GRADUATE SEMINAR REPORT
ON

BOUNDED SYMMETRIC, UNITARY, AND NORMAL TRANSFORMATIONS OF HILBERT SPACE

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CONTENTS

PREFACE

Bounded symmetric transformations are applicable in solving differential and integral equations. This report is compiled from different reading materials written on bounded symmetric, unitary and Normal transformations which are listed in the references.

This report is a compilation of the two semminars I have delivered, in partial fulfilment, for the M.Sc. degree in Mathematics. The centeral theme of the paper is proving the spectral decomposition Theorem for bounded symmetric, unitary and Normal transformations of Hilbert Spaces. It has three parts. The first part deals on a review of definitions and properties of Hilbert Spaces, linear transformations and the stielties integral which are vital through out the paper.

In the second part, proofs of some elementary properties, the general schwarz inequality, the convergent of bounded sequences of symmetric transformations, the uniqueness of positive square root of a positive symmetric transformation and the inverse of a symmetric transformation are discussed.

In the third part, followed by a lemma which asserts that every trigonometric polynomial can be represented by the square of the absolute value of another trigonometric polynomial, the spectral decomposition Theorem of a unitary transformation is provid. Also the theorem on the spectral decomposition of a normal transformation is proved.

Finally, I would like to express my deepest gratitude to my advisor Dr. Seid Mohammed for his unreserved guidance and valuable suggestions he offered me through out the preparation and presentation of the paper. Further more, my gratitude also goes to W/t Bizuayehu Adafre for her assistance in typing and printing the paper.

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1. PRELIMINARIES:

Definition 1: (a) A complex vector space H is called an **inner product space**, if there is a function $<.,.>: H\times H \longrightarrow \mathbb{C}$ such that

(i) $<x+y,z> = <x,z> + <y,z>$

(ii) $<x,y> = \overline{<y,x>}$ (The bar denoting complex conjugate)

(iii) $<\alpha x,y> = \alpha<x,y>$, $\alpha\in\mathbb{C}$

(iv) $<x,x> \geq 0$ and $<x,x> = 0$ if and only if $x = 0$, for $x,y,z\in H$.

Property: i) $<x,y+z> = <x,y> + <x,z>$

ii) $<x,\alpha y> = \bar{\alpha}<x,y>$

iii) $<x,x>\in\mathbb{R}$, for $x,y,z\in H$

Every inner product space can be normed by defining

$$\|x\| = <x,x>^{1/2} = \sqrt{<x,x>}$$

b) A normed space H is called **complete** if and only if every cauchy-sequence in H has a limit in H. Where a sequence (x_n) in H is said to be a **cauchy-sequence**, if given $\epsilon > 0$, there is a positive integer $N(\epsilon)$ such that

$$\|x_n - x_m\| < \epsilon \text{ , for all } n,m \geq N.$$

c) A complete inner product space H is called a **Hibert space**.

Example: 1. The space $H = l^2(n)$, with $<x,y> = \sum_{i=1}^{n} x_i\bar{y}_i$

and $\|x\| = \left[\sum_{i=1}^{n} |x_i|^2\right]^{1/2}$, where $x = (x_1,x_2,\ldots,x_n)$

$y = (y_1,y_2,\ldots,y_n)$ in $l^2(n)$

2. The space $H = L^2[a,b]$

with $<x,y> = \int_a^b x(t)\,\overline{y(t)}\,dt$ and $\|x\| = \left[\int_a^b |x(t)|^2 dt\right]^{1/2}$,

$x,y\in H$ are Hilbert spaces.

Definition 2: Let H be a Hilbert space. A transformation $T: H\longrightarrow H$ is called

a) **Linear** if it is i) Additive: $T(h_1+h_2) = Th_1 + Th_2$

ii) Homogeneous: $T(\alpha h) = \alpha Th$, $\alpha \in \mathbb{C}$, $h, h_1, h_2 \in H$.

b) **Bounded:** If $\exists M > 0$ such that $\|Th\| \le M\|h\|$. The smallest M is called the norm of T dented by $\|T\|$ and defined by

$$\|T\| = \sup_{h \neq 0} \frac{\|Th\|}{\|h\|}$$

c) **Continuous:** if the sequence (h_n) in H converging to h implies Th_n converges to Th.

Convention: When we say a linear transformation, it is bounded.

Remark: A linear transformation is continuous and conversely a transformation which is additive, homogeneous and continuous is bounded.

Definition 3: A function f is **upper semi-continuous** at a point $x_o \in D$ if and only if $\overline{\lim_{x \to x_o}} f(x) \le f(x_o)$.

. A function f is lower semi-continuous at a point $x_o \in D$ if and only if $\overline{\lim_{x \to x_o}} f(x) \ge f(x_o)$.

. f is **semi-continuous** at a point if it is either upper semi-continuous or lower semi-continuous.

Definition 4: Let (A_n) be a sequence of transformations. Then

. (A_n) is said to be monotone increasing (monotone decreasing) if and only if $A_1 \le A_2 \le \ldots \le A_k \le \ldots$ $(A_1 \ge A_2 \ge \ldots \ge A_k \ge \ldots)$.

. (A_n) is said to be bounded below (above) if and only if there is a transformation B such that $B \le A_n$ $(A_n \le B)$ for all $n \in \mathbb{N}$.

. A sequence is said to be bounded if and only if it is bounded below and bounded above.

The stielties Integral:

Given, on $a \le x \le b$, a continuous function $f(x)$ and a non decreasing function $\alpha(x)$, we define the stieltjes integral of f with respect to α, denoted by

$$\int_a^b f d\alpha = \int_a^b f(x)\ d\alpha(x),$$

to be the limit of the sums

$$\Sigma = \sum_{k=1}^{n} f(\xi_k)[\alpha(x_k) - \alpha(x_{k-1})]$$

where $a = x_o < x_1 < \ldots < x_n = b$; $x_{k-1} \le \xi_k \le x_k$

and when $\max\ (x_k, x_{k-1}) \longrightarrow 0$.

2. BONDED SUMMETRIC TRANSFORMATIONS

2.1. Symmetric Transformations

Definition 5: Let $T: H \longrightarrow H$ be linear, where H is a Hibert space. The transformation T^* such that $\langle Tf, g\rangle = \langle f, T^* g\rangle$, $f, g \in H$ is called the **adjoint transformation** of T.

It can be shown that T^* is linear, continuous and $\|T^*\| = \|T\|$

To show, $\|T^*\| = \|T\|$

$$|\langle f, T^* g\rangle| = |\langle Tf, g\rangle| \le \|T\|.\|f\|.\|g\|$$

implies, $\|T^* g\| \le \|T\|.\|g\|$, hence T^* is bounded

implies, $\|T^*\| \le \|T\| \qquad (*)$

Since we can find for every element f of H an element g of H such that $g = Tf$ and $\langle Tf, g\rangle = \|Tf\|$ and $\|g\| = 1$, we have

$$\|Tf\| = \langle Tf, g\rangle = \langle f, T^* g\rangle \le \|f\|.\|T^*\|.\|g\| = \|f\|.\|T^*\|$$

implies $\|T\| \le \|T^*\| \qquad (**)$

From (*) and (**), we get $\|T\| = \|T^*\|$

Definition 6: The linear transformation $T: H \longrightarrow H$ for which $\langle Tf, g\rangle = \langle f, Tg\rangle$ $f, g \in H$ is called a **symmetric Transformation.**

Proposition 1: A linear transformation $A: H \longrightarrow H$ is symmetric if and only if $\langle Af, f\rangle \in \mathbb{R}$.

Proof: Let A be symmetric, then for $f \in H$ we have

$$\langle Af, f\rangle = \langle f, Af\rangle = \overline{\langle Af, f\rangle}$$

hence, $\langle Af, f\rangle \in \mathbb{R}$

To show the converse, we start with the identify valid for every linear transformation A

$$\langle A(f+g), f+g\rangle - \langle A(f-g). f-g\rangle + i\langle A(f+ig\rangle, f+ig)\rangle - i\langle A(f-ig), f-ig\rangle = 4\langle Af, g\rangle \quad (1)$$

$$\langle f+g, A(f+g)\rangle - \langle f-g, A(f-g)\rangle + i\langle f+ig, A(f+ig)\rangle - i\langle f-ig, A(f-ig)\rangle = 4\langle f, Ag\rangle.$$

(1) which follows from (1) by interchanging f and g and taking the complex conjugates. But, if the quadratic from $\langle Af, f\rangle$ is always real-valued, we have $\langle f, Af\rangle = \overline{\langle Af, f\rangle} = \langle Af, f\rangle$ and thus the first members of (1) and (1′) are equal and hence $\langle Af, g\rangle = \langle f, Ag\rangle$ which shows that A is symmetric.

The proposition just proved, as well as relations (1) and (1′) holds only for complex Hilbert spaces and not for real Hilbert spaces.

Theorem 1: Let A be a symmetric transformaton. Then

$$\|A\| = \sup_{\|f\| = 1} |\langle Af, f\rangle|$$

Proof: We consider the quadratic form $\langle Af, f\rangle$ and $\|f\| = 1$.

$$|\langle Af, f\rangle \le \|Af\| \|f\| \le \|A\| \|f\|^2 = \|A\|$$

$$\text{then, } N_A = \sup_{\|f\| = 1} |\langle Af, f\rangle| \le \|A\| \qquad (*)$$

Again we have to show $N_A \ge \|A\|$. Let $\lambda > 0$, $\lambda \in \mathbb{R}$. Then

$$\|Af\|^2 = \frac{1}{4}\left[\langle A(\lambda f + \tfrac{1}{\lambda}Af),\ \lambda f + \tfrac{1}{\lambda}Af\rangle - \langle A(\lambda f - \tfrac{1}{\lambda}Af),\ \lambda f - \tfrac{1}{\lambda}Af\rangle\right]$$

$$= \frac{1}{4}\left[\langle A\frac{\lambda f + \frac{1}{\lambda}Af}{\|\lambda f + \frac{1}{\lambda}Af\|}, \frac{\lambda f + \frac{1}{\lambda}Af}{\|\lambda f + \frac{1}{\lambda}Af\|}\rangle \|\lambda f + \tfrac{1}{\lambda}Af\|^2 - \langle A\frac{\lambda f - \frac{1}{\lambda}Af}{\|\lambda f - \frac{1}{\lambda}Af\|}, \frac{\lambda f - \frac{1}{\lambda}Af}{\|\lambda f - \frac{1}{\lambda}Af\|}\rangle \|\lambda f - \tfrac{1}{\lambda}Af\|^2\right.$$

$$\le \frac{1}{4}\left[N_A \|\lambda f + \tfrac{1}{\lambda}Af\|^2 - N_A \|\lambda f - \tfrac{1}{\lambda}Af\|^2\right]$$

$$= \frac{1}{4}\left[N_A\left(\|\lambda f+\frac{1}{\lambda}Af\|^2 - \|\lambda f-\frac{1}{\lambda}Af\|^2\right)\right]$$

$$= \frac{1}{4}N_A\left[<\lambda f+\frac{1}{\lambda}Af,\ \lambda f+\frac{1}{\lambda}Af> - <\lambda f - \frac{1}{\lambda}Af,\ \lambda f - \frac{1}{\lambda}Af>\right]$$

$$= \frac{1}{4}N_A\left[\lambda^2<f,f>+<f,Af>+<Af,f>+\frac{1}{\lambda^2}<Af,Af>+\lambda^2<f,f>-<f,Af>-<Af,f>\right.$$

$$\left.+\frac{1}{\lambda^2}<Af,Af>\right]$$

$$= \frac{1}{4}\ N_A\left[2\lambda^2\|f\|^2 + \frac{2}{\lambda^2}\ \|Af\|^2\right]$$

$$= \frac{1}{2}\ N_A\ \left[\lambda^2\|f\|^2 + \frac{1}{\lambda^2}\ \|Af\|^2\right]$$

$$= \frac{1}{2}\ N_A\ \left[\lambda^2 + \frac{\|Af\|^2}{\lambda^2}\right]$$

which implies

$$\|Af\|^2 \le \frac{1}{2}\ N_A\ \left[\lambda^2 + \frac{1}{\lambda^2}\ \|Af\|^2\right]$$

Case 1: If $\|Af\| \ne 0$. Then for $\lambda^2 = \|Af\|$, we have

$$\|Af\|^2 \le \frac{1}{2}\ N_A\ \left[\|Af\| + \|Af\|\right] = N_A\ \|Af\|$$

hence, $\|Af\| \le N_A$.

Case 2: If $\|Af\| = 0$, then $0 = \|Af\|^2 \le \frac{1}{2}\ N_A.\lambda^2$, which imphis $\|Af\| \le N_A$.

Hence $\|A\| = \sup\limits_{\|f\| = 1} \|Af\| \le N_A$ (**)

From (*) and (**) we get,

$$\|A\| = \sup_{\|f\| = 1} |<Af,f>|$$

Definition 7: The linear transformations A and B are said to be **permutable** if AB = BA

We define an order relation among the symmetric transformation by:

$A \ge B$ when $\langle Af,f\rangle \ge \langle Bf,f\rangle$

(reflexive, antisymmetric and transitive)

Property: Let $A, B, C, A_n, n\in\mathbb{N}$ be symmetric. Then

1. $A + B$ is symmetric
2. If $AB = BA$, then AB is symmetric
3. $A \ge B$ implies $A + C \ge B + C$ and $cA \ge cB$, $c\in\mathbb{R}$ and $c > 0$
4. $\langle Af,f\rangle = \langle Bf,f\rangle$ implies $A = B$
5. $Af = \lim\limits_{n\to\infty} A_n f$ implies A is symmetric

Linear transformation A such that $\langle Af,f\rangle \ge 0$ is said to be **positive** and we write $A \ge 0$. Since every linear transformation A for which the quadratic form $\langle Af,f\rangle$ assumes only real values, is symmetric, then every positive linear transformation of complex Hilbert space is symmetric.

Proposition 2: (The Generalized Schwarz Inequality)

For every positive symmetric transformation A and for any two element f and g of H, we have

$$|\langle Af,g|^2 \le \langle Af,f\rangle\ \langle Ag,g\rangle.$$

Proof: For every real value of λ and for $h_\lambda = f+\lambda\langle Af,g\rangle g$ we have

$$0 \le \langle Ah_\lambda,h_\lambda\rangle = \langle A(f+\lambda\langle Af,g\rangle g),\ f+\lambda\langle Af,g\rangle g\rangle$$

$$= \langle Af,f\rangle+2\lambda|\langle Af,g\rangle|^2+\lambda^2\langle Ag,g\rangle|\langle Af,g\rangle|^2$$

Implies $\langle Ag,g\rangle|\langle Af,g\rangle|^2\lambda^2+2|\langle Af,g\rangle|^2\lambda+\langle Af,f\rangle \ge 0$, which is a second degree polynomial in λ and has real coefficients and cannot have two real distinct zeros, which implies $[2|\langle Af,g\rangle|^2]^2-4\langle Ag,g\rangle|\langle Af,g\rangle|^2\langle Af,f\rangle \le 0$ implies $|\langle Af,g\rangle|^2 \le \langle Af,f\rangle\langle Ag,g\rangle$,

Definition 8: The greatest lower bound and the least upper bounds of the symmetric transformaiton A are defined as the largest real number m and the smallest real number M for which $m\langle f,f\rangle\le\langle Af,f\rangle\le M\langle f,f\rangle$ or $mI \le A \le MI$, where I denotes the identity transformation. In other wards m is the greatest lower bound and M is the least upper bound of the quadratic form $<Af,f>$, where f varies under the condition $\|f\| = 1$. But under this condition the least upper bound of $|<Af,f>|$ is equal to the norm of A and consequently, $\|A\| = \max\{|m|, |M|\}$.

Theorem 2 Every bounded monotone sequence of symmetric transformations A_n converges to a symmetric transformation A.

Proof: If suffices to consider the case where

$$0 \le A_1 \le A_2 \le \dots \le I.$$

For m<n, we have $A_{mn} = A_n - A_m \ge 0$ and $\|A_{mn}\| \le 1$
Now, using the generalized schwarz inequality, for every f we have

$$\|A_{mn}f\|^4 = <A_{mn}f, A_{mn}f>^2 \le <A_{mn}f,f><A_{mn}^2f, A_{mn}f>$$

since $0 \le A_{mn} \le I$ and $\|A_{mn}\| \le 1$, we get

$$\|A_nf-A_nf\|^4 = \|A_{mn}f\|^4 \le <A_{mn}f,f><A_{mn}^2f,A_{mn}f>$$

$$\le <(A_n-A_m)f,f>\|A_{mn}^2f\|.\|A_{mn}f\|$$

$$\le (<A_nf,f>-<A_mf,f>)\|A_{mn}^2\|.\|f\|.\|A_{mn}\|\|f\|$$

$$\le (<A_nf,f> - <A_mf,f>)\|A_{mn}\|^3.\|f\|^2$$

$$\le (<A_nf,f> - <A_mf,f> .\|f\|^2$$

Hence, $.\|A_nf-A_mf\|^4 \le (<A_mf,f> - <A_mf,f>).\|f\|^2$

But the numerical sequence $\{<A_nf,f>\}$ is bounded and increasing, hence convergent, which implies, by the inequality obtained, the sequence of elements A_nf is cauchy and hence convergent. Define $Af = \lim_n A_nf$, which is true for all f in H. The transformation A is obvioiusly linear and also symmetric by the above property.

Note: The square of a symmetric transformation is always a positive transformation that is $<A^2f,f> = <Af,Af> = \|Af\|^2 \ge 0$.

Now we consider the converse problem, which is to find, for a given positive symmetric transformation A, a square root, which is a symmetric transformation T suchthat $T^2 = A$.

Theorem 3: Every symmetric $A \geq 0$ possesses a unique positive symmetric square root, denoted by $A^{1/2}$. It can be represented as the limit of a sequence of polynomials in A and consequently is permutable with all transformations which are permutable with A.

Proof: w.l.o.g we can assume $A \leq I$

set $A = I-B$, where $0 \leq B \leq I$ and $T = I-Y$. Then solve the equation $Y = \frac{1}{2}[B + Y^2]$ (1)

let us try the method of successive approximation.

put $Y_o = 0$, $Y_1 = \frac{1}{2}B$ and ingeneral

$$Y_{n+1} = \frac{1}{2}[B + Y_n^2], \quad n \geq 0 \tag{2}$$

and we show the sequence (Y_n) converges to a solution of (1). First we show that Y_n and $Y_n - Y_{n-1}$ are polynomial in B with non-negative real coefficents which we prove by induction on n. These proposition are obviously true for n = 1 and trivially Y_n is apolynomial in B with non-negative real coefficients. To show the second, the formula

$$Y_{n+1} - Y_n = \frac{1}{2}(B+Y_n^2) - \frac{1}{2}(B+Y_{n-1}^2) = \frac{1}{2}(Y_n^2 - Y_{n-1}^2)$$

$$= \frac{1}{2}(Y_n + Y_{n-1})(Y_n - Y_{n-1})$$

since $B \geq 0$ implies $B^n \geq 0$ for $n = 2,3,\ldots$, we have $Y_n \geq 0$ and $Y_n - Y_{n-1} \geq 0$. Finally, $\|Y_n\| \leq 1$ for all n. This is true for n = 0 and is proved by induction with the recurrence formula (2).

Hence the sequence (Y_n) is bounded monotone sequence, and then by theorem 2 is convergent.

let $Y = \lim_{n\to\infty} Y_n$, then Y satisfies equation (1), since

$$Y = \lim_{n\to\infty} Y_{n+1} = \lim_{n\to\infty} \frac{1}{2} [B + Y_n^2) = \frac{1}{2} [B + Y^2)$$

Hence, $Y = \frac{1}{2} [B + Y^2]$

We also have $\|Y\| \le 1$ and consequently $Y \le I$, $T = I - Y \ge 0$

thus, $T^2 - (I-Y)^2 = I - 2IY + Y^2$

$$= I - 2\left[\frac{1}{2} (B+Y^2)\right]+Y^2 = I-B-Y^2-Y^2 = I-B = A$$

therefore, $T^2 = A$.

Hence, we have constructed a solution of $T^2 = A$, $T = I-Y$ which is symmetric and positive ; furthermore it is the limit of polynomials in A. To pove that T is unique, let T′ be also a positive symmetric transformation such that $T'^2 = A$ and claim $T' = T$.

$T'A = (T')^3 = AT'$ implies T′ is permutable with A and with polynomials in A as well as with their limits and in particular with T.

Now let Z and Z′ be the positive symmetric square roots of T and T′ respectively, obtained by the above procedure, but starting with T and T′ instead of A (that is $Z^2 = T$ and $Z'^2 = T'$).

Let f be an arbitrary element of H and let $g = (T-T')f$. Then we have

$$\|zg\|^2+\|z'g\|^2 = \langle z^2g,g\rangle + \langle z'^2g,g\rangle = \langle Tg,g\rangle + \langle T'g,g\rangle$$

$$= \langle (T+T')g,g\rangle = \langle (T+T')(T-T')f,g\rangle = \langle (T^2-T'^2)f,g\rangle = \langle (A-A)f,g\rangle = 0,$$

which yields $zg = z'g = 0$; hence $Tg = zzg = 0$ and $T'g = z'z'g = 0$, that means $Tg-T'g = (T-T')g = 0$. If follows

$$\|(T-T')f\|^2 = \langle (T-T')^2f,f\rangle = \langle (T-T')g,f\rangle = 0$$, and hence

$(T-T')f = 0$. Since this holds for all f, we have $T' = T$.

Corollary 1: The product of two permutable, positive symmetric transformations is also a positive symmetric transformation.

Proof: Let $A,B \geq 0$ be symmetric and $AB = BA$. Then by Theorem 3, $AB^{1/2} = B^{1/2}A$. Now for an arbitrary element f, we have $\langle ABf,f\rangle = \langle AB^{1/2}B^{1/2}f,f\rangle = \langle B^{1/2}AB^{1/2}f,f\rangle = \langle AB^{1/2}f,B^{1/2}f\rangle \geq 0$ since $A \geq 0$ and $B^{1/2}$ is symmetric. Hence $AB \geq 0$.

Corollary 2: Let $A,B,C \geq 0$ be symmetric and $AC = CA, BC = CB$. If $A \geq B$, then $AC = CA \geq BC = CB$.

Proof: $A \geq B$ implies $A - B \geq 0$

But $C(A-B) = CA - CB = AC - BC = (A-B)C$, hence C and A-B permutes. Then, by corollary 1, $C(A-B) \geq 0$, which implies

$$CA = AC \geq CB = BC$$

Now we consider the inverse of a symmetric transformation A. If A^{-1} exists it is also symmetric , since

$$\langle A^{-1}f,g\rangle = \langle A^{-1}f,AA^{-1}g\rangle = \langle AA^{-1}f,A^{-1}g\rangle = \langle f,A^{-1}g\rangle$$

Lemma: If the Neumann series $\sum_{n=o}^{\infty} A^n$ converges uniformly, then $(I-A)^{-1}$ exists and is continuous.

Proof: $\sum_{n=o}^{\infty} A^n$ converges uniformly means there exists a transformation

$$S = \sum_{n=o}^{\infty} A^n. \quad \text{Then}$$

$$A.S = A. \sum_{n=o}^{\infty} A^n = A. \lim_{m\to\infty} \sum_{n=o}^{m} .A^n = \lim_{m\to\infty} \sum_{n=o}^{m} A.A^n$$

$$= \lim_{m\to\infty} \sum_{n=o}^{m} A^{n+1} = \sum_{n=o}^{\infty} A^{n+1}$$

similarily ,

$$S.A = \sum_{n=o}^{\infty} A^{n+1}$$

Then, $A.S = \sum_{n=o}^{\infty} A^{n+1} = A + A^2 + A^3 + \ldots = S-I$

$$S.A = \sum_{n=o}^{\infty} A^{n+1} = A + A^2 + A^3 + \ldots = s-I,$$ which implies

$I = S-A.S = (I-A).S$ and $I = S-S.A = S.(I-A)$ which implies $I = (I-A).S = S.(I-A)$

Hence, $(I-A)^{-1}$ exist and $(I-A)^{-1} = S = \sum_{n=o}^{\infty} A^n$

Now, if we take $I-A$ instead of A, the lemma implies

$$A^{-1} = [I-(I-A)]^{-1} = \sum_{n=o}^{\infty} (I-A)^n .$$

Thus, A^{-1} exist in particular if $\|I-A\| < 1$, which can be represented by the Neumman series.

$$A^{-1} = \sum_{n=o}^{\infty} (I-A)^n$$ and it converges in norm, as shown below:

$$\|[I-(I-A)]^{-1} - \sum_{k=o}^{n-1} (I-A)^k\| \le \sum_{m=n}^{\infty} \|I-A\|^m = \frac{\|I-A\|^n}{I-\|I-A\|} \xrightarrow{n\to\infty} o ,$$ as $\|I-A\| < 1$.

Note that this representation is also valid for non-symmetric transformations.

For A symmetric, the condition $\|I-A\| < 1$ is equivalent to

$$0 < m \le M < 2 \qquad (3)$$

where m and M are the greatest lower and least upper bounds of A.

If we assume only that $m > 0$, we can always find a positive quantity c such that the bounds m', M' of the transformation $A' = cA$ satisfy condition (3), we have only to take $c < \frac{2}{M}$. Consequently we have the development,

$$A^{-1} = (c^{-1}A')^{-1} = c(A')^{-1} = c(cA)^{-1} = c \sum_{n=o}^{\infty} (I-cA)^n.$$ Since the terms

of this series are positive transformations, the same is true of the sum A^{-1}. Multiplying the inequalities $m.I \le A \le MI$ by A^{-1}, we obtain $mA^{-1} \le I \le MA^{-1}$, hence

$$M^{-1}I \le A^{-1} \le m^{-1}I.$$

heorem 4: In order that the linear transformation T of Hilbert space possess an inverse, it is necessary and sufficient that the symmetric transformations T^*T and TT^* have positive greatest lower bounds.

Proof: Assume that the transformation TT^* and T^*T, which are summetric have positive greatest lower bounds. This assures the existence of $(T^*T)^{-1}$ and $(TT^*)^{-1}$. Then $(T^*T)^{-1}T^*T = I$ and $TT^*(TT^*)^{-1} = I$ implies T has a left inverse $(T^*T)^{-1}T^*$ and a right inverse $T^*(TT^*)^{-1}$ and also $(T^*T)^{-1}T^* = T^*(TT^*)^{-1}$.

Hence, T^{-1} exists.

Assume T^*T does not have a positive greatest lower bound. Then there exist a sequence (f_n) such that $\|f_n\| = 1$ and

$\|Tf_n\|^2 = <Tf_n, Tf_n> = <T^*Tf_n, f_n> \longrightarrow 0$. If T^{-1} exists, it follows that

$\|f_n\| = \|T^{-1}Tf_n\| \le \|T^{-1}\|\|Tf_n\| \longrightarrow 0$ which contradicts $\|f_n\| = 1$

Hence T has no inverse.

Similarily, if TT^* does not have a positive greatest lower bound, T^* does not have an inverse, and hence T can not have an inverse either.

2.2. Orthogonal projections

The simplest symmetric transformation are the orthogonal projections on to subspaces of the Hilbert space H.

Definition 9: Let H be a Hilbert space. Then

i) The transformation $P: H \longrightarrow H$ is called a **projection** in H if $P^2 = P$

ii) let M be a subspace of H. The subspace $M^{\perp} = \{f \in H | < f, g> = 0, \forall g \in M\}$ such that $H = M \oplus M^{\perp}$ is called the **orthogonal complement** of M.

Remark: If $H = M\oplus M^{\perp}$, then every element f in H can be uniquely decomposed into $f = g + h$, where $g\in M$, $h\in M^{\perp}$.

iii) let $H = M\oplus M^{\perp}$. The projection $P: H\longrightarrow M$ defined by $Pf = g$, which is linear is called the **orthogonal projection** onto M along $M^{\perp}$. The projection on to $M^{\perp}$ is $I-P$.

Proposition 3: Let $H = M\oplus M^{\perp}$ and let P be a projection onto M. Then, P is orthogonal if and only if P is symmetrical.

Proof: Let P be orthogonal. Let $f = g+h$ and $f' = g'+h'$ be decompositions of f and f' with respect to the subspaces M and $M^{\perp}$. Then we have $\langle g,h'\rangle = 0$ and $\langle h,g'\rangle = 0$ and $Pf = g, Pf' = g'$

Hence, $\langle Pf,f'\rangle = \langle g,f'\rangle = \langle g,f'\rangle-\langle g,h'\rangle = \langle g,f'-h'\rangle$

$= \langle g,g'\rangle = \langle g,g'\rangle+\langle h,g'\rangle=\langle g+h,g'\rangle=\langle f,Pf'\rangle$

implies $\langle Pf,f'\rangle = \langle f,Pf'\rangle$ and hence P is symmetric.

Again, let P be symmetric, for all $f\in H$, $f = Pf+(f-Pf)$, where $Pf\in M$ and $f-Pf$ is orthogonal to M. That is

$\langle f-Pf,Pg\rangle = \langle Pf-P^2f,g\rangle = \langle Pf-Pf,g\rangle = 0$ for all g

Hence, P is orthogonal.

We have $\langle Pf,f\rangle = \langle P^2f,f\rangle = \langle Pf,Pf\rangle = \|Pf\|^2$, for all $f\in H$. It follows in particular $0 \le P \le I$; then $P = 0$ if M consist of the single element 0, and $P = I$ if M coincides with H.

Prposition 4: For two symmetric projections P_1 and P_2, $P_1P_2 = P_2$ and $P_1 \ge P_2$ are equivalent.

Proof: Let $P_1P_2 = P_2$, then $P_1-P_2 \ge 0$, $P_1 \ge P_2$

If $P_1 \ge P_2$, then $I-P_1 \le I-P_2$ and consequently

$$\|(I-P_1)P_2f\|^2 = \langle (I-P_1)P_2f,\ (I-P_1)P_2f\rangle = \langle (I-P_1)^2P_2f,P_2f\rangle$$

$$= \langle (I-P_1)P_2f,P_2f\rangle \le \langle (I-P_2)P_2f,P_2f\rangle = \langle 0,P_2f\rangle = 0$$

therefore, $(I,P_1)P_2f = 0$ for all f

Hence $(I-P_1)P_2 = 0$ or $P_1P_2 = P_2$

Theorem 5: If the orthogonal projections P and Q onto the subspaces B and D satisfy the condition $\|P-Q\| < 1$, then B can be mapped linearly and isometrically onto D.

Proof: $\|P-Q\| < 1$ implies $\|P(Q-P)P\| < 1$ and that consequently the symmetric transformation $A = I + P(Q-P)P$ has a greatest lower bound. Hence by theorem 4, A^{-1} exists and $A^{-1/2} = (A^{-1})^{1/2}$ exists by theorem 3, which are symmetric and positive.

Now consider the transformations:

$U = QA^{1/2}P$ and $U^{*} = PA^{-1/2}Q$. Since $PA = AP$, we have $PA^{-1/2} = A^{-1/2}P$ and since further $PQQP = PQP = P+P(Q-P)P = PA$, it follows that $U^{*}U = PA^{-1/2}QQA^{-1/2}P = A^{-1/2}PQQPA^{-1/2} = A^{-1/2}PAA^{-1/2} = PA^{-1/2}AA^{-1/2} = PI = P$. This implies that for elements of B, $U: B \longrightarrow D$ is isometric, that is for $f,g \in B$, $\langle Uf,Ug\rangle = \langle U^{*}Uf,g\rangle = \langle Pf,g\rangle = \langle f,g\rangle$. Since B is a closed set its image under U say D′ is closed and is a subspace of D. Since U is zero for the elements orthogonal to B, then D′ is also the image under U of H (the entire space).

Now, let h be an element orthogonal to D′, that is $\langle h,Uf\rangle = 0$, for all f in H. Then $U^{*}h = 0$ (since $0 = \langle h,Uf\rangle = \langle U^{*}h,U^{*}Uf\rangle = \langle U^{*}h,f\rangle$, for all f in H), hence

$PQh = A^{1/2}A^{-1/2}PQh = A^{1/2}PA^{-1/2}Qh = A^{1/2}U^{*}h = 0$ and consequently, $(Q-P)Qh = Qh$. But since $\|Q-P\| < 1$, This equation is possible only if $Qh = 0$, that is only if h is also orthogonal to D. Hence $D' = D$ and $U: B \longrightarrow D$ is obviously linear. Thus, the theorem.

2.3. Functions of Bounded Symmetric Transformations.

For a symmetric transformation A, we assign to the polynomial with real coefficients $p(\lambda) = a_o + a_1\lambda + a_2\lambda^2 + \dots + a_n\lambda^n$ the symmetric transformation

$$P(A) = a_o + a_1A + a_2A^2 + \dots + a_nA^n$$

This correspondence is obviously homogeneous, additive and multiplicative, that is, the transformations $cP(A)$, $P(A) + q(A)$, $P(A).q(A)$ corresponds to $cP(\lambda)$, $P(\lambda)+q(\lambda)$, $p(\lambda).q(\lambda)$ respectively.

Moreover, this correspondence is positive type, that is if $p(\lambda) \geq 0$ for $m \leq \lambda \leq M$, where m and M denote the greatest lower and least upper bounds of A, we also have $p(A) \geq 0$. Inorder to see this, we decompose $p(\lambda)$ in the form

$$p(\lambda) = c \prod_i (\lambda-\alpha_i) \prod_j (\beta_j-\lambda) \prod_k[(\lambda-\gamma_k)^2 + \delta_k^2], \text{ where } c \geq 0,\ \alpha_i \leq m,$$

$\beta_j \leq M$, and where the quadratic factors correspond to complex conjugate zeros and to real zeros between m and M, the latter being necessarily of even multiplicity. Replacing λ by A, all the factors will be positive transformations, and since they are also permutable, the same will be true of the product $p(A)$.

More generally, the inequality $p(\lambda) \geq q(\lambda)$ for $m \leq \lambda \leq M$ implies that $p(A) \geq q(A)$, to see this, we consider the difference $p(\lambda)-q(\lambda) \geq 0$ and apply the above. Hence the correspondence is monotonic.

To extend this correspondence to larger classes of functions where the above properties are preserved, we consider monotonic sequence which converge every where (It will be a little more convienient to consider decreasing rather than increasing sequences). For this let us consider:

1. The class C_1 of non-negative real-valued functions defined in the interval $m \le \lambda \le M$ which are continuus or at least upper-semicontinuous.

To a function $u(\lambda)$ of this class we can find a decreasing sequence of polynomials $(p_n(\lambda))$ which converges to $u(\lambda)$. The sequence of transformations $p_n(A)$ is then also decreasing and bounded below by 0, hence it is convergent. The limit is a symmetric transformation which we assign to $u(\lambda)$ and denote by $u(A)$.

This definition is unique; that is , $u(A)$ does not depend on the particular choice of the sequence $(p_n(\lambda))$: if $(q_n(\lambda))$ is another sequence of the same type, then $\lim_n p_n(A) = \lim_n q_n(A)$.

Proof: for every integer r, and for s sufficiently large we have $p_s(\lambda) \le q_r(\lambda) + \frac{1}{r}$, $q_s(\lambda) \le p_r(\lambda) + \frac{1}{r}$, in the entire interval $m \le \lambda \le M$, which is a consequence of Dini Theorem on monotomic sequences of continuous functions or also of the Borel covering Theorem. We then have

$p_s(A) \le q_r(A) + \frac{1}{r} I$, $q_s(A) \le p_r(A) + \frac{1}{r} I$, and taking limit (first for $s \longrightarrow \infty$ and then $r \longrightarrow \infty$). We get,

$$\lim_s p_s(A) \le \lim_r q_r(A) , \lim_s q_s(A) \le \lim_r p_r(A)$$

Hence $\lim_n p_n(A) = \lim_n q_n(A)$

It follows by the same reasoning, that if $u_1(\lambda) \ge u_2(\lambda)$, $m \le \lambda \le M$, then $u_1(A) \ge u_2(A)$. Hence the correspondence extended is therefore monotonic, and further it is positively homogeneous, additive and multiplicative since , if the sequences $(p_n(\lambda))$, and $(q_n(\lambda))$ decreases to $u(\lambda)$ and $v(\lambda)$, then the sequences $((p_n(\lambda))$, $c> 0$, $(p_n(\lambda) + q_n(\lambda))$ and $(P_n(\lambda)q_n(\lambda))$ decreases to $cu(\lambda)$, $u(\lambda) + v(\lambda)$ and $u(\lambda)v(\lambda)$ respectively.

2. Now, let us consider the class C_2 of functions admitting a decomposition in to the difference of two functions of class C_1.

We assign to the function $w(\lambda) = u(\lambda) - v(\lambda)$ the transformation $w(A) = u(A)-v(A)$. This definition is unique, since $u_1(\lambda)-v_1(\lambda) = u_2(\lambda) - v_2(\lambda)$ implies $u_1(A) - v_1(A) = u_2(A) - v_2(A)$, which we obtain by writing the two equations in the form $u_1 + v_2 = v_1 + u_2$ and applying additivity for the class C_1.

For the class C_2, the correspondence is homogeneous, additive and multiplicative, a consequence of the corresponding properties of the class C_1 and the following decompositions:

$$c(u-v) = cu - cv \;, \text{ for } c > 0 \,.$$

$$c(u-v) = (-c)v - (-c)u \quad , \text{ for } c < 0 \,,$$

$$(u_1-v_1) + (u_2-v_2) = (u_1 + u_2) - (v_1 + v_2)$$

$$(u_1-v_1)(u_2-v_2) = (u_1u_2 + v_1v_2) - (u_1v_2 + u_2v_1)$$

Finally the correspondence for class C_2 is also monotonic, since $u_1(\lambda)-v_1(\lambda)\geq u_2(\lambda)-v_2(\lambda)$ implies $u_1(A)-v_1(A) \geq u_2(A) - v_2(A)$, that is $u_1(\lambda) - v_1(\lambda) \geq u_2(\lambda) - v_2(\lambda)$ means $u_1(\lambda) + v_2(\lambda) \geq v_1(\lambda) + u_2(\lambda)$

implies $u_1(A) + v_2(A) \geq v_1(A) + u_2(A)$, monotonocity of C_1.

implies $u_1(A) - v_1(A) \geq u_2(A) - v_2(A)$.

2.4. Spectral Decomposition of a Bounded Symmetric Transformation.

Among "the functions" of the symmetric transformation A which we have just defined there are projections, namely those which correspond to the function $e(\lambda)$ taking only the values 0 and 1. Since we have $[e(\lambda)]^2 = e(\lambda)$, then also $[e(A)]^2 = e(A)$.

Theorem 6: To every symmetric transformation A in a Hilbert space, with greatest lower and least upper bounds equal to m and M, we can assign a "spectral family" on the interval $[m,M]$, that is , a family of projections $\{E_\lambda\}$ depending on the real parameter λ such that

a) $E_\lambda \le E_\mu$, or equivalently , $E_\lambda E_\mu = E_\lambda$ for $\lambda \le \mu$,

b) $E_{\lambda+o} = E_\lambda$

c) $E_\lambda = 0$ for $\lambda < m$ and $E_\lambda = I$ for $\lambda \ge M$, in such a way that we have

$$A = \int_{m-o}^{M} \lambda dE_\lambda. \qquad (4)$$

Moreover , these properties uniquely determine the family $\{E_\lambda\}$. For every fixed value of the parameter , E_λ is the limit of a sequence of polynomials in A.

<u>**Proof:**</u> Let us consider in particular the function $e_\mu(\lambda)$, depending on the real paramenter μ, where

$$e_\mu(\lambda) = \begin{cases} 1 & \text{if } \lambda \le \mu \\ 0 & \text{if } \lambda > \mu \end{cases}, \; e_\mu(\lambda) \text{ belongs to class } C_1.$$ Hence

there correspond to it a transformation $e_\mu(A)$ which is projective and denoted by E_μ. Since , $e_\mu(\lambda)\, e_v(\lambda) = e_\mu(\lambda)$ for $\mu < v$, we have $E_\mu E_v = E_\mu = E_v E_\mu$ and hence $E_\mu \le E_v$ and since on the segment $m \le \lambda \le M$, $e_\mu(\lambda) = 0$ if $\mu < m$ and $e_\mu(\lambda) = 1$ if $\mu \ge M$, we have $E_\mu = 0$ if $\mu < m$ and $E_\mu = I$ if $\mu \ge M$, also $E_{\mu+o} = E_\mu$.

Moreover, as a function of μ, E_μ is continuous from the right. Inorder to see this, we fix μ and construct a sequence of polynomials $p_n(\lambda)$ which decreases in $[m,M]$ to $e_\mu(\lambda)$ and satisfying $p_n(\lambda) \ge e_{\mu+\frac{1}{n}}(\lambda)$ Then we have $p_n(A) \ge E_{\mu+\frac{1}{n}} \ge E_\mu$

Since $p_n(A) \longrightarrow E_\mu$, it follows that $E_{\mu+\frac{1}{n}} \longrightarrow E_\mu$ for $n \longrightarrow \infty$, and since E_μ is monotonic function of μ, this implies

$E_{\mu+\epsilon} \longrightarrow E_\mu$ for $0 < \epsilon \longrightarrow 0$

Hence, E_μ is continuous from the right.

For $\mu < v$ we obviously have

$$\mu[e_v(\lambda)-e_\mu(\lambda)] \le \lambda[e_v(\lambda)-e_\mu(\lambda)] \le v[e_v(\lambda)-e_\mu(\lambda)],$$ which implies

$$\mu[E_v-E_\mu] \le A[E_v-E_\mu] \le v[E_v-E_\mu] \qquad (5)$$

Now, consider a sequence of points $\mu_o,\mu_1,\dots,\mu_n$ such that $\mu_o < m < \mu_1 < \mu_2 < \dots < \mu_{n-1} < M \le \mu_n$. Writting (5) with $\mu = \mu_{k-1}$, $v = \mu_k$, $k = 1,2,\dots,n$ and taking the sum, we get

$$\sum_{k=1}^{n} \mu_{k-1} \left[E_{\mu_k} - E_{\mu_{k-1}}\right] \le A \sum_{k=1}^{n} \left[E_{\mu_k} - E_{\mu_{k-1}}\right] \le \sum_{k=1}^{n} \mu_k \left[E_{\mu_k} - E_{\mu_{k-1}}\right]$$

The middle term, $A \sum_{k=1}^{n} \left[E_{\mu_k} - E_{\mu_{k-1}}\right] \le A\left[E_{\mu_n} - E_{\mu_o}\right] = A[I-0] = A$ and if $\max\limits_{k=1}^{n} (\mu_k - \mu_{k-1}) \le \epsilon$, we have

$$\sum_{k=1}^{n} \mu_k [E_{\mu_k} - E_{\mu_{k-1}}] - \sum_{k=1}^{n} \mu_{k-1} [E_{\mu_k} - E_{\mu_{k-1}}] = \sum_{k=1}^{n} \left[\mu_k - \mu_{k-1}\right]\left[E_{\mu_k} - E_{\mu_{k-1}}\right]$$

$$\le \epsilon \sum_{k=1}^{n} [E_{\mu_k} - E_{\mu_{k-1}}] = \epsilon I \qquad (*)$$

If λ_k is between μ_{k-1} and μ_k, that is

$$\sum_{k=1}^{n} \mu_{k-1} [E_{\mu_k} - E_{\mu_{k-1}}] \le \sum_{k=1}^{n} \lambda_k [E_{\mu_k} - E_{\mu_{k-1}}] \le \sum_{k=1}^{n} \mu_k [E_{\mu_k} - E_{\mu_{k-1}}],$$ then

$$\left\| A - \sum_{k=1}^{n} \lambda_k [E_{\mu_k} - E_{\mu_{k-1}}] \right\| \le \left\| \sum_{k=1}^{n} \mu_k [E_{\mu_k} - E_{\mu_{k-1}}] - \sum_{k=1}^{n} \mu_{k-1} [E_{\mu_k} - E_{\mu_{k-1}}] \right\| \le \epsilon,$$

from (*).

If we increase indefinitely the number n of decomposition intervals (μ_{k-1},μ_k), so that the maximum length tends to zero, we get

$$\sum_{k} \lambda_k [E_{\mu_k} - E_{\mu_{k-1}}] \longrightarrow A \text{ in norm.}$$

Since E_λ is constant for $\lambda \ge M$ and $\lambda < m$, we can express this result by writting, in analogy with ordinary stieltjes integral,

$$A = \int_{-\infty}^{\infty} \lambda dE_\lambda = \int_{m-o}^{M} \lambda dE_\lambda .$$

Moreover for every integer $r > 0$ we have

$A^r = \int_{m-o}^{M} \lambda^r dE_\lambda$, since $\left[\sum_k \lambda_k (E_{\mu_k} - E_{\mu_{k-1}})\right]^r = \sum_k \lambda_k^r (E_{\mu_k} - E_{\mu_{k-1}})$, this follow from the fact that , as a consequence of the relation $E_v E_\mu = E_{\min\{v,u\}}$ the difference $E_{\mu_k} - E_{\mu_{k-1}}$ are pairwise orthogonal projections.

If this relation remains valid for r = 0 , we have for every polynomial p(λ):

$$p(A) = \int_{m-o}^{M} p(\lambda)\, dE_\lambda$$

Secondly, let us consider an arbitrary continuous function u(λ) in [m,M]. Given any $\epsilon > 0$, we can find a polynomial p(λ) such that

$-\frac{\epsilon}{3} \le u(\lambda) - p(\lambda) \le \frac{\epsilon}{3}$ in [m,M] , we have also

$-\frac{\epsilon}{3} I \le u(A) - p(A) \le \frac{\epsilon}{3} I$, hence $\|u(A)-p(A)\| \le \frac{\epsilon}{3}\|I\| = \frac{\epsilon}{3}$ ... (i)

on the otherhand , for every decomposition of the λ-axis and for the sum

$S_u = \sum_k u(\lambda_k)(E_{\mu_k} - E_{\mu_{k-1}})$ and $S_p = \sum_k p(\lambda_k)(E_{\mu_k} - E_{\mu_{k-1}})$, we have

$S_u - S_p = \sum_k [u(\lambda_k) - p(\lambda_k)][E_{\mu_k} - E_{\mu_{k-1}}] \le \frac{\epsilon}{3} \sum_k (E_{\mu_k} - E_{\mu_{k-1}}) = \frac{\epsilon}{3} I$, then

we have $-\frac{\epsilon}{3} I \le S_u - S_p \le \frac{\epsilon}{3} I$ implies $\|S_u - S_p\| \le \|\frac{\epsilon}{3} I\| = \frac{\epsilon}{3}$ . . . (ii)

When the decomposition is sufficiently fine so that

$\|p(A) - s_p\| \le \frac{\epsilon}{3}$ . . . (iii)

From (i),(ii) and (iii), we have

$\|u(A) - S_u\| \le \|u(A) - p(A)\| + \|p(A) - S_p\| + \|S_p - S_u\| \le \frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{\epsilon}{3} = \epsilon$

Therefore , for every continuou function u(λ) ,

$U(A) = \int_{m-o}^{M} u(\lambda)\, dE_\lambda$ in the sense of convergence in the norm of some stieltjes type. It follows for every f, g

$$\langle u(A)f, g\rangle = \int_{m-o}^{M} u(\lambda)\, d\langle E_\lambda f, g\rangle, \text{ in the ordinary stieltjes sense.} \qquad (6)$$

Uniqueness: Since $\{E_\lambda\}$ is a spectral family on the interval $[m, M]$ and satisfies (4) we conclude as above that it also satisfies (6) and inparticular satisfies (6) for all continuous functions $u(\lambda)$. But, since the first member of (6) doesnot depend on $\{E_\lambda\}$, it follows that for every pair of elements f, g, the function $\langle E_\lambda f, g\rangle$ is determined upto an additive constant by the relation (6) at its point of continuity and at $m-o$ and M. Since the function is continuous from the right and has the fixed value $\langle f, g\rangle$ at the point M it is therefore uniquely determined every where.

3. UNITARY AND NORMAL TRANSFORMATIONS

3.1. Unitary Transformations

Definition 10: Let H be a Hilbert Space. The linear transformation $U: H \longrightarrow H$ is said to be isometric if it leaves scalar products invariant, $\langle Uf, Ug\rangle = \langle f, g\rangle$, $f, g \in H$ or equivalently, if $U^* U = I$.

If the linear transformation $U: H \longrightarrow H$ is onto, U is said to be unitary. Since in this case the equation $Ug = f$ has a solution g for an arbitrary given $f \in H$, we have

$$UU^* f = U(U^* Ug) = U(Ig) = Ug = f$$

Hence, $UU^* = I$

The two equations $U^* U = I$ and $UU^* = I$ or the equivalent equation $U^* = U^{-1}$ are obviously characterstic for unitary transformations.

Example: $U: l^2 \longrightarrow l^2$ defined by $U(x_1, x_2, x_3, x_4, \ldots) = (x_2, x_1, x_4, x_3, \ldots)$ is unitary,

Remark: In a finite dimensional space evey isometric transformation is unitary. By contrast, in an infinite - dimensional space there are isometric transformation which are not unitary. For example the transformation

$u(x_1, x_2, \ldots) = (0, x_1, x_2, \ldots)$ is isometric but not unitary, since it is not onto.

Lemma: Every positive trigonometric polynomial can be represented by the square of the absolute value of another trigonometric polynomial.

Proof:- Let $t(x) = \sum_{k=-n}^{n} c_k e^{ikx} \geq 0.$

It would be sufficient to consider only strictly positive polynomials $t(x)$, since the general case can be reduced to this case by adding to $t(x)$ the constant term $\epsilon > 0$, whcih then is made to tend to 0. Now let us consider the polynomial

$$t(x) = \sum_{k=-n}^{n} c_k e^{ikx} > 0.$$

We observe first that $c_k = \overline{c_{-k}}$ and that we can assume $c_n \neq 0$, this is because $t(x) > 0$ means $t(x) \in \mathbb{R}$ (since no order in the complex case). Then we have

$$\sum_{k=-n}^{n} c_k e^{ikx} = t(x) = \overline{t(x)} = \sum_{k=-n}^{n} \bar{c}_k e^{-ikx}$$

which implies by the property of polynomials $c_k = \bar{c}_{-k}$.

Now consider the polynomial

$$p(z) = c_{-n} + c_{-n+1} z + \ldots + c_{n-1} z^{2n-1} + c_n z^{2n}$$

It obviously satisfies the relation

$$p(z) = z^{2n} \, \overline{p\left(\frac{1}{\bar{z}}\right)} \qquad (7)$$

as shown below

$$p(z) = c_{-n} + c_{-n+1}z + \dots + c_{n-1}z^{2n-1} + c_n z^{2n}$$

$$= z^{2n}\left[c_{-n}\frac{1}{z^{2n}} + c_{-n+1}\cdot\frac{1}{z^{2n-1}} + \dots + c_{n-1}\cdot\frac{1}{z} + c_n \right]$$

$$= z^{2n}\left[\bar{c}_n\cdot(\tfrac{1}{z})^{2n} + \bar{c}_{n-1}(\tfrac{1}{z})^{2n-1} + \dots + \bar{c}_{-n+1}\cdot\frac{1}{z} + \bar{c}_{-n} \right]$$

$$= z^{2n}\left[\overline{c_n\cdot(\frac{1}{\bar{z}})^{2n} + c_{n-1}(\frac{1}{\bar{z}})^{2n-1} + \dots + c_{-n+1}\frac{1}{\bar{z}}} + \bar{c}_{-n} \right]$$

$$= z^{2n}\,\overline{p(\frac{1}{\bar{z}})}.$$

Moreover, since

$$e^{-inx}p(e^{ix}) = e^{-inx}\left[c_{-n+1}e^{ix} + \dots + c_n e^{2inx}\right]$$

$$= c_{-n}e^{inx} + c_{-n+1}e^{i(-n+1)x} + \dots + c_n e^{inx} = t(x)\ ,\ p(z)$$

can have no zeros on the circle $|z| = 1$. Denote by $\alpha_1, \alpha_2, \dots$ its zeros in the interior and by $\beta_1, \beta_2, \dots$ those in the exterior of the unit circle. Let the multiplicities of these zeros be equal to $r_1, r_2, \dots$ and $s_1, s_2, \dots$ respectively. Then we have the factorization

$$p(z) = c\prod_k (z-\alpha_k)^{r_k}\, z^S \prod_j \left(\frac{1}{z} - \frac{1}{\bar{\beta}_j}\right)^{s_j}$$

where $S = \sum s_j$.

Relation (7) shows that if β is a zero exterior to the unit circle, $\alpha = \frac{1}{\bar{\beta}}$ is a zero interior to this circle, of the same multiplicity as β, and conversly. Hence we can enumerate the β so that we have $\beta_k = \frac{1}{\bar{\alpha}_k}$; then $r_k = s_k$ and

$$s = \sum s_k = \sum r_k = \frac{1}{2}\sum(s_k + r_k) = \frac{2n}{2} = n$$

We shall therefore have

$$t(x) = e^{-inx}p(e^{ix}) = c\prod_k (e^{ix}-\alpha_k)^{r_k}\prod_k (e^{-ix} - \bar{\alpha}_k)^{r_k}$$

$$= c\prod_k (e^{ix}-\alpha_k)^{r_k}\,\overline{\prod_k (e^{ix}-\alpha_k)^{r_k}}$$

$$= c\left|\prod_k (e^{ix}-\alpha_k)^{r_k}\right|^2$$

The coefficient c is necessarily positive, and the trigonometric polynomial

$q(X) = \sqrt{c}\,\prod_k (e^{ix}-\alpha_k)^{r_k}$ satisfies the requirment of the lemma.

Theorem 7: Every unitary transformation U has a spectral decomposition

$$U = \int_0^{2\pi} e^{i\varphi} dE_\varphi , \qquad (8)$$

where $\{E_\varphi\}$ is a spectral family over the segment $o \le \varphi \le 2\pi$. We can require that E_φ be continuous at the point $\varphi = 0$, that is $E_o = 0$; $\{E_\varphi\}$ will then be determined uniquely by U. Moreover, E_φ is the limit of a sequence of polynomials in U and U^{-1}.

Proof:- To begin, we assign to the trigonometric polynomial

$$p(e^{i\varphi}) = \sum_{k=-n}^{n} c_k e^{ik\varphi}$$

the transformation

$p(U) = \sum_{k=-n}^{n} c_k U^k$; here we admit complex coefficient c_k. The correspondence is obviously homogeneous, additive, multiplicative and such that the transformation corresponding to the conjugate polynomial

$$\overline{p(e^{i\varphi})} = \sum_{k=-n}^{n} \bar{c}_k e^{-ik\varphi}$$

is the adjoint of the one corresponding to $p(e^{i\varphi})$, to show this:

Let $q(e^{i\varphi}) = \overline{p(e^{i\varphi})} = \sum_{k=-n}^{n} \bar{c}_k e^{-ik\varphi} \Rightarrow q(U) = \sum_{k=-n}^{n} \bar{c}_k (U^k)^{-1} \equiv \overline{p(u)}$

claim: $q(U) = p(U)^* = \overline{p(U)}$

$$< \sum_{k=-n}^{n} c_k U^k f, g> = \sum_{k=-n}^{n} c_k <U^k f, g> = \sum_{k=-n}^{n} c_k <f, (U^k)^* g>$$

$$= <f, \sum_{k=-n}^{n} \bar{c}_k (U^*)^k g> = <f, \sum_{k=-n}^{n} \bar{c}_k (U^{-1})^k g>$$

$$= <f, \sum_{k=-n}^{n} \bar{c}_k (U^k)^{-1} g>$$

implies $<p(U)f,g> = <f,q(U)q>$

Then $p(U)^* = q(U) = \overline{p(U)}$

If $p(e^{i\varphi})$ is real value, $p(U)$ is symmetric, since $p(e^{i\varphi}) \in \mathbb{R}$ implies $p(e^{i\varphi}) = \overline{p(e^{i\varphi})}$ and hence the corresponding transformations $p(U) = \overline{p(U)} = p(U)^*$, hence $p(U)$ is symmetric. The correspondence is also positive type. that is, if $p(e^{i\varphi}) \geq 0$, then $p(U) \geq 0$. To see this we use the above lemma. By the lemma $p(e^{i\varphi}) \geq 0$ implies there is $q(e^{i\varphi})$ such that

$$p(e^{i\varphi}) = |q(e^{i\varphi})|^2 = \overline{q(e^{i\varphi})}.\ q(e^{i\varphi}),$$

we have, $p(U) = q(U)^*.q(U)$,

hence, $<p(U)f,f> = <q(U)f,q(U)f> \geq 0$ and therefore $p(U) \geq 0$.

The correspondence established in this manner for trigonometric polynomials extends to more general functions of period 2π, namely, first to functions which are limits of decreasing sequences of positive trigonometric polynomials, then to linear combinations of these functions (with real or complex coefficients); the method analogous to what we followed for symmetric transformations. The correspondence thus extended continues to be linear, multiplicative, and of positive type, and the transformations which correspond to two conjugate functions are adjoint to one another.

The class of these functions includes, in particular, the function $e_\psi(\varphi)$ which depend on the real parameter $\psi (0 \leq \psi \leq 2\pi)$ and are defined as follows:

$$e_o(\varphi) = 0,\ e_{2\pi}(\varphi) = 1,\ \text{and for } 0 < \psi < 2\pi,$$

$$e_\psi(\varphi) = \begin{cases} 1 & \text{when } 2k\pi < \varphi \leq 2k\pi + \psi \\ 0 & \text{when } 2k\pi + \psi < \varphi \leq 2(k+1)\pi \end{cases},\ k = 0, \pm 1, \pm 2, \ldots$$

Since these functions are equal to their squares, the corresponding transformations E_ψ will be projections. We shall have, in particular, $E_o = 0$ and $E_{2\pi} = I$, and since $e_\psi(\varphi) \leq e_\chi(\varphi)$ for $\psi \leq \chi$, we, have $E_\psi \leq E_\chi$.

Moreover, E_ψ is a function of ψ which is continuous on the right. To see this we consider first , for $0 \le \psi \le 2\pi$, the function

$$e'_\psi(\varphi) = e_\psi(\varphi) + e'_o(\varphi), \text{ where } e'_o(\varphi) = \begin{cases} 1 & \text{if } \varphi = 2k\pi \\ 0 & \text{else where .} \end{cases}$$

These functions are upper semi-continuous ; there fore, we can construct, for each fixed ψ, a sequence of trigonometric polynomials $p_n(e^{i\varphi})$ which decreases to $e'_\psi(\varphi)$, and has the additional property that for n sufficiently large

$$p_n(e^{i\varphi}) \ge e'_{\psi+\frac{1}{n}}(\varphi) .$$

This implies for the corresponding transformation that $E'_{\psi+\frac{1}{n}} \longrightarrow E'_\psi$, hence also that $E_{\psi+\frac{1}{n}} \longrightarrow E_\psi$ $(n \longrightarrow \infty)$, and more generally, that $\lim\limits_{\chi\to\psi+o} E_\chi = E_\psi$.

The transformations E_ψ therefore form a spectral family over the segment $[0,2\pi]$; moreover $E_o = 0$. By its construction, E_ψ is the limit of polynomials in U and in $U^* = U^{-1}$.

To show (8) , we consider a decomposition of the segment $[0,2\pi)$ by means of the points

$$0 = \psi_o < \psi_1 < \ldots < \psi_n = 2\pi$$

such that $\max\,(\psi_k - \psi_{k-1}) \le \epsilon$. We choose an arbitrary point φ_k in each of the intervals $[\psi_{k-1}, \psi_k]$. For $\psi_{h-1} < \varphi \le \psi_h$, we have

$$\left|e^{i\varphi} - \sum_{k=1}^{n} e^{i\varphi_k}\left[e_{\psi_k}(\varphi) - e_{\psi_{k-1}}(\varphi)\right]\right| = \left|e^{i\varphi} - e^{i\varphi_h}\right| \le |\varphi - \varphi_h| \le \epsilon \qquad (*) \text{ and}$$

analogous result for $\varphi = 0$. This is because in the decomposition of the interval $[0,2\pi]$, for $\psi_{h-1} < \varphi \le \psi_h$ we have

$$e_{\psi_k}(\varphi) - e_{\psi_{k-1}}(\varphi) = \begin{cases} 0 & \text{if } k \ne h \\ 1 & \text{if } k = h \end{cases}$$

Then $\sum_{k=1}^{n} e^{i\varphi_k}\left[e_{\psi_k}(\varphi) - e_{\psi_{k-1}}(\varphi)\right] = e^{i\varphi_h}$

and

$$|e^{i\varphi}-e^{i\varphi_h}| = |-i\int_{\varphi}^{\varphi_h} e^{it}dt| \le |-i|\int_{\varphi}^{\varphi_h}|e^{it}|dt = \int_{\varphi}^{\varphi_h}dt = \varphi_h-\varphi$$

But $\varphi_h-\varphi = |\varphi-\varphi_h| \le \epsilon$ and hence (*).
Hence for every value of φ,

$$0 \le \left[e^{i\varphi} - \sum_{k=1}^{n} e^{i\varphi_k}\left[e_{\psi_k}(\varphi) - e_{\psi_{k-1}}(\varphi)\right]\right]\left[e^{i\varphi} - \sum_{k=1}^{n} e^{i\varphi_k}\left[e_{\psi_k}(\varphi) - e_{\psi_{k-1}}(\varphi)\right]\right]$$

$$= \left| e^{i\varphi} - \sum_{k=1}^{n} e^{i\varphi_k}\left(e_{\psi_k}(\varphi) - e_{\psi_{k-1}}(\varphi)\right)\right|^2 \le \epsilon^2$$

passing to the corresponding transformations, it follows that

$$0 \le \left[U - \sum_{k=1}^{n} e^{i\varphi_k}\left[E_{\psi_k} - E_{\psi_{k-1}}\right]\right]^*\left[U - \sum_{k=1}^{n} e^{i\varphi_k}\left[E_{\psi_k} - E_{\psi_{k-1}}\right]\right] \le \epsilon^2 I$$

hence that, $\|U - \sum_{k=1}^{n} e^{i\varphi_k}\left(E_{\psi_k} - E_{\psi_{k-1}}\right)\| \le \epsilon$, which implies by stielties integral, for a finer subdivision of the interval $[0,2\pi]$,

$$U = \int_{0}^{2\pi} e^{i\varphi}dE\varphi .$$

In as much as the projections $E_{\psi_k}-E_{\psi_{k-1}}$ are pairwise orthogonal, we also have , for every integer $r \ge 0$,

$$\sum_{k} e^{ir\varphi_k}\left[E_{\psi_k}-E_{\psi_{k-1}}\right] = \left[\sum_{k} e^{i\varphi_k}\left[E_{\psi_k}-E_{\psi_{k-1}}\right]\right]^r \Rightarrow U^r,$$

$$\sum_{k} e^{-ir\varphi_k}\left[E_{\psi_k}-E_{\psi_{k-1}}\right] = \left[\sum_{k} e^{-i\varphi_k}\left[E_{\psi_k}-E_{\psi_{k-1}}\right]\right]^r \Rightarrow (U^*)^r = U^{-r}$$

Hence, $\int_{0}^{2\pi} e^{in\varphi}dE\varphi = U^n$, $n = 0,\pm1,\pm2,\ldots$

It follows that for every trigonometric polynomial and even for every continuous function $u(e^{i\varphi})$,

$$u(U) = \int_0^{2\pi} u(e^{i\varphi})dE\varphi \ , \tag{9}$$

in the sense of convergence in norm of the sums of stiltjes type. The same formula, interpreted in the sense of weak convergence, is also valid for the other function $u(e^{i\varphi})$ for which the correspondence is established, that is, we have $\langle u(U)f,g\rangle = \int_0^{2\pi} u(e^{i\varphi})d\,\langle E_\varphi f,g\rangle$, where the integral is taken in the stieltjes-lebesque sense. This formula, being a consequence of (7), is valid for an arbitrary spectral family $\{F_\varphi\}$ over $[0,2\pi]$ for which (7) holds. Taking, in particular $u(e^{i\varphi}) = e_\psi(\varphi)$, we obtain $\langle E_\psi f,g\rangle = \int_0^{2\pi} e_\psi(\varphi)\ d\langle F_\varphi f,g\rangle = \int_0^{\psi} d\langle F_\varphi f,g\rangle = \langle (F_\psi - F_0)f,g\rangle$. When we have in addition $F_0 = 0$, it follows that $E_\psi = F_\psi$. This proves the uniquenese of the spectral family corresponding to U.

3.2. Normal Transformations

Definition 11: linear transformations N which are permutable with their adjoints, that is $NN^* = N^*N$ are called normal transformations. symmetric and unitary transformation are particular types of normal transformations.

Every normal transformation N can be written in the form

$$N = X + iY \tag{10}$$

where X and Y are permutable symmetric transformations. We have only to set

$X = \frac{1}{2}(N + N^*)$, $Y = \frac{1}{2i}(N-N^*)$ which are symmetric. it is clear that $\|X\| \le \|N\|$, $\|Y\| \le \|N\|$.

Another type of decomposition, which is less immediate is

$$N = RU = UR \tag{11}$$

where R is a positive symmetric transformation and U is a unitary transformation.

The decomposition (10) is analogous to the decomposition of a complex number into its real and imaginary parts: $z = x+iy$, the decomposition (11) is the analogous of z into the product of its modulus and a factor of unit modulus: $z = re^{i\varphi}$.

Theorem 8: Every normal transformation N of a Hilbert space can be written in the form UR where R is a positive symmetric transformation and U is a unitary transformation and such that U and R permutes with one another and with all linear transformations which permutes with N and N^*.

Proof:- In order to obtain the decomposition $N = RU = UR$, we take for R the positive square root of the positive transformation $NN^* = N^*N$; since R is the limit of a sequence of polynomials in N^*N, it is permutable with N and N^* (by Theorem 3), that is, $RN = NR$, $RN^* = N^*R$. we have for every element f:

$$\|Rf\|^2 = \langle Rf,Rf\rangle = \langle R^2f,f\rangle = \begin{cases} \langle N^*Nf,f\rangle = \langle Nf,Nf\rangle = \|Nf\|^2 \\ \langle NN^*f,f\rangle = \langle N^*f,N^*f\rangle = \|N^*f\|^2 \end{cases}$$

hence

$$\|Nf\| = \|N^*f\| = \|Rf\| \tag{12}$$

We denote by S the subspace of H consisting of elements of the form Rf and of their limits, that is,

$S = \left\{ Rf \text{ and } \lim_n Rf_n \,\middle|\, f, f_n \in H \right\}$, and let M be its orthogonal complement, that is,

$M = S^{\perp} = \{Rh \mid \langle Rh,Rf\rangle = 0,\ \forall\, Rf \in S\}$. M obviously consists of the elements h for which $Rh = 0$ (i.e. $M = \{h \mid Rh = 0\}$, or equivalently, by

(12), those for which $Nh = 0$ or $N^* h = 0$. But the set of elements h such that $N^* h = 0$ is the orthogonal complement of the subspace S′ which consists of the elements of the form Nf and of their limits, that is,

$$S' = \left\{ Nf \text{ and } \lim_n Nfn \,\middle|\, f, fn \in H \right\}$$

consequently $S = S'$

We assign to each element of the form $g = Rf \in S$, the element $Ug = Nf$; the latter is uniquely determined, because if $Rf = Rf'$ we have $R(f-f') = 0$, hence by (12), $N(f-f') = 0$ and $Nf = Nf'$. This correspondence is obviously homogeneous, additive and moreover isometric : $\|Ug\| = \|Nf\| = \|Rf\| = \|g\|$.

We can extend it by continuity to all elements of S and we thus obtain an isometric transformation U of S into itself ; it will even be unitary , because the elements of the form Nf and their limits fill the subspace entirely. We can extend the transformation U to the entire space $H = S+M$ in such away that it remains unitary: we have only to define U in the complementary subspace M by an arbitrary unitary transformation of M into itself (for example by the identity transformation) and then define it in the entire space $H = S+M$ by linearity, that is

$$U: H \longrightarrow H \text{ by } Ug = \begin{cases} Nf = URf & \text{if } g \in S \\ g & \text{if } g \in M \end{cases}$$

The equation $Nf = Ug = URf$ is verified for elements of the form Rf by the very definition of the transformation U. As for the equation $Nf = RUf$, it is obvious for elements f of M, that is $f \in M$

implies $Uf = f$ and $Rf = 0$

implies $Nf = URf = o = RUf$ and it also holds for elements

g of S, since these elements are of the form

$g = \lim_n Rf_n$ and thus we have

$$Ng = N \lim_{n} Rf_n = \lim_{n} NRf_n = \lim_{n} RNf_n = R \lim_{n} Nf_n = R \lim_{n} URf_n =$$

$$= RU \lim_{n} Rf_n = RUg;$$

the equation is therefore true for all elements of H.

Hence, $N = UR = RU$.

We observe further that the transformation R is obviously, permutable with every linear transformation A which is permutable with N and N^* (Theorem 3). The same is true of U, if it is defined to be the identity in the subspace M. Infact, we have on one hand

$$AURf = ANf = NAf = URAf = UARf,$$

hence $AUg = UAg$, for all elements of the form Rf and consequently for all elements of S. On the other hand, for g beloging to M, Ag also belongs to M, since $RAg = ARg = 0$; therfore,

$$AUg = Ag = UAg.$$

Hence $AU = UA$

Remark: We remark that part of this argument is also applicable to the case of an arbitrary linear transformation T instead of a normal transformation N. we can form the positive symmetric transformation $R = (T^*T)^{1/2}$ and we have further $\|Tf\| = \|Rf\|$ for all elements of f, from which it follows as above that the transformation U, defined for elements of the form $g = Rf$ by $Ug = Tf$, is homogeneous, additive and isometric and that it can be extended by continuity to the entire subspace S with the preservation of these properties.

However, the elements Ug will not ingeneral belong to S and we shall not be able to extend U to a unitary transformation of H. But by setting $Ug = 0$ for elements g of the orthogonal complement M, we extend U to a partially isometric transformation of the space H; this is a linear transformation of H which is isometric for the elements of a certain subspace of H and is zero for the elements of the orthogonal complement of subspace.

3.3. The Spectral Decomposition of Normal Transformations

Theorem 9: To every normal transformation N there corresponds a family $\{E(\delta)\}$ of projections such that $E(\delta)$ is an additive and multiplicative rectangle function,

$$N = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} z\, E(dxdy) \quad , \quad N^{*} = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \bar{z}\, E(dxdy)$$

and more generally,

$$q(N,N^{*}) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} q(z,\bar{z})\, E(dxdy),$$

Where $q(z,\bar{z})$ is an arbitrary polynomial in $z = x+iy$ and $\bar{z} = x-iy$; for every fixed rectangle δ, $E(\delta)$ is the limit of a sequence of polynomials in N and N^{*}.

Proof:- From the decomposition $N = X + iY$ a spectral decomposition of the normal transformation N can be obtained in the following manner:

Let $\left\{ E^{X}_{\lambda} \right\}$ and $\left\{ E^{Y}_{\lambda} \right\}$ be spectral families of the symmetric transformations X and Y over the segment $-\|N\| \leq \lambda \leq \|N\|$ (by theorem 6). For every fixed value of x and of y, E^{X}_{x} and E^{Y}_{y} are limits of polynomials in X and in Y respectively (Theorem 6), and consequently the limits of polynomials in N and N^{*}. It follows in particular, that $E^{X}_{x}E^{Y}_{y} = E^{Y}_{y}E^{X}_{x}$ (since $NN^{*} = N^{*}N$). We have

$$N = X + iY = \int_{-\infty}^{\infty} xdE^{X}_{x} \cdot \int_{-\infty}^{\infty} dE^{Y}_{y} + i\int_{-\infty}^{\infty} dE^{X}_{x} \cdot \int_{-\infty}^{\infty} ydE^{Y}_{y}$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} (x+iy)\, dE^{X}_{x}dE^{Y}_{y} \tag{13}$$

in the sense that the sum

$$\sum_{h,k} Z_{hk} \left(E^{X}_{x_h} - E^{X}_{x_{h-1}}\right)\left(E^{Y}_{y_k} - E^{Y}_{y_{k-1}}\right)$$

corresponding to a decomposition of the complex plane into rectangles

$$\delta_{hk} = [x_{h-1} < x \leq x_h,\ y_{k-1} < y \leq y_k]$$

and arbitrary points $Z_{hk} = X_{hk} + iy_{hk}$ of δ_{hk}, converges in norm to the transformation N as the decomposition becomes arbitrarly fine. Since $E^X_x E^Y_y = E^Y_y E^X_x$, the products $E(\delta_{hk}) = \left[E^X_{x_h} - E^X_{x_{h-1}}\right]\left[E^Y_{y_k} - E^Y_{y_{k-1}}\right]$ are also projections ; moreover, they are pairwise orthogonal, and consequently they define a decomposition of the entire space into the vector sum of mutually orthogonal subspaces.

we have, by analogy with (13)

$$N^* = X - iY = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}(x-iy)\, dE^X_x\, dE^Y_y \tag{14}$$

More generally, it follow from the relations

$$X^r Y^s = \int_{-\infty}^{\infty} x^r\, dE^X_x \int_{-\infty}^{\infty} y^s\, dE^Y_y = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x^r y^s\, dE^X_x\, dE^Y_y \quad (r, s = 0, 1, 2, \dots)$$

that $P(X,Y) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} p(x,y)\, dE^X_x\, dE^Y_y$ for every polynomial $p(x,y) = \sum_{r,s} C_{rs} X^r Y^s$ and for the corresponding transformation.

$$P(X,Y) = \sum_{r,s} C_{rs} X^r Y^s$$

Equivalently, we have

$$q(N, N^*) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} q(z,\bar{z})\, dE^X_x\, dE^Y_y \tag{15}$$

for every polynomial

$q(z,\bar{z}) = \sum_{r,s} d_{rs} z^r (\bar{z})^s$ in $z = x+iy$ and $\bar{z} = x-iy$, and for the corresponding transformation

$$q(N, N^*) = \sum_{r,s} d_{rs} N^r (N^*)^s$$

formula (15) obviously includes (13) and (14)

The projections E(δ), which is a function of the variable rectangle δ, is **additive** and **multiplicative** in the sense that for two adisjoint rectangles δ_1, δ_2 whose union is a rectangle we have

$$E(\delta_1) + E(\delta_2) = E(\delta_1 \cup \delta_2),$$

and for two arbitrary rectangles δ_1, δ_2 we have.

$$E(\delta_1).E(\delta_2) = E(\delta_1 \cap \delta_2)$$

where we set the second member equal to 0 if the set $\delta_1 \cap \delta_2$ is empty. If δ includes the closed rectangle

$$\Delta = [m_x \leq x \leq M_X \ , \ m_Y \leq y \leq M_Y]$$

where $m_x, M_X, m_Y . M_Y$ denotes the greatest lower and least upper bounds of X and Y, we have $E(\delta) = I$; consequently $E(\delta) = 0$ if the rectangle δ lies entirely in the exterior of Δ. In all these statements, which are easier to verify, we see a certain advantage in half open rectangles

$$\delta = [x_1 < x \leq x_2, \ y_1 < y \leq y_2],$$

since the intersection of two rectangles of this type in either a rectangle of the same type or empty.

Moreover, it is easy to extend the definition of $E(\delta)$ to rectangles of other types. For example, for an open rectangle $\delta = [x_1 < x \leq x_2, \ y_1 < y \leq y_2]$ we set

$$E(\delta) = \left\{E^X_{x_2 - o} - E^X_{x_1 - o}\right\} \left\{E^Y_{y_2 - o} - E^Y_{y_1}\right\}$$

and for a closed rectangle $\delta = [x_1 \leq x \leq x_2, y_1 \leq y \leq y_2]$ we set

$$E(\delta) = \left\{E^X_{x_2} - E^X_{x_1 - o}\right\} \left\{E^Y_{y_2} - E^Y_{y_1 - o}\right\}.$$

The additive and multiplicative properties remain preserved and the function $E(\delta)$ will even be denumerably additive, or equivalently, it will be continuous in the following sense:

$$\text{if } \delta_1 \subset \delta_2 \subset \delta_3 \subset \ldots, \text{ then } \lim_n E(\delta_n) = E(\bigcup_n \delta_n)$$

$$\text{and if } \delta_1 \supset \delta_2 \supset \delta_3 \supset \ldots, \text{ then } \lim_n E(\delta_n) = E(\bigcap_n \delta_n)$$

In formulas (13)-(15) the rectangles are taken with respect to this additive and multiplicative rectangle function $E(\delta)$, a fact we can conveniently express by writing $E(d_x d_y)$ instead of $dE_x^X\, dE_y^Y$.

Remark:- The domain of integration can be restricted to the rectangle Δ, or rather to the disc $x^2+y^2 = z\bar{z} \le \|N\|^2$, since for every rectangle δ lying in the exterior of the disc and at a distance $\epsilon > 0$ from it we have $E(\delta) = 0$.

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