\frac{\partial G(x,x^-)}{\partial x} = \frac{\partial G(x^+,x)}{\partial x}$$

Hence

$$u\prime\prime(x) = \int_a^b G_{xx}(x,\zeta)f(\zeta)d\zeta + [G_x(x^+,x) - G_x(x-,x)]f(x)$$

$$= \int_a^b G_{xx}(x,\zeta)f(\zeta)d\zeta + \frac{1}{p(x)}f(x)$$

By substitute in the given equation, we have

$$L_\lambda(u) = \int_a^b [p(x)G_{xx}(x,\zeta) + p\prime(x)G_x(x,\zeta) + g(x)G(x,\zeta) + \lambda G(x,\zeta)]f(\zeta)d\zeta + \frac{p(x)f(x)}{p(x)}$$

$$= \int_a^b L_\lambda G(x,\zeta)f(\zeta)G(x,\zeta) + f(x)$$

$$L_\lambda(u) = f(x)$$

□

Theorem 2.5.3. *Let (μ,ν) be an eigenpair of boundary value problem*

$$L_\lambda(y) = 0$$

subject to boundary condition

$$l_i(y) = 0, i = 1,2.$$

Then the nonhomogeneous boundary value problem

$$L_\lambda(y) = f(x)$$

subject to boundary condition

$$l_i(y) = 0, i = 1,2.$$

has a solution if and only if

$$\int_a^b f(x)v(x)dx = 0.$$

Proof. Assume that u is the solution of the nonhomogeneous boundary problem $L_\lambda(y) = f(x)$. Since u and v satisfy the homogeneous boundary conditions, we have

$$P(u,v) = p(x)(v\prime u - u\prime v)$$

From Green's formula, we get

$$\int_a^b vfdx = \int_a^b [0 - vf]dx$$

$$= \int_a^b [uL_\lambda(v) - vL_\lambda(u)]dx = 0$$

$$\Rightarrow \int_a^b vfdx = 0$$

On the other hand suppose $\int_a^b vfdx = 0$ and choose u to the initial value problem solve $L_\lambda(u) = f(x)$ with the initial conditions

$$u(a) = v(a), u\prime(a) = v\prime(a).$$

Then $l_1(u) = 0$ since $l_1(v) = 0$. Now we need to show $l_2(u) = b_1u(b) + b_2u\prime(b) = 0$ □

From Green's formula, we have

$$\int_a^b vf dx = \int_a^b [uL_\lambda(v) - vL_\lambda(u)]dx = p(x)[uv\prime - vu\prime]|_a^b$$

Since $l_1(u) = 0$,

$$\int_a^b vf dx = p(b)[u(b)v\prime(b) - v(b)u\prime(b)]$$

Because of $\int_a^b vf dx = 0$, we get

$$\Rightarrow p(b)[u(b)v\prime(b) - v(b)u\prime(b)] = 0$$

Since $b_1^2 + b_2^2 \neq 0$, without loss of generality suppose $b_1 \neq 0$. If $b_2 = 0$, then $b_1 v(b) = 0$ implies that $v(b) = 0$ and hence $l_2(u) = 0$. If $b_2 \neq 0$, then $\frac{b_1}{b_2} v(b) = v\prime(b)$. Since v is a nontrivial $L_\lambda(y) = 0, v(b) \neq 0$ and $v\prime(b) \neq 0$. Implies that

$$u(b) = \frac{v(b)u\prime(b)}{v\prime(b)} \ and$$

$$b_1 u(b) + b_2 u\prime(b) = b_1[\frac{v(b)u\prime(b)}{v\prime(b)}] + b_2 u\prime(b)$$

$$b_1 u(b) + b_2 u\prime(b) = u\prime(b)(b_1[\frac{v(b)}{v\prime(b)}] + b_2) = 0$$

Hence, $l_2(u) = 0$.
Therefore, u is the solution of the above nonhomogeneous boundary value problem.

2.6 Modified Green's Function

Theorem 2.6.1. *(Fredholm Alternative)*
For a nonhomogeneous boundary condition

$$Ly(x) = f(x),$$

with boundary conditions

$$y(0) - h_0 y\prime(0) = 0$$

$$y(1) - h_1 y\prime(1) = 0$$

where h_0 and h_1 are arbitrary constants, either of the following statement is holds:

1. *$y = 0$ is the only homogeneous solution (i.e $\lambda = 0$ is not an eigenvalue), in the case of nonhomogeneous problem has a unique solution*

2. *There are nontrivial homogeneous solution $\rho_n(x)$(i.e $\lambda = 0$ is an eigenvalue), in which cases the homogeneous problem has no solution or an infinite number of solutions.*

If the linear homogeneous differential equation has nontrivial solution that satisfies boundary conditions, then a Green's function for the differential equation with these boundary conditions doesn't exist. In these cases we find the generalized Green's function (modified Green's function).

Example 2.6.1. *Take*

$$y'' + y = f(x), \text{ subject to } y(0) = y(\pi) = 0.$$

$y(x) = \beta_1 \sin x + \beta_2 \cos x$ is the general solution of homogeneous problem $y'' + y = 0$. From boundary condition $y(0) = y(\pi) = 0$ if and only if $\beta_2 = 0$ which implies $y(x) = \beta_1 \sin x$, where c_1 is arbitrary constant. Hence $y(x) = \sin x$ is a solution of $y''+y = 0$, subject to $y(0) = y(\pi) = 0$.. Thus, the Green's function doesn't exist. Define Green's function $G(x, \zeta), 0 \le x \le \pi, 0 \le \zeta \le \pi$. From properties (1) *and* (3)*, we obtain*

$$G(x, \zeta) = \begin{cases} \alpha_1 \sin x + \alpha_2 \cos x, & x < \zeta \\ \beta_1 \sin x + \beta_2 \cos x, & x > \zeta. \end{cases}$$

and

$$(\beta_1 - \alpha_1) \cos \zeta - (\beta_2 - \alpha_2) \sin \zeta = 1$$

The property G satisfies the boundary conditions that u satisfies, which is,

$$G_1(0, \zeta) = G_2(\pi, \zeta) = 0$$

implies that $\alpha_2 = 0$ and $\beta_2 = 0$. Hence $(\beta_1 - \alpha_1) \cos \zeta - (\beta_2 - \alpha_2) \sin \zeta = 1$ implies $(\beta_1 - \alpha_1) \cos \zeta = 1$, a contradiction since there is no constant $(\beta_1 - \alpha_1)$ which make $(\beta_1 - \alpha_1) \cos \zeta = 1$ for all ζ. In this case we use the modified Green's function $H(x, \zeta), 0 \le x \le \pi, 0 \le \zeta \le \pi$ define as

$$H(x, \zeta) = \begin{cases} H_1(x, \zeta), & x < \zeta \\ H_2(x, \zeta), & x > \zeta. \end{cases}$$

such that $H(x, \zeta)$ satisfies

1. *$H(x, \zeta)$ is continuous on $0 \le x \le \pi, 0 \le \zeta \le \pi$. In particular, at $x = \zeta$, $H_2(x, \zeta) - H_1(x, \zeta) = 0$, and also$H_1(x, \zeta)$ and $H_1(x, \zeta)$ satisfy*

   $$\frac{\partial^2}{\partial x^2} H(x, \zeta) + H(x, \zeta) = Cu(x)u(\zeta)$$

   where $u(x)$ is a solution of $y'' + y = 0$ subject to $y(0) = y(\pi) = 0$ and C is arbitrary constant.

2. *$H(x, \zeta)$ satisfies the boundary conditions of the given differential equation, that is*

   $$H_2(\pi, \zeta) = H_1(0, \zeta) = 0$$

3. $\frac{\partial}{\partial x}H(x,\zeta)$ *has a jump of continuity of magnitude one at* $x=\zeta$ *i.e*

$$\frac{\partial}{\partial x}H_2(\zeta,\zeta)-\frac{\partial}{\partial x}H_2(\zeta,\zeta)=1$$

4. $H(x,\zeta)$ *satisfies the condition*

$$\int_0^{\pi} H(x,\zeta)u(x)dx=0$$

Hence, we will find a modified Green's function $H(x,\zeta)$ *using the above four properties. From* (1)

$$\frac{\partial^2}{\partial x^2}H(x,\zeta)+H(x,\zeta)=C\sin x\sin\zeta$$

The general solution of $\frac{\partial^2}{\partial x^2}H(x,\zeta)+H(x,\zeta)=0$ *is* $H_g=\alpha\sin x+\beta\cos x$ *and using the method of variation of parameter we obtain the particular solution of*

$$\frac{\partial^2}{\partial x^2}H(x,\zeta)+H(x,\zeta)=\sin x\sin\zeta,$$

$$H_p=\frac{-Cx}{2}\cos x\sin\zeta.$$

Thus,

$$H(x,\zeta)=\begin{cases}\alpha_1\sin x+\alpha_2\cos x-\frac{Cx}{2}\cos x\sin\zeta, & x<\zeta\\ \beta_1\sin x+\beta_2\cos x-\frac{Cx}{2}\cos x\sin\zeta, & x>\zeta.\end{cases}$$

From (1) *and* (3) *we get*

$$(\beta_1-\alpha_1)\sin\zeta+(\beta_2-\alpha_2)\cos\zeta=0$$

$$(\beta_1-\alpha_1)\cos\zeta-(\beta_2-\alpha_2)\sin\zeta=1$$

Since the Wronskian of y_1 *and* y_2 *is different from zero, the above linear system determines* $(\beta_1-\alpha_1)$ *and* $(\beta_2-\alpha_2)$. *Using Cramer's rule we obtain* $(\beta_1-\alpha_1)=\cos\zeta$ *and* $(\beta_2-\alpha_2)=-\sin\zeta$. *From* (2), $H(0,\zeta)=0$ *implies* $\alpha_2=0$ *and* $H(\pi,\zeta)=0$ *implies* $\beta_2=\frac{C\pi}{2}\sin\zeta$ *Hence*

$$-\sin\zeta=\frac{C\pi}{2}\sin\zeta$$

This imply that $C=\frac{-2}{\pi}$. *Now* $H(x,\zeta)$ *become*

$$H(x,\zeta)=\begin{cases}\alpha_1\sin x+\frac{x}{\pi}\cos x\sin\zeta, & x<\zeta\\ (\alpha_1+\cos\zeta)\sin x+(\frac{x}{\pi}-1)\cos x\sin\zeta, & x>\zeta.\end{cases}$$

From property (4), *we obtain*

$$\int_0^x[\alpha_1\sin x+\frac{x}{\pi}\cos x\sin\zeta]\sin xdx\ +$$

$$\int_x^{\pi}[(\alpha_1+\cos\zeta)\sin x+(\frac{x}{\pi}-1)\cos x\sin\zeta]\sin xdx=0$$

This implies

$$\alpha_1 = \frac{(\sin\zeta - 2\sin^3\zeta - 2\pi\cos\zeta + 2\zeta\cos\zeta - \sin 2\zeta\cos\zeta)}{2\pi}$$

Therefore, the required modified Green's function is:

$$H(x,\zeta) = \begin{cases} \alpha_1 \sin x + \frac{x}{\pi}\cos x \sin\zeta, & x < \zeta \\ (\alpha_1 + \cos\zeta)\sin x + (\frac{x}{\pi} - 1)\cos x \sin\zeta, & x > \zeta. \end{cases}$$

where

$$\alpha_1 = \frac{(\sin\zeta - 2\sin^3\zeta - 2\pi\cos\zeta + 2\zeta\cos\zeta - \sin 2\zeta\cos\zeta)}{2\pi}$$

Furthermore, consider

$$Lu(x) = f(x), 0 < x < l$$

with homogeneous boundary condition. If $\lambda = 0$ is not an eigenvalue, then we can find Green's function $G(x,s)$ by solving

$$LG(x,s) = \delta(x-s)$$

Let $\lambda = 0$ be an eigenvalue. Then there are nontrivial homogeneous solutions v_h, that is

$$Lv_h = 0$$

and to have a solution for $Lu(x) = f(x), 0 < x < l$, we must have

$$\int_0^l f(x)v_h dx = 0$$

Since $\delta(x-s)$ is not orthogonal to $u(x)$, in fact

$$\int_0^l v_h(x)\delta(x-s)dx = v_h(s) \neq 0$$

hence we can not expect to get a solution for

$$LG(x,s) = \delta(x-s)$$

that is we cannot get $G(x,s)$. To overcome the problem, introduce

$$\delta(x-s) + cv_h(x)$$

is orthogonal to $v_h(x)$ if we select c an appropriately, such that

$$c = \frac{v_h(s)}{\int_0^l v_h^2(x)dx}$$

Introduce the new Green's function as

$$\hat{G}(x,s) = G(x,s) + \beta v_h(x) v_h(s)$$

satisfies

$$L\hat{G}(x,s) = \delta(x-s) - \frac{v_h(x)v_h(s)}{\int_0^l v_h^2(x)dx}$$

The modified Green's function is also symmetric. To solve

$$Lu(x) = f(x), 0 < x < l$$

using the modified Green's function, we use Green's theorem as $v = \hat{G}$, such that

$$\int_0^l (uL\hat{G} - \hat{G}Lu)dx = 0$$

Since u and $\hat{G}$ satisfies the homogeneous boundary conditions

$$\int_0^l f(s)\hat{G}(x,s)ds + \frac{\int_0^l u(s)v_h(s)ds}{\int_0^l v_h^2(x)dx} v_h(x)$$

Since

$$\frac{\int_0^l u(s)v_h(s)ds}{\int_0^l v_h^2(x)dx} v_h(x)$$

is a multiple of the homogeneous solution, we get a particular solution for the nonhomogeneous

$$Lu(x) = f(x), 0 < x < l$$

$$u(x) = \int_0^l f(s)\hat{G}(x,s)ds$$

Chapter 3

Green's Function for Partial Differential Equations

3.1 Heat Equation

Consider one dimensional heat equation,

$$u_t = u_{xx}, x \in (0,1), t > 0$$

with boundary and initial conditions

$$u(x,0) = f(x), x \in (0,1)$$

and

$$u(0,t) = u(1,t) = 0.$$

From the separation of variable method we get the solution

$$u(x,t) = \sum_{n=1}^{\infty} a_n \sin(n\pi x) e^{-(n\pi)^2 t}$$

where a_n are the Fourier coefficients of the expansion of $f(x)$ in Fourier sine series such that

$$a_n = 2\int_0^1 f(x)\sin(n\pi x)dx$$

$$\Rightarrow u(x,t) = \sum_{n=1}^{\infty} 2[\int_0^1 f(x)\sin(n\pi x)dx]\sin(n\pi x)e^{-(n\pi)^2 t}]$$

$$\Rightarrow u(x,t) = \int_0^1 f(\zeta)\sum_{n=1}^{\infty}[2\sin(n\pi\zeta)\sin(n\pi x)e^{-(n\pi)^2 t}]d\zeta$$

An expression

$$\sum_{n=1}^{\infty}[2\sin(n\pi\zeta)\sin(n\pi x)e^{-(n\pi)^2 t}$$

is known as influence function for the initial condition. An influence function shows the fact that the temperature at a point x at a time t is due to the initial temperature f at a point ζ. So by integrate (sum) the influence of all initial points ζ we get the temperature $u(x,t)$.
Consider a nonhomogeneous problem

$$u_t = u_{xx} + P(x,t), x \in (0,1), t > 0$$

with boundary and initial conditions

$$u(x,0) = f(x), x \in (0,1)$$

and

$$u(0,t) = u(1,t) = 0.$$

We expand a function u and P in an eigenfunctions $\sin(\pi nx)$ such that

$$P(x,t) = \sum_{n=1}^{\infty} p_n(t)\sin(n\pi x)$$

where

$$p_n(t) = 2\int_0^1 P(x,t)\sin(n\pi x)dx$$

and let

$$u(x,t) = \sum_{n=1}^{\infty} u_n(t)\sin(n\pi x)$$

by differentiating $u(x,t)$, we get

$$u_t = \sum_{n=1}^{\infty} u_n\prime(t)\sin(n\pi x)$$

$$u_x = \sum_{n=1}^{\infty}(n\pi)u_n(t)\cos(n\pi x)$$

$$u_{xx} = \sum_{n=1}^{\infty} -(n\pi)^2 u_n(t)\sin(n\pi x)$$

Hence, by substitution we obtain the nonhomogeneous first order linear ordinary differential equation

$$u_n\prime(t) + (n\pi)^2 u_n(t) = p_n(t).$$

Solving above equation using variation of parameter, we get

$$u_n(t) = u_n(0)e^{-(n\pi)^2 t} + \int_0^t p_n(\gamma)e^{-(n\pi)^2 t}d\gamma$$

where

$$u_n(0) = a_n = 2\int_0^1 f(x)\sin(n\pi x)dx$$

By substitute $u_n(t), u_n(0)$ and $p_n(t)$ in an equation

$$u(x,t) = \sum_{n=1}^{\infty} u_n(t)\sin(n\pi x)$$

we obtain

$$u(x,t) = \sum_{n=1}^{\infty}[u_n(0)e^{-(n\pi)^2 t} + \int_0^t p_n(\gamma)e^{-(n\pi)^2 t}d\gamma]\sin(n\pi x)$$

$$= \int_0^1 f(\zeta)[\sum_{n=1}^{\infty} 2\sin(n\pi\zeta)\sin(nx\pi)e^{-(n\pi)^2 t}]d\zeta +$$

$$\int_0^1\int_0^t P(\zeta,\gamma)[\sum_{n=1}^{\infty} 2\sin(n\pi\zeta)\sin(nx\pi)e^{-(n\pi)^2(t-\gamma)}]d\gamma d\zeta.$$

Introducing the Green's function as

$$G(x;s,t-\gamma) = \sum_{n=1}^{\infty} 2\sin(n\pi\zeta)\sin(nx\pi)e^{-(n\pi)^2(t-\gamma)}$$

we get the solution of the given problem

$$u(x,t) = \int_0^1 f(\zeta)G(x;s,t-\gamma)]d\zeta + \int_0^1\int_0^t P(\zeta,\gamma)G(x;s,t-\gamma)d\gamma d\zeta$$

The Green's function depends only on the elapsed time $t-\gamma$.

Consider the heat equation on an infinite domain

$$\frac{\partial u}{\partial t} - k\nabla^2 u = P(\overline{x},t)$$

subjected to initial condition

$$u(\overline{x},0) = g(\overline{x})$$

where $\overline{x} = (x_1, x_2, x_3)$ and $\overline{x} \in R^3$ that is the spatial domain is an infinite. Let $\overline{x} = \overline{x_0} = (x_1, x_2, x_3)$ and $t = t_0$ be concentrated source. Take the Green's function $G(\overline{x}, t; \overline{x_0}, t_0)$ is the solution of an equation

$$\frac{\partial G}{\partial t} - k\nabla^2 G = \delta(\overline{x} - \overline{x_0})\delta(t-t_0)$$

From causality principle, we have

$$G(\overline{x}, t; \overline{x_0}, t_0) = 0 \; for \; t < t_0$$

by translate the variable t to the origin, we get

$$G(\overline{x}, t; \overline{x_0}, t_0) = G(\overline{x}, t-t_0; \overline{x_0}, 0)$$

Because of lack of boundary condition, we will find the solution of Green's function $G(\overline{x}, t; \overline{x_0}, t_0)$ by applying Fourier transform. Take the Fourier transforms of the Green's function $G(\overline{x}, t; \overline{x_0}, t_0)$ as $\overline{G}(\overline{w}, t; \overline{x_0}, t_0)$ for infinite domain, we get an ordinary differential equation of the form

$$\frac{\partial \overline{G}(\overline{w}, t; \overline{x_0}, t_0)}{\partial t} + kw^2 \overline{G}(\overline{w}, t; \overline{x_0}, t_0) = \frac{e^{i\overline{w}.\overline{x_0}}}{(2\pi)^3} \delta(t - t_0)$$

subject to

$$\overline{G}(\overline{w}, t; \overline{x_0}, t_0) = 0 \; if \; t < t_0 \; where \; w^2 = \overline{w}.\overline{w}$$

For $t > t_0$, we have

$$\frac{\partial \overline{G}(\overline{w}, t; \overline{x_0}, t_0)}{\partial t} + kw^2 \overline{G}(\overline{w}, t; \overline{x_0}, t_0) = 0$$

Thus, the Fourier transform of the Green's function $G(\overline{x}, t; \overline{x_0}, t_0)$ is

$$\overline{G} = \begin{cases} 0, & \text{if } t < t_0 \\ Ae^{-kw^2(t-t_0)}, & t > t_0. \end{cases}$$

Integrating ordinary differential equation from t_0^- to t_0^+, we obtain

$$\overline{G}(t_0^+) - \overline{G}(t_0^-) = \frac{e^{i\overline{w}.\overline{x_0}}}{(2\pi)^3}$$

since $\overline{G}(t_0^-) = 0$, we get

$$A = \frac{e^{i\overline{w}.\overline{x_0}}}{(2\pi)^3}$$

Hence

$$\overline{G}(\overline{w}, t; \overline{x_0}, t_0) = \frac{e^{i\overline{w}.\overline{x_0}}}{(2\pi)^3} e^{-kw^2(t-t_0)}$$

By using inverse of Fourier transform, we obtain

$$G = \begin{cases} 0, & \text{if } t < t_0 \\ \int_{-\infty}^{\infty} \frac{e^{-kw^2(t-t_0)}}{(2\pi)^3} e^{-i\overline{w}.(\overline{x}-\overline{x_0}} d\overline{w}, & t > t_0. \end{cases}$$

writing Fourier transform $\overline{G}(\overline{w}, t; \overline{x_0}, t_0)$ as a Gaussian, we have

$$G = \begin{cases} 0, & \text{if } t < t_0 \\ \int_{-\infty}^{\infty} \frac{1}{(2\pi)^3} \left[\frac{\pi}{p(t-t_0)}\right]^{\frac{3}{2}} e^{\frac{|\overline{x}-\overline{x_0}|^2}{4k(t-t_0)}}, & t > t_0. \end{cases}$$

We find the solution of heat equation

$$\frac{\partial u}{\partial t} - k\nabla^2 u = P(\overline{x}, t)$$

by using the solved Green's function as

$$L = \frac{\partial}{\partial t} - k\nabla^2, \; where \; L = L_1 - kL_2$$

and

$$L_1 = \frac{\partial}{\partial t}, L_2 = \nabla^2.$$

From Green's formula for L_2 we have

$$\int\int\int [uL_2v - vL_2u]d\overline{x} = \int\int (u\nabla v - v\nabla u).\overline{n}ds.$$

Since L_1 is not self adjoint, we cannot use the Green's formula for L_1. So by applying integration by part, we get

$$\int_{t_1}^{t_2} uL_1vdt = uv|_{t_1}^{t_2} - \int_{t_1}^{t_2} vL_1udt$$

Hence if we introduce self-adjoint linear operator $L_1^* = -\frac{\partial}{\partial t}$, we get

$$\int_{t_1}^{t_2} [uL_1^*v - vL_1u]dt = -uv|_{t_1}^{t_2}$$

Since

$$Lu = P(\overline{x}, t), LG = \delta(\overline{x} - \overline{x_0})\delta(t - t_0)$$

define $L_1^* = -\frac{\partial}{\partial t} - k\nabla^2$.

$$\int_{t_1}^{t_2}\int\int\int [uL^*v - vL^*u]d\overline{x}dt = -\int\int\int uv|_{t_1}^{t_2}d\overline{x} + k\int_{t_1}^{t_2}\int\int (v\nabla u - u\nabla v).\overline{n}dsdt$$

Using Green's function G to find $u(\overline{x}, t)$, consider the source varying Green's function, that using translation

$$G(\overline{x}, t_1; \overline{x_1}, t) = G(\overline{x}, -t_1; \overline{x}, -t_1)$$

By causality principle

$$G(\overline{x}, t_1; \overline{x_1}, t) = 0, t > t_1$$

Thus,

$$[\frac{\partial}{\partial t} - k\nabla^2]G(\overline{x}, t_1; \overline{x_1}, t) = \delta(\overline{x} - \overline{x_0})\delta(t - t_0)$$

$$\Rightarrow L^*[G(\overline{x}, t_1; \overline{x_1}, t)] = \delta(\overline{x} - \overline{x_0})\delta(t - t_0)$$

where $G(\overline{x}, t_1; \overline{x_1}, t)$ is the adjoint Green's function. Moreover

$$G^*(\overline{x}, t_1; \overline{x_1}, t) = G(\overline{x}, t_1; \overline{x_1}, t),$$

$$G^*(\overline{x}, t_1; \overline{x_1}, t) = G(\overline{x}, t_1; \overline{x_1}, t) = 0, t > t_1$$

Suppose that $u(\overline{x}, t)$ be the solution to heat equation

$$Lu(\overline{x}) = P(\overline{x}, t)$$

with

$$u(\overline{x}, 0) = g\overline{x},\ and\ v = G(\overline{x}, t_0, \overline{x_0}, t)$$

be the varying source Green's function obeying

$$\Rightarrow L^* v = \delta(\overline{x} - \overline{x_0})\delta(t - t_0)$$

$$G(\overline{x}, t_0; \overline{x_0}, t) = 0, t_0 > t.$$

Hence Green's formula becomes

$$\int_0^{t_0^+} \int\int\int [u\delta(\overline{x} - \overline{x_0})\delta(t - t_0) - G(\overline{x}, t_0; \overline{x_0}, t)P(\overline{x}, t)]d\overline{x}dt = \int\int\int (u(\overline{x}, 0)G(\overline{x}, t_0; \overline{x_0}, 0)d\overline{x}$$

$$+ \int_0^{t_0^+} \int\int [G(\overline{x}, t_0; \overline{x_0}, t)\nabla u - u\nabla G(\overline{x}, t_0; \overline{x_0}].\overline{n}dsdt.$$

Because of $G(\overline{x}, t_0; \overline{x_0} = 0$ for $t > t_0$, to solve $u(\overline{x}, t)$, replacing the upper limit of integration t_0^+ with t_0, and by reciprocity, we get

$$u(\overline{x}, t) = \int_0^t \int\int\int G(\overline{x}, t; \overline{x_0}, t_0)P(\overline{x_0}, t_0)]d\overline{x}dt_0 + \int\int\int G(\overline{x}, t; \overline{x_0}, 0)g(\overline{x_0})d\overline{x_0}$$

$$+ \int_0^t \int\int [G(\overline{x}, t; \overline{x_0}, t_0)\nabla_{\overline{x_0}} u(\overline{x_0}, t_0) - u(\overline{x_0}, t_0)\nabla_{\overline{x_0}} G(\overline{x}, t; \overline{x_0}].\overline{n}ds_0 dt_0.$$

Therefore, the solution of heat equation on an infinite domain written as

$$u(\overline{x}, t) = \int_0^t \int\int\int [\frac{1}{4\pi k(t - t_0)}]^{\frac{3}{2}} e^{\frac{|\overline{x} - \overline{x_0}|^2}{4k(t - t_0)}} P(\overline{x_0}, t_0)d\overline{x_0}dt\overline{0}$$

$$+ \int\int\int f(\overline{x_0})[\frac{1}{4\pi kt}]^{\frac{3}{2}} e^{\frac{|\overline{x} - \overline{x_0}|^2}{4kt}} d\overline{x_0}$$

3.2 Poisson's Equation

Consider a poisson's equation

$$\nabla^2 u = f(\overrightarrow{r})$$

with homogeneous boundary conditions. The Green's function $G(\overrightarrow{r}, \overrightarrow{r_0})$ must satisfy

$$\nabla^2 G(\overrightarrow{r}, \overrightarrow{r_0}) = \delta(x - x_0)\delta(y - y_0)$$

where $\overrightarrow{r} = (x, y)$, with the same homogeneous boundary conditions. Then the solution of

$$\nabla^2 u = f(\overrightarrow{r})$$

is given by

$$u(\overrightarrow{r}) = \int\int f(\overrightarrow{r_0}) G(\overrightarrow{r}, \overrightarrow{r_0}) dr_0$$

Using one dimensional eigenfunctions, we get Green's function $G(\overrightarrow{r}, \overrightarrow{r_0})$. Suppose the problem is on a rectangular domain

$$\nabla^2 G(\overrightarrow{r}, \overrightarrow{r_0}) = \delta(x - x_0)\delta(y - y_0), 0 < x < L, 0 < y < J$$

$G(\overrightarrow{r}, \overrightarrow{r_0}) = 0$, on all sides of the rectangle. Then the eigenfunction expansion for $G(\overrightarrow{r}, \overrightarrow{r_0})$ becomes

$$G(\overrightarrow{r}, \overrightarrow{r_0}) = \sum_{n=1}^{\infty} g_n(y) \sin(\frac{n\pi x}{L})$$

where $g_n(y)$ obeys

$$\frac{d^2 g_n(y)}{dy^2} - (\frac{n\pi}{L})^2 g_n = \frac{2}{L}\sin(\frac{n\pi x_0}{L})\delta(y - y_0), g_n(0) = g_n(H) = 0$$

This implies that

$$\frac{L}{2\sin(\frac{n\pi x_0}{L})}\frac{d^2 g_n(y)}{dy^2} - \frac{L}{2\sin(\frac{n\pi x_0}{L})}(\frac{n\pi}{L})^2 g_n = \delta(y - y_0)$$

which is the second order linear ordinary differential equation. Hence

$$g_n(y) = \begin{cases} c_n \sinh\frac{n\pi}{L} y \sinh\frac{n\pi}{L}(y_0 - H), & y < y_0 \\ c_n \sinh\frac{n\pi}{L}(y - H)\sinh\frac{n\pi}{L}(y_0), & y > y_0. \end{cases}$$

and from the jump condition, we have

$$\frac{dg_n}{dy}|_{y_0-}^{y_0+} = \frac{2}{L}\sin(\frac{n\pi x_0}{L})$$

where $\frac{1}{p} = \frac{2}{L}\sin(\frac{n\pi x_0}{L})$ p is the coefficient of $\frac{d^2 g_n}{dy^2}$. Now we have

$$c_n\frac{n\pi}{L}[\sinh\frac{n\pi}{L}y_0 \cosh\frac{n\pi}{L}(y_0 - H) - \sinh\frac{n\pi}{L}(y_0 - H)\cosh\frac{n\pi}{L}y_0] = \frac{2}{L}\sin(\frac{n\pi x_0}{L})$$

Implies

$$c_n \frac{n\pi}{L}[\sinh \frac{n\pi}{L}(y_0 - (y_0 - H)) = \frac{2}{L}\sin(\frac{n\pi x_0}{L})$$

Hence

$$c_n = \frac{2\sin(\frac{n\pi x_0}{L})}{n\pi \sinh \frac{n\pi}{L}(H)}$$

Thus,

$$G(\overrightarrow{r}, \overrightarrow{r_0}) = \sum_{n=1}^{\infty} \frac{2\sin(\frac{n\pi x_0}{L})}{n\pi \sinh \frac{n\pi}{L}(H)} \sin(\frac{n\pi x}{L}) \begin{cases} \sinh \frac{n\pi}{L} y \sinh \frac{n\pi}{L}(y_0 - H), & y < y_0 \\ \sinh \frac{n\pi}{L}(y - H) \sinh \frac{n\pi}{L}(y_0), & y > y_0. \end{cases}$$

The Green's function $G(\overrightarrow{r}, \overrightarrow{r_0})$ is symmetric and we could have used Fourier sine series in y, but we can replace it by

$$G(\overrightarrow{r}, \overrightarrow{r_0}) = \sum_{n=1}^{\infty} h_n(y) \sin(\frac{n\pi x}{H})$$

To get the solution of Poisson's equation subject to nonhomogeneous boundary conditions

$$\nabla^2 u = f(\overrightarrow{r}), u = h(\overrightarrow{r}), \ \textit{on the boundary}$$

consider the Green's function $G(\overrightarrow{r}, \overrightarrow{r_0})$ such that

$$\nabla^2 G(\overrightarrow{r}, \overrightarrow{r_0}) = \delta(x - x_0)\delta(y - y_0), 0 < x < L, 0 < y < J$$

subject to homogeneous boundary condition

$$G(\overrightarrow{r}, \overrightarrow{r_0}) = 0$$

By applying Green's formula

$$\int\int [u\nabla^2 G(\overrightarrow{r}, \overrightarrow{r_0}) - G(\overrightarrow{r}, \overrightarrow{r_0})\nabla^2 u] dxdy = \oint (u\nabla G - G\nabla u).\overrightarrow{n} ds$$

we get

$$\int\int [u(\overrightarrow{r})\delta(\overrightarrow{r} - \overrightarrow{r_0}) - f(\overrightarrow{r})G(\overrightarrow{r}, \overrightarrow{r_0})] dxdy = \oint (h(\overrightarrow{r})\nabla G).\overrightarrow{n} ds$$

From Dirac delta property, we get

$$u(\overrightarrow{r_0}) = \int\int f(\overrightarrow{r})G(\overrightarrow{r}, \overrightarrow{r_0}) dxdy + \oint (h(\overrightarrow{r})\nabla G).\overrightarrow{n} ds$$

where $\nabla_{\overrightarrow{r_0}}$ is the gradient at (x_0, y_0) which is called dipole source. To solve a poisson's equation

$$\nabla^2 u = f(\overrightarrow{r}), \ \textit{on an infinite domain}$$

Take the Green's function $G(\overrightarrow{r}, \overrightarrow{r_0})$ which satisfies

$$\nabla^2 G(\overrightarrow{r}, \overrightarrow{r_0}) = \delta(x - x_0)\delta(y - y_0)$$

on an infinite space with no boundary. So we find the two dimensional case of Green's function. Since symmetry G depends only on the distance $r = |\overrightarrow{r} - \overrightarrow{r_0}|$,

$$\nabla^2 G(\overrightarrow{r}, \overrightarrow{r_0}) = \delta(x - x_0)\delta(y - y_0)$$

becomes

$$\frac{1}{r}\frac{d}{dr}(r\frac{dG}{dr}) = \delta(r)$$

When we solve the above equation, we get

$$G(r) = c_1 \ln r + c_2$$

To get the constants $c_i, i = 1, 2$, we integrate the equation

$$\nabla^2 G(\overrightarrow{r}, \overrightarrow{r_0}) = \delta(x - x_0)\delta(y - y_0)$$

over a circle of radius r containing the point (x_0, y_0) that is

$$\int\int \nabla^2 G dxdy = \int\int \nabla\delta(x - x_0)\delta(y - y_0)dxdy = 1$$

$$\Rightarrow \int\int \nabla^2 G dxdy = 1$$

Applying Green's formula, we have

$$\int\int \nabla^2 G dxdy = \oint \nabla G.\overrightarrow{n}\, ds = \oint \frac{\partial G}{\partial r} ds = \frac{\partial G}{\partial r} 2\pi r$$

implies that

$$\frac{\partial G}{\partial r} r = \frac{1}{2\pi}$$

$$\Rightarrow r\frac{\partial}{\partial r}[c_1 \ln r + c_2] = \frac{1}{2\pi}$$

$$\Rightarrow c_1 = \frac{1}{2\pi}$$

For convenience, let $c_2 = 0$. Then we get the Green's function

$$G(r) = \frac{1}{2\pi} \ln r$$

3.3 Wave Equation

Take the wave equation on an infinite domains

$$\frac{\partial^2 u}{\partial t^2} - c^2\nabla^2 u = P(\overrightarrow{x}, t)$$

subject to initial conditions

$$u(\overrightarrow{x}, 0) = f(\overrightarrow{x})$$

$$u_t(\overrightarrow{x}, 0) = g(\overrightarrow{x})$$

where $\overrightarrow{x} = (x_1, x_2, x_3)$. that is the spatial domain $\overrightarrow{x} \in R^3$.
Consider the point $\overrightarrow{x} = \overrightarrow{x_0} = (x_{01}, x_{02}, x_{03})$ be a concentrated source. The Green's function $G(\overrightarrow{x}, t; \overrightarrow{x_0}, t_0)$ obeys an equation

$$\frac{\partial^2 G}{\partial t^2} - c^2\nabla^2 G = \delta(\overrightarrow{x} - \overrightarrow{x_0})\delta(t - t_0).$$

Since the variable t increases in one direction from causality principle, we have

$$G(\overrightarrow{x}, t; \overrightarrow{x_0}, t_0) = 0, t < t_0$$

and also we translate the variable t to the origin we have

$$G(\overrightarrow{x}, t; \overrightarrow{x_0}, t_0) = G(\overrightarrow{x}, t - t_0; \overrightarrow{x_0}, 0)$$

Since the domain is infinite we will solve the Green's function $G(\overrightarrow{x}, t; \overrightarrow{x_0}, t_0)$ using Fourier transforms. Using Fourier transforms, we get

$$f(x_1, x_2, x_3) = \int\int\int F(w_1, w_2, w_3)e^{-i(w_1,w_2,w_3)(x_1,x_2,x_3)}dw_1dw_2dw_3$$

$$\Rightarrow f(\overrightarrow{x}) = \int\int\int F(\overrightarrow{w})e^{-i\overrightarrow{w})(\overrightarrow{x})}d\overrightarrow{w}$$

and

$$F(\overrightarrow{x}) = \frac{1}{(2\pi)^3}\int\int\int f(\overrightarrow{x})e^{(i\overrightarrow{w})(\overrightarrow{x})}d\overrightarrow{x}$$

For δ-function, we get

$$F(\delta(\overrightarrow{x} - \overrightarrow{x_0})) = \frac{1}{(2\pi)^3}\int\int\int \delta(\overrightarrow{x} - \overrightarrow{x_0})e^{(i\overrightarrow{w})(\overrightarrow{x})}d\overrightarrow{x} = \frac{e^{(i\overrightarrow{w})(\overrightarrow{x})}}{(2\pi)^3}$$

and

$$\delta(\overrightarrow{x} - \overrightarrow{x_0}) = \frac{1}{(2\pi)^3}\int\int\int e^{(-i\overrightarrow{w})(\overrightarrow{x})}e^{(i\overrightarrow{w})(\overrightarrow{x_0})}d\overrightarrow{w}$$

$$= \frac{1}{(2\pi)^3}\int\int\int e^{-i\overrightarrow{w}[\overrightarrow{x}-\overrightarrow{x_0}]}d\overrightarrow{w}$$

From the equation

$$\frac{\partial^2 G}{\partial t} - c^2\nabla^2 G = \delta(\overrightarrow{x} - \overrightarrow{x_0})\delta(\overrightarrow{t} - \overrightarrow{t_0}).$$

$$G(\overrightarrow{x}, t; \overrightarrow{x_0}, t_0) = 0, t < t_0$$

take a Fourier transform of $G(\overrightarrow{x}, t; \overrightarrow{x_0,}, t_0)$ and solve for $\overline{G}(\overrightarrow{w}, t; \overrightarrow{x_0,}, t_0)$, we get an ordinary differential equation

$$\frac{\partial^2 \overline{G}}{\partial t^2} + c^2 w^2 \overline{G} = \frac{e^{i\overrightarrow{w}\overrightarrow{x_0}}}{(2\pi)^3}\delta(t - t_0)$$

$$\overline{G}(\overrightarrow{w}, t; \overrightarrow{x_0}, t_0) = 0, t < t_0$$

where $\overline{G}$ is Fourier transform of G.
For $t > t_0$

$$\frac{\partial^2 \overline{G}}{\partial t^2} + c^2 w^2 \overline{G} = 0$$

Thus, the transform of the Green's function $\overline{G}(\overrightarrow{w}, t; \overrightarrow{x_0}, t_0)$ is given as

$$\overline{G}(\overrightarrow{w}, t; \overrightarrow{x_0}, t_0) = \begin{cases} 0, & \text{if } t < t_0 \\ A\cos cw(t - t_0) + B\sin cw(t - t_0), & t > t_0. \end{cases}$$

Since $\overline{G}$ is continuous at $t = t_0, A = 0$, to calculate B, we integrate the above ODE

$$\int_{t_0^-}^{t_0^+} \frac{\partial^2 \overline{G}}{\partial t^2} + \int_{t_0^-}^{t_0^+} c^2 w^2 \overline{G} = \int_{t_0^-}^{t_0^+} \frac{e^{i\overrightarrow{w}\overrightarrow{x_0}}}{(2\pi)^3}\delta(t - t_0)dt$$

$$\Rightarrow \frac{\partial^2 \overline{G}}{\partial t^2}\Big|_{t_0^-}^{t_0^+} = \frac{e^{i\overrightarrow{w}\overrightarrow{x_0}}}{(2\pi)^3}$$

Since,

$$\frac{\partial^2 \overline{G}}{\partial t^2}\Big|_{t_0^-} = 0$$

we get

$$cwB\cos cw(t - t_0)\big|_{t_0^+} = \frac{e^{i\overrightarrow{w}\overrightarrow{x_0}}}{(2\pi)^3}$$

$$\Rightarrow B = \frac{e^{i\overrightarrow{w}\overrightarrow{x_0}}}{cw(2\pi)^3}$$

Therefore,

$$\overline{G} = \frac{e^{i\overrightarrow{w}\overrightarrow{x_0}}}{cw(2\pi)^3}\sin cw(t - t_0)$$

Applying inverse Fourier transform on $\overline{G}$ we get

$$G(\overrightarrow{x}, t, \overrightarrow{x_0}, t_0) = \begin{cases} 0, & \text{if } t < t_0 \\ \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{e^{i\overrightarrow{w}\overrightarrow{x_0}}}{cw(2\pi)^3}\sin cw(t - t_0)d\overrightarrow{w} & t > t_0. \end{cases}$$

where $w = |\overrightarrow{w}|$

Hence to solve

$$\frac{1}{(2\pi)^3}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{e^{i\overrightarrow{w}\overrightarrow{x_0}}}{cw}\sin cw(t-t_0)d\overrightarrow{w}$$

we introduce spherical coordinates with $\overrightarrow{w} = 0, \phi = 0$ corresponding to the w_3 axis, and we integrate in the direction $(\overrightarrow{x} - \overrightarrow{x_0})$. This implies $\overrightarrow{w}.(\overrightarrow{x} - \overrightarrow{x_0}) = |\overrightarrow{w}||(x - x_0)|\cos\phi$, let $\rho = |(x - x_0)|$ we get $\overrightarrow{w}.(\overrightarrow{x} - \overrightarrow{x_0}) = w\rho\cos\phi$ with an angle θ measured from the positive w_1 axis, the volume differential becomes

$$d\overline{w} = dw_1 dw_2 dw_3 = w^2 \sin\theta d\phi d\theta dw$$

Hence the integration limits for an infinite space becomes $0 < \phi < \pi, 0 < \theta < 2\pi, 0 < w < \infty$. Based on our selection of coordinate, the integrand is independent of θ, implies

$$G(\overline{x}, t, \overline{x_0}, t_0) = \frac{1}{(2\pi)^2}\int_0^{\infty}\int_0^{\pi}\frac{e^{-iw\rho\cos\phi}}{cw}\sin cw(t-t_0)w^2 sin\phi d\phi dw$$

Integrating in terms of ϕ

$$G(\overline{x}, t, \overline{x_0}, t_0) = \frac{1}{ic\rho(2\pi)^2}\int_0^{\infty}\sin cw(t-t_0)\int_0^{\pi}e^{-iw\rho\cos\phi}(iw\rho\sin\phi)d\phi dw$$

$$= \frac{1}{ic\rho(2\pi)^2}\int_0^{\infty}\sin cw(t-t_0)[e^{-iw\rho\cos\phi}]|_0^{\pi}dw$$

$$= \frac{1}{ic\rho(2\pi)^2}\int_0^{\infty}\sin cw(t-t_0)[e^{iw\rho} - e^{-iw\rho}]dw$$

since, $[e^{iw\rho} - e^{-iw\rho}] = 2i\sin w\rho$ we get

$$G(\overline{x}, t, \overline{x_0}, t_0) = \frac{2}{c\rho(2\pi)^2}\int_0^{\infty}\sin\rho w\sin cw(t-t_0)dw$$

$$= \frac{2}{c\rho(2\pi)^2}\int_0^{\infty}\cos w[\rho - c(t-t_0)] - \cos w[\rho + c(t-t_0)]dw$$

because of $\int_{-\infty}^{\infty}[e^{-iwz}]dw = \delta(z)$, and using the real part of e^{-iwz}, the evenness of cosine function, we have

$$\int_0^{\infty}\cos wz = \delta(z)$$

Thus,

$$G(\overline{x}, t, \overline{x_0}, t_0) = \frac{2}{c\rho(2\pi)^2}(\delta[\rho - c(t-t_0)] - \delta[\rho + c(t-t_0)])\ for\ t > t_0$$

Since $\rho + c(t - t_0) > 0$, we obtain

$$G(\overline{x}, t, \overline{x_0}, t_0) = \begin{cases} 0, & \text{if } t < t_0 \\ \frac{2}{c\rho(2\pi)^2}(\delta[\rho - c(t-t_0)] & t > t_0. \end{cases}$$

To get the solution of the wave equation

$$\frac{\partial^2 u}{\partial t^2} = c^2\nabla^2 u + P(\overrightarrow{x}, t), u(\overline{x}, 0) = f(\overline{x}), u_t(\overline{x}, 0) = g(\overline{x})$$

applying Green's function, define a linear operator L as

$$L = \frac{\partial^2}{\partial t^2} - c^2\nabla^2$$

$$\Rightarrow L = L_1 - c^2 L_2,\ where\ L_1 = \frac{\partial^2}{\partial t^2}, L_2 = \nabla^2.$$

The Green's formula for L_1 and L_2 are given by

$$\int_{t_1}^{t_2} (uL_1(v) - vL_1(u))dt = uv_t - vu_t|_{t_1}^{t_2}$$

and

$$\int\int\int (uL_2(v) - vL_2(u))d\overline{x} = \int\int (u\nabla v - v\nabla u).\overline{n}ds$$

Since $Lu = P(\overline{x}, t), LG = \delta(\overline{x} - \overline{x_0})\delta(\overline{t} - \overline{t_0})$ and
$uL(v) - vL(u) = uL_1(v) - vL_1(u) - c^2(uL_2(v) - vL_2(u))$,

$$\int_{t_1}^{t_2}\int\int\int (uL(v) - vL(u))d\overline{x}dt = \int\int\int uv_t - vu_t|_{t_1}^{t_2} d\overline{x} - c^2\int_{t_1}^{t_2}\int\int (u\nabla v - v\nabla u).\overline{n}dsdt$$

This shows that Maxwell's reciprocity holds spatially for the Green's function, provided the elapsed times between the points $\overline{x}$ and $\overline{x_0}$ are the same for the infinite domain.

$$G(\overline{x}, t, \overline{x_0}, t_0) = \begin{cases} 0, & \text{if } t < t_0 \\ \frac{2}{c|\overline{x}-\overline{x_0}|(2\pi)^2}(\delta[|\overline{x} - \overline{x_0}| - c(t - t_0)] & t > t_0. \end{cases}$$

$$= \begin{cases} 0, & \text{if } t < t_0 \\ \frac{2}{c|\overline{x_0}-\overline{x}|(2\pi)^2}(\delta[|\overline{x_0} - \overline{x}| - c(t - t_0)] & t > t_0. \end{cases}$$

$$= G(\overline{x_0}, t, \overline{x}, t_0)$$

Let $u(\overline{x}, t)$ be the solution of a wave equation

$$Lu = P(\overline{x}, t)$$

with initial conditions

$$u(\overline{x}, 0) = f(\overline{x}), u_t(\overline{x}, 0) = g(\overline{x})$$

and $v = G(\overline{x}, t_0, \overline{x_0}, t)$ be a solution to

$$Lv = \delta(\overline{x} - \overline{x_0})\delta(\overline{t} - \overline{t_0})$$

with homogeneous boundary condition and the causality principle

$$G(\overline{x}, t_0, \overline{x_0}, t) = 0 \ for \ t_0 < t$$

Integrate in time from $t_1 = 0$ to $t_2 = t_0^+$(a point just beyond the appearance of our point source $t = t_0$), we obtain

$$\int_0^{t_0^+} \int\int\int (uL(v) - vL(u)) d\overline{x} dt = \int_0^{t_0^+} \int\int\int (u(\overline{x}, t)\delta(\overline{x} - \overline{x_0})\delta(\overline{t} - \overline{t_0}) - G(\overline{x}, t_0, \overline{x_0}, t)P(\overline{x}, t)) d\overline{x} dt$$

$$= \int\int\int uG_t - Gu_t|_0^{t_0^+} d\overline{x} - c^2 \int_0^{t_0^+} [\int\int (u\nabla G - G\nabla u).\overline{n} ds] dt$$

At point $t = t_0, G = G_t = 0$. And using reciprocity, we obtain

$$u(\overline{x_0}, t_0) = \int_0^{t_0^+} \int\int\int G(\overline{x}, t_0, \overline{x_0}, t)P(\overline{x}, t)) d\overline{x} dt$$

$$+ \int\int\int u(\overline{x}, 0)G(\overline{x_0}, t_0, \overline{x}, 0) - u_t(\overline{x}, 0)G_t(\overline{x_0}, t_0, \overline{x}, 0) d\overline{x} -$$

$$c^2 \int_0^{t_0^+} [\int\int (u(\overline{x}, t)\nabla G(\overline{x_0}, t_0, \overline{x}, t) - G(\overline{x_0}, t_0, \overline{x}, t)\nabla u(\overline{x}, t))].\overline{n} ds] dt$$

As $t_0^+ \to t$ and interchange $(\overline{x_0}, t_0)$ with $(\overline{x}, t)$, we obtain

$$u(\overline{x}, t) = \int_0^t \int\int\int G(\overline{x}, t, \overline{x_0}, t_0)P(\overline{x_0}, t_0)) d\overline{x_0} dt_0$$

$$+ \int\int\int [g(\overline{x_0})G(\overline{x}, t, \overline{x_0}, 0) - f(\overline{x_0})G_{t_0}(\overline{x}, t, \overline{x_0}, 0)] d\overline{x_0} -$$

$$c^2 \int_0^t [\int\int (u(\overline{x_0}, t_0)\nabla_{x_0} G(\overline{x}, t, \overline{x_0}, t_0) - G(\overline{x}, t, \overline{x_0}, t_0)\nabla_{x_0} u(\overline{x_0}, t_0))].\overline{n} ds_0] dt_0$$

where ∇_{x_0} stands for the gradient in terms of the source location $\overline{x_0}$. The three algebraic expression given in above are the contributions due to the source, the initial conditions and boundary conditions respectively. For our infinite domain, an expression

$$c^2 \int_0^t [\int\int (u(\overline{x_0}, t_0)\nabla_{x_0} G(\overline{x}, t, \overline{x_0}, t_0) - G(\overline{x}, t, \overline{x_0}, t_0)\nabla_{x_0} u(\overline{x_0}, t_0))].\overline{n} ds_0] dt_0$$

is removed. Therefore, the complete solution of wave equation with infinite domain is given as:

$$u(\overline{x}, t) = \frac{1}{2\pi^2 c} \int_0^t \int\int\int \frac{1}{|\overline{x} - \overline{x_0}|} \delta[|\overline{x} - \overline{x_0}| - c(t - t_0)] P(\overline{x_0}, t_0) d\overline{x_0} dt_0 +$$

$$\frac{1}{2\pi^2 c} \int\int\int (\frac{g(x_0)}{|\overline{x} - \overline{x_0}|} \delta[|\overline{x} - \overline{x_0}| - c(t - t_0)] - \frac{f(x_0)}{|\overline{x} - \overline{x_0}|} \frac{\partial}{\partial t_0} \delta[|\overline{x} - \overline{x_0}| - c(t - t_0)]) d\overline{x_0}$$

Conclusion

In this thesis a number of boundary value problems have been solved in an integral form which has a function as a kernel of the integral. This function is called Green's function. If the Green's function of the problem is known then the solution for the problem is found. Thus, we are interested in finding Green's function of the problem. It all started in 1828 when George Green first found Green's function for the solution of a potential equation. In order to find Green's function, a definition of the function is needed. From this study, knowing its properties is very helpful in finding the function. The results in section (2.1) are on the solution of the BVP (2.1) subject to (2.2) and the results in section (2.2) are the techniques to construct the Green's functions to solve the linear ordinary differential equations. Theorem (3) gives $y(x) = y_1(x) + y_2(x)$ is the solution of the BVP (2.1) subject to (2.2) when $y_1(x)$ and $y_2(x)$ are the solution of associated homogeneous of DE (2.1) subject to (2.2) and DE (2.1) subject to (2.3) respectively. Further, Theorem (4) states the existences and uniqueness of Green's function of homogeneous boundary value problem (2.1) subject to (2.3), and

$$y(x) = \int_a^b G(x, \zeta) f(\zeta) d\zeta$$

is a unique solution of BVP (2.1) subject to (2.3).
From section (3.1) - (3.4), we have the following results:

1. Solution of one dimensional heat equation and heat equation on an infinite domains applying Green's functions.

2. The complete solution of Poisson's equation and wave equation on an infinite domains using Green's functions.

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