INVESTIGATION OF QUANTUM PROPERTIES OF OUTPUT RADIATION FROM SINGLE-MODE SUBHARMONIC GENERATOR

By

Besie Mengesha Gayo

A PROJECT SUBMITTED IN PARTIAL FULFILLMENT OF THE
REQUIREMENTS FOR THE DEGREE OF
MSC. IN PHYSICS (QUANTUM OPTICS)
AT
ADISS ABABA UNIVERSITY
COLLEGE OF NATURAL AND COMPUTATIONAL SCIENCES
ADISS ABABA, ETHIOPIA
AUGUST, 2018

ADISS ABABA UNIVERSITY
COLLEGE OF NATURAL AND COMPUTATIONAL SCIENCES
DEPARTMENT OF PHYSICS

The undersigned hereby certify that they have read and recommend to the College of Natural and Computational Sciences for acceptance a project entitled "**INVESTIGATION OF QUANTUM PROPERTIES OF OUTPUT RADIATION FROM SINGLE-MODE SUBHARMONIC GENERATOR**" by **Besie Mengesha Gayo** in partial fulfillment of the requirements for the degree of **MSc. in Physics (Quantum Optics)**.

Approved by the Examination Committee:

Advisor: ______________________________
Dr. Deribe Hirpo

Examiners: ______________________________
Dr. Fesseha Kassahun

____
Dr. Belayneh Mesfin

Chairman: ______________________________
Dr. Chenet Amente

ADISS ABABA UNIVERSITY

Date: **August, 2018**

Author: **Besie Mengesha Gayo**

Title: **INVESTIGATION OF QUANTUM PROPERTIES OF OUTPUT RADIATION FROM SINGLE-MODE SUBHARMONIC GENERATOR**

Department: **College of Natural and Computational Sciences**
Department of Physics

Degree: **MSc.**
Convocation: **August**
Year: **2018**

Permission is here with granted to Adiss Ababa University to circulate and to have copied for non-commercial purposes, at its discretion, the above title upon the request of individuals or institutions.

Signature of Author

To my Son Dan Bessie Mengesha!

Table of Contents

List of Figures

Abstract

In this project we studied the squeezing as well as the statistical properties of the output light produced by single-mode subharmonic generating system coupled to vacuum reservoir. We obtained the c-number Langevin equations employing the master equation for the system under consideration. Applying the solutions of these equations, we determined the anti-normally ordered characteristic function which enables us to obtain the Q function. Using the Q function, we have obtained the mean and variance of the photon number and photon number distribution for cavity mode. Moreover, with the aid of the solutions of c-number Langevin equations and input-output relation, we obtain the mean photon number, power spectrum, quadrature variance and quadrature squeezing for the output mode. Finally, for cavity and output mode we found that squeezing occurs in the plus quadrature. In addition, when single-mode subharmonic generator is coupled to vacuum reservoir, the output light generated by the system is in squeezed state and we found that there is 40% squeezing of the output light and 50% squeezing of the cavity light below the vacuum level at steady state and threshold.

Acknowledgements

First of all I would like to thank the almighty God, my strength, power and leading me to this position.

I love and respect my mother Kebebush Ejeta because of the take care of me starting from uterus up to now, I will not compensate to all of you in such practical natural phenomenon.

My endless heart feel thankful goes to my advisor Dr. Deribe Hirpo for his marvelous and remarkable advise, suggestions and constant support in academic and social affairs. He is always on the bottom of my heart in gravitating me to study and love quantum optics. I wish to him forever success, peace and grace throughout his life.

Next I thank and respect Dr. Fesseha Kassahun because of his studies and materials have been my resources to carry out this project work.

My strong appreciation and respect goes to physics department of Addis Ababa university, Ministry of Education, all my instructors and my families for facilitating conditions for my education and financial support.

I would also like to thank my friends Lema Tirfie, Getahun Chane and Takele Aychew for providing encouragement and support.

Chapter 1

Introduction

Optical single-mode subharmonic generator is one of the most interesting and well studied devices in the nonlinear quantum optics [1, 2, 3]. Nonlinear optics is the study of nonlinear response of material to interaction of light. Therefore polarization P responds nonlinearly to the electric field of light. An electric field applied to a nonlinear material induces a macroscopic dipole moment called polarization. The amount of polarization of a material depends on the applied electric field linearly or nonlinearly. The proportionality which relates these two quantities is the susceptibility of the medium. Susceptibility refers how materials respond to an applied field. There are two types of susceptibilities. These are linear and nonlinear susceptibilities. Liner susceptibility is a constant which relates the induced polarization and electric field in linear media. The induced polarization depends on the applied electric field linearly as follows [4, 5, 6].

$$P(t) = \varepsilon_0 \chi^{(1)} E(t) \tag{1.1}$$

where $\chi^{(1)}$ is linear susceptibility, $P(t)$ is polarization of the medium, ε_0 the permittivity of free space and $E(t)$ is electric field of the light.

In nonlinear susceptibility polarization P responds nonlinearly to the electric field of electromagnetic radiation. That is

$$P(t) = \varepsilon_0 \chi^{(2)} E^2(t) \tag{1.2}$$

where $\chi^{(2)}$ is second order nonlinear susceptibility. When electromagnetic radiation is incident on a nonlinear medium, a response other than the driving frequency occurs and is often observed. The response with frequencies less than the driving frequency is called subharmonics and the response with frequencies greater than the driving frequency is called super harmonics. Some examples of nonlinear interactions are subharmonic generation, second harmonic generation and sum frequency generation. In second harmonic generation two photons of the same frequency interacts with nonlinear material and effectively combine to generate a photon with twice the frequency of the initial photon [7, 8]. In a single-mode subharmonic generating system, a pump photon of frequency 2ω is down converted by nonlinear crystal into a pair of signal photons each of frequency ω. Due to this inherent two-photon nature of the interaction, the subharmonic generator is found to be a good source of a squeezed light. Squeezed state of light has an applicable area in optical communications and optical measurements as it has less noise [9, 10].

In this project, we study the statistical and squeezing properties of output light produced by single-mode subharmonic generator coupled to vacuum reservoir. Employing the master equation for the system under consideration, we obtain the c-number Langevin equations for the cavity mode operator. Applying the solutions of these equations, we determine the anti-normally ordered characteristic function which enables us to obtain the Q function. Using the Q function and the solutions of c-number Langevin equations, we obtain the mean and variance of the photon number and photon number distribution for cavity mode. We next determine the mean photon number, power spectrum, quadrature variance and quadrature squeezing for the output radiation by employing the solutions of c-number Langevin equations and input-output relation.

Chapter 2

C-number Langevin Equations and The Q Function

2.1 C-number Langevin equations

Figure 2.1: Schematic diagram for single-mode subharmonic generator.

The single-mode subharmonic generating system can be described in the interaction picture upon treating the pump radiation classically, by the Hamiltonian of the form

$$\hat{H} = \frac{i\varepsilon}{2}\left(\hat{a}^2 - \hat{a}^{\dagger 2}\right) \tag{2.1}$$

where ε is proportional to the amplitude of the external coherent radiation which is taken to be a real-positive constant and $\hat{a}$ is the annihilation operator for the signal mode. We consider the case in which a continuum mode of squeezed vacuum centered at frequency ω is allowed to enter the cavity through one of the coupler mirrors. In this case, the master equation describing the single-mode subharmonic generating system coupled to squeezed vacuum reservoir in the interaction picture is described as [1, 2]

$$\begin{aligned}\frac{d\hat{\rho}}{dt} &= \frac{\varepsilon}{2}\big(\hat{\rho}\hat{a}^{\dagger 2} - \hat{\rho}\hat{a}^2 + \hat{a}^2\hat{\rho} - \hat{a}^{\dagger 2}\hat{\rho}\big)\\ &+ \frac{\kappa(N+1)}{2}\big(2\hat{a}\hat{\rho}\hat{a}^\dagger - \hat{a}^\dagger\hat{a}\hat{\rho} - \hat{\rho}\hat{a}^\dagger\hat{a}\big)\\ &+ \frac{\kappa N}{2}\big(2\hat{a}^\dagger\hat{\rho}\hat{a} - \hat{a}\hat{a}^\dagger\hat{\rho} - \hat{\rho}\hat{a}\hat{a}^\dagger\big)\\ &- \frac{\kappa M}{2}\big(\hat{a}^2\hat{\rho} - 2\hat{a}\hat{\rho}\hat{a} + \hat{\rho}\hat{a}^2 - 2\hat{a}^\dagger\hat{\rho}\hat{a}^\dagger + \hat{a}^{\dagger 2}\hat{\rho} + \hat{\rho}\hat{a}^{\dagger 2}\big),\end{aligned} \tag{2.2}$$

where N and M represent the squeezed vacuum reservoir, with $N = sinh^2(r)$, $M = sinh(r)cosh(r)$, κ is the cavity damping constant and r is squeeze parameter.
Using time evolution expectation value of an operator $\hat{A}$ in the Schroodinger picture

$$\frac{d}{dt}\langle\hat{A}(t)\rangle = Tr(\frac{d\hat{\rho}(t)}{dt}\hat{A}), \tag{2.3}$$

along with the master equation described by Eq. (2.2), we can write

$$\begin{aligned}\frac{d}{dt}\langle\hat{a}(t)\rangle = Tr(\frac{d\hat{\rho}}{dt}\hat{a}) &= Tr\frac{\varepsilon}{2}\big(\hat{\rho}\hat{a}^{\dagger 2}\hat{a} - \hat{\rho}\hat{a}^3 + \hat{a}^2\hat{\rho}\hat{a} - \hat{a}^{\dagger 2}\hat{\rho}\hat{a}\big)\\ &+ \frac{\kappa(N+1)}{2}Tr\big(2\hat{a}\hat{\rho}\hat{a}^\dagger\hat{a} - \hat{a}^\dagger\hat{a}\hat{\rho}\hat{a} - \hat{\rho}\hat{a}^\dagger\hat{a}^2\big)\\ &+ \frac{\kappa N}{2}Tr\big(2\hat{a}^\dagger\hat{\rho}\hat{a}^2 - \hat{a}\hat{a}^\dagger\hat{\rho}\hat{a} - \hat{\rho}\hat{a}\hat{a}^\dagger\hat{a}\big)\\ &- \frac{\kappa M}{2}Tr\big(\hat{a}^2\hat{\rho}\hat{a} - 2\hat{a}\hat{\rho}\hat{a}^2 + \hat{\rho}\hat{a}^3 - 2\hat{a}^\dagger\hat{\rho}\hat{a}^\dagger\hat{a} + \hat{a}^{\dagger 2}\hat{\rho}\hat{a} + \hat{\rho}\hat{a}^{\dagger 2}\hat{a}\big),\end{aligned} \tag{2.4}$$

so that using cyclic property of trace operation, we obtain

$$\begin{aligned}Tr(\frac{d\hat{\rho}}{dt}\hat{a}) =& Tr\frac{\varepsilon}{2}\big(\hat{\rho}\hat{a}^{\dagger 2}\hat{a} - \hat{\rho}\hat{a}^3 + \hat{\rho}\hat{a}^3 - \hat{\rho}\hat{a}\hat{a}^{\dagger 2}\big)\\ &+ \frac{\kappa(N+1)}{2}Tr\big(2\hat{\rho}\hat{a}^\dagger\hat{a}^2 - \hat{\rho}\hat{a}\hat{a}^\dagger\hat{a} - \hat{\rho}\hat{a}^\dagger\hat{a}^2\big)\\ &+ \frac{\kappa N}{2}Tr\big(2\hat{\rho}\hat{a}^2\hat{a}^\dagger - \hat{\rho}\hat{a}^2\hat{a}^\dagger - \hat{\rho}\hat{a}\hat{a}^\dagger\hat{a}\big)\\ &- \frac{\kappa M}{2}Tr\big(\hat{\rho}\hat{a}^3 - 2\hat{\rho}\hat{a}^3 + \hat{\rho}\hat{a}^3 - 2\hat{\rho}\hat{a}^\dagger\hat{a}\hat{a}^\dagger + \hat{\rho}\hat{a}\hat{a}^{\dagger 2} + \hat{\rho}\hat{a}^{\dagger 2}\hat{a}\big),\end{aligned} \tag{2.5}$$

simplifying Eq. (2.5), we have

$$\begin{aligned}\frac{d}{dt}\langle\hat{a}(t)\rangle =& \frac{\varepsilon}{2}Tr\big(\hat{\rho}\hat{a}^{\dagger 2}\hat{a} - \hat{\rho}\hat{a}\hat{a}^{\dagger 2}\big) + \frac{\kappa(N+1)}{2}Tr\big(\hat{\rho}\hat{a}^\dagger\hat{a}^2 - \hat{\rho}\hat{a}\hat{a}^\dagger\hat{a}\big)\\ &+ \frac{\kappa N}{2}Tr\big(\hat{\rho}\hat{a}^2\hat{a}^\dagger - \hat{\rho}\hat{a}\hat{a}^\dagger\hat{a}\big)\\ &- \frac{\kappa M}{2}Tr\big(-2\hat{\rho}\hat{a}^\dagger\hat{a}\hat{a}^\dagger + \hat{\rho}\hat{a}\hat{a}^{\dagger 2} + \hat{\rho}\hat{a}^{\dagger 2}\hat{a}\big).\end{aligned} \tag{2.6}$$

Now taking into account the commutation relation

$$[\hat{a}, \hat{a}^\dagger] = 1, \tag{2.7}$$

Eq. (2.6) can be simplified as

$$\begin{aligned}
Tr(\hat{\rho}\hat{a}^{\dagger 2}\hat{a} - \hat{\rho}\hat{a}\hat{a}^{\dagger 2}) &= Tr\big(\hat{\rho}\hat{a}^\dagger(\hat{a}\hat{a}^\dagger - 1)) - \hat{\rho}\hat{a}\hat{a}^{\dagger 2})\big) \\
&= Tr(\hat{\rho}\hat{a}^\dagger\hat{a}\hat{a}^\dagger - \hat{\rho}\hat{a}^\dagger - \hat{\rho}\hat{a}\hat{a}^{\dagger 2}) \\
&= Tr\big(\hat{\rho}(\hat{a}\hat{a}^\dagger - 1)\hat{a}^\dagger - \hat{\rho}\hat{a}^\dagger - \hat{\rho}\hat{a}\hat{a}^{\dagger 2}\big) \\
&= Tr(\hat{\rho}\hat{a}\hat{a}^{\dagger 2} - \hat{\rho}\hat{a}^\dagger - \hat{\rho}\hat{a}^\dagger - \hat{\rho}\hat{a}\hat{a}^{\dagger 2}) \\
&= -2Tr(\hat{\rho}\hat{a}^\dagger),
\end{aligned} \tag{2.8}$$

$$\begin{aligned}
Tr(\hat{\rho}\hat{a}^\dagger\hat{a}^2 - \hat{\rho}\hat{a}\hat{a}^\dagger\hat{a}) &= Tr\big(\hat{\rho}\hat{a}^\dagger\hat{a}^2 - \hat{\rho}(1 + \hat{a}^\dagger\hat{a})\hat{a}\big) \\
&= Tr(\hat{\rho}\hat{a}^\dagger\hat{a}^2 - \hat{\rho}\hat{a} - \hat{\rho}\hat{a}^\dagger\hat{a}^2) \\
&= -Tr(\hat{\rho}\hat{a}),
\end{aligned} \tag{2.9}$$

$$\begin{aligned}
Tr(\hat{\rho}\hat{a}^2\hat{a}^\dagger - \hat{\rho}\hat{a}\hat{a}^\dagger\hat{a}) &= Tr(\hat{\rho}\hat{a}^2\hat{a}^\dagger - \hat{\rho}\hat{a}(\hat{a}\hat{a}^\dagger - 1)) \\
&= Tr(\hat{\rho}\hat{a}^2\hat{a}^\dagger - \hat{\rho}\hat{a}^2\hat{a}^\dagger + \hat{\rho}\hat{a}) \\
&= Tr(\hat{\rho}\hat{a}),
\end{aligned} \tag{2.10}$$

$$\begin{aligned}
&Tr(-2\hat{\rho}\hat{a}^\dagger\hat{a}\hat{a}^\dagger + \hat{\rho}\hat{a}\hat{a}^{\dagger 2} + \hat{\rho}\hat{a}^{\dagger 2}\hat{a}) \\
&= Tr\big(-2\hat{\rho}\hat{a}^\dagger\hat{a}\hat{a}^\dagger + \hat{\rho}(\hat{a}^\dagger\hat{a} + 1)\hat{a}^\dagger + \hat{\rho}\hat{a}^\dagger(\hat{a}\hat{a}^\dagger - 1)\big) \\
&= Tr(-2\hat{\rho}\hat{a}^\dagger\hat{a}\hat{a}^\dagger + \hat{\rho}\hat{a}^\dagger\hat{a}\hat{a}^\dagger + \hat{\rho}\hat{a}^\dagger + \hat{\rho}\hat{a}^\dagger\hat{a}\hat{a}^\dagger - \hat{\rho}\hat{a}^\dagger) \\
&= 0.
\end{aligned} \tag{2.11}$$

Substituting Eqs. (2.8) - (2.11) in to Eq. (2.6), we get

$$\frac{d}{dt}\langle\hat{a}(t)\rangle = -\frac{\kappa}{2}\langle\hat{a}(t)\rangle - \varepsilon\langle\hat{a}^\dagger(t)\rangle. \tag{2.12}$$

In a similar manner one readily obtains

$$\frac{d}{dt}\langle\hat{a}^\dagger(t)\rangle = -\frac{\kappa}{2}\langle\hat{a}^\dagger(t)\rangle - \varepsilon\langle\hat{a}(t)\rangle. \tag{2.13}$$

Moreover, with the aid of Eqs. (2.2) and (2.3), we have

$$\begin{aligned}\frac{d}{dt}\langle\hat{a}^\dagger(t)\hat{a}(t)\rangle =&\frac{\varepsilon}{2}Tr\big(\hat{\rho}\hat{a}^{\dagger 3}\hat{a}-\hat{\rho}\hat{a}^2\hat{a}^\dagger\hat{a}+\hat{a}^2\hat{\rho}\hat{a}^\dagger\hat{a}-\hat{a}^{\dagger 2}\hat{\rho}\hat{a}^\dagger\hat{a}\big)\\ &+\frac{\kappa(N+1)}{2}Tr(2\hat{a}\hat{\rho}\hat{a}^{\dagger 2}\hat{a}-\hat{a}^\dagger\hat{a}\hat{\rho}\hat{a}^\dagger\hat{a}-\hat{\rho}\hat{a}^\dagger\hat{a}\hat{a}^\dagger\hat{a})\\ &+\frac{\kappa N}{2}Tr(2\hat{a}^\dagger\hat{\rho}\hat{a}\hat{a}^\dagger\hat{a}-\hat{a}\hat{a}^\dagger\hat{\rho}\hat{a}^\dagger\hat{a}-\hat{\rho}\hat{a}\hat{a}^{\dagger 2}\hat{a})\\ &-\frac{\kappa M}{2}Tr(\hat{a}^2\hat{\rho}\hat{a}^\dagger\hat{a}-2\hat{a}\hat{\rho}\hat{a}\hat{a}^\dagger\hat{a}+\hat{\rho}\hat{a}^2\hat{a}^\dagger\hat{a}-2\hat{a}^\dagger\hat{\rho}\hat{a}^{\dagger 2}\hat{a}\\ &+\hat{a}^{\dagger 2}\hat{\rho}\hat{a}^\dagger\hat{a}+\hat{\rho}\hat{a}^{\dagger 3}\hat{a}),\end{aligned}\tag{2.14}$$

again applying cyclic property of trace operation, we get

$$\begin{aligned}Tr(\frac{d\hat{\rho}}{dt}\hat{a}^\dagger\hat{a}) =&\frac{\varepsilon}{2}Tr(\hat{\rho}\hat{a}^{\dagger 3}\hat{a}-\hat{\rho}\hat{a}^2\hat{a}^\dagger\hat{a}+\hat{\rho}\hat{a}^\dagger\hat{a}^3-\hat{\rho}\hat{a}^\dagger\hat{a}\hat{a}^{\dagger 2})\\ &+\frac{\kappa(N+1)}{2}Tr(2\hat{\rho}\hat{a}^{\dagger 2}\hat{a}^2-\hat{\rho}\hat{a}^\dagger\hat{a}\hat{a}^\dagger\hat{a}-\hat{\rho}\hat{a}^\dagger\hat{a}\hat{a}^\dagger\hat{a})\\ &+\frac{\kappa N}{2}Tr(2\hat{\rho}\hat{a}\hat{a}^\dagger\hat{a}\hat{a}^\dagger-\hat{\rho}\hat{a}^\dagger\hat{a}^2\hat{a}^\dagger-\hat{\rho}\hat{a}\hat{a}^{\dagger 2}\hat{a})\\ &-\frac{\kappa M}{2}Tr(\hat{\rho}\hat{a}^\dagger\hat{a}^3-2\hat{\rho}\hat{a}\hat{a}^\dagger\hat{a}^2+\hat{\rho}\hat{a}^2\hat{a}^\dagger\hat{a}-2\hat{\rho}\hat{a}^{\dagger 2}\hat{a}\hat{a}^\dagger\\ &+\hat{\rho}\hat{a}^\dagger\hat{a}\hat{a}^{\dagger 2}+\hat{\rho}\hat{a}^{\dagger 3}\hat{a}),\end{aligned}\tag{2.15}$$

up on use of Eq. (2.7), the traces in Eq. (2.15) takes the form

$$\begin{aligned}&Tr\big(\hat{\rho}\hat{a}^{\dagger 3}\hat{a}-\hat{\rho}\hat{a}^2\hat{a}^\dagger\hat{a}+\hat{\rho}\hat{a}^\dagger\hat{a}^3-\hat{\rho}\hat{a}^\dagger\hat{a}\hat{a}^{\dagger 2}\big)\\ &=Tr(\hat{\rho}\hat{a}^{\dagger 2}\hat{a}^\dagger\hat{a}-\hat{\rho}\hat{a}^2\hat{a}^\dagger\hat{a}+\hat{\rho}\hat{a}^\dagger\hat{a}\hat{a}^2-\hat{\rho}\hat{a}^\dagger\hat{a}\hat{a}^\dagger\hat{a}^\dagger)\\ &=Tr\big(\hat{\rho}\hat{a}^{\dagger 2}(\hat{a}\hat{a}^\dagger-1)-\hat{\rho}\hat{a}^2(\hat{a}\hat{a}^\dagger-1)+\hat{\rho}(\hat{a}\hat{a}^\dagger-1)\hat{a}^2-\hat{\rho}\hat{a}^\dagger(\hat{a}^\dagger\hat{a}+1)\hat{a}^\dagger\big)\\ &=Tr(\hat{\rho}\hat{a}^{\dagger 2}\hat{a}\hat{a}^\dagger-\hat{\rho}\hat{a}^{\dagger 2}-\hat{\rho}\hat{a}^3\hat{a}^\dagger+\hat{\rho}\hat{a}^2+\hat{\rho}\hat{a}\hat{a}^\dagger\hat{a}^2-\hat{\rho}\hat{a}^2-\hat{\rho}\hat{a}^{\dagger 2}\hat{a}\hat{a}^\dagger-\hat{\rho}\hat{a}^{\dagger 2}\\ &=Tr\big(-2\hat{\rho}\hat{a}^{\dagger 2}-\hat{\rho}\hat{a}^2(\hat{a}^\dagger\hat{a}+1)+\hat{\rho}\hat{a}(\hat{a}\hat{a}^\dagger-1)\hat{a}\big)\\ &=Tr(-2\hat{\rho}\hat{a}^{\dagger 2}-\hat{\rho}\hat{a}^2\hat{a}^\dagger\hat{a}-\hat{\rho}\hat{a}^2+\hat{\rho}\hat{a}^2\hat{a}^\dagger\hat{a}-\hat{\rho}\hat{a}^2)\\ &=-2Tr(\hat{\rho}\hat{a}^{\dagger 2}+\hat{\rho}\hat{a}^2),\end{aligned}\tag{2.16}$$

$$\begin{aligned}Tr\big(2\hat{\rho}\hat{a}^{\dagger 2}\hat{a}^2-2\hat{\rho}\hat{a}^\dagger\hat{a}\hat{a}^\dagger\hat{a}\big)&=Tr\big(2\hat{\rho}\hat{a}^{\dagger 2}\hat{a}^2-2\hat{\rho}\hat{a}^\dagger(1+\hat{a}^\dagger\hat{a})\hat{a}\big)\\ &=Tr(2\hat{\rho}\hat{a}^{\dagger 2}\hat{a}^2-2\hat{\rho}\hat{a}^\dagger\hat{a}-2\hat{\rho}\hat{a}^{\dagger 2}\hat{a}^2)\\ &=-2Tr(\hat{\rho}\hat{a}^\dagger\hat{a}),\end{aligned}\tag{2.17}$$

$$
\begin{aligned}
&Tr\big(2\hat{\rho}\hat{a}\hat{a}^{\dagger}\hat{a}\hat{a}^{\dagger} - \hat{\rho}\hat{a}^{\dagger}\hat{a}^{2}\hat{a}^{\dagger} - \hat{\rho}\hat{a}\hat{a}^{\dagger 2}\hat{a}\big)\\
&= Tr\big(2\hat{\rho}\hat{a}\hat{a}^{\dagger}\hat{a}\hat{a}^{\dagger} - \hat{\rho}\hat{a}^{\dagger}\hat{a}\hat{a}\hat{a}^{\dagger} - \hat{\rho}\hat{a}\hat{a}^{\dagger 2}\hat{a}\big)\\
&= Tr\big(2\hat{\rho}\hat{a}\hat{a}^{\dagger}(\hat{a}^{\dagger}\hat{a}+1) - \hat{\rho}(\hat{a}\hat{a}^{\dagger}-1)(\hat{a}^{\dagger}\hat{a}+1) - \hat{\rho}\hat{a}\hat{a}^{\dagger 2}\hat{a}\big)\\
&= Tr(2\hat{\rho}\hat{a}\hat{a}^{\dagger^{2}}\hat{a} + 2\hat{\rho}\hat{a}\hat{a}^{\dagger} - \hat{\rho}\hat{a}\hat{a}^{\dagger 2}\hat{a} - \hat{\rho}\hat{a}\hat{a}^{\dagger} + \hat{\rho}\hat{a}^{\dagger}\hat{a} + \hat{\rho} - \hat{\rho}\hat{a}\hat{a}^{\dagger 2}\hat{a})\\
&= Tr(\hat{\rho}\hat{a}\hat{a}^{\dagger} + \hat{\rho}\hat{a}^{\dagger}\hat{a} + \hat{\rho})\\
&= Tr\big(\hat{\rho}(\hat{a}^{\dagger}\hat{a}+1) + \hat{\rho}\hat{a}^{\dagger}\hat{a} + \hat{\rho}\big)\\
&= Tr(\hat{\rho}\hat{a}^{\dagger}\hat{a} + \hat{\rho} + \hat{\rho}\hat{a}^{\dagger}\hat{a} + \hat{\rho})\\
&= 2Tr(\hat{\rho}\hat{a}^{\dagger}\hat{a} + \hat{\rho}),
\end{aligned}
\tag{2.18}
$$

$$
\begin{aligned}
&Tr\big(\hat{\rho}\hat{a}^{\dagger}\hat{a}^{3} - 2\hat{\rho}\hat{a}\hat{a}^{\dagger}\hat{a}^{2} + \hat{\rho}\hat{a}^{2}\hat{a}^{\dagger}\hat{a} - 2\hat{\rho}\hat{a}^{\dagger 2}\hat{a}\hat{a}^{\dagger} + \hat{\rho}\hat{a}^{\dagger}\hat{a}\hat{a}^{\dagger 2} + \hat{\rho}\hat{a}^{\dagger 3}\hat{a}\big)\\
&= Tr\big(\hat{\rho}(\hat{a}\hat{a}^{\dagger}-1)\hat{a}^{2} - 2\hat{\rho}\hat{a}\hat{a}^{\dagger}\hat{a}^{2} + \hat{\rho}\hat{a}(\hat{a}^{\dagger}\hat{a}+1)\hat{a} - 2\hat{\rho}\hat{a}^{\dagger 2}\hat{a}\hat{a}^{\dagger} + \hat{\rho}\hat{a}^{\dagger}(\hat{a}^{\dagger}\hat{a}+1)\hat{a}^{\dagger}\\
&+ \hat{\rho}\hat{a}^{\dagger 2}(\hat{a}\hat{a}^{\dagger}-1)\big)\\
&= Tr(\hat{\rho}\hat{a}\hat{a}^{\dagger}\hat{a}^{2} - \hat{\rho}\hat{a}^{2} - 2\hat{\rho}\hat{a}\hat{a}^{\dagger}\hat{a}^{2} + \hat{\rho}\hat{a}\hat{a}^{\dagger}\hat{a}^{2} + \hat{\rho}\hat{a}^{2} - 2\hat{\rho}\hat{a}^{\dagger 2}\hat{a}\hat{a}^{\dagger} + \hat{\rho}\hat{a}^{\dagger 2}\hat{a}\hat{a}^{\dagger} + \hat{\rho}\hat{a}^{\dagger 2}\\
&+ \hat{\rho}\hat{a}^{\dagger 2}\hat{a}\hat{a}^{\dagger} - \hat{\rho}\hat{a}^{\dagger 2})\\
&= 0.
\end{aligned}
\tag{2.19}
$$

Substituting Eqs. (2.16) - (2.19) in to Eq. (2.15), we get

$$
\frac{d}{dt}\langle\hat{a}^{\dagger}(t)\hat{a}(t)\rangle = -\kappa\langle\hat{a}^{\dagger}(t)\hat{a}(t)\rangle - \varepsilon[\langle\hat{a}^{2}(t)\rangle + \langle\hat{a}^{\dagger 2}(t)\rangle] + \kappa N. \tag{2.20}
$$

In a similar manner one readily obtains

$$
\frac{d}{dt}\langle\hat{a}^{2}(t)\rangle = -\kappa\langle\hat{a}^{2}(t)\rangle - 2\varepsilon\langle\hat{a}^{\dagger}(t)\hat{a}(t)\rangle - \varepsilon - \kappa M \tag{2.21}
$$

and

$$
\frac{d}{dt}\langle\hat{a}^{\dagger 2}(t)\rangle = -\kappa\langle\hat{a}^{\dagger 2}(t)\rangle - 2\varepsilon\langle\hat{a}^{\dagger}(t)\hat{a}(t)\rangle - \varepsilon - \kappa M. \tag{2.22}
$$

We notice that the operators in Eqs. (2.12), (2.13), (2.20), (2.21) and (2.22) are in the normal order. The corresponding c-number variable associated with the normal ordering by replacing $\hat{a}$ with α and $\hat{a}^{\dagger}$ with α^{*}, take the form

$$
\frac{d}{dt}\langle\alpha(t)\rangle = -\kappa\langle\alpha(t)\rangle - \varepsilon\langle\alpha^{*}(t)\rangle, \tag{2.23}
$$

$$\frac{d}{dt}\langle\alpha^*(t)\rangle = -\kappa\langle\alpha^*(t)\rangle - \varepsilon\langle\alpha(t)\rangle, \tag{2.24}$$

$$\frac{d}{dt}\langle\alpha^*(t)\alpha(t)\rangle = -\kappa\langle\alpha^*(t)\alpha(t)\rangle - \varepsilon[\langle\alpha^2(t)\rangle + \langle\alpha^{*2}(t)\rangle] + \kappa N, \tag{2.25}$$

$$\frac{d}{dt}\langle\alpha^2(t)\rangle = -\kappa\langle\alpha^2(t)\rangle - 2\varepsilon\langle\alpha^*(t)\alpha(t)\rangle - \varepsilon - \kappa M, \tag{2.26}$$

$$\frac{d}{dt}\langle\alpha^{*2}(t)\rangle = -\kappa\langle\alpha^{*2}(t)\rangle - 2\varepsilon\langle\alpha^*(t)\alpha(t)\rangle - \varepsilon - \kappa M. \tag{2.27}$$

On the basis of Eqs. (2.23) and (2.24), we can write

$$\frac{d}{dt}\alpha(t) = -\frac{\kappa}{2}\alpha(t) - \varepsilon\alpha^*(t) + f(t), \tag{2.28}$$

$$\frac{d}{dt}\alpha^*(t) = -\frac{\kappa}{2}\alpha^*(t) - \varepsilon\alpha(t) + f^*(t). \tag{2.29}$$

where $f(t)$ is a noise force of which the mean and correlation properties remain to be determined. It is not difficult to see that Eq. (2.23) will be equal to the expectation value of Eq. (2.28) provided that

$$\langle f(t)\rangle = 0. \tag{2.30}$$

Employing the relation

$$\frac{d}{dt}\langle\alpha^2(t)\rangle = \langle\frac{d\alpha(t)}{dt}\alpha(t)\rangle + \langle\alpha(t)\frac{d\alpha(t)}{dt}\rangle, \tag{2.31}$$

along with Eq. (2.28), we see that

$$\frac{d}{dt}\langle\alpha^2(t)\rangle = -\kappa\langle\alpha^2(t)\rangle - 2\varepsilon\langle\alpha^*(t)\alpha(t)\rangle + \langle f(t)\alpha(t)\rangle + \langle\alpha(t)f(t)\rangle. \tag{2.32}$$

Comparing Eqs. (2.32) and (2.26), we get

$$\langle f(t)\alpha(t)\rangle + \langle\alpha(t)f(t)\rangle = -(\kappa M + \varepsilon). \tag{2.33}$$

The solution of Eq. (2.28) can be written as

$$\alpha(t) = \alpha(0)e^{-\frac{\kappa t}{2}} - \varepsilon\int_0^t e^{-\frac{\kappa}{2}(t-t')}\alpha^*(t')dt' + \int_0^t e^{-\frac{\kappa}{2}(t-t')}f(t')dt' \tag{2.34}$$

Multiplying Eq. (2.34) on the left by $f(t)$ and taking the expectation value of the resulting expression, we obtain

$$\begin{aligned}\langle f(t)\alpha(t)\rangle =&\langle f(t)\alpha(0)\rangle e^{-\frac{\kappa t}{2}} - \varepsilon\int_0^t e^{-\frac{\kappa}{2}(t-t')}\langle f(t)\alpha^*(t')\rangle dt' \\ &+ \int_0^t e^{-\frac{\kappa}{2}(t-t')}\langle f(t)f(t')\rangle dt'. \end{aligned} \tag{2.35}$$

Since the noise operator at some time should not affect the system variable at earlier times, we can write

$$\langle f(t)\alpha(0)\rangle = \langle f(t)\rangle\langle\alpha(0)\rangle = 0, \tag{2.36}$$

$$\langle f(t)\alpha^*(t')\rangle = \langle f(t)\rangle\langle\alpha^*(t')\rangle = 0. \tag{2.37}$$

Taking in to account Eqs. (2.36) and (2.37), we can reduce Eq. (2.35) to

$$\langle f(t)\alpha(t)\rangle = \int_0^t e^{-\frac{\kappa}{2}(t-t')}\langle f(t)f(t')\rangle dt'. \tag{2.38}$$

Multiplying Eq. (2.34) on the right by $f(t)$ and taking the expectation value of the resulting expression, we get

$$\begin{aligned}\langle \alpha(t)f(t)\rangle =&\langle \alpha(0)f(t)\rangle e^{-\frac{\kappa t}{2}} - \varepsilon\int_0^t e^{-\frac{\kappa}{2}(t-t')}\langle \alpha^*(t')f(t)\rangle dt' \\ &+ \int_0^t e^{-\frac{\kappa}{2}(t-t')}\langle f(t')f(t)\rangle dt'. \end{aligned} \tag{2.39}$$

Because the noise operator at some time has no effect on the system variable at earlier times, we can write

$$\langle \alpha(0)f(t)\rangle = \langle\alpha(t)\rangle\langle f(t)\rangle = 0, \tag{2.40}$$

$$\langle \alpha^*(t')\langle f(t)\rangle = \langle\alpha^*(t')\rangle\langle f(t)\rangle = 0. \tag{2.41}$$

Taking into account Eqs. (2.40) and (2.41), we can put Eq. (2.39) as

$$\langle \alpha(t)f(t)\rangle = \int_0^t e^{-\frac{\kappa}{2}(t-t')}\langle f(t')f(t)\rangle dt'. \tag{2.42}$$

Employing Eqs. (2.38) and (2.42) in to Eq. (2.33), we get

$$\int_0^t e^{-\frac{\kappa}{2}(t-t')}\langle f(t)f(t')\rangle dt' + \int_0^t e^{-\frac{\kappa}{2}(t-t')}\langle f(t')f(t)\rangle dt' = -(\kappa M + \varepsilon), \tag{2.43}$$

which on assuming

$$\langle f(t')f(t)\rangle = \langle f(t)f(t')\rangle, \tag{2.44}$$

reduces to

$$\int_0^t e^{-\frac{\kappa}{2}(t-t')}\langle f(t)f(t')\rangle dt' = \frac{-(\kappa M+\varepsilon)}{2}. \tag{2.45}$$

Using the relations [1, 2]

$$\int_0^t e^{-S(t-t')}\langle f(t)g(t')\rangle dt' = D, \tag{2.46}$$

we assert that

$$\langle f(t)g(t')\rangle = 2D\delta(t-t'). \tag{2.47}$$

where S and D are constants or D may be as a function of time.

Based on Eqs. (2.46) and (2.47) we can write Eq. (2.45) as

$$\langle f(t)f(t')\rangle = -(\kappa M+\varepsilon)\delta(t-t'). \tag{2.48}$$

Employing Eq. (2.29) in the relation

$$\frac{d}{dt}\langle \alpha^*(t)\alpha^*(t)\rangle = \Big\langle \frac{d\alpha^*(t)}{dt}\alpha^*(t)\Big\rangle + \Big\langle \alpha^*(t)\frac{d\alpha^*(t)}{dt}\Big\rangle, \tag{2.49}$$

we obtain

$$\frac{d}{dt}\langle \alpha^{*2}(t)\rangle = -\kappa\langle \alpha^{*2}(t)\rangle - 2\varepsilon\langle \alpha^*(t)\alpha(t)\rangle + \langle f^*(t)\alpha^*(t)\rangle + \langle \alpha^*(t)f^*(t)\rangle. \tag{2.50}$$

Eq. (2.50) and (2.27) will have the same form if

$$\langle f^*(t)\alpha^*(t)\rangle + \langle \alpha^*(t)f^*(t)\rangle = -(\kappa M+\varepsilon). \tag{2.51}$$

The formal solution of Eq. (2.29) can be written as

$$\alpha^*(t) = \alpha^*(0)e^{-\frac{\kappa t}{2}} - \varepsilon\int_0^t e^{-\frac{\kappa}{2}(t-t')}\alpha(t')dt' + \int_0^t e^{-\frac{\kappa}{2}(t-t')}f^*(t')dt'. \tag{2.52}$$

Multiplying Eq. (2.52) on the left by $f^*(t)$ and taking the expectation value of the resulting expression, we get

$$\begin{aligned}\langle f^*(t)\alpha^*(t)\rangle =& \langle f^*(t)\alpha^*(0)\rangle e^{-\frac{\kappa t}{2}} - \varepsilon\int_0^t e^{-\frac{\kappa}{2}(t-t')}\langle f^*(t)\alpha(t')\rangle dt' \\ &+ \int_0^t e^{-\frac{\kappa}{2}(t-t')}\langle f^*(t)f^*(t')\rangle dt'.\end{aligned} \tag{2.53}$$

Because the noise force at some time should not affect the system variable at earlier times, we can write

$$\langle f^*(t)\alpha^*(0)\rangle = \langle f^*(t)\rangle\langle\alpha^*(0)\rangle = 0, \tag{2.54}$$

$$\langle f^*(t)\alpha(t')\rangle = \langle f^*(t)\rangle\langle\alpha(t')\rangle = 0. \tag{2.55}$$

Employing Eqs. (2.54) and (2.55), we can rewrite Eq. (2.53) as

$$\langle f^*(t)\alpha^*(t)\rangle = \int_0^t e^{-\frac{\kappa}{2}(t-t')}\langle f^*(t)f^*(t')\rangle dt' \tag{2.56}$$

Multiplying Eq. (2.52) on the right by $f^*(t)$ and taking the expectation value of the resulting expression, one obtains

$$\begin{aligned}\langle\alpha^*(t)f^*(t)\rangle =& \langle\alpha^*(0)f^*(t)\rangle e^{-\frac{\kappa t}{2}} - \varepsilon\int_0^t e^{-\frac{\kappa}{2}(t-t')}\langle\alpha(t')f^*(t)\rangle dt' \\ &+ \int_0^t e^{-\frac{\kappa}{2}(t-t')}\langle f^*(t')f^*(t)\rangle dt'\end{aligned} \tag{2.57}$$

As the noise force at any time has no effect on the system variable at earlier times, we note that

$$\langle\alpha^*(0)f^*(t)\rangle = \langle\alpha^*(0)\rangle\langle f^*(t)\rangle = 0, \tag{2.58}$$

$$\langle\alpha(t')f^*(t)\rangle = \langle\alpha(t')\rangle\langle f^*(t)\rangle = 0. \tag{2.59}$$

On the basis of Eqs. (2.58) and (2.59), we can reduce Eq. (2.57) to

$$\langle\alpha^*(t)f^*(t)\rangle = \int_0^t e^{-\frac{\kappa}{2}(t-t')}\langle f^*(t')f^*(t)\rangle dt'. \tag{2.60}$$

Substituting Eqs. (2.56) and (2.60) in to (2.51) leads to

$$\int_0^t e^{-\frac{\kappa}{2}(t-t')}\langle f^*(t)f^*(t')\rangle dt' + \int_0^t e^{-\frac{\kappa}{2}(t-t')}\langle f^*(t')f^*(t)\rangle dt' = -(\kappa M+\varepsilon), \tag{2.61}$$

and assuming

$$\langle f^*(t)f^*(t')\rangle = \langle f^*(t')f^*(t)\rangle, \tag{2.62}$$

we obtain

$$\int_0^t e^{-\frac{\kappa}{2}(t-t')}\langle f^*(t')f^*(t)\rangle dt' = \frac{-(\kappa M+\varepsilon)}{2}, \tag{2.63}$$

which in view of Eqs. (2.46) and (2.47) leads to

$$\langle f^*(t')f^*(t)\rangle = -(\kappa M + \varepsilon)\delta(t - t'). \tag{2.64}$$

Furthermore, using the relation

$$\frac{d}{dt}\langle \alpha^*(t)\alpha(t)\rangle = \Big\langle \frac{d\alpha^*(t)}{dt}\alpha(t)\Big\rangle + \Big\langle \alpha^*(t)\frac{d\alpha(t)}{dt}\Big\rangle \tag{2.65}$$

along with Eqs. (2.28) and (2.29), we get

$$\begin{aligned}\frac{d}{dt}\langle \alpha^*(t)\alpha(t)\rangle =& -\kappa\langle \alpha^*(t)\alpha(t)\rangle - \varepsilon(\langle \alpha^{*2}(t)\rangle + \langle \alpha^2(t)\rangle)\\ &+ \langle f^*(t)\alpha(t)\rangle + \langle \alpha^*(t)f(t)\rangle.\end{aligned} \tag{2.66}$$

Up on comparing Eqs. (2.25) and (2.66), we have

$$\langle f^*(t)\alpha(t)\rangle + \langle \alpha^*(t)f(t)\rangle = \kappa N. \tag{2.67}$$

Multiplying Eq. (2.34) on the left by $f^*(t)$ and taking the expectation value of the resulting expression, one can obtain

$$\begin{aligned}\langle f^*(t)\alpha(t)\rangle =&\langle f^*(t)\alpha(0)\rangle e^{-\frac{\kappa t}{2}} - \varepsilon\int_0^t e^{-\frac{\kappa}{2}(t-t')}\langle f^*(t)\alpha^*(t')\rangle dt'\\ &+ \int_0^t e^{-\frac{\kappa}{2}(t-t')}\langle f^*(t)f(t')\rangle.\end{aligned} \tag{2.68}$$

On account of Eq. (2.30) and with the fact that at earlier times the noise force does not affect the systems variable, one can write

$$\langle f^*(t)\alpha(0)\rangle = \langle f^*(t)\rangle\langle \alpha(0)\rangle = 0, \tag{2.69}$$

$$\langle f^*(t)\alpha^*(t')\rangle = \langle f^*(t)\rangle\langle \alpha^*(t')\rangle = 0. \tag{2.70}$$

In view of Eqs. (2.69) and (2.70), we put Eq. (2.68) as

$$\langle f^*(t)\alpha(t)\rangle = \int_0^t e^{-\frac{\kappa}{2}(t-t')}\langle f^*(t)f(t')\rangle dt'. \tag{2.71}$$

Multiplying Eq. (2.52) on the right by $f(t)$ and taking the expectation value of the resulting expression, one can obtain the following.

$$\begin{aligned}\langle \alpha^*(t)f(t)\rangle =&\langle \alpha^*(0)f(t)\rangle e^{-\frac{\kappa t}{2}} - \varepsilon\int_0^t e^{-\frac{\kappa}{2}(t-t')}\langle \alpha(t')f(t)\rangle dt'\\ &+ \int_0^t e^{-\frac{\kappa}{2}(t-t')}\langle f^*(t')f(t)\rangle dt'.\end{aligned} \tag{2.72}$$

Taking in account of Eq. (2.30) and with the fact that at earlier times the noise force does not affect the systems variable, one can write

$$\langle \alpha^*(0) f(t) \rangle = \langle \alpha^*(0) \rangle \langle f(t) \rangle = 0, \tag{2.73}$$

$$\langle \alpha(t') f(t) \rangle = \langle \alpha(t') \rangle \langle f(t) \rangle = 0. \tag{2.74}$$

In view of Eq. (2.73) and (2.74), we can put (2.72) as

$$\langle \alpha^*(t) f(t) \rangle = \int_0^t e^{-\frac{\kappa}{2}(t-t')} \langle f^*(t') f(t) \rangle dt'. \tag{2.75}$$

Substituting Eqs. (2.71) and (2.75) in to (2.67), we get

$$\int_0^t e^{-\frac{\kappa}{2}(t-t')} \langle f^*(t) f(t') dt' \rangle + \int_0^t e^{-\frac{\kappa}{2}(t-t')} \langle f^*(t') f(t) \rangle dt' = \kappa N, \tag{2.76}$$

which on assuming

$$\langle f^*(t) f(t') \rangle = \langle f^*(t') f(t) \rangle, \tag{2.77}$$

reduces to

$$\int_0^t e^{-\frac{\kappa}{2}(t-t')} \langle f^*(t) f(t') \rangle dt' = \frac{\kappa N}{2}. \tag{2.78}$$

In view of Eqs. (2.46) and (2.47), we get

$$\langle f^*(t) f(t') \rangle = \kappa N \delta(t - t'). \tag{2.79}$$

From the above, Eqs. (2.30), (2.48), (2.64) and (2.79) represent the mean and correlation properties of the noise force.

We next proceed to obtain the solution of Eq. (2.28) by taking the sum of Eqs. (2.28) and (2.29) as follow.

Adding Eqs. (2.28) and (2.29), we get

$$\frac{d}{dt}\big(\alpha^*(t) + \alpha(t)\big) = -\frac{\kappa}{2}\big(\alpha^*(t) + \alpha(t)\big) - \varepsilon\big(\alpha^*(t) + \alpha(t)\big) + f^*(t) + f(t), \tag{2.80}$$

$$\begin{aligned} \frac{d}{dt}\alpha_+(t) &= -\frac{\kappa}{2}\alpha_+(t) - \varepsilon\alpha_+(t) + F_+(t) \\ &= -\frac{1}{2}(\kappa + 2\varepsilon)\alpha_+(t) + F_+(t) \\ &= -\frac{1}{2}\lambda_+\alpha_+(t) + F_+(t), \end{aligned} \tag{2.81}$$

where

$$\alpha_+(t) = \alpha^*(t) + \alpha(t), \tag{2.82}$$

$$\lambda_+ = \kappa + 2\varepsilon, \tag{2.83}$$

$$F_+(t) = f^*(t) + f(t). \tag{2.84}$$

Subtracting Eq. (2.29) from (2.28) gives

$$\frac{d}{dt}\big(\alpha^*(t) - \alpha(t)\big) = -\frac{\kappa}{2}\big(\alpha^*(t) - \alpha(t)\big) + \varepsilon\big(\alpha^*(t) - \alpha(t)\big) + f^*(t) - f(t), \tag{2.85}$$

$$\begin{aligned} \frac{d}{dt}\alpha_-(t) &= -\frac{\kappa}{2}\alpha_-(t) + \varepsilon\alpha_-(t) + F_-(t) \\ &= -\frac{1}{2}(\kappa - 2\varepsilon)\alpha_-(t) + F_-(t) \\ &= -\frac{1}{2}\lambda_-\alpha_-(t) + F_-(t), \end{aligned} \tag{2.86}$$

where

$$\alpha_-(t) = \alpha^*(t) - \alpha(t), \tag{2.87}$$

$$\lambda_- = \kappa - 2\varepsilon, \tag{2.88}$$

$$F_-(t) = f^*(t) - f(t). \tag{2.89}$$

We note from Eqs. (2.81) and (2.86) that

$$\frac{d}{dt}\alpha_\pm(t) = -\frac{1}{2}\lambda_\pm\alpha_\pm(t) + F_\pm(t), \tag{2.90}$$

where

$$\lambda_\pm = \kappa \pm 2\varepsilon, \tag{2.91}$$

$$\alpha_\pm(t) = \alpha^*(t) \pm \alpha(t), \tag{2.92}$$

$$F_{\pm}(t) = f^*(t) \pm f(t). \tag{2.93}$$

Noting that Eq. (2.86) has well-behaved solution for $\kappa - 2\varepsilon > 0$ we take $\kappa = 2\varepsilon$ as threshold. The formal solution of Eqs. (2.81) and (2.86) for $\kappa > 2\varepsilon$ can be written as

$$\alpha_+(t) = \alpha_+(0)e^{\frac{-\lambda_+ t}{2}} + \int_0^t e^{\frac{-\lambda_+(t-t')}{2}} F_+(t')dt', \tag{2.94}$$

$$\alpha_-(t) = \alpha_-(0)e^{\frac{-\lambda_- t}{2}} + \int_0^t e^{\frac{-\lambda_-(t-t')}{2}} F_-(t')dt'. \tag{2.95}$$

In view of Eqs. (2.82) and (2.87), Eqs. (2.94) and (2.95) take the form

$$\alpha^*(t) + \alpha(t) = \big(\alpha^*(0) + \alpha(0)\big)e^{\frac{-\lambda_+ t}{2}} + \int_0^t e^{\frac{-\lambda_+(t-t')}{2}} F_+(t')dt', \tag{2.96}$$

$$\alpha^*(t) - \alpha(t) = \big(\alpha^*(0) - \alpha(0)\big)e^{\frac{-\lambda_- t}{2}} + \int_0^t e^{\frac{-\lambda_-(t-t')}{2}} F_-(t')dt'. \tag{2.97}$$

Adding Eqs. (2.96) and (2.97), we have

$$\begin{aligned}\alpha^*(t) = &\frac{1}{2}\alpha^*(0)\big(e^{\frac{-\lambda_+ t}{2}} + e^{\frac{-\lambda_- t}{2}}\big) + \frac{1}{2}\alpha(0)\big(e^{\frac{-\lambda_+ t}{2}} - e^{\frac{-\lambda_- t}{2}}\big)\\ &+ \frac{1}{2}\Big(\int_0^t e^{\frac{-\lambda_+(t-t')}{2}} F_+(t')dt' + \int_0^t e^{\frac{-\lambda_-(t-t')}{2}} F_-(t')dt'\Big),\end{aligned} \tag{2.98}$$

after subtracting Eqs. (2.97) from (2.96), we get

$$\begin{aligned}\alpha(t) = &\frac{1}{2}\alpha^*(0)\big(e^{\frac{-\lambda_+ t}{2}} - e^{\frac{-\lambda_- t}{2}}\big) + \frac{1}{2}\alpha(0)\big(e^{\frac{-\lambda_+ t}{2}} + e^{\frac{-\lambda_- t}{2}}\big)\\ &+ \frac{1}{2}\Big(\int_0^t e^{\frac{-\lambda_+(t-t')}{2}} F_+(t')dt' - \int_0^t e^{\frac{-\lambda_-(t-t')}{2}} F_-(t')dt'\Big).\end{aligned} \tag{2.99}$$

or

$$\begin{aligned}\alpha(t+\tau) = &\frac{1}{2}\big[\alpha^*(t)(e^{\frac{-\lambda_+ \tau}{2}} - e^{\frac{-\lambda_- \tau}{2}}) + \frac{1}{2}\alpha(t)(e^{\frac{-\lambda_+ \tau}{2}} + e^{\frac{-\lambda_- \tau}{2}})\big]\\ &+ \frac{1}{2}\big[e^{\frac{-\lambda_+ \tau}{2}} \int_0^\tau e^{\frac{\lambda_+ \tau'}{2}} F_+(t+\tau')d\tau'\\ &- e^{\frac{-\lambda_- \tau}{2}} \int_0^\tau e^{\frac{-\lambda_- \tau'}{2}} F_-(t+\tau')d\tau'\big].\end{aligned} \tag{2.100}$$

One can rewrite Eqs. (2.98), (2.99) and (2.100) as

$$\alpha^*(t) = A_+(t)\alpha^*(0) + A_-(t)\alpha(0) + B_+(t) + B_-(t), \tag{2.101}$$

$$\alpha(t) = A_+(t)\alpha(0) + A_-(t)\alpha^*(0) + B_+(t) - B_-(t), \tag{2.102}$$

$$\alpha(t+\tau) = \alpha^*(t)A_-(\tau) + \alpha(t)A_+(\tau) + B_+(t+\tau) - B_-(t+\tau). \tag{2.103}$$

in which

$$A_\pm(t) = \frac{1}{2}\big[e^{\frac{-\lambda_+ t}{2}} \pm e^{\frac{-\lambda_- t}{2}}\big], \tag{2.104}$$

$$B_\pm(t) = \frac{1}{2}\int_0^t e^{\frac{-\lambda_\pm(t-t')}{2}} F_\pm(t')dt', \tag{2.105}$$

$$B_\pm(t+\tau) = \frac{1}{2}\int_0^\tau e^{\frac{-\lambda_\pm \tau'}{2}} F_\pm(t+\tau')d\tau'. \tag{2.106}$$

It perhaps worth mentioning that Eqs. (2.101) - (2.106) are applied in calculating various quantities of interest. We also realize that these solutions would be well-behaved functions at steady state, if $\kappa > 2\varepsilon$. Hence we designate $\kappa = 2\varepsilon$ at threshold.
Now assuming the cavity mode to be initially in the vacuum state and taking into account Eq. (2.30), we note from Eq. (2.102) that

$$\langle \alpha(t) \rangle = 0 \tag{2.107}$$

2.2 The Q function

The Q function for a single-mode light is expressible as [1]

$$Q(\alpha^*, \alpha, t) = \frac{1}{\pi^2}\int d^2z \phi_a(z^*, z, t) e^{\alpha z^* - z\alpha^*}, \tag{2.108}$$

where

$$\phi_a(z^*, z, t) = Tr\big(\hat{\rho}(0) e^{-z^*\hat{a}(t)} e^{z\hat{a}^\dagger(t)}\big), \tag{2.109}$$

is anti-normally ordered characteristic function in the Heisenberg picture.
Employing the identity [1, 2]

$$e^{\hat{A}}e^{\hat{B}} = e^{\hat{B}}e^{\hat{A}}e^{[\hat{A},\hat{B}]}, \tag{2.110}$$

we can write Eq. (2.109) as

$$\phi_a(z^*, z, t) = e^{-z^* z} Tr(\hat{\rho}(0) e^{z\hat{a}^\dagger(t)} e^{-z^*\hat{a}(t)})$$
$$= e^{-z^* z} \langle e^{z\hat{a}^\dagger(t)} e^{-z^*\hat{a}(t)} \rangle. \tag{2.111}$$

The c-number function corresponding to Eq. (2.111) can be written as

$$\phi_a(z^*, z, t) = e^{-z^* z} \langle e^{z\alpha^*(t)} e^{-z^*\alpha(t)} \rangle. \tag{2.112}$$

On the basis of Eq. (2.107) and (2.23), we see that $\alpha(t)$ is a Gaussian variable with a vanishing mean. We can thus write [2]

$$\langle e^{z\alpha^*(t)} e^{-z^*\alpha(t)} \rangle = exp\frac{1}{2}[\langle (z\alpha^*(t) - z^*\alpha(t))^2 \rangle]$$
$$= exp\frac{1}{2}(z^2\langle \alpha^{*2}(t)\rangle - 2z^*z\langle \alpha^*(t)\alpha(t)\rangle + z^{*2}\langle \alpha^2(t)\rangle). \tag{2.113}$$

In view of Eq. (2.114), we can put (2.113) as

$$\phi_a(z^*, z, t) = e^{-z^* z} exp\frac{1}{2}(z^2\langle \alpha^{*2}(t)\rangle - 2z^*z\langle \alpha^*(t)\alpha(t)\rangle + z^{*2}\langle \alpha^2(t)\rangle). \tag{2.114}$$

We now wish to evaluate the expectation values in Eq. (2.114) employing the solution of c-number Langevin equations. To this end, making use of Eqs. (2.101) and (2.102), we see that

$$\langle \alpha^{*2}(t)\rangle = \langle (\alpha^*(0)A_+(t) + \alpha(0)A_-(t) + B_+(t) + B_-(t))^2 \rangle$$
$$= \langle \alpha^{*2}(0)\rangle A_+^2(t) + \langle \alpha^*(0)\alpha(0)\rangle A_+(t)A_-(t) + \langle \alpha^*(0)B_+(t)\rangle A_+(t)$$
$$+ \langle \alpha^*(0)B_-(t)\rangle A_+(t) + \langle \alpha(0)\alpha^*(0)\rangle A_-(t)A_+(t) + \langle \alpha^2(0)\rangle A_-^2(t)$$
$$+ \langle \alpha(0)B_+(t)\rangle A_-(t) + \langle \alpha(0)B_-(t)\rangle A_-(t) + \langle \alpha^*(0)B_+(t)\rangle A_+(t)$$
$$+ \langle \alpha(0)B_+(t)\rangle A_-(t) + \langle B_+^2(t)\rangle + \langle B_+(t)B_-(t)\rangle + \langle \alpha^*(0)B_-(t)\rangle A_+(t)$$
$$+ \langle \alpha(0)B_-(t)\rangle A_-(t) + \langle B_-(t)B_+(t)\rangle + \langle B_-^2(t)\rangle. \tag{2.115}$$

For a system initially assumed to be in a vacuum state, we see that

$$\langle \alpha^*(0)\rangle = \langle \alpha^*(0)\alpha(0)\rangle = \langle \alpha^2(0)\rangle = \langle \alpha(0)\alpha^*(0)\rangle = 0. \tag{2.116}$$

Moreover, since the noise force at some time does not affect the systems variable at an earlier time, we can write

$$\langle\alpha^*(0)B_+(t)\rangle = \langle\alpha^*(0)\rangle\langle B_+(t)\rangle = 0, \langle\alpha^*(0)B_-(t)\rangle = \langle\alpha^*(0)\rangle\langle B_-(t)\rangle = 0 \tag{2.117}$$

and

$$\langle\alpha(0)B_+(t)\rangle = \langle\alpha(0)\rangle\langle B_+(t)\rangle = 0, \langle\alpha(0)B_-(t)\rangle = \langle\alpha(0)\rangle\langle B_-(t)\rangle = 0. \tag{2.118}$$

Employing Eqs. (2.116) - (2.118), we can write Eq. (2.115) as

$$\langle\alpha^{*2}(t)\rangle = \langle B_+^2(t)\rangle + \langle B_-^2(t)\rangle + \langle B_+(t)B_-(t)\rangle + \langle B_-(t)B_+(t)\rangle. \tag{2.119}$$

Furthermore, taking into account Eqs. (2.101) and (2.102), we can write

$$\begin{aligned}\langle\alpha^*(t)\alpha(t)\rangle =& \langle\alpha^*(0)\alpha(0)\rangle A_+^2(t) + \langle\alpha^{*2}(0)\rangle A_+(t)A_-(t) + \langle\alpha^*(0)B_+(t)\rangle A_+(t)\\ &- \langle\alpha^*(0)B_-(t)\rangle A_+(t) + \langle\alpha^2(0)\rangle A_-(t)A_+(t) + \langle\alpha(0)\alpha^*(0)\rangle A_-^2(t)\\ &+ \langle\alpha(0)B_+(t)\rangle A_-(t) - \langle\alpha(0)B_-(t)\rangle A_-(t) + \langle\alpha(0)B_+(t)\rangle A_+(t)\\ &+ \langle\alpha^*(0)B_+(t)\rangle A_-(t) + \langle B_+^2(t)\rangle - \langle B_+(t)B_-(t)\rangle + \langle\alpha(0)B_-(t)\rangle A_+(t)\\ &+ \langle\alpha^*(0)B_-(t)\rangle A_-(t) + \langle B_-(t)B_+(t)\rangle - \langle B_-^2(t)\rangle.\end{aligned} \tag{2.120}$$

With the aid of Eqs. (2.116) - (2.118), Eq. (2.120) reduces to

$$\langle\alpha^*(t)\alpha(t)\rangle = \langle B_+^2(t)\rangle - \langle B_-^2(t)\rangle + \langle B_-(t)B_+(t)\rangle - \langle B_+(t)B_-(t)\rangle. \tag{2.121}$$

With the same procedure, we get

$$\langle\alpha^2(t)\rangle = \langle B_+^2(t)\rangle + \langle B_-^2(t)\rangle - \langle B_+(t)B_-(t)\rangle - \langle B_-(t)B_+(t)\rangle. \tag{2.122}$$

The expectation values in Eqs. (2.119), (2.121) and (2.122) can be evaluated using Eqs. (2.105) along with Eqs. (2.84) and (2.89). We thus have

$$\begin{aligned}\langle B_+^2(t)\rangle &= \frac{1}{4}e^{-\lambda_+ t}\int_0^t\int_0^t e^{\frac{\lambda_+(t'+t'')}{2}}\langle\big(f^*(t'')+f(t'')\big)\big(f^*(t')+f(t')\big)\rangle dt'dt''\\ &= \frac{1}{4}e^{-\lambda_+ t}\int_0^t\int_0^t e^{\frac{\lambda_+(t'+t'')}{2}}[\langle f^*(t'')f^*(t')\rangle + \langle f^*(t'')f(t')\rangle\\ &+ \langle f(t'')f^*(t')\rangle + \langle f(t'')f(t')\rangle].\end{aligned} \tag{2.123}$$

On account of Eqs. (2.48), (2.64) and (2.79), we get

$$\langle B_+^2(t)\rangle = \frac{1}{4}e^{-\lambda_+ t}\int_0^t\int_0^t e^{\frac{\lambda_+(t'+t'')}{2}}[2(\kappa N-\kappa M-\varepsilon)\delta(t'-t'')dt'dt'']$$

$$= \frac{(\kappa M+\kappa N-\varepsilon)}{2}e^{-\lambda_+ t}\int_0^t e^{\frac{\lambda_+ t'}{2}}[\int_0^t e^{\frac{\lambda_+ t''}{2}}\delta(t'-t'')dt'']dt'. \quad (2.124)$$

Carrying out the integration over t'' and employing the relation [1, 2]

$$\int f(x)\delta(x-a)dx = f(a), \quad (2.125)$$

we obtain

$$\langle B_+^2(t)\rangle = \frac{(\kappa N-\kappa M-\varepsilon)}{2}e^{-\lambda_+ t}\int_0^t e^{\lambda_+ t'}dt'$$

$$= \frac{(\kappa N-\kappa M-\varepsilon)}{2}e^{-\lambda_+ t}\frac{e^{\lambda_+ t'}}{\lambda_+}|_0^t$$

$$= \frac{(\kappa N-\kappa M-\varepsilon)}{2\lambda_+}e^{-\lambda_+ t}(e^{\lambda_+ t}-1)$$

$$= \frac{(\kappa N-\kappa M-\varepsilon)}{2\lambda_+}(1-e^{-\lambda_+ t}). \quad (2.126)$$

At steady state, Eq. (2.126) reduces to

$$\langle B_+^2(t)\rangle_{ss} = \frac{(\kappa N-\kappa M-\varepsilon)}{2\lambda_+}. \quad (2.127)$$

With the same procedure, we get

$$\langle B_-^2(t)\rangle_{ss} = \frac{-(\kappa M+\kappa N+\varepsilon)}{2\lambda_-}. \quad (2.128)$$

Moreover using Eq. (2.105), we have

$$\langle B_+(t)B_-(t)\rangle = \frac{1}{4}e^{\frac{-1}{2}(\lambda_+ t+\lambda_- t)}\int_0^t\int_0^t e^{\frac{\lambda_+ t''+\lambda_- t'}{2}}\langle f^*(t'')f^*(t')\rangle - \langle f^*(t'')f(t')\rangle$$

$$+ \langle f(t'')f^*(t')\rangle - \langle f(t'')f(t')\rangle dt''dt'. \quad (2.129)$$

Employing Eqs. (2.48), (2.64) and (2.79), we can put Eq. (2.129) as

$$\langle B_+(t)B_-(t)\rangle = \frac{1}{4}e^{\frac{-1}{2}(\lambda_+ t+\lambda_- t)}\int_0^t\int_0^t e^{\frac{\lambda_+ t+\lambda_- t'}{2}}\big[(-\kappa M-\varepsilon)-\kappa N$$

$$+ \kappa N\delta(t-t') - (-\kappa M-\varepsilon)\big)\big]\delta(t-t')dt''dt'$$

$$= 0. \quad (2.130)$$

With the same procedure

$$\langle B_-(t)B_+(t)\rangle = 0. \tag{2.131}$$

Taking in to account Eq. (2.83) and (2.88) it is possible to put Eqs. (2.127) and (2.128) at steady state in the form

$$\langle B_+^2\rangle_{ss} = \frac{(\kappa N - \kappa M - \varepsilon)}{2(\kappa + 2\varepsilon)}, \tag{2.132}$$

$$\langle B_-^2\rangle_{ss} = \frac{-(\kappa M + \kappa N + \varepsilon)}{2(\kappa - 2\varepsilon)}. \tag{2.133}$$

In view of Eq. (2.130) - (2.133), we can put Eqs. (2.119), (2.121) and (2.122) in the form

$$\langle \alpha^{*2}\rangle = \frac{-(\kappa^2 M + 2\varepsilon\kappa N + \varepsilon\kappa)}{\kappa^2 - 4\varepsilon^2}, \tag{2.134}$$

$$\langle \alpha^*\alpha\rangle = \frac{(2\varepsilon^2 + \kappa^2 N + 2\varepsilon\kappa M)}{\kappa^2 - 4\varepsilon^2}, \tag{2.135}$$

$$\langle \alpha^2\rangle = \frac{-(\kappa^2 M + 2\varepsilon\kappa N + \varepsilon\kappa)}{\kappa^2 - 4\varepsilon^2}. \tag{2.136}$$

In view of Eqs. (2.134) - (2.136), we can express the anti-normally ordered characteristic function specified by Eq. (2.114) as

$$\begin{aligned}
\phi_a(z^*, z, t) &= e^{-z^*z} e^{\frac{1}{2}\left[\frac{(-2\varepsilon\kappa N - \kappa^2 M - \varepsilon\kappa)}{\kappa^2 - 4\varepsilon^2}(z^{*2}+z^2) + \frac{2z^*z(2\varepsilon^2 + \kappa^2 N + 2\varepsilon\kappa M)}{\kappa^2 - 4\varepsilon^2}\right]} \\
&= e^{-z^*z(1+\frac{2\varepsilon^2 + 2\kappa N + 2\varepsilon\kappa M}{\kappa^2 - 4\varepsilon^2}) - \frac{(2\varepsilon\kappa N + \kappa^2 M + \varepsilon\kappa)}{2(\kappa^2 - 4\varepsilon^2)}(z^{*2}+z^2)} \\
&= e^{-z^*z\frac{(\kappa^2 - 2\varepsilon^2 + \kappa^2 N + 2\varepsilon\kappa M)}{\kappa^2 - 4\varepsilon^2} - \frac{(2\varepsilon\kappa N + \kappa^2 M + \varepsilon\kappa)}{2(\kappa^2 - 4\varepsilon^2)}(z^{*2}+z^2)} \\
&= e^{-az^*z - \frac{b}{2}(z^{*2}+z^2)}.
\end{aligned} \tag{2.137}$$

in which

$$a = \frac{\kappa^2 - 2\varepsilon^2 + \kappa^2 N + 2\varepsilon\kappa M}{\kappa^2 - 4\varepsilon^2}, \tag{2.138}$$

$$b = \frac{2\varepsilon\kappa N + \kappa^2 M + \varepsilon\kappa)}{\kappa^2 - 4\varepsilon^2}. \tag{2.139}$$

Employing Eq. (2.137) in Eq. (2.108), the Q function for the cavity light can be written as

$$Q(\alpha^*, \alpha, t) = \frac{1}{\pi^2}\int d^2 z exp\Big(-az^*z + \alpha z^* - z\alpha^* - \frac{b}{2}(z^{*2} + z^2)\Big), \tag{2.140}$$

and integrating using the relation [1, 2]

$$\int \frac{1}{\pi} d^2 z exp\Big(-az^*z + bz + cz^* + Az^2 + Bz^{*2}\Big) = \bigg(\frac{1}{a^2 - 4AB}\bigg)^{\frac{1}{2}} e^{\frac{abc + Ac^2 + Bb^2}{\alpha^2 - 4AB}} \tag{2.141}$$

the Q function is finally expressible as

$$\begin{aligned} Q(\alpha^*, \alpha, t) &= \frac{1}{\pi}\bigg(\frac{1}{a^2 - b^2}\bigg)^{\frac{1}{2}} exp\bigg(\frac{-a\alpha^*\alpha - \frac{b}{2}(\alpha^{*2} + \alpha^2)}{a^2 - b^2}\bigg) \\ &= \frac{(u^2 - v^2)^{\frac{1}{2}}}{\pi} exp\Big(-u\alpha^*\alpha - \frac{v}{2}(\alpha^{*2} + \alpha^2)\Big). \end{aligned} \tag{2.142}$$

in which

$$u = \frac{a}{a^2 - b^2} \tag{2.143}$$

and

$$v = \frac{b}{a^2 - b^2}. \tag{2.144}$$

Chapter 3

Photon Statistics

The statistical properties of the light produced by a single-mode subharmonic generating system is described by the mean and variance of the photon number and photon number distribution. In this section we apply the Q function and the solutions of c-number Langevin equations to find the mean and variance of the photon number and photon number distribution for cavity mode. We next use the solutions of c-number Langevin equations and input-output relation to obtain the mean photon number and power spectrum for output mode

3.1 Mean photon number

The mean photon number for a single-mode subharmonic generating system is calculated using the Q function as [1, 2]

$$\langle \hat{a}^\dagger \hat{a} \rangle = \int d^2\alpha Q(\alpha^*, \alpha, t) A_a(\alpha) \tag{3.1}$$

A_a is c-number function corresponding to $\hat{a}^\dagger \hat{a}$ in the anti-normal order. Employing Eq. (2.142) and $A_a = \alpha^*\alpha - 1$, we have

$$\begin{aligned} \bar{n} &= \frac{(u^2 - v^2)^{\frac{1}{2}}}{\pi} \int d^2\alpha exp\big(- u\alpha^*\alpha - \frac{v}{2}(\alpha^{*2} + \alpha^2)\big)(\alpha^*\alpha - 1) \\ &= \frac{(u^2 - v^2)^{1/2}}{\pi}\big[\int d^2\alpha exp(-u\alpha^*\alpha - \frac{v}{2}(\alpha^{*2} + \alpha^2))\alpha^*\alpha \\ &- \int d^2\alpha exp(-u\alpha^*\alpha - \frac{v}{2}(\alpha^{*2} + \alpha^2))\big], \end{aligned} \tag{3.2}$$

or

$$\bar{n} = m_1 - m_2 \tag{3.3}$$

where

$$m_1 = \frac{(u^2 - v^2)^{\frac{1}{2}}}{\pi} \int d^2\alpha e^{-u\alpha^*\alpha - \frac{v}{2}(\alpha^{*2}+\alpha^2)} \alpha^*\alpha, \tag{3.4}$$

$$m_2 = \frac{(u^2 - v^2)^{\frac{1}{2}}}{\pi} \int d^2\alpha e^{-u\alpha^*\alpha - \frac{v}{2}(\alpha^{*2}+\alpha^2)}. \tag{3.5}$$

Carrying out the integration and using the relation given in Eq. (2.141)

$$\begin{aligned} m_2 &= \frac{(u^2 - v^2)^{\frac{1}{2}}}{\pi} \int d^2\alpha e^{-u\alpha^*\alpha - \frac{v}{2}(\alpha^{*2}+\alpha^2)} \\ &= \frac{(u^2 - v^2)^{\frac{1}{2}}}{\pi} \int d^2\alpha e^{-u\alpha^*\alpha - \frac{v}{2}(\alpha^{*2}+\alpha^2)+\eta\alpha+\gamma\alpha^*}|_{\eta=\gamma=0} \\ &= \big(\frac{1}{u^2 - v^2}\big)^{1/2}(u^2 - v^2)^{1/2} e^{\frac{u\eta\gamma - v/2(\eta^2+\gamma^2)}{u^2-v^2}}|_{\eta=\gamma=0} \\ &= 1 \end{aligned} \tag{3.6}$$

and

$$\begin{aligned} m_1 &= \frac{(u^2 - v^2)^{\frac{1}{2}}}{\pi} \int d^2\alpha e^{-u\alpha^*\alpha - \frac{v}{2}(\alpha^{*2}+\alpha^2)} \alpha^*\alpha|\eta = \gamma = 0 \\ &= \frac{(u^2 - v^2)^{\frac{1}{2}}}{\pi} \int d^2\alpha e^{-u\alpha^*\alpha - \frac{v}{2}(\alpha^{*2}+\alpha^2)+\eta\alpha+\gamma\alpha^*} \alpha^*\alpha|_{\eta=\gamma=0} \\ &= \frac{(u^2 - v^2)^{1/2}}{\pi} \frac{d}{d\eta}\frac{d}{d\gamma} \int d^2\alpha e^{-u\alpha^*\alpha - \frac{v}{2}(\alpha^{*2}+\alpha^2)+\eta\alpha+\gamma\alpha^*}|_{\eta=\gamma=0}, \end{aligned} \tag{3.7}$$

carrying out the integration

$$\begin{aligned} m_1 &= \frac{d}{d\eta}\frac{d}{d\gamma} e^{\frac{u\eta\gamma - v/2(\eta^2+\gamma^2)}{u^2-v^2}}|_{\eta=\gamma=0} \\ &= \frac{d}{d\eta}\frac{u\eta - v\gamma}{u^2 - v^2} e^{\frac{u\eta\gamma - \frac{v}{2}(\eta^2+\gamma^2)}{u^2-v^2}}|_{\eta=\gamma=0} \\ &= \big[\frac{u}{u^2 - v^2} e^{\frac{u\eta\gamma - \frac{v}{2}(\eta^2+\gamma^2)}{u^2+v^2}}\big] + \big[\big(\frac{u\eta - v\gamma}{u^2 - v^2}\big)\big(\frac{u\gamma - v\eta}{u^2 - v^2}\big)\big] e^{\frac{u\eta\gamma - \frac{v}{2}(\eta^2+\gamma^2)}{u^2-v^2}}|_{\eta=\gamma=0} \\ &= \frac{u}{u^2 - v^2}. \end{aligned} \tag{3.8}$$

Substituting Eq. (3.6) and (3.8) in to Eq. (3.3) along with Eqs. (2.143) and (2.144), we get

$$\begin{aligned} \bar{n} =& \frac{u}{u^2 - v^2} - 1 \\ &= a - 1. \end{aligned} \tag{3.9}$$

In view of Eq. (2.138), we have

$$\bar{n} = \frac{2\varepsilon^2}{\kappa^2 - 4\varepsilon^2} + \frac{\kappa(\kappa N + 2\varepsilon M)}{\kappa^2 - 4\varepsilon^2}. \tag{3.10}$$

3.2 Mean photon number for the output mode

The mean photon number for the output mode is expressible as

$$\bar{n}_{out}(t) = \langle \hat{a}^\dagger_{out}(t)\hat{a}_{out}(t)\rangle \tag{3.11}$$

Using input-output relation [1, 2]

$$\hat{a}_{out}(t) = \sqrt{\kappa}\hat{a}(t) - \hat{a}_{in}(t), \tag{3.12}$$

we see that

$$\hat{a}^\dagger_{out}(t) = \sqrt{\kappa}\hat{a}^\dagger(t) - \hat{a}^\dagger_{in}(t), \tag{3.13}$$

$$\hat{a}_{out}(t+\tau) = \sqrt{\kappa}\hat{a}(t+\tau) - \hat{a}_{in}(t+\tau), \tag{3.14}$$

where $\hat{a}_{in}$ is annihilation operator for the reservoir mode.
Multiplying Eqs. (3.12) and (3.13), we have

$$\begin{aligned}\langle \hat{a}^\dagger_{out}(t)\hat{a}_{out}(t)\rangle &= \big\langle (\sqrt{\kappa}\hat{a}^\dagger(t) - \hat{a}^\dagger_{in}(t))(\sqrt{\kappa}\hat{a}(t) - \hat{a}_{in}(t))\big\rangle \\ &= \kappa\langle \hat{a}^\dagger(t)\langle\hat{a}(t)\rangle - \sqrt{\kappa}(\langle \hat{a}^\dagger_{in}(t)\hat{a}(t)\rangle + \langle \hat{a}^\dagger(t)\hat{a}_{in}(t)\rangle) \\ &+ \langle \hat{a}^\dagger_{in}(t)\hat{a}_{in}(t)\rangle. \end{aligned} \tag{3.15}$$

In view of the relation [2],

$$\langle \hat{a}^\dagger_{in}(t)\hat{a}(t)\rangle + \langle \hat{a}^\dagger(t)\hat{a}_{in}(t)\rangle = \sqrt{\kappa}N, \tag{3.16}$$

in which, $\langle \hat{a}^\dagger_{in}(t)\hat{a}_{in}(t)\rangle = N$, we can put Eq. (3.15) in the form

$$\langle \hat{a}^\dagger_{out}(t)\hat{a}_{out}(t)\rangle = N(1-\kappa) + \kappa\langle \hat{a}^\dagger(t)\hat{a}(t)\rangle. \tag{3.17}$$

The corresponding c-number function to Eq. (3.17) can be written as

$$\langle \alpha^*_{out}(t)\alpha_{out}(t)\rangle = N(1-\kappa) + \kappa\langle \alpha^*(t)\langle\alpha(t)\rangle. \tag{3.18}$$

With the aid of of Eqs. (2.135), the mean photon number for output mode becomes

$$\bar{n}_{out} = N(1-\kappa) + \frac{\kappa(2\varepsilon^2 + \kappa^2 N + 2\varepsilon\kappa M)}{\kappa^2 - 4\varepsilon^2} \tag{3.19}$$

For vacuum reservoir ($N = M = 0$), we notice from Eqs. (3.10) and (3.19) that mean photon number for cavity mode is greater than output mode.

3.3 The variance of the photon number

The variance of the photon number in is defined by[1, 2]

$$\begin{aligned}(\Delta n)^2 &= \langle \hat{n}^2 \rangle - \langle \hat{n} \rangle^2 \\ &= \langle \hat{a}^\dagger \hat{a} \hat{a}^\dagger \hat{a} \rangle - \bar{n}^2\end{aligned} \tag{3.20}$$

This can be put in the anti-normal order using Eq. (2.7). We then see that

$$\begin{aligned}(\Delta n)^2 &= \langle (\hat{a}\hat{a}^\dagger - 1)(\hat{a}\hat{a}^\dagger - 1) \rangle - \bar{n}^2 \\ &= \langle \hat{a}\hat{a}^\dagger \hat{a}\hat{a}^\dagger \rangle - 2\langle \hat{a}\hat{a}^\dagger \rangle + 1 - \bar{n}^2 \\ &= \langle \hat{a}(\hat{a}\hat{a}^\dagger - 1)\hat{a}^\dagger \rangle - 2\langle \hat{a}\hat{a}^\dagger \rangle + 1 - \bar{n}^2 \\ &= \langle \hat{a}^2 \hat{a}^{\dagger 2} \rangle - 3\langle \hat{a}\hat{a}^\dagger \rangle + 1 - \bar{n}^2.\end{aligned} \tag{3.21}$$

The c-number function corresponding to Eq. (3.21) becomes

$$(\Delta n)^2 = \langle \alpha^{*2}\alpha^2 \rangle - 3\langle \alpha^* \alpha \rangle + 1 - \bar{n}^2. \tag{3.22}$$

The expectation value of $\alpha^{*2}\alpha^2$ can be obtained employing the Q function given in Eq. (2.142) as

$$\begin{aligned}\langle \alpha^{*2}\alpha^2 \rangle &= \int d^2\alpha Q(\alpha^*, \alpha, t)\alpha^{*2}\alpha^2 \\ &= \frac{(u^2 - v^2)^{1/2}}{\pi} \int d^2\alpha e^{-u\alpha^*\alpha - \frac{v}{2}(\alpha^{*2} + \alpha^2)} \alpha^{*2}\alpha^2 \\ &= \int d^2\alpha \frac{d^2}{d\eta^2} \frac{d^2}{d\gamma^2} \frac{(u^2 - v^2)^{1/2}}{\pi} e^{-u\alpha^*\alpha + \eta\alpha + \gamma\alpha^* - \frac{v}{2}(\alpha^{*2} + \alpha^2)} |_{\eta=\gamma=0}.\end{aligned} \tag{3.23}$$

Applying the relation in Eq. (2.141), we have

$$\begin{aligned}
\langle\alpha^{*2}\alpha^2\rangle &= (u^2-v^2)^{1/2}\frac{d^2}{d\eta^2}\frac{d^2}{d\gamma^2}\frac{1}{(u^2-v^2)^{1/2}}e^{\frac{u\eta\gamma-\frac{v}{2}(\eta^2+\gamma^2)}{u^2-v^2}}|_{\eta=\gamma=0}\\
&= \frac{d^2}{d\eta^2}\frac{d^2}{d\gamma^2}e^{\frac{u\eta\gamma-\frac{v}{2}(\eta^2+\gamma^2)}{u^2-v^2}}|_{\eta=\gamma=0}\\
&= \frac{d^2}{d\eta^2}\frac{d}{d\gamma}(\frac{u\eta-v\gamma}{u^2-v^2})e^{\frac{u\eta\gamma-\frac{v}{2}(\eta^2+\gamma^2)}{u^2-v^2}}|_{\eta=\gamma=0}\\
&= \frac{d^2}{d\eta^2}(\frac{u\eta-v\gamma}{u^2-v^2})^2e^{\frac{u\eta\gamma-\frac{v}{2}(\eta^2+\gamma^2)}{u^2-v^2}} - \frac{v}{u^2-v^2}e^{\frac{u\eta\gamma-\frac{v}{2}(\eta^2+\gamma^2)}{u^2-v^2}}|_{\eta=\gamma=0}\\
&= \frac{d}{d\eta}\left(\frac{u\eta-v\gamma}{u^2-v^2}\right)^2(\frac{u\gamma-v\eta}{u^2-v^2})e^{\frac{u\eta\gamma-\frac{v}{2}(\eta^2+\gamma^2)}{u^2-v^2}} + \frac{2u(u\gamma-v\eta)}{(u^2-v^2)}e^{\frac{u\eta\gamma-\frac{v}{2}(\eta^2+\gamma^2)}{u^2-v^2}}\\
&- (\frac{v}{u^2-v^2})(\frac{u\gamma-v\eta}{u^2-v^2})e^{\frac{u\eta\gamma-\frac{v}{2}(\eta^2+\gamma^2)}{u^2-v^2}}|\eta=\gamma=0\\
&= (\frac{u\eta-v\gamma}{u^2-v^2})^2(\frac{u\gamma-v\eta}{u^2-v^2})^2e^{\frac{u\eta\gamma-\frac{v}{2}(\eta^2+\gamma^2)}{u^2-v^2}} + \frac{2u(u\eta-v\gamma)(u\gamma-v\eta)}{(u^2-v^2)^3}e^{\frac{u\eta\gamma-\frac{v}{2}(\eta^2+\gamma^2)}{u^2-v^2}}\\
&- \frac{v(u\eta-v\gamma)}{u^2-v^3}e^{\frac{u\eta\gamma-\frac{v}{2}(\eta^2+\gamma^2)}{u^2-v^2}} + \frac{2u^2}{(u^2-v^2)^2}e^{\frac{u\eta\gamma-\frac{v}{2}(\eta^2+\gamma^2)}{u^2-v^2}}\\
&+ \frac{2u(u\eta-v\gamma)}{u^2-v^2}(\frac{u\gamma-v\eta}{u^2-v^2})e^{\frac{u\eta\gamma-\frac{v}{2}(\eta^2+\gamma^2)}{u^2-v^2}} + \frac{v^2}{(u^2-v^2)^2}e^{\frac{u\eta\gamma-\frac{v}{2}(\eta^2+\gamma^2)}{u^2-v^2}}\\
&- \frac{v(u\gamma-v\eta)^2}{(u^2-v^2)^3}e^{\frac{u\eta\gamma-\frac{v}{2}(\eta^2+\gamma^2)}{u^2-v^2}}|_{\eta=\gamma=0}\\
&= \frac{2u^2}{(u^2-v^2)^2} + \frac{v^2}{(u^2-v^2)^2}. \qquad (3.24)
\end{aligned}$$

On account of Eqs. (2.143) and (2.144) and in view of $\langle\alpha^*\alpha\rangle = \int d^2\alpha Q(\alpha^*,\alpha,t)\alpha^*\alpha$ described in Eq. (3.8) along with Eq. (3.9), we can rewrite Eq. (3.22) as

$$\begin{aligned}
(\Delta n)^2 &= \frac{2u^2}{(u^2-v^2)^2} + \frac{v^2}{(u^2-v^2)^2} - 3\frac{u}{u^2-v^2} + 1 - \bar{n}^2\\
&= 2a^2 + b^2 - 3a + 1 - \bar{n}^2\\
&= a^2 + b^2 - a\\
&= a(a-1) + b^2, \qquad (3.25)
\end{aligned}$$

in view of Eq. (3.9), we get

$$(\Delta n)^2 = \bar{n}a + b^2. \qquad (3.26)$$

We notice from this relation that, the variance of the photon number is greater than the mean photon number and therefore, the photon statistic is said to be super-Poissonian.

3.4 Photon number distribution

The photon number distribution for a single-mode radiation is expressible as [1, 3]

$$P(n) = \frac{\pi}{n!}\frac{\partial^{2n}}{\partial\alpha^{*n}\partial\alpha^{n}}[Q(\alpha^*\alpha)e^{\alpha^*\alpha}]|_{\alpha^*=\alpha=0} \tag{3.27}$$

Using Eq. (2.141) we have

$$\begin{aligned} P(n) &= \frac{\pi}{n!}\frac{(u^2-v^2)^{\frac{1}{2}}}{\pi}\frac{\partial^{2n}}{\partial\alpha^{*n}\partial\alpha^{n}}e^{-u\alpha^*\alpha-\frac{v}{2}(\alpha^{*2}+\alpha^2)}e^{\alpha^*\alpha}|_{\alpha^*=\alpha=0} \\ &= \frac{(u^2-v^2)^{\frac{1}{2}}}{n!}\frac{\partial^{2n}}{\partial\alpha^{*n}\partial\alpha^{n}}e^{\alpha^*\alpha(1-u)-\frac{v}{2}(\alpha^{*2}+\alpha^2)}|_{\alpha^*=\alpha=0}, \end{aligned} \tag{3.28}$$

applying the power series expansion, we have

$$e^{(1-u)\alpha^*\alpha} = \sum_{k=0}^{\infty}\frac{(1-u)^k(\alpha^*\alpha)^k}{k!}, \tag{3.29}$$

$$e^{\frac{v}{2}(\alpha^{*2}+\alpha^2)} = \sum_{k=0}^{\infty}(\frac{-v}{2})^{l+m}\frac{\alpha^{*2l}\alpha^{2m}}{l!m!}. \tag{3.30}$$

Up on substituting Eq. (3.29) and (3.30) in to (3.28), one obtains

$$P(n) = \frac{(u^2-v^2)^{\frac{1}{2}}}{n!}\sum_{klm}\frac{(-1)^{l+m}v^{l+m}(1-u)^k}{2^{l+m}k!l!m!}\frac{\partial^n}{\partial\alpha^*}\alpha^{*2l+k}\frac{\partial}{\partial\alpha^n}\alpha^{2m+k}|_{\alpha^*=\alpha=0}. \tag{3.31}$$

Using the relation [2]

$$\frac{\partial\chi^m}{\partial\chi^n} = \frac{m!}{(m-n)!}\chi^{(m-n)}, \tag{3.32}$$

we can write

$$\frac{\partial^n}{\partial\alpha^{*n}}\alpha^{*2l+k} = \frac{(2l+k)!}{(2l+k-n)!}\alpha^{*2l+k-n}, \tag{3.33}$$

$$\frac{\partial^n}{\partial\alpha^{n}}\alpha^{2m+k} = \frac{(2m+k)!}{(2m+k-n)!}\alpha^{2m+k-n}. \tag{3.34}$$

Making use of Eqs. (3.33) and (3.34), we put (3.31) as

$$\begin{aligned} P(n) &= \frac{(u^2-v^2)^{\frac{1}{2}}}{n!}\sum_{klm}(1-u)^k v^{l+m}\frac{(-1)^{l+m}}{2^{l+m}k!l!m!} \\ &\times \big(\frac{(2l+k)!}{(2l+k-n)!}\alpha^{*2l+k-n}\frac{(2m+k)!}{(2m+k-n)}\alpha^{2m+k-n}\big)|_{\alpha^*=\alpha=0}. \end{aligned} \tag{3.35}$$

Applying the conditions $\alpha^* = \alpha = 0$, we assert that

$$\alpha^{*2l+k-n}|\alpha^* = 0 = \delta_{2l+k,n}\,, \tag{3.36}$$

$$\alpha^{2m+k-n}|\alpha = 0 = \delta_{2m+k,n}\,. \tag{3.37}$$

In view of Eq. (3.36) and (3.37), we can put Eq. (3.35) as

$$P(n) = \frac{(u^2-v^2)^{\frac{1}{2}}}{n!}\sum_{klm}\frac{(1-u)^k}{k!}\frac{(\frac{-v}{2})^{l+m}}{l!m!}\frac{(2l+k)!}{(2l+k-n)!}\delta_{2l+k,n} \times \frac{(2m+k)!}{(2m+k-n)!}\delta_{2l+k,n}. \tag{3.38}$$

From the property of kronecker delta function we see that $k+2m = n$ and $k+2l = n$, and thus $l = m$, $k = n-2l$.

Hence

$$P(n) = (u^2-v^2)^{\frac{1}{2}}\sum_{l} n!(-1)^{2l}\frac{v^{2l}(1-u)^{n-2l}}{2^{2l}(n-2l)!l!^2}, \tag{3.39}$$

since $n-2l \geq 0$, then $l \leq \frac{n}{2}$ and Eq. (3.39) becomes

$$P(n) = (u^2-v^2)^{\frac{1}{2}}\sum_{l=0}^{[n]} n!\frac{v^{2l}(1-u)^{n-2l}}{2^{2l}(n-2l)!l!^2}. \tag{3.40}$$

where, $[n] = \frac{n}{2}$ for even n, $[n] = \frac{n-1}{2}$ for odd n and we notice that there is a finite probability for counting odd number of photons outside the cavity, even though single-mode subharmonic generator generates pairs of photons. This could be related to the fact that odd number of photons can escape through the coupler mirror. Due to this there is a possibility to observe an odd number of signal photons in the cavity.

3.5 Power spectrum

The power spectrum for the output radiation can be expressed in terms of the c-number variables associated with the normal ordering as [1, 2]

$$P^{out}(\omega) = \frac{1}{\pi} Re \int_0^\infty \langle \alpha_{out}^*(t)\alpha_{out}(t+\tau)\rangle_{ss} e^{i(\omega-\omega_0)\tau} d\tau \tag{3.41}$$

where, ω_0 is central frequency.

Multiplying Eqs. (3.13) and (3.14), we have

$$\begin{aligned}\langle \hat{a}_{out}^\dagger(t)\hat{a}_{out}(t+\tau)\rangle &= \langle (\sqrt{\kappa}\hat{a}^\dagger(t) - \hat{a}_{in}^\dagger(t))(\sqrt{\kappa}\hat{a}(t+\tau) - \hat{a}_{in}(t+\tau))\rangle \\ &= \kappa\langle \hat{a}^\dagger(t)\hat{a}(t+\tau)\rangle - \sqrt{\kappa}(\langle \hat{a}^\dagger(t)\hat{a}_{in}(t+\tau)\rangle \\ &+ \langle \hat{a}_{in}^\dagger(t)\hat{a}(t+\tau)\rangle) + \langle \hat{a}_{in}^\dagger(t)\hat{a}_{in}(t+\tau)\rangle.\end{aligned} \tag{3.42}$$

In view of Eq. (3.16), we see that

$$\langle \hat{a}^\dagger(t)\hat{a}_{in}(t+\tau)\rangle + \langle \hat{a}_{in}^\dagger(t)\hat{a}(t+\tau)\rangle = \sqrt{\kappa}N\delta(\tau). \tag{3.43}$$

Upon combining Eqs. (3.42) and (3.43), we obtain

$$\langle \hat{a}_{out}^\dagger(t)\hat{a}_{out}(t+\tau)\rangle = \kappa\langle \hat{a}^\dagger(t)\hat{a}(t+\tau)\rangle + N(1-\kappa)\delta(\tau). \tag{3.44}$$

The corresponding c-number function for Eq. (3.44) becomes

$$\langle \alpha_{out}^*(t)\alpha_{out}(t+\tau)\rangle = \kappa\langle \alpha^*(t)\alpha(t+\tau)\rangle + N(1-\kappa)\delta(\tau). \tag{3.45}$$

Multiplying Eq. (2.100) from the left by $\alpha^*(t)$ and taking the expectation value of the resulting expression and taking in to account the fact that a noise force at some time should not affect system variables at earlier time, we obtain

$$\begin{aligned}\langle \alpha^*(t)\alpha(t+\tau)\rangle_{ss} =& \frac{1}{2}\langle \alpha^{*2}(t)\rangle_{ss}(e^{\frac{-\lambda_+\tau}{2}} - e^{\frac{-\lambda_-\tau}{2}}) + \frac{1}{2}\langle \alpha^*(t)\alpha(t)\rangle_{ss}(e^{\frac{-\lambda_+\tau}{2}} + e^{\frac{-\lambda_-\tau}{2}}) \\ =& \frac{1}{2}e^{\frac{-\lambda_+\tau}{2}}\big(\langle \alpha^{*2}(t)\rangle_{ss} + \langle \alpha^*(t)\alpha(t)\rangle_{ss}\big) \\ &+ \frac{1}{2}e^{\frac{-\lambda_-\tau}{2}}\big(\langle \alpha^*(t)\alpha(t)\rangle_{ss} - \langle \alpha^{*2}(t)\rangle_{ss}\big).\end{aligned} \tag{3.46}$$

Substituting Eq. (3.46) in to Eq. (3.45), we have

$$\begin{aligned}\langle \alpha_{out}^*(t)\alpha_{out}(t+\tau)\rangle_{ss} =& \frac{\kappa}{2}e^{\frac{-\lambda_+\tau}{2}}\big(\langle \alpha^{*2}(t)\rangle_{ss} + \langle \alpha^*(t)\alpha(t)\rangle_{ss}\big) \\ &+ \frac{\kappa}{2}e^{\frac{-\lambda_-\tau}{2}}\big(\langle \alpha^*(t)\alpha(t)\rangle_{ss} - \langle \alpha^{*2}(t)\rangle_{ss}\big) + N(1-\kappa)\delta(\tau).\end{aligned} \tag{3.47}$$

On account of Eqs. (2.134) and (2.135), we can put (3.47) as

$$\langle \alpha^*_{out}(t)\alpha_{out}(t+\tau)\rangle_{ss} = \frac{\kappa}{2\pi}e^{-\frac{\lambda_-}{2}\tau}\Big(\frac{\varepsilon+\kappa(N+M)}{\kappa-2\varepsilon}\Big) + \frac{\kappa}{2\pi}e^{-\frac{\lambda_+}{2}\tau}\Big(\frac{\kappa(N-M)-\varepsilon}{\kappa+2\varepsilon}\Big) + \frac{1}{\pi}N(1-\kappa)\delta(\tau). \tag{3.48}$$

Substituting Eq. (3.48) in to (3.41), we have

$$\begin{aligned} P^{out}(\omega) =& \frac{1}{2\pi}\frac{\kappa(\varepsilon+\kappa(N+M))}{\kappa-2\varepsilon} Re\int_0^\infty e^{-(\frac{\lambda_-}{2}-i(\omega-\omega_0)\tau}d\tau \\ &+ \frac{1}{2\pi}\frac{\kappa(\kappa(N-M)-\varepsilon)}{\kappa+2\varepsilon} Re\int_0^\infty e^{-(\frac{\lambda_+}{2}-i(\omega-\omega_0)\tau}d\tau \\ &+ \frac{1}{\pi}Re\int_0^\infty N(1-\kappa)e^{i(\omega-\omega_0)\tau}\delta(\tau)d\tau, \end{aligned} \tag{3.49}$$

integrating Eq. (3.49) and using the relation [1, 2],

$$\int f(x)\delta(x-a)dx = f(a), \tag{3.50}$$

the power spectrum for the output light becomes

$$\begin{aligned} P^{out}(\omega) =& \frac{1}{2\pi}\frac{\kappa(\varepsilon+\kappa(N+M))}{\kappa-2\varepsilon} Re\Bigg(\frac{-1}{\frac{\lambda_-}{2}-i(\omega-\omega_0)}\Bigg)e^{-(\frac{\lambda_-}{2}-i(\omega-\omega_0)\tau}|_0^\infty \\ &+ \frac{1}{2\pi}\frac{\kappa(\kappa(N-M)-\varepsilon)}{\kappa+2\varepsilon} Re\Bigg(\frac{-1}{\frac{\lambda_+}{2}-i(\omega-\omega_0}\Bigg)e^{-(\frac{\lambda_+}{2}-i(\omega-\omega_0)\tau}|_0^\infty \\ &+ \frac{1}{\pi}N(1-\kappa). \end{aligned} \tag{3.51}$$

Simplifying Eq. (3.51) and taking real part only, one can obtain power spectrum for the output radiation as

$$P^{out}(\omega) = \frac{N(1-\kappa)}{\pi} + \frac{\kappa}{\pi}\Bigg[\frac{\varepsilon+\kappa(N+M)}{(\kappa-2\varepsilon)^2+4(\omega-\omega_0)^2} + \frac{\kappa(N-M)-\varepsilon}{(\kappa+2\varepsilon)^2+4(\omega-\omega_0)^2}\Bigg]. \tag{3.52}$$

With the same procedure the power spectrum for the cavity radiation is found to be

$$P(\omega) = \frac{1}{\pi}\left(\frac{\varepsilon + \kappa(N+M)}{(\kappa - 2\varepsilon)^2 + 4(\omega-\omega_0)^2} + \frac{\kappa(N-M) - \varepsilon}{(\kappa + 2\varepsilon)^2 + 4(\omega-\omega_0)^2}\right). \tag{3.53}$$

As clearly shown in Fig. 3.1 below, the power spectrum for cavity radiation is more picked.

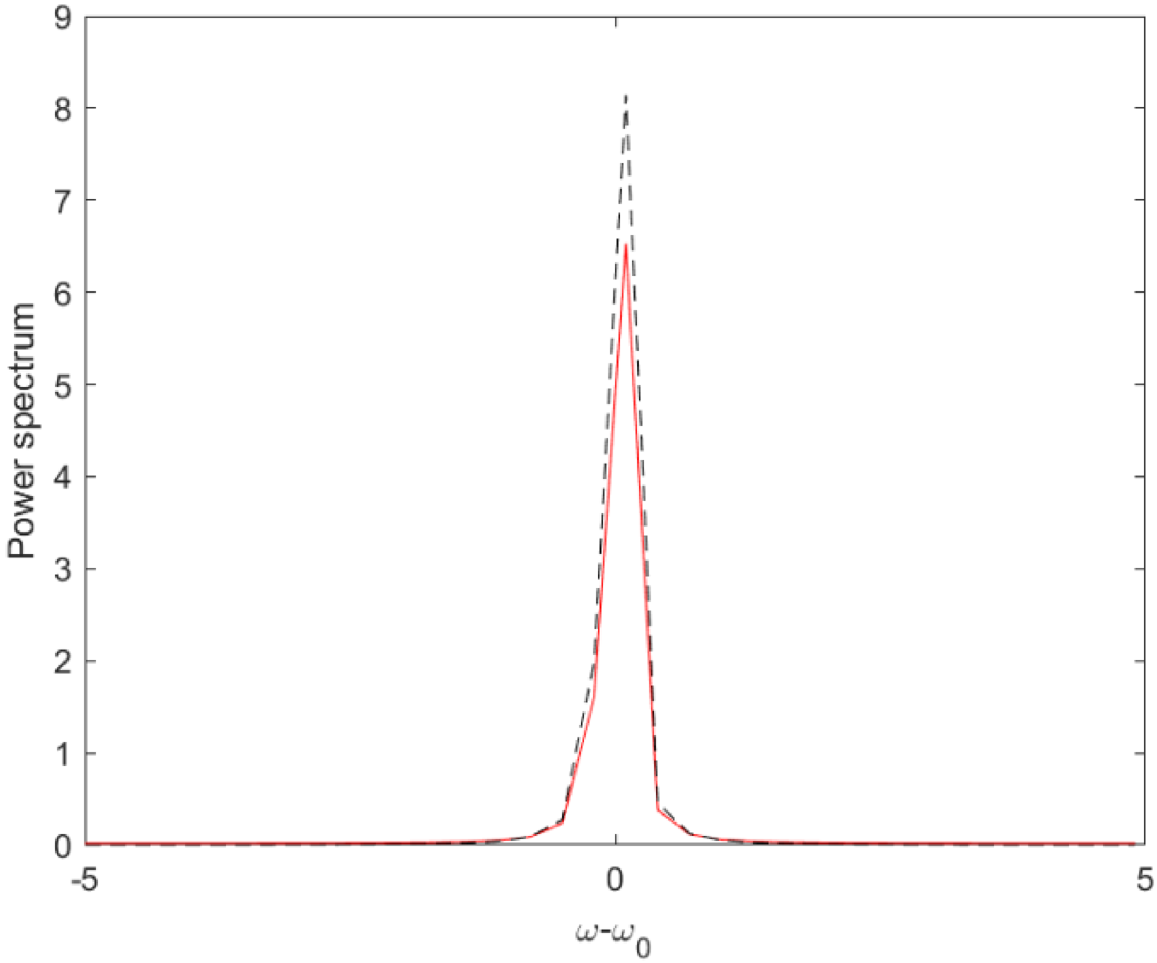

Figure 3.1: Plots for the power spectrum of the output radiation (Red) [Eq. (3.52)] and cavity radiation (dotted line) [Eq. (3.53)] versus ω at steady state for $\kappa = 0.8$, $M = 0.52$ and $N = 0.27$.

Chapter 4

Quadrature Variance

The squeezing properties of a single-mode output radiation can be described with the aid of the quadrature operators defined by [1, 2]

$$\hat{a}_{+}^{out} = \hat{a}_{out}^{\dagger} + \hat{a}_{out}, \tag{4.1}$$

and

$$\hat{a}_{-}^{out} = i(\hat{a}_{out}^{\dagger} - \hat{a}_{out}). \tag{4.2}$$

In view of Eqs. (4.1) and (4.2), the plus and minus quadrature variance can be calculated as

$$(\Delta a_{+}^{out})^2 = \langle \hat{a}_{+}^{out2} \rangle - \langle \hat{a}_{+}^{out} \rangle^2, \tag{4.3}$$

$$(\Delta a_{-}^{out})^2 = \langle \hat{a}_{-}^{out2} \rangle - \langle \hat{a}_{-}^{out} \rangle^2, \tag{4.4}$$

Substituting Eqs. (4.1) and (4.2) in to (4.3) and (4.4) along with Eq. (2.7), we get

$$\begin{aligned}(\Delta a_{+}^{out})^2 &= \langle (\hat{a}_{out}^{\dagger} + \hat{a}_{out})(\hat{a}_{out}^{\dagger} + \hat{a}_{out}) \rangle - \big(\langle (\hat{a}_{out}^{\dagger} + \hat{a}_{out}) \rangle \big)^2 \\ &= \langle \hat{a}_{out}^{\dagger 2} \rangle + \langle \hat{a}_{out}^{\dagger} \hat{a}_{out} \rangle + \langle \hat{a}_{out} \hat{a}_{out}^{\dagger} \rangle + \langle \hat{a}_{out}^{2} \rangle - (\langle \hat{a}_{out}^{\dagger} \rangle^2 + 2\langle \hat{a}_{out}^{\dagger} \rangle \langle \hat{a}_{out} \rangle + \langle \hat{a}_{out} \rangle^2) \\ &= 1 + \langle \hat{a}_{out}^{\dagger 2} \rangle + \langle \hat{a}_{out}^{2} \rangle + 2\langle \hat{a}_{out}^{\dagger} \hat{a}_{out} \rangle - \big(2\langle \hat{a}_{out}^{\dagger} \rangle \langle \hat{a}_{out} \rangle + \langle \hat{a}_{out}^{\dagger} \rangle^2 + \langle \hat{a}_{out} \rangle^2 \big), \end{aligned} \tag{4.5}$$

$$
\begin{aligned}
(\Delta a_-^{out})^2 &= -\langle(\hat{a}_{out}^\dagger - \hat{a}_{out})(\hat{a}_{out}^\dagger - \hat{a}_{out})\rangle - \big(i\langle(\hat{a}_{out}^\dagger - \hat{a}_{out})\rangle\big)^2 \\
&= -\big(\langle\hat{a}_{out}^{\dagger 2}\rangle - \langle\hat{a}_{out}^\dagger\hat{a}_{out}\rangle - \langle\hat{a}_{out}\hat{a}_{out}^\dagger\rangle + \langle\hat{a}_{out}^2\rangle\big) + \big(\langle\hat{a}_{out}^\dagger\rangle^2 - 2\langle\hat{a}_{out}^\dagger\rangle\langle\hat{a}_{out}\rangle + \langle\hat{a}_{out}\rangle^2\big) \\
&= \langle\hat{a}_{out}^\dagger\hat{a}_{out}\rangle + \langle\hat{a}_{out}\hat{a}_{out}^\dagger\rangle - \langle\hat{a}_{out}^{\dagger 2}\rangle - \langle\hat{a}_{out}^2\rangle - \big(2\langle\hat{a}_{out}^\dagger\rangle\langle\hat{a}_{out}\rangle - \langle\hat{a}_{out}\rangle^2 - \langle\hat{a}_{out}^\dagger\rangle^2\big) \\
&= 1 + 2\langle\hat{a}_{out}^\dagger\hat{a}_{out}\rangle - \langle\hat{a}_{out}^{\dagger 2}\rangle - \langle\hat{a}_{out}^2\rangle - \big(2\langle\hat{a}_{out}^\dagger\rangle\langle\hat{a}_{out}\rangle - \langle\hat{a}_{out}\rangle^2 - \langle\hat{a}_{out}^\dagger\rangle^2\big) \\
&= 1 - \langle\hat{a}_{out}^{\dagger 2}\rangle - \langle\hat{a}_{out}^2\rangle + 2\langle\hat{a}_{out}^\dagger\hat{a}_{out}\rangle - \big(2\langle\hat{a}_{out}^\dagger\rangle\langle\hat{a}_{out}\rangle - \langle\hat{a}_{out}^\dagger\rangle^2 - \langle\hat{a}_{out}\rangle^2\big), \qquad (4.6)
\end{aligned}
$$

From Eqs. (4.5) and (4.6), one can write the variance of the quadrature operators in the form

$$
\begin{aligned}
(\Delta a_\pm)_{out}^2 &= 1 \pm \langle\hat{a}_{out}^{\dagger 2}(t)\rangle \pm \langle\hat{a}_{out}^2(t)\rangle + 2\langle\hat{a}_{out}^\dagger(t)\hat{a}_{out}(t)\rangle \\
&\mp\langle\hat{a}_{out}^\dagger(t)\rangle^2 \mp \langle\hat{a}_{out}(t)\rangle^2 - 2\langle\hat{a}_{out}^\dagger(t)\rangle\langle\hat{a}_{out}(t)\rangle. \qquad (4.7)
\end{aligned}
$$

We notice that the operators in Eq. (4.7) are in the normal order. Hence the corresponding expression in terms of the c-number variables associated with the normal ordering would be

$$
\begin{aligned}
(\Delta a_\pm)_{out}^2 &= 1 \pm \langle\alpha_{out}^{*2}(t)\rangle \pm \langle\alpha_{out}^2(t)\rangle + 2\langle\alpha_{out}^*(t)\alpha_{out}(t)\rangle \\
&\mp\langle\alpha_{out}^*(t)\rangle^2 \mp \langle\alpha_{out}(t)\rangle^2 - 2\langle\alpha_{out}^*(t)\rangle\langle\alpha_{out}(t)\rangle. \qquad (4.8)
\end{aligned}
$$

which on account of Eq. (2.107), Eq. (4.8) reduces to

$$
(\Delta a_\pm)_{out}^2 = 1 + 2\langle\alpha_{out}^*(t)\alpha_{out}(t)\rangle \pm [\langle\alpha_{out}^{*2}(t)\rangle + \langle\alpha_{out}^2(t)\rangle]. \qquad (4.9)
$$

Using Eqs. (3.12) and (3.13), one can obtain

$$
\begin{aligned}
\langle\hat{a}_{out}^2(t)\rangle &= \langle(\sqrt{\kappa}\hat{a}(t) + \hat{a}_{in}(t))(\sqrt{\kappa}\hat{a}(t) + \hat{a}_{in}(t))\rangle \\
&= \kappa\langle\hat{a}^2(t)\rangle + \sqrt{\kappa}(\langle\hat{a}(t)\hat{a}_{in}(t)\rangle + \langle\hat{a}_{in}(t)\hat{a}(t)\rangle) + \langle\hat{a}_{in}^2(t)\rangle. \qquad (4.10)
\end{aligned}
$$

Assuming that the cavity mode and input mode operators are not correlated, we can write $\langle\hat{a}(t)\hat{a}_{in}(t)\rangle = \langle\hat{a}(t)\rangle\langle\hat{a}_{in}(t)\rangle = 0$. Moreover, using the relation $\langle\hat{a}_{in}^2(t)\rangle = -M$, we can write Eq. (4.10) as

$$
\langle\hat{a}_{out}^2(t)\rangle = \kappa\langle\hat{a}^2(t)\rangle - M, \qquad (4.11)
$$

In a similar procedure, it can be verified that

$$
\langle\hat{a}_{out}^{\dagger 2}(t)\rangle = \kappa\langle\hat{a}^{\dagger 2}(t)\rangle - M. \qquad (4.12)
$$

The corresponding c-number variables for Eqs. (4.11) and (4.12) become

$$\langle \alpha_{out}^2(t) \rangle = \kappa \langle \alpha^2(t) \rangle - M \tag{4.13}$$

and

$$\langle \alpha_{out}^{*2}(t) \rangle = \kappa \langle \alpha^{*2}(t) \rangle - M. \tag{4.14}$$

On account of Eqs. (3.17), (4.13) and (4.14) along with Eqs. (2.134) - (2.136), we can calculate the plus and minus quadrature variance for the output radiation as

$$\begin{aligned}
(\Delta a_+)^2_{out} &= 1 + 2\big(N(1-\kappa) - M\big) + \frac{2\varepsilon^2\kappa + \kappa^3 N + 2\varepsilon\kappa^2 M}{\kappa^2 - 4\varepsilon^2} - \frac{2\kappa(\kappa^2 M + 2\varepsilon\kappa N + \varepsilon\kappa}{\kappa^2 - 4\varepsilon^2} \\
&= 1 + 2\big(N(1-\kappa) - M\big) + \frac{4\varepsilon^2\kappa + 2\kappa^3 N + 4\varepsilon\kappa^2 M - 2\kappa^3 M - 4\varepsilon\kappa^2 N - 2\varepsilon\kappa^2}{\kappa^2 - 4\varepsilon^2} \\
&= 1 + 2\big(N(1-\kappa) - M\big) + \frac{4\varepsilon(\varepsilon\kappa + \kappa^2 M - \kappa^2 N) + 2\kappa(\kappa^2 N - \kappa^2 M - \varepsilon\kappa)}{(\kappa + 2\varepsilon)(\kappa - 2\varepsilon)} \\
&= 1 + 2\big(N(1-\kappa) - M\big) + \frac{2(2\varepsilon - \kappa)(\varepsilon\kappa + \kappa^2 M - \kappa^2 N)}{(\kappa + 2\varepsilon)(\kappa - 2\varepsilon)} \\
&= 1 + 2\big(N(1-\kappa) - M\big) - \frac{2\kappa(\varepsilon + \kappa M - \kappa N)}{\kappa + 2\varepsilon}.
\end{aligned} \tag{4.15}$$

With the same procedure, it can be verified as

$$(\Delta a_-)^2_{out} = 1 + 2\big(N(1-\kappa) + M\big) + \frac{2\kappa(\varepsilon + \kappa M + \kappa N)}{\kappa - 2\varepsilon}. \tag{4.16}$$

For the case of vacuum reservoir ($N = M = 0$), the output plus and minus quadrature variances can be put as

$$(\Delta a_+)^2_{out} = 1 - \frac{2\kappa\varepsilon}{\kappa + 2\varepsilon} \tag{4.17}$$

and

$$(\Delta a_-)^2_{out} = 1 + \frac{2\kappa\varepsilon}{\kappa - 2\varepsilon}. \tag{4.18}$$

At threshold ($\kappa = 2\varepsilon$), Eq. (4.17) reduces to

$$(\Delta a_+)^2_{out} = 1 - \varepsilon, \tag{4.19}$$

and for $\kappa = 2\varepsilon = 0.8$, the plus quadrature variance for the output radiation becomes

$$(\Delta a_+)^2_{out} = 0.6. \tag{4.20}$$

Moreover, the squeezing properties of the cavity radiation are described employing the plus and minus quadrature operators defined as [1, 2]

$$\hat{a}_+ = \hat{a}^\dagger + \hat{a} \tag{4.21}$$

and

$$\hat{a}_- = i(\hat{a}^\dagger - \hat{a}). \tag{4.22}$$

The plus and minus quadrature variances can be written as

$$(\Delta a_+)^2 = \langle \hat{a}_+^2 \rangle - \langle \hat{a}_+ \rangle^2, \tag{4.23}$$

$$(\Delta a_-)^2 = \langle \hat{a}_-^2 \rangle - \langle \hat{a}_- \rangle^2. \tag{4.24}$$

Substituting Eqs. (4.21) and (4.22) in to (4.23) and (4.24) and employing the commutation relation described in Eq. (2.7), we get

$$\begin{aligned}(\Delta a_+)^2 &= \langle (\hat{a}^\dagger + \hat{a})(\hat{a}^\dagger + \hat{a}) \rangle - \big(\langle (\hat{a}^\dagger + \hat{a}) \rangle\big)^2 \\ &= \langle \hat{a}^{\dagger 2} \rangle + \langle \hat{a}^\dagger \hat{a} \rangle + \langle \hat{a}\hat{a}^\dagger \rangle + \langle \hat{a}^2 \rangle - (\langle \hat{a}^\dagger \rangle^2 + 2\langle \hat{a}^\dagger \rangle \langle \hat{a} \rangle + \langle \hat{a} \rangle^2) \\ &= 1 + \langle \hat{a}^{\dagger 2} \rangle + \langle \hat{a}^2 \rangle + 2\langle \hat{a}^\dagger \hat{a} \rangle - \big(2\langle \hat{a}^\dagger \rangle \langle \hat{a} \rangle + \langle \hat{a}^\dagger \rangle^2 + \langle \hat{a} \rangle^2\big),\end{aligned} \tag{4.25}$$

$$\begin{aligned}(\Delta a_-)^2 &= -\langle (\hat{a}^\dagger - \hat{a})(\hat{a}^\dagger - \hat{a}) \rangle - \big(i\langle (\hat{a}^\dagger - \hat{a}) \rangle\big)^2 \\ &= -\big(\langle \hat{a}^{\dagger 2} \rangle - \langle \hat{a}^\dagger \hat{a} \rangle - \langle \hat{a}\hat{a}^\dagger \rangle + \langle \hat{a}^2 \rangle\big) + \big(\langle \hat{a}^\dagger \rangle^2 - 2\langle \hat{a}^\dagger \rangle \langle \hat{a} \rangle + \langle \hat{a} \rangle^2\big) \\ &= \langle \hat{a}^\dagger \hat{a} \rangle + \langle \hat{a}\hat{a}^\dagger \rangle - \langle \hat{a}^{\dagger 2} \rangle - \langle \hat{a}^2 \rangle - \big(2\langle \hat{a}^\dagger \rangle \langle \hat{a} \rangle - \langle \hat{a} \rangle^2 - \langle \hat{a}^\dagger \rangle^2\big) \\ &= 1 + 2\langle \hat{a}^\dagger \hat{a} \rangle - \langle \hat{a}^{\dagger 2} \rangle - \langle \hat{a}^2 \rangle - \big(2\langle \hat{a}^\dagger \rangle \langle \hat{a} \rangle - \langle \hat{a} \rangle^2 - \langle \hat{a}^\dagger \rangle^2\big) \\ &= 1 - \langle \hat{a}^{\dagger 2} \rangle - \langle \hat{a}^2 \rangle + 2\langle \hat{a}^\dagger \hat{a} \rangle - \big(2\langle \hat{a}^\dagger \rangle \langle \hat{a} \rangle - \langle \hat{a}^\dagger \rangle^2 - \langle \hat{a} \rangle^2\big).\end{aligned} \tag{4.26}$$

From Eqs. (4.25) and (4.26), one can write the variance of the quadrature operators in the form

$$\begin{aligned}(\Delta a_\pm)^2 &= 1 \pm \langle \hat{a}^{\dagger 2}(t) \rangle \pm \langle \hat{a}^2(t) \rangle + 2\langle \hat{a}^\dagger(t)\hat{a}(t) \rangle \\ &\quad \mp \langle \hat{a}^\dagger(t) \rangle^2 \mp \langle \hat{a}(t) \rangle^2 - 2\langle \hat{a}^\dagger(t) \rangle \langle \hat{a}(t) \rangle.\end{aligned} \tag{4.27}$$

The associated c-number variables with normal ordering for Eq. (4.26) will be

$$
\begin{aligned}
(\Delta a_{\pm})^2 &= 1 \pm \langle \alpha^{*2}(t)\rangle \pm \langle \alpha^2(t)\rangle + 2\langle \alpha^*(t)\alpha(t)\rangle \\
&\quad \mp \langle \alpha^*(t)\rangle^2 \mp \langle \alpha(t)\rangle^2 - 2\langle \alpha^*(t)\rangle\langle \alpha(t)\rangle. \qquad (4.28)
\end{aligned}
$$

which on account of Eq. (2.107), reduces to

$$
(\Delta a_{\pm})^2 = 1 + 2\langle \alpha^*(t)\alpha(t)\rangle \pm [\langle \alpha^{*2}(t)\rangle + \langle \alpha^2(t)\rangle]. \qquad (4.29)
$$

In view of Eqs. (2.134) - (2.136), the variance of the quadrature operator for cavity mode becomes

$$
\begin{aligned}
(\Delta a_{+})^2 &= 1 + \frac{2(2\varepsilon^2 + \kappa^2 N + 2\varepsilon\kappa M)}{\kappa^2 - 4\varepsilon^2} - \frac{2(\kappa^2 M + 2\varepsilon\kappa N + \varepsilon)}{\kappa^2 - 4\varepsilon^2} \\
&= 1 + \frac{4\varepsilon^2 + 2\kappa^2 N + 4\varepsilon\kappa M - 2\kappa^2 M - 4\varepsilon\kappa N - 2\varepsilon\kappa}{\kappa^2 - 4\varepsilon^2} \\
&= 1 + \frac{4\varepsilon(\varepsilon + \kappa M - \kappa N) + 2\kappa(\kappa N - \kappa M - \varepsilon)}{(\kappa + 2\varepsilon)(\kappa - 2\varepsilon)} \\
&= 1 + \frac{2(2\varepsilon - \kappa)(\varepsilon + \kappa M - \kappa N)}{(\kappa + 2\varepsilon)(\kappa - 2\varepsilon)} \\
&= 1 - \frac{2(\varepsilon + \kappa M - \kappa N)}{\kappa + 2\varepsilon}. \qquad (4.30)
\end{aligned}
$$

In a similar procedure it can be verified as

$$
(\Delta a_{-})^2 = 1 + \frac{2(\varepsilon + \kappa N + \kappa M)}{\kappa - 2\varepsilon}. \qquad (4.31)
$$

For the case of vacuum reservoir, Eqs. (4.30) and (4.31) take the form

$$
(\Delta a_{+})^2 = 1 - \frac{2\varepsilon}{\kappa + 2\varepsilon}, \qquad (4.32)
$$

$$
(\Delta a_{-})^2 = 1 + \frac{2\varepsilon}{\kappa - 2\varepsilon}. \qquad (4.33)
$$

We observe that Eq. (4.32) reduces, at threshold to

$$
(\Delta a_{+})^2 = 0.5. \qquad (4.34)
$$

For the case where $\varepsilon = N = M = 0$, we get from Eqs. (4.15) and (4.31) that

$$
(\Delta a_{+})^2_{out} = (\Delta a_{+})^2 = 1. \qquad (4.35)
$$

This represents the quadrature variance of the output and cavity vacuum light modes. When we compare Eqs. (4.15) and (4.16) as well as (4.30) and (4.31), squeezing occurs in the plus quadrature for both output and cavity modes.

The quadrature squeezing of a single-mode radiation is defined as [1, 2]

$$S = \frac{(\Delta a_+)^2_{vac} - (\Delta a_+)^2}{(\Delta a_+)^2_{vac}} \tag{4.36}$$

Employing Eqs. (4.15), (4.30) and (4.35) in Eq. (4.36), one can obtain the quadrature squeezing for output and cavity light produced by single-mode subharmonic generating system as

$$S^{out} = 2(M + N(\kappa - 1)) + \frac{2\kappa(\varepsilon + \kappa M - \kappa N)}{\kappa + 2\varepsilon} \tag{4.37}$$

and

$$S = \frac{2(\varepsilon + \kappa M - \kappa N)}{\kappa + 2\varepsilon}. \tag{4.38}$$

For vacuum reservoir, the quadrature squeezing for the output and cavity light modes reduce to

$$S^{out} = \frac{2\kappa\varepsilon}{\kappa + 2\varepsilon} \tag{4.39}$$

and

$$S = \frac{2\varepsilon}{\kappa + 2\varepsilon}. \tag{4.40}$$

Now at threshold, we observe that there is 40% squeezing of the output light and 50% squeezing of the cavity light below the vacuum level when single-mode subharmonic generator is coupled to vacuum reservoir.

Chapter 5

Conclusion

In this project we studied the squeezing and statistical properties of the output radiation generated by a single-mode subharmonic generating system coupled to vacuum reservoir. Employing the master equation, we have obtained the c-number Langevin equations corresponding to normally ordered operators. With the aid of the solutions of these equations we determined anti-normally ordered characteristic function which enables us to obtain the Q function. Using the Q function and c-number Langevin equations, we have obtained the mean and variance of the photon number and the photon number distribution for cavity mode. We have found that the light generated by subharmonic generating system is super-Poissonian. From the photon number distribution we have seen that there is a probability of finding odd number of signal photons out side the cavity, even though single-mode subharmonic generator generates pairs of photons. This is due to the fact that, odd number of photons can escape through the coupler mirror.

Finally, with the aid of c-number Lanagavin equations and input-output relation, we determined the mean photon number, power spectrum, quadrature variance and quadrature squeezing for output mode. Moreover, we have found that power spectrum for cavity radiation is greater than output radiation near $\omega = 0$. We have also seen that the light generated by the system is in a squeezed state and squeezing occurs in the plus quadrature. Furthermore, we found that there is 40% squeezing of the output light for $\kappa = 0.8$ and 50% squeezing of the cavity light below the vacuum level at steady state and threshold when single-mode subharmonic generator is coupled to vacuum reservoir.

Bibliography

[1] Sintayehu Tesfa, *Output radiation from a degenerate parametric oscillator*, (arXiv: quant - ph/0702108V1, 12 Feb 2007).

[2] Fesseha Kassahun, *Fundamentals of Quantum Optics*, (Lulu, North Caroline, 2008).

[3] Geoffrey New, *Introduction to Nonlinear Optics*, (Cambridge University Press, New York, 2011).

[4] B. Daniel and K. Fesseha, *Opt. Commun*, **151**, 384 (1998).

[5] J.Anwar and M.S.Zubairy, *Phys*, Rev.A **45**, 1804 (1992).

[6] K. Fesseha, *Opt. Commun*, **156**, 145 (1998).

[7] B. Teklu, *Opt. Commun*, **261**, 310 (2006).

[8] M. J. Collett and C. W. Gardiner, *Phys*, Rev. A **30**, 1386 (1984).

[9] L. A. Wu, M. Xiao, and H. J. Kimble, *J. Opt*, Soc. Am. B **4**, 1465 (1987).

[10] C. W. Gardiner and M. J. Collett, *Phys*, Rev. A **31**, 3761 (1985).

DECLARATION

I, hereby declare that this project is a review of previous works and that all sources of materials have been duly acknowledged.

Name: Besie Mengesha

Signature:————————————————.

e-mail:-dbessiem87@gmail.com

Place and time of submission:

Addis Ababa University

Department of physics

August, 2018

This Project has been submitted for examination with my approval as University advisor.

Name: Dr. Deribe Hirpo

Signature:————————————————.

e-mail:-deribeh@gmail.com