THE MAXIMUM MODULUS THEOREM

By

DEREJE LEGESSE

DEPARTMENT OF MATHEMATICS

ADDIS ABABA UNIVERSITY

ADVISOR: MENGISTU GOA SANGAGO (Ph.D)

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DECLARATION

I declare that this project has been composed by me and that no part of the project has formed the basis for the award of any Degree, Diploma, Associate ship, Fellowship, or any other similar title to me.

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February 2013 **DEREJE LEGESSE**

PERMISSION

This is to certify that this project is compiled by **Mr. Dereje Legesse** in the Department of Mathematics, Addis Ababa University, under my supervision. I here by also confirm that the project can be submitted for evaluation by examiners and eventual defense.

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February 2013 **MENGISTU GOA SANGAGO (PhD)**

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ABSTRACT

The maximum Modulus Theorem expressing one of the basic properties of the modulus of analytic function. The purpose of this paper is to present some variants of the maximum modulus theorem and present the application of maximum modulus theorem. The convex functions form a special class of function, defined on convex subset of the real line (that is intervals), and having the geometric property. Their importance in varies fields of analysis is steadily growing. We are also concerned with two interesting themes, “convexity and continuity” and “convexity and differentiability.”

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INTRODUCTION.

Complex variable is a subject which has something for all mathematicians. In addition to having applications to other parts of analysis, it can rightly claim to be an ancestor of many areas of mathematics (e.g., homotopy theory, manifolds).

This project has two chapters in which each of the chapter was subdivided by section.

The first chapter deals with elementary properties and definition of analytic functions, definition of some topological concepts in the set of complex plane. In this chapter one of the section was some applications of Cauchy's Integral Theorem was given. It is shown how to count the number of zeros inside a curve; also, using some information on the existence of roots of an analytic equation, it will be proved that a non-constant analytic function on a region maps open sets onto open sets which leads us to proof The Open Mapping Theorem.

The second chapter, covering the required point maximum modulus principle and Mobius transformations. This chapter continues the study of a property of analytic functions seen in the first chapter. In the first section maximum modulus principle is presented again with a second proof, and other versions of it are also given. The remainder of the section is devoted to various extensions and applications of this maximum principle.

The last section presents some results of E. Phragmen and E. Lindelof (published in 1908) which extend the Maximum Principle by easing the requirement of boundedness on the boundary. The Phragmen-Lindelof Theorem bears a relation to the Maximum Modulus Theorem which is analogous to the relationship Liouville’s theorem.

CHAPTER ONE

PRELIMINARY TO COMPLEX ANALYTIC FUNCTIONS.

1.1 Elementary Properties and Definition of Analytic Functions

1.1.1 Definition. The system of complex numbers denoted $\mathbb{C}$,is the set of order paier $\mathbb{R}^2$ together with the usual rules of vector addition and scalar multiplication by a real number a, namely

$$(x_1, y_1) + (x_2 + y_2) = (x_1 + x_2, y_1 + y_2)$$

$$a(x, y) = (ax, ay)$$

and with the operation of complex multiplication defined by

$$(x_1, y_1)(x_2, y_2) = (x_1x_2 - y_1y_2, x_1y_2 + y_1x_2)$$

1.1.2 Definition. A metric space is a pair (X,d) where X is a set and d is a function from XxX into $\mathbb{R}$ called a distance function or metric, which satisfies the following conditions for x, y and z in X :

i. $d(x, y) \geq 0$

ii. $d(x, y) = 0$ if and only if $x = y$

iii. $d(x, y) = d(y, x)$ (Symmetry)

iv. $d(x,z) \leq d(x, y) + d(y,z)$ (Triangle inequality)

If x and $r > 0$ are fixed then define $B(x,r) = \{y \in X : d(x, y) < r\}$

$$\bar{B}(x,r) = \{y \in X : d(x, y) \leq r\}$$

$B(x,r)$ and $\bar{B}(x,r)$ are called the open and closed balls, respectively, with center x and radius r .

1.1.3 Definition. For a metric space (X,d) a set $G \subseteq X$ is open if for each z_0 in G there is $\in > 0$ such that $B(z_0;\in) \subseteq G$. Thus, a set in $\mathbb{C}$ is open if it has no "edge."

1.1.4 Definition. A metric space (X,d) is connected if the only subset of X which are both open and closed are □ and X . If $A \subseteq X$ then A is a connected subset of X if the metric space (A,d) is connected.

1.1.5. Definition. Let A be a subset of X. Then the interior of A, int A is the set $\bigcup\{G: G$ is open and $G \subseteq A\}$. The closure of A, $\overline{A}$ is the set $\cap\ \{F: F$ is closed and $F \supseteq A\}$ or for any $A \subseteq \mathbb{C}$, the closure of A is the set $\overline{A} = \{z:$ there is some sequence (a_n) in A such that $a_n \to z\}$.

Remark. $\overline{A}$ may be X and $intA$ may be empty.

1.1.6 Definition. The boundary of a set $A \subseteq \mathbb{C}$ is the set $A \equiv \{z: \forall r > 0, B(z,r) \cap A \neq \square$, $and\ B(z,r) \cap \mathbb{C} - A \neq \square\}$. In other words a point z belongs to the boundary of the set A provided that every open disc around z contains both points of A and point's not in A.

1.1.7 Definition. Let $A \subseteq \mathbb{C}$, then A is called a bounded set if there exists a real number K such that $|z| \leq K$ for all $z \in A$.

1.1.8 Definition. Let (X, d) be a metric space and K is a subset of X. A collection $\Im$ of open set in X satisfying the property ;

$$K \subseteq \cup\ \{G: G \in \Im\} \tag{1.1}$$

is called a cover of K; if each member of $\Im$ is an open set it is called an open cover of K.

1.1.9 Definition. A subset K of a metric space X is compact if for every collection $\Im$ of open sets in X with the property (1.1) there is a finite number of sets $G_1, G_2, G_3, \dots, G_n$ in $\Im$ such that

$$K \subseteq G_1 \cup G_2 \cup G_3 \cup \dots \cup G_n.$$

1.1.10 Definition. A path in a region $G \subseteq \mathbb{C}$ is a continuous function $\Upsilon: [a, b] \to G$ for some interval $[a,b]$ in $\mathbb{R}$. If Υ' exists for each t in $[a,b]$ and $\gamma': [a,b] \to \mathbb{C}$ is continuous then γ is called a **smooth path**. Also Υ is called **piecewise smooth** if there is a partition of $[a,b]$, $a = t_0 < t_1 < t_2 < t_3 < \dots < t_n = b$, such that Υ is smooth on each subinterval $[t_{j-1}, t_j]$, $1 \leq j \leq n$

1.1.11 Definition. If $\Upsilon: [a, b] \to \mathbb{C}$ is a path then the set $\{\Upsilon(t): a \leq t \leq b\}$ is called the trace of Υ and is denoted it by $\{\Upsilon\}$. Υ is a rectifiable path if Υ is a function of bounded variation. If P is a partition of $[a,b]$ then $V(\Upsilon; P)$ is exactly the sum of lengths of line segments connecting points on the trace of Υ. To say that Υ is rectifiable is to say that Υ has finite length and its length is $V(\Upsilon)$.

Remark. The trace of a path is always a compact set.

1.2 Zero of analytic function

1.2.1 Definition. A power series about z_0 is an infinite series of the form $\sum_{n=0}^{\infty} a_n (z - z_0)^n$ where $z \in \mathbb{C}$ and $a_0, a_1, a_3, \dots \in \mathbb{C}$.

1.2.2 Definition. A function $f(z)$ is analytic in $|z-z_0|<R$ is said to have zero of order $m \in z^+$ (positive integer) at the point z_0 if; $f(z_0)=f'(z_0)=...=f^{(m-1)}(z_0)=0$ and $f^{(m)}(z_0)\neq 0$. For $m=1$, z_0 is a simple zero of f.

Suppose $f(z)$ is analytic at z_0 , then we have for all

$$|z-z_0|<R, \quad f(z)=\frac{f(z_0)}{0!}+\frac{f'(z_0)}{1!}(z-z_0)+...+\frac{f^{(n)}(z_0)}{n!}(z-z_0)^n+...$$

1.2.3 Theorem. Let a function f be analytic at a point z_0. It has a zero of order m at z_0 if and only if there is a function g , which is analytic and non-zero at z_0, such that $f(z)=(z-z_0)^m g(z)$.

Proof. To prove both direction of this theorem, we use the fact that if a function is analytic at a point z_0 , then it must have a Taylor series representation in powers of $z-z_0$ which is valid throughout a neighborhood $|z-z_0|<\in$ of z_0.

($\Leftarrow$) Assume $f(z)=(z-z_0)^m g(z)$. Since $g(z)$ is analytic at z_0, $g(z_0)\neq 0$, it has a Taylor series representation ; $g(z)=\dfrac{g(z_0)}{0!}+\dfrac{g'(z_0)}{1!}(z-z_0)+...+\dfrac{g^{(n)}(z_0)}{n!}(z-z_0)^n+...$ (1.2)

in some neighborhood $|z-z_0|<\in$ of z_0. Therefore if we multiply both sides by $(z-z_0)^m$ we get

$(z-z_0)^m g(z)=\dfrac{g(z_0)}{0!}(z-z_0)^m+\dfrac{g'(z_0)}{1!}(z-z_0)^{m+1}+\dfrac{g''(z_0)}{2!}(z-z_0)^{m+2}+...$. Since $f(z)=(z-z_0)^m g(z)$

then we have $f(z)=\dfrac{g(z_0)}{0!}(z-z_0)^m+\dfrac{g'(z_0)}{1!}(z-z_0)^{m+1}+...$ when $|z-z_0|<\in$. But this is actually a Taylor series expansion for $f(z)$, it follows that;

$$f(z_0)=f'(z_0)=...=f^{(m-1)}(z_0)=0 \text{ and } f^{(m)}(z_0)=m!g(z_0)\neq 0 \qquad (1.3)$$

Hence z_0 is the zero of order m of f.

($\Rightarrow$)If we assume that f has a zero of order m at z_0,the analyticity of f at z_0 and the fact that equation (1.3) hold tells as that in some neighborhood $|z-z_0|<\in$, there is a Taylor series;

$$f(z)=\frac{f(z_0)}{0!}+\frac{f'(z_0)}{1!}(z-z_0)+\frac{f''(z_0)}{2!}(z-z_0)^2+...+\frac{f^{(m)}(z_0)}{m!}(z-z_0)^m+\frac{f^{(m+1)}(z_0)}{(m+1)!}(z-z_0)^{(m+1)}...$$

This implies $f(z)=\dfrac{f^{(m)}(z_0)}{m!}(z-z_0)^m+\dfrac{f^{(m+1)}(z_0)}{(m+1)!}(z-z_0)^{(m+1)}...=\displaystyle\sum_{n=m}^{\infty}\frac{f^{(n)}(z_0)}{n!}(z-z_0)^n$

$f(z) = (z - z_0)^m \left[\frac{f^{(m)}(z_0)}{m!} + \frac{f^{(m+1)}(z_0)}{(m+1)!} + \ldots \right]$. Consequently $f(z) = (z - z_0)^m g(z)$ Where; $g(z) = \left[\frac{f^{(m)}(z_0)}{m!} + \frac{f^{(m+1)}(z_0)}{(m+1)!} + \frac{g^{(m+2)}(z_0)}{(m+2)!} + \ldots \right]$ for $|z - z_0| < \in$. That is $f(z) = (z - z_0)^m g(z)$. Since $g(z_0) = \frac{f^{(m)}(z_0)}{m!} \neq 0$, so $g(z_0) \neq 0$ and also $g(z)$ is analytic at z_0 and also since it is represented by a power series which converges in $|z - z_0| < \in$.■

Example. The polynomial function $f(z) = z^3 - 8 = (z-2)(z^2 + 2z + 4)$ has zero of order $m = 1$ at $z_0 = 2$, Since $f(z) = (z-2)g(z)$ where $g(z) = z^2 + 2z + 4$ and because f and g are entire function and $g(2) = 12 \neq 0$.Note how the fact that $z_0 = 2$ is a zero of order $m = 1$ of f also follows from the observation that f is entire function and that $f(2) = 0$ and $f'(2) = 12 \neq 0$

1.2.4 Lemma. The zeros of an analytic function are isolated.

Proof. Suppose $f(z)$ has a zero of order m at z_0. In view of theorem (1.2.3), we can write $f(z) = (z - z_0)^m g(z)$ where $g(z_0) \neq 0$ and $g(z)$ is analytic at z_0.

Claim. If a function $g(z)$ is continuous at z_0 and $g(z_0) \neq 0$, then there exists a neighborhood of z_0 in which $g(z)$ is nonzero.

Proof. Given that $g(z)$ is continuous at z_0 then, for given $\in > 0$ there exists a $\delta > 0$ such that

$$|g(z) - g(z_0)| < \in \text{ whenever } |z - z_0| < \delta \tag{1.4}$$

Also again given $g(z_0) \neq 0$, that is $|g(z_0)| \neq 0$. Suppose $g(z) = 0$ at some point in $N_\delta(z_0)/\{z_0\}$. Then, from (1.4), $|g(z)| < \in$, since $\in > 0$ is arbitrary, on choosing $\epsilon = \frac{|g(z_0)|}{2}$ this is contradiction .

This concludes $g(z) \neq 0, \forall z \in N_\delta(z_0)$. So from this claim the function $g(z)$ is analytic at z_0 and $g(z_0) \neq 0$, it follows that there exists a neighborhood of z_0, say $|z - z_0| < \delta$ in which the function is nonzero. Now $f(z) = (z - z_0)^m g(z) \neq 0$, $0 < |z - z_0| < \delta$. Thus, a deleted neighborhood of z_0 is available in which $g(z)$ is nonzero and since z_0 is arbitrary the assertion is proved. ■

1.3 Counting zeros; the Open Mapping Theorem

In this section some applications of Cauchy's Integral Theorem are given. It is shown how to count the number of zeros inside a curve; also, using some information on the existence of roots of an analytic equation, it will be proved that a non-constant analytic function on a region maps open sets onto open sets.

1.3.1 Definition. If γ is a closed rectifiable curve in $\mathbb{C}$ then for $a \notin \{\gamma\}$, $n(\gamma;a)=\frac{1}{2\pi i}\int_\gamma \frac{1}{z-a}dz$

is called the index of γ with respect to the point a. It is also sometimes called the winding number of γ around a.

1.3.2 Definition. If γ is a closed rectifiable curve in G then γ is homotopic to zero $(\gamma \sim 0)$ if γ is homotopic to a constant curve.

1.3.3 Definition. If G is an open set then γ is homologous to zero, in symbols $\gamma \approx 0$, if $n(\gamma;w)=0$ for all w in $\mathbb{C}-G$.

1.3.4 Theorem. Let G be a region and let f be an analytic function on G with zeros $a_1,a_2,a_3,a_4,\ldots a_m$ (repeated according to multiplicity). If γ is a closed rectifiable curve in G which does not pass through any point a_k and if $\gamma \approx 0$ in G then;

$$\frac{1}{2\pi i}\int_\gamma \frac{f'(z)}{f(z)}dz=\sum_{k=1}^{m} n(\gamma;a_k).$$

Proof. Suppose G is a region and let f be analytic in G with zeros at $a_1,a_2,a_3,a_4,\ldots a_m$ (where some of the a_k may be repeated according to the multiplicity of the zero multiplicity). So we can write

$$f(z)=(z-a_1)(z-a_2)(z-a_3)\ldots(z-a_m)g(z) \qquad (1.5)$$

where g is analytic on G and $g(z)\neq 0$ for any $z\in G$. So if $g(z)\neq 0$ for any $z\in G$ then $\frac{g'(z)}{g(z)}$ is analytic in G; since $\gamma \approx 0$,by Cauchy's theorem we have $\int_\gamma \frac{g'(z)}{g(z)}=0$. Now from the above equation (1.5) $f(z)=(z-a_1)(z-a_2)\ldots(z-a_m)g(z)$, applying the formula for differentiating a product;

$$f'(z)=(z-a_2)(z-a_3)\ldots(z-a_m)g(z)+(z-a_1)(z-a_3)\ldots(z-a_m)g(z)+(z-a_1)(z-a_2)\ldots(z-a_m)g(z) \\ +\ldots(z-a_1)(z-a_2)\ldots(z-a_m)g'(z)$$

$$\frac{f'(z)}{f(z)}=\frac{(z-a_2)(z-a_3)\ldots(z-a_m)g(z)}{(z-a_1)(z-a_2)\ldots(z-a_m)g(z)}+\frac{(z-a_1)(z-a_3)\ldots(z-a_m)g(z)}{(z-a_1)(z-a_2)\ldots(z-a_m)g(z)}+\frac{(z-a_1)(z-a_2)\ldots(z-a_m)g(z)}{(z-a_1)(z-a_2)\ldots(z-a_m)g(z)} \\ +\ldots+\frac{(z-a_1)(z-a_2)\ldots(z-a_m)g'(z)}{(z-a_1)(z-a_2)\ldots(z-a_m)g(z)}.$$

$$\Rightarrow \frac{1}{2\pi i}\int_{\gamma}\frac{f'(z)}{f(z)}dz = \frac{1}{2\pi i}\int_{\gamma}\frac{1}{z-a_1}dz + \frac{1}{2\pi i}\int_{\gamma}\frac{1}{z-a_2}dz + \frac{1}{2\pi i}\int_{\gamma}\frac{1}{z-a_3}dz + \ldots \frac{1}{2\pi i}\int_{\gamma}\frac{g'(z)}{g(z)}dz.$$

$$\Rightarrow \frac{1}{2\pi i}\int_{\gamma}\frac{f'(z)}{f(z)}dz = n(\gamma;a_1)+n(\gamma;a_2)+n(\gamma;a_3)+n(\gamma;a_4)+\ldots+n(\gamma;a_m) = \sum_{k=1}^{m} n(\gamma;a_k) \blacksquare$$

1.3.5 Corollary. Suppose G is a region and let f be analytic in G with $a_1, a_2, a_3, a_4, \ldots a_m$ are the points in G that satisfies $f(z)=\alpha$ and if γ is a closed rectifiable curve in G which does not pass through any point a_k and if $\gamma \approx 0$ in G then

$$\frac{1}{2\pi i}\int_{\gamma}\frac{f'(z)}{f(z)-\alpha}dz = \sum_{k=1}^{m} n(\gamma;a_k)$$

Proof: Suppose G is a region and let f be analytic in G and $a_1, a_2, a_3, a_4, \ldots a_m$ are the points in G satisfies the equation $f(z)=\alpha$. So we can write

$$f(z)-\alpha = (z-a_1)(z-a_2)(z-a_3)\ldots(z-a_m)g(z) \qquad (1.6)$$

where g is analytic on G and $g(z)\neq 0$ for any $z\in G$. If $g(z)\neq 0$ for any $z\in G$ then $\frac{g'(z)}{g(z)}$ is analytic in G; since $\gamma \approx 0$, by Cauchy's theorem we have $\int_{\gamma}\frac{g'(z)}{g(z)} = 0$. Now from above (1.6), $f(z)-\alpha = (z-a_1)(z-a_2)(z-a_3)\ldots(z-a_m)g(z)$, we have

$$f'(z) = (z-a_2)(z-a_3)\ldots(z-a_m)g(z)+(z-a_1)(z-a_3)\ldots(z-a_m)g(z)+(z-a_1)(z-a_2)\ldots(z-a_m)g(z)+\ldots$$
$$\ldots(z-a_1)(z-a_2)\ldots(z-a_m)g'(z)$$

$$\frac{f'(z)}{f(z)-\alpha} = \frac{(z-a_2)(z-a_3)\ldots(z-a_m)g(z)}{(z-a_1)(z-a_2)\ldots(z-a_m)g(z)} + \frac{(z-a_1)(z-a_3)\ldots(z-a_m)g(z)}{(z-a_1)(z-a_2)\ldots(z-a_m)g(z)} + \frac{(z-a_1)(z-a_2)\ldots(z-a_m)g(z)}{(z-a_1)(z-a_2)\ldots(z-a_m)g(z)} + \ldots$$
$$\ldots\frac{(z-a_1)(z-a_2)\ldots(z-a_m)g'(z)}{(z-a_1)(z-a_2)\ldots(z-a_m)g(z)}$$

$$\Rightarrow \frac{1}{2\pi i}\int_{\gamma}\frac{f'(z)}{f(z)-\alpha}dz = \frac{1}{2\pi i}\int_{\gamma}\frac{1}{z-a_1}dz + \frac{1}{2\pi i}\int_{\gamma}\frac{1}{z-a_2}dz + \frac{1}{2\pi i}\int_{\gamma}\frac{1}{z-a_3}dz + \ldots \frac{1}{2\pi i}\int_{\gamma}\frac{g'(z)}{g(z)}dz$$

$$\Rightarrow \frac{1}{2\pi i}\int_{\gamma}\frac{f'(z)}{f(z)-\alpha}dz = n(\gamma;a_1)+n(\gamma;a_2)+n(\gamma;a_3)+n(\gamma;a_4)+\ldots+n(\gamma;a_m) = \sum_{k=1}^{m} n(\gamma;a_k) \blacksquare$$

1.3.6 Theorem. Suppose f is analytic in $B(a,R)$ and let $\alpha = f(a)$. If $f(z)-\alpha$ has a zero of order m at $z=a$ then there is an $\in > 0$ and $\delta > 0$ such that for $|\xi-\alpha|<\delta$, the equation $f(z)=\xi$ has exactly m simple roots in $B(a;\in)$.

Proof. Since the condition that $f(z)-\alpha$ have a zero of finite multiplicity guarantees that f is not constant. And again since the zeros of analytic function are isolated, we can choose $\in > 0$ such that $\in < \frac{R}{2}$, $f(z)=\alpha$ has no solution with $0<|z-a|<2\in$ and $f'(z)\neq 0$ if $0<|z-a|<2\in$. (If $m\geq 2$ then $f'(a)=0$). Let $\gamma(t)=a+\in e^{(2\pi i t)}, 0\leq t\leq 1$, and Put $\sigma = fo\gamma$. Since f is analytic on G and γ is a rectifiable curve in G, so σ is a rectifiable curve. Now $\alpha \notin \{\sigma\}$; so there is a $\delta > 0$ such that $B(\alpha;\delta)\cap\{\sigma\} = \square$. Thus, $B(\alpha;\delta)$ is contained in the same component of $\mathbb{C}-\{\sigma\}$;

That is, $|\alpha-\xi|<\delta \Rightarrow n(\sigma;\alpha)=n(\sigma;\xi)$

But we have; $n(\sigma;\alpha)=\frac{1}{2\pi i}\int_\sigma \frac{dw}{f(w)-\alpha}=\frac{1}{2\pi i}\int_\gamma \frac{f'(z)}{f(z)-\alpha}dz=\sum_{k=1}^{m} n(\gamma;z_k(\alpha))$ and

$$n(\sigma;\xi)=\frac{1}{2\pi i}\int_\sigma \frac{dw}{f(w)-\xi}=\frac{1}{2\pi i}\int_\gamma \frac{f'(z)}{f(z)-\xi}dz=\sum_{k=1}^{m} n(\gamma;z_k(\xi))$$

So if $|\alpha-\xi|<\delta$ then $n(\sigma;\alpha)=n(\sigma;\xi)$ or $\sum_{k=1}^{m} n(\gamma;z_k(\alpha)=\sum_{k=1}^{m} n(\gamma;z_j(\xi)$ where $z_k(\alpha)$ and $z_j(\xi)$ are the points in G that satisfies $f(z)=\alpha$ and $f(z)=\xi$ respectively. But since $n(\gamma;z)$ must be either zero or one, we have that there are exactly m solutions to the equation $f(z)=\xi$ inside $B(a;\in)$ .Since $f'(z)\neq 0$ for $0<|z-a|<\in$, each of these roots (for $\xi\neq\alpha$) must be simple; by theorem 1.3.4 above. ■

1.3.7 Open Mapping Theorem. Let G be a region and suppose that f is a non-constant analytic function on G. Then for any open set U in G, $f(U)$ is open.

Proof. Let U be open in G. We want to show that $f(U)$ is open. Let $\alpha\in f(U)$, then since U is open in G there is some $a\in U$ such that $f(a)=\alpha$. Since f is not constant, the point a is an isolated zero of $f(z)-\alpha$ which means that there is $\in > 0$ such that $B(a,\in)\subseteq U$ and $f(z)-\alpha$ has no zero in the punctured disk $B'(a,\in)$. In particular $f(z)-\alpha$ does not vanish on the circle $|z-a|=\frac{\in}{2}$. Let γ be the curve given by $\gamma(t)=\frac{\in}{2}e^{2\pi i t}, 0\leq t\leq 1$. By the preceding theorem 1.3.6, there is some $\delta > 0$ such that $f(z) - \xi$ certainly has zeros inside γ whenever $\xi \in B(\alpha, \delta)$. But this is simply means that $\xi \in B(\alpha, \delta) \subseteq f(U)$. That is $B(f(a), \delta)) \subseteq f(U)$. Therefore $f(U)$ is open. ■

CHAPTER TWO

THE MUXIMUM MODULUS THEOREM

INTRODUCTION

One of the most powerful consequences of the Cauchy integral formula is The Maximum Modulus Theorem, also called The Maximum Modulus Principle. It states that if f is a non-constant analytic function on a region G, then $|f|$ cannot have a local maximum anywhere inside G, It can attain a maximum only on the boundary of G. This theorem and the Cauchy integral formula will be used to develop some of the important properties of harmonic function. This chapter continues the study of a property of analytic functions and is devoted to various extensions and applications of this maximum principle.

2.1 The Maximum Principle

Let Ω be any subset of $\mathbb{C}$ and suppose α is in the interior of Ω. We can, therefore, choose a positive number ρ such that $B(\alpha;\rho)\subseteq\Omega$; it readily follows that there is a point ξ in Ω with $|\xi|>|\alpha|$. To state this in another way, if α is a point in Ω with $|\alpha|\geq|\xi|$ for each ξ in the set Ω then α belongs to $\partial\Omega$.

2.1.1 Maximum Modulus Theorem-First Version.

If f is analytic in a region G and a is a point in G with $|f(a)|\geq|f(z)|$ for all z in G then f must be a constant function.

Proof. Let $\Omega = f(G)$ and $\alpha = f(a)$ for a point a in G. From the hypothesis we have that $|\alpha|\geq|\xi|$ for each ξ in Ω ; as in the discussion preceding the theorem α is in $\partial\Omega\cap\Omega$. Therefore $\partial\Omega\cap\Omega\neq\phi$. In particular, the set Ω cannot be open. Hence the Open Mapping Theorem says that f must be constant. ■

2.1.2 Corollary. Suppose f is analytic and non-constant in a region G. Then for any $z_0\in G$ there is $z\in G$ such that $|f(z)|>|f(z_0)|$.

Proof. Given f is analytic and non-constant in a region G. We went to show that $|f(z)|>|f(z_0)|$. Since $z_0\in G$, $f(z_0)\in f(G)$ and by Open Mapping Theorem $f(G)$ is open. Then there is some $\rho>0$ such that $B(f(z_0),\rho)\subseteq f(G)$.

Suppose that $f(z_0)=\mathrm{Re}^{i\theta}$, for θ is real, where $|f(z_0)|=|\mathrm{Re}^{i\theta}|=R|e^{i\theta}|=R$. Then for any $0<r<\rho$, $f(z_0)+re^{i\theta}\in B(f(z_0),\rho)\subseteq f(G)$. Hence there is $z\in G$ such that $f(z)=f(z_0)+re^{i\theta}$. But $|f(z)|=|f(z_0)+re^{i\theta}|=|\mathrm{Re}^{i\theta}+re^{i\theta}|=R+r$. That is $|f(z)|=R+r=|f(z_0)|+r>|f(z_0)|$.

Therefore we get $|f(z)|>|f(z_0)|$.■

2.1.3 Maximum Modulus Theorem-Second Version.

Let G be a bounded region and suppose that $f : \bar{G} \to \mathbb{C}$ is continuous function that is analytic in G. Then

$$\max\{|f(\xi)| : \xi \in \overline{G}\} = \max\{|f(\xi)| : \xi \in \partial G\}$$

Proof. If f is constant on $\overline{G}$, since G is bounded for every point $a \in \overline{G}$ such that $|f(a)| = |f(z)|$ for all z in $\overline{G}$. Suppose f is not constant. Since G is bounded, its closure $\overline{G}$ is bounded and closed. Therefore, $\overline{G}$ is compact. The continuity of f on $\overline{G}$ implies that $|f|$ is continuous real-valued function on $\overline{G}$ and which therefore is bounded on $\overline{G}$ and achieves its maximum; that is $|f|$ achieves its maximum at some point $\xi \in \overline{G}$ such that $|f(\xi)| = \max\{|f(z)| : z \in \overline{G}\}$. By first version of The Maximum Modulus Theorem $|f|$ does not attain its maximum value in G, $|f(\xi)| = \max\{|f(z)| : z \in \overline{G}\}$ for all $z \in G$. It follows that $\xi \in \overline{G} - G = \partial G$. Therefore

$$\max\{|f(z)| : z \in \overline{G}\} = |f(\xi)| = \max\{|f(z)| : z \in \partial G\}$$

Remark. For unbounded region the above result may be false.

Example1. Let G be the infinite horizontal strip $G = \{z: \frac{-\pi}{2} < \operatorname{Im} z < \frac{\pi}{2}\}$ and $f(z) = \exp(\exp z) : z \in \mathbb{C}$. Then f is an entire function and so it is continuous on $\overline{G}$.

Claim. f is bounded on the boundary of G. To see this, let $\xi \in \partial G$, so that $\xi = x \pm i\frac{\pi}{2}$ for some $x \in \mathbb{R}$. We have $f(\xi) = \exp(\exp(x \pm i\frac{\pi}{2})) = \exp(e^x e^{\pm i\pi/2}) = e^{\pm ie^x} = \operatorname{Cos} e^x \pm i\sin e^x$.

So $|f(\xi)| = \operatorname{Cos}^2(e^x) + \sin^2(e^x) = 1$ .That is $|f(\xi)| = 1$ for every $\xi \in \partial G$, the boundary of G. Is $|f(\xi)| < 1$ for all $\xi \in G$? The answer is no. For example, suppose that $\mathbf{z = x \in \mathbb{R} \subseteq G}$. Then we find that $f(z) = f(x) = \exp(\exp z) = e^{e^x}$ and clearly $|f(z)| = e^{e^x} \to \infty$ as $x \to \infty$ and so $|f(z)|$ is not even bounded on G never mind being less than 1. Thus, the hypothesis that G is bounded is essential in The Maximum Modulus Theorem-Second Version.

Example (2) Let $f(z) = e^{iz^2} = e^{i(x^2 - y^2)}e^{-2xy}$, so that $|f(z)| = e^{-2xy}$

If $G = \{z \in \mathbb{C}: \operatorname{Re} z > 0, \operatorname{Im} z < 0\}$ then, for points on the boundary ∂G, either $x = 0$ or $y = 0$ and so $|f(z)| = 1$ on ∂G. However, for $y = -x$ $(x > 0)$ we have $|f(z)| = e^{2x^2} \to \infty$ as $x \to \infty$.

Then both examples (1) and (2) show that the modulus of an analytic function need not attain its maximum on the boundary ∂G and the maximum of the modulus on the boundary may not be the maximum value inside the domain G unless G is bounded.

Example (3) Find the maximum modulus of $f(z)=3z-2i$ on $|z|\leq 3$.

To do this, we compute $|f(z)|^2 = |3z-2i| = 9|z|^2 - 12\,\mathrm{Im}\,z + 4$. By Maximum Modulus Theorem-second version, $\max_{|z|\leq 3}|f(z)|$ occurs on the boundary |z| =3.Therefore, on |z| =3,

$$\begin{aligned}|f(z)| &= \sqrt{9|z|^2 - 12\,\mathrm{Im}\,z + 4} \\ &= \sqrt{9(3)^2 - 12\,\mathrm{Im}\,z + 4} \\ &= \sqrt{85 - 12\,\mathrm{Im}\,z}\end{aligned}$$

The last expression attains its maximum when Imz attains its minimum on |z| = 3, namely, at the point z = −3i. Thus $\max_{|z|\leq 3}|f(z)| = \sqrt{85-12(-3)} = \sqrt{121} = 11$

2.1.4 Definition. By the extended complex number system $\mathbb{C}_\infty$,we shall mean the complex plane $\mathbb{C}$ along with a symbol ∞,which satisfies the following properties:

a. If $z \in \mathbb{C}$, then we have $z+\infty = z-\infty = \infty, \frac{z}{\infty} = 0$

b. If $z \in \mathbb{C}$,but $z \neq 0$,then $z.\infty = \infty$ and $\frac{z}{0} = \infty$

c. $\infty + \infty = \infty.\infty = \infty$ and $\frac{\infty}{z} = \infty$ ($z \neq 0$)

The set $\mathbb{C} \cup \{\infty\}$ is called the extended complex plane and will be denoted by $\mathbb{C}_\infty$.

2.1.5 Definition. If $f\colon G \to \mathbb{R}$ *and* $a \in \bar{G}$ or $a=\infty$,then the limit superior of $f(z)$ as z approaches a ,denoted by $\lim_{z\to a} Sup|f(z)|$ is defined by,

$$\lim_{z\to a} Sup|f(z)| = \lim_{r\to o^+} Sup\{|f(z)| : z \in G \cap B(a,r)\}$$

(If $a=\infty$, $B(a;r)$ is the ball in the metric of $\mathbb{C}_\infty$, where $\mathbb{C}_\infty = \mathbb{C}$ U $\{\infty\}$. Similarly the limit inferior of $f(z)$ as z approaches a , denoted by $\lim_{z\to a}\inf f(z)$ is defined by,

$$\lim_{z\to a}\inf|f(z)| = \lim_{r\to 0^+}\inf\{|f(z)| : z \in G \cap B(a,r)\},$$

and so $\lim_{z\to a} f(z)$ exists and equals α iff $\alpha = \lim_{z\to a} Sup|f(z)| = \lim_{z\to a}\inf|f(z)|$.

If $G \subseteq \mathbb{C}$ then let $\partial_\infty G$ denote the boundary of G in $\mathbb{C}_\infty$ and call it the extended boundary of G.

$$\partial_\infty G = \partial G \text{ if G is bounded and } \partial_\infty G = \partial G \cup \{\infty\} \text{ if } G \text{ is unbounded.}$$

2.1.6 Maximum Modulus Theorems -Third Versions.

Let G be a region in $\mathbb{C}$ and f is analytic function on G. Suppose there is a constant M such that $\limsup_{z\to a} |f(z)| \le M$ for all a in $\partial_\infty G$ .Then

$$|f(z)| \le M \quad \text{for all } z \text{ in } G.$$

Proof. Let $\delta > 0$ be arbitrary and put $H = \{z \in G : |f(z)| > M + \delta\}$. The theorem will be demonstrated if H is proved to be empty. Since $|f|$ is continuous, H is open.

Since $\limsup_{z\to a} |f(z)| \le M$ for each a in $\partial_\infty G$, there is a ball $B(a;r)$ such that $|f(z)| < M + \delta$ for all z in $G \cap B(a;r)$. Hence $\overline{H} \subseteq G$. Since this condition also holds if G is unbounded and $a = \infty$, H must be bounded. Thus, $\overline{H}$ is compact. So the second version of the Maximum Modulus Theorem applies. That is $|f|$ must attain its maximum value somewhere on the ∂H .

But for z in ∂H , $|f(z)| = M + \delta$. Since $\overline{H} \subseteq \{z : |f(z)| \ge M + \delta$.

Therefore, $H = \square$ or f is a constant. But the hypothesis implies that $H = \square$ if f is a constant. ■

2.2 Functions on the Unit Disc

2.2.1 Schwarz's Lemma. Let $D=\{z:|z|<1\}$ and suppose f is analytic in $|z|<1$ with

(a) $|f(z)| \leq 1$ for all z in $|z| < 1$ b) $f(0) = 0$ (2.1)

Then $|f^n(0)| \leq n!$ and $|f(z)| \leq |z|^n$ for all z in the disk D . (2.2)

Moreover if $|f^{(n)}(0)| = n!$ or $|f(z)| = |z|^n$ for some $z \neq 0$ then there is a constant w, $|w|=1$ such that $f(z) = wz^n$ for all z in disk D.

Proof. Suppose f is analytic in $|z| \leq 1$. Then from Taylor expansion and using the fact that

$f(0) = f'(0) = f''(0) = \ldots = f^{(n-1)}(0) = 0$. It follows that

$$f(z) = \frac{f^{(n)}(0)}{n!} z^n + \frac{f^{(n+1)}(0)}{(n+1)!} z^{(n+1)} + \frac{f^{(n+2)}(0)}{(n+2)!} z^{(n+2)} + \ldots + \ldots \quad ,|z|<1$$

For $z=0$, equation (2.1) is the hypothesis, since $|f(0)| = |0|^n = 0 \leq 1$ and $|f^n(0)| = 0! = 1 \leq 1$ the conclusion is true. Let $z \neq 0$, then from the above expansion;

$$\frac{f(z)}{z^n} = \frac{f^{(n)}(0)}{n!} + \frac{f^{(n+1)}(0)}{(n+1)!} z + \frac{f^{(n+2)}(0)}{(n+2)!} z^2 + \ldots + \ldots \qquad 0<|z|<1$$

The series on the right side converges for $z=0$. Let us define g from $g: G \to \mathbb{C}$ by

$$g(z) = \begin{cases} \dfrac{f(z)}{z^n}, & z \neq 0 \\ \dfrac{f^n(0)}{n!}, & z = 0 \end{cases}$$

Now $g(z)$ is continuous in the disk D. Let C be a circle $|z|=r$ where $0<r<1$. Then by The Maximum Modulus Theorem,

$$|g(z)| \leq \max_{|z|=r} |g(z)| = \max_{|z|=r} \left| \frac{f(z)}{z^n} \right| \leq \frac{1}{r^n} \quad , |z|=r$$

The above inequality is true for all $r<1$. Letting $r \to 1$, we find that

$$|g(z)| = \frac{|f(z)|}{|z|^n} \leq 1 \quad \text{Or} \quad |f(z)| \leq |z|^n .$$

Since $g(0)=\frac{f^{(n)}(0)}{n!}$ and $|g(z)|\le 1, \forall z$ in $|z|=r$, therefore $|g(0)|\le 1$. This implies that

$$|f^{(n)}(0)|\le n!$$

In case $|f(z)|=|z|^n$ for some $z\ne 0$ or $|f^{(n)}(0)|=n!$ then $|g(z)|$ assumes its maximum value inside D. Thus again applying The Maximum Modulus Theorem, this is the only possible when $g(z)$ is constant, say $g(z)=w$ or $\boldsymbol{f(z)=wz^n}$ for some constant w with $|g(z)|=|w|=1$.■

Remark: Originally, the Schwarz's lemma was proved for $n=1$

2.2.2 Definition. A mapping $f(z)$ is said to be Conformal at a point z_0 if it preserves the angle between the oriented curves passing through z_0 in magnitude as well as sense.

Example. If $f(z)=e^z$ then f is conformal throughout $\mathbb{C}$.

2.2.3 Definition. A mapping of the form $\varphi(z)=\frac{az+b}{cz+d}$ is called a linear fractional transformation. If $a,b,c,$ and d also satisfy $ad-bc\ne 0$ then $\varphi(z)$ is called a möbius transformation.

Aim. Our main theme in defining φ_a is to utilize this function in conjunction with Schwarz's Lemma for determining the estimates of $|\boldsymbol{f(z)}|$ and $|\boldsymbol{f'(z)}|$.

We can use Schwarz's Lemma to classify those mappings of the open unit disc $D(0,1)$ onto itself which are analytic, one-one and with analytic inverse.

2.2.4 Proposition. If $|a|<1$ and $D=\{z:|z|<1\}$ such that $\varphi_a:D\to D$ defined a Mobius transformation by $\varphi_a(z)=\frac{z-a}{1-\bar{a}z}$ then the function $\varphi_a(z)=\frac{z-a}{1-\bar{a}z}$

- **I.** is a one-one map of $D=\{z:|z|<1\}$ onto itself.
- **II.** $\varphi_{-a}(\varphi_a(z))=z=\varphi_a(\varphi_{-a}(z))$, if φ_a is the inverse of φ_{-a} .
- **III.** $\varphi_a(a)=0$ and $\varphi_a(0)=-a$
- **IV.** $\varphi_{-a}(0)=a$ and $\varphi_{-a}(a)=\frac{2a}{1+|a|^2}$
- **V.** $\varphi'_a(0)=1-|a|^2$ and $\varphi'_a(a)=\frac{1}{1-|a|^2}$

VI. $\varphi'_{-a}(a) = \dfrac{1-|a|^2}{(1+|a|^2)^2}$ and $\varphi'_{-a}(0) = 1 - a\bar{a}$

VII. Furthermore, φ_a maps ∂D onto ∂D.

Proof. Let $\varphi_a(z) = \dfrac{z-a}{1-\bar{a}z}$, since φ_a is analytic for $D = \{z : |z| < |a|^{-1}\}$, so that it is analytic in an open disk containing the closure of $D = \{z : |z| < 1\}$

I. To show one to one ; let $\varphi_a(z_1) = \dfrac{z_1 - a}{1-\bar{a}z_1}$ and $\varphi_a(z_2) = \dfrac{z_2 - a}{1-\bar{a}z_2}$ where $|z_1| < 1$ and $|z_2| < 1$

Let $\varphi_a(z_1) = \varphi_a(z_2)$.

Now we want to show $z_1 = z_2$. Therefore we have ;

$$\varphi_a(z_1) = \varphi_a(z_2) \Rightarrow \frac{z_1 - a}{1-\bar{a}z_1} = \frac{z_2 - a}{1-\bar{a}z_2}$$

$$\Rightarrow (z_1 - a)(1-\bar{a}z_2) = (1-\bar{a}z_1)(z_2 - a)$$

$$\Rightarrow z_1(1-\bar{a}z_2) - a(1-\bar{a}z_2) = 1(z_2 - a) - \bar{a}z_1(z_2 - a)$$

$$\Rightarrow z_1 - \bar{a}z_1z_2 - a + a\bar{a}z_2 = z_2 - a - \bar{a}z_1z_2 + \bar{a}az_1$$

$$\Rightarrow z_1 - \bar{a}az_1 = z_2 - a\bar{a}z_2$$

$$\Rightarrow z_1(1-\bar{a}a) = z_2(1-a\bar{a})$$

$\Rightarrow z_1 = z_2$, so φ_a is a one to one mapping.

II. Let $\varphi_{-a}(z)$ be the inverse of $\varphi_a(z)$ and let $w = \varphi_{-a}(z)$ then by letting $w = \dfrac{z-a}{1-\bar{a}z}$ solve for z interims of w.

That is $w - \bar{a}zw = z - a \quad \Rightarrow \quad w - \bar{a}zw = z - a$

$$\Rightarrow w + a = z + \bar{a}zw$$

$\Rightarrow z = \dfrac{w+a}{1+\bar{a}w}$. Therefore $\varphi_{-a}(z) = \dfrac{z+a}{1+\bar{a}z}$

$$\text{Now}\ \varphi_{-a}(\varphi_a(z)) = \varphi_{-a}\left(\frac{z-a}{1-\bar{a}z}\right) = \frac{\frac{z-a}{1-\bar{a}z} + a}{1 + \bar{a}\frac{z-a}{1-\bar{a}z}} = \frac{z - a + a - a\bar{a}}{1 - \bar{a}z + \bar{a}z - \bar{a}a} = z\left(\frac{1-|a|^2}{1-|a|^2}\right) = z,$$

$$\text{Similarly}\ \ \varphi_a(\varphi_{-a}(z)) = \varphi_a\left(\frac{z+a}{1+\bar{a}z}\right) = \frac{\frac{z+a}{1+\bar{a}z} - a}{1 + \bar{a}\frac{z+a}{1+\bar{a}z}} = \frac{z + a - a - a\bar{a}}{1 + \bar{a}z - \bar{a}z - \bar{a}a} = z\left(\frac{1-|a|^2}{1-|a|^2}\right) = z\ .$$

Therefore $\varphi_{-a}(\varphi_a(z)) = z = \varphi_a(\varphi_{-a}(z))$. So we conclude that φ_a has its own inverse .

III. Since $\varphi_a(z)=\dfrac{z-a}{1-\bar{a}z}$, we have $\varphi_a(a)=\dfrac{a-a}{1-\bar{a}a}=0$ and $\varphi_a(0)=\dfrac{0-a}{1-\bar{a}0}=-a$

IV. Since $\varphi_{-a}(z)=\dfrac{z+a}{1+\bar{a}z}$,then we have $\varphi_{-a}(a)=\dfrac{0+a}{1+\bar{a}0}=a$, and $\varphi_{-a}(a)=\dfrac{a+a}{1+\bar{a}a}=\dfrac{2a}{1+|a|^2}$

V. Let $\varphi_a(z)=\dfrac{z-a}{1-\bar{a}z}$ then $\varphi_a'(z)=\dfrac{1-\bar{a}z-(-\bar{a}(z-a))}{(1-\bar{a}z)^2}=\dfrac{1-a\bar{a}}{(1-\bar{a}z)^2}=\dfrac{1-|a|^2}{(1-\bar{a}z)^2}$. Therefore we have ;

$$\varphi_a'(a)=\frac{1-|a|^2}{(1-\bar{a}a)^2}=\frac{1}{1-|a|^2} \quad \text{and} \quad \varphi_a'(0)=\frac{1-|a|^2}{(1-0)^2}=1-|a|^2$$

VI. Let $\varphi_{-a}(z)=\dfrac{z+a}{1+\bar{a}z}$ then $\varphi_{-a}'(z)=\dfrac{1+\bar{a}z-\bar{a}(z+a)}{(1+\bar{a}z)^2}=\dfrac{1-|a|^2}{(1+\bar{a}z)^2}$. Therefore we have ;

$$\varphi_{-a}'(0)=\frac{1-|a|^2}{(1+\bar{a}0)^2}=1-|a|^2 \quad \text{and} \quad \varphi_{-a}'(a)=\frac{1-|a|^2}{(1+\bar{a}0)^2}=\frac{1-|a|^2}{(1+|a|^2)^2}$$

VII. Let θ be a real number and let $z=e^{i\theta}$ then $|\varphi_a(z)|=|\varphi_a(e^{i\theta})|=\left|\dfrac{e^{i\theta}-a}{1-\bar{a}e^{i\theta}}\right|=\left|\dfrac{e^{i\theta}-a}{e^{i\theta}-\bar{a}}\right|\left|e^{-i\theta}\right|=1.$

Therefore we conclude that φ_a maps ∂D onto ∂D.■

2.2.5 Corollary. Let f (z) be analytic in $|z|<1$ and $|f(z)|\le 1$ then

i. $\left|\dfrac{f(z)-f(z_0)}{1-\overline{f(z_0)}f(z)}\right| \le \left|\dfrac{z-z_0}{1-\overline{z_0}z}\right|$

ii. $|f'(z)| \le \dfrac{1-|f(z)|^2}{1-|z|^2}$ for any z, z_0 inside the unit disk (2.3)

Proof

i. From the above proposition 2.2.4, we have that; $\varphi_{-z_0}(z)=\dfrac{z+z_0}{1+\overline{z_0}z}$ and $\varphi_{z_0}(z)=\dfrac{z-z_0}{1-\overline{z_0}z}$.

Consider a function $g(z)=\varphi_{f(z_0)}f\varphi_{-z_0}(z)$ and also suppose $|z_0|<1$, $f(z_0)=\alpha$ (so $|\alpha|<1$ unless f is constant.) Note that g(z) is a function that maps D into D and is analytic such that $g(0)=\varphi_{f(z_0)}f\varphi_{-z_0}(0)=\varphi_{f(z_0)}f(z_0)=0$,since $\varphi_{-z_0}(0)=z_0$. Therefore $g(0)=0$. Thus Schwartz's lemma is applicable to g(z). So that $|g(z)|\le|z|$ for $|z|<1$ and $|g'(0)|\le 1$. But by definition of g for any z , we have $g(z)=\varphi_{f(z_0)}f\varphi_{-z_0}(z)$. That is

$$g(z)=\varphi_{f(z_0)}f\left(\frac{z+z_0}{1+\overline{z_0}z}\right)=\frac{f\left(\frac{z+z_0}{1+\overline{z_0}z}\right)-f(z_0)}{1-\overline{f(z_o)}f\left(\frac{z+z_0}{1+\overline{z_0}z}\right)}$$

Thus $|g(z)|=\left|\varphi_{f(z_0)}f\left(\frac{z+z_0}{1+\overline{z_0}z}\right)\right|=\left|\frac{f\left(\frac{z+z_0}{1+\overline{z_0}z}\right)-f(z_0)}{1-\overline{f(z_o)}f\left(\frac{z+z_0}{1+\overline{z_0}z}\right)}\right|\leq|z|$ (2.4)

Let $\alpha=\frac{z+z_0}{1+\overline{z_0}z}$ where $|\alpha|<1$ then $z+z_0=\alpha+\alpha\overline{z_0}z\Leftrightarrow z=\frac{\alpha-z_0}{1-\alpha\overline{z_0}}$ (2.5)

From equation (2.4) and (2.5) we have $\left|\frac{f(\alpha)-f(z_0)}{1-\overline{f(z_o)}f(\alpha)}\right|\leq\left|\frac{\alpha-z_0}{1-\alpha\overline{z_0}}\right|$. Replacing α by z we have

$$\left|\frac{f(z)-f(z_0)}{1-\overline{f(z_0)}f(z)}\right|\leq\left|\frac{z-z_0}{1-z\overline{z_0}}\right|.$$

The equality will occur when $|g(z)|=|z|$ or $|g'(0)|=1$, this means

$$\frac{f(z)-f(z_0)}{1-\overline{f(z_0)}f(z)}=c\frac{z-z_0}{1-z\overline{z_0}}\ ,\ |c|=1.$$

Claim: $|g(z)|=|z|$ and $f(z)=\varphi_{-f(z_0)}(c\varphi_{z_0}(z))$

Consider again equation $g(z)=\varphi_{f(z_0)}f\varphi_{-z_0}(z)$. Note that g(z) is a function from $|z|<1$ to $|z|<1$ and is analytic such that $g(0)=\varphi_{f(z_0)}f\varphi_{-z_0}(0)=\varphi_{f(z_0)}f(z_0)=0$, since $\varphi_{-z_0}(0)=z_0$. Therefore g(0) = 0. Thus Schwartz's lemma is applicable to g and

$$|g(z)|\leq|z| \tag{2.6}$$

But g^{-1} is also analytic and bijective on $|z|<1$, $g^{-1}(0)=0$, applying Schwarz's Lemma to g^{-1} we obtain ;

$$|z|=|g^{-1}(g(z))|\leq|g(z)| \tag{2.7}$$

From (2.6) and (2.7) we have $|g(z)|=|z|$ on $|z|<1$. Thus by Schwarz's Lemma

$$g(z)=cz \text{ for all z in } |z|<1. \tag{2.8}$$

To show the second part; from equation (2.8) we have $g(z) = cz$ and from above

$g(z) = \varphi_{f(z_o)} f \varphi_{-z_o}(z) = cz$. Now put $\alpha = f(z_0) \neq 0$ we get $f\varphi_{-z_0}(z) = \varphi_{-\alpha}(cz)$

$$\Leftrightarrow \varphi_{-z_0} f(z) = c\varphi_{-\alpha}(z)$$

$$\Leftrightarrow f(z) = \varphi_{z_0}(\varphi_{-\alpha} c(z)) = \varphi_{z_0}(c\varphi_{-f(z_0)}(z)) \qquad (2.9)$$

ii. From (i) above we have $|g'(0)| \leq 1$. Now obtain an explicit formula for $g'(0)$, using chain rule; for any z from $g(z) = \varphi_{f(z_0)} f\varphi_{-z_0}(z)$ we get $g'(z) = \varphi_{f(z_0)}'(f(\varphi_{-z_0}(z))).f'(\varphi_{-z_0}(z)).\varphi_{-z_0}'(z)$

$g'(0) = \varphi_{f(z_0)}'(f(\varphi_{-z_0}(0))).f'(\varphi_{-z_0}(0)).\varphi_{-z_0}'(0) = \dfrac{1-|z_0|^2}{1-|f(z_0)|^2}.f'(z_0)$, since $\varphi'_{-z_0}(0) = 1 - z_0\overline{z_0} = 1 - |z_0|^2$, $f'(\varphi_{-z_0}(0)) = f'(z_0)$, $f(\varphi_{-z_0}(0)) = f(z_0)$ and

$\varphi_{f(z_0)}'(f(\varphi_{-z_0}(0))) = \varphi_{f(z_0)}'(f(z_0)) = \dfrac{1}{\left(1-|f(z_0)|^2\right)^2}$. Therefore

$$g'(0) = \frac{1-|z_0|^2}{1-|f(z_0)|^2}.f'(z_0) \text{ and } g'(0) \leq 1.$$

This implies that $|f'(z_0)| \leq \dfrac{1-|f(z_0)|^2}{1-|z_0|^2}$. Since z_0 is arbitrary in $|z| < 1$, so

$$|f'(z)| \leq \frac{1-|f(z)|^2}{1-|z|^2} \qquad (2.10)$$

Moreover equality will occur exactly when $|g'(0)| = 1$ or by virtue of Schwarz's Lemma when there is a constant c with $|c| = 1$ and $f(z) = \varphi_{z_0}(c\varphi_{\alpha}(z))$ for all $|z| < 1$.

Note that if $|c| = 1$ and $|a| < 1$ then $c\varphi_a$ defines a one to one analytic map of the open unit disk D onto itself. ■

2.2.6 Theorem. Let $f : D \to D$ be a One – One analytic map of D onto itself and suppose that $f(z_0) = 0$. Then there is a complex number c with $|c| = 1$ such that $f(z) = c\varphi_{z_0}(z)$.

Proof. Since f is one to one and onto there is an analytic function $g : D \to D$ such that

$g(f(z))=z$ for $|z|<1$. From the above lemma 2.2.5 (ii) we have $|f'(z_0)|\le\frac{1-|f(z_0)|^2}{1-|z_0|^2}$ for arbitrary z_0 in $|z|<1$, where $f(z_0)=\alpha$. Applying this inequality to both f and g on equation (2.10) in this theorem $f(z_0)=0$ we have; $|f'(z_0)|\le\frac{1-|f(z_0)|^2}{1-|z_0|^2}=\frac{1}{1-|z_0|^2}$ and $|g'(0)|\le\frac{1-|g(0)|^2}{1-|0|^2}=1-|z_0|^2$, since $g(f(z_0))=z_0$.So,

$$g(f(z_0))=g(0)=z_0.$$

But $g(f(z))=z\Rightarrow g'(f(z)).f'(z)=1$ .So we get $g'(0).f'(0)=1$. Therefore $|f'(z_0)|=\frac{1}{1-|z_0|^2}$.

Again from the above equation (2.9) we have $f(z)=\varphi_{-f(z_0)}(c\varphi_{z_0}(z))$, applying this formula we have $\varphi_{f(z_0)}f(z)=c\varphi_{z_0}(z))$

$$\Leftrightarrow\frac{f(z)-f(z_0)}{1-\overline{f(z_0)}f(z)}=c\frac{z-z_0}{1-z\overline{z_0}}$$. Then solve for $f(z)$.

$$\Leftrightarrow(f(z)-f(z_0))(1-z\overline{z_0})=(1-\overline{f(z_0)}f(z))c(z-z_0)$$

$$\Leftrightarrow f(z)[1-z\overline{z_0}+f(z_0)c(z-z_0)]=c(z-z_0)+f(z_0)(1-z\overline{z_0})$$

Therefore $f(z)=\frac{c(z-z_0)}{1-\overline{z_0}z}=c\varphi_{z_0}(z)$ ■

2.2.7. Corollary. Suppose $|f(z)|\le 1$ for $|z|<1$ and f is a non-constant analytic function. If $g:D\to D$ defined by $g(z)=\frac{f(z)-a}{1-\overline{a}f(z)}$ where $a=f(0)$,then

$$\frac{|f(0)|-|z|}{1+|f(0)||z|}\le|f(z)|\le\frac{|f(0)|+|z|}{1-|f(0)||z|}\ \text{, for }\ |z|<1$$

Proof. We claim that $|g(z)|\le 1$ and $g(0)=0$. It is easy to check that $g(0)=0$, since

$g(0)=\frac{f(0)-a}{1-\overline{a}f(0)}=\frac{a-a}{1-\overline{a}a}=0$. We also have $|g(z)|=\frac{|f(z)-a|}{|1-\overline{a}f(z)|}\le 1$, since $|f(z)|\le 1$, now apply Schwarz lemma $|g(z)|\le|z|$ for $|z|<1$ if and only if $|f(z)-a|\le|z||1-\overline{a}f(z)|$

$$\Leftrightarrow\big||f(z)|-|a|\big|\le|f(z)-a|\le|z||1-\overline{a}f(z)|\le|z|+|z|.|a||f(z)|$$

$$\Leftrightarrow \left| |f(z)|-|a| \right| \le |z|+|z|.|a||f(z)|$$

$$\Leftrightarrow -|z|-|z|.|a||f(z)| \le |f(z)|-|a| \le |z|+|z|.|a||f(z)|$$

$$\Leftrightarrow -|z|-|z|.|a||f(z)| \le |f(z)|-|a| \text{ and } |f(z)|-|a| \le |z|+|z|.|a||f(z)|$$

Case (1) If $-|z|-|z|.|a||f(z)| \le |f(z)|-|a|$ then $|a|-|z| \le |f(z)|+|z|.|a||f(z)|$

$$\Rightarrow |a|-|z| \le |f(z)|(1+|z|.|a|) \text{. Therefore } \frac{|a|-|z|}{1+|z|.|a|} \le |f(z)| \qquad (2.11)$$

Case 2 If $|f(z)|-|a| \le |z|+|z|.|a||f(z)|$ then $|f(z)|-|z|.|a||f(z)| \le |a|+|z|$

$$\Rightarrow |f(z)|(1-|z|.|a|) \le |a|+|z| \text{.Therefore}$$

$$|f(z)| \le \frac{|a|+|z|}{1-|z|.|a|} \qquad (2.12)$$

Then from (2.11) and (2.12) above we get $\dfrac{|a|-|z|}{1+|a||z|} \le |f(z)| \le \dfrac{|a|+|z|}{1-|a||z|}$

Set $a=f(0)$, So we get $\dfrac{|f(0)|-|z|}{1+|f(0)||z|} \le |f(z)| \le \dfrac{|f(0)|+|z|}{1-|f(0)||z|}$ ■

Example.

Show that there does not exist an analytic function $f: D \to D$ with $f\left(\frac{1}{2}\right)=\frac{3}{4}$ and $f'\left(\frac{1}{2}\right)=\frac{2}{3}$

Solution. Assume there exist an analytic function $f: D \to D$ with $f\left(\frac{1}{2}\right)=\frac{3}{4}$. According to the definition, we have f is analytic on D with $|f(z)| \le 1, f(a)=\alpha$. In our case, we have $a=\frac{1}{2}$ and $f(a)=\frac{3}{4}$. We know $f'(a) \le \dfrac{1-|\alpha|^2}{1-|a|^2}$ and therefore we must have;

$$\left|f'\left(\frac{1}{2}\right)\right| \le \frac{1-\left(\frac{3}{4}\right)^2}{1-\left(\frac{1}{2}\right)^2} = \frac{7/16}{3/4} = \frac{7}{12} = 0.58\bar{3} \text{. But } f'\left(\frac{1}{2}\right)=\frac{2}{3}=0.\bar{6} \text{ which is not possible.}$$

2.3 Convex functions and Hadamard's Three Circles Theorem

2.3.1 Convex functions

In this section we will study convex functions and logarithmically convex functions and show that such functions appear in connection with the study of analytic functions.

2.3.1.1 **Definition.** If $[a, b]$ is an interval in the real line, a function $f : [a, b] \to \mathbb{R}$ is convex if for any two points x_1 and x_2 in $[a, b]$;

$$f(\lambda x_2 + (1-\lambda)x_1) \le \lambda f(x_2) + (1-\lambda) f(x_1), \text{ whenever } 0 \le \lambda \le 1.$$

If we look at the graph of f, the convexity inequality can be formulated geometrically by saying that each point on the chord between $(x_1, f(x_1))$ and $(x_2, f(x_2))$ is above the graph of f. Observe that for any two points $x_1 < x_2$ in (a,b) each point x in (x_1, x_2) may be expressed as $x = \lambda x_1 + (1-\lambda)x_2$ where $\lambda = \frac{x_2 - x}{x_2 - x_1}$. Thus the convexity inequality may be written as

$$f(x) \le \left(\frac{x_2 - x}{x_2 - x_1}\right) f(x_1) + \left(\frac{x - x_1}{x_2 - x_1}\right) f(x_2) \text{ for } x_1 < x < x_2.$$

$$\Leftrightarrow f(x)(x_2 - x_1) \le x_2 f(x_1) - x f(x_1) + x f(x_2) - x_1 f(x_2)$$

$$\Leftrightarrow x_2 f(x) - x_1 f(x) \le x_2 f(x_1) - x f(x_1) + x f(x_2) - x_1 f(x_2)$$

$$\Leftrightarrow x_2 f(x) - x_2 f(x_1) - x f(x) + x f(x_1) \le x f(x_2) - x_1 f(x_2) - x f(x) + x_1 f(x)$$

$$\Leftrightarrow x_2 (f(x) - f(x_1)) - x (f(x) - f(x_1)) \le (x - x_1) f(x_2) - (x - x_1) f(x)$$

$$\Leftrightarrow (x_2 - x)(f(x) - f(x_1)) \le (x - x_1)(f(x_2) - f(x))$$

Rearranging, this inequality may also be written as $\frac{f(x) - f(x_1)}{x - x_1} f(x_1) \le \frac{f(x_2) - f(x)}{x_2 - x}$ for $x_1 < x < x_2$. Therefore convexity may also be formulated geometrically by saying that for

$x_1 < x < x_2$, the slope of the chord from $(x_1, f(x_1))$ to $(x, f(x))$ is no longer than the slope of the chord from $(x, f(x))$ to $(x_2, f(x_2))$.

A function is convex if and only if the portion of the plane lying above the graph of the function is a convex set. A set G is star shaped if there is a point a in G such that for each z in G, the line segment $[a, z]$ lies entirely in G.

2.3.1.2 **Proposition**. A function $f: [a , b] \to \mathbb{R}$ is convex iff the set

$$A = \{(x, y): a \leq x \leq b \text{ and } f(x) \leq y\} \text{ is convex.}$$

Proof (⇒) Suppose $f: [a , b] \to \mathbb{R}$ is a convex function and let (x_1, y) and (x_2, y) be Points in A. That is $x_1, x_2 \in A \subseteq [a, b]$. Thus $f(x_1) \leq y$ and $f(x_2) \leq y$ for $y \in \mathbb{R}$. Let $0 \leq \lambda \leq 1$ and $x = \lambda x_1 + (1 - \lambda)x_2 \epsilon [a, b]$ then by definition of convex function;

$$f(x) \leq \lambda f(x_1) + (1-\lambda) f(x_2) \leq \lambda y + (1-t)y = y$$

$$\Rightarrow f(x) \leq y$$

Thus x is in A. So A is convex set.

(⇐) Suppose A is convex set. Let x_1, x_2 be two points in [a, b] then

$(tx_2 + (1-t)x_1, tf(x_2) + (1-t) f(x_1))$ lies in A if $0 \leq t \leq 1$ by virtue of its convexity. But the definition of A gives that $f(tx_2 + (1-t)x_1) \leq tf(x_2) + (1-t) f(x_1)$; that is, f is convex. ■

2.3.1.3 **Theorem.** A differentiable function f on $[a,b]$ is convex if and only if f' is increasing

Proof (⇒) Assume that f is convex. We want to show f' is increasing.

Let $a \leq x < y \leq b$ and suppose that $0 < t < 1$. Then $f(x + t(y-x)) \leq (1-t) f(x) + tf(y)$.

That is $\dfrac{f(x+t(y-x)) - f(x)}{t(y-x)} \leq \dfrac{f(y) - f(x)}{(y-x)}$. Then letting $t \to 0^+$ we get

$$f'(x) \leq \frac{f(y) - f(x)}{(y-x)} \tag{2.13}$$

Similarly, since $f\big((1-t)x + ty\big) - f(y) \leq (1-t)f(x) + tf(y) - f(y)$

$$\Leftrightarrow \frac{f((1-t)x + ty) - f(y)}{(1-t)(y-x)} \leq \frac{(1-t)(f(x) - f(y))}{(1-t)(y-x)}$$

$$\Leftrightarrow \frac{f((1-t)x + ty) - f(y)}{(1-t)(y-x)} \leq \frac{f(x) - f(y)}{(y-x)}$$

$$\Leftrightarrow \frac{f((1-t)x + ty) - f(y)}{(1-t)(x-y)} \geq \frac{f(y) - f(x)}{(y-x)}$$. Then letting $t \to 1$ we get,

$$f'(y) \geq \frac{f(y) - f(x)}{y-x} \tag{2.14}$$

From equation (2.13) and (2.14) we have $f'(x) \leq f'(y)$. So $\boldsymbol{f'}$ is increasing.

Conversely. Suppose that f' is increasing and let $x < u < y$. Apply the mean value theorem for differentiation to find r and s with $x < r < u < s < y$. That is $\frac{f(u)-f(x)}{u-x} = f'(r)$ and

$\frac{f(y)-f(u)}{y-u} = f'(s)$. Since f' is increasing, $f'(r) \le f'(s)$

$$\Rightarrow \frac{f(u)-f(x)}{u-x} \le \frac{f(y)-f(u)}{y-u}, \quad \text{whenever} \quad x < u < y.$$

In particular by letting $u = (1-t)x + ty$ where $0 < t < 1$ we have

$$\frac{f(u)-f(x)}{(1-t)x+ty-x} \le \frac{f(y)-f(u)}{y-(1-t)x-ty} \Leftrightarrow \frac{f(u)-f(x)}{x-tx+ty-x} \le \frac{f(y)-f(u)}{y-x+tx-ty}$$

$$\Leftrightarrow \frac{f(u)-f(x)}{t(y-x)} \le \frac{f(y)-f(u)}{(1-t)(y-x)}$$

Hence, $\left(f(u)-f(x)\right)(1-t) \le (f(y)-f(u))t$

$$\Rightarrow f(u) - tf(u) - f(x) + tf(x) \le tf(y) - tf(u)$$

$$\Rightarrow f(u) \le f(x) - tf(x) + tf(y)$$

$$\Rightarrow f(u) \le (1-t)f(x) + tf(y)$$

This shows that f must be convex. ■

2.3.1.4. Corollary. If $f: (a, b) \to \mathbb{R}$ is convex then f is continuous. But this does not remain true if f is defined on the closed interval $[a, b]$.

Proof. Since f is convex on (a,b) and if $a < s < t < u < b$. We want to show that

$$\frac{f(t)-f(s)}{t-s} \le \frac{f(u)-f(s)}{u-s} \le \frac{f(u)-f(t)}{u-t} \tag{2.15}$$

Let $a < s < t < u < b$,$t = \lambda s + (1-\lambda)u$, with $\lambda = \frac{u-t}{u-s} \in (0,1)$. Since f is convex on (a,b) we get

$$f(t) = f(\lambda s + (1-\lambda)u) \le \lambda f(s) + (1-\lambda) f(u) \Rightarrow f(t) \le \frac{u-t}{u-s} f(s) + (1 - \frac{u-t}{u-s}) f(u)$$

$$\Leftrightarrow (u-s) f(t) \le (u-t) f(s) + (t-s) f(u) \quad ; \text{since } u-s > 0$$

$$\Leftrightarrow uf(t) - sf(t) \le uf(s) - tf(s) + tf(u) - sf(u)$$

$$\Leftrightarrow -uf(t)+sf(t)+uf(s)-tf(s)+tf(u)-sf(u)\geq 0$$

$$\Leftrightarrow tf(u)-tf(s)-sf(u)-uf(t)+uf(s)+sf(t)\geq 0 \qquad (2.16)$$

$$\Leftrightarrow tf(u)-tf(s)-sf(u)+sf(s)-uf(t)+uf(s)+sf(t)-sf(s)\geq 0$$

$$\Leftrightarrow t(f(u)-f(s))-s(f(u)-f(s))+u(f(s)-f(t))+s(f(t)-f(s))\geq 0$$

$$\Leftrightarrow (t-s)(f(u)-f(s))+(u-s)(f(s)-f(t))\geq 0$$

$$\Leftrightarrow \frac{f(u)-f(s)}{u-s}\geq \frac{f(t)-f(s)}{t-s}$$

Therefore $$\frac{f(t)-f(s)}{t-s}\leq \frac{f(u)-f(s)}{u-s} \qquad (2.17)$$

Similarly using equation (2.16) above $tf(u)-tf(s)-sf(u)-uf(t)+uf(s)+sf(t)\geq 0$ we get

$$-sf(u)-uf(t)+sf(t)+uf(s)+tf(u)-tf(s)\geq 0$$

$$\Leftrightarrow uf(u)-sf(u)-uf(t)+sf(t)-uf(u)+uf(s)+tf(u)-tf(s)\geq 0$$

$$\Leftrightarrow (u-s)f(u)-(u-s)f(t)-u(f(u)-f(s))+t(f(u)-f(s))\geq 0$$

$$\Leftrightarrow (u-s)(f(u)-f(t))-(t-u)(f(u)-f(s))\geq 0$$

$$\Leftrightarrow \frac{f(u)-f(t)}{-(t-u)}\leq \frac{f(u)-f(s)}{u-s}$$

Therefore $$\frac{f(u)-f(t)}{u-t}\leq \frac{f(u)-f(s)}{u-s} \qquad (2.18)$$

Thus from equation (2.17) and (2.18) we get $\frac{f(t)-f(s)}{t-s}\leq \frac{f(u)-f(s)}{u-s}\leq \frac{f(u)-f(t)}{u-t}$

Next. Given $x\in(a,b)$ choose $\delta>0$ such that $[x-\delta,x+\delta]\subseteq(a,b)$

Claim. $$\frac{f(x)-f(x-\delta)}{\delta}\leq \frac{f(z)-f(x)}{z-x}\leq \frac{f(x+\delta)-f(x)}{\delta} \quad \text{for all} \quad z\in(x-\delta,x+\delta)$$

Proof. Consider a point $z\in(x-\delta,x)$ and applying the second inequality of equation (2.15), or let $a<x-\delta<z<x<b$ and $z=\lambda(x-\delta)+(1-\lambda)x$ with $\lambda=\frac{x-z}{\delta}\in(0,1)$. Since f is convex,

$f(z)=f(\lambda(x-\delta)+(1-\lambda)x)\leq \lambda f(x-\delta)+(1-\lambda)f(x)$

$$\Leftrightarrow\ f(z)\le \lambda f(x-\delta)+(1-\lambda)f(x)=\lambda f(x-\delta)+f(x)-\lambda f(x)$$

$$\Leftrightarrow\ f(z)\le \lambda f(x-\delta)+(1-\lambda)f(x)=\frac{x-z}{\delta}f(x-\delta)+f(x)-\frac{x-z}{\delta}f(x)$$

$$\Leftrightarrow\ \delta f(z)\le (x-z)f(x-\delta)+\delta f(x)-(x-z)f(x)$$

$$\Leftrightarrow\ \delta f(z)\le xf(x-\delta)-zf(x-\delta)+\delta f(x)-xf(x)+zf(x)$$

$$\Leftrightarrow\ xf(x-\delta)-zf(x-\delta)+\delta f(x)-xf(x)+zf(x)-\delta f(z)\ge 0$$

$$\Leftrightarrow\ zf(x)-xf(x)-zf(x-\delta)+xf(x-\delta)\ge \delta f(z)-\delta f(x)$$

$$\Leftrightarrow\ (z-x)f(x)-(z-x)f(x-\delta)\ge \delta(f(z)-f(x))$$

$$\Leftrightarrow\ (z-x)(f(x)-f(x-\delta))\ge \delta(f(z)-f(x))$$

$$\frac{f(x)-f(x-\delta)}{\delta}\le\frac{f(z)-f(x)}{z-x}\quad \text{Since } z-x<0 \qquad (2.19)$$

Secondly consider a point $z\in(x,x+\delta)$ and applying the first inequality of equation (2.15), or if $a<x<z<x+\delta<b$, Put $z=\lambda x+(1-\lambda)(x+\delta)$ with $\lambda=\frac{x+\delta-z}{\delta}\in(0,1)$, since f is convex; $f(z)\le \lambda f(x)+(1-\lambda)f(x+\delta)$ if and only if $f(z)\le\frac{x+\delta-z}{\delta}f(x)+(1-\frac{x+\delta-z}{\delta})f(x+\delta)$

$$\Leftrightarrow\ \delta f(z)\le (x+\delta)f(x)-zf(x)+zf(x+\delta)-xf(x+\delta)$$

$$\Leftrightarrow\ zf(x+\delta)-xf(x+\delta)+(x+\delta)f(x)-zf(x)-\delta f(z)\ge 0$$

$$\Leftrightarrow\ zf(x+\delta)-xf(x+\delta)+(x+\delta)f(x)-zf(x)-xf(z)-\delta f(z)+xf(z)\ge 0$$

$$\Leftrightarrow\ zf(x+\delta)-xf(x+\delta)+(x+\delta)f(x)-zf(x)-zf(x)-(x+\delta)f(z)+xf(z)\ge 0$$

$$\Leftrightarrow\ zf(x+\delta)-zf(x)-xf(x+\delta)+xf(x)-(x+\delta)f(z)-(x+\delta)f(x)+xf(z)-xf(x)\ge 0$$

$$\Leftrightarrow\ z(f(x+\delta)-f(x))-x(f(x+\delta)-f(x))-(x+\delta)(f(z)-f(x))+x(f(z)-xf(x))\ge 0$$

$$\Leftrightarrow\ (z-x)(f(x+\delta)-f(x))+(-(x+\delta)+x)(f(z)-f(x))\ge 0$$

$$\Leftrightarrow\ (z-x)(f(x+\delta)-f(x))-\delta(f(z)-f(x))\ge 0$$

$$\Leftrightarrow\ (z-x)(f(x+\delta)-f(x))\ge \delta(f(z)-f(x))$$

$$\Leftrightarrow \frac{f(x+\delta)-f(x)}{\delta} \geq \frac{f(z)-f(x)}{z-x}$$

Therefore $\frac{f(z)-f(x)}{z-x} \leq \frac{f(x+\delta)-f(x)}{\delta}$ (2.20)

Thus from equation (2.19) and (2.20) we get:

$$\frac{f(x)-f(x-\delta)}{\delta} \leq \frac{f(z)-f(x)}{z-x} \leq \frac{f(x+\delta)-f(x)}{\delta} \text{ for all } z \in (x-\delta, x+\delta).$$

From this equivalent we have:

$$\frac{f(x)-f(x-\delta)}{\delta}(z-x) \leq f(z)-f(x) \leq \frac{f(x+\delta)-f(x)}{\delta}(z-x)$$

Taking the limit as $z \to x$ we have that

$$\frac{f(x)-f(x-\delta)}{\delta}(z-x) \to 0 \quad \text{and} \quad \frac{f(x+\delta)-f(x)}{\delta} \to 0.$$

Thus $(z) - f(x) \to 0$. That is $|z-x| < \delta \Rightarrow |f(z)-f(x)| < \in$ which shows that f is continuous. ■

The statement is not true if f is defined on the closed interval $[a,b]$. Here is a counter example define $f: [a, b] \to \mathbb{R}$ by

$$f(x) = \begin{cases} 0, if, x \in [a,b) \\ 1, if, x = b \end{cases},$$

f is convex on $[a,b]$ and f is continuous on (a,b) .But f is not continuous at $x = b$.

2.3.1.5 Lemma. A function $f: [a, b] \to \mathbb{R}$ is convex iff any of the following equivalent conditions satisfied:

a) $a \leq x < u < y \leq b$ gives $\det\begin{pmatrix} f(u) & u & 1 \\ f(x) & x & 1 \\ f(y) & y & 1 \end{pmatrix} \geq 0$

b) $a \leq x < u < y \leq b$ gives $\frac{f(u)-f(x)}{u-x} \leq \frac{f(y)-f(x)}{y-x}$

c) $a \leq x < u < y \leq b$ gives $\frac{f(u)-f(x)}{u-x} \leq \frac{f(y)-f(u)}{y-u}$

Proof. (⇒) Let a function $f:[a,b]\to\mathbb{R}$ is convex, and then we want to show a, b, c is true.

(a) Since f is convex on (a,b) and $a\le x<u<y\le b$. We went to show that $\det\begin{pmatrix} f(u) & u & 1\\ f(x) & x & 1\\ f(y) & y & 1\end{pmatrix}\ge 0$

. Since $\det\begin{pmatrix} f(u) & u & 1\\ f(x) & x & 1\\ f(y) & y & 1\end{pmatrix}=f(u)(x-y)-u(f(x)-f(y))+yf(x)-xf(y)$. So we have to show that $xf(u)-yf(u)-uf(x)+uf(y)+yf(x)-xf(y)\ge 0$ .Let $a\le x<u<y\le b$, Put $u=\lambda x+(1-\lambda)y$ with $\lambda=\dfrac{y-u}{y-x}\in(0,1)$, since f is convex on $[a,b]$ we get $f(u)=f(\lambda x+(1-\lambda)y)\le\lambda f(x)+(1-\lambda)f(y)$.

This implies $f(u)\le\lambda f(x)+(1-\lambda)f(y)$

$$\Leftrightarrow\ f(u)\le\frac{y-u}{y-x}f(x)+(1-\frac{y-u}{y-x})f(y)=\frac{(y-u)f(x)}{y-x}+(\frac{y-x-y+u}{y-x})f(y)$$

$$\Leftrightarrow\ f(u)\le\frac{(y-u)f(x)}{y-x}+\frac{(-x+u)f(y)}{y-x}$$

$$\Leftrightarrow\ (y-x)f(u)\le yf(x)-uf(x)-xf(y)+uf(y)$$

$$\Leftrightarrow\ yf(x)-uf(x)-xf(y)+uf(y)\ge yf(u)-xf(u)$$

$$\Leftrightarrow yf(x)-uf(x)-xf(y)+uf(y)-yf(u)+xf(u)\ge 0$$

$\Leftrightarrow xf(u)-yf(u)-uf(x)+uf(y)+yf(x)-xf(y)\ge 0$.This is the proof for (a)

Next we have to show $\dfrac{f(u)-f(x)}{u-x}\le\dfrac{f(y)-f(x)}{y-x}$ if $a\le x<u<y\le b$.

Since from (a) we have $xf(u)-yf(u)-uf(x)+uf(y)+yf(x)-xf(y)\ge 0$, Rearranging we get $uf(y)-uf(x)-xf(y)-yf(u)+yf(x)+xf(u)\ge 0$

$$\Rightarrow uf(y)-uf(x)-xf(y)+xf(x)-yf(u)+yf(x)+xf(u)-xf(x)\ge 0$$

$$\Leftrightarrow u(f(y)-f(x))-x(f(y)-f(x))-y(f(u)-f(x))+x(f(u)-f(x))\ge 0$$

$$\Leftrightarrow (u-x)(f(y)-f(x))+(-y+x)(f(u)-f(x))\ge 0$$

$\Leftrightarrow \dfrac{f(y)-f(x)}{y-x}\ge\dfrac{f(u)-f(x)}{u-x}$ is the required one.

Therefore if $a \le x < u < y \le b$ then $\frac{f(u)-f(x)}{u-x} \le \frac{f(y)-f(x)}{y-x}$ (2.21)

Finally to show if $a \le x < u < y \le b$ then $\frac{f(u)-f(x)}{u-x} \le \frac{f(y)-f(u)}{y-u}$. Again from (a) we have

$-xf(y)-yf(u)+xf(u)+yf(x)+uf(y)-uf(x) \ge 0$

$$\Leftrightarrow\ yf(y)-xf(y)-yf(u)+xf(u)+yf(x)+uf(y)-uf(x)-yf(y) \ge 0$$

$$\Leftrightarrow\ (y-x)f(y)-f(u)(y-x)+(y-u)f(x)+(u-y)f(y) \ge 0$$

$$\Leftrightarrow\ (y-x)(f(y)-f(u))+(y-u)(f(x)-f(y)) \ge 0$$

$$\Leftrightarrow\ \frac{f(y)-f(u)}{y-u} \ge \frac{f(y)-f(x)}{y-x}$$

Therefore $\frac{f(y)-f(x)}{y-x} \le \frac{f(y)-f(u)}{y-u}$ (2.22)

We conclude from (2.21) and (2.22) if $a \le x < u < y \le b$ then $\frac{f(u)-f(x)}{u-x} \le \frac{f(y)-f(u)}{y-u}$

Conversely. Suppose (b) is true. That is $a \le x < u < y \le b$ gives $\frac{f(u)-f(x)}{u-x} \le \frac{f(y)-f(x)}{y-x}$

Put $u = \lambda x + (1-\lambda)y$ where $x < u < y$ and $0 < \lambda < 1$, $\frac{f(u)-f(x)}{u-x} \le \frac{f(y)-f(x)}{y-x}$ this implies that

$$\frac{f(u)-f(x)}{\lambda x+(1-\lambda)y-x} \le \frac{f(y)-f(x)}{y-x} \Leftrightarrow \frac{f(u)-f(x)}{\lambda x+x-\lambda y-x} \le \frac{f(y)-f(x)}{y-x}$$

$$\Leftrightarrow f(u)-f(x) \le \lambda(f(y)-f(x))$$

$$\Leftrightarrow f(\lambda x+(1-\lambda)y) \le \lambda f(y)+(1-\lambda)f(x)$$ .So f is Convex.

Suppose (c) is true. That is $a \le x < u < y \le b$ gives $\frac{f(u)-f(x)}{u-x} \le \frac{f(y)-f(u)}{y-u}$.

Put $u = \lambda y + (1-\lambda)x$ where $x < u < y$ and $0 < \lambda < 1$, $\frac{f(u)-f(x)}{u-x} \le \frac{f(y)-f(u)}{y-x}$

$$\Leftrightarrow \frac{f(u)-f(x)}{\lambda y+x-\lambda x-x} \le \frac{f(y)-f(u)}{y-\lambda y-x+\lambda x}$$

$$\Leftrightarrow \frac{f(u)-f(x)}{\lambda(y-x)} \leq \frac{f(y)-f(u)}{(1-\lambda)(y-x)}$$

$$\Leftrightarrow f(u)-\lambda f(u)-f(x)+\lambda f(x) \leq \lambda f(y)-\lambda f(u)$$

$$\Leftrightarrow f(u) \leq \lambda f(y)+f(x)-\lambda f(x)$$

$$\Leftrightarrow f(\lambda y+(1-\lambda)x) \leq \lambda f(y)+f(x)-\lambda f(x).\text{So } f \text{ is convex. } \blacksquare$$

2.3.2 Hadamard's Three Circles Theorem

In actuality we will mostly be concerned with functions which are not only convex, but which are logarithmically convex; that is, logf(x) is convex. Ofcourse this assumes that f(x) $>$ 0 for each x. It is easy to see that logarithmically convex function is convex, but not conversely.

2.3.2.1 Lemma. Let $a<b$ and let G be the vertical strip $G=\{x+iy: a<x<b; y \epsilon \mathbb{R}\}$. Suppose $f:\overline{G}\to\mathbb{C}$ is continuous and f is analytic in G. If we define;

$$M:[a,b]\to\mathbb{R} \text{ by } M(x)=sup\{|f(x+iy)|: -\infty<y<\infty\}$$

and if $|\text{f(z)}| \leq$ B for all z in G, further suppose that $|\text{f(z)}| \leq 1$ for all $z \in \partial G$. Then

$$|f(z)| \leq 1 \text{ for all } z \in G.$$

Proof: For each $\in>0$, let $g_{\in}(z)=\dfrac{1}{1+\in(z-a)}$ for each $z\in\overline{G}$ .Then for $z=x+yi$ in $\overline{G}$,

Since $\text{Re}\{1+\in(z-a)\}=1+\in(x-a)\geq 1$ in $\overline{G}$ we have;

$$|g_{\in}(z)|=\left|\frac{1}{1+\in(z-a)}\right| \leq \left|\text{Re}\left(\frac{1}{1+\in(z-a)}\right)\right| \leq \frac{1}{1+\in(x-a)} \leq 1.$$

Since $|f(z)|\leq 1$ and $|g_{\in}(z)|\leq 1$ for z in $\overline{G}$, then for z in ∂G, $|f(z)g_{\in}(z)|\leq 1$. Also, since f is bounded by B in G, $|f(z)g_{\in}(z)|\leq B.\left|\dfrac{1}{1+\in(z-a)}\right|\leq\dfrac{B}{\in|\text{Im}\,z|}$. So if $R=\left\{x+yi: a\leq x\leq b, |y|\leq\dfrac{B}{\in}\right\}$; then $|f(z)g_{\in}(z)|\leq B\left|\dfrac{1}{1+\in(z-a)}\right|\leq\dfrac{B}{\in|\text{Im}\,z|}\leq 1$. That is $|f(z)g_{\in}(z)|\leq 1$, for all $z\in\partial R$. It follows from the maximum modulus theorem that $|f(z)g_{\in}(z)|\leq\max\{|f(\zeta)g_{\in}(\zeta)|:\zeta\in\partial R\}\leq 1$ for z in R. If $|\text{Im}\,z|>\dfrac{B}{\in}$

again; $|f(z)g_{\in}(z)| \leq \frac{B}{\in|\operatorname{Im} z|} = \frac{B}{\in|y|} \leq \frac{B}{\in}.\frac{\in}{B} = 1$. That is for all z in G, $|f(z)g_{\in}(z)| \leq 1$. Thus for all z in G, $|f(z)| \leq \frac{1}{|g_{\in}(z)|} = |1+\in(z-a)|$. Letting $\in$ approach zero, we have that

$$|f(z)| \leq 1 \text{ for all } z \in G. ■$$

2.3.2.2. Theorem / The three lines theorem/ Let $a<b$ and let G be the vertical strip $G = \{x+iy: a<x<b; y \in \mathbb{R}\}$. Suppose $f:\overline{G} \to \mathbb{C}$ is continuous and f is analytic in G. If we define $M:[a,b] \to \mathbb{R}$ by

$$\mathrm{M(x) = sup\{|f(x+iy)|: -\infty < y < \infty\}}$$

and if $|f(z)| \leq B$ for all z in G then $\log M(x)$ is a convex function.(That is if $|f(a+iy)| \leq M(a)$ and $|f(b+iy)| \leq M(b)$ for all $y \in \mathbb{R}$ then; $|f(x+iy)| \leq M(u) \leq M(a)^{\frac{b-u}{b-a}} M(b)^{\frac{u-a}{b-a}}$ for $y \in \mathbb{R}$, $a \leq u \leq b$). In other words, $\mathrm{Sup\log|f(x+iy)|}, y \in \mathbb{R}$ is a convex function of u.)

Proof. Note that to say that $\log M(x)$ is convex means (Lemma 2.3.1.5) that for $a \leq x < u < y \leq b$, if we put $u = \lambda x + (1-\lambda)y$ with $\lambda = \frac{y-u}{y-x} \in (0,1)$ then;

$$\log M(u) = \log M(\lambda x + (1-\lambda)y) \leq \lambda \log M(x) + (1-\lambda)\log M(y)$$

$$\Rightarrow \log M(u) \leq \frac{y-u}{y-x}\log M(x) + (1-\frac{u-x}{y-x})\log M(y)$$

$$\Rightarrow \log M(u)^{(y-x)} \leq \log M(x)^{(y-u)} + \log M(y)^{(u-x)} = \log\left(M(x)^{(y-u)} M(y)^{(u-x)}\right)$$

$$\Rightarrow \log M(u)^{(y-x)} \leq \log\left(M(x)^{(y-u)} M(y)^{(u-x)}\right)$$

Taking the exponential of both sides gives,

$$M(u)^{(y-x)} \leq M(x)^{(y-u)} M(y)^{(u-x)} \text{ Whenever } a \leq x < u < y \leq b.$$

Also, since $\log M(x)$ is convex we have that $\log M(x)$ is bounded by $\mathrm{Max\{\log M(a), \log M(b)\}}$. That is for $a \leq x \leq b$, $\mathrm{M(x) \leq Max\{\log M(a), \log M(b)\}}$. So to prove this theorem the above lemma is used. Thus from the above lemma (2.3.2.1)for $a \leq x < u < y \leq b$, $M(u)^{(y-x)} \leq M(x)^{(y-u)} M(y)^{(u-x)}$. To prove this theorem we need only established

$$M(u)^{(b-a)} \leq M(a)^{(b-u)} M(b)^{(u-a)} \text{ for } a < u < b.$$

To do this recall that for a constant $A>0$, $A^z = e^{(z\log A)}$ is an entire function of z with no zeros. So $g(z)$ define by $g(z)=M(a)^{\frac{b-z}{b-a}}M(b)^{\frac{z-a}{b-a}}$ is an entire, never vanishes, (because $|A^z| = A^{\mathrm{Re} z}$) for $z=x+yi$, so

$$|g(z)| = \left|M(a)^{\frac{b-z}{b-a}}M(b)^{\frac{z-a}{b-a}}\right| = M(a)^{\frac{b-x}{b-a}}M(b)^{\frac{x-a}{b-a}} \tag{2.23}$$

(It is assumed here that $M(a)$ and $M(b)$ is nonzero. However, if either $M(a)$ or $M(b)$ is zero then $f\equiv 0$.) Since $M(a)^{\frac{b-x}{b-a}}M(b)^{\frac{x-a}{b-a}}$ is continuous for x in [a, b] and never vanishes, $\frac{1}{|g|}$ must be bounded in $\overline{G}$. Also $|g(a+yi)| = \left|M(a)^{\frac{b-a-yi}{b-a}}M(b)^{\frac{a+yi-a}{b-a}}\right| = M(a)^{\frac{b-a}{b-a}}M(b)^{\frac{0}{b-a}} = M(a)$ and

$$|g(b+yi)| = \left|M(a)^{\frac{b-b-yi}{b-a}}M(b)^{\frac{b+yi-a}{b-a}}\right| = M(a)^{\frac{0}{b-a}}M(b)^{\frac{b-a}{b-a}} = M(b)$$

So that $\left|\frac{f(z)}{g(z)}\right| \le 1$ for z in ∂G and $\frac{f}{g}$ satisfies the hypothesis of the above lemma (2.3.2.1). Thus $|f(z)| \le |g(z)|, z\in G$. Using equation (2.23) for $a<u<b$, $|f(z)| \le |g(z)| = M(a)^{\frac{b-u}{b-a}}M(b)^{\frac{u-a}{b-a}}$

$$\Rightarrow \sup\{|f(z)| : -\infty < y < \infty\} \le M(a)^{\frac{b-u}{b-a}}M(b)^{\frac{u-a}{b-a}} \text{, for } a<u<b$$

Since $M(u) = \sup\{|f(z)| : -\infty < y < \infty\} \le M(a)^{\frac{b-u}{b-a}}M(b)^{\frac{u-a}{b-a}}$, for $a<u<b$

Therefore $M(u) \le M(a)^{\frac{b-u}{b-a}}M(b)^{\frac{u-a}{b-a}}$ for $a<u<b$. ■

2.3.2.3 Theorem / Hadamard's three circle theorem/

Let $f(z)$ be analytic in the annular region $r_1 \le |z| \le r_3$ and assume $0<r_1<r_2<r_3$ .If $M(r_1), M(r_2), M(r_3)$ are respectively the maximum of $|f(z)|$ on the three circles $|z| = r_1, r_2, r_3$ then;

$$M(r_2)^{\log\left(\frac{r_3}{r_1}\right)} \le M(r_3)^{\log\left(\frac{r_2}{r_1}\right)} M(r_1)^{\log\left(\frac{r_3}{r_2}\right)}$$

Proof. Consider $\phi(z) = z^{\lambda} f(z)$ where λ is a real constant then $\phi(z)$ is analytic function in the region $r_1 \le |z| \le r_3$

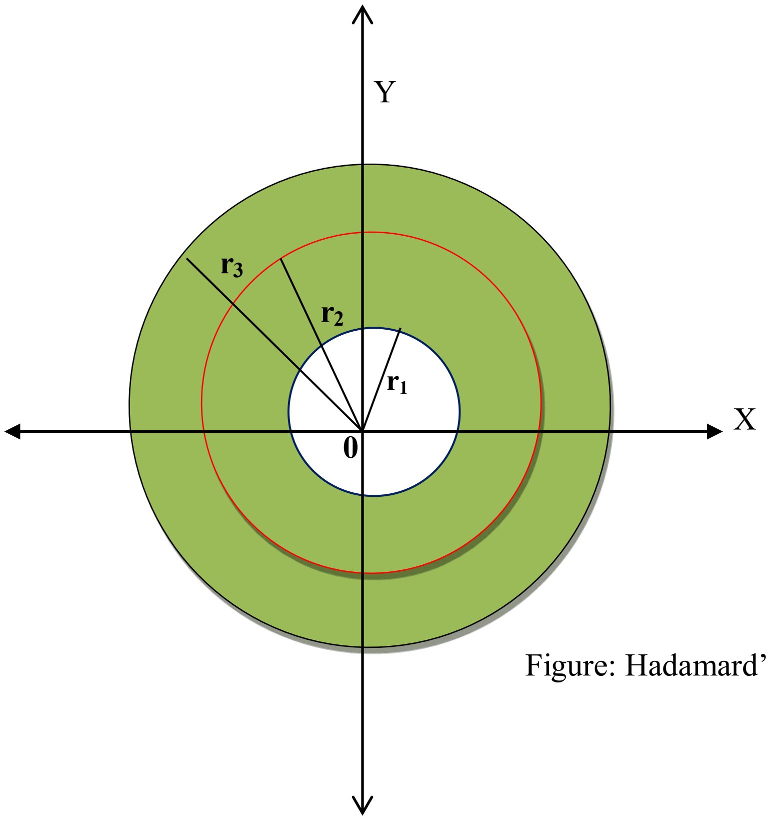

Figure: Hadamard's three circle theorem

The function $\phi(z)$ is not, in general, single valued. So for convenience, we choose the Principal branch. For this if we cut the annulus along the negative real axis there will be a domain in which the principal branch of the function $\phi(z)$ is analytic. By Maximum Modulus Theorem the function $\phi(z)$ attains its maximum value on the boundary of the cut annulus. Consider a branch of this function which is analytic in the region for which $\frac{\pi}{2} \le \arg z \le \frac{3\pi}{2}$.

We observe the principal value cannot attain its maximum modulus on the cut and therefore maximum modulus must be attained on one of the boundary circles of the annulus. That is by The Maximum Modulus Theorem $|\phi(z)|$ is less than or equal to its value on the boundary of the closed annulus. Thus $|\phi(z)| = |z^{\lambda} f(z)| \le Max\{r_1^{\lambda} M(r_1), r_3^{\lambda} M(r_3)\}$ if $r_1 < |z| < r_3$. Hence

$$r_2^{\lambda} M(r_2) \le Max\{r_1^{\lambda} M(r_1), r_3^{\lambda} M(r_3)\} \text{ , if } r_1 < r_2 < r_3 \qquad (2.24)$$

Since λ is arbitrary, choose λ so that $r_1^{\lambda} M(r_1) = r_3^{\lambda} M(r_3)$. Taking log of both sides, we get $\log(r_1^{\lambda} M(r_1)) = \log(r_3^{\lambda} M(r_3))$

$$\Rightarrow \log r_1^{\lambda} + \log M(r_1) = \log r_3^{\lambda} + \log M(r_3)$$

$$\Leftrightarrow \lambda \log r_1 + \log M(r_1) = \lambda \log r_3 + \log M(r_3)$$

$$\Leftrightarrow \lambda(\log r_1 - \log r_3) = \log M(r_3) - \log M(r_1)$$

$$\Leftrightarrow \lambda = \frac{\log M(r_3) - \log M(r_1)}{\log r_1 - \log r_3}.$$

That is $\lambda = -\dfrac{\log\left(M(r_3)/M(r_1)\right)}{\log {r_3}/{r_1}}$ (2.25)

From (2.24) and (2.25) we have $r_2^{\lambda} M(r_2) \le Max\left\{ r_1^{\frac{\log(M(r_3)/M(r_1))}{-\log r_3/r_1}} M(r_1), r_3^{\frac{\log(M(r_3)/M(r_1))}{-\log r_3/r_1}} M(r_3) \right\}$

So $r_2^{\lambda} M(r_2) \le r_1^{\lambda} M(r_1)$

$$\Leftrightarrow M(r_2) \le \frac{r_1^{\lambda} M(r_1)}{r_2^{\lambda}} = \left(\frac{r_1}{r_2}\right)^{\lambda} M(r_1) = \left(\frac{r_2}{r_1}\right)^{-\lambda} M(r_1)$$

Now, $M(r_2) \le \left(\frac{r_2}{r_1}\right)^{-\lambda} M(r_1)$, then $M(r_2)^{\log\left(\frac{r_3}{r_1}\right)} \le \left(\frac{r_2}{r_1}\right)^{-\lambda \log\left(\frac{r_3}{r_1}\right)} M(r_1)^{\log\left(\frac{r_3}{r_1}\right)}$.

If we substitute the value of λ we get

$$M(r_2)^{\log\left(\frac{r_3}{r_1}\right)} = \left(\frac{r_2}{r_1}\right)^{\frac{\log\left(\frac{M(r_3)}{M(r_1)}\right)}{\log\left(\frac{r_3}{r_1}\right)} \log\left(\frac{r_3}{r_1}\right)} M(r_1)^{\log\left(\frac{r_3}{r_1}\right)} = \left(\frac{r_2}{r_1}\right)^{\log\left(\frac{M(r_3)}{M(r_1)}\right)} M(r_1)^{\log\left(\frac{r_3}{r_1}\right)}$$

$$\Rightarrow M(r_2)^{\log\left(\frac{r_3}{r_1}\right)} \le \left(\frac{M(r_3)}{M(r_1)}\right)^{\log\left(\frac{r_2}{r_1}\right)} M(r_1)^{\log\left(\frac{r_3}{r_1}\right)} \quad . \text{(Since } a^{\ln b} = b^{\ln a}\text{)}$$

$$M(r_2)^{\log\left(\frac{r_3}{r_1}\right)} \le \frac{M(r_3)^{\log\left(\frac{r_2}{r_1}\right)}}{M(r_1)^{\log\left(\frac{r_2}{r_1}\right)}} . M(r_1)^{\log\left(\frac{r_3}{r_1}\right)} = M(r_3)^{\log\left(\frac{r_2}{r_1}\right)} M(r_1)^{\log\left(\frac{r_3}{r_1}\right) - \log\left(\frac{r_2}{r_1}\right)} = M(r_3)^{\log\left(\frac{r_2}{r_1}\right)} M(r_1)^{\log\left(\frac{r_3}{r_1}\right)\left(\frac{r_1}{r_2}\right)}$$

Therefore $M(r_2)^{\log\left(\frac{r_3}{r_1}\right)} \le M(r_3)^{\log\left(\frac{r_2}{r_1}\right)} M(r_1)^{\log\left(\frac{r_3}{r_2}\right)}$. Equality achieved when $\phi(z)$ is constant. ■

2.3.2.4 Theorem / Hadamard's three circle theorem as convexity/

Let $f(z)$ be analytic in the annulus region $r_1 \le |z| \le r_3$ and $0 \le r_1 \le r_2 \le r_3$. If $M(r_i), i = 1,2,3$ are respectively the maximum of $|f(z)|$ on the three circles; $|z| = r_1$, $|z| = r_2$, $|z| = r_3$, then

$$M(r_2) \le \frac{\log r_3 - \log r_2}{\log r_3 - \log r_1} \log M(r_1) + \frac{\log r_2 - \log r_1}{\log r_3 - \log r_1} \log M(r_3).$$

Proof. Using Hadamard's three circle theorem, we have $M(r_2)^{\log\left(\frac{r_3}{r_1}\right)} \le M(r_3)^{\log\left(\frac{r_2}{r_1}\right)} M(r_1)^{\log\left(\frac{r_3}{r_2}\right)}$

Taking log of both sides of the above inequality we get;

$$\log M(r_2)^{\log\left(\frac{r_3}{r_1}\right)} \le \log\left(M(r_3)^{\log\left(\frac{r_2}{r_1}\right)} M(r_1)^{\log\left(\frac{r_3}{r_2}\right)} \right) = \log M(r_3)^{\log\left(\frac{r_2}{r_1}\right)} + \log M(r_1)^{\log\left(\frac{r_3}{r_2}\right)}$$

$$\Leftrightarrow \log\left(\frac{r_3}{r_1}\right) \log M(r_2) \le \log\left(\frac{r_2}{r_1}\right) \log M(r_3) + \log\left(\frac{r_3}{r_2}\right) \log M(r_1)$$

$$\Leftrightarrow (\log r_3 - \log r_1) \log M(r_2) \le (\log r_2 - \log r_1) \log M(r_3) + (\log r_3 - \log r_2) \log M(r_1)$$

$$\Leftrightarrow M(r_2) \le \frac{\log r_3 - \log r_2}{\log r_3 - \log r_1} \log M(r_1) + \frac{\log r_2 - \log r_1}{\log r_3 - \log r_1} \log M(r_3).$$

Thus we can express Hadamard's three circle theorem by saying that $\log M(r)$ is a convex function of $\log r$. ■

2.4 The Phragmen-Lindelof Theorem

Introduction.

This section presents some results of E. Phragmen and E. Lindelof (published in 1908) which extend the Maximum Principle by easing the requirement of boundedness on the boundary.

The Phragmen-Lindelof Theorem bears a relation to the Maximum Modulus Theorem which is analogous to the relationship of the following result to Liouville's theorem. If f is entire and $|f(z)| \le 1+|z|^{\frac{1}{2}}$ then f is a constant function. So it is not necessary to assume that an entire function is bounded in order to prove that it is constant; it is sufficient to assume that its growth as $z \to \infty$ is restricted by $1+|z|^{\frac{1}{2}}$. The Phragmen-Lindelof Theorem places a growth restriction on an analytic function $f : G \to \mathbb{C}$ as z nears a point on the extended boundary. Nevertheless, the conclusion, like that of the Maximum Modulus Theorem, is that f is bounded.

2.4.1 Definition. An open set G is simply connected if G is connected and every closed curve in G is homotopic to zero.

2.4.2 Theorem. If G is simply connected and $f: G \to \mathbb{C}$ is analytic in G then f has a Primitive in G.

Proof. Fix a point z_0 in G and define F on G by $F(z) = \int_{[z_0,z]} f$ for $z \in G$. Since $[z_0, z] \subseteq G$, F is well defined. We claim that F is differentiable and $F' = f$ everywhere in G. To show this, let $z \in G$ be given. Since G is open, there is $r > 0$ such that $B(z,r) \subseteq G$. We wish to show that

$$\frac{F(z+\zeta) - F(z)}{\zeta} - f(z) \to 0\,, as\ \zeta \to 0$$

We assumed that $F(z+\zeta)$ is defined for all ζ with $|\zeta| < r$, since $B(z,r) \subseteq G$, as noted above. Suppose that $|\zeta| < r$ from now on. We will apply to Cauchy's theorem to rewrite $F(z+\zeta) - F(z)$ in another form.

$$\text{Indeed} \qquad F(z+\zeta) - F(z) = \int_{z_0}^{z+\zeta} f - \int_{z_0}^{z} f \quad .$$

Let T denote the triangle with vertices $z_0\,, z+\zeta\,, z$. Since $z_0 \in G$ and since $[\, z+\zeta\,, z\,]$, the line segment from $z+\zeta$ to z, lies in $B(z,r) \subseteq G$. It follow that $T \subseteq G$.

Indeed, any point w on the line segment $[\, z+\zeta\,, z\,]$, lies in G and so, therefore the line segment $[z_0, w]$ also lies in G. By varying w, we exhaust the triangle T. Now by Cauchy's theorem

$$\int_{\partial T} f = 0$$

However, the contour integral around a triangle is equal (by definition) to the sum of the integrals along its sides. Hence we have $\int_{z_0}^{z+\zeta} f + \int_{z+\zeta}^{z} f + \int_{z}^{z_0} f = 0$

Reversing the direction of the contour is equivalent to a change in sign of the integral, and therefore we may rewrite the above equation as $\int_{z_0}^{z+\zeta} f - \int_{z_0}^{z} f = \int_{z}^{z+\zeta} f$. Interims of F , this becomes

$$F(z+\zeta) - F(z) = \int_{z}^{z+\zeta} f(\xi)d\xi$$

Hence $\frac{F(z+\zeta)-F(z)}{\zeta} - f(z) = \frac{1}{\zeta}\int_{z}^{z+\zeta} f(\xi)d\xi - f(z) = \int_{z}^{z+\zeta} \frac{f(\xi)-f(z)}{\zeta} d\xi$.

Since $f(z)\int_{z}^{z+\zeta} d\xi = f(z)\zeta$. It remains to estimate this last integral. To do this, we use the continuity of f at z . Let $\varepsilon > 0$ be given. Then there is a $\delta > 0$ such that $|f(\xi) - f(z)| < \varepsilon$ whenever $|\xi - z| < \delta$. It follows that if $|\zeta| < \min\{\delta, r\}$ then $|\xi - z| < \delta$ for every $\xi \in [z, z+\zeta]$ so that $|f(\xi) - f(z)| < \varepsilon$ for such ξ. Therefore

$$\begin{aligned}\left|\frac{F(z+\zeta)-F(z)}{\zeta} - f(z)\right| &= \frac{1}{|\zeta|}\left|\int_{z}^{z+\zeta}\big(f(\xi) - f(z)\big)d\xi\right| \\ &\le \frac{1}{|\zeta|}\int_{z}^{z+\zeta}|f(\xi) - f(z)|\,d\xi \\ &\le \frac{1}{|\zeta|}\,\varepsilon L([z, z+\zeta]) \\ &= \frac{1}{|\zeta|}\,\varepsilon|\zeta| \\ &= \varepsilon\end{aligned}$$

Since ε is arbitrary letting $\varepsilon \to 0$, we have $\frac{F(z+\zeta)-F(z)}{\zeta} - f(z) \to 0$. Therefore

$$F'(z) = f(z) . \blacksquare$$

2.4.3 Phragmen-Lindelof Theorem. Let G be a simply connected region and let f be an analytic function on G . Suppose there is an analytic function $\varphi: G \to \mathbb{C}$ which never vanishes and is bounded on G . If M is a constant and $\partial_\infty G = A \cup B$ such that :

(a) for every a in A, $\limsup_{z\to a}|f(z)| \le M$;

(b) for every b in B, and $\eta > 0$, $\limsup_{z\to b}|f(z)||\varphi(z)|^{\eta} \le M$,then $|f(z)| \le M$ for all z in G .

Proof. Let $|\varphi(z)| \le k$ for all z in G . Also because G is simply connected there is an analytic branch of $\log\varphi(z)$ on G (Theorem 2.4.2). Hence $g(z) = e^{\eta\log\varphi(z)}$ is an analytic branch of $\varphi(z)^{\eta}$ for $\eta > 0$ and $|g(z)| = |\varphi(z)|^{\eta}$. Define $F: G \to \mathbb{C}$ by $F(z) = f(z)g(z)k^{-\eta}$; then F is analytic on G and $|F(z)| = |f(z)g(z)k^{-\eta}| = |f(z)||g(z)||k^{-\eta}| \le |f(z)|\frac{k^{\eta}}{k^{\eta}} = |f(z)|$,since $|\varphi(z)| \le k$ for all z in G. That is $|F(z)| \le |f(z)|$ for all z in G. But then, by conditions (a) and (b) on $\partial_\infty G$, F satisfies the hypothesis of The Maximum Modulus Theorem - third version. Thus $|F(z)| \le \max(M, k^{-\eta}M)$ for all z in G

Since $F(z) = f(z)g(z)k^{-\eta}$ and $|F(z)| = |f(z)g(z)k^{-\eta}| = |f(z)||g(z)||k^{-\eta}|$.

So $|f(z)g(z)k^{-\eta}| = |f(z)||\varphi(z)|^{\eta}|k^{-\eta}| = |F(z)| \le Max(M, k^{-\eta}M)$ for all z in G .This gives, $|f(z)| \le |\varphi(z)|^{-\eta} Max(M, k^{-\eta}M)$ for all z in G and for all $\eta > 0$. Letting $\eta \to 0^{+}$ gives that

$$|f(z)| \le M \text{ for all } z \text{ in } G. \blacksquare$$

2.4.4 Corollary. Let $a \ge \frac{1}{2}$ and put $G = \left\{ z : |\arg z| < \frac{\pi}{2a} \right\}$. Suppose that f is analytic on G and there is a constant M such that $\limsup_{z \to w} |f(z)| \le M$ for all w in ∂G. If there are positive constants P and $b < a$ such that

$$|f(z)| \le Pe^{|z|^{b}} \tag{2.26}$$

for all z with $|z|$ sufficiently large, then $|f(z)| \le M$ for all z in G.

Proof. Let $b < c < a$ and put $\varphi(z) = \exp(-z^{c})$ for z in G .If $z = re^{i\theta}$, $|\theta| < \frac{\pi}{2a}$ then

$\operatorname{Re} z^{c} = r^{c} Cosc\theta$. So for z in G, $|\varphi(z)| = |\exp(-z^{c})| = e^{-r^{c} Cosc\theta}$ when $z = re^{i\theta}$.

Since $c < a$, $Cosc\theta \ge \rho > 0$ for all z in G. This gives that φ is bounded on G. Also, if $\eta > 0$ and $z = re^{i\theta}$ is sufficiently large,

$$
\begin{aligned}
|f(z)||\varphi(z)| &\le Pe^{(|z|^{b})}|\varphi(z)|^{\eta} = Pe^{|re^{i\theta}|^{b}} e^{-\eta r^{c} Cosc\theta} \\
&= Pe^{r^{b}|Cos\theta + i\sin\theta|^{b}} e^{-\eta r^{c} Cosc\theta} \\
&\le Pe^{(r^{b} - \eta r^{c}\rho)}
\end{aligned}
$$

But $r^{b} - \eta r^{c}\rho = r^{c}(r^{b-c} - \eta\rho)$ .Since $b < c$, $r^{b-c} \to 0^{+}$ as $r \to \infty$, so that $r^{b} - \eta r^{c}\rho \to -\infty$ as $r \to \infty$. Thus $Lim\sup |f(z)||\varphi(z)|^{\eta} = 0$. Hence, f and φ satisfy the hypothesis of the Phragmen-Lindelof theorem so that;

$$|f(z)| \le M \text{ for all } z \text{ in } G. \blacksquare$$

2.4.5 Corollary. Let $a \ge \frac{1}{2}$, $G = \{z : |\arg z| < \frac{\pi}{2a}\}$ and suppose that for every w in ∂G , $\limsup_{z \to w} |f(z)| \le M$. Moreover, assume that for every $\delta > 0$ there is a constant P (which may depend on δ) such that

$$|f(z)| \le Pe^{(\delta |z|^{a})} \tag{2.27}$$

for z in G, and $|z|$ sufficiently large. Then $|f(z)| \le M$ for all z in G.

Proof. Define $F : G \to \mathbb{C}$ by $F(z) = f(z)e^{(-\in z^a)}$ where $\in > 0$ is arbitrary. If $x > 0$ and δ is chosen with $0 < \delta < \in$ then there is a constant P with

$$|F(x)| = \left|f(z)e^{(-\in z^a)}\right| \le Pe^{(\delta|z|^a)}e^{(-\in x^a)} = Pe^{[(\delta-\in)x^a]} = \frac{P}{e^{[(\in-\delta)x^a]}} \ .$$

But then $|F(x)| \to 0$ as $x \to \infty$ in $\mathbb{R}$; so $M_1 = Sup\{|F(x)| : 0 < x < \infty\} < \infty$.

Define $M_2 = \max\{M_1, M\}$ and $\quad H_+ = \left\{z \in G : 0 < \arg z < \frac{\pi}{2a}\right\}$,

$$H_- = \left\{z \in G : 0 > \arg z > \frac{-\pi}{2a}\right\}$$

then $\limsup_{z \to w} |f(z)| \le M$ for all z in ∂H_+ and ∂H_-. Using hypothesis 2.27 ,Corollary 2.4.4 gives $|F(z)| \le M_2$ for all z in H_+ and H_-. Hence, $|F(z)| \le M_2$ for all z in G.

We claim that $M_2 = M$.

In fact, if $M_2 = M_1 = M$ then $|F|$ assumes its maximum value in G at some point x, $0 < x < \infty$ (because $|F(x)| \to 0$ as $x \to \infty$ and $\limsup_{x \to 0} |f(x)| = \limsup_{x \to 0} |F(x)| \le M < M_1$). This would give that F is a constant by the Maximum Principle and so $M = M_1$. Thus, $M_2 = M$ and $|F(z)| \le M$ for all z in G; that is, $|f(z)| \le Me^{(\in \mathrm{Re}\, z^a)}$ for all z in G; since M is independent of $\in$, we can let $\in \to 0$ and get $|f(z)| \le M$ for all z in G.■

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