Graduate Seminar Report

On

Some Basic properties of Theory of Hardy Space

By

Hunegnaw Dessie

Advisor
Dr. Seid Mohammed

School of Graduate Studies
Addis Ababa University

June 2003

Page

Preface

The theory of H^p *spaces* has its origins in discoveries made forty or fifty years ago by such mathematicians as G.H. Hardy, J. Riesz, V. Smirnov, and G.Szegö. Most of this early work is concerned with the properties of individual functions of class H^p, and is classical in spirit. In recent years, the development of functional analysis has stimulated new interest in the H^p classes as linear spaces. This point of view has suggested a variety of natural problems and has provided new methods of attack, leading to important advances in the theory.

This seminar report contains two parts. In the first part we try to develop the definition and some properties of harmonic functions including the Mean value property, the maximum principle, and the Poisson integral formula in general and in particular in unit disk D and upper half plane H. The main object of this part is to show the representation of the harmonic in the unit disk D and upper half plane H.

In the second part we introduce sub harmonic functions with example, and some elementary properties, and some properties on the theory of Hardy space with its definition, and completeness of Hardy space (H^p *spaces*), and also finally we introduce the use of Blashke products to reduce the problem to the case of non vanishing analytic functions.

I would like to thank my advisor Dr. Seid Mohammed for his help in identifying the topic and stimulating advice during the preparation.

Finally my thanks goes to my parents, and spouse, Meaza Melkam for their help and encouraging in doing this seminar.

To my father, Kess Dessie Assres and
my mother, Mntwuab Wondie

Part I Harmonic function

1.1 Definition and examples

Definition: A real valued function U on G which is open, connected subset of C is said to be harmonic if it has continuous second partial derivatives and satisfies the partial differential equation, $\Delta u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$ which is known as Laplace`s equation

Example1: Let u: C⇒R by u (x,y)= $3xy^3 - x^3$, u has continuous second partial derivatives $\frac{\partial u}{\partial x} = 3y^2 - 3x^2$, $\frac{\partial^2 u}{\partial x^2} = -6x$, $\frac{\partial u}{\partial y} = 6xy$, $\frac{\partial^2 u}{\partial y^2} = 6x$, $\Delta u = 0$. Hence u is harmonic function on C.

Example2: Let g: C⇒R by g (x+iy) = x^2-y^2, g is harmonic function on C.

Theorem1: Let U be an open disk with center (a, b). Suppose f and g be differentatiable on U; let $\frac{\partial f(z)}{\partial y} = \frac{\partial g(z)}{\partial x}$, for all. There is a function h which is in C^2 (U) such that $\frac{\partial h}{\partial x} = f$ and $\frac{\partial h}{\partial y} = g$

Proof for each z =x+iy Define; h (x , y) $= \int_a^x f(t,b)dt + \int_b^y g(x,s)ds$

Then by fundamental theorem of calculus:

$$\frac{\partial h(x,y)}{\partial y} = g(x,y) \text{ and } \frac{\partial h}{\partial x} = f(x,b) \qquad (1.1)$$

Since g $\in C^1$, differentiation under the integral sign is possible. then

$$\frac{\partial}{\partial x}\int_b^y g(x,s)ds = \int_b^y \frac{\partial}{\partial x} g(x,s)ds = \int_b^y \frac{\partial}{\partial y} f(x,s)ds$$

$$=f(x , y)-f(x ,b)$$

Now, $$\frac{\partial h}{\partial x}=\frac{\partial}{\partial x}\int_a^x f(t,b)dt + \frac{\partial}{\partial x}\int_b^y g(x,s)ds = f(x,b)+f(x,y)-f(x,b)=f(x,y) \quad (1.2)$$

And hence from (1.1) and (1.2), $\frac{\partial h}{\partial x}=f$ and $\frac{\partial h}{\partial y}=g$ but f,g$\in C^1$ this implies that h$\in C^2$

. Therefore h is the required function.

Corollary Let u be harmonic function on unit disc D.Then there is an analytic function H on D such that ReH = u

Proof Let f $=\frac{\partial u}{\partial y}$ and g = $-\frac{\partial u}{\partial x}$ this implies that $\frac{\partial f}{\partial y}=\frac{\partial^2 u}{\partial y^2}$ and $\frac{\partial g}{\partial x}=-\frac{\partial^2 u}{\partial x^2}$. Since u is harmonic function, we have $\frac{\partial^2 u}{\partial y^2}+\frac{\partial^2 u}{\partial x^2}=0$, from this $\frac{\partial f}{\partial y}=\frac{\partial g}{\partial x}$. Then by the above theorem there exists v$\in C^2$ such that $\frac{\partial v}{\partial x}=f(x,y)$ and $\frac{\partial v}{\partial y}=g(x,y)$ which is equivalent to $\frac{\partial v}{\partial x}=\frac{\partial u}{\partial y}$ and $\frac{\partial v}{\partial y}=-\frac{\partial u}{\partial x}$ and H(z)=u(z)+iv(z) satisfies the cauchy Rieman equation on D. Therefore u is a real part of H which is analytic. The function v is called a harmonic conjugate for u on D.If u is a harmonic function on D, we can find an analytic function H such that u is the real part of H.But H is infinitely many times differentiable and hence u is infinitely many times differentiable

1.2 Mean Value property and its principle and maximum principle

Definition: A continuous function u: G⇒R is said to have the mean value property (MVP) if whenever $\overline{B}$ (a, r)$\subseteq$G, then

$$u(a) = \frac{1`}{2\pi} \int_0^{2\pi} u\left(a + re^{i\theta}\right) d\theta$$

The fact that harmonic function is the real part of analytic has a number of important consequences; one of these is a property of harmonic that is analogues to the cauchy integral formula. It allows us to ascertain the value of harmonic function u at the center of a disk from its value on the boundary.

Theorem2: (The mean value theorem)

Suppose that u: G⇒R is harmonic on G ⊆C and that $\overline{B}$ (a, r) for some r>0 then

$$u(a)= \frac{1}{2\pi} \int_0^{2\pi} u(a+re^{i\theta})d\theta$$

Proof. Let f be analytic function on $\overline{B}(a,r)$ such that Ref = u,

By the cauchy integral formula;

$f(a) = \frac{1}{2\pi i} \int_\gamma \frac{f(\varepsilon)}{\varepsilon - a} d\varepsilon$ Whenever γ is the circle $|z-a| =$ r , for $\varepsilon = a + re^{i\theta}$ then

$$f(a) = \frac{1}{2\pi i} \int \frac{f(a+re^{i\theta})}{a+re^{i\theta} - a} rie^{i\theta} d0 = \frac{1}{2\pi} \int f(a+re^{i\theta})d\theta \quad = u(a) \ + iv(a)$$

$$= \frac{1}{2\pi} \int u(a+re^{i\theta})d\theta + i\frac{1}{2\pi} \int v(a+re^{i\theta})d\theta$$

Therefore $u(a) = \frac{1}{2\pi} \int u(a+re^{i\theta})d\theta$

Theorem3: (`Maximum principle)

Let G be an open, connected set and suppose that u is a continuous real valued function in G with MVP. If there is a point G such that u(a)≥u(z) for all z in G then u is a constant function i.e. u(z) = u(a) for all in G.

Proof Let A= $\{z \in G : u(a) = u(z)\}$

Claim A = G From continuity of u, the set A is closed subset of G as

$$A = \left(u^{-1}((-\infty, u(a) \cup (u(a), \infty)\right)'$$

Let z_0 be in G. since G is open set we can choose r>0 such that $\overline{B}(z_{0,}, r)$ is contained in G. suppose there is a point b in B(z_0, r) such that u(b)$\neq$u(a) then u(b)<u(a). This implies that by continuity taking ε= u(a)-u(b) we get u(z)<u(a) = u(z_0) for all z in a neighborhood of b .In particular if p = $|z_0 - b|$ and b =z_0+ p$e^{i\beta}$, 0≤β<2π,then there is a proper interval I of [0,2π] such that β in I and u(z_0+p$e^{i\beta}$)<u(z_0) for all θ in I.

Hence by the MVP, u(z_0) $= \frac{1}{2\pi}\int_0^{2\pi} u(z_0 + \rho e^{i\theta})d\theta < u(z_0)$ which is a contradiction. So B(z_0,r)⊆A and A is also open .By the connectedness of G , A=G //

According to MP, if u (z) is analytic on a domain D.then $|f(z)|$ cannot have a maximum anywhere in D unless f (z) is constant and if f (z) be analytic in a bounded region D and let $|f(z)|$ be continuous in the closed region $\overline{D}$ then $|f(z)|$ assumes its maximum on the boundary of the region.

1.3 Poisson integrals

Let us begin the discussion of the Poisson integrals by stating the following fact;

Theorem 4: For each real number r,0≤ r<1, the two sided infinite series

P(r , t)= $\sum_{n=-\infty}^{\infty} r^{|n|} e^{int}$ converges uniformly for t in R that is the sequence S_N given by:

S_N (t)= $\sum_{n=-N}^{N} r^{|n|} e^{int}$ converges uniformly on R

Proof To see this, we write $S_N(t) = 1+2\sum_{n=1}^{N} r^n (e^{int} + e^{-int})$

$$=1+2\sum_{n=1}^{N} r^n \cos nt$$

But $|S_N(t)| \le 1+\sum_{n=1}^{N} 2r^n$ as $\left|r^n (e^{int} + e^{-int})\right| \le 2r^n$. Therefore it converges uniformly. But $\lim S_N(t) = 1+\sum r^n (e^{int} + e^{-int})$ which converges uniformly on R.

Definition : P(r, t) = $\sum_{n=-\infty}^{\infty} r^{|n|} e^{int}$, 0≤ r<1 and t in R is called the poison kernel. .

Note 1 . $P(r,t) = \sum_{n=1}^{\infty} r^n (e^{int} + e^{-int}) = 1+\sum_{n=1}^{\infty} 2r^n \cos nt$,

Since $\text{Cosnt} = \dfrac{e^{int}+e^{-int}}{2}$

2 Let $z = re^{i\theta}$ then $P(r,\theta-t) = \sum_{n=-\infty}^{\infty} r^{|n|} e^{in(\theta-t)}$

$$= \text{Re}\left[\frac{e^{it} + re^{i\theta}}{e^{i\theta} - re^{i\theta}}\right] = \frac{1-r^2}{1-2r\cos(\theta-t)+r^2}$$

Now, $P(r,\theta-t) = 1+2\sum_{n=1}^{\infty} r^n \cos n(\theta-t) = \text{Re}\left[1+2\sum_{n=1}^{\infty} r^n e^{in(\theta-t)}\right]$

$$= \text{Re}\left[1+2\sum_{n=1}^{\infty} (re^{i\theta}.e^{-it})^n\right]$$

$$= \text{Re}[1 + 2\sum (ze^{-it})^n = \text{Re}[1 + \frac{2ze^{-it}}{1 - ze^{it}}]$$

$$= \text{Re}[\frac{e^{it} + re^{i\theta}}{e^{it} - re^{i\theta}}]$$

But P(r,θ-t) $=1+ \sum_{n=1}^{\infty} r^n [e^{in(\theta - t)} + e^{-in(\theta - t)}]$

$$=1+ \sum_{n=1}^{\infty} r^n e^{in(\theta - t)} + \sum_{n=1}^{\infty} r^n e^{-in(\theta - t)}$$

$$= 1+ \frac{re^{i(\theta - t)}}{1 - re^{i(\theta - t)}} + \frac{re^{-i(\theta - t)}}{1 - re^{-i(\theta - t)}}$$

$$= \frac{1 - r^2}{1 - 2r\cos(\theta - t) + r^2} = P(r, \theta - t) = P_{z_0}(\theta)$$

For each fixed t in R, f(z) = $\frac{e^{it} + z}{e^{it} - z}$ is analytic function which satisfies CauchyRemain equation. Hence Ref(z) is harmonic. Notice that P(r, t) > 0 and $\frac{1}{2\pi} \int_0^{2\pi} P(r, t) dt$ =1 indeed,

$$\frac{1}{2\pi} \int_0^{2\pi} P(r, t) dt = \frac{1}{2\pi} \int_0^{2\pi} dt + \frac{1}{2\pi} \int_0^{2\pi} 2\cos ntdt = 1 + 0 = 1$$

Definition: Let f be continuous on $\partial D = \{z : |z| = 1\}$ then,

Pf (z)= $\frac{1}{2\pi} \int_0^{2\pi} P(r, \theta - t) f(e^{it}) dt$ is said to be the Poisson integral of f. From the definition of pf (z) and the fact that P(r,θ-t) = Ref (z) of the above, we can conclude that pf(z) is harmonic function.

1.4 Poisson integrals in D and H (upper half plane)

From the mean value theorem, if u is harmonic on B (a, r), then u(a) is the mean value of u on the circle $|z-a|=r$ now, if a=0 and r =1 we get a unit disc D=$\{z:|z|=1\}$=B(0,1). Hence if u is continuous on the closed disk D and is harmonicon the open disc D $\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=0$.i.e. satisfies Laplace`s equation then

$$u(0)=\frac{1}{2\pi}\int u(e^{i\theta})d\theta$$

Theorem5: Suppose that u is continuous on the closed unit disk $\overline{D}$ and is harmonic on the open unit disk D. let $z_0 = re^{i\theta_0}$ and then

$$u(z_0)=\frac{1}{2\pi}\int_0^{2\pi} P_{z_0}(\theta)u(e^{i\theta})d\theta$$

where $P_{z_0}(\theta)$ is the Poisson kernel of z_0.

Proof: consider the map $\tau(z)=\frac{z-z_0}{1-\overline{z}_0 z}$: D⇒$\overline{D}$ which is a *möbius* transformation and $\tau^{-1}(z)$ exists which we will be given by $\tau^{-1}(z)=\frac{z+z_0}{1+\overline{z}_0 . z}$, for z on ∂D we get $|\tau(z)|=1$ i.e. if $z=re^{i\theta}$ then $\left|\frac{e^{i\theta}-z_0}{1-\overline{z}_0 . e^{i\theta}}\right|=1$ this implies that $\tau(z)\in\partial D$. Therefore the unit circle ∂D is invariant under τ and hence $\tau(z)=e^{i\varphi}$ for some φ in R.

Then $e^{i\varphi}=\frac{e^{i\theta}-z_0}{1-\overline{z}_0 e^{i\theta}}$

Differentiation gives us:

$$\frac{d\varphi}{d\theta}=\frac{1-|z_0|^2}{\left|e^{i\theta}-z_0\right|^2}=\frac{1-r^2}{1-2r\cos(\theta-\theta_0)+r^2}=p_{z_0}(\theta-\theta_0)=P(r,\theta-\theta_0)$$

Which is known as Poisson kernel of z_0 in D. Now consider τ^{-1} and τ which are continuous on $\overline{D}$ and harmonic on D. The composition of u and τ^{-1} i.e. uo τ^{-1} is continuous on $\overline{D}$ and harmonic on D. Then by the Mean Value property (MVP)

$$u o \tau^{-1}(0)=u(z_0)=\frac{1}{2\pi}\int (uo\tau^{-1})(e^{i\varphi})d\varphi \quad =\frac{1}{2\pi}\int u(\tau^{-1}(e^{i\varphi}))d\varphi \quad .$$
$$=\frac{1}{2\pi}\int u(e^{i\theta})P_{z_0}(\theta)d\theta$$

$$u(z_0)=\frac{1}{2\pi}\int_0^{2\pi} P_{z_o}(\theta)u(e^{i\theta})d\theta$$ which is known as <u>Poisson integral formula</u>.

From this we conclude that if u is continuous on $\overline{D}$ and harmonic in D, for each z_0 in D, u(z_0) can be represented by;

$$u(z_0)=\frac{1}{2\pi}\int u(e^{i\theta})P_{z_0}(\theta)$$

Suppose the map z: D→H be given by z(w) = $\frac{i(1-w)}{1+w}$, clearly z is a conformal map between D and H such that z(i) =1, z(1) = 0 z(-i) = -1 and z(-1) = ∞. If w on ∂D and $w\neq -1$ then z(w) = t in R, from the disc, we get

$$p_z(\theta)=P_z(e^{i\theta})=\mathrm{Re}[\frac{e^{i\theta}+z}{e^{i\theta}-z}] \quad .$$

Hence, $p_w(e^{i\theta})=\mathrm{Re}[\frac{e^{i\theta}+w}{e^{i\theta}-w}]$, let $z_0=z(w_0)$, $z_0=x_0+iy_0$

Under the inverse map of z above i.e. z : $H\to D$: $z^{-1}(z_0)=\frac{i-z_0}{i+z_0}=w_0$,each t in R maps to ∂D

For t in **R** we have, $z^{-1}(t) = \frac{i-t}{i+t} = e^{i\theta}$, for some θ .

$$P_{w_0}(e^{i\theta}) = \mathrm{Re}\left[\frac{e^{i\theta}+w_0}{e^{i\theta}-w_0}\right] = P_{w_0}(z^{-1}(t))$$

$$= \mathrm{Re}\left[\frac{\frac{i-t}{i+t}+\frac{i-z_0}{i+z_0}}{\frac{i-t}{i+t}+\frac{i-z_0}{i+z_0}}\right] = \frac{y_0(1+t^2)}{(x_0-t)^2+y_0^{\ 2}} \qquad (*)$$

And hence normalizing the above equation (*) yields:

$$P_{w_0}(e^{i\theta}).\frac{d\theta}{2\pi} = \frac{y_0(1+t^2)}{\pi.(x_0-t)^2+y_0^{\ 2})(1+t^2)}dt = \frac{y_0}{\pi(x_0-t)^2+y_0^{\ 2}}dt,$$

From this we get:

$$P_{w_0}(e^{i\theta})\frac{d\theta}{dt} = \frac{1}{\pi}\frac{y_0}{(x_0-t)^2+y_0^{\ 2}}$$ Which is called the Poisson kernel

of z_0 in the upper half plane. Based on the above information, if u is continuous on $H \cup \{\infty\}$ and harmonic on H, then $uo\,z^{-1}$ is continuous on the closed unit disc and harmonic on the open disc and hence

$$uoz^{-1}(w) = \frac{1}{2\pi}\int_0^{2\pi}(u(z^{-1}(e^{i\theta})P_w(e^{i\theta})d\theta = \int_{-\infty}^{\infty}u(t)\frac{y}{(x-t)^2+y^2}dt = P_z(t)$$

Which is known as Poisson integral in the upper half plane. When t in R is fixed the Poisson kernel for the upper half plane is harmonic function of z because,

$$P_z(t) = \frac{1}{\pi}\mathrm{Im}\left(\frac{1}{t-z}\right) = P_y(t)$$

From its defining formula we see that $P_z(t) \le \frac{c_z}{1+t^2}$ where c_z is a constant depending on z, consequently if $1 \le q \le \infty$ then $P_z(t) \in L^q(R)$ and the function

u(z) $= \int P_z(t)\,f(t)\,dt$ is harmonic on H whenever $f \in L^p(R), 1 \le p \le \infty$, moreover , since $P_z(t)$ is continuous function , the above integral will still produce a harmonic function u(z) if f(t)dt is replaced by a finite measure dμ(t) or by a positive measure dμ(t) such that $\int \frac{1}{1+t^2} d\mu(t) < \infty$ (so that $\int p_z(t) d\mu(t)$ converges.)

The following are some of the properties of Poisson integral

i. It is non negative and $\int P_y(t)dt = 1$

ii $P_y(t) \le \frac{1}{\pi y}$ and $Sup_{|t|>\delta} P_y(t) \to 0$ as y→0

iii . P_y is even, and decreasing function of t>0

v. $\int_{|t|>\delta} P_y(t)dt \to 0$ as y→0 for any δ>0.

The proof of these is the consequence of the definition.

An important tool for studying integrals like u(z) $= \int p_y(t) f(t)dt$ is the Minikoski inequality for integrals which is stated as follows:

If μ and v are σ-finite measures, $1 \le p < \infty$ and F (x, t) is $v x \mu$ measurable then;

$$\left\| \int F(x,t) dv(x) \right\|_{L^P(\mu)} \le \int \|F(x,t)\|_{L^P(\mu)} dv(x).$$

As a result of this inequality, if u(x,y) = $\int p_y(x-t) f(t)dt$ .then

$\left(\int |u(x,y)|^P dx \right)^{1/p} \le \|f\|_p, 1 \le p < \infty$ and If u(x,y) = $\int p_y(x-t) d\mu(t)$ where μ is a finite measure on R. Then $\int |u(x,y)dx| \le \int |d\mu|$.

Lemma1: Assume f$\in L^P, 1 \le p \le \infty$ and f is continuous *at* x_0 *let u(x,y)=* $\int p_y(t) f(x-t)dt$

then $\lim_{(x,y)->x_0} u(x,y) = f(x_0)$

Proof: we are going to show that given ε>0 there exists δ>0 such that $|x - x_o| \Rightarrow |u(x,y) - f(x_0)| < \varepsilon$, indeed since f is Continuous at x_0 there exists δ>0 such that $|x - x_0| < \delta \Rightarrow |f(x) - f(x_0)| < \varepsilon$, if x-t $\in (x_0 - \delta, x_0 + \delta)$ then $|(x-t) - x_0| < \delta$ implies that

$|f(x-t)-f(x_0)|<\varepsilon$, but

$$|u(x,y)-f(x_0)| = \int_{|t|<\delta} p_y(t)|f(x-t)-f(x_0)|dt + \int_{|t|\geq\delta} p_y(t)|f(x-t)-f(x_0)|dt$$

Hence for the fixed δ from the above $\int_{|t|\geq\delta} p_y(t)|f(x-t)-f(x_0)|dt \longrightarrow 0$ as $y \longrightarrow 0$ and hence with δ small and $|x-x_0|$ small, $\int_{|t|<\delta} p_y(t)|f(x-t)-f(x_0)|dt$ is small. Therefore the limit converges to f(x_0).

Lemma2 If u(z) is harmonic on H and bounded and continuous on $\overline{H}$

Then $u(z) = \int p_y(x-t)u(t)dt$

Proof if u is continuous at ∞, then it is a consequence of the definition

But u may n`t be continuous at ∞

Let $U(z) = u(z) - \int p_y(x-t)u(t)dt$

Claim U(z) = 0 , clearly U is harmonic on H and continuous in $\overline{H}$, and by the above lemma , for x in R,U(z) =0

Set $V(z) = \begin{cases} U(z) & y \geq 0 \\ -U(\overline{z}) & y<0 \end{cases}$ $\quad$ z = x +iy

V is bounded harmonic function on the complex plane on. Then by Lioville theorem which says a bounded analytic on the complex plane is constant, V(z)=V(0) =U(0) = 0, hence V(z) = 0 for all z in C .This implies that:

$$u(z) = \int p_y(x-t)u(t)dt$$

Thereom5: Let u be a harmonic on the upper half plane H. Then

(a) let 1≤p≤∞ ,u is the Poisson integral of a function in L^p if and only if

$$Sup_y \|u(x,y)\|_{L^P(dx)} < \infty$$

(b) u is the Poisson integral of a finite measure on R if and only if $Sup_y \int u(x,y)dx < \infty$

(c) u is positive if and only if

$$u(z) = cy + \int p_y(x-t)d\mu(t) c \geq 0, \mu \geq 0, \int \frac{d\mu(t)}{1+t^2} < \infty$$

Proof let u be the Poisson integral of a function f in L^p i.e.

$$u(z) = \int p_y(x-t)f(t)dt = p_y * f(x)$$

Then by minikoski inequality for integrals we have $\left(\int |u(x,y)|^p dx\right)^p \leq \|f\|_p, 1 \leq p < \infty$

This implies that $\|u(x,y)\|_{L^p(dx)} \leq \|f\|_p$, taking the suprumum on both sides we get the required result.

Conversely assume $Sup_y \|u(x,y)\|_{L^p(dx)} < \infty$, we need to show that

$u(z) = p_y * f(x) = \int p_y(x-t)f(t)dt$ To prove this let's prove this inequality;

$$|u(z)| \leq \left(\frac{2}{\pi y}\right)^{1/p} Sup_{\beta \succ 0} \|u(x,\beta)\|_{L^p(dx)}, y > 0$$

Let $\omega = \alpha + i\beta$. Consider B(z,y).

Then by the MVP: $u(z) = \frac{1}{2\pi} \int u(z + re^{i\theta})d\theta$.Integrating from 0 to y i.e.

$$\int_0^y u(z)rdr = \frac{1}{2\pi} \int_0^y \int_0^{2\pi} u(z + re^{i\theta})rdrd\theta = \frac{1}{2\pi} \iint_{B(z,y)} u(\omega)d\alpha d\beta$$

From this we get $u(z) = \frac{1}{\pi y^2} \iint u(\omega)d\alpha d\beta$ and

$$|u(z)| = \frac{1}{\pi y^2} \left| \iint u(\omega)d\alpha d\beta \right|; \quad \textit{applying Hölder's inequality}$$

$$|u(z)| \leq \left(\frac{1}{\pi y^2} \iint_{B(\omega,y)} |u(\omega)|^p d\alpha d\beta\right)^{1/p}$$

$\left(\frac{1}{\pi y^2} \int_0^{2y} \int_{-\infty}^{\infty} |u(\alpha + i\beta|^p d\alpha d\beta\right)^{1/p} \leq \left(\frac{2}{\pi y}\right)^{1/p} Sup_{\beta \succ 0} \left| \int |u(\alpha + i\beta)^p d\alpha d\beta < \infty \right|$ u(z) is bounded in y> y_n >0. For y> 0 such that y> $y_n > 0$ by the above lemma

$$u(\ +i\,y_n) = \int p_y(x-t)u(t+iy_n)dt, \quad if 1 < p \le \infty$$

Let $f_n(t) = u(t+iy_n), f_n(t)$ is bounded in L^p then by the Banach Alague theorem which says the closed unit ball of the dual of a banach space is compact in the weak star topology, $\{f_n\}$ has a weak star accumulation point $f \in L^p$, since Poisson kernels are in $L^q, q = p/p-1$

we have:

$$u(z) = \lim u(z+i\,y_n) = \lim \int p_y(x-t)f_n(t)dt = \int p_y(x-t)\lim f_n(t)dt = \int p_y(x-t)f(t)dt$$

but f$\in L^p$. Therefore u is the Poisson Integra of f.

(b) u is the poisons integral of a finite measure v on R. Then by the minikoski integral inequality we have $\int |u(x+iy)|dx \le \int |v| < \infty$; $Sup_y \int |u(x+iy)dx \le \int \| v \| < \infty$:

The proof of the converse is the same as the converse of (a) except the measures $u(t+i\,y_n)dt$ which have the bounded norms ,converges weak star to finite measure on R i.e. let $d\mu_n(t) = u(t+iy_n)dt$, μ_n has a finite measure and a bounded norms and converges to a finite measure $d\mu(t)$ in the weak star topology .

Hence $u(z) = \lim u(t+iy_n) = \int p_y(x-t)\lim u(t+iy_n) = \int p_y(x-t)d\mu(t)$.

The easiest proof (c) involves mapping H back to D, using the analoge of (b) for harmonic functions on the disc, and then returning to H. A harmonic function u(z) on D is the Poisson integral of a finite measure v in ∂D if and only if $\underset{r}{Sup} \int |v(re^{i\theta})| d\theta < \infty$. The measure v is then the limit of the measures $u(re^{i\theta})/2\pi$ in the weak star topology on measures on ∂D. If $u(z) \ge 0$, then the measures $u(re^{i\theta})d\theta$ are positive and bounded since $$u(0) = \frac{1}{2\pi} \int u(re^{i\theta}) d\theta .$$

And so the limit v exists and v is a positive measure. That proves the disc version of (c) . Now, map D to H by w→z(w) = $\frac{i(1-w)}{1+w}$. The harmonic function u on H is positive is and only if the harmonic function u (z(w)), which is positive , is Poisson integral of a positive measure v on ∂D. Consider first the case when v is supported on the point w = -1 , which corresponds to z = ∞ , then u(z(w)) = v({-1}) $P_w(-1) = v(\{-1\}) \frac{1-|w|^2}{|1+w|^2}$ = v({-1})Imz = v ({-1})y. Now assume v({-1}) = 0 . The map z moves v onto a finite positive measure $\bar{v}$ on R, and for t = $z(e^{i\theta})$. $P_w(\theta) = \pi(1+t^2)P_z(t)$. In this case we have $u(z) = \int P_y(x-t) d\mu(t) \quad where\ \mu = \pi(1+t^2)\bar{v}$.

Part II Sub harmonic function and H^p spaces (Hardy spaces)

2.1 Definitions

Definition: A real valued function f defined in a set G, subset of C is said to be upper semi continuous (u.s.c) in G if

i. $-\infty \le f(x) < \infty$

ii. The set {x : x in G, f(x)<a } is open in G for each a in R . This definition is equivalent to the conditions given x_0 in G and $f(x_0) <$ k there exists δ>0 such that if $|x - x_0| < \delta$ and x in G then f(x) ≤k , indeed if f is upper semi continuous and $f(x_0) < k$ then the set A = $\{x : x \in G, f(x) < k\}$ is open and $x_0 \in A$ hence there exists δ>0 such that the ball $B(x_0, \delta) \subseteq A$ i.e. if $|x - x_0| < \delta$ and x in G then f(x) ≤ k. We can observe that if f is finite then f is continuous on G iff f and –f are u.s.c

We have seen that harmonic functions can be defined in terms of the MVP if we replace equality by inequality in this relation, we obtain the sub harmonic functions.

Definition2: Let G be an open set in the plane a function u is said to be sub harmonic (s.h) on G if

(i) u is u.s.c on G (ii.) For each $z_0 \in G$, there is r = $r(z_0)$>0 such that the ball $B(z_0, r) \subseteq G$ and for every $\rho < r$,

$$u(z_0) \le \frac{1}{2\pi} \int u(z_0 + \rho e^{i\theta}) d\theta$$

Example; every harmonic function is sub harmonic function

Facts:

(i) If f is analytic function on G then $u(z) = \begin{cases} \log|f(z)| & if \;\; f(z) \neq 0 \\ -\infty & if \;\; f(z) = 0 \end{cases}$ is s.h.

Proof It is immediate that u is u.s.c and

$-\infty = u(z_0) \leq \frac{1}{2\pi} \int u(z_0 + re^{i\theta})d\theta$ if f (z) = 0 and if f(z) ≠ 0 then $\log|f(z)|$ is analytic and u(z) = Relog f(z) = log $|f(z)|$ is harmonic . Thus u is sub harmonic in G .

(ii) If u_1, u_2 are sub harmonic in G then u = max { u_1, u_2} is sub harmonic and hence if u is sub harmonic max{u, 0) is sub harmonic . Now let 's state the Jensen's inequality to prove the following theorem

(iii). (Jensen's inequality) ; let (X,μ) be measurable space such that 0<μ(X)<∞. Let v be a real valued μ-integrable function and let φ(t) be a convex function on R Then

$$\varphi\left[\frac{1}{\mu(X)} \int v d\mu\right] \leq \frac{1}{\mu(X)} \int \varphi(v) d\mu$$

Proof the convexity of φ means that φ (t) is the Supremum of functions form a t+b lying below φ(t):

$\varphi(t_0) = Sup\{at_0 + b : at + b \leq \varphi(t), t \in \Re\}$, Whenever at+b≤φ(t) we have

$$a[\frac{1}{\mu(X)} \int v d\mu] + b = \frac{1}{\mu(X)}[a \int v d\mu + \mu(X) b].$$

$$= \frac{1}{\mu(X)} \int [(av + b) d\mu] \leq \frac{1}{\mu(X)} \int \varphi(v) d\mu$$

Taking the Suprumum we get the following $\varphi[\frac{1}{\mu(X)}\int v d\mu] \leq \frac{1}{\mu(X)}\int \varphi(v) d\mu$.

Remark Jensen's inequality is also true if $\int v d\mu = -\infty$ provided that φ is defined at

t = -∞ and increasing on [-∞, ∞).

(iii) If u is u.s.c on a set G , then there exists a decreasing sequence $\{f_n\}$ of continuous function on G Such that $f_n(x) \to f(x)\ as\ n \to \infty$

(iv) If f is u.s.c on a compact set G , then f attains its maximum on G .

Theorem1 . Let v be a s.h function in G and let φ(t) be increasing convex function on [-∞,∞) Continuous at t = - ∞ . Then φov is s.h in G.

Proof Since every convex function is continuous on R , φ is continuous on [∞, ∞). It follows that φov is u.s.c on G , if $z_0 \in G$ and if r <r(z_0), then because φ is increasing ;

$$\varphi(v(z_0)) \leq \varphi[\frac{1}{2\pi}\int v(z_0 + re^{i\theta}) d\theta,$$

by Jensen's inequality:

$$\varphi(v(z_0)) \leq \frac{1}{2\pi}\int (\varphi o v)(z_0 + re^{i\theta}) d\theta$$, Therefore $\varphi o v$ is s.h in G

Corollary1 : If u is s.h in a domain G, then $e^{\lambda u}, for \lambda > 0$ is sub harmonic function .

Proof Since e^x is convex and increasing in (-∞, ∞) , and for λ>0, λu is sub harmonic , then by the above theorem $e^{\lambda u}$ is s.h in G .

Corollary2: If f is analytic in G then $|f|^\lambda$ is s.h.

Proof since f is analytic u = $\log|f|$ is s.h then by corrollary1 $|f|^{\lambda} = e^{\lambda u}$ is s.h in G.

The semi continuity of u in the definition of subhamonic guarantees that u is measurable and bounded above on any compact subset of G . Therefore the integral in the above definition either converges or diverges to -∞ .

Theorem2: let V: G$\rightarrow[-\infty,\infty]$ be an u.s.c function. Then v is s.h in G iff the following conditions holds. If u is a harmonic function on a bounded open set W of G and if

$$\overline{\lim_{z\to\varepsilon}}[v(z)-u(z)]\le 0, \textit{forall}\varepsilon \in \partial W, \text{ Then } v(z)\le u(z), \forall z \in W$$

Proof Assume v is s.h in G .let u and W be as in the above statement. Then $\varphi(z)=v(z)-u(z)$ is s.h in W and $\overline{\lim_{z\to\varepsilon}}\varphi(z)\le 0, \forall\varepsilon\in\partial W$, z in W. We show $\phi(z)\le 0$, on W . Assume W is connected. Let a = $\underset{z\in W}{Sup}\,\varphi(z)$ and suppose a >0.let $\{z_n\}$ be a sequence in W such that $\varphi(z_n)\to a$.. Since a>0 , the z_n can't accumulate on ∂W , and there is a limit point z in W as $\overline{W}$ is compact . By the semi- continuity $\varphi(z)=a$ and the set E=$\{z\in G:\varphi(z)=a\}$ is non-empty. The set E is closed because φ is u.s.c and has a maximum value a.

If $z_0 \in E$, then because $\varphi(z)\le a$ on W , $\varphi(z)=a$ a.e on $B(z_0,r)$ for some r>0. Hence E is dense in $B(z_0,r)$.

Since E is closed this means $B(z_0,r)\subseteq E$ and E is open. Since W is a ssumed t o b e connected, we have a contradiction and we conclude $a\le 0$.

Conversely, let $z_0\in G$ and let $\overline{B}(z_0,r)\subseteq G$. since v is u.s.c there are continuous functions u_n decreasing to v on $\partial B(z_0,r)$ as n$\to\infty$. For each n in N solve the Dirichlet for u_n on $B(z_0,r)$. Let ψ_n be the harmonic function on $B(z_0,r)$ having the same value with u_n on the boundary [ψ_n is the Poisson integral of u_n]. Then ψ_n is continuous on $\overline{B}(z_0,r)$ and by hypothesis $v(z_0) \le \psi_n(z_0)$

Hence we have $v(z_0) \le \lim_{n\to\infty} \frac{1}{2\pi} \int u_n(z_0 + re^{i\theta})d\theta$

$= \frac{1}{2\pi} \int v(z_0 + re^{i\theta})d\theta$ by monotone convergence. Thus we have $v(z_0) \le \frac{1}{2\pi} \int v(z_0 + re^{i\theta})d\theta$, v is sub harmonic at z_0.

Since z_0 is arbitrary v is s.h.

Corollary : If G is a connected open set and if v a sub harmonic function on G such that $v(z) \neq -\infty$, then whenever $\overline{B}(z_0,r) \subseteq G$ then

$$-\infty < \frac{1}{2\pi} \int v(z_0 + re^{i\theta})d\theta.$$

Proof let $u_n(z)$ be continuous functions decreasing to v(z) on $\partial\overline{B}(z_0,r)$, and let $U_n(z)$ denote the harmonic extension of u_n to $B(z_0,r)$. If $\frac{1}{2\pi} \int v(z_0 + re^{i\theta})d\theta = -\infty$, then since v is bounded above and since Poisson kernels are bounded and positive we have $\frac{1}{2\pi} \int P_z(\theta) v(z_0 + re^{i\theta})d\theta = -\infty, |z| < 1$, Consequently $U_n(z) \to -\infty$ for each z in $B(z_0,r)$ and hence by the above theorem $v \equiv -\infty$ on $B(z_0,r)$

The non empty set $\{ z \in G : v(z) \equiv -\infty$ on a neighborhood of $z\}$ is then open and closed, we again have a contradiction.

Definition: The sub harmonic v(z) on G is said to have a harmonic majorant if there is a harmonic function U(z) such that $v(z) \le U(Z)$, for all z in G and the least harmonic majorant u(z) is a harmonic majorant such that $u(z) \le U(z)$ for every harmonic majorant U(z) of v(z).

Thereom3: let v be a sub harmonic function in the unit disc D, Assume v(z) $\neq$ -∞ .

For 0<r<1,let

$$v_r(z) = \begin{cases} v(z) & |z| \geq r \\ \frac{1}{2\pi} \int P_{z/r}(\theta) v(re^{i\theta}) d\theta & |z| < r \end{cases} .$$

$v_r(z)$ is sub harmonic on D and it is harmonic on $|z| < r, v(z) \leq v_r(z), z \in D$ and v_r is an increasing function of r. and

$$v_r(0) = \frac{1}{2\pi} \int v(re^{i\theta}) d\theta .$$

Proof by the above corollary $v_r(z)$ is finite and clearly v_r is harmonic on B (0,r) = {|z|< r}. To see that $v_r(z)$ is upper semi continuous at a point $z_0 \in \partial B(0,r)$.

we must show: $$\overline{\lim_{\substack{z \to z_0 \\ |z|<r}}} v_r(z) \leq v(z_0).$$

This follows from the approximate identity properties of the Poisson kernel and from semi continuity of v . Write $z_0 = re^{i\theta}$, *for* $\varepsilon > 0$, there is δ >0 such that $v(re^{i\theta}) < v(z_0) + \varepsilon$ *if* $|\theta - \theta_0| < \delta$. Then if |z| <r and if $|z - z_0|$ is small,

$$v_r(z) = \frac{1}{2\pi} \int_{|\theta-\theta_0| \leq \delta} P_{z/r}(\theta)(v(z_0 + \varepsilon))\, d\theta + \frac{1}{2\pi}\left(\sup_{\theta} v(re^{i\theta}) \right) \int_{|\theta-\theta_0|>\delta} P_{z/r}(\theta)\, d\theta \leq v(z_0) + 2\varepsilon$$

Hence v_r is upper semi continuous .If we again take continuous functions $u_n(z)$ decreasing to v(z) on $\partial B(0,r)$, then by the same proof of the corollary we have v(z) ≤ $v_r(z)$. Because v is a sub harmonic, this inequality shows the mean value inequality at each point z_0 with $|z_0|$ = r i.e

$$v_r(z_0) \le \frac{1}{2\pi}\int v(z_0 + \rho e^{i\theta})\, d\theta \quad , \rho < r$$

Consequently $v_r(z)$ is a sub harmonic function on D.

Corollary: if v is a sub harmonic function on D , then

$$\text{m(r)} = \frac{1}{2\pi}\int v(re^{i\theta})\, d\theta \text{ is an increasing function of r.}$$

Theoreom4: Let v be a s.h in the unit disc D then v has a harmonic majorant if and only if

$$\sup_{0\le r<1} \frac{1}{2\pi}\int v(re^{i\theta})d\theta = Sup_r v_r(0) < \infty$$

And the least harmonic majorant of v is then

$$\text{u(z)} = \lim_{r\to 1} \frac{1}{2\pi}\int p_{z/r}(\theta)v(re^{i\theta})d\theta = \lim_{r\to 1} v_r(z)$$

Poof Suppose $Sup v_r(0) < \infty$ then by Haranck's theorem the function $v_r(z)$ increases to a finite harmonic function u(z) on D . Then by above theorem v(z)≤ $v_r(z)$ an $v_r(z) \le u(z)$, *and* hence v(z) ≤u(z) ,for all z. Therefore u is a harmonic majorant of v.

Conversely let U be a harmonic on D and let v(z) ≤U(z) , ∀z in D .This implies that $v_r(z) \le U(z), \forall r$. Consequently $Sup_r v_r(0) < \infty$, and again $u(z) = \lim_{r\to 1} v_r(z)$ is finite and harmonic . Since $v_r(z) \le U(z)$, we have u(z) ≤U(z) and hence u is the least harmonic majorant . From continuity $u(z) = \lim_{r\to 1} u(rz)$. The least harmonic majorant of v can also be written:

$$u(z) = \lim_{r\to 1}\int P_z(\theta)v(re^{i\theta})d\theta / 2\pi .$$

Theorem 5: Let v be a sub harmonic function in the upper half plane H.

If $\underset{y}{Sup} \int |v(x+iy)| dx = M < \infty$,

Then v has a harmonic majorant in H of the form $u(z) = \int P_y(x-t)d\mu(t)$ where μ is a finite signed measure on R.

Proof From theorem 5 of first part, we proved this inequality;

$|u(z)| \le (2/\pi y)^{1/p} \underset{\beta>0}{Sup} \|u(x,\beta)\|_{L^P(dx)}$ And similar proof gives us

$v(z) \le \frac{2}{\pi y} \underset{\eta}{Sup} \int |v(\varepsilon + i\eta)| d\varepsilon$, z = x+iy , y>0 for sub harmonic function v in the upper half plane H.

Now fix y>0 and consider the harmonic function

$u(z) = u_{y_0}(z) = \int P_{y-y_0}(z-t)v(t,t_0)dt$ defined on the upper half plane $\{y > y_0\}$

Claim v(z)≤u(z) .on $y > y_0$ .To scc this lct ε >0 and A >0 be large. Let $u_n(t)$ continuous functions decreasing $v(t+iy_0)$ on [-A, A] and let $U_n(z) = \int_{-A}^{A} P_{y-y_0}(x-t)u_n(t)dt, y > y_0$ be the Poisson integral of u_n. The function V(z) $= v(z) - \varepsilon \log|z+i| - U_n(z)$ is sub harmonic on $y > y_0$ with ε fixed we have $\lim_{z\to\infty} V(z) = -\infty$ and by theorem1 and if A is large we have $\overline{\lim_{z\to(t,y_0)}} V(z) \le 0, for |t| \ge A$.

If |t| < A,

$$\overline{\lim_{z\to(t,y_0)}} V(z) \le v(t,y_0) - u_n(t,y_0) \le 0.$$

Then by theorem 4 and a conformal mapping V(z)≤0 on $y > y_0$. Sending n→∞ , then A→∞ and then ε→0, we obtain v(z)≤u(z) on $y > y_0$. The measures $v(t,y_0)dt$ remain bounded as $y_0 \to 0$, and if $d\mu(t)$ is a weak –star cluster point , then

$\lim_{y_0 \to 0} u_{y_0}(z) = \int P_y(x-t)d\mu(t)$ is a harmonic majorant of v(z) and hence the function u is actually the least harmonic majorant.

The classical theory of the Hardy space H^P is a mixture of real and complex analysis. In this short chapter,. We are going to see the sub harmonicity of $|f|^p$ and $\log|f|$ for analytic function f and the use of Blaschke products to reduce the problems to the case of non-vanishing analytic functions. There are two H^P theories one for the disc and for the upper half plane. We introduce these twin theories simultaneously.

Definition: Let 0<p<∞ and let f(z) be analytic function on D,

We say $f \in H^p(D)$, $\quad$ if $\sup_{0 \le r < 1} \frac{1}{2\pi} \int |f(re^{i\theta}|^p d\theta = \|f\|^P{}_{H^P} < \infty$.

If p = ∞ we say $f \in H^\infty$ if f(z) is a bounded analytic on D and we write

$$\|f\|_\infty = \sup_{z \in D} |f(z)|.$$

Example: Let f(z)= z = $re^{i\theta}$, z in D , 0≤r <1 then |f(z)| = r, 0<p<∞.

$$\sup_{0 \le r < 1} \frac{1}{2\pi} \int |f(re^{i\theta})|^p d\theta = \sup_{0 \le r 1} \frac{1}{2\pi} \int r^p d\theta \le 1.$$

Hence $f \in H^p(D)$ and also clearly $f \in H^\infty(D)$. Based on the theorem 4 above the analytic function $f(z) \in H^p(D)$ if and only if the sub harmonic function $|f(z)|^p$ has a harmonic majorant and that for p<∞ , $\|f\|^p{}_{H^p}$ is the value of the least harmonic majorant at z = 0, indeed by theorem4. the least harmonic majorant of $|f(z)|^p$ is of the form $u(z)=\lim_{r\to 1}\frac{1}{2\pi}\int P_{z/r}(\theta)\left|f(re^{i\theta})\right|^p d\theta$, now if $z=0,\ P_{z/r}(\theta)=1$, this implies

that $u(0)=\lim_{r\to 1}\frac{1}{2\pi}\int\left|f(re^{i\theta})\right|^p d\theta$,

But it can be shown that:

$$\sup_{0\le r<1}\frac{1}{2\pi}\int\left|f(re^{i\theta})\right|^p d\theta=\lim_{r\to 1}\frac{1}{2\pi}\int\left|f(re^{i\theta})\right|^p d\theta=\|f\|^p{}_{H^p}=u(0).$$

Definition: Let f be analytic function on the upper half plane H, for 0<p<∞ we say

$$f\in H^P(dt),\quad \text{if } \sup_y\int|F(x+iy)^p\,dx=\|f\|^P H^P<\infty.$$

When p =∞ we write $f\in H^\infty$ for the bounded analytic functions on H , and we give

$$\|f\|_\infty=\sup_H|f(z)|.$$

Note that the definition of $H^p(dt)$ involves all y, 0<y<∞, instead of small values of y, like, say 0<y<1, For example, if $g(z)=\frac{e^{-iz/p}}{(i+z)^{2/p}}$,

$$|g(x+iy)|^p=\frac{\left|e^{y-ix}\right|}{|x+(1+y)i|^2}=\frac{e^y}{x^2+(y+1)^2},$$

This implies that: $\int_{-\infty}^{\infty}\frac{e^y}{x^2+(y+1)^2}dx=\frac{e^y\pi}{y+1}$,

For 0<y<1, $\sup_{0<y<1}\frac{\pi e^y}{y+1}\le\sup_{0<y<1}\pi e^y=\pi e$, but $g\notin H^p(dt)$.

Let $z = \tau(w) = \dfrac{i(1-w)}{1+w}$ be the conformal mapping of D onto H.

Clearly $fo\tau \in H^{\infty}(D)$ if and only if $f \in H^{\infty}(dt)$, indeed, if $fo\tau \in H^{\infty}(D)$ then

$$\|fo\tau\|_{\infty} = \sup_{w\in D}|fo\tau(w)| = \sup_{w\in D}|f(\tau(w))| < \infty$$

But for each z in H, there exists w in D such that τ(w) = z, and hence

$$\sup_{z\in H}|f(z)| = \sup_{w\in D}|f(\tau(w))| < \infty .$$

This implies that $\sup_{z\in H}|f(z)| < \infty$ and hence $f \in H^{\infty}(dt)$. Conversely suppose $f \in H^{\infty}(dt)$, then for all w in D, $\sup|fo\tau(w)| < \infty$ but τ(z) is in H $\sup_{z\in H}|f(z)| < \infty$ and hence $fo\tau \in H^{\infty}(dt)$.

However, for p< ∞, $H^{p}(D)$ and $H^{p}(dt)$ are unfortunately not transformed into each other , for example $H^{p}(D)$ contains non zero constants , but $H^{p}(dt)$ doesn't contain them. In order to treat $H^{p}(D)$ and $H^{p}(dt)$ together we are going to prove the following to lemmas.

Lemma1; if 0<p<∞ and if f∈ $H^{p}(dt)$ then the sub harmonic function $|f(z)|^{p}$ has harmonic majorant u(z) in H and

$$u(i) \le \frac{1}{\pi}\|f\|^{p}{}_{H^{p}}$$

Proof since f∈ $H^{p}(dt)$ then by theorem 5 of part two, $|f|^{p}$ has a harmonic majorant and the least harmonic is of the form $u(z) = \int P_y(x-t)|f(z)|^p\,dt$.

Hence $u(i) = \int \frac{1}{\pi(t^2+1)}|f(t)|^p\,dt \le \frac{1}{\pi}\int |f(t)|^p\,dt$,

This implies that: $u(i) \le \frac{1}{\pi}\int |f(t)|^p\,dt \le \frac{1}{\pi}\|f\|^p$.

Therefore $u(i) \le \frac{1}{\pi}\|f\|^p{}_{H^p(dt)}$

Lemma 2: if 0<p<∞ and if f is analytic function in the upper half plane such that the sub harmonic $|f(z)|^p$ has a harmonic majorant, then

$$F(z) = \frac{\pi^{-1/p}}{(z+i)^{2/p}} f(z)$$

is in $H^p(dt)$ and $\|F\|^p{}_{H^p} \le u(i)$ where u is the least harmonic majorant of $|f(z)|^p$.

Proof let u be the least harmonic majorant of $|f(z)|^p$. The positive harmonic function has the form :

$$u(z) = cy + \int P_y(x-t)\,d\mu(t)$$

Where $c \ge 0$ and μ is a positive measure on R such that $\int (1+t^2)^{-1}d\mu(t) < \infty$.

Consequently $|F(z)|^p = \frac{1}{\pi(x^2+(y+1)^2)}|f(z)|^p \le \frac{1}{\pi(1+x^2)}u(z)$ and hence

$$|F(z)|^p \le \frac{cy}{\pi(1+x^2)} + \frac{1}{\pi(1+x^2)}\int P_y(x-t)\,d\mu(t).$$

Using Fubini theorem

$$\int \left| F(x+iy)^p \, dx \right| \leq \text{cy} + 1/\pi \int (\int \frac{1}{1+x^2} P_y(x-t)dx) d\mu(t)$$

but $\int \frac{1}{1+x^2} P_y(x-t)dx = \frac{y+1}{t^2+(y+1)^2} = P_{y+1}(t)$. This implies that

$$\int \left| F(x+iy) \right|^p dx \leq cy + \int P_{y+1}(t) d\mu(t) = \text{-c} + \text{u}((1+\text{y})\text{i}) < \infty.$$

Taking the suprumum we get $\|F\|^p{}_{H^p} < \infty$. But $\int \left| F(x+iy) \right|^p dx$ is a decreasing function of y. hence $\|F\|^p{}_{H^p} = \lim_{y \to 0} \int \left| F(x+iy) \right|^p dx$.

From this we have:

$$\|F\|^p{}_{H^p} \leq u(i).$$

For p≥1, H^p is a normed linear space, for p<1, the inequality $|z_1 + z_2|^p \leq |z_1|^p + |z_2|^p$ shows that H^p is a metric space with metric

$$\text{d(f,g)} = \|f - g\|^p{}_{H^p}.$$

Theorem 6: for 0<p≤∞, H^p is complete.

Proof Assume p<∞ , we give the proof in the upper half plane ; the reasing for the disc is very similar .

We already proved that, for each y>0 $|f(x+iy)| \leq (2/\pi y)^{1/p} \|f\|_{H^p}$, this shows that any H^p cauchy sequences $\{f_n\}$ converges point wise on H to analytic function f(z)

By Fatou's lemma:

$$\int |f(x+iy) - f_n(x+iy)|^p dx \leq \underline{\lim}_{m\to\infty} \int |f_m(x+iy) - f_n(x+iy)|^p dx \leq \underline{\lim}_{m\to\infty} \|f_m - f_n\|^p{}_{H^p}$$

Hence $\|f - f_n\|^p{}_{H^p} \leq \underline{lim}_{m\to 0} \|f_m - f_n\|^p{}_{H^p}$.

Therefore H^p is a complete space.

2.3 Blaschke Products

Definition: A finite Blaschke product is a function of the form

$$\text{B(z)} = e^{i\varphi} \prod_{j=1}^{n} \frac{z - z_j}{1 - \bar{z}_j z}, \ |z_j| < 1 \ \ j = 1,2\ldots n.$$

The function B has the properties:

(i) B is continuous across ∂D.

(ii) |B| = 1 on ∂D, indeed since $|z_1 z_2| = |z_1||z_2|$. We have

$$|\text{B(z)}| = \prod \frac{|z - z_j|}{|1 - \bar{z}_j\ z|} \ \ but \ \ \frac{|z - z_j|}{|1 - \overline{z}_j z|} = 1 \ \ \forall z \in \partial D \ \ and \ |z_j| < 1.$$ Hence |B| = 1

(iii) B has finitely many zeros in D.

These properties determine B up to a constant factor of modulus one, indeed, if an analytic function f(z) has (i)→(iii) , and if B(z) is the finite Blashke Product with the same zeros , then by the maximum principle |f/B|≤ 1 and |B/f| ≤1 on D , and so f/B is constant . The degree of B is its number of zeros. A Blashke product of degree 0 is a constant function of absolute value 1.Now we are going to see $\{z_n\}$ of a non zero H^p function on the disc satisfy Blaschke's condition:

$$\sum(1-|z_n|)<\infty.$$

Theorem1: Let f be an analytic function on the disc, f≠ 0 and let $\{z_n\}$ be the zeros of f(z) . If $\log|f(z)|$ has a harmonic majorant, then

$$\sum(1-|z_n|)<\infty.$$

If f(0) ≠0 and if u(z) is the least harmonic majorant of $\log|f(z)|$, then

$$\sum(1-|z_n|)\le u(0)-\log|f(0)|.$$

Proof Assume $f(0)\neq 0$, then by theorem3 $\underset{r}{Sup}\frac{1}{2\pi}\int\log|f(re^{i\theta})|d\theta<\infty$ and if u is the least harmonic majorant of log|f(z)| then

$$u(0)=\underset{r}{Sup}\frac{1}{2\pi}\int\log|f(re^{i\theta})|d\theta=\lim_{r\to 1}\frac{1}{2\pi}\int\log|f(re^{i\theta})|d\theta.$$

Fix r<1 so that $|z_n|\neq$ r for all n, and let z_1 , z_2 z_3,...z_n be those zeros with $|z_j|<r$. Then f(rz) has zeros z_1/r , z_2/r ,...z_n/r.

Let $B_r(z)=\prod_j^n\frac{(z-z_j/r)}{(1-\bar{z}_j\,z/r)}$, a finite Blaschke product with the same zeros as f(rz). And let g(z) $=f(rz)/B_r(z)$, g is analytic and zero free on $\bar{D}$, so that

$$\log|g(0)| = \frac{1}{2\pi}\int \log|g(e^{i\theta})|d\theta .$$

Since $|g(e^{i\theta})| = \frac{|f(re^{i\theta})|}{|B_r(z)|} = |f(re^{i\theta})|$. This gives the familiar Jensen

formula $$\log|f(0)| + \sum_{|z_j|<r} \log\frac{r}{|z_j|} = \frac{1}{2\pi}\int \log|f(re^{i\theta})|d\theta .$$

Letting r tend to 1 yields:

$$\sum \log\frac{1}{|z_j|} \le \lim_{r\to 1}\frac{1}{2\pi}\int \log|f(re^{i\theta})|d\theta - \log|f(0)| = u(0) - \log|f(0)| .$$

But $1-|z_j| \le \log\frac{1}{|z_j|}$ $\forall j$, this implies that:

$$\sum_j^n (1-|z_j|) \le \sum_{\substack{j \\ |j|<r}}^n \frac{1}{|z_j|} \le u(0) - \log|f(0)| \text{ and hence } \sum(1-|z_j|) < \infty .$$

If $f \in H^p(D)$ then $\log|f| \le \frac{1}{p}|f|^p$ and log|f| has a harmonic majorant, hence if

$f \in H^p(D)$ or if f(w) = F(z(w)), $F \in H^p(dt)$ *then* $\sum(1-|z_n|) < \infty$

Theorem2: Let $\{z_n\}$ be a sequence of points in D such that $\sum(1-|z_n|) < \infty$. Let m be the number of z_n equal to 0, Then the Blaschke product

$$B(z) = z^m \prod_{|z_n| \neq 0} \frac{-\bar{z}_n}{|z_n|} \frac{z - z_n}{1 - \bar{z}_n z} \text{ converges on D .} \qquad (*)$$

The function B(z) is in $H^p(D)$ and the zeros of B(z) are precisely the points z_n, each zero having multiplicity equal to the number of times it occurs in the sequence $\{z_n\}$. Moreover $|B(z)| \leq 1$ *and* $|B(e^{i\theta})| = 1$ almost everywhere.

Proof. By definition a Blaschke product on D is a function of the form (*) above.

Suppose $|z_n| > 0$ *for all n let* $b_n(z) = \dfrac{\bar{z}_n \, z - z_n}{|z_n|(1 - \bar{z}_n \, z)}$.

Claim: $\prod b_n$ converges on D.

$\prod b_n$ converges on D to analytic function having $\{z_n\}$ for zeros if and only if $\sum |1 - b_n(z)|$ converges uniformly on each compact subset of D .

By calculation:

$$|1 - b_n(z)| = \frac{|z_n + z \,|\, z_n \,||}{|\, z_n \,|(|1 - \bar{z}_n z)}(1 - |\, z_n \,|) \leq \frac{1 + |\, z \,|}{1 - |\, z \,|}(1 - |\, z_n \,|) < \infty .$$

From this we get that $|1 - b_n(z)| \leq \dfrac{1 + |\, z \,|}{1 - |\, z \,|}(1 - |\, z_n \,|)$ and therefore

$$\sum |\,(1 - b_n(z)\,| \leq \frac{1 + |\, z \,|}{1 - |\, z \,|} \sum (1 - |\, z_n \,|) < \infty .$$

This means that $\sum |\,(1 - b_n(z)\,|$ converges uniformly on each compact sub set of D. Then $\prod b_n$ converges on D.Since $|\, b_n(z) \,| \leq 1$, it is clear that $B(z) \in H^\infty(D)$ and $|B(z)| \leq 1$. The bounded harmonic function B(z) has non tangential limits $|B(e^{i\theta})| \leq 1$ *almost everywhere* .To see $|B(e^{i\theta})| = 1$ *a.e.*

Let $B_n(z) = \prod_{k=1}^{n} b_k(z)$. Then B/B_n is an other Blaschke Product and

$$\left|\frac{B(0}{B_n(0)}\right| \le \frac{1}{2\pi}\int\frac{|B(e^{i\theta})|}{|B_n(e^{i\theta})|}d\theta = \frac{1}{2\pi}\int|B(e^{i\theta})|\,d\theta\ .$$

letting $n \to \infty$ *we get* $\frac{1}{2\pi}\int|B(e^{i\theta})|\,d\theta = 1$

Hence $|B(e^{i\theta})| = 1$ $a.e.$

Based on the above two theorems we conclude that if f is analytic function on D then f(z) has a factorization f(z) = B(z)g(z) where B(z) is a Blaschke product and g has no zeros on D if and only log|f(z)| has a harmonic majorant .

Theorem 3: (F. Riesz) let 0<p< ∞ let f∈ $H^p(D)$, $f \ne 0$ *let* $\{z_n\}$ be the zeros of f(z) and let B(z) be the Blaschke product with zeros $\{z_n\}$. Then g(z) = $\frac{f(z)}{B(z)}$ is in

$$H^p(D)\ \ and\ \ \|g\|_{H^p} = \|f\|_{H^p}\ .$$

Proof if f∈ H^p then B (z) converges. Let B_n be the finite Blashcke product with zeros

z_1 , z_2 , z_3 ,… z_n *and let* $g_n = f/B_n$, $g_n \to g$ *as* $n \to \infty$ *and* g_n *is* increasing function of r.

Fix r< 1. Then

$$\int|g_n(re^{i\theta})|^p d\theta/2\pi \le \lim_{R\to 1}\int\frac{|f(\mathrm{Re}^{i\theta})|^p}{|B_n(\mathrm{Re}^{i\theta})|^p}\frac{d\theta}{2\pi} \quad but \quad \lim_{R\to 1}|B_n(\mathrm{Re}^{i\theta})| = |B_n(e^{i\theta})| = 1.$$

If 1-R is small then:

$\left|B(\mathrm{Re}^{i\theta})\right| > 1-\varepsilon$ *so that* $\int\left|g_n(re^{i\theta})\right|^p \frac{d\theta}{2\pi} \le \lim_{R\to 1}\int\left|f(\mathrm{Re}^{i\theta})\right|\frac{d\theta}{2\pi} = \|f\|^p{}_{H^p}$. This implies that

$$\|g\|^p{}_{H^p} = \lim_{n\to\infty}\int\left|g_n(re)\right|^p d\theta \le \|f\|^p{}_{H^p} \quad \text{hence } \|g\| \le \|f\| \qquad (1) .$$

Since $|g_n|$ is increasing to $|g|$ and $|g| \ge |f|$ we have $\|f\|_{H^p} \le \|g\|_{H^p}$ (2) .

From (1) and (2) we get $\|f\|_{H^p} = \|g\|_{H^p}$

The above theorem is also true for $f \in H^p(dt)$ because the proof of theorem 5 shows that $\sup_y \int\left|f(x+iy)\right|^p dx = \lim_{y\to 0}\int\left|f(x+iy)\right|^p dx$.

Blaschke products have a simple characterization in terms of harmonic majorants.

Theorem 4: $f \in H^\infty(D), \|f\| \le 1$. The following are equivalent.

(a) f(z) = λB(z) , where λ is a constant , |λ| = 1 and B(z) is a Blaschke product

(b) $\lim_{r\to 1}\int\log\left|f(re^{i\theta})\right|\frac{d\theta}{2\pi} = 1$

(c) The least harmonic majorant of log|f(z)| is 0.

Proof by theorem 4 (b) and (c) are equivalent. Suppose that f(z) is the Blaschke product with zero $\{z_n\}$, and let ε >0 , we may divided f(z) by a finite Blaschke product $B_n(z)$ so that $\left|(\frac{f}{B})(0)\right| > 1-\varepsilon$ .Since B_n is continuous on $\overline{D}$ and $\left|B_n(e^{i\theta})\right| = 1$.

$\lim_{r\to 1}\int\log\left|f(re^{i\theta})\right| d\theta = \lim_{r\to 1}\int\log\left|\frac{f}{B_n}(re^{i\theta})\right| d\theta$. But since $\log\left|\frac{f}{B_n}\right|$ is sub harmonic and negative log (1-ε) $\le \int\log\left|\frac{f}{B_n}(re^{i\theta})\right|\frac{d\theta}{2\pi} \le 0$. Hence (b) holds. Suppose (c) holds.

Let g(z) =f(z)/B(z), where B(z) is the Blashcke product formed from the zeros of f(z) . Then because $\|f\| \leq 1$, $\log|f(z)| \leq \log|g(z)| \leq 0$. Since log|g(z)| is a harmonic majorant of log |f(z)| , (c) implies that log |g(z)| = 0. Hence g(z) = λ where λ is a constant and |λ| = 1, and so (a) holds.

Reference

1. J. B Garnett, Bounded analytic functions
Academic Press, Inc, 1981

1 J.B Conway, functions of one complex variables
Spring-Verlag in New York, 1981.

2 Rudin, Real and complex analysis, McGraw- Hill, Inc, 1966.

3 Petter L. Duren, Theory of $H^p spaces$, Academic press, Inc, 1970.