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On the Möbius Function of Pointed Partitions and Exponential Pointed Structures

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Abstract

This thesis is concerned with partial orders on set and integer partitions and related structures. The study of set and integer partitions dates back to Euler and Sylvester [28]. Over the course of time it has become apparent that the combinatorics of partitions encodes important mathematical structures in a variety of fields. Results on partially ordered sets of partitions involving Möbius numbers, homology and homotopy of order complexes emerged in the works of Rota and Stanley [56]. In this work, we use pointed partition structures where a part of an integer and set partition is marked. These have been introduced in the works of Ehrenborg and Readdy [60], Ehrenborg and Jung [63] and Ziegler [29]. We exhibit new enumerative and geometric properties of the partial orders of pointed partitions studied by Ehrenborg and Readdy [60]. In particular, we compute the Möbius numbers and homotopy types of lower intervals.

In addition, we investigate their ordered counterparts and provide analogous results in this setting. Then we use the pointed set partition lattice to introduce exponential pointed structures which are the pointed analog of exponential structures introduced by Stanley [55]. We show that this concept encompasses many examples introduced before. In particular, we introduce pointed decompositions of lattices and study their enumerative and geometric structure. We also show that exponential pointed structures satisfy pointed analogs of Stanley's compositional and exponential formulas.

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Chapter 1

Introduction

We find the study of partitions of integers and sets, and partially ordered sets in mathematical works of Boole and Dedekind as early as in the nineteenth century. Sylvester [28] studied the poset of partitions of set $[n]$ where every block has even size. Sylvester proved that the Möbius function of a partition lattice in which each block size is divisible by 2 is $(-1)^{n/2}E_n$, where E_n is the n^{th} Euler number which enumerates the number of alternating permutations. Following the publication of Gian-Carlo Rota's seminal paper on the Möbius function, Stanley studied the order structure of partitions and generalized Sylvester's result to the d-divisible partition lattice, that is, the collection of partitions of $\{1, 2, 3, ..., n\}$ where each block size is divisible by d and he showed that its Möbius number is up to the sign of the number of permutations in S_{n-1} with descent set $\{d, 2d, \ldots, n-d\}$ [57].

Another important work in this area was Günter M. Ziegler's paper published in 1986 on the poset of partitions of an integer, where he disproved the conjecture that the poset P_n of partitions of an integer n, ordered by refinement, is Cohen-Macaulay for all n, by showing that the Möbius function on the intervals does not alternate in sign

in general [29]. In the same year, Hanlon, Calderbank, and Robinson [7] extended Stanley's result of 1978 to the action of the symmetric group S_n on the top homology group of the order complex of the d-divisible partition lattice $\prod_n^d - \{\hat{1}\}$ without the maximum element. They showed that this action is the Specht module on the border strip corresponding to the descent set $\{d, 2d, \ldots, n-d\}$.
In 1996, Wachs [46] showed that the d-divisible partition lattice has an EL-shelling(Edge Lexicographically shalling). Hence as a consequence she obtained that the homotopy type is a wedge of spheres of dimension $\frac{n}{d} - 2$. She then gave a more explicit proof of the S_n representation on the top homology of $\prod_n^d - \{\hat{1}\}$. In 2006, Ehrenborg and Readdy [60] continued the exploration in the direction of partition lattices and permutation statistics. They computed the Möbius number of filters in the partition lattice formed by restricting to partitions by type. That is, to each set partition of $[n]$ assign as a type the integer partition consisting of the multiset of cardinalities of the blocks of a set partition. Then select set partitions of a given type. So, given a set F of integer partitions it is possible to find the Möbius number of the poset of set partitions whose type belongs to F. They then defined the notion of a knapsack partition and for a filter of a knapsack partition they showed that its Möbius number is a sum of descent set statistics. Later on Ehrenborg and Jung [38] extended these results of Ehrenborg and Readdy [60] topologically and showed that the associated order complex is a wedge of spheres.

This dissertation studies partial orders on set and integer partitions and related structures by building on results about partially ordered sets of partitions from enumerative, algebraic and geometric combinatorics involving Möbius numbers, homology of order complex and homotopy of order complexes. We use pointed partition structures, where one part of an integer and set partition is marked. These have been introduced in the works of Ehrenborg and Readdy [60], Ehrenborg and Jung [63] and Ziegler [29]. We exhibit new enumerative and geometric properties of the partial orders of pointed partitions studied by Ehrenborg and Readdy. In particular, we compute the Möbius numbers and homotopy types of lower intervals. In addition, we investigate their ordered counterparts and provide analogous results in this setting.

Stanley introduced the notion of exponential structures, that is, a sequence of posets that behave, in many respects like the partition lattice. Ehrenborg and Readdy [61] extended Stanleys notion of exponential structures to that of exponential Dowling structures. In this thesis, we extend Stanley's notion of exponential structures to that of exponential pointed structures. We show that this concept encompasses many examples introduced before. In particular, we introduce pointed decompositions of lattices and study their enumerative and geometric structure. We derive an analog of Stanley's compositional and exponential formula for pointed structures.

1.1 Organization of the Dissertation

This dissertation has two parts. The first part is concerned with posets of partitions of an integer n, of a set $[n] = \{1, 2, \cdots, n\}$ and of vector space V_n decomposition ordered by refinement. We compute Möbius numbers using combinatorial, as well as by topological methods. In the second part of this dissertation we extend Stanley's notion of exponential structures to that of exponential pointed structures and derive compositional and exponential formulas for pointed structures. It is an attempt to generalize Stanley's notion of exponential structures to that of pointed graded lattices.

Accordingly the content of this dissertation can be described as follows:

Chapter 1, contains an overview of the structure of the dissertation and the historical development of the objects studied in the dissertation. **Chapter 2** is devoted to background results and notations that are used in the dissertation. It also reviews the main results from recent works by Ehrenborg and Readdy [60], and Ehrenborg and Jung [63]. The proof of theorems and definitions of terms follow the same outline as in [63] and [60] except when we feel that there is an ambiguity. We also investigate how to describe the Möbius number of the poset of set partitions whose type is some filter, say F. We do so by constructing a set of permutations on n objects having descent set equal to a given set F with the help of the inclusion-exclusion principle. In **Chapter 3**, we compute the Möbius function of pointed integer partition and pointed ordered set partition posets using topological and analytic methods, and we show that the associated order complex is homotopy equivalent to a wedge of spheres in either case. As a consequence we obtain the reduced homology groups for each subposet. Quillen's order homotopy theorem forms our most powerful tool for proving theorems about the homotopy type of posets. In **Chapter 4**, we show how to compute the *Möbius* function of pointed graded lattices and use this general method to compute the Möbius function of pointed direct sum decomposition of a vector space. We also extend Stanley's notion of exponential structures to that of exponential pointed structures and derive compositional and exponential formulas for pointed structures. We do so by reviewing and extending some known results about exponential structure and then restate these results for the pointed structures. The chapter also contains a summary of the work done in this thesis and a discussion of some open problems that can be explored further in this area.

Chapter 2

Partially Ordered Sets and the Möbius Function

In this chapter, we give basic notions concerning partially ordered sets (posets) using the notations of Stanley [53] and [54]. In addition, we give a summary of some concepts [7], [43] and[46] we used in proving our main results.

2.1 Definitions and Basic Concepts

Definition 2.1.1. A **partially ordered set** (or poset) P is a set together with a binary relation denoted by $\leq$ satisfying the following axioms:

1. for all x in $P,\ x \leq x$. (reflexivity)
2. for all $x,$ and y in P if $x \leq y$ and $y \leq x$ then $x = y$.(antisymmetry)
3. for all $x,\ y,$ and z in P if $x \leq y$ and $y \leq z$ then $x \leq z$. (transitivity)

The notation $x \geq y$ is used to mean $y \leq x$, $x < y$ to mean $x \leq y$ and $x \neq y\ and\ x > y\ to\ mean\ y < x$. Two elements x and y of P are **comparable** if $x \leq y\ or\ y \leq x$; otherwise x and y are **incomparable**. Two posets P and Q are isomorphic if there exists an order-preserving bijection $\phi : P \rightarrow Q$ whose inverse is also order-preserving.

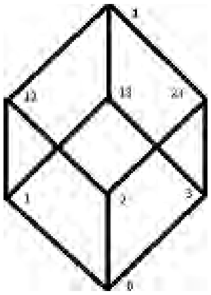
That is,

$$x \leq y \text{ in } P\text{if and only if } \phi(x) \ \ \leq \phi(y) \text{ in } Q.$$

We assume that all posets are finite throughout our discussion.

If P is a poset ordered by $\leq$, Q is a poset ordered by $\preceq$ and $Q \subseteq P$ then Q is a weak subposet of P if $x \preceq y$ in Q implies $x \leq y$ in P. If Q is a weak subposet of P with $P = Q$ as sets, then we call P a refinement of Q. By an induced subposet of P, we mean a subset Q of P and a partial ordering $\preceq$ of Q such that for $x, y \in \ Q$ we have $x \preceq y \ \in \ Q$ if and only if $x \leq y \; in \; P$.

The **closed interval** $[x, y] = \{z \in P : \ x \leq z \leq y\}$, where $x \leq y$ in P, is a special type of induced subposet of P. An induced subposet Q of P is said to be convex if $y \in Q$ such that $x < y < z$ in P and $x, \ z \in Q$. Similarly the **open interval** (x, y) can be defined as $(x, y) = \{z \in P : x < z < y\}$. If $x, y \in P$, then we say y covers x if $x < y$ and if no $z \in P$ satisfies $x < z < y$. Thus y covers x if and only if $x < y$ and $[x, y] = \{x, y\}$. Finite posets are represented graphically by the Hasse diagram, which is drawn using elements of P as vertices and the cover relation as edge (directed from below). For instance the Hasse diagram of poset B_3 consisting of the subsets of the set $[3] = \{1, 2, 3\}$ ordered by inclusion is



We say that a poset P has a **minimal element** denoted by $\hat{0}$ if there exists an element $\hat{0} \in P$ such that $x \geq \ \hat{0}$ for all x in P. Similarly, P has a **maximal element** denoted by $\hat{1}$ if there exists $\hat{1}$ in P such that $x \leq \hat{1}$ for all $x in P$.

A **chain** (or totally ordered set or linearly ordered set) is a poset in which any two elements are comparable. A subset C of P is called a chain if C is a chain when regarded as an induced subposet of P. The **length** $\ell(C)$ of a finite chain is defined by

$$\ell(C) = |C| - 1, \text{ where } |C| \text{ is the cardinality of } C.$$

The length (or rank) of a finite poset P is defined by $\ell(P) := \max\{\ell(C) : C \text{ is a chain of } P\}$. The length of an interval $[x, y]$ of P is denoted by $\ell(x, y)$. A chain $\widehat{0} = x_0 < x_1 < \ldots < x_n = \widehat{1}$ in the poset P is maximal or saturated if x_{i+1} covers x_i for $0 \leq i \leq n$, where x_0 is minimal element and x_n is maximal element. If every maximal chain of P has the same length n, then we say that P is **graded** and of rank n. In this case there is a unique rank function $\rho : P \longrightarrow \{0, 1, 2, \cdots, n\}$ such that $\rho(x) = 0$ if x is a minimal element of P, and $\rho(y) = \rho(x) + 1$ if y covers x in P.

A **multichain** of the poset P is a chain with repeated elements; that is, a multiset whose underlying set is a chain of P. An **antichain** (or Sperner family or clutter) is a subset A of a poset P such that any two distinct elements of A are incomparable. An **order ideal** (or down-set or decreasing subset) of P is a subset I of P such that if $x \in I$ *and* $y \leq x$, then $y \in I$. Similarly a **dual ideal** (or a filter) is a subset F of P such that if $x \in I$ and $y \geq x$, then $y \in I$. The set complement $P \backslash I$ of an ideal (respectively a filter) is a filter (respectively an ideal). The set of all order ideals of P, ordered by inclusion, forms a poset. Note that if f is an order preserving map from a poset P to a poset Q and F is a filter of Q then the inverse image $f^{-1}(F)$ is a filter of P.

A **lattice** L is a poset for which every pair of elements has a least upper bound and a greatest lower bound [58]. Since L is finite it always has minimal and maximal elements denoted by $\hat{0}$ and $\hat{1}$ respectively. For x and y in a lattice L, an upper bound of x and y is an element $z \in L$ satisfying $z \geq x$ and $z \geq y$. Then the **join** of x and y (denoted by $x \vee y$) is the least upper bound of two elements x and y, that is, $x \vee y \leq z$ for every upper bound z of x and y. The greatest lower bound of two elements x and y is dually denoted by $x \wedge y$, and it satisfies the condition that $x \wedge y \geq z$ for every lower bound z of x and y. Following [58], we refer to the greatest lower bound as a **meet**. If every pair of elements of a poset P has a meet (respectively, join), we say that P is a **meet-semi lattice** (respectively, **join-semi lattice**). In checking whether a (finite) poset is a lattice, it is sometimes easy to see that meets (respectively joins) exist, but the existence of joins (respectively meets) is not so clear. For that we use the the next proposition (its proof is given in [53] page 103).

Proposition 2.1.1. *Let P be a finite meet-semilattice with* $\hat{1}$. *Then P is a lattice. (Of course, dually a finite join-semi lattice with* $\hat{0}$ *is a lattice.)*

A finite lattice L is **modular** if and only if for all x, y, z in L such that $x \leq y$, we have $x \vee \ (y \wedge z) = (x \vee y) \wedge z$. A lattice L with $\hat{0}$ and $\hat{1}$ is complemented if for all x in L there is a y in L such that $x \wedge y = \hat{0}$ and $x \vee y = \hat{1}$. If for all x in L the complement y of x is unique, then L is uniquely complemented. If every interval $[x, y]$ of L is itself complemented then L is relatively complemented. An **atom** of a finite lattice L is an element covering $\hat{0}$, and L is said to be **atomic** (or a point lattice) if every element of L is the join of atoms. Dually, a **coatom** is an element that is covered by $\hat{1}$.

2.2 The Principle of Inclusion-Exclusion, The Möbius Function and the Möbius Inversion Formula

The Principle of Inclusion-Exclusion is one of the fundamental tools of enumerative combinatorics. To make use of this principle we first approximate our answer by over count, then subtract off an over counted approximation to our original error, and so on until after finitely many steps we have ′converged′ to the correct answer. The usual combinatorial situation involving the principle can be described as follows. Suppose you are given a set O of objects and P of properties (or conditions), where each object satisfies some subset of the set of conditions. Now for any subset $S \subseteq P$, let $f(S)$ be the number of objects in O that have at least the conditions in S and $g(S)$ be the number of objects in O that have exactly the conditions in S. The Principle of Inclusion-Exclusion (PIE) is a formula that allows one to find the function g from the function f. Clearly then

$$f(S) = \sum_{S \subseteq T \subseteq P} g(T); \tag{2.1}$$

this is because an object is counted by $f(S)$ if and only if it is counted by $g(T)$ for some $T \supseteq S$.

Theorem 2.2.1. [***PIE***] *If f and g satisfy (2.1), then*

$$g(S) = \sum_{S \subseteq T \subseteq P} (-1)^{|T|-|S|} f(T) \tag{2.2}$$

The dual form of The Principle of Inclusion-Exclusion can be obtained by interchanging $\cap$ with $\cup$, $\subseteq$ with $\supseteq$, and so on, throughout. The dual form of theorem (2.2.1) states that

Theorem 2.2.2. [***PIE, dual form***] *If f and g satisfy*

$$f(T) = \sum_{S \supseteq T} g(S), \text{ then} \tag{2.3}$$

$$g(T) = \sum_{S \supseteq T} (-1)^{|S|-|T|} f(S) \tag{2.4}$$

Definition 2.2.1. The **Möbius function** μ is a function which assigns to each order relation $x < y$ in a poset P an integer and its recursive formulation is given as [58]:

$$\mu(x, y) = \begin{cases} 1, & \text{for all } x = y, \\ -\sum_{z : x \leq z < y} \mu(x, z), & \text{for all } x < y. \end{cases}$$

We can also define the **Möbius function** μ as inverse of the **zeta** function which is defined by

$$\zeta(x, y) = \begin{cases} 1, & \text{for all } x \leq y \text{ in } P, \\ 0, & \text{otherwise.} \end{cases}$$

Thus, the relation $\mu\zeta = \delta$ is equivalent to the recursive formulation of the **Möbius function** μ defined above in 2.2.1, where

$$\delta_{xy} = \delta(x, y) = \begin{cases} 1, & \text{for all } x = y \text{ in } P, \\ 0, & x \neq y. \end{cases}$$

Proposition 2.2.3. [***The Product Theorem***] *Let P and Q be locally finite posets, and let $P \times Q$ be their direct product. If $(s,\ t) \leq (s_0,\ t_0)$ in $P \times Q$ then*

$$\mu_{P \times Q}((s,\ t), (s_0,\ t_0)) = \mu_P(s, s_0)\mu_Q(t, t_0).$$

Proof. Let $(s,\ t) \leq (s_0,\ t_0)$. We have

$$\begin{aligned} \sum_{(s,t) \leq (u,v) \leq (s',t')} \mu_P(s, u)\mu_Q(t, v) &= \left(\sum_{s \leq u \leq s'} \mu_P(s, u) \right) \left(\sum_{t \leq v \leq t'} \mu_Q(t, v) \right) && (2.5) \\ &= \delta_{ss'}\delta_{tt'} && (2.6) \\ &= \delta_{(s,t),(s',t')}. && (2.7) \end{aligned}$$

Comparing with equation in 2.2.1, which determines μ uniquely, completes the proof.

□

Proposition 2.2.4. [***Möbius Inversion Formula***] *Let P be a poset for which every principal order ideal I_x is finite. Let $f,\ g : P \longrightarrow K$,where K is a field. Then*

$$g(x) = \sum_{y \leq x} f(y), for \ \ all\ y\ in\ P, \tag{2.8}$$

if and only if

$$f(x) = \sum_{y \leq x} g(y)\mu(y,x), for \ \ all\ x\ in\ P. \tag{2.9}$$

Proof. Assume (2.8) holds, we have (for fixed $x \in P$)

$$\begin{aligned} \sum_{y \leq x} g(y)\mu(y,x) &= \sum_{y \leq x} \mu(y,x) \sum_{u \leq y} f(u) && (2.10)\\ &= \sum_{u \leq x} f(u) \sum_{u \leq y \leq x} \mu(y,x) && (2.11)\\ &= \sum_{u \leq x} f(u)\delta(u,x) && (2.12)\\ &= f(x), && (2.13) \end{aligned}$$

which is (2.9). A completely analogous argument shows that (2.8) follows from (2.9). □

A dual formulation of the Möbius inversion formula is sometimes convenient.

Proposition 2.2.5. [***Möbius Inversion Formula, dual form***] *Let P be a poset with principal order ideal F_x. Let $f,\ g : P \longrightarrow K$, where K is a field. Then*

$$g(x) = \sum_{y \geq x} f(y), for \ \ all\ x\ in\ P, \tag{2.14}$$

if and only if

$$f(x) = \sum_{y \geq x} \mu(x,y)g(y), for \ \ all\ x\ in\ P. \tag{2.15}$$

Proof. Exactly as above, except now we change the order. □

As in the Principle of Inclusion-Exclusion, the purely abstract statement of the Möbius inversion formula as given above is just a trivial observation in linear algebra. That is, Möbius inversion formula is a simplified version of Principle of the Inclusion-Exclusion under appropriate circumstances. But, in applications Möbius inversion formula is reaching further than the Principle of Inclusion-Exclusion.

Example 2.2.6. Let $P = B_n$, the boolean algebra of rank n. Now $B_n \cong \mathbf{2}^n$, and the Möbius function of the chain $\mathbf{2} = 1,\ 2$ is given by $\mu(1,1) = \mu(2,2) = 1$, $\mu(1,2) = -1$. Hence if we identify B_n with the set of all subsets of an $n-$set X, we conclude from the product theorem that $\mu(T,S) = (-1)^{|S-T|}$. Since $|S-T|$ is the length $l(S,T)$ of the interval $[S,T]$, in purely order-theoretic term we have

$$\mu(T,S) = (-1)^{l(S,T)}. \tag{2.16}$$

The Möbius inversion formula for B_n becomes the following statement.

Let $f,\ g:\ B_n \to \mathcal{C}$; then

$$g(S) = \sum_{T \subseteq S} f(T), \text{ for all } S \subseteq X, \tag{2.17}$$

if and only if

$$f(S) = \sum_{T \subseteq S} (-1)^{|S-T|} g(T), \text{ for all } S \subseteq X. \tag{2.18}$$

This is just equation (2.2). Hence we can say that Möbius inversion on a boolean algebra is equivalent to the Principle of Inclusion-Exclusion.

2.3 New Posets from Old

If P and Q are posets on disjoint sets, then the ordinal sum of P and Q is the poset $P \oplus Q$ on the union $P \cup Q$ such that $s \leq t$ in $P \oplus Q$ if

(a) $s, t \in P$ and $s \leq t$ in P, or

(b) $s, t \in Q$ and $s \leq t$ in Q, or

(c) $s \in P$ and $t \in Q$.

Now we give some simple examples of posets:

- ► We denote the trivial poset consisting of a single element by $\mathbf{1} = \bullet$.
- ► The disjoint union of n copies of P is denoted by nP. An $n-$element antichain (a subset A of a poset P such that any two distinct elements of A are incomparable) is isomorphic to $n\mathbf{1}$ and an n-element chain is the ordinal sum $\underbrace{\mathbf{1} \oplus \mathbf{1} \oplus \ldots \oplus \mathbf{1}}_{n \text{ times}}$ of n trivial posets.

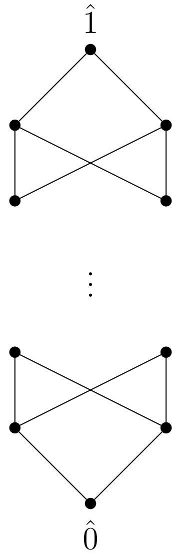


Figure 1: Hasse Diagram of nP, for $P = \mathbf{11}$.

- ► We denote the adjoin of a poset P and $\{\hat{0}, \hat{1}\}$ by $\hat{P}$, and similarly $\widehat{P \oplus Q} = P \oplus Q \cup \{\hat{0}, \hat{1}\}$. An example of the Hasse diagram of, $\widehat{P \oplus Q}$ is shown in Figure 3.

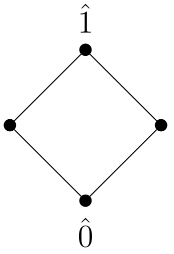


Figure 2: Hasse Diagram of $\hat{P}$, for $P = \mathbf{11}$.

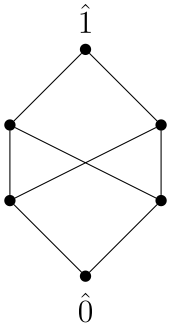


Figure 3: Hasse Diagram of $\widehat{P \oplus Q}$, for $P = \mathbf{11}$ and $Q = \mathbf{11}$.

Proposition 2.3.1. *Let P and Q be posets. Then*

$$\mu_{\widehat{P \oplus Q}}(\hat{0}, \hat{1}) = -\mu_{\hat{P}}(\hat{0}, \hat{1}) \cdot \mu_{\hat{Q}}(\hat{0}, \hat{1}).$$

Proof. We use induction on the cardinality of Q.

If $Q = \emptyset$ then $\hat{Q} = \{\hat{0}, \hat{1}\}$ and $\mu_{\hat{Q}}(\hat{0}, \hat{1}) = -1$. It also holds that $P \oplus Q = P$. Hence

$$\mu_{\widehat{P \oplus Q}}(\hat{0}, \hat{1}) = \mu_{\hat{P}}(\hat{0}, \hat{1}) = -\mu_{\hat{P}}(\hat{0}, \hat{1}) \cdot \mu_{\hat{Q}}(\hat{0}, \hat{1}).$$

Let $q \in Q$ thus we can have that, $(P \oplus Q)_{\leq q} = P \oplus Q_{\leq q}$, where $Q_{\leq q}$ denotes the set $\{x \in Q | x \leq q\}$. $(P \oplus Q)_{\leq q}$ is defined the same way.

Now assume $|Q| \geq 1$ and that the hypothesis has been verified for all posets of smaller cardinality.

$$\begin{aligned}
\mu_{\widehat{P\oplus Q}}(\hat{0},\hat{1}) &= -\sum_{\hat{0}\leq x<\hat{1}} \mu_{\widehat{P\oplus Q}}(\hat{0},x) && (2.19)\\
&= -\mu_{\widehat{P\oplus Q}}(\hat{0},\hat{0}) - \sum_{p\in P}\mu_{\widehat{P\oplus Q}}(\hat{0},p) - \sum_{q\in Q}\mu_{\widehat{P\oplus Q}}(\hat{0},q) && (2.20)\\
&= -\sum_{p\in P}\mu_{\widehat{P\oplus Q}}(\hat{0},p) - \sum_{q\in Q}\mu_{\widehat{P\oplus Q_{<q}}}(\hat{0},q) - 1 && (2.21)\\
&= -\sum_{p\in P}\mu_{\widehat{P\oplus Q}}(\hat{0},p) + \sum_{q\in Q}\mu_{\hat{P}}(\hat{0},\hat{1})\cdot\mu_{\hat{Q}}(\hat{0},q) - 1 && (2.22)\\
&= -(\sum_{p\in P\cup\{\hat{0}\}}\mu_{\widehat{P\oplus Q}}(\hat{0},p)) + \mu_{\hat{P}}(\hat{0},\hat{1})\cdot\sum_{q\in Q}\mu_{\hat{Q}}(\hat{0},q) && (2.23)\\
&= \mu_P(\hat{0},1)(1+\sum_{q\in Q}\mu_{\hat{Q}}(\hat{0},q)) && (2.24)\\
&= -\mu_{\hat{P}}(\hat{0},1)\cdot\mu_{\hat{Q}}(\hat{0},1). && (2.25)
\end{aligned}$$

□

Corollary 2.3.2. *Let* $P =$ ***11***.*Then* $\mu_{\widehat{P\oplus\cdots\oplus P}}(\hat{0},\hat{1}) = (-1)^n$.

A mapping $t \to \bar{t}$ on a poset P is called a **closure operator** (or closure) if for all $s, t \in P$,

- $t \leq \bar{t}$
- $s \leq t \Rightarrow \bar{s} \leq \bar{t}$
- $\bar{\bar{t}} = \bar{t}$

An element t of P is called **closed** if $t = \bar{t}$. The set of closed elements with orders induced by P is denoted $\overline{P}$, called **the quotient** of P relative to the closure.

Proposition 2.3.3. [***Stanley*** [*58*]] *Let P be a poset with closure operator $t \to \bar{t}$ and quotient $\overline{P}$. Then for all $s,\ t \in P$,*

$$\sum_{u\in P, \bar{u}=\bar{t}} \mu(s,u) = \begin{cases} \mu_{\overline{P}}(\bar{s},\bar{t}) & \text{if } s = \bar{s}, \\ 0 & \text{if } s < \bar{s}. \end{cases}$$

Proposition 2.3.4. [***Stanley*** [*58*]] *Let P be a poset with $\hat{0}$ and $\hat{1}$ and $x \in P \setminus \{\hat{1}\}$ such that $\mu_P(\hat{0},x) = 0$. Then $\mu_P(\hat{0},\hat{1}) = \mu_{P\setminus\{x\}}(\hat{0},\hat{1})$.*

Proof. We use induction on the cardinality l of the longest chain between x and $\hat{1}$. Assume $l = 0$, that is, x is coatom (an element covered by $\hat{1}$.)Then we have

$$\begin{aligned} \mu(\hat{0},\hat{1}) &= -\sum_{\hat{0}\le y<\hat{1}} \mu_P(\hat{0},y) && (2.26)\\ &= -\sum_{\hat{0}\le y<\hat{1}, y\neq x} \mu_P(\hat{0},y) - \mu_P(\hat{0},x) && (2.27)\\ &= -\sum_{\hat{0}\le y<\hat{1}, y\neq x} \mu_P(\hat{0},y) && (2.28)\\ &= -\sum_{\hat{0}\le y<\hat{1}, y\neq x} \mu_{P\setminus\{x\}}(\hat{0},y) = \mu_{P\setminus\{x\}}(\hat{0},\hat{1}). && (2.29) \end{aligned}$$

Now assume $l > 0$. Then

$$\begin{aligned} \mu_P(\hat{0},\hat{1}) &= -\sum_{\hat{0}\le y<\hat{1}} \mu_P(\hat{0},y) && (2.30)\\ &= -\sum_{\hat{0}\le y<\hat{1}, x\nleq y} \mu_P(\hat{0},y) - \sum_{x<y<\hat{1}} \mu_P(\hat{0},y) && (2.31)\\ &= -\sum_{\hat{0}\le y<\hat{1}, x\nleq y} \mu_P(\hat{0},y) - \sum_{x<y<\hat{1}} \mu_{P\setminus\{x\}}(\hat{0},y) && (2.32)\\ &= -\sum_{\hat{0}\le y<\hat{1}, x\nleq y} \mu_{P\setminus\{x\}}(\hat{0},y) && (2.33)\\ &= \mu_{P\setminus\{x\}}(\hat{0},\hat{1}). && (2.34) \end{aligned}$$

Therefore, $\mu_P(\hat{0}, \hat{1}) = \mu_{P\setminus\{x\}}(\hat{0}, \hat{1})$ □

2.4 Poset Topology and The Homotopy type

In this section, we use the notations introduced by Wachs [46], and refer to the works in [3],[4], [36], [38] and [45].

Definition 2.4.1. An abstract simplicial complex $\triangle$ on finite vertex set V is a nonempty collection of subsets of V satisfying:

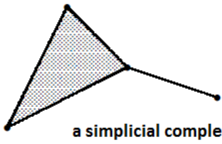
(i) If $v \in V$, then $\{v\} \in \triangle$, and

(ii) If $G \in \triangle$ and $F \subseteq G$, then $F \in \triangle$.

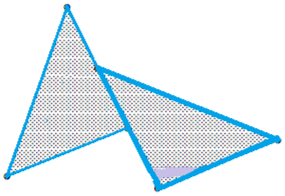
The elements of $\triangle$ are called **faces** (or simplices) of $\triangle$ and the maximal faces are called **facets**. We say that a face F has dimension d and write $dim\ F = d$, when $d = |F| - 1$. Faces of dimension d are referred to as $d - faces$. The dimension of $\triangle$, denoted by $dim\ \triangle$ is defined to be

$$dim\triangle = \max_{F\in\triangle}(dimF) \tag{2.35}$$

We also allow the $(-1)-$dimensional complex $\{\emptyset\}$, which we refer to as the empty simplicial complex. We say $\triangle$ is **pure** if all facets have the same dimension.



a simplicial complex



not a simplicial complex

A $d-$dimensional geometric simplex in $\mathbb{R}^n$ is defined to be the convex hull of $d+1$ affinely independent points or vertices in $\mathbb{R}^n$. The convex hull of any subset of the vertices is called a face of the geometric simplex. A geometric simplicial complex Γ in $\mathbb{R}^n$ is a nonempty collection of geometric simplices in $\mathbb{R}^n$ such that

(i) every face of a simplex in Γ is in Γ, and

(ii) the intersection of any two simplices of Γ is a face of both of them. From a geometric simplicial complex Γ, one gets an abstract simplicial complex $\triangle(\Gamma)$ by letting the faces of $\triangle(\Gamma)$ be the vertex sets of the simplices of Γ. Every abstract simplicial complex $\triangle$ can be obtained in this way, i.e., given a geometric simplicial complex Γ we can have that $\triangle(\Gamma) = \triangle$. Although Γ is not unique, the underlying topological space, obtained by taking the union of the simplices of Γ under the usual topology on $\mathbb{R}^n$, is unique up to homeomorphism [2]. We refer to this space as the geometric realization of $\triangle$ (methods for turning abstract simplical complex into topological space) and denote it by $\|\triangle\|$. We will usually drop the $\|\ \|$ and let $\triangle$ denote an abstract simplicial complex as well as its geometric realization.

To every poset P, we can associate an abstract simplicial complex $\triangle(P)$ called the **order complex** of P. The vertices of $\triangle(P)$ are the elements of P and the $d-$faces of $\triangle(P)$ are the $d-$chains (i.e., totally ordered subsets) of P. Note that the order complex of the empty poset is the empty simplicial complex $\{\emptyset\}$. Thus, for this simplicial complex we can always have a topological space called geometric realization. Conversely to every simplicial complex $\triangle$, one can associate a poset $P(\triangle)$ called the **face poset** of $\triangle$, which is defined to be the poset of nonempty faces ordered by inclusion. The face lattice $L(\triangle)$ is $P(\triangle)$ with a smallest element $\hat{0}$ and a largest element $\hat{1}$ attached.

If Δ is finite, then let f_i denote the number of $i-$dimensional faces of Δ. Define the **reduced Euler characteristic** $\widetilde{\chi}(\Delta)$ by

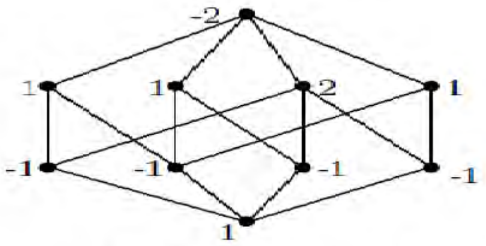
$$\widetilde{\chi}(\Delta) = -f_{-1} + f_0 - f_1 + \cdots$$

Note that $f_{-1} = 1$ unless $\Delta = \emptyset$. The simplicial complexes $\Delta_1 = \emptyset$ and $\Delta_2 = \{\emptyset\}$ are not the same; in particular, $\widetilde{\chi}(\Delta_1) = 0$ and $\widetilde{\chi}(\Delta_2) = -1$. Recall that the **ordinary Euler characteristic** $\chi(\Delta)$ is defined as $f_0 - f_1 + f_2 - \cdots$. Hence

$$\widetilde{\chi}(\Delta) = \chi(\Delta) - 1.$$

Recall that the **Möbius function** μ is a function which assigns to each interval in a poset P an integer and its recursive formulation is given by:

$$\mu(x,y) = \begin{cases} 1 & \text{for all } x = y, \\ -\sum_{z: x \leq z < y} \mu(x,z) & \text{for all } x < y. \end{cases}$$



The values of $\mu(\hat{0}, x)$ are shown for each element x of the poset

In Figure above the values of $\mu(\hat{0}, x)$ are shown for each element x of the poset. Philip Hall's formula to calculate the Möbius function μ on a poset P is given by

$$\mu(x,y) = \sum_{x = z_0 < z_1 < \cdots < z_i = y} (-1)^i$$

for all $x < y$ in P [58].

Its connection to the reduced Euler characteristic $\widetilde{\chi}(\Delta)$ is given by Philip Hall as below.

Proposition 2.4.1. [***Philip Hall Theorem***] *For any poset P,*

$$\mu(\hat{P}) = \widetilde{\chi}(\Delta(P)).$$

As stated in [42], there is a standard result of algebraic topology [21] that the Euler characteristic of a complex can be computed from its homology groups. Thus

$$\mu(P) = \sum_n (-1)^n \; rank \; \widetilde{H_n}(\Delta(P)),$$

where $\widetilde{H_n}(\Delta(P))$ represents reduced simplicial homology with integer coefficients. This relationship between Möbius numbers and homology is one of the main reasons for studying the geometric realizations of posets.

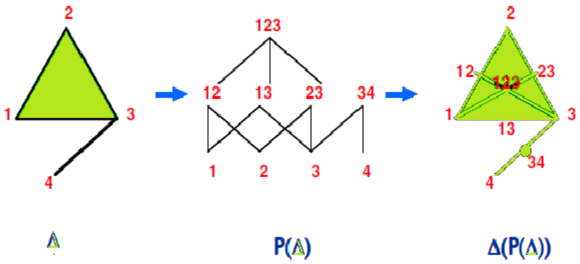
Let Δ_0 be a simplicial complex, and let $'a'$ be a vertex not in Δ_0. The cone,

$$Cone(\Delta_0) = \; Cone_a(\Delta_0)$$

over Δ_0 is the simplicial complex obtained from Δ_0 by adding $\sigma \cup \{a\}$ for each $\sigma \in \Delta_0$. Equivalently, Δ is a cone with apex a if $\sigma \cup \{a\}$ is a face of Δ whenever σ is a face of Δ. In particular, if a poset P has some element which is comparable to every other element, then $\Delta(P)$ is a cone. It is well known that any realization of a cone is contractible (complex which is homotopy equivalent to a point). Since a homotopy equivalence induces homology isomorphism, any contractible space is acyclic (has trivial homology groups). It follows that if P is acyclic, $\mu(P) = 0$.

If we start with a simplicial complex $\triangle$, take its face poset $P(\triangle)$, and then take the order complex $\triangle(P(\triangle))$, we get a simplicial complex known as the(first)

barycentric subdivision of $\triangle$; see Figure below. The geometric realizations are always homeomorphic, i.e., $\triangle \cong \triangle(P(\triangle))$. Therefore, from a topological point of view simplicial complexes and posets can be considered to be essentially equivalent notions. Thus, when we attribute a topological property to a poset, we mean that the geometric realization of the order complex of the poset has that property. For instance, if we say that the poset P is homeomorphic to the $n-$sphere $\mathbb{S}^n$ we mean that $\|\triangle(P)\|$ is homeomorphic to $\mathbb{S}^n$. Then for a poset P we have that $\mu_P(\widehat{0},\widehat{1}) = \widetilde{\chi}(P(\triangle))$.



An example of barycentric subdivision.

Let A and B be two finite sets such that $A \cap B = \emptyset$. Let Δ be a family of subsets of A, and let Γ be a family of subsets of B. The **join** of Δ and Γ is the family $\Delta * \Gamma = \{\delta \cup \gamma : \delta \in \Delta, \gamma \in \Gamma\}$. Let Δ be a simplicial complex, and let $\sigma \in \Delta$. The deletion of Δ with respect to σ is the subcomplex $del_{\triangle}(\sigma) = \{\tau \in \Delta : \tau \cap \sigma = \emptyset\}$. The **link** of Δ with respect to σ is the subcomplex $lk_{\triangle}(\sigma) = \{\tau \in \Delta : \tau \cap \sigma = \emptyset,\ and\ \tau \cup \sigma \in \Delta\}$, and star $st_{\triangle}(\sigma) = \{\tau \in \Delta : \tau \cup \sigma \in \Delta\}$. Clearly, $del_{\triangle}(\sigma) \cap st_{\triangle}(\sigma) = lk_{\triangle}(\sigma)$ and $\sigma * lk_{\triangle}(\sigma) = st_{\triangle}(\sigma)$.

An (order-preserving) poset map between two posets $P = (A, \leq)$ and $Q = (B, \leq)$ is a function $f : A \longrightarrow B$ such that $f(x) \leq_Q f(y)$ whenever $x \leq_P y$. We will often write $f : P \longrightarrow Q$. We obtain a poset structure on a simplicial complex $\triangle$ of sets by defining $\sigma \leq \tau$ whenever $\sigma \subseteq \tau$.

A simplicial map f from a simplicial complex $\triangle$ to a poset P sends vertices of $\triangle$ to elements of P and faces of the simplicial complex $\triangle$ to chains of P.

For two continuous maps f and g of the space X into the space Y, if there exists a continuous map $F : X \times [0,1] \longrightarrow Y$ such that $F(x,0) = f(x)$ and $F(x,1) = g(x)$ for every x in X, then we call the map F a homotopy between two maps f and g. When a homotopy F between f and g exists, we write $f \approxeq g$ and say that f is homotopic to g. The spaces X and Y are of the same homotopy type (or are called **homotopy equivalent**) if there exist mappings $f : X \longrightarrow Y$ and $g : Y \longrightarrow X$ such that the map $f \circ g : X \longrightarrow X$ is homotopic to the identity map of the space X and the map $g \circ f : Y \longrightarrow Y$ is homotopic to the identity map of the space Y. Each of the maps f and g is called a homotopy equivalence. If there exists a homotopy equivalence between two spaces X and Y then we say the spaces X and Y are homotopy equivalent, and two spaces X and Y are said to have the same **homotopy type**. A space with the homotopy type of a point is **contractible**. Equivalently, X is contractible if and only if identity map of the space X is homotopic to a constant map. A homotopy from identity map of the space X to a constant is called a **contraction** of X. If two spaces X and Y have the same homotopy type then they have the same homology groups. Hence the Euler characteristic χ, which is defined as the sum $\chi(\triangle) = f_0 - f_1 + f_2 - \ldots$, where f_i is the number of $i-$faces of the complex $\triangle$, is an invariant under homotopy.

Theorem 2.4.2. [***Order Homotopy Theorem, Quillen 1978***]
Let $f, g : \Delta \longrightarrow P$ be simplicial maps from a simplicial complex Δ to a poset P. If $f(x) \leq g(x)$ for every x in ground set of Δ, then f and g are homotopic.

Corollary 2.4.3. *Let $f : P \longrightarrow P$ be an order-preserving map such that $f(x) \geq x$*

for all $x \in P$*. Then* f *induces homotopy equivalence between* P *and* $f(P)$*.*

Theorem 2.4.4. [***Quillen's Fiber Lemma***]

Let f *be a simplicial map from the simplicial complex* Γ *to the poset* P *such that for all elements* x *in the poset* P*, the subcomplex* $\Delta(f^{-1}(P_{\geq x}))$ *is contractible. Then the order complex* $\Delta(P)$ *and the simplicial complex* Γ *are homotopy equivalent.*

Chapter 3

On the Möbius Function of Pointed Integer and Set Partitions

3.1 Introduction

In this chapter we compute the Möbius function of posets of pointed integer partitions and pointed set partitions using topological and analytic methods, and show that in each case the associated order complex is a wedge of spheres. By this we can also compute the associated reduced homology group for each subposet. We also discuss pointed partitions, pointed compositions and knapsack partitions of integers and sets, and we give an overview of the Möbius function of restricted partitions mainly based on the works of Richard Ehrenborg and Margaret A. Readdy.

3.2 Pointed Partitions, Pointed Compositions and Knapsack Partitions

Let n be a non-negative integer. A multiset $u = \{u_1, u_2, \ldots, u_r\}$ of integers is an integer partition of n provided that

$$\sum_{i=1}^{r} u_i \ = n \ \text{ and } \ u_i \geq \ 1, \ for \ \ i = 1, 2, \ldots r. \tag{3.1}$$

Hence a partition of n is a representation of n as a sum of integers where the order of the terms (or parts) is irrelevant. We use multiplicities as a superscript of each u_i in their decreasing order to describe the multiset u. Thus, for instance, for the integer partition $\{6, 4, 4, 3, 2, 2, 1, 1\}$ of 23 we write $\{6, 4^2, 3, 2^2, 1^2\}$.

Definition 3.2.1. A pair $\{u,\ \underline{m}\} = \{u_1,\ u_2,\ \ldots,\ u_r,\ \underline{m}\}$ is called a **pointed integer partition** of n if $u = \ \{u_1,\ u_2,\ \ldots,\ u_r\}$ is an integer partition of $n - m$, where m is a non-negative integer.

Let $Z_n^{\bullet}$ denote the set of all pointed integer partitions of the non-negative integer n. On this set we define an order relation as the transitive closure of the following cover relations: For some $1 \leq \ i \leq r$,

$$\{u_1,\ u_2,\ \ldots,\ u_{i-1}, u_i, \ldots, u_r,\ \underline{m}\} \ \leq \ \{u_1,\ u_2,\ \ldots, u_{i-2},\ u_{i-1}{+}u_i,\ \ldots,\ u_r,\ \underline{m}\} \tag{3.2}$$

and

$$\{u_1,\ u_2,\ \ldots,\ u_{r-1}, u_r,\ \underline{m}\} \ \leq \ \{u_1,\ u_2,\ \ldots,\ u_{r-1},\ \underline{u_r + \ m}\}. \tag{3.3}$$

In words, it is defined by adding any two neighbouring parts together and going up in the order so that if one of the part is the pointed part then the sum becomes the pointed part in the new pointed partition.

Example 3.2.1. Consider the case $n = 3$. Thus $Z_3^\bullet = \{111\underline{0},\ 12\underline{0},\ 11\underline{1},\ 2\underline{1},\ 1\underline{2},\ 3\underline{0},\ \underline{3}\}$. It can be easily shown that $(Z_n^\bullet,\ \leq)$ is a poset, however, $Z_n^\bullet$ is not a lattice for $n \geq 3$, since there is no unique least upper bound for each pair of elements.

Let $[n] = \{1,\ 2,\ 3, \ldots,\ n\}$ and for $i \leq j$ let $[i,j] = \{i,\ i+1,\ \ldots,\ j\}$. A **pointed set partition** S of the set $[n]$ is a pair $(\tau, \mathbf{B})$,where $\mathbf{B}$ is a subset of $[n]$ and $\tau = \{B_1,\ B_2,\ \ldots,\ B_k\}$ is a set partition of the set difference $[n] - \mathbf{B}$. The pointed partition S can be written as $S = \{B_1,\ B_2,\ \ldots,\ B_k,\ \underline{B}\}$ where the underlined set B is called the zero block of the pointed set partition S. We denote the number of blocks of S (including **the zero block**) by $|S|$ and write $S = B_1|\ B_2|\ \ldots|\ B_k|\underline{B}$ for the pointed partition. For instance, we write $125|34|6|\underline{789}$ for $\{\{1,2,5\},\{3,4\},\{6\},\underline{\{7,8,9}\}\}$.

Let $\prod_n^\bullet$ denote the set of all pointed set partitions on the set $[n]$ and order $\prod_n^\bullet$ by refinement. That is, for two pointed set partitions S_1 and S_2, we have that $S_1 \leq S_2$ if every block of S_1 is contained in some block (possibly the zero block) of S_2 and the zero block of S_1 in the zero block of S_2. Hence the cover relations are given by

$$\{B_1,\ B_2,\ \ldots,\ B_k, \underline{B}\}\ \leq \{B_1 \cup\ B_2,\ \ldots,\ B_k,\ \underline{B}\} \tag{3.4}$$

and

$$\{B_1,\ B_2,\ \ldots,\ B_k, \underline{B}\}\ \leq \{B_1,\ \ldots,\ B_{k-1},\ \underline{B_k \cup \mathbf{B}}\}. \tag{3.5}$$

The bijection $\{B_1,\ B_2,\ \ldots,\ B_k, \underline{B}\} \to \{B_1,\ B_2,\ \ldots,\ B_k, \underline{B \cup \{n+1}\}\}$ shows that this lattice is isomorphic to the partition lattice $\prod_{n+1}$. Thus, $\prod_n^\bullet$ is a lattice. By defining the **type of a pointed set partition** $S = \{B_1, B_2, \ldots, B_k, \underline{B}\}$ to be the pointed integer partition, $type(S) = \{|B_1|, |B_2|, \ldots, |B_k|, \underline{|B|}\}$, we relate the pointed set partitions to the notion of pointed integer partition.

Definition 3.2.2. Let n be a non-negative integer. A pointed integer composition of n is $\overrightarrow{c} = (c_1, \ \cdots, \ c_{k-1}, \ \underline{c_k})$ of non-negative integers with sum $c_1+\cdots+c_{k-1}+c_k = n$ where c_1 through c_{k-1} are required to be positive. The only part allowed to be 0 is the last entry $\underline{c_k}$.

The type of the composition $\overrightarrow{c} = (c_1, \ \ldots, \ c_{k-1}, \ \underline{c_k})$ is the pointed integer partition $\textit{type}(\overrightarrow{c}) = \{c_1, \ldots, \ c_{k-1}, \ \underline{c_k}\}$.

Let $C_n^\bullet$ denote the collection of all pointed compositions of n. Define an order relation on $C_n^\bullet$ by the cover relations:

$$(c_1, \ldots, c_{j-1}, c_j, c_{j+1}, \ldots, c_{r-1}, \underline{c_r}) \leq (c_1, \ldots, c_{j-1}, c_j + c_{j+1}, \ldots, c_{r-1}, \underline{c_r}) \tag{3.6}$$

and

$$(c_1, \ldots, c_{r-1}, \underline{c_r}) \leq (c_1, \ldots, c_{r-2}, \underline{c_{r-1} + c_r}). \tag{3.7}$$

That is, the cover relation occurs by adding two adjacent entries of the composition. Note that the poset $C_n^\bullet$ is isomorphic to the Boolean algebra on n elements and the maximal and minimal elements are the two compositions $(\underline{n})$ and $(1, \ 1, \ \ldots, \ 1, \ \underline{0})$, respectively.

Remark 3.2.1. The three posets $Z_n^\bullet$, $\prod_n^\bullet$, $C_n^\bullet$ are graded, and we will denote the rank function of a graded poset by ρ and use $\rho(x, y)$ to denote the rank difference $\rho(x, y) = \rho(y) - \rho(x)$.

Let $u = \{u_1^{m_1}, u_2^{m_2}, \ldots, u_r^{m_r}\}$ be an integer partition. Since there are $\prod_{i=1}^r (m_i + 1)$ multi-subsets λ of u we have that

$$\left| \left\{ \sum_{e \in \lambda} e : \lambda \subseteq u \right\} \right| \leq \prod_{i=1}^{r} (m_i + 1) \tag{3.8}$$

Definition 3.2.3. We call an integer partition $u = \{u_1^{m_1}, u_2^{m_2}, \ldots, u_r^{m_r}\}$ a **knapsack partition** if equality holds in (3.8). Which is to mean that each integer in the set on the left hand side of (3.8) has a unique representation as a sum of elements from the multiset u.

Definition 3.2.4. A pointed integer partition $\{u,\ \underline{m}\}$ is called a **pointed knapsack partition** if the partition u is a knapsack partition.

Let $\alpha = \alpha_1 \cdots \alpha_n$ be a permutation in the symmetric group $\mathbf{S}_n$. Then the descent set of α, denoted by $\mathbf{D}(\alpha)$ is defined to be the set $\mathbf{D}(\alpha) = \{i \in [n-1] : \alpha_i > \alpha_{i+1}\}$. Thus $\mathbf{D}(\alpha) \subseteq [n-1]$ and corresponds to compositions of n. Hence we define the descent composition of the permutation α to be the composition
$\mathbf{D}_{\overrightarrow{c}}(\alpha) = (s_1, s_2 - s_1, \ldots, s_{k-1} - s_{k-2}, n - s_{k-1})$, where the descent set of α is the set $\{s_1, s_2, \ldots, s_{k-1}\}$ with $s_1 < s_2 < \ \cdots \ < s_{k-1}$.
For a pointed composition $\overrightarrow{c} = (c_1,\ \ldots,\ c_{k-1},\ \underline{c_k})$ of n with $c_k > 0$, let $\beta(\overrightarrow{c})$ denote the number of permutations in the symmetric group $\mathbf{S}_n$ with descent composition $\overrightarrow{c}$, that is, with the descent set $\{c_1, c_1 + c_2, \ldots, c_1 + \ldots + c_{k-1}\}$. If $c_k = 0$, let $\beta(\overrightarrow{c}) = 0$ for $k \geq 2$ and $\beta(\overrightarrow{c}) = 1$ for $\overrightarrow{c} = (\underline{0})$.

3.3 The Möbius Function of Restricted Partitions

Let F be a filter in the pointed integer partition poset $Z_n^\bullet$. Let $\prod_n^\bullet(F)$ be the filter of the pointed set partition lattice $\prod_n^\bullet$ consisting of all set partitions having their types belonging to F, that is, $\prod_n^\bullet(F) = \{\pi \in \prod_n^\bullet : \mathit{type}(\pi) \in F\}$. Similarly, define $C_n^\bullet(F)$ to be the filter of pointed compositions having types belonging to F, that is, $C_n^\bullet(F) = \{\vec{c} \in C_n^\bullet(F) : \mathit{type}(\vec{c}) \in F\}$. Since for any two elements there is a maximum obtained by taking their join, $\prod_n^\bullet(F)$ and $C_n^\bullet(F)$ are join semi-lattices. Hence after adjoining a minimal element $\hat{0}$ we have that both $\prod_n^\bullet(F) \cup \{\hat{0}\}$ and $C_n^\bullet(F) \cup \{\hat{0}\}$ are lattices.

Theorem 3.3.1. [***Ehrenborg, Readdy [62]***] *Let F be a filter of the pointed integer partition poset $Z_n^\bullet$. Then the Möbius function of the filter $\prod_n^\bullet(F)$ with a minimal element $\hat{0}$ adjoined is given by* $\mu(\prod_n^\bullet(F) \cup \hat{0})$

$$= \sum_{\vec{c} \in C_n^\bullet(F)} (-1)^{\rho(\vec{c},1)} \mu_{C_n^\bullet(F) \cup \{\hat{0}\}}(\hat{0}, \vec{c}) \beta(\vec{c}). \tag{3.9}$$

Although $C_n^\bullet(F) \cup \{\hat{0}\}$ is not necessarily graded, the expression $\rho(\vec{c}, 1)$ appearing in the statement of Theorem 3.3.1 is well-defined since the interval $(\vec{c}, 1)$ is itself graded. For a pointed composition $\vec{c} = (c_1, \ \ldots \ , c_{k-1}, \ \underline{c_k})$ of a non-negative integer n recall the multi-nomial coefficient

$$\binom{n}{\vec{c}} = \frac{n!}{c_1! \cdots c_k!}.$$

We note that an ordered set partition is a partition in $\prod_n^\bullet$ with an ordering of the blocks such that the last block is the zero block. The type of an ordered set partition is the pointed composition where one lists the sizes of the blocks in their given order. Hence we note that given a composition $\vec{d}$ of n there are $\binom{n}{\vec{d}}$ ordered set partitions with $\vec{d}$ as its type.

Observe that $\binom{n}{\vec{c}}$ counts the number of permutations in S_n having descent set contained in the set $\{c_1, c_1 + c_2, \ldots, c_1 + \cdots + c_{k-1}\}$.

By the principle of inclusion and exclusion, we have that

$$\beta(\vec{c}) = \sum_{\vec{c} \leq \vec{d}} (-1)^{\rho(\vec{c},1)} \binom{n}{\vec{c}}.$$

Now using this expression for $\beta(\vec{c})$ the right-hand side of (3.3.1) becomes

$$\sum_{\vec{c}\in C_n^{\bullet}(F)} (-1)^{\rho(\vec{c},1)} \mu_{C_n^{\bullet}(F)\cup\{\hat{0}\}}(\hat{0},\vec{c})\beta(\vec{c}) = \sum_{\vec{d}\in C_n^{\bullet}(F)} (-1)^{\rho(\vec{d},\hat{1})} \binom{n}{\vec{d}} \sum_{\hat{0}<\vec{c}\leq\vec{d}} \mu_{C_n^{\bullet}(F)\cup\{\hat{0}\}}(\hat{0},\vec{c})$$
$$= - \sum_{\vec{d}\in C_n^{\bullet}(F)} (-1)^{\rho(\vec{d},\hat{1})} \binom{n}{\vec{d}} .$$

The last sum can be viewed as an ordered set partition in $\prod_n^{\bullet}$ with an ordering of the blocks such that the last block is the empty block. The type of an ordered set partition is the pointed composition where one lists the size of each block in their given order. Hence we note that given a composition $\vec{d}$ of n there are $\binom{n}{\vec{d}}$ ordered set partitions with $\vec{d}$ as its type. Hence we have

$$- \sum_{\vec{d}\in C_n^{\bullet}(F)} (-1)^{\rho(\vec{d},\hat{1})} \binom{n}{\vec{d}} = \sum_{type\ (\pi)\in\ C_n^{\bullet}(F)} (-1)^{|\pi|}\ (where\ \ \pi\ is\ an\ ordered\ set\ partition\ )$$
$$= \sum_{\pi\in\prod_n^{\bullet}(F)} (-1)^{|\pi|}(|\pi|-1)! \tag{3.10}$$
$$= - \sum_{\pi\in\prod_n^{\bullet}(F)} \mu_{\prod_n^{\bullet}(F)\cup\{\hat{0}\}}(\pi,\hat{1}) \tag{3.11}$$
$$= \mu(\textstyle\prod_n^{\bullet}(F)\cup\{\hat{0}\}).$$

Corollary 3.3.2. [***Ehrenborg, Readdy*** [*62*]]

Let $n = rp + m$. *Let* $\prod_n^{\bullet,r,m}$ *be all the partitions in* $\prod_n^{\bullet}$ *where the zero block has cardinality at least* m *and the remaining blocks have cardinality divisible by* r. *Then the Möbius function of* $\prod_n^{\bullet,r,m}$ *is given by* $\mu(\prod_n^{\bullet,r,m}\cup\{\hat{0}\}) = (-1)^{p+1}\cdot\beta(r,r,\ldots,r,\underline{m})$.

Corollary 3.3.3. [***Stanley*** [*55*]]

Let $n = rp$ *and let* $\prod_n^{r}$ *denote the* $r-$*divisible lattice,that is, all partitions on* n *elements where the cardinality of each block is divisible by* r *and a minimal element* $\hat{0}$

is adjoined. Then $\mu(\prod_n^r)$ *is given by the sign* $(-1)^p$ *times the number of permutations* α *in* S_n *with descent set* $\{r, 2r, \ldots, r(p-l)\}$ *and* $\alpha(n) = n$.

Proof. Consider $\mathbf{D}_{\vec{c}}(\alpha) = (r,\ r,\ \ldots,\ r-1)$. Let $m = r-1$ to use (corollary 3.3.2) and also use the fact that $\prod_n^\bullet \cong \prod_{n+1}$. □

For a pointed knapsack partition $\{u,\ \underline{m}\}$ of n, define F to be the filter in the poset of compositions of n generated by compositions $\vec{c}$ such that *type*$(\vec{c}) = \{u,\ \underline{m}\}$. Now define $V(u,\ \underline{m})$ to be the collection of all pointed compositions $\vec{c} = (c_1, c_2, \ldots, c_r)$ in the filter F such that each c_i, $1 \ \leq \ i \leq \ r-1$, is a sum of distinct parts of the partition u and $c_r = m$. The goal here is to determine $\mu(C_n^\bullet(F) \cup \{\hat{0}\})$ and an explicit formula for $\mu(\prod_n^\bullet(F) \cup \{\hat{0}\})$. The procedure to attain this goal is using the set $V(u,\ \underline{m})$ defined as above.

For the pointed knapsack partition $\{2, 2, 2, 6, \underline{m}\}$, the set V of pointed compositions is

$$\text{V} = \{(2,\ 2,\ 2,\ 6,\underline{\text{m}}),\ (2,\ 2,8,\underline{\text{m}}),\ (2,\ 2,\ 6,\ 2,\underline{\text{m}}),\ (2,\ 8,\ 2,\underline{\text{m}}),\ (2,\ 6,\ 2,\ 2,\underline{\text{m}}),\ (8,\ 2,\ 2,\underline{\text{m}}),\ (6,\ 2,\ 2,\ 2,\underline{\text{m}})\}.$$

Observe the composition $(4, 2, 6, \underline{m})$ does not belong to V since 4 is the sum of two equal parts. If we also consider the pointed knapsack partition $\{r, r, \ldots, r, \underline{m}\}$ the set V only consists of the pointed composition $(r, r, \ldots, r, \underline{m})$.

Theorem 3.3.4. [***Ehrenborg, Readdy [62]***]

Let F *be the filter in the pointed integer partition poset* $Z_n^\bullet$ *generated by the pointed knapsack partition* $\{u,\ \underline{m}\} = \{u_1,\ u_2,\ \ldots,\ u_r,\ \underline{m}\}$ *of the integer* n.

1. *Let* $\vec{c} = (c_1, \ldots, c_{k-1}, c_k)$ *be a pointed composition in the filter* $C_n^\bullet(F)$*. Then*

$$\mu_{C_n^\bullet(F)\cup\{\hat{0}\}}(\hat{0}, \vec{c}) = \begin{cases} (-1)^{r-k} & \text{if } \vec{c} \in V \\ 0 & \text{otherwise.} \end{cases}$$

2. *The Möbius function* $\mu(\prod_n^\bullet(F) \cup \{\hat{0}\})$ *with a minimal element* $\hat{0}$ *adjoined is given by* $\mu(\prod_n^\bullet(F) \cup \{\hat{0}\}) = (-1)^{r-1} \sum_{\vec{c}\in V} \beta(\vec{c})$*, that is,* $(-1)^{r-1}$ *times the number of permutations in* S_n *whose descent composition belongs to the set* V*.*

So far we have discussed how to compute the Möbius number of filters in the partition lattice formed by restricting to partitions by type. That is, to each set partition of $[n]$ we have assigned as a type the integer partition consisting of the multiset of cardinalities of the blocks of a set partition, then selecting set partitions of given type. So, given a set I of integer partitions it is possible to find the Möbius number of the poset of set partitions whose type belongs to I. Then we have seen the notion of a knapsack partition and for a filter of a knapsack partition we showed that its Möbius number is a sum of descent set statistics.

In the remaining section of this chapter, we will study partial orders on set and integer partitions and related structures by building on results about partially ordered sets of partitions from enumerative, algebraic and geometric combinatorics involving Möbius numbers, homology of order complex and homotopy of order complexes. We will use pointed partition structures, where one part of an integer and set partition is marked. Pointed partition structures have been introduced in the works of Ehrenborg and Readdy [60], Ehrenborg and Jung [63] and Ziegler [29]. In this dissertation we show new enumerative and geometric properties of the partial orders of pointed partitions studied by Ehrenborg and Readdy. In particular, we will compute the

Möbius numbers and homotopy types of lower intervals topologically and show that the associated order complex is a wedge of spheres. In addition, we will investigate their ordered counterparts and provide analog results in this setting.

3.4 Pointed Integer and Set Partitions

Let n be a non-negative integer. A multiset $u = \{u_1, u_2, \cdots, u_r\}$ of integers is an integer partition of n provided that either $n = 0$ *and* $u = \{0\}$ or $n \geq 1$ and

$$(i)\ \sum_{i=1}^{r} u_i \ = n$$

$$(ii)\ u_i \geq\ 1,\ for\ all\ i = 1, 2, \ldots, r.$$

Hence a partition of n is a representation of n as a sum of positive integers where the order of the terms (or parts) is irrelevant. We use multiplicities and write $\{i_1^{u_1} \cdots i_e^{u_e}\}$ where $i_1 > \cdots > i_e$ *and* $u_1 \cdots u_e \geq 1$, as a superscript of each u_i in their decreasing order to give the multiset u.

Let $I_n^{\bullet}$ denote the set of all pointed integer partitions of the non-negative integer n. On this set we define an order relation by the two cover relations:

$$\{u_1,\ u_2,\ \ldots,\ u_i, \ldots, u_j, \ldots, u_r,\ \underline{m}\}\ \leq\ \{u_1,\ u_2,\ \ldots,\ \hat{u}_i, \ldots, \hat{u}_j,\ \ldots,\ u_r,\ u_i{+}u_j,\ \underline{m}\} \tag{3.12}$$

and

$$\{u_1,\ u_2,\ \ldots,\ u_i, \ldots, u_r,\ \underline{m}\}\ \leq\ \{u_1,\ u_2,\ \ldots,\ \hat{u}_i,\ \ldots,\ u_r,\ \underline{u_i +\ m}\}. \tag{3.13}$$

Where $\hat{u}_i$ *and* $\hat{u}_j$ means that the corresponding elements are omitted.

For instance consider the case $n = 3$. Thus $I_3^{\bullet} = \{111\underline{0},\ 12\underline{0},\ 11\underline{1},\ 2\underline{1},\ 1\underline{2},\ 3\underline{0},\ \underline{3}\}$. It can be easily shown that $(I_n^{\bullet},\ \leq)$ is a poset.

Proposition 3.4.1. *Let $I_n^\bullet$ denote the set of all pointed integer partitions of the non-negative integer n. Then*

(i) It is graded.

(ii) It has unique minimal and maximal elements $\hat{0}$ and $\hat{1}$ respectively.

(iii) It is not lattice for $n \geq 3$.

Proof. Since the intervals in $I_n^\bullet$ are graded, then $I_n^\bullet$ is by itself graded. As we have two order relations on $I_n^\bullet$ there is no unique maximal element for every pairs of elements for $n \geq 3$. □

The Möbius function μ is a function which assigns to each interval in a poset P an integer and its recursive formulation is given as:

$$\mu(x,y) = \begin{cases} 1 & \text{for all } x = y, \\ -\sum_{z: x \leq z < y} \mu(x,z) & \text{for all } x < y. \end{cases}$$

Example 3.4.2. Let us consider when $n = 3$, whose Hasse Diagram is shown in *Figure* 4 and $I_3^\bullet = \{111\underline{0}, 12\underline{0}, 11\underline{1}, 3\underline{0}, 2\underline{1}, 1\underline{2}, \underline{3}\}$. We compute the Möbius number for $I_3^\bullet$ recursively as follow.

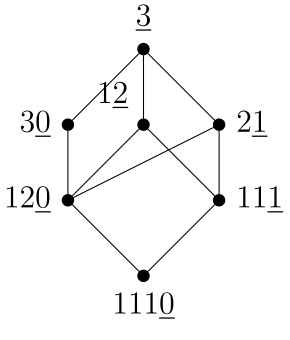


Figure 4: Hasse Diagram of $I_3^\bullet$

For $x < y$ in $I_3^\bullet$,

$$\mu(x,y) = -\sum_{z:x\leq z<y} \mu(x,z). \tag{3.14}$$

Thus we have that,

$$\mu(\hat{0},\hat{0}) = 1, \mu(\hat{0},12\underline{0}) = -1 = \mu(\hat{0},11\underline{1}), \tag{3.15}$$

$$\mu(\hat{0},1\underline{2}) = -(\mu(\hat{0},\hat{0}) + \mu(\hat{0},12\underline{0}) + \mu(\hat{0},11\underline{1})) = -(1+(-1)+(-1)) = 1 = \mu(\hat{0},2\underline{1}).$$

Similarly,

$$\mu(\hat{0},3\underline{0}) = -(\mu(\hat{0},\hat{0}) + \mu(\hat{0},12\underline{0})) = -(1+(-1)) = 0 \tag{3.16}$$

Therefore,

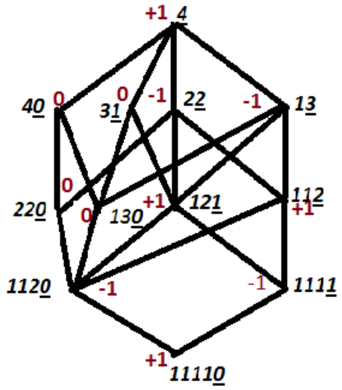
$$\begin{aligned} \mu(I_3^\bullet) &= \mu(\hat{0},\hat{1}) && (3.17)\\ \mu(0,\hat{1}) &= -(\mu(\hat{0},\hat{0}) + \mu(\hat{0},1\underline{2}) + \mu(\hat{0},2\underline{1}) + \mu(\hat{0},3\underline{0}) + \mu(\hat{0},12\underline{0}) + \mu(\hat{0},11\underline{1})) \\ &= -(1+1+1+0-1-1) = -1 = (-1)^3 && (3.18) \end{aligned}$$

Hence, $\mu(I_3^\bullet) = (-1)^3$.

Continuing in this manner we see that, for $n = 4$, whose Hasse Diagram is shown as below:

$I_4^\bullet = \{1111\underline{0}, 112\underline{0}, 13\underline{0}, 4\underline{0}, 22\underline{0}, 111\underline{1}, 12\underline{1}, 11\underline{2}, 3\underline{1}, 2\underline{2}, 1\underline{3}, \underline{4}\}$. We get

$$\mu(I_4^\bullet) = 1 = (-1)^4.$$



$\mu(I_4^\bullet) = \ 1 \ = (-1)^4.$

Thus, we have shown $\mu(I_3^\bullet) = -1 = (-1)^3$ and $\mu(I_4^\bullet) = \ 1 \ = (-1)^4$. We would like to generalize this result to $I_n^\bullet$, and show that $\mu(I_n^\bullet) = (-1)^n$.

Theorem 3.4.3. [***Samuel Asefa, 2014***]
For $n \geq 1$, $\mu(I_n^\bullet) = (-1)^n$ where $I_n^\bullet$ is the set of all pointed integer partitions of a non-negative integer n.

3.5 Proof of Theorem 3.4.3

We start with a simple observation.

Lemma 3.5.1. *For $R_n = \{1\cdots 12\underline{i} : 0 \leq i \leq n-1\} \cup \{1\cdots 1\underline{i} : 0 \leq i \leq n-1\} \subseteq I_n^\bullet$. Then, $\mu_{R_n}(\hat{0}, \hat{1}) = (-1)^n$.*

This follows from Proposition 2.3.1 since R_n is isomorphic to the ordinal sum of $n-1$ copies of $P = $ ***11***.

Proposition 3.5.2. *For $\lambda \in \overline{I_n^\bullet} = \{\mu \in I_n^\bullet : \mu \notin R_n\}$, then $\mu_{\overline{I_n^\bullet}}(\hat{0}, \lambda) = 0$.*

To prove Proposition 3.5.2 , first we observe the following:

Lemma 3.5.3. *If* $\lambda \in \overline{I_n^\bullet}$ *, then* $\{\eta \in R_n : \eta \leq \lambda\}$ *has a unique maximal element.*

For instance if we take $I_4^\bullet = \{1111\underline{0}, 112\underline{0}, 13\underline{0}, 4\underline{0}, 22\underline{0}, 111\underline{1}, 12\underline{1}, 11\underline{2}, 3\underline{1}, 2\underline{2}, 1\underline{3}, \underline{4}\}$.

It is easy to check that,

$$\begin{aligned} \bar{I}_4^\bullet &= \{22\underline{0}, 13\underline{0}, 4\underline{0}, 3\underline{1}\}. \\ R_4 &= \{1111\underline{0}, 112\underline{0}, 111\underline{1}, 12\underline{1}, 11\underline{2}, 2\underline{2}, 1\underline{3}, \underline{4}\}. \end{aligned}$$

Now, if we take $13\underline{0} \in I_4^\bullet$,

$$\{\eta \in R_4 : \eta \leq 13\underline{0}\} = \{1111\underline{0}, 112\underline{0}\} = \{\eta \in R_4 : \eta \leq 22\underline{0}\} = \{\eta \in R_4 : \eta \leq 4\underline{0}\}$$

For this set the unique maximal element is $112\underline{0} \in R_4$. Take $3\underline{1} \in \bar{I}_4^\bullet$ such that $\{\eta \in R_4 : \eta \leq 3\underline{1}\} = \{1111\underline{0}, 112\underline{0}, 111\underline{1}, 12\underline{1}\}$. The unique maximal element for this set is $12\underline{1}$.

Proof of Lemma 3.5.3:

Case 1: The pointed part of $\lambda \in \overline{I_n^\bullet}$ *is* 0 .

In this case the set $\{\eta \in R_n : \eta \leq \lambda\}$ contains only two elements, namely, $11\cdots 1\underline{0}$, and $11\cdots 12\underline{0}$ of which $\eta = 11\cdots 12\underline{0}$ is the maximal element.

Case 2: The pointed part of $\lambda \in \overline{I_n^\bullet}$ is non zero.

To determine the unique maximal element of $\{\eta \in R_n : \eta \leq \lambda\}$, note that

$$\begin{aligned} R_n &= \{1\cdots 12\underline{i} : 0 \leq i \leq n-1\} \cup \{1\cdots 1\underline{i} : 0 \leq i \leq n\} \\ &= \{1\cdots 1\underline{0}, \ldots, 1\underline{n-1}, \underline{n}, 1\cdots 12\underline{0}, \ldots, 2\underline{n-2}\} \end{aligned}$$

Let $\lambda = \lambda_1 \cdots \lambda_n \underline{m} \notin R_n$, where $1 \leq m \leq n-3$ *and* $n \geq 4$. Then there exists i such that

$$\lambda_i \geq 3 \text{ or there exists, } 1 \leq i < j \leq n \text{ and } \lambda_i, \ \lambda_j \geq 2. \tag{3.19}$$

If $11\cdots1\underline{i} < \lambda_1\ldots\lambda_n\underline{m}$ and $11\cdots2\underline{i} < \lambda_1\ldots\lambda_n\underline{m}$, then $i \leq m$.

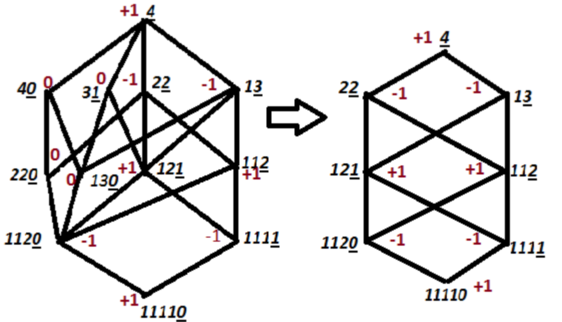
Now, $\{11\cdots1\underline{i}, 11\cdots12\underline{i} : 0 < i \leq m\} \subseteq R_n$ has $11\cdots2\underline{m}$ as its unique maximal element. Using equation (3.19) above, $11\cdots12\underline{m} < \lambda_1\cdots\lambda_n\underline{m}$, and this shows that $11\cdots12\underline{m}$ is the unique maximal element. Therefore, if $\lambda \in \bar{I}_n^\bullet$. Thus, it follows that $\{\eta \in R_n : \eta \leq \lambda\}$ has a unique maximal element. This ends the proof of the Lemma.

Now to go a head with the proof of Theorem 3.4.3, we use induction on the rank of λ to prove that $\mu_{\overline{I_n^\bullet}}(\hat{0}, \lambda) = 0$, which is exactly the proof of Proposition 3.5.2.

If λ is of minimal rank, then the set $\{\eta \in I_n^\bullet : \eta \leq \lambda\} = \{\eta \in R_n : \eta \leq \lambda\} \cup \{\lambda\}$, then $\mu_{\overline{I_n^\bullet}}(\hat{0}, \lambda) = 0$.

Assume λ is not of minimal rank. By induction hypothesis for all $\eta \in \overline{I_n^\bullet}$ and $\eta < \lambda$ we have $\mu_{I_n^\bullet}(\hat{0}, \eta) = 0$. Then, $\mu_{I_n^\bullet}(\hat{0}, \lambda) = \mu_{R_n \cup \{\lambda\}}(\hat{0}, \lambda)$, but $\{\eta \leq \lambda : \eta \in R_n\}$ has a unique maximal element η_{Top} (the pointed partition at the top).

Example 3.5.4. For example, you can consider $\mu(I_4^\bullet) = 1 = (-1)^4$ and $\mu(R_4^\bullet) = 1 = (-1)^4$ whose Hasse diagram is given below.



$\mu(I_4^\bullet) = 1 = (-1)^4.$ $\mu(\bar{I}_4^\bullet) = \mu(R_4) = 1 = (-1)^4$

We now use topological methods and provide a proof of Theorem 3.4.3. For the notation of Poset Topology and background concept we follow papers by Björner [3], Björner and Wachs [4], Jonsson [36], Jung [38] and Wachs [45]. In this section we determine $\widetilde{H}_i(\Delta(I_n^\bullet))$ and show that $\Delta(O_n^\bullet)$ is contractible, in particular $\mu_{O_n^\bullet}(\hat{0},\hat{1}) = 0$

Theorem 3.5.5. [***Samuel Asefa, 2014***]

Let $I_n^\bullet$ denote the set of all pointed integer partitions of a non-negative integer n. Then the order complex of $I_n^\bullet$, $(\Delta(I_n^\bullet))$, has the homotopy type of a sphere of dimension $n-1$.

Proof. Let $f : I_n^\bullet \longrightarrow I_n^\bullet$ be an order-preserving map such that $f(\lambda) = \max\{\tau \in R_n : \tau \leq \lambda\}$ and $R_n = \{1\cdots 12\underline{i} : 0 \leq i \leq n-1\} \cup \{1\cdots 1\underline{i} : 0 \leq i \leq n-1\} \ \subseteq I_n^\bullet$. By Proposition 3.5.3 $\{\tau \in R_n : \tau \leq \lambda\}$ has a unique maximal element, and hence f is well defined.

Thus, $f(\lambda) \leq \lambda$ for all $\lambda \in I_n^\bullet$ and $f(I_n^\bullet) = R_n$. Therefore, by Corollary 2.4.3 $I_n^\bullet \simeq R_n$. Now it follows that $\Delta(I_n^\bullet) \simeq \Delta(R_n)$. Using the fact that $\Delta(\bullet\bullet) * \Delta(\bullet\bullet) = S^0 * S^0 = S^1$ and proof of Lemma 3.5.1 $\Delta(I_n^\bullet)$ has homotopy type of a join of n $0-$spheres. Thus, $\Delta(I_n^\bullet) = S^{n-1}$. □

Corollary 3.5.6. *Let $I_n^\bullet$ denote the set of all pointed integer partitions of a non-negative integer n. The Möbius function of the poset $I_n^\bullet$ is given by $(-1)^n$.*

Proof. Let $\widetilde{\chi}(\Delta(I_n^\bullet))$ be the reduced Euler characteristic of the order complex of $I_n^\bullet$. Thus by Hall's theorem we have that $\mu_{I_n^\bullet}(\hat{0},\hat{1}) = \widetilde{\chi}(\Delta(I_n^\bullet))$. Recall that, given two simplical complexes Δ and Γ, we have that $\mid \Delta \mid \simeq \mid \Gamma \mid \Longrightarrow \widetilde{\chi}(\Delta) = \widetilde{\chi}(\Gamma)$. Hence, $\mu_{I_n^\bullet}(\hat{0},\hat{1}) = \widetilde{\chi}(\Delta(I_n^\bullet)) = (-1)^{(n-1)-1} = (-1)^n$. This completes proof of Theoerm 3.4.3. □

Corollary 3.5.7. *Let $I_n^\bullet$ denote the set of all pointed integer partitions of non-negative integer n. Then $\widetilde{H_i}(\Delta(I_n^\bullet)) = Z$ if $i = n-1$, and 0 otherwise.*

Proof. Recall that, given two simplical complexes Δ and Γ, if $|\ \Delta\ | \simeq |\ \Gamma\ |$ we have that $\widetilde{\chi}(\Delta) = \widetilde{\chi}(\Gamma)$ and then$\widetilde{H_i}(\Delta, Z) = \widetilde{H_i}(\Gamma, Z)$. Thus, $\widetilde{H_i}(S^{n-1}, Z) = Z, \quad if \quad i = n-1$, 0 otherwise. ☐

3.6 The Möbius Function of Pointed Set Partition

A pointed ordered set partition π of the set $[n]$ is a list of blocks $(B_1, \ldots, \underline{B_m})$ where the blocks are subsets of the set $[n]$ satisfying

(i) All blocks except possibly the last block are non-empty,

(ii) $B_i \cap B_j = \emptyset$ for all i and j with $1 \leq i < j \leq m$, and

(iii) $\cup_{i=1}^m B_i = [n]$

We can associate a pointed ordered set partition π to the pointed composition $c = (|B_1|, \ldots, |\underline{B_m}|)$, where $|B_i|$ for each i denotes the number of parts of the block i. Let $O_n^\bullet$ be the collection of the all pointed ordered set partition of $[n]$. Partially order the elements of $O_n^\bullet$ by the cover relations:

(i) $(B_1, \ldots, B_{j-1}, B_j, \ldots, B_{m-1}, \underline{B_m}) \leq (B_1, \ldots, B_{j-1}, B_j \cup B_{j+1}, B_{j+2}, \ldots, B_{m-1}, \underline{B_m})$,

and

(ii) $(B_1, \ldots, B_{m-2}, B_{m-1}, \underline{B_m}) \leq (B_1, \ldots, B_{m-2}, \underline{B_{m-1} \cup \underline{B_m}})$.

Clearly, $(O_n^\bullet, \leq)$ is a poset.

Theorem 3.6.1. [***Samuel Asefa, 2014***]

Let $O_n^\bullet$ *be the collection of all pointed ordered set partitions of* $[n]$*. Then* $\Delta(O_n^\bullet)$ *is contractible, and* $\mu_{O_n^\bullet}(\hat{0},\hat{1}) = 0$. .

Proof. Let $\overline{O_n^\bullet}$ be the pointed ordered partition of $[n]$ whose pointed part is $\emptyset$. $\overline{O_n^\bullet}$ is a subposet of $O_n^\bullet$. Then the simplicial complex $\Delta(\overline{O_n^\bullet})$ is a cone with apex $[n]/\emptyset$. Hence contractible.

Recall that for Δ_0 be a simplicial complex, and $'a'$ be a vertex not in Δ_0. The cone, $Cone(\Delta_0) = \ Cone_a(\Delta_0)$ over Δ_0 with apex $'a'$ is the simplicial complex obtained from Δ_0 by adding $\sigma \cup \{a\}$ for each $\sigma \in \Delta_0$. Equivalently, Δ is a cone with apex a if $\sigma \cup \{a\}$ is a face of Δ whenever σ is a face of Δ.

Let $f : O_n^\bullet \longrightarrow O_n^\bullet$ be the map defined by

$$f(\pi) = \max_{\pi \in O_n^\bullet}\{B_1 \mid \ldots B_r \mid \emptyset : B_1 \mid \ldots B_r \mid \emptyset \leq \pi\}$$

For any $\pi = \pi_1 \mid \ldots \mid \pi_r \mid \underline{\tau}$, pointed ordered set partition $\pi' = \pi_1 \mid \ldots \mid \pi_r \mid \tau \mid \underline{\emptyset}$ clearly satisfies $\pi' \leq \pi$. On the other hand if $\pi'' = \pi_1'' \mid \ldots \mid \pi_s'' \mid \underline{\emptyset} \leq \pi$, then by definition of the cover relation on $O_n^\bullet$ $\pi'' = \pi_1'' \mid \ldots \mid \pi_s''$ must be a refinement of $\pi = \pi_1 \mid \ldots \mid \pi_r$. Hence f is well defined.

If $\pi \leq \sigma$then $\max\{B : B \leq \pi\} \leq \max\{B' : B' \leq \sigma\}$ and then $f(\pi) \leq f(\sigma)$. Thus, f is a poset map.

By definition of f, we have that $f(\pi) \leq \pi$ for all $\pi \in O_n^\bullet$ and $f(O_n^\bullet) = \overline{O_n^\bullet}$. Therefore, by Corollary 2.4.3 $O_n^\bullet \simeq \overline{O_n^\bullet}$. Hence, $O_n^\bullet$ is contractible. From which we get that $\widetilde{H_n}(\Delta(O_n^\bullet)) = 0$ for all n. It follows that it is $Z-$acyclic. Thus, $\widetilde{\chi}(\Delta(O_n^\bullet)) = 0$. Therefore, by Philip Hall's theorem $\mu_{O_n^\bullet}(\hat{0},\hat{1}) = 0$. $\square$

Chapter 4

On the Möbius Function of Pointed Graded Lattices and Exponential Pointed Structures

4.1 Möbius Function for Pointed Graded Lattices

Let V_n be the vector space of n-tuples of elements of the field $\mathbb{F}_q$. We define the lattice of linear subspaces of V_n ordered by inclusion as follows.

Definition 4.1.1. Define L_n to be the lattice of linear non-empty subspaces of V_n partially ordered by inclusion. The meet $V \wedge W$ is given by the intersection $V \cap W$ and the join $V \vee W$ by the sum $V + W = span(V \cup W)$.

Definition 4.1.2. A **flag** (also called a **complete flag**) is a maximal chain in V_n.

For a chain to be maximal, it must contain $n+1$ subspaces, whose dimensions start at 0 and count up through n, that is,

$\{0\} = V_0 \subset V_1 \subset \cdots \subset V_n = V_n$, where $dim(V_i) = i$.

Since one can specify a flag by choosing subspaces $V_0, V_1, \ldots, V_n$ in a sequence, we determine the number of flags in V_n by counting the choices for each V_i, $i = 0, 1, 2, \ldots, n$.

V_0: No choice, must be $\{0\}$.

V_1: choosing a line through the origin. It suffices to choose any nonzero point $u \in V_n \backslash \{0\} = \mathbb{F}_q \backslash \{0\}$ and let V_1 be the subspace $\langle u \rangle$ it spans, and there are $q^n - 1$ ways to do this. But, since $\langle u \rangle = \langle \lambda u \rangle$ for any nonzero scalar λ of $\mathbb{F}_q \backslash \{0\}$, we are overcounting by a factor of $q-1$. So, there are $\frac{q^n-1}{q-1}$ choices.

V_2: Extending $V_1 = \langle u_1 \rangle$ by adding a new vector so that $V_2 = \langle u_1, u_2 \rangle$. Any $u_2 \in \mathbb{F}_q^n - \langle u_1 \rangle$ works, of which there are $q^n - q$. But since $\langle u_1, u_2 \rangle = \langle u_1, \lambda u_2 + w \rangle$ for any $\lambda \in \mathbb{F}_q \backslash \{0\}$ and $w \in V_1$, there are $\frac{q^n-q}{q(q-1)} = \frac{q^{n-1}-1}{q-1}$ choices.

V_k: In general, having chosen $V_{k-1} = \langle u_1, u_2, \ldots, u_{k-1} \rangle$, there are $q^n - q^{k-1}$ choices of $u_k \in \mathbb{F}_q^n - V_{k-1}$, and since $\langle u_1, u_2, \cdots, u_{k-1}, u_k \rangle = \langle u_1, u_2, \cdots, u_{k-1}, \lambda u_k + w \rangle$ for $\lambda \in \mathbb{F}_q - \{0\}$ and $w \in V_{k-1}$, there are $\frac{q^n - q^{k-1}}{(q-1)q^{k-1}}$ choices at this step.

Multiplying out the number of choices at each step, we obtain that that

$$\left(1+q+\cdots+q^{n-1}\right)\left(1+q+\cdots+q^{n-2}\right)\cdots\left(1+q+q^2\right)(1+q)(1),$$

which is a polynomial in q. We note that if we plug in $q=1$ this gives $n!$ which is the number of maximal chains of the Boolean algebra B_n. More precisely , if we consider the product

$$\left(1+q+\cdots+q^{n-1}\right)\left(1+q+\cdots+q^{n-2}\right)\cdots\left(1+q+q^2\right)(1+q)$$

and $n! = n \times (n-1) \times \cdots \times 2 \times 1$, then we can identify each number k with its q-analog $1+q+\cdots+q^{k-1}$, which we abbreviate as $[k]_q$.

The q-analogue of $\binom{n}{k}$, denoted by $\begin{bmatrix} n \\ k \end{bmatrix}_q$ is the number of k-dimensional subspaces of $\mathbb{F}_q^n = V_n$ (k-subpaces for short). One can easily show that $\begin{bmatrix} n \\ k \end{bmatrix}_q$ is related to $[n]!$ in the same way as $\binom{n}{k}$ to factorials [58].

A graded lattice L is a lattice with the minimal element $\widehat{0}$, maximal element $\widehat{1}$, and a rank function $r_k : L \longrightarrow \{0, 1, 2, \ldots, n\}$ satisfying $r_k(x) = 0$ if x is a minimal element of L, and $r_k(y) = r_k(x) + 1$ if y covers x in L. The rank of a graded lattice L is defined as the rank of the maximal element $\widehat{1}$, that is, $r_k(\widehat{1})$.

Let L be graded lattice with rank function r_k , minimal element $\widehat{0}$ and maximal element $\widehat{1}$. Consider the set $D(L^\bullet)$ with minimal element denoted by $\widehat{0}_{D(L^\bullet)}$, maximal element denoted by $\widehat{1}$ and $r_k(\widehat{1}) = r_k(A_1) + \cdots + r_k(A_1)$.

$$D(L^\bullet) = \Big\{ \; A_1 \mid A_2 \mid \cdots \mid A_r \mid \underline{A_{r+1}} \;\; : A_1 \vee \cdots \vee A_{r+1} = \widehat{1}, \quad \Big\} \cup \Big\{ \widehat{0}_{D(L^\bullet)} \Big\} .$$

With the order relation defined by:

1. $A_1 \mid \cdots \mid A_r \mid \underline{A_{r+1}} \leq A_1 \mid A_2 \mid \ldots \mid \hat{A_j} \mid \hat{A_{j+1}} \mid \cdots \mid A_r \mid A_j \vee A_{j+1} \mid \underline{A_{r+1}}$, and
2. $A_1 \mid A_2 \mid \cdots \mid A_r \mid \underline{A_{r+1}} \;\leq A_1 \mid A_2 \mid \cdots \mid A_{r-1} \mid \underline{A_r \vee A_{r+1}}$.

Where $\hat{A_i}$ and $\hat{A_j}$ in (1) above means that the corresponding elements are omitted and the bar $\mid$ as in partitions stands for the defined order relation with respect to a given ground set. Clearly, $(D(L^\bullet), \leq)$ is a graded lattice. However it is not a lattice in general. For instance, if we take a partition $L = \prod_n^r$ of all partitions on n elements where the cardinality of each block is divisible by r. Clearly this lattice does not have a minimal element if $r > 1$. However, it is join-semi lattice. Thus $\prod_n^r$ joined by an artificial new minimal element $\widehat{0}$, is a lattice. Hence, although $\prod_n^r$ lacks a minimal element, it is still called a lattice.

Example 4.1.1. $L = B_n = \{A \subseteq [n]\}, \quad r_k(A) \;\; = |A|$. Clearly, $\widehat{1} = [n], \widehat{0} = \emptyset$ and $D(B_n^\bullet) = \Big\{A_1 \mid \cdots \mid A_r \mid \underline{A_{r+1}} : \;\; A_1 \cup \cdots \cup A_{r+1} = [n] \textit{ and } n = |A_1| + \cdots + |A_{r+1}|\Big\} \cup \{\emptyset\}$. Then $D(B_n^\bullet)$ is pointed ordered graded lattice which can be identified with the pointed partition lattice $\prod_n^\bullet$.

Example 4.1.2. $L_n = \{u : \ u \text{ is a subspace of } \mathbb{F}_q^n, \ and \ r_k(u) = dim(u)\}$.

$$D(L_n^\bullet) = \left\{ \ A_1 \mid \cdots \mid A_r \mid \underline{A_{r+1}} \quad : A_1 \oplus \cdots \oplus A_{r+1} = \mathbb{F}_q^n, \ \right\} \cup \left\{ \widehat{0}_{D(L_n^\bullet)} \right\}.$$

Then $D(L_n^\bullet)$ is the poset of pointed vector space decompositions of $\mathbb{F}_q^n$.

Theorem 4.1.3. [***Samuel Asefa, 2014***]

Let L be a graded lattice. Then $D(L^\bullet)$ is contractible, that is, it is a simplicial complex which is homotypy equivalent to a point. In particular, $\widetilde{H_n}(\Delta(D(L^\bullet)) = 0$ for all n and $\mu(D(L^\bullet)) = 0$.

Proof. For $\tau \in D(L^\bullet) \setminus \widehat{0}_{D(L^\bullet)}$ we prove by induction on rank τ that $\Delta\left(\left[\widehat{0}_{D(L^\bullet)}, \tau\right]\right)$ is contractible.

If $\tau = A_1 \mid A_2 \mid \cdots \mid A_r \mid \underline{A_{r+1}}$ for $A_{r+1} \neq \widehat{0}$, then the minimal rank of such a τ occurs when $A_1, \ A_2, \ \cdots, \ A_r, \ A_{r+1}$ are of rank 1 in L. Then

$$\left[\widehat{0}_{D(L^\bullet)}, \tau\right] = \left\{\widehat{0}_{D(L^\bullet)} < A_1 \mid A_2 \mid \cdots \mid A_r \mid A_{r+1} \mid \underline{\emptyset} < \tau\right\},$$

and hence $\Delta\left(\left[\widehat{0}_{D(L^\bullet)}, \tau\right]\right)$ is contractible.

Now assume $\tau = A_1 \mid A_2 \mid \cdots \mid A_r \mid \underline{A_{r+1}}$ with $A_{r+1} \neq \widehat{0}$ is not minimal with this property. But for all $\tau' < \tau$ with $\tau' = A_1' \mid A_2' \mid \cdots \mid A_s' \mid \underline{A_{s+1}}'$ for $A_{s+1}' \neq \widehat{0}$ we have that $\Delta\left(\left[\widehat{0}_{D(L^\bullet)}, \tau'\right]\right)$ is contractible. Set

$$\overline{\left[\widehat{0}_{D(L^\bullet)}, \tau\right]} = \left[\widehat{0}_{D(L^\bullet)}, \tau\right] \setminus \left\{\tau' = A_1' \mid A_2' \mid \cdots \mid A_s' \mid \underline{A_{s+1}}' : A_{s+1}' \neq \widehat{0} \ and \ \tau' \neq \tau\right\}$$

Consider $f : \left[\widehat{0}_{D(L^\bullet)}, \tau\right] \longrightarrow \overline{\left[\widehat{0}_{D(L^\bullet)}, \tau\right]}$ which sends $A_1 \mid A_2 \mid \cdots \mid A_s \mid A_{s+1} \mid \underline{\emptyset}$ to $A_1 \mid A_2 \mid \cdots \mid A_s \mid A_{s+1} \mid \underline{\emptyset}$, $A_1 \mid A_2 \mid \cdots \mid A_s \mid \underline{A_{s+1}} \neq \tau$ with $A_{s+1} \neq \widehat{0}$ to $A_1 \mid A_2 \mid \cdots \mid A_s \mid A_{s+1} \mid \underline{\emptyset}$, τ to τ and $\widehat{0}_{D(L^\bullet)}$ to $\widehat{0}_{D(L^\bullet)}$. Then f is a poset map that sends the open interval $(\widehat{0}_{D(L^\bullet)}, \tau)$ to $\overline{(\widehat{0}_{D(L^\bullet)}, \tau)} := \overline{[\widehat{0}_{D(L^\bullet)}, \tau]} \setminus \left\{\widehat{0}_{D(L^\bullet)}, \tau\right\}$

For $\tau' = A'_1 \mid A'_2 \mid \cdots \mid A'_s \mid \underline{\emptyset} \in \overline{\left(\widehat{0}_{D(L^\bullet)}, \tau\right)}$ we have $f^{-1}\left(\overline{\left(\widehat{0}_{D(L^\bullet)}, \tau\right)}_{\geq \tau'}\right)$ has τ' as its unique minimal element and hence contractible.

Therefore, by Quillen's Fiber lemma (Theorem 2.4.4), we see that

$$\Delta\left(\left[\widehat{0}_{D(L^\bullet)}, \tau\right]\right) \cong \Delta\left(\overline{\left[\widehat{0}_{D(L^\bullet)}, \tau\right]}\right).$$

But, $f(\tau) = A_1 \mid A_2 \mid \cdots \mid A_r \mid A_{r+1} \mid \underline{\emptyset}$ is the unique maximal element of $\overline{\left[\widehat{0}_{D(L^\bullet)}, \tau\right]}$ and hence $\Delta\left(\overline{\left[\widehat{0}_{D(L^\bullet)}, \tau\right]}\right)$ is contractible.

In particular by Corollary 2.4.3 $\Delta\left(\left[\widehat{0}_{D(L^\bullet)}, \tau\right]\right)$ is contractible. From this we get that $\widetilde{H_n}(\Delta(D(L^\bullet)) = 0$ for all n. Thus, $\widetilde{\chi}(\Delta(D(L^\bullet)) = 0$. Consequently, by Philip Hall's theorem $\mu_{D(L^\bullet)}(\hat{0}_{D(L^\bullet)}, \hat{1}_{D(L^\bullet)}) = \mu(D(L^\bullet)) = 0$. □

As stated in [54] we can also order the set $D(V_n)$ of direct sum decomposition of V_n by refinement. This means that for two direct sum $V = V_1 \oplus \cdots \oplus V_k$ and; $U = U_1 \oplus \cdots \oplus U_l$ of V_n the inequality $V \leq U$ holds if and only if for all, $1 \leq j \leq k$, there exists an i, $1 \leq i \leq l$ such that $V_j \leq U_i$. Since in general there does not exist a direct sum decomposition which refines all the others, we always need to add a least element $\widehat{0}$ to $D(V_n)$ in order to make $D(V_n)$ into a bounded poset with top element $\widehat{1} = V_n$.

Let $V_n^\bullet$ be the set of all pointed direct sum decompositions of V_n.

Then $D(V_n^\bullet) = (V_n^\bullet, \leq)$ is a poset with the order relation defined as follows:

$$\begin{aligned} U_1 \oplus \cdots \oplus U_j \oplus U_{j+1} \oplus \cdots \oplus U_r \oplus \underline{U_{r+1}} \;&\leq\; U_1 \oplus \ldots \oplus \widehat{U}_j \oplus \widehat{U}_{j+1} \oplus \cdots \oplus U_r \\ &\oplus < U_j, U_{j+1} > \oplus \underline{U_{r+1}},\ \textit{and} \\ U_1 \oplus \cdots \oplus U_i \oplus \cdots \oplus U_r \oplus \underline{U_{r+1}} \;&\leq\; U_1 \oplus \cdots \oplus U_i \oplus \ldots \oplus U_{r-1} \oplus < \underline{U_r + U_{r+1}} > . \end{aligned} \tag{4.1}$$

Where $\widehat{U}_j$ and $\widehat{U}_{j+1}$ means that the corresponding elements are omitted and

$< U_j, U_{j+1} >$ denotes the vector space generated by U_j and U_{j+1}.

Corollary 4.1.4. *The Möbius number* $\mu(V_n^\bullet)$ *of the poset* $D(V_n^\bullet)$ *is zero.*

Proof. This follows from Theorem 4.1.3 since $V_n^\bullet$ is a graded lattice. □

4.2 Exponential Pointed Structures

Stanley introduced the notion of an exponential structure (for example see in [54]). In this section we generalize Stanleys notion of exponential structures to that of exponential pointed structures. We derive the compositional and exponential formula for exponential pointed structures analog to Stanleys theorem on the compositional and exponential formula for exponential structures [54].
First we recall the definition of exponential structures from [54].

Definition 4.2.1. An **exponential structure** $Q = (Q_1, Q_2, \ldots)$ is a sequence of posets such that

1. For every n, the poset Q_n is finite and has a unique maximal element $\widehat{1}_n$ (denoted simply by $\widehat{1}$), and every maximal chain in a graded poset Q_n contains n elements (or length $n-1$).

2. For an element π in Q_n of rank k, the interval $[\pi, \widehat{1}]$ is isomorphic to the partition lattice $\prod_k$ on k elements.

3. For $\pi \in Q_n$ and ρ a minimal element of Q_n satisfying $\rho \leq \pi$, if ρ' is another minimal element of Q_n satisfying $\rho' \leq \pi$, then $[\rho, \pi] \cong [\rho', \pi]$.

Remark 4.2.1. For $\pi \in Q_n$ if ρ a minimal element of Q_n satisfying $\rho \leq \pi$, then $[\rho, \widehat{1}] \cong \prod_n$. Hence, $[\rho, \pi] \cong \prod_1^{a_1} \times \cdots \times \prod_n^{a_n}$ and
$\{\sigma \in Q_n : \sigma \leq \pi\} \cong Q_1^{a_1} \times \cdots \times Q_n^{a_n}$, where $a_1 \cdots a_n \in N$.
If $Q = (Q_1,\ Q_2,\ \ldots)$ is an exponential structure, then let $M(n)$ denote the number of minimal elements of Q_n from which all the basic combinatorial properties of Q can be deduced. We call the sequence $M = (M(1),\ M(2),\ \cdots)$ the denominator sequence of Q.

Example 4.2.1. $Q_n = \prod_n$ is an example in which $M(n) = 1$

Example 4.2.2. Let r be a positive integer, and let S be an $n-$set.
An $r-$partition of S is a set

$$\pi = \{(B_{11}, B_{12}, \ldots, B_{1r}), (B_{21}, B_{22}, \ldots, B_{2r}), \ldots, (B_{k1}, B_{k2}, \ldots, B_{kr})\} \tag{4.2}$$

satisfying:

(i) For each $j \in [r]$, the set $\pi_j = \{B_{1j}, B_{2j}, \ldots, B_{kj}\}$ forms a partition of S(into k blocks), and

(ii) For fixed i, $\mid B_{i1} \mid = \mid B_{i2} \mid = \ldots = \mid B_{ir} \mid$.

The set $Q_n = Q_n(S)$ of all $r-$partitions of S has an obvious partial ordering by componentwise refinement which makes $Q_1, Q_2, \ldots$ into an exponential structure with $M(n) = n!^{r-1}$.

Definition 4.2.2. An **exponential pointed structure** $Q^\bullet = (Q_1^\bullet, Q_2^\bullet, \ldots)$ is a sequence of posets such that

1. For every n,the poset $Q_n^\bullet$ has a unique maximal element $\widehat{1}_n$ (denoted simply by $\widehat{1}$) and every maximal chain in a graded poset $Q_n^\bullet$ contains $n+1$ elements (or length n).

2. For an element $\pi^\bullet$ in $Q_n^\bullet$ of rank k, the interval $[\pi^\bullet, \widehat{1}]$ is isomorphic to the pointed partition lattice $\prod_{n-k}^\bullet$ on $n-k$ elements.

3. For $\pi^\bullet \in Q_n^\bullet$ and $\rho^\bullet$ is a minimal element of $Q_n^\bullet$ satisfying $\rho^\bullet \leq \pi^\bullet$, if $\rho_0^\bullet$ is another minimal element of $Q_n^\bullet$ satisfying $\rho_0^\bullet \leq \pi^\bullet$, then $[\rho^\bullet, \pi^\bullet] \cong [\rho_0^\bullet, \pi^\bullet]$.

Remark 4.2.2. For $\pi^\bullet \in Q_n^\bullet$ and $\rho^\bullet$ is a minimal element of $Q_n^\bullet$ satisfying $\rho^\bullet \leq \pi^\bullet$, then $[\rho^\bullet, \widehat{1}] \cong \prod_n^\bullet$. Hence $[\rho^\bullet, \pi^\bullet] \cong \prod_1^{a_1} \times \cdots \times \prod_n^{a_n} \times \prod_m^\bullet$ and $\{\sigma^\bullet \in Q_n^\bullet : \sigma^\bullet \leq \pi^\bullet\} \cong Q_1^{a_1} \times \cdots \times Q_n^{a_n} \times Q_m^\bullet$, where $a_1, \ \ldots, \ a_n \in N$.

Note that the poset $Q_n^\bullet$ has $M^\bullet(n)$ minimal elements, we call the sequence $(M^\bullet(1), M^\bullet(2), \ldots)$ **the denominator sequence** of $Q_n^\bullet$.

Example 4.2.3. Let $\prod_n^\bullet$ be the collection of all pointed set partition of $[n]$. $Q_n^\bullet = O_n^\bullet$ (the pointed ordered partition of $[n]$) is an example of an exponential pointed structure with $M^\bullet(n) = 1$.

Proposition 4.2.4. [***Samuel Asefa, 2014***]

Let $D(V_n^\bullet)$ be the poset of pointed direct sum decomposition.

1. *If $V = V_1 \oplus V_2 \oplus \cdots \oplus V_r \oplus \underline{V_{r+1}} \in D(V_n^\bullet) \setminus \{\widehat{0}\}$, then $(\widehat{0}, V] \cong D(V_1) \setminus \{\widehat{0}\} \times D(V_2) \setminus \{\widehat{0}\} \times \cdots D(V_{r+1}^\bullet) \setminus \{\widehat{0}\}$, where $D(V_i) \setminus \{\widehat{0}\}$ is the subposet determined by each V_i and $[V, \hat{0}|\underline{V_n}] \cong \prod_r^\bullet$.*

2. *Let $\rho, \pi \in D(V_n^\bullet)$, then $[\rho, \pi] \cong D(V_1)^{a_1} \times D(V_2)^{a_2} \times \cdots \times D(V_{r+1}^\bullet)$, where a_i is the number of the subposets determined by each V_i.*

3. *Let $\sigma \leq \tau \in \prod_n^\bullet$ such that $\sigma = \sigma_1 \cup \sigma_2 \cup \cdots \cup \sigma_r \cup \underline{\sigma_{r+1}}$, and $\tau = \tau_1 \cup \tau_2 \cup \cdots \cup \tau_k \cup \underline{\tau_{k+1}}$, then $[\sigma, \tau] \cong \prod_1^{a_1} \times \cdots \times \prod_n^{a_n} \times \prod_m^\bullet$, where $a_1, \ldots, a_n \in \{1, 2, \ldots\}$ and $m \in \{0, 1, 2, \ldots, n\}$.*

Proof. Let $V = V_1 \oplus V_2 \oplus \cdots \oplus V_r \oplus \underline{V_{r+1}}$. Consider the map $T : (\widehat{0}, V] \longrightarrow D(V_1) \setminus \{\widehat{0}\} \times D(V_2) \setminus \{\widehat{0}\} \times \cdots \times D(V^{\bullet}_{r+1}) \setminus \{\widehat{0}\}$ such that for $W \in (\widehat{0}, V]$; $W = W_1 \oplus W_2 \oplus \cdots \oplus W_s \oplus \underline{W_{s+1}}$ and after a possible re-indexing

$$T(W) = \left(\bigoplus_{j=1}^{l_1} W_j, \bigoplus_{j=l_1+1}^{l_2} W_j, \cdots, \bigoplus_{j=l_{r-1}+1}^{s} W_j, \bigoplus_{j=l_r+1}^{s+1} \underline{W_j} \right)$$

where

$$\begin{aligned} W_1 \oplus W_2 \oplus \cdots \oplus W_{l_1} &= V_1 \text{ and hence } W_1 \oplus W_2 \oplus \cdots \oplus W_{l_1} \in D(V_1), \qquad (4.3)\\ W_{l_1+1} \oplus W_{l_1+2} \oplus \cdots \oplus W_{l_2} &= V_2 \text{ and hence } W_{l_1+1} \oplus W_{l_1+2} \oplus \cdots \oplus W_{l_2} \in D(V_2)\\ \vdots &= \vdots\\ W_{l_{r-1}+1} \oplus W_{l_{r-1}+2} \oplus \cdots \oplus W_{l_r} &= V_r \text{ and hence } W_{l_{r-1}+1} \oplus W_{l_{r-1}+2} \oplus \cdots \oplus W_{l_r} \in D(V_r),\\ W_{l_r+1} \oplus W_{l_r+2} \oplus \cdots \oplus \underline{W_{s+1}} &= V_{r+1} \text{ and hence } W_{l_r+1} \oplus W_{l_r+2} \oplus \cdots \oplus \underline{W_{s+1}} \in D(V^{\bullet}_{r+1}). \end{aligned}$$

Such a choice exists since $W \leq V$, and T is a well defined injective map. Similarly, we see that T^{-1} is also a well defined injective map. So, any inclusion $W \leq V$ corresponds bijectively to a $k - tuple$ of elements of the direct sum decomposition posets $D(V_i) \setminus \{\widehat{0}\}$ of the subposet determined by each V_i.

Hence, the interval $[\widehat{0}, V] \setminus \{\widehat{0}\}$ in $D(V^{\bullet}_n)$ for $V = V_1 \oplus V_2 \oplus \cdots \oplus V_r \oplus \underline{V_{r+1}}$ is isomorphic to the direct product $D(V_1) \setminus \{\widehat{0}\} \times D(V_2) \setminus \{\widehat{0}\} \times \cdots \times D(V^{\bullet}_{r+1}) \setminus \{\widehat{0}\}$.

For $V = V_1 \oplus V_2 \oplus \cdots \oplus V_r \oplus \underline{V_{r+1}}$, to show $[V, 0|\underline{V_n}] \cong \prod^{\bullet}_r$,

we let $W_1 \oplus W_2 \oplus \cdots \oplus W_s \oplus \underline{W_{s+1}} \longmapsto I_1 \cup I_2 \cup \cdots \cup I_s \cup \underline{I_{s+1}}$ where, $I_1, \cdots, I_{s+1}$ are chosen such that

$$\bigoplus_{j \in I_i} V_j = W_i. \qquad (4.4)$$

If $V_1 \oplus V_2 \oplus \cdots \oplus V_r \oplus \underline{V_{r+1}} \leq W_1 \oplus W_2 \oplus \cdots \oplus W_s \oplus \underline{W_{s+1}}$, such a choice of $I_1, \cdots, I_{s+1}$ exists. Moreover, for every $1 \leq i, j \leq s+1$ we must have $I_i \cap I_j = \emptyset$, $\bigcup I_i = [r+1]$.

Hence, this map is well defined and it is an isomorphism between the interval $[V, 0|\underline{V_n}]$ and $\prod_r^\bullet$.

Proof of the claim in (2):

Since $\rho \leq \pi$ there exists sets $I_1, \cdots, I_{r+1} \subseteq [n+1]$, such that

$$\bigoplus_{j \in I_i} \rho_j = \pi_i, \text{where} \tag{4.5}$$

$$\begin{aligned} \rho &= \rho_1 \oplus \rho_2 \oplus \cdots \oplus \rho_n \oplus \underline{\rho_{n+1}} && (4.6) \\ \pi &= \pi_1 \oplus \pi_2 \oplus \cdots \oplus \pi_r \oplus \underline{\pi_{r+1}}. && (4.7) \end{aligned}$$

Now if we define a map

$\psi : [\rho, \pi] \longrightarrow D(V_1) \times D(V_2) \times \cdots D(V_{r+1}^\bullet)$ such that for

$$\begin{aligned} W &= W_1 \oplus W_2 \oplus \cdots \oplus W_s \oplus \underline{W_{s+1}} \in [\rho, \pi] && (4.8) \\ \psi(W) &= \left(\bigoplus_{j=1}^{l_1} W_j, \bigoplus_{j=l_1+1}^{l_2} W_j, \ldots, \bigoplus_{j=l_{r-1}+1}^{s} W_j, \bigoplus_{j=l_r+1}^{s+1} \underline{W_j} \right). && (4.9) \end{aligned}$$

Thus ψ is clearly a well defined poset map which is an isomorphism.

Statement (3) can be shown in the same way. □

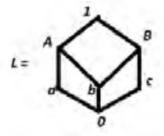
Theorem 4.2.5. [***Samuel Asefa, 2015***]

Let $V_n^\bullet$ be the set of all pointed direct sum decompositions of V_n and $D(V_n^\bullet)$ the poset of pointed direct sum decompositions of $V_n^\bullet$ with the order relation defined earlier. Then $D(V_n^\bullet) \setminus \{\hat{0}_{D(L^\bullet)}\}$ is a pointed exponential structure.

Proof. The proof follows from Proposition 4.2.4 above. □

The same proof as the one for Theorem 4.2.5 shows that if L is a geometric lattice then $D(L^\bullet) \setminus \{\hat{0}_{D(L^\bullet)}\}$ is a pointed exponential structure. But this is not true for all graded lattices L.

Example 4.2.6. For the lattice L give as below $D(L^{\bullet})\backslash\widehat{0}_{D(L^{\bullet})}$ has lower intervals which is not always isomorphic to $\prod_n^{\bullet}$



Next, we state and prove the analog of the compositional and exponential formula for Exponential Structures [54].

Proposition 4.2.7. [***Samuel Asefa, 2014***]

Let $Q^{\bullet} = (Q_1^{\bullet}, Q_2^{\bullet}, \ldots)$ be a an exponential pointed structure with associated exponential structure $Q = (Q_1, Q_2, \ldots)$. The number of elements in $Q_n^{\bullet}$ of type $(a_1, \ldots, a_n, m)$ is given by

$$\frac{M^{\bullet}(n)n!}{1!^{a_1}\cdots n!^{a_n}a_1!\cdots a_n!\boldsymbol{m!}M(1)^{a_1}\cdots M(n)^{a_n}\boldsymbol{M^{\bullet}}(m)},$$

where $M^{\bullet}(m)$ denotes the number of minimal elements of $Q_m^{\bullet}$.

Proof. Let $N =\mid \{(\rho, \pi) : \rho \in Q_n^{\bullet}, \rho \le \pi \text{ and type of } \pi = (1!^{a_1}, \ldots, n!^{a_n}, m)\} \mid$

$$\begin{aligned} &= \begin{pmatrix} n \\ m \end{pmatrix} (\text{the number of set partition of n-m elements of type } (1!^{a_1}, \ldots, n!^{a_n})) \\ &= \frac{M^{\bullet}(n).n!}{1!^{a_1}\cdots n!^{a_n}a_1!\cdots a_n!m!}. \end{aligned} \tag{4.10}$$

i.e.; we pick ρ in $M^{\bullet}(n)$ ways and $\pi \in \Pi_n^{\bullet}$ of type $(a_1, \ \cdots, \ a_n, \ m)$ in

$$\begin{pmatrix} n \\ m \end{pmatrix}\begin{pmatrix} n-m \\ 1, 1, \cdots, 1, \cdots, n, n, \cdots, n \end{pmatrix}\frac{1}{a_1!\cdots a_n!} \text{ ways.} \tag{4.11}$$

On the other hand, if K is the desired number of $\pi \in Q_n^{\bullet}$ of type $(a_1, \ \ldots, \ a_n, \ m)$, then we pick π in K ways and then choose $\rho \le \pi$. Since $Q_n^{\bullet}$ has $M^{\bullet}(n)$ minimal elements, the subposet $\Lambda_\pi^\bullet \cong Q_1^{a_1} \times \cdots \times Q_n^{a_n} \times Q_m^\bullet$ has $M(1)^{a_1} \cdots M(n)^{a_n} M^\bullet(m)$ minimal elements. Hence there are $M(1)^{a_1} \cdots M(n)^{a_n} M^\bullet(m)$ choices for ρ.

Thus, $N = K \cdot M(1)^{a_1} \cdot \ldots \cdot M(n)^{a_n} \cdot M^\bullet(m)$. From which the proposition follows. $\square$

Theorem 4.2.8. [***Samuel Asefa, 2014***] *[The Compositional Formula for Exponential Pointed Structures] Let $Q^\bullet = (Q_1^\bullet, Q_2^\bullet, \ldots)$ be an exponential pointed structure with denominator sequence $(M^\bullet(1), M^\bullet(2), \ldots)$ and associated exponential structure $Q = (Q_1, Q_2, \ldots)$ with denominator sequence $(M_1, M_2, \ldots)$. Given functions $f : \mathbb{P} \longrightarrow \mathbf{K}$, $g : \mathbf{N} \longrightarrow \mathbf{K}$ with $g(0) = 1$, define a new function $h^\bullet : \mathbf{N} \longrightarrow \mathbf{K}$ by*

$$h^\bullet(n) = \sum_{\pi^\bullet = \pi|\underline{B} \in Q_n^\bullet} f(1)^{a_1} \cdots f(n)^{a_n} \cdot f(m) \cdot g(|\pi|) \tag{4.12}$$

$$\textit{where type } \pi^\bullet = (a_1, \ldots, a_n, m), |\pi| = \sum_{i=1}^{n} i a_i, \textit{ and } m = |B|. \tag{4.13}$$

Define formal power series $F, G, H^\bullet \in K[[x]]$ by

$$F(x) = \sum_{n \geq 1} f(n) \frac{x^n}{n! M(n)},$$

$$G(x) = \sum_{n \geq 1} g(n) \frac{x^n}{n!}, \textit{ and}$$

$$H^\bullet(x) = \sum_{n \geq 1} h^\bullet(n) \frac{x^n}{n! M^\bullet(n)}$$

Then $H^\bullet(x) = G(F(x))(1 + F(x))$.

Proof. $\left[\frac{x^n}{n!}\right] G(F(x))(1 + F(x))$

$$= \left[\frac{x^n}{n!}\right] G(F(x)) + \left[\frac{x^n}{n!}\right] G(F(x)) F(x) \tag{4.14}$$

$$= \sum_{\pi|\underline{\emptyset} \in \prod_n^\bullet} \left(\frac{f(1)}{M(1)}\right)^{a_1} \cdots \left(\frac{f(n)}{M(n)}\right)^{a_n} g(|\pi|) + \left[\frac{x^n}{n!}\right] G(F(x)) F(x). \tag{4.15}$$

(Note that for a power series $F(x) = \sum f(n)x^n$ we set $[cx^n]F(x)$ as $\frac{f(n)}{c}$ for $c \neq 0$.) Applying the compositional formula and product formula on $G(F(x))F(x)$ we see that

$$\left[\tfrac{x^n}{n!}\right] G(F(x))F(x)$$

$$= \sum_{\pi|\underline{|B|}\in\prod^\bullet_n, |B|\geq 1} \left(\frac{f(1)}{M(1)1!}\right)^{a_1} \cdots \left(\frac{f(n-|B|)}{M(n-|B|)(n-|B|)!}\right)^{a_{n-|B|}} g(|\pi|)\frac{f(|B|)}{M^\bullet(|B|)}.$$

$$\left[\tfrac{x^n}{n!}\right] G(F(x))$$

$$= \sum_{\pi\in\prod^\bullet_n} \left(\frac{f(1)}{M(1)}\right)^{a_1} \cdots \left(\frac{f(n)}{M(n)}\right)^{a_n} g(|\pi|) \tag{4.16}$$

$$= \sum_{\pi|\underline{\emptyset}\in\prod^\bullet_n} \left(\frac{f_1}{M(1)1!}\right)^{a_1} \cdots \left(\frac{f(n)}{M(n)n!}\right)^{a_n} g(|\pi|)\frac{f(0)}{M^\bullet(0)}. \tag{4.17}$$

$$\left[\tfrac{x^n}{n!}\right] G(F(x))(1+F(x))$$

$$= \sum_{\pi|\underline{\emptyset}\in\prod^\bullet_n} \left(\frac{f(1)}{M(1)}\right)^{a_1} \cdots \left(\frac{f(n)}{M(n)}\right)^{a_n} g(|\pi|)\frac{f(0)}{M^\bullet(0)}$$

$$+ \sum_{\pi|\underline{|B|}\in\prod^\bullet_n, |B|\geq 1} \left(\frac{f(1)}{M(1)1!}\right)^{a_1} \cdots \left(\frac{f(n)}{M(n)(n)!}\right)^{a_n} g(|\pi|)\frac{f(|B|)}{M^\bullet(|B|)}$$

$$= \sum_{\pi|\underline{|B|}\in\prod^\bullet_n} \left(\frac{f(1)}{M(1)1!}\right)^{a_1} \cdots \left(\frac{f(n)}{M(n)(n-|B|)!}\right)^{a_n} g(|\pi|)\frac{f(|B|)}{M^\bullet(|B|)}.$$

$$Thus, h^\bullet(n) = \left[\frac{x^n}{n!M^\bullet(n)}\right] G(F(x))(1+F(x)) \tag{4.18}$$

$$= \sum_{\pi|\underline{B}\in Q^\bullet_n} f(1)^{a_1}\cdots f(n)^{a_n} g(|\pi|) f(|B|). \tag{4.19}$$

□

Corollary 4.2.9 (The Pointed Exponential Formula). *Let* $Q^\bullet = (Q^\bullet_1, Q^\bullet_2, \ldots)$ *be an exponential pointed structure with denominator sequence* $(M^\bullet(1), M^\bullet(2), \ldots)$ *and associated exponential structure* $Q = (Q_1, Q_2, \ldots)$ *with denominator sequence* $(M_1, M_2, \ldots)$.

Given function $f : \mathbb{P} \longrightarrow \mathbf{K}$*, define a new function* $h^\bullet : \mathbf{N} \longrightarrow \mathbf{K}$ *by*

$$h^\bullet(n) \quad = \sum_{\pi^\bullet = \pi|\underline{B} \in Q_n^\bullet} f(1)^{a_1} \cdots f(n)^{a_n} g(|\pi|) f(|B|) \textit{ where type}\pi^\bullet = (a_1, \ldots, a_n, m).$$

Define

$$F(x) \quad = \quad \sum_{n \geq 1} f(n) \frac{x^n}{n! M(n),} \tag{4.20}$$

$$H^\bullet(x) \quad = \quad \sum_{n \geq 1} h^\bullet(n) \frac{x^n}{n! M^\bullet(n)}. \tag{4.21}$$

$$\textit{Then} \quad H^\bullet(x) \quad = \quad exp(F(x)) f(|B|). \tag{4.22}$$

Example 4.2.10. Let $Q^\bullet = (Q_1^\bullet, Q_2^\bullet, \ldots)$ be an exponential pointed structure with denominator sequence $(M^\bullet(1), M^\bullet(2), \ldots)$ and associated exponential structure $Q = (Q_1, Q_2, \ldots)$ with denominator sequence $(M_1, M_2, \ldots)$. Let $V_n(t)$ be the polynomial $V_n(t) = \sum_{x \in Q_n} t^{rk(x, \hat{1})}$. In Example 5.5.6. in [54] Stanley obtains the generating function

$$\sum_{n \geq 0} V_n(t) \frac{x^n}{n! M(n)} = exp\left(t \sum_{n \geq 1} \frac{x^n}{n! M(n)}\right) = \left(exp\left(\sum_{n \geq 1} \frac{x^n}{n! M(n)}\right)\right)^t,$$

by setting $f(n) = 1$ and $g(n) = t^n$ in Theorem 5.5.4. in [54]. Similarly, defining $U_n(t)$ by

$$U_n(t) = \sum_{x \in Q_n^\bullet} t^{rk(x, \hat{1})},$$

we obtain

$$\sum_{n \geq 0} U_n(t) \frac{x^n}{n! M^\bullet(n)} \quad = \quad \left(\sum_{n \geq 0} \frac{x^n}{n! M^\bullet(n)}\right) exp\left(t \sum_{n \geq 1} \frac{x^n}{n! M(n)}\right) \tag{4.23}$$

$$= \quad \left(\sum_{n \geq 0} \frac{x^n}{n! M^\bullet(n)}\right) \cdot \left(exp\left(\sum_{n \geq 1} \frac{x^n}{n! M(n)}\right)\right)^t, \tag{4.24}$$

by setting $f(n) = t$ and $g(n) = t^n$ in Theorem 4.2.8.

4.3 Summary and Open Problems

In this thesis we have computed the Möbius function of pointed integer partition and pointed set partition using topological and analytic methods, and we also have shown that the associated order complex is homotopy equvalent to a wedge of spheres. From the relation,

$$\mu(P) = \sum_{n} (-1)^n \; rank \; \widetilde{H_n}(\Delta(P)) \tag{4.25}$$

where $\widetilde{H_n}(\Delta(P))$ represents reduced simplicial homology with integer or field coefficients, we have computed the associated reduced homology group for each subposet. In addition we have computed the Möbius function of pointed graded lattice and used this general method to compute the Möbius function of pointed direct sum decomposition of vector space.

However, we have not dealt with the question whether these posets are EL-shellable (Edge Lexicographically shallable). Let P be a bounded poset, with edge set $E(P)$. The edge labeling is an EL-shellable, if for every interval $[x, y]$ in P

- There is a unique increasing maximal chain C in $[x, y]$, and
- $C \leq_L C'$ for all other maximal chains C' in $[x, y]$

Let $R_n = \{1 \cdots 12\underline{i} : 0 \leq i \leq n-1\} \cup \{1 \cdots 1\underline{i} : 0 \leq i \leq n-1\} \subseteq I_n^{\bullet}$ 3.5.1. One can conjuncture that the Hasse diagram of R_n admit an EL-labeling which is EL-shellable. We have tried to verify for $3 \leq n \leq 8$ by labeling the right side edge set numbers from 1 through n starting from the most bottom edge, to the left side edge set assign numbers 1 to n starting from the most top edge, and label the remaining right and left inclined edges following the steps below:

1. assign numbers from $n-1$ to 2 to those left inclined edge set.
2. to those edge set inclined to the right start labeling by assigning the number $\frac{n}{2}$ (if $n = 4, 6, 8$) or $\frac{n+1}{2}$ (if $n = 3, 5, 7$)to the top most right inclined edge set and continue it till you assign 1 to an edge, after that you will remain either one edge or two edge un assigned based on whether n is even or odd for which you can assign any number less than or equal to $\frac{n}{2}$(or $\frac{n+1}{2}$).

Hence, this labeling gives us an EL-labeling which is an EL-shellable.

Moreover, we have extended Stanley's notion of exponential structures to that of exponential pointed structures and derived similar compositional and exponential formulas for pointed structures. However, from this theory of exponential pointed structures, one can go back and give the generating function to the Möbius function of exponential pointed structures as an application for the theory.

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