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COLLEGE OF NATURAL AND COMPUTATIONAL SCIENCE
DEPARTMENT OF MATHEMATICS

ON STURM-LIOUVILLE THEORY AND ITS APPLICATIONS

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A Thesis Submitted in Partial Fulfillment of the Requirement for the Degree of Master of Science in Mathematics

ADDIS ABABA UNIVERSITY
DEPARTMENT OF MATHEMATICS

The undersigned hereby certify that they have read and recommend to the department of Mathematics for acceptance of this project entitled **"On Sturm-Liouville Theory and Its Applications "** by **Wondimagegnehu Sahelu** in partial fulfillment of the requirements for the degree of Master of Science in Mathematics (Differential Equation).

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Acknowledgments

First,I would like to express my deepest gratitude to God for giving me patience; I am also grateful to my advisor Dr.TSEGAYE GEDIF for his helpful discussion, comments and providing the necessary materials in the preparation of this paper.

Secondly, I would like to extend my thank to my Brother ASIKE KAWUSA (Urban Engineer, MSC) by giving computer and help by hosting and all who encouraged me to complete my project and their heart full help while writing the paper. At the last ,but not the least I would like thanks to Department of mathematics, Addis Ababa University Dr. TILAHUN ABEBAW giving training to write in latex with a great endurance and Dr.TADESSE ABDI for giving the necessary materials to the preparation of this paper.

Abstract

In this paper we present Inner product space ,orthogonality ,series ,Fourier series ,parameter form of Bessel equation and the Sturm-Liouville Theory to find eigenvalue and eigenfunction ,eigenfunction expansion and at last apply Sturm-Liouville Theory to the oscillation of hanging chain ,elastic vibrations and buckling beams.

Notations

Through out this project we will use the following notations:

1) C=complex numbers.

2) C^1=$Continuous\ Differentiable\ functions$.

3) $C^0 = Continuousfunctions$

4) $\Re^n$ =n-tuple of Real number.

5) C^n =n-tuple of complex numbers.

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Introduction

Our development of the theory of Sturm-Liouville in this paper involve some from linear algebra that are some what abstract but simple to explain in the context of a finite dimensional vector space. For this reason we start our discussion by reviewing some basic concepts from linear algebra.

We will encounter ordinary differential equations Bessel's equations,whose solutions are called Bessel functions. The full implementation of the separation of variables method will leads us to study expansions of functions in terms of Bessel functions.

The main focus of this paper is to develop a general theory that encompasses the specific expansions theorems considered previously Bessel expansions. This theory is named after Sturm and Liouville , who developed it early part of the nineteenth century.

Beyond its esthetic appeal, Sturm-Liouville theory has many applications in applied mathematics , physics and engineering . We will use it to obtain further expansion results rather than developing them a case by case basis. Some classical applications are presented in this paper including the problems of hanging chain and the vibrating beam.

Chapter 1

Preliminaries

1.1 Inner product space

Definition 1. *Let X be a vector space over F . A function $X \times X \longrightarrow F$ is called an inner product in X, if for any pair $x, y \in X$, the inner product $(x, y) \mapsto \langle x, y\rangle \in F$ satisfies the following conditions .*

1. $\langle x, y\rangle = \overline{\langle y, x\rangle}$ for all $x, y \in X$
2. $\langle ax + by, z\rangle = a\langle x, z\rangle + b\langle y, z\rangle$ for all $a, b \in F$ $x, y, z \in X$
3. $\langle x, y\rangle \geq 0$ for all $x, y \in X$
4. $\langle x, x\rangle = 0 \Longleftrightarrow x = 0$

A vector space on which an inner product is defined is called an inner product space.
The symbol in $\overline{\langle y, x\rangle}$ in (1) denotes the complex conjugate of $\langle y, x\rangle$, so that $\langle x, y\rangle = \langle y, x\rangle$ if X is a real vector space. Note also that (1) and (2) imply $\langle x, ay\rangle = \overline{\langle ay, x\rangle} = \overline{a}\langle x, y\rangle$.
* We now define the norm of a vector x as $\|x\| = \sqrt{\langle x, x\rangle}$.
* In $\Re^n$ we define the inner product of the vectors
$x = (x_1, x_2, ..., x_n)$, $y = (y_1, y_2, ..., y_n)$ by

$$\langle x, y\rangle = x_1y_1 + x_2y_2 + ... + x_ny_n \tag{1.1.1}$$

which implies $\|x\| = \sqrt{{x_1}^2 + {x_2}^2 + ... + {x_n}^2}$.
* In C^n we define

$$\langle z, w\rangle = z_1\overline{w_1} + z_2\overline{w_2} + ... + z_n\overline{w_n} \tag{1.1.2}$$

For any pair $z, w \in C^n$. Consequently,
$\|z\| = \sqrt{|z_1|^2 + |z_2|^2 + ... + |z_n|^2}$
A natural choice for the definition of inner product on C([a,b]),by analogy with (1.1.1) is

$$\langle f, g\rangle = \int_a^b f(x)\overline{g(x)}dx \quad f, g \in C([a, b]) \tag{1.1.3}$$

Sothat $\|f\| = [\int_a^b |f(x)|^2 dx]^{\frac{1}{2}}$

Definition 2. *(i) A pair of non-zero vectors x and y in the inner product space X is said to be orthogonal if $\langle x, y\rangle = 0$. Symbolically expressed by writing $x \perp y$. A set of non-zero vectors ν in X is orthogonal if every pair in ν is orthogonal.*
(ii) An orthogonal set $\nu \subseteq X$ is said to be orthonormal if $\|x\| = 1$ for every $x \in \nu$.

1.1.1 Inner products and orthogonality of functions

In defining inner product of two vectors, we summed the product of their components. To define the inner product of two functions; we will integrate their product as follows. Let f and g be real-valued functions defined on an interval (a, b). The inner product of f and g denoted by (f, g), is the number $(f, g) = \int_a^b f(x)g(x)dx$. If f and g are complex valued, then define

$$(f, g) = \int_a^b f(x)\overline{g(x)}dx \tag{1.1.4}$$

The same properties as the inner product vector space (1.1.4) satisfies.

1. $(af, g) = a(f, g)$, for any number a.
2. $(f + g, h) = (f, h) + (g, h)$.
3. $(f, f) \geq 0$, for any function f.

Definition 3. *The functions* f *and* g *are called the orthogonal on the interval* (a, b) *if* $(f, g) = \int_a^b f(x)g(x)dx = 0$. *In the case of complex-valued functions* $(f, g) = \int_a^b f(x)\overline{g(x)}dx = 0$. *We define the norm of* f, *denote* $\|f\|$, *by* $\|f\| = \sqrt{(f, f)} = [\int_a^b |f(x)|^2 dx]^{\frac{1}{2}}$

Note how both definitions, othogonality and norm, are based on the notion of inner products as they were in the finite dimensional case.
A set of functions $\{f_1, f_2, f_3, ...\}$ defined on the interval (a, b) is called an orthogonal set if $\|f_n\| \neq 0$ for all n ,and each distinct pair of functions from the set is orthogonal , that is $(f_n, f_m) = 0$ for $n \neq m$. If, in addition, the norm of each f_n is 1, the set is called an orthonormal set. Hence, if we divide each function in an orthogonal set by its norm we obtain an orthonormal set.

Example 1. *Orthogonal functions*
Show that the sequence of functions $f_n(x) = \sin nx$ $(n = 1, 2, 3, ...)$ *is orthogonal* $[-\pi, \pi]$ *on the interval and obtain the corresponding orthonormal set.*

solution 1. *We need to show that* $\|\sin nx\| \neq 0$ *and each distinct pair of functions in the given set is orthogonal. We have*

$$\|\sin nx\|^2 = \int_{-\pi}^{\pi} \sin^2 nx dx = \int_{-\pi}^{\pi} \frac{1 - \cos 2nx}{2} dx = \pi$$

Thus the norm of $\sin nx$ *is* $\|\sin nx\| = \sqrt{\pi} \neq 0$.
For $m \neq n$, *we have*

$$\int_{-\pi}^{\pi} \sin mx \sin nx dx = \frac{1}{2}[\int_{-\pi}^{\pi} \cos{(m-n)x} - \cos{(m+n)x} dx] = 0$$

The corresponding orthonormal set is $\frac{\sin x}{\sqrt{\pi}}, \frac{\sin 2x}{\sqrt{\pi}}, \frac{\sin 3x}{\sqrt{\pi}}, ...$

1.1.2 Generalized fourier series

Theorem 1.1.1. *Generalized fourier series*
If $f_1, f_2, f_3, ...$ *is a set of orthogonal functions on* (a, b) *and if* f *can be represented as a series in the form*

$$f(x) = \sum_{j=1}^{\infty} a_j f_j(x) \tag{1.1.5}$$

then

$$f(x) = \sum_{j=1}^{\infty} \frac{(f, f_j)}{\|f_j\|^2} f_j(x) \tag{1.1.6}$$

Proof. Let A be a class of functions on (a, b) . Given $f_1, f_2, f_3, ...$ is an orthogonal set of function A and a function f in A. From the hypothesis f expressed as a series of the form

$$f(x) = \sum_{j=1}^{\infty} a_j f_j(x)$$

multiplying both sides of (1.1.5) by f_k and integrate term by term on the interval (a, b).
This gives

$$\int_a^b f(x) f_k(x) dx = \sum_{j=1}^{\infty} a_j \int_a^b f_j(x) f_k(x) dx$$

Because of orthogonality, the k^{th} term is the only non-zero term on the right side and so $\int_a^b f(x) f_k(x) dx = a_k \int_a^b f_k^2(x) dx$. The left side is the inner product of f and f_k and the integral on the right side is the square of the norm of f_k, so $(f, f_k) = a_k \|f_k\|^2$. Thus

$$a_k = \frac{(f, f_k)}{\|f_k\|^2} = \frac{1}{\|f_k\|^2} \int_a^b f(x) f_k(x) dx \tag{1.1.7}$$

Hence

$$f(x) = \sum_{j=1}^{\infty} \frac{(f, f_j)}{\|f_j\|^2} f_j(x)$$

□

1.1.3 Orthogonality with respect to a weight

If f and g are real-valued functions on (a, b), we define their inner product with respect to the weight w to be the number

$$(f, g) = \int_a^b f(x) g(x) w(x) dx$$

We assume that $w(x)$ is a non-negative piecewise continuous function on $[a,b]$ that is not identically 0 on any sub-interval of $[a,b]$. The corresponding definition of orthogonality is as follows

Definition 4. *The functions f and g are orthogonal with respect to the weight function w on the interval $[a,b]$ if $(f,g)=\int_a^b f(x)g(x)w(x)dx=0$. The norm of f with respect to the weight w is*

$$\|f\| = [\int_a^b |f(x)|^2 w(x)dx]^{\frac{1}{2}} \tag{1.1.8}$$

The following is analog of theorem (1.1.1) for expansions in series with respect to functions that are orthogonal with respect to a weight.

Theorem 1.1.2. *If $f_1, f_2, f_3, \dots$ is a set of orthogonal functions with respect to a the weight w on $[a,b]$ and if f can be represented as a series in the form (1.1.5), then*

$$f(x) = \sum_{j=1}^{\infty} \frac{(f, f_j)}{\|f_j\|^2} f_j(x) \tag{1.1.9}$$

We end this section with a statement of Parseval's identity for complete orthogonal systems.

Theorem 1.1.3. *Parseval's identity*
Let $f_1, f_2, f_3, \dots$ be a complete set of orthogonal functions with respect to the weight w on $[a,b]$ and let f be such that its norm as given by (1.1.8) is finite (That is, f is square integrable with respect to the w.) Then

$$\int_a^b |f(x)|^2 w(x)dx = \sum_{j=1}^{\infty} \frac{|(f, f_j)|^2}{\|f_j\|^2} \tag{1.1.10}$$

1.2 Bessel's Equation and Bessel Functions

We will see in this section that Bessel's equation of order $p \geq 0$,

$$x^2 y^{''} + xy^{'} + (x^2 - p^2)y = 0, x > 0 \tag{1.2.1}$$

Note that Bessel's equation is a whole family of differential equations, one of each value of p. Note also the unfortunate clash of terminology - Bessel's

equation of order p is a differential of order 2.
Historically, the equation with $p = 0$ was first encountered and solved by Daniel Bernoulli in 1732 in his study of the hanging chain problem. Similar equation appeared later in 1770 in the work of Lagrange on astronomical problems .In 1824, while investigating the problem of elliptic planetary motion, the great German astronomer F.W.Bessel encountered a special form of (1.2.1). Influenced by the monumental work of Fourier that had just appeared in 1822, Bessel conducted a systematic study of (1.2.1).

Solution of Bessels Equation
We will apply the method of Frobenius. It is easy to show that $x = 0$ is a singular point of Bessels equation. So, as suggested by the method of Frobenius, we try for a solution.

$$y = \sum_{m=0}^{\infty} a_m x^{r+m} \tag{1.2.2}$$

Where $a_0 \neq 0$. substituting this into (1.2.1) yields

$$\sum_{m=0}^{\infty} a_m (r+m)(r+m-1)x^{r+m} + \sum_{m=0}^{\infty} a_m (r+m)x^{r+m} +$$

$$\sum_{m=2}^{\infty} a_{m-2} x^{r+m} - p^2 \sum_{m=0}^{\infty} a_m x^{r+m} = 0$$

writing the terms corresponding to m=0 and m=1 separately gives

$$a_0(r^2-p^2)x^2 + a_1[(r+1)^2-p^2]x^{(r+1)} + \sum_{m=2}^{\infty}(a_m[(r+m)^2-p^2]+a_{m-2})x^{r+m} = 0$$

Equating coefficients of the series to zero gives

$$a_0(r^2-p^2) = 0(m=0) \tag{1.2.3}$$

$$a_1[(r+1)^2-p^2] = 0(m=1) \tag{1.2.4}$$

$$a_m[(r+m)^2-p^2] + a_{(m-2)} = 0(m \geq 2) \tag{1.2.5}$$

From (1.2.3), since $a_0 \neq 0$, we get the indicial equation $(r+p)(r-p) = 0$ with indicial roots $r = p$ and $r = -p$.

First solution on Bessel's Equation
Setting r =p in (1.2.5) gives the recurrence relation

$$a_m = \frac{-1}{m(m+2p)} a_{m-2}, m \geq 2$$

(1.2.4) becomes

$$a_1[(p+1)^2 - p^2] = 0$$

which implies that $a_1 = 0$ (recalling that $p \geq 0$ in (1.2.1)), and so $a_3 = a_5 = \cdots = 0$. To make it easier to find a pattern for the even indexed terms we rewrite the recurrence relation with $m = 2k$ and get

$$a_{2k} = \frac{-1}{2^2 k(k+p)} a_{2(k-1)}, k \geq 1$$

This gives

$$a_2 = \frac{-1}{2^2(1+p)} a_0,$$

$$a_4 = \frac{-1}{2^2 2(2+p)} a_2 = \frac{1}{2^4 2!(1+p)(2+p)} a_0$$

$$a_6 = \frac{-1}{2^2 3(3+p)} a_4 = \frac{-1}{2^6 3!(1+p)(2+p)(3+p)} a_0$$

and so on. Substituting these coefficient into (1.2.2) gives one solution to Bessel's equation

$$y = a_0 \sum_{k=0}^{\infty} \frac{(-1)^k}{2^{2k} k!(1+p)(2+p)...(k+p)} x^{2k+p} \tag{1.2.6}$$

Where $a_0 \neq 0$ is arbitrary. We choose

$$a_0 = \frac{1}{2^p \Gamma(p+1)}$$

Definition 5. *The gamma function is defined for $x > 0$ by*

$$\Gamma(x) = \int_0^\infty t^{x-t} e^{-t} dt.$$

This integral is improper and converges for all $x > 0$.The basic property of the gamma function is

$$\Gamma(x+1) = x\Gamma(x).$$

and simplify the terms in the series using the basic property of the gamma function, as follows:

$$\begin{aligned}\Gamma(1+p)[(1+p)(2+p)...(k+p)] &= \Gamma(2+p)[(2+p)...(k+p)\\ &= \Gamma(3+p)[(3+p)...(k+p)]\\ &= ... = \Gamma(k+p+1)\end{aligned}$$

After this simplification, (1.2.6) yields the first solution, denoted by J_P and called the Bessel function of order p,

$$J_p(x) = \sum_{k=0}^{\infty} \frac{(-1)^k}{k!\Gamma(k+p+1)} (\frac{x}{2})^{2k+p} \tag{1.2.7}$$

When $p = n$, we have $\Gamma(k+p+1) = (k+n)!$, and so the Bessel function of order n is

$$J_n(x) = \sum_{k=0}^{\infty} \frac{(-1)^k}{k!(k+n)!} (\frac{x}{2})^{2k+n}$$

1.3 Bessel series expansions

In this section we explore some recurrence relations, orthogonality properties of Bessel functions and expansions of functions in Bessel series. Many of these properties are used in solving the boundary value problems.

Identities Involving Bessel Functions
We start with two basic identities. For any $p \geq 0$

$$\frac{d}{dx}[x^p J_p(x)] = x^p J_{p-1}(x) \tag{1.3.1}$$

$$\frac{d}{dx}[x^{-p} J_p(x)] = -x^{-p} J_{p+1}(x) \tag{1.3.2}$$

Note that for p = 0, the second identity yields

$$\frac{d}{dx}[J_0(x)] = -J_1(x)$$

Many other useful identities follow from (1.3.1) and (1.3.2). We list some of the most commonly used ones:

$$xJ_p^{'}(x) + pJ_p(x) = xJ_{p-1}(x) \tag{1.3.3}$$

$$xJ_p^{'}(x) - pJ_p(x) = -xJ_{p+1}(x) \tag{1.3.4}$$

$$J_{p-1}(x) - J_{p+1}(x) = 2J_p^{'}(x) \tag{1.3.5}$$

$$J_{p-1}(x) + J_{p+1}(x) = \frac{2p}{x}J_p(x) \tag{1.3.6}$$

To prove (1.3.3), we expand the left side of (1.3.1) using the product rule and get

$$x^pJ_p^{'}(x) + px^{p-1}J_p(x) = x^pJ_{p-1}(x)$$

Dividing through by x^{p-1} gives (1.3.3). Identity (1.3.4) is proved similarly by starting with (1.3.2) and expanding using the product rule. Adding (1.3.3) and (1.3.4) and simplify yields (1.3.5). Subtracting (1.3.4) from (1.3.3) and simplifying yields (1.3.6).
There are similar identities involving integrals of Bessel functions. For example, the identities

$$\int x^{p+1}J_p(x)dx = x^{p+1}J_{p+1}(x) + c \tag{1.3.7}$$

$$\int x^{-p+1}J_p(x)dx = -x^{-p+1}J_{p-1}(x) + c \tag{1.3.8}$$

1.3.1 Orthogonality of Bessel Functions

To understand the orthogonality relations of Bessel functions, let us recall the familiar example of the functions $sinn\pi x$,$n = 1, 2, 3, ...$ We know that these functions are orthogonal on the interval $[0, 1]$, in the sense that $\int_0^1 sinn\pi x sinm\pi x dx = 0$ if $n \neq m$. In constructing systems of orthogonal Bessel functions, we will proceed in a similar way by using a single Bessel function and its zeros. Fix an order $p \geq 0$, the Bessel function J_p has infinitely many zeros on the positive x-axis $x > 0$ (just like $sinx$). We denote these zeros in ascending order $0 < \alpha_{p1} < \alpha_{p2} < ... < \alpha_{pj} < ....$ Hence α_{pj} denotes the j^{th} positive zero of J_p.

Unlike the case of the sine function, where the zeros are easily determined by $n\pi$, there is no formula for the positive zeros of the Bessel functions. Since the numerical values of these zeros are very important in applications, they are found in most mathematical tables and computer systems. For later use, we list in table 1 the first five positive zeros of J_0, J_1 and J_2.

j	1	2	3	4	5
α_{0j}	2.40483	5.52008	8.65373	11.7915	14.9309
α_{1j}	3.83171	7.01559	10.1735	13.3237	16.4706
α_{2j}	5.13562	8.41724	11.6198	14.796	18.9801

Table1: positive zeros of J_0 **,**J_1 **and** J_2.

Let a be a positive number. To generate orthogonal functions on the interval $[0, a]$ from J_p, we proceed as in the case of the sine function, using α_{pj}, the zeros of the Bessel function. We obtain the functions

$$J_p(\frac{\alpha_{pj}}{a}x), j = 1, 2, 3, ... \tag{1.3.9}$$

To simplify the notation, we let

$$\lambda_{pj} = \frac{\alpha_{pj}}{a}, j = 1, 2, 3, ... \tag{1.3.10}$$

So λ_{pj} is the value of the j^{th} positive zero of J_p scaled by a fixed factor $\frac{1}{a}$. We are now in a position to state some fundamental identities.

Theorem 1.3.1. ***Orthogonality of Bessel functions with respect to a weight***

Fix $p \geq 0$ *and* $a > 0$. *Let* $J_p(\lambda_{pj}x)(j = 1, 2, ...)$ *be as in (1.3.9) and (1.3.10). Then*

$$\int_0^a J_p(\lambda_{pj}x)J_p(\lambda_{pk}x)xdx = 0 \ for \ j \neq k \tag{1.3.11}$$

and

$$\int_0^a J_p^2(\lambda_{pj}x)xdx = \frac{a^2}{2}J_{p+1}^2(\alpha_{pj}) \ for \ j = 1, 2, ... \tag{1.3.12}$$

Note that (1.3.12) involvesλ_{pj} and α_{pj}. Property (1.3.11) is described by saying that the functions $J_p(\lambda_{pj}x)$, j =1, 2 , . . . are orthogonal on the interval $[0, a]$ with respect to the weight x. The phrase "with respect to the weight x " refers to the presence of the function x in the integrand in (1.3.11).

On the interval $[0, 1]$, that is when $a = 1$ formulas (1.3.11) and (1.3.12) take on a simpler form

$$\int_0^1 J_p(\alpha_{pj}x)J_p(\alpha_{pk}x)xdx = 0 \text{ for } j \neq k \tag{1.3.13}$$

$$\int_0^a J_p^2(\alpha_{pj}x)xdx = \frac{1}{2}J_{(p+1)}^2(\alpha_{pj}) \text{ for } j = 1, 2, .. \tag{1.3.14}$$

1.3.2 Bessel series and Bessel-Fourier coefficients

Now we will see how we can expand functions using Bessel series. More precisely, a given function f on the interval $[0, a]$ can be expressed as a series

$$f(x) = \sum_{j=1}^{\infty} A_j J_p(\lambda_{pj}x) \tag{1.3.15}$$

called the Bessel series of order p of f. Multiplying both sides of (1.3.15) by$J_p(\lambda_{pk}x)$ and integrating term by term on the interval $[0, a]$ gives

$$\int_0^a f(x)J_p(\lambda_{pk}x)xdx = \sum_{j=1}^{\infty} A_j \int_0^a J_p(\lambda_{pj}x)J_p(\lambda_{pk}x)xdx \tag{1.3.16}$$

The orthogonality property (1.3.11) shows that all the terms on the right sides of (1.3.16) are 0 except when j = k. Canceling the zero terms and using (1.3.12), we get

$$A_j = \frac{\int_0^a f(x)J_p(\lambda_{pj}x)xdx}{\int_0^a J_p^2(\lambda_{pj}x)xdx} = \frac{2}{a^2 J_{p+1}^2(\alpha_{pj})} \int_0^a f(x)J_p(\lambda_{pj}x)xdx \tag{1.3.17}$$

The number A_j is called the j^{th} Bessel coefficient of the function f. The next theorem gives conditions under which the Bessel series expansion of a function is valid.

Theorem 1.3.2. *: **Bessel series of order p***
If f is piecewise smooth on $[0, a]$, then f has a Bessel series expansion of order p on the interval $(0, a)$ given by

$$f(x) = \sum_{j=1}^{\infty} A_j J_p(\lambda_{pj} x)$$

Where $\lambda_{p1}, \lambda_{p1}, \ldots$ are the scaled positive zeros of the Bessel function J_p given by (1.3.10), and A_j is given by (1.3.17). In the interval $(0, a)$, the series converges to $f(x)$, Where f is continuous and converges to the average

$$\frac{f(x+) + f(x-)}{2}$$

at the points of discontinuity.

Example 2. *Find the Bessel series expansion of order 0 of the function $f(x) = 1$,$0 < x < 1$. (Note : We will drop the index p and write α_j and λ_j instead of α_{pj} and λ_{pj}.)*
***Solution** :*

$$f(x) = \sum_{j=1}^{\infty} A_j J_0(\alpha_j x)$$

Where α_j is the j^{th} positive zero of J_0 and

$$A_j = \frac{2}{J_1^2(\alpha_j)} \int_0^1 f(x) J_0(\alpha_j x) x dx$$

$$= \frac{2}{J_1^2(\alpha_j)} \int_0^1 J_0(\alpha_j x) x dx$$

$$= \frac{2}{\alpha_j^2 J_1^2(\alpha_j)} \int_0^{\alpha_j} J_0(t) t dt, (t = \alpha_j x)$$

$$= \frac{2}{\alpha_j^2 J_1^2(\alpha_j)} J_1(t) t\Big|_0^{\alpha_j}, (by (1.3.7) \ with \ p = 0)$$

$$= \frac{2}{\alpha_j J_1(\alpha_j)}$$

Theorem (1.3.2) asserts that the Bessel series converges to $f(x)$ at all points in the interval. Thus

$$1 = \sum_{j=1}^{\infty} \frac{2}{\alpha_j J_1(\alpha_j)} J_0(\alpha_j x), 0 < x < 1$$

j	*1*	*2*	*3*	*4*	*5*
α_j	*2.4048*	*5.5201*	*8.6537*	*11.7915*	*14.9309*
$J_1(\alpha_j)$	*0.5191*	*-0.3403*	*0.2714*	*-0.2325*	*0.2065*
$\frac{2}{\alpha_j J_1(\alpha_j)}$	*1.6020*	*-1.0648*	*0.8514*	*-0.7295*	*0.6487*

Table 2: Numerical data for example above

Using the numerical data provided by table 1 and 2, we can write explicitly the first few terms of the series:

$1 = 1.6020J_0(2.4048x) - 1.0648J_0(5.5201x) + 0.8514J_0(8.6537x) - 0.7295J_0(11.7915x) + 0.6487J_0(14.9309x) + ...$

It is worth noticing that the Bessel coefficients tend to 0 as $j \to \infty$. This is a property that holds in general.

1.3.3 Parametric Form of Bessel's Equation

Theorem 1.3.3. ***Parametric Form of Bessel's Equation***
Let $p \geq 0, a > 0$, and let α_{pj} denote the j^{th} positive zero of $J_p(x)$. For $j = 1, 2, ...$, the functions $J_p(\frac{\alpha_{pj}}{a}x)$ are solutions of the parametric form of Bessel's equation of order p.

$$x^2 y^{''}(x) + x y^{'}(x) + (\lambda^2 x^2 - p^2) y(x) = 0, \qquad (1.3.18)$$

Together with the boundary conditions

$$y(0)\ finite, y(a), \qquad (1.3.19)$$

When $\lambda = \lambda_{pj} = \frac{\alpha_{pj}}{a}$, and these are the only solutions of (1.3.18), aside from scalar multiples, with these properties. Moreover, these solutions satisfy (1.3.11) and (1.3.12) and so they are orthogonal on the interval $[0, a]$ with respect to the weight function x.

Chapter 2

Sturm-Liouville theory , Hanging chain , Elastic Vibration and Buckling Beams

2.1 Sturm-Liouville Theory

2.1.1 Sturm-Liouville Theory

Definition 6. *A homogeneous boundary value problem of the second order linear differential equation that can be written in the form*

$$(p(x)y^{'})^{'} + (q(x) + \lambda r(x))y = 0, a \leq x \leq b \tag{2.1.1}$$

$$c_1 y(a) + c_2 y^{'}(a) = 0 \tag{2.1.2a}$$

$$d_1 y(b) + d_2 y^{'}(b) = 0 \tag{2.1.2b}$$

with $q, r \in C^0[a, b]$, $p \in C^1[a, b]$, $(c_1, c_2) \neq (0, 0)$, $(d_1, d_2) \neq (0, 0)$ *and* λ *a real constant is called Sturm-Liouville problem.*

Equation (2.1.1) is said to be a regular Sturm-Liouville problem with $p(x) > 0$ and $r(x) > 0$ for $a \leq x \leq b$. A singular Sturm-Liouville problem is Sturm-Liouville problem in which at least one of the regularity properties fails or on an infinite interval. In this case the boundary conditions are not always described by sets of equations like (2.1.2a) and (2.1.2b). Typically, a Sturm-Liouville problem is singular either because it occurs on an infinite interval, or because one or more of the coefficients goes to 0 or ∞ at an end point of the interval or both. Indeed, it is convenient to require that $p(x)$, $p'(x)$,$q(x)$and $r(x)$ are continuous on the interval $a < x < b$, with $p(x) > 0$ and $r(x) > 0$ for $a < x < b$. This will be the case for all singular problems encountered. Clearly $y = 0$ is a solution of every Sturm-Liouville problem. The non-zero solutions of a Sturm-Liouville problem are called the eigenfunctions of the problem and those values of λ for which non-zero solutions can be found are called the Eigenvalues.

Example 3. *Find the Eigenvalues and corresponding Eigenfunctions of the Sturm-Liouville problem*

$$y^{''} + \lambda y = 0, y(0) = y(\pi) = 0$$

Solution:

This differential equation fits the form of (2.1.1) with $p(x) = 1$, $q(x) = 0$ and $r(x) = 1$. In the boundary conditions, $a = 0$ and $b = \pi$, with $c_1 = d_1 = 1$ and $c_2 = d_2 = 0$, so this is a regular Sturm-Liouville problem.

We seek non-zero solutions of the problem. As is often the case with Sturm-Liouville problems, the nature of the solution depends on the sign of λ, so we consider three cases

`Case 1:-` $\lambda < 0$. Let us write $\lambda = -\alpha^2$ where $\alpha > 0$, Then the equation becomes $y^{''} - \alpha^2 y = 0$. Its characteristic equation is $\lambda^2 - \alpha^2 = 0$ with characteristic roots $\lambda_1 = \alpha$ and $\lambda_2 = -\alpha$. Thus a fundamental set of solutions is $\{e^{\alpha x}, e^{-\alpha x}\}$. Their Wronskian is

$$W(e^{\alpha x}, e^{-\alpha x}) = \begin{vmatrix} e^{\alpha x} & e^{-\alpha x} \\ \alpha e^{\alpha x} & -\alpha e^{-\alpha x} \end{vmatrix} = -2\alpha \neq 0$$

Thus $e^{\alpha x}$ and $e^{-\alpha x}$ are linearly independent. The general solution of the differential equation is of the form $y = c_1 e^{\alpha x} + c_2 e^{-\alpha x}$; and so any solution is a linear combination of the functions $e^{\alpha x}$ and $e^{-\alpha x}$, which form a fundamental set of solutions. It is worth while to note that the fundamental set of solutions is not unique. In fact, we now describe another set that is more convenient in some applications.

Recall the definition of the hyperbolic functions

$$\cosh \alpha x = \frac{e^{\alpha x} + e^{-\alpha x}}{2}$$

and

$$\sinh \alpha x = \frac{e^{\alpha x} - e^{-\alpha x}}{2}$$

Since these are linear combinations of functions from the fundamental set of solutions, they are themselves solutions of the differential equation. You could also verify the last assertion directly by using the derivative formulas.

$$\frac{d}{dx} \cosh \alpha x = \alpha \sinh \alpha x$$

and

$$\frac{d}{dx} \sinh \alpha x = \alpha \cosh \alpha x$$

Computing the Wronskian of $\cosh \alpha x$ and $\sinh \alpha x$. We find

$$W(\cosh \alpha x, \sinh \alpha x) = \begin{vmatrix} \cosh \alpha x & \sinh \alpha x \\ \alpha \sinh \alpha x & \alpha \cosh \alpha x \end{vmatrix} = \alpha(\cosh^2 \alpha x - \sinh^2 \alpha x) = \alpha \neq 0$$

We conclude that $\cosh \alpha x$ and $\sinh \alpha x$ form a fundamental set of solutions and so the general solution of the differential equation is the form $y = c_1 \cosh \alpha x + c_2 \sinh \alpha x$. We need $y(0) = 0$, so substituting into the general solution gives $c_2 = 0$. Now using the condition $y(\pi) = 0$, we get $0 = c_1 \sinh \alpha x$, and since $\sinh \alpha x \neq 0$ unless $x = 0$, we infer that $c_1 = 0$. Thus there are no non-zero solution in this case.

`Case2:-` $\lambda = 0$. Here the general solution of the differential equation is $y = c_1 x + c_2$, and as in `case 1` the boundary conditions force $c_1$and c_2 to be 0. Thus again there is no non-zero solution.

`Case 3:-` $\lambda > 0$. In this case we can write $\lambda = \alpha^2$ with $\alpha > 0$, and so the equation becomes $y'' + \alpha^2 y = 0$. Its characteristic equation is $\lambda^2 + \alpha^2 = 0$. The characteristic roots are $\lambda_1 = i\alpha$ and $\lambda_2 = -i\alpha$.
Thus a fundamental set of solutions is $\{e^{i\alpha x}, e^{-i\alpha x}\}$.
We now describe an alternative fundamental set of real-valued solutions. The

construction is similar to the one we used in constructing the hyperbolic cosine and sine solutions in `case 1`. Recall Eulers identity

$$e^{i\alpha x} = \cos \alpha x + i \sin \alpha x$$

Consider the following solution, which is formed by taking a particular linear combination from the fundamental set of solutions.

$$y_1 = \frac{e^{i\alpha x} + e^{-i\alpha x}}{2}$$

$$= \frac{1}{2}(\cos \alpha x + i \sin \alpha x + \cos \alpha x - i \sin \alpha x)$$

$$= \cos \alpha x$$

Thus $y_1 = \cos \alpha x$ is a solution of the differential equation, which is a fact that we can verify directly. Similarly, the linear combination

$$y_2 = \frac{e^{i\alpha x} - e^{-i\alpha x}}{2i} = \frac{1}{2i}(\cos \alpha x + i \sin \alpha x - \cos \alpha x + i \sin \alpha x) = \sin \alpha x$$

Computing the Wronskian of $\cos \alpha x$ and $\sin \alpha x$. We find

$$W(\cos \alpha x, \sin \alpha x) = \begin{vmatrix} \cos \alpha x & \sin \alpha x \\ -\alpha \sin \alpha x & \alpha \cos \alpha x \end{vmatrix} = \alpha(\cos^2 \alpha x + \sin^2 \alpha x) = \alpha \neq 0$$

Clearly, y_1 and y_2 are real-valued and linearly independent. Thus the general solution of $y'' + \alpha^2 y = 0$ is $y = c_1 \cos \alpha x + c_2 \sin \alpha x$. From $y(0) = 0$ we get $0 = c_1 \cos 0 + c_2 \sin 0$, or $0 = c_1$. Thus $y = c_2 sin\alpha x$. Now we substitute the other boundary condition to get $0 = c_2 \sin \alpha\pi$. Since we are seeking non-zero solutions, we take $c_2 \neq 0$. Thus we must have $\sin \alpha\pi = 0$, and hence $\alpha = 1, 2, 3, ...$ This means that, since $\lambda = \alpha^2$, the problem has eigenvalues $\lambda_1 = 1, \lambda_2 = 4, \lambda_3 = 9, ...$ and corresponding eigenfunctions. $y_1 = \sin x$, $y_2 = \sin 2x$,$y_3 = \sin 3x, ...$ We have let the constant c_2 be 1 in each case. All other eigenfunctions will be non-zero multiple of these.

We now describe some fundamental properties of eigenvalues and eigenfunctions of regular Sturm-Liouville problems, all of which are illustrated by the solution to the above example. In that example, the eigenvalues form an increasing sequence of real numbers. Moreover, to each eigenvalues there corresponds just one linearly independent eigenfunction.

Theorem 2.1.1. *The eigenvalues of regular Sturm-Liouville problem are all real and form an increasing sequence* $\lambda_1 < \lambda_2 < \lambda_3 < ...$ *where* $\lambda_j \to \infty$ *as* $j \to \infty$.

Our next result deals with the orthogonality of eigenfunctions. This property holds for regular as well as some singular Sturm-Liouville problems. To understand the reason behind the orthogonality property, we start by deriving some consequences of the boundary condition (2.1.2), in Sturm-Liouville problems.

Let λj and λ_k be two distinct eigenvalues of the regular sturm-liouville problem (2.1.1)-(2.1.2) and let y_j and y_k denote their corresponding eigenfunctions. From (2.1.2a) we have

$c_1 y_j(a) + c_2 y_j^{'}(a) = 0$
$c_1 y_k(a) + c_2 y_k^{'}(a) = 0$
In matrix form becomes

$$\begin{bmatrix} y_j(a) & y_j^{'}(a) \\ y_k(a) & y_k^{'}(a) \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Since not both c_1 and c_2 are 0, we must have that

$$\begin{vmatrix} y_j(a) & y_j^{'}(a) \\ y_k(a) & y_k^{'}(a) \end{vmatrix} = 0$$

Or equivalently, $y_k(a)y_j^{'}(a) - y_j(a)y_k^{'}(a) = 0$. Similarly, since the eigenfunctions satisfy (2.1.2b), we get that
$y_k(b)y_j^{'}(b) - y_j(b)y_k^{'}(b) = 0$.
Combining these two identities and the fact that $p(a)$ and $p(b)$ are finite, we infer that

$$p(b)(y_k(b)y_j^{'}(b) - y_j(b)y_k^{'}(b)) - p(a)(y_k(a)y_j^{'}(a) - y_j(a)y_k^{'}(a)) = 0 \quad (2.1.3)$$

In singular problems, since we may be dealing with infinite intervals, or with functions that may be unbounded near the end points, instead of (2.1.2) we will require a condition similar to (2.1.3), but stated using limits as follows

Condition for orthogonality in singular Sturm-Liouville problem:

Suppose that y_1 and y_2 are eigenfunctions of a Sturm-Liouville problem, corresponding to two distinct eigenvalues λ_1 and λ_2, respectively. We require that

$$\lim_{x\to b} p(x)(y_1(x)y_2^{'}(x) - y_2(x)y_1^{'}(x)) - \lim_{x\to a} p(x)(y_1(x)y_2^{'}(x) - y_2(x)y_1^{'}(x)) = 0 \tag{2.1.4}$$

As we just noticed, (2.1.4) reduces to (2.1.3) for regular Sturm-Liouville problems, and thus it holds for regular Sturm-Liouville problems. It also holds in many important singular problems, such as Legendres and Bessels equations.

Theorem 2.1.2. *:-**Uniqueness and Orthogonality of Eigenfunctions***
(a) Each eigenvalue of a regular Sturm-Liouville problem has just one linearly independent eigenfunction corresponding to it.
(b) Eigenfunctions corresponding to different eigenvalues of a regular Sturm-Liouville problem are orthogonal with respect to the weight function r(x). This assertion is also valid for the other Sturm-Liouville problems allowed by condition (3.1.4).

2.1.2 Eigenfunction Expansions

From theorem (2.1.2) it follows that if $\lambda_1 < \lambda_2 < \lambda_3 < ...$ is the set of eigenvalues for a regular Sturm-Liouville problem, then a corresponding set of eigenfunctions $\{y_1, y_2, y_3, ...\}$ is orthogonal with respect to the weight function $r(x)$. Thus, as in section (1.1), we can find orthogonal expansions for suitable functions in terms of $y_1, y_2, y_3, ...$ More precisely, we have the following fundamental result in Sturm-Liouville theory. Recall that the inner product (y_j, y_k) with weight $r(x)$ is defined as $\int_a^b y_j y_k r(x)dx$ and that the norm $\|y_j\|$ is $\sqrt{(y_j, y_j)}$.

Theorem 2.1.3. *:- **Eigenfunction Expansions***
Let $y_1, y_2, y_3, ...$ be the collection of eigenfunctions for a regular Sturm-Liouville problem on an interval $[a, b]$. If f is piecewise smooth on the interval $[a, b]$, then we have

$$f(x) = \sum_{j=1}^{\infty} A_j y_j(x),$$

where

$$A_j = \frac{(f, y_j)}{\|y_j\|^2} = \frac{\int_a^b f(x) y_j(x) r(x) dx}{\int_a^b y_j^2(x) r(x) dx}$$

For $a < x < b$, the series converges to $f(x)$ if f is continuous at x, and to

$$\frac{f(x+) + f(x-)}{2}$$

Otherwise.

The series expansion is called the eigenfunction expansion of the function f, and the coefficients A_j are called generalized Fourier coefficients. Eigenfunction expansions arise naturally in the solution of applied problems.

Example 4. : Eigenvalues and eigenfunctions
Find the eigenvalues and eigenfunctions of Sturm-Liouville problem

$$X^{''} + \lambda X = 0, X^{'}(0) = 0, X(1) + X^{'}(1) = 0$$

Solution:
If $\lambda = 0$, the general solution of the differential equation is $X = ax + b$. It is easy to check that the only way to satisfy the boundary conditions is to take $a = b = 0$. Thus $\lambda = 0$ is not an eigenvalue since no non-trivial solutions exist.
If $\lambda < 0$, the general solution of the differential equation is
$X = c_1 \cosh \sqrt{-\lambda} x + c_2 \sinh \sqrt{-\lambda} x$. It is a straight forward exercise to check that no non-trivial solution of this form will satisfy the boundary conditions. Thus there are no negative eigenvalues.
When $\lambda > 0$, for convenience, we set $\lambda = \mu^2$ and find that the general solution of the differential equation is

$$X = A \cos \mu x + B \sin \mu x$$

We now apply the boundary conditions:

$$X^{'}(0) = 0 \Rightarrow B = 0$$

$$X(1) + X^{'}(1) = 0 \Rightarrow A(\cos \mu - \mu \sin \mu) = 0$$

To ensure that we get non-zero eigenfunctions, we take
$A = 1$ and set $\cos\mu - \mu\sin\mu = 0$
Equivalently,

$$\cot\mu = \mu \tag{2.1.5}$$

Thus the eigenvalues $\lambda = \mu^2$ correspond to the positive root μ of this equation. If we plot the graphs of $y = \cot\mu$ and $y = \mu$, we see that these graphs intersect infinitely. Thus, (2.1.5) has infinitely many roots. Although we cannot compute these roots in simple form, we can find their numerical values and use them in our subsequent computations. For now, we denote the roots by $\mu_1, \mu_2, ..., \mu_n, ...$ and conclude that the eigenfunctions are

$$X = X_n = \cos\mu_n x, n = 1, 2, 3, ...$$

The eigenvalues are $\mu_1^2, \mu_2^2, ..., \mu_n^2, ...$

Example 5. :- Eigenfunction expansions
(a) Compute the first five eigenfunctions $X_1(x), X_2(x), ..., X_5(x)$ *in the above example explicitly.*
(b) Given $f(x) = x(1-x), 0 < x < 1$ *. What is the eigenfunction expansion of* f *?*

Solution :-
(a) According to the solution of above example, to find the eigenvalues ,we must solve the equation $cot\mu = \mu$ with the help of a computer system, we find the first five solutions to be approximately.
$\mu_1 = 0.860, \mu_2 = 3.426, \mu_3 = 6.437, \mu_4 = 9.529, \mu_5 = 12.645$
Thus the five eigenfunctions are
$X_1(x) = \cos(0.860x), X_2(x) = \cos(3.426x), X_3(x) = \cos(6.437x), X_4(x) = \cos(9.529x), X_5(x) = \cos(12.645x)$
(b) By Theorem (2.1.3) the eigenfunction expansion of f is

$$f(x) = \sum_{j=1}^{\infty} A_j \cos\mu_j x$$

Where

$$A_j = \frac{\int_0^1 x(1-x)\cos\mu_j x dx}{\int_0^1 \cos^2\mu_j x dx}$$

with the numerical values of the μ_j's given in (a). We evaluate these coefficients with the help of a computer and find
$A_1 = 0.189, A_2 = -0.032, A_3 = -0.091, A_4 = -0.001, A_5 = -0.025$

Thus the eigenfunction expansion of f is

$$f(x) = 0.189\cos(0.860x) - 0.032\cos(3.426x) - 0.091\cos(6.437x) - 0.001\cos(9.529x) - 0.025\cos(12.645x) + ...$$

2.2 The Hanging Chain

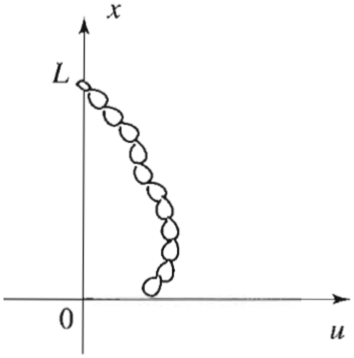

Figure 2.1: Hanging chain

Having studied Sturm-Liouville theory for second order equations, we illustrate the theory as it applies to the oscillations of the hanging chain. This problem played an important role in the development of the theory of partial differential equations. It was while solving this problem that Daniel Bernoulli first discovered Bessel functions in 1732. Although we link the solution to general Sturm-Liouville theory, our presentation contains all the necessary details to solve this problem based on the properties of Bessel functions (in section1.3 Bessel series expansion).
To describe the equation governing the motion of the hanging chain, we place the x-axis in a vertical position, pointing upward. Consider a chain of length L, hanging down with one end fastened at x=L (Figure 2.1). The small transverse oscillations of the chain are described by the boundary value problem.

$$\frac{\partial^2 u}{\partial t^2} = g[x\frac{\partial^2 u}{\partial x^2} + \frac{\partial u}{\partial x}], 0 < x < L, t > 0 \tag{2.2.1}$$

$$u(L,t) = 0, t > 0,$$

$$u(x,0) = f(x), u_t(x,0) = v(x)$$

Here g is the gravitational acceleration, $f(x)$ is the initial displacement of the chain, and $v(x)$ its intial velocity. We will solve this boundary value problem using separation of variables. The method will lead to Bessel's equation, and the solution will involve a form of Bessel series.

Separation of variables
We assume a product solution of the form $u(x,t) = X(x)T(t)$.
Plugging this into (2.2.1) and separating the variables, we get

$$X\ddot{T} = gT(xX^{''} + X^{'}),$$

$$\frac{1}{g}\frac{\ddot{T}}{T} = \frac{xX^{''} + X^{'}}{X}$$

Since the variables are separated, for the last equality to hold, each side must be equal to a constant. Thus

$$\frac{1}{g}\frac{\ddot{T}}{T} = \lambda$$

and

$$\frac{xX^{''} + X^{'}}{X} = \lambda$$

The boundary condition implies that $T(t)X(L) = 0$ for all $t > 0$. Hence to avoid the trivial solution, we require that $X(L) = 0$. Thus we arrive at the equations
$\ddot{T} - \lambda g T = 0$ and $xX^{''} + X^{'} - \lambda X = 0, X(L) = 0$

Solving the separated equations
If $\lambda \geq 0$, the solutions of the differential equation in T are either linear or

exponential functions. We discard these solutions for obvious practical reasons. Thus λ must be negative. To simplify notation, we will simply replace λ in the equations by $-\mu^2$ and get

$$\ddot{T} + \mu^2 g T = 0 \tag{2.2.2}$$

and

$$xX^{''} + X^{'} + \mu^2 X = 0, X(L) = 0 \tag{2.2.3}$$

The general solution of (2.2.2) is

$$T(t) = A\cos(\sqrt{g}\mu t) + B\sin(\sqrt{g}\mu t) \tag{2.2.4}$$

Next we consider the differential equation in X from the point of view of Sturm-Liouville theory. By rewriting (2.2.3), we see that the differential equation can be put in the Sturm-Liouville form

$$(xX^{'})^{'} + \mu^2 X = 0$$

Comparing this to (2.1.1) from section (2.1.1), we find that $p(x) = x$, $q(x) = 0$ and $r(x) = 1$ and also that $a = 0$ and $b = L$. Since $p(0) = 0$,this is a singular Sturm-Liouville problem. We will show that this problem has infinitely many eigenvalues and a corresponding complete set of eigenfunctions, and by Theorem (2.1.2b) section (2.1.1), it follows that these eigenfunctions are orthogonal with respect to the weight function $r(x) = 1$. It will then follow that we can construct the solution $u(x,t)$ to the hanging chain problem by superposing the corresponding product solutions. In this particular example it is possible to be much more concrete since (2.2.3) is closely related to Bessel's equation. To see this, we make the change of variables $s = 2\sqrt{x}$. Using the chain rule, we verify that

$$X^{'} = \frac{dX}{dx} = \frac{dX}{ds}\frac{1}{\sqrt{x}}$$

$$X^{''} = \frac{d^2X}{dx^2} = \frac{1}{x}\frac{d^2X}{ds^2} - \frac{1}{2x^{\frac{3}{2}}}\frac{dX}{ds}$$

Substituting in (2.2.3) and simplifying, we get

$$s^2\frac{d^2X}{ds^2} + s\frac{dX}{ds} + \mu^2 s^2 X = 0, X(2\sqrt{L}) = 0,$$

which is precisely the parametric form of Bessel's equation of order 0 (theorem 1.3.3,subsection 1.3.3). Since we are only interested in the bounded solutions of this equation, we can apply (subsection1.3.3, theorem 1.3.3) (with $p = 0$) and that

$$\mu = \mu_j = \frac{\alpha_j}{2\sqrt{L}}$$

and

$$X(s) = X_j(s) = J_0(\frac{\alpha_j}{2\sqrt{L}}s) \tag{2.2.5}$$

Where $j = 1, 2, 3, ...$ and α_j denotes the j^{th} zero of J_0. Moreover, these functions are orthogonal on the interval $0 \leq s \leq 2\sqrt{L}$ with respect to the weight functions. The orthogonality relations expressed by (1.3.11) and (1.3.12) of section (1.3.1) (with $a = 2\sqrt{L}$)become

$$\int_0^{2\sqrt{L}} J_0(\frac{\alpha_j}{2\sqrt{L}}s)J_0(\frac{\alpha_k}{2\sqrt{L}}s)sds = 0, j \neq k, \tag{2.2.6}$$

and

$$\int_0^{2\sqrt{L}} J_0^2(\frac{\alpha_j}{2\sqrt{L}}s)sds = 2LJ_1^2(\alpha_j), j = 1, 2, ... \tag{2.2.7}$$

Substituting $s = 2\sqrt{x}$ back into (2.2.5), (2.2.6), (2.2.7), we find the solutions

$$X_j(2\sqrt{x}) = J_0(\alpha_j\sqrt{\frac{x}{L}}), j = 1, 2, 3, ... \tag{2.2.8}$$

and their orthogonality relations

$$\int_0^L J_0(\alpha_j\sqrt{\frac{x}{L}})J_0(\alpha_k\sqrt{\frac{x}{L}})dx = 0, j \neq k$$

$$\frac{1}{LJ_1^2(\alpha_j)}\int_0^L J_0^2(\alpha_j\sqrt{\frac{x}{L}})dx = 1, j = 1, 2, ...$$

Having solved the equation in X, we combine (2.2.4) and (2.2.8) and conclude that a product solution of (2.2.1) satisfying the accompanying boundary condition is

$$u_j(x,t) = J_0(\alpha_j\sqrt{\frac{x}{L}})[A_j\cos(\sqrt{\frac{g}{L}}\frac{\alpha_j}{2}t) + B_j\sin(\sqrt{\frac{g}{L}}\frac{\alpha_j}{2}t)] \tag{2.2.9}$$

The functions $u_j (j = 1, 2, ...)$ are called the normal modes of the chain. For $j = 1$ we get the fundamental mode of the chain. From (2.2.9), we obtain the frequency of the j^{th} normal mode

$$\nu_j = \frac{\alpha_j}{4\pi}\sqrt{\frac{g}{L}}$$

Bessel series solution of the entire problem

By superposing all the product solutions, we get

$$u(x,t) = \sum_{j=1}^{\infty} J_0(\alpha_j\sqrt{\frac{x}{L}})[A_j \cos(\sqrt{\frac{g}{L}}\frac{\alpha_j}{2}t) + B_j \sin(\sqrt{\frac{g}{L}}\frac{\alpha_j}{2}t)] \qquad (2.2.10)$$

Setting $t = 0$ and using the initial condition, we find that

$$f(x) = u(x,0) = \sum_{j=1}^{\infty} A_j J_0(\alpha_j\sqrt{\frac{x}{L}})$$

To determine the A_j's we multiply both sides of the equality by $J_0(\alpha_k\sqrt{\frac{x}{L}})$ and then integrate with respect to x from 0 to L. By the orthogonality relations, all the terms with $j \neq k$ are equal to zero , and when $j = k$ we get

$$A_j = \frac{1}{LJ_1^2(\alpha_j)}\int_0^L f(x)J_0(\alpha_j\sqrt{\frac{x}{L}})dx, j = 1, 2, ... \qquad (2.2.11)$$

To determine B_j we proceed in a similar way using the initial velocity $v(x)$. Differentiating u with respect to t and then setting $t = 0$, we get

$$v(x) = u_t(x,0) = \sum_{j=1}^{\infty} B_j\sqrt{\frac{g}{L}}\frac{\alpha_j}{2}J_0(\alpha_j\sqrt{\frac{x}{L}}).$$

Multiplying both sides by $J_0(\alpha_k\sqrt{\frac{x}{L}})$ and then integrating with respect to x from 0 to L, the orthogonality properties yield

$$B_j = \frac{2}{\alpha_j J_1^2(\alpha_j)}\frac{1}{\sqrt{gL}}\int_0^L v(x)J_0(\alpha_j\sqrt{\frac{x}{L}})dx, j = 1, 2, ... \qquad (2.2.12)$$

This completely determines the solution in terms of the initial conditions.

Example 6. *: Vibrating chain*
A chain of length 1meter is hanging from one end. An initial displacement of the chain is done by pulling the center 0.005m while keeping the lower end fixed. (see figure 2.2) The chain is then left to vibrate freely
(a) What are the frequencies of the first three normal modes?
(Take $g = 9.8\frac{m}{sec^2}$ *)*
(b) Find the first three normal modes $u_1(x,t), u_2(x,t), u_3(x,t)$ *at* $t = 0$*.*
(c) Determine the motion of the chain by finding $u(x,t)$*.*

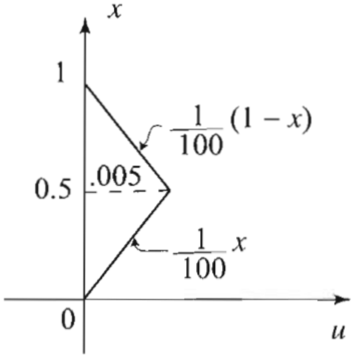

Figure 2.2: Initial shape of the chain

Solution:
(*a*) Note that $B_j = 0$ for all j because the initial velocity is zero. Hence from (2.2.9), the j^{th} normal mode is

$$u_j(x,t) = A_j J_0(\alpha_j \sqrt{x}) \cos(\sqrt{9.8}\frac{\alpha_j}{2}t)$$

Where α_j is the j^{th} zero of J_0. The frequency of the j^{th} normal mode is $\nu_j = \frac{\alpha_j}{4\pi}\sqrt{g}$. Using the numerical values of α_j from table below, we find

$$\nu_1 = 0.5990, \nu_2 = 1.375, \nu_3 = 2.1558.$$

(b) At time $t = 0$ we have $u_1(x,0) = A_1 J_0(2.4048\sqrt{x})$
$u_2(x,0) = A_2 J_0(5.5201\sqrt{x})$ and $u_3(x,0) = A_3 J_0(8.6537\sqrt{x})$
(c) The function $u(x,t)$ is given by (2.2.10), where the coefficients B_j are all equal to 0. It remains to compute the A_j's.

For this purpose, we use (2.2.11) and the equation of the initial displacement $f(x)$ given in figure. We get

$$A_j = \frac{1}{J_1^2(\alpha_j)}[\int_0^{\frac{1}{2}} \frac{1}{100} x J_0(\alpha_j \sqrt{x}) dx + \int_{\frac{1}{2}}^1 \frac{1}{100}(1-x) J_0(\alpha_j \sqrt{x}) dx]$$

These integrals are computed with the help of the following identities.

$$\int x J_0(\alpha_j \sqrt{x}) dx = \frac{2}{\alpha_j} x^{\frac{3}{2}} J_1(\alpha_j \sqrt{x}) - \frac{4}{\alpha_j^2} x J_2(\alpha_j \sqrt{x}) + c;$$

$$\int J_0(\alpha_j \sqrt{x}) dx = \frac{2}{\alpha_j} x^{\frac{1}{2}} J_1(\alpha_j \sqrt{x}) + c,$$

which can be derived from (1.3.7), section 1.3 Bessel with $p = 0$), after making the substitution $t = \alpha_j \sqrt{x}$. Apply these formulas and simplify, we get

$$A_j = \frac{J_2(\alpha_j) - J_2(\frac{\alpha_j}{\sqrt{2}})}{25\alpha_j^2 J_1^2(\alpha_j)}$$

Numerical values of the A_j's can be approximated with the help of a computer. They are presented table along with other pertinent numerical data. Plugging the numerical data into (2.2.10), we get

$$u(x,t) = 0.003847 J_0(2.4048\sqrt{x}) \cos(3.76415t) - 0.005787 J_0(5.52008\sqrt{x}) \cos(8.64029t) +$$
$$0.002371 J_0(8.65373\sqrt{x}) \cos(13.5452t) - ...$$

j	1	2	3
α_j	2.40483	5.52008	8.65373
ν_j	0.5990	1.375	2.1558
A_j	0.003847	-0.005787	0.002371

Table :*Numerical data for the example above*

2.3 Elastic Vibration and Buckling Beams

The applications that we consider in this section lead to partial differential equations that are fourth order in the space variables. We consider the transverse vibrations of an elastic homogeneous beam with various supports at its ends.

2.3.1 Transverse vibration of a Beam: Simply Supported Case

The free vertical vibrations of a uniform beam of length L are described by the equation

$$u_{tt} = -c^2 u_{xxxx} \tag{2.3.1}$$

Where $c^2 = \frac{EI}{\rho A}$,$E =$ Young's constant (determined by the constitutive material of the beam),$I =$ moment of inertia of a cross section of the beam with respect to an axis through its center of mass and perpendicular to the $(x, u)-$ plane, $\rho = \textit{density}$ (mass per unit volume), and $A =$ area of cross section. It is assumed that the beam is of uniform density throughout, that the cross sections are constant, and that in its equilibrium position the centers of mass of the cross sections lie on the x- axis (see figure 2.3). The variable $u(x, t)$ represents the displacement of the point on the beam corresponding to position x at time t.

It is customary in engineering to show a simple beam resting on a pin and a roller and not, say, on two pins. In this manner, the beam is said to be statically determinate. If the beam were to have two pins as supports, it would become statically indeterminate and develop horizontal reactions, which is still acceptable; however, it would not constitute the simplest possible beam. As far as the boundary conditions are concerned, both prevent vertical translation (completely) and both allow rotation (unlike a fixed or clamped support, which locks also rotation). The only difference between a pin and a roller is that a pin prevents horizontal movement, whereas a roller does not (it can roll horizontally). However, both allow rotation and both prevent vertical translation as we said. A structure in equilibrium must also be stable. A simply supported beam is the simplest structure that is also stable. If the supports were to be two rollers, the beam would be unstable (it would not resist any perturbation in the horizontal direction). On the other hand, if it is to rest on two pins, it would have one additional reaction in excess of being a stable structure. Therefore, a pin and a roller are the simplest possible supports for a beam to be both in equilibrium and stable.

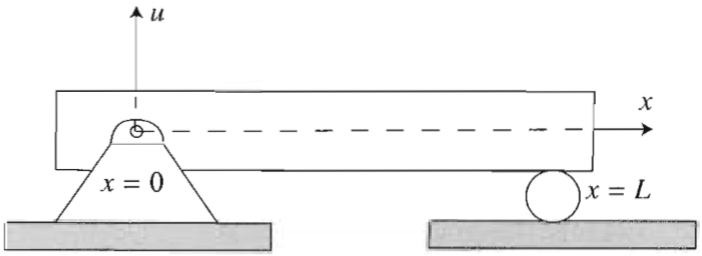

Figure 2.3: A simply supported beam is pinned at one end and roller supported at the other. The ends can rotate freely but do not move vertically.

The boundary conditions that acompany equation (2.3.1) depend on the end supports of the beam. The case under consideration, that of simply supported ends, is described by

$$u(0,t)=0, u(L,t)=0, u_{xx}(0,t)=0, u_{xx}(L,t)=0 \tag{2.3.2}$$

The fact that the beam is prevented from translating vertically at its supports is expressed by the first two equations in (2.3.2). To understand the remaining equations in (2.3.2). Recall that $u_{xx}(x,t)$ represents the curvature or concavity of the beam at any x. It is a fact from the theory of strength of materials that curvature is proportional to bending moment for an elastic beam. The last two equations in (2.3.2) state that the moments at the supports are zero, as is the case for a simply supported beam, since the beam can rotate at its ends.

To complete the description of the initial value, we specify the initial conditions

$$u(x,0)=f(x), u_t(x,0)=g(x), 0<x<L \tag{2.3.3}$$

Which give the initial displacement and velocity of the beam. To solve (2.3.1)-(2.3.3), we use the method of separation of variables. This leads to the equations

$$\frac{X^{(4)}}{X}=-\frac{T''}{c^2T}=\alpha^4 \tag{2.3.4}$$

We have chosen the separation constant to be positive because we expect periodic behavior in t, and for reasons that will become apparent momentarily it is convenient to denote this constant by α^4. For the sake of completeness, we note that it can be shown that the boundary value problem $x^{(4)} - \lambda x = 0$ together with boundary conditions implied by (2.3.2) has nontrivial solutions only for positive choices of the separation constant λ. We begin by considering the boundary value problem for X:

$$X^{(4)} - \alpha^4 X = 0, X(0) = 0, X(L) = 0, X^{''}(0) = 0, X^{''}(L) = 0 \qquad (2.3.5)$$

Note that this a fourth-order Sturm-Liouville problem , with $\lambda = \alpha^4, p(x) = 1, q(x) = 0$ and $r(x) = 1$. Since (2.3.5) is a linear equation with constant coefficients, we solve it by passing to the characteristic equation . We find

$$r^4 - \alpha^4 = 0 \Rightarrow r = \pm\alpha, andr = \pm i\alpha$$

Thus the general solution is

$$X(x) = A\cosh\alpha x + B\sinh\alpha x + C\cos\alpha x + D\sin\alpha x$$

Where A, B, C and D are arbitrary constants. The first and the third boundarys imply that $A + C = 0$,$A - C = 0 \Rightarrow A = C = 0$. The second and the fourth conditions then imply

$$\begin{cases} B\sinh\alpha L + D\sin\alpha L &= 0 \\ B\sinh\alpha L - D\sin\alpha L &= 0 \end{cases}$$

These are equivalent to $B\sinh\alpha L = 0$ and $D\sin\alpha L = 0$, from which it follows that
$B = 0$ ($\sinh\alpha L \neq 0$ for $\alpha > 0$) and $\alpha = \alpha_n = \frac{n\pi}{L}$, $n = 1, 2, ...$
We thus obtain the solutions in X:

$$X_n(x) = \frac{\sin n\pi}{L}x, n = 1, 2, ...$$

Going back to (2.3.4) and solving for T with $\alpha = \alpha_n$, we obtain the corresponding solutions

$$T_n(t) = A_n\cos c\alpha_n^2 t + A_n^*\sin c\alpha_n^2 t$$

Forming the product solutions and superposing, we find the general form of the solution :

$$u(x,t) = \sum_{n=1}^{\infty}\sin\frac{n\pi}{L}x(A_n\cos c\alpha_n^2 t + A_n^*\sin c\alpha_n^2 t) \qquad (2.3.6)$$

Using the initial conditions (2.3.3), we get

$$f(x) = \sum_{n=1}^{\infty} A_n \sin \frac{n\pi}{L} x$$

and

$$g(x) = \sum_{n=1}^{\infty} A_n^* c\alpha_n^2 \sin \frac{n\pi}{L} x.$$

Since these should just be the sine expansions of f and g on the interval $0 < x < L$, it follows directly from next theorem (2.3.1) that

$$A_n = \frac{2}{L} \int_0^L f(x) \sin \frac{n\pi}{L} x dx \tag{2.3.7}$$

$$A_n^* = \frac{2}{c\alpha_n^2 L} \int_0^L g(x) sin \frac{n\pi}{L} x dx, n = 1, 2, 3, ... \tag{2.3.8}$$

The solution of (2.3.1)-(2.3.3) is thus given by the series (2.3.6) with the coefficients determined by (2.3.7) and (2.3.8).

Theorem 2.3.1. *: Eigenfunction Expansions*
If f is piecewise smooth on the interval $[a, b]$, then we have the eigenfunction expansion

$$f(x) = \sum_{j=1}^{\infty} A_j X_j(x),$$

where

$$A_j = \frac{\int_a^b f(x) X_j(x) r(x) dx}{\int_a^b X_j^2(x) r(x) dx}$$

For $a < x < b$, the eigenfunction expansion converges to $f(x)$ at the points of continuity, and otherwise it converges to

$$\frac{f(x+) + f(x-)}{2}.$$

2.3.2 The transverse Vibrations of a Beam: clamped case

If the ends of the beam in the previous example are clamped (Figure 2.4), the boundary conditions become

$$u(0,t)=0, u(L,t)=0, u_x(0,t)=0, u_x(L,t)=0 \tag{2.3.9}$$

For all $t > 0$. While the first two conditions in (2.3.9) are the same as those for the simply supported case, the last two impose a restraint against end rotation, since the beam does resist moments at its ends. The initial conditions remain

$$u(x,0)=f(x), u_t(x,0)=g(x), 0<x<L \tag{2.3.10}$$

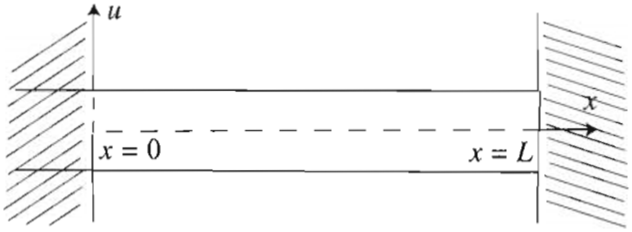

Figure 2.4: A beam with two clamped ends

We solve the boundary value problem (2.3.1),(2.3.9),(2.3.10). After separating the variables, we arrive at the fourth order Sturm-Liouville problem

$$X^{(4)}-\lambda X=0, X(0)=0, X(L)=0, X^{'}(0)=0, X^{'}(L)=0 \tag{2.3.11}$$

and the general solution is

$$X(x)=A\cosh\alpha x+B\sinh\alpha x+C\cos\alpha x+D\sin\alpha x \tag{2.3.12}$$

To determine the eigenvalues and the eigenfunctions, we must find all the non-zero functions of the form (2.3.12) that satisfy the boundary conditions, we see that
The first and the third conditions imply that

$A + C = 0$,$B + D = 0$,
The second and the fourth conditions then imply

$$A \cosh \alpha L + B \sinh \alpha L + C \cos \alpha L + D \sin \alpha L = 0$$

$$A \sinh \alpha L + B \cosh \alpha L - C \sin \alpha L + D \cos \alpha L = 0$$

The first two conditions imply that $A = -C$,$B = -D$.
Substituting in the last two equations gives

$$\left.\begin{array}{l} A(\cosh \alpha L - \cos \alpha L) + B(\sinh \alpha L - \sin \alpha L) = 0 \\ A(\sinh \alpha L + \sin \alpha L) + B(\cosh \alpha L - \cos \alpha L) = 0 \end{array}\right\} \qquad (2.3.13)$$

For this system to have non-zero solutions A and B its determinant must be zero. In this case, A can assume any value and

$$B = -A \frac{(\cosh \alpha L - \cos \alpha L)}{(\sinh \alpha L - \sin \alpha L)}$$

Computing the determinant and setting it to zero, we arrive at the equation

$$\cosh \alpha L \cos \alpha L = 1$$

$$or$$

$$\cos \alpha L = \frac{1}{\cosh \alpha L} \qquad (2.3.14)$$

Thus the admissible values of α are the roots of this equation. There are infinitely many roots $\alpha_1, \alpha_2, ..., \alpha_n, ...$ for each value of α_n we take $A = 1$ it then follows that

$$C = -1, B = -\frac{(\cosh \alpha_n L - \cos \alpha_n L)}{(\sinh \alpha_n L - \sin \alpha_n L)}, D = -B$$

The corresponding eigenfunctions are obtained from (12) :

$$X_n(x) = \cosh \alpha_n x - \cos \alpha_n x - \frac{(\cosh \alpha_n L - \cos \alpha_n L)}{(\sinh \alpha_n L - \sin \alpha_n L)} (\sinh \alpha_n x - \sin \alpha_n x)$$

where the α_n's are the positive roots of the equation. Using the solution for T from the previous example and superposing product solutions, we arrive at

$$u(x) = \sum_{n=1}^{\infty}\{\cosh\alpha_n x - \cos\alpha_n x - \frac{(\cosh\alpha_n L - \cos\alpha_n L)}{(\sinh\alpha_n L - \sin\alpha_n L)}(\sinh\alpha_n x - \sin\alpha_n x)\}$$
$$\times\{A_n\cos c\alpha_n^2 t + A_n^*\sin c\alpha_n^2 t\} \qquad (2.3.15)$$

We now determine the coefficients by using the initial conditions and appealing to theorem(2.3.1). We get

$$A_n = \frac{1}{k_n}\int_0^L f(x)X_n(x)dx, \qquad A_n^* = \frac{1}{\alpha_n^2 c k_n}\int_0^L g(x)X_n(x)dx$$
$$where, k_n = \int_0^L X_n^2(x)dx \qquad (2.3.16)$$

and the α_n's are the positive roots of (2.3.14). We illustrate this problem numerically

Example 7. ***: vibrating elastic beam :clamped case***
consider the boundary value problem (2.3.1), (2.3.9), (2.3.10), with
$L = 1, c = 1,$
$f(x) = \sin^2 \pi x, g(x) = 0$

The solution of this problem is given by (2.3.15), where A_n and A_n^* are given by (2.3.16) and α_n is the n^{th} positive root of (2.3.14), with $L = 1$, the eigenfunctions are

$$X_n(x) = \cosh\alpha_n x - \cos\alpha_n x - \frac{(\cosh\alpha_n - \cos\alpha_n)}{(\sinh\alpha_n - \sin\alpha_n)}(\sinh\alpha_n x - \sin\alpha_n x)$$

Since g is identically zero, we have $A_n^* = 0$.
Given $L = 1$ and compute the first five positive roots of the equation $\cos\alpha = \frac{1}{\cosh\alpha}$ with the help of a compute algebra system these and other relevant numerical values are shown in table 1

n	1	2	3	4	5
α_n	4.7300	7.8532	10.9956	14.1372	17.2788
α_n^2	22.3729	61.6727	120.9032	199.8604	298.5569
A_n	0.612	0	-0.022	0	-0.002

Table 1*: numerical data for example 1*

Using the data from table 1, we find

$$u(x,t) = 0.612X_1(x)\cos\alpha_1^2 t - 0.022X_3(x)\cos\alpha_3^2 t$$
$$-0.002X_5(x)\cos\alpha_5^2 t + ...$$

Here u is approximated up to the fifth partial sum,

$$u(x,t) \approx 0.612X_1(x)\cos\alpha_1^2 t - 0.022X_3(x)\cos\alpha_3^2 t - 0.002X_5(x)\cos\alpha_5^2 t.$$

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