ADDIS ABABA UNIVERSITY
SCHOOL OF GRADUATE STUDIES
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GRADUATE SEMINAR REPORT

ON

Pólya's Counting

(Submitted in partial fulfillment of M.Sc. Degree in
Mathematics)

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1. Introduction

This seminar report is prepared for the partial fulfillment of M.Sc degree at Addis Ababa University. The seminar report is devoted to Pólya's Counting. In this seminar report we will:-

- Examine a special class of counting problems.
- See the development of a special formula and the illustration of a technique for counting inequivalent colorings in the presence of symmetric and when any physical motion (rotation or reflection) is allowed.
- Also obtain a generating function that gives a pattern inventory of the distinct colorings.

At the beginning the seminar report is about equivalence and symmetric group. Following this we explain how to determine inequivalent colorings using Burnside's Theorem. The last section shows Pólya's Counting formula using generating function, cycle index and Pólya's Theorem.

The motivation in developing formulas for counting distinct colorings when motions are allowed, comes from a problem in chemistry, the enumeration of isomers.

The motivation, derivation and application of Pólya's Theorem has been the primary purpose of this seminar report.

2. Equivalence and symmetric group

Consider the way of coloring the corners of an equilateral triangle. How many different colorings are there, If we have two colors red and blue?

The answer is $2^3 = 8$, these are RRR, RRB, RBR, BRR, BBR, BRB, RBB and BBB since an equilateral triangle has three corners. But should we regard all of the 8 colorings to be different? If the triangle is fixed in space, then each corner is distinguished from the others by its position and it matters which color each corner gets. Thus in this case all 8 colorings are different. Suppose however that rotating the triangle about its center through the angles 0°, 120° and 240° or reflecting the triangle about the lines joining corners and midpoints of opposite sides are possible, then because it is so symmetrical it matters not which corners are colored red and which are colored blue. The only way two colorings can be distinguished from one another is by the number of corners of each color.

Thus there is

1 coloring with all red corners

1 coloring with 2 red corners and 1 blue corner

1 coloring with 2 blue corners and 1 red corner

1 coloring with all blue corners.

A total of 4 different colorings. These are RRR, RRB, BBR and BBB.

Similarly if we color the 4 corners of a square with colors red and blue, we have $2^4 = 16$ different colorings provided the square is regarded as fixed in position. These are RRRR, RRRB, RRBR, RBRR, BRRR, RRBB, RBBR, BRRB, BBRR, BRBR, RBRB, RBBB, BRBB, BBRB, BBBR, and BBBB.

However if we allow the square to rotate about its center through the angles 0°, 90°, 180o and 270° or if we reflect its corners through the lines joining opposite corners or through the lines joining midpoints of opposite sides, the 16 ways to

However if we allow the square to rotate about its center through the angles 0°, 90°, 180o and 270° or if we reflect its corners through the lines joining opposite corners or through the lines joining midpoints of opposite sides, the 16 ways to color its corners are partitioned in to parts in such a way that two colorings in the same part are regarded as the same (the colorings are equivalent) and two colorings in different parts are regarded as different (the colorings are inequivalent).The number of inequivalent colorings is thus the number of different parts. Therefore when we color the 4 corners of a square with red and blue, we have;

1 coloring with all red corners
1 coloring with three red corners
2 colorings with two red corners
1 coloring with three blue corners
1 coloring with no red corners

Giving a total of 6 different (inequivalent) colorings. These **are** RRRR, RRRB, RRBB, RBRB, BBBR **and** BBBB.

The difficulty in determining different colorings (inequivalent colorings) comes from the geometric symmetries of the figure being colored. With a little more work, we can obtain a generating function that gives a pattern inventory of the distinct colorings.

For example the pattern inventory of red-blue colorings of the corners of

- an equilateral triangle and
- a square

when the above rotations and reflections are possible are

$$r^3 + r^2b + rb^2 + b^3 \quad \text{and}$$
$$r^4 + r^3b + 2r^2b^2 + rb^3 + b^4$$

respectively where the coefficient of $r^i b^j$ is the number of non-equivalent colorings with i- red corners and j-blue corners.

From the list of inequivalent colorings or from the pattern inventory of red-blue colorings of the corners of an equilateral triangle and the corners of a square, we can see that there are four different 2- colorings of the triangle and there are six different 2-colorings of the square when they are rotated and reflected as mentioned above. However we seek a theory and a formula to explain why there are four such distinct 2-colorings of an equilateral triangle and six such distinct 2-colorings of a square.

Let X be a finite set, with out loss of generality, we take X to be the set {1, 2, . . ., n} consisting of the first n-positive integers. We denote the set of all n! permutations of {1, 2, 3, . . ., n} by $S_{n.}$. Since permutations are functions they can be combined using composition. That is if f and g are two permutations of {1, . . ., n} then their composition, in the order f followed by g is a permutation.

Definition: A group of permutations of X, for short, a permutation group is defined to be a non-empty subset G of permutations in S_n satisfying the following three properties.

i. Closure under composition: for all permutations f and g in G, fog is also in G

ii. Identity: the identity permutation i of S_n belongs to G.

iii. Closure under inverse: for each permutation f in G the inverse f^{-1} is also in G.

The set S_n of all permutations of X = {1, 2, …, n} is a permutation group, called the symmetric group of order n!.

Let Ω be a geometric figure. A symmetry of Ω is a geometric motion (rotation or reflection) that brings the figure Ω on to it self.

For instance, the symmetries of a square are divided in to two classes rotations - circular motion in the plan and reflections. That is

- 4 rotations about the center of the square through the angles of 0^0, 90^0, 180^0 and 270^0 and

- 4 reflections about the lines joining opposite corners and the lines joining the mid-point of opposite sides.

In a regular n-gon the smallest rotation is $\left(\frac{360}{n}\right)$. There are n-rotations in all.

There are two types of reflections for a regular even n-gon, flipping about two opposite sides and flipping about two opposite corners. Since there are $\frac{n}{2}$ pairs of opposite sides and $\frac{n}{2}$ pairs of opposite corners, a regular even n-gon will have $\frac{n}{2} + \frac{n}{2} = n$ reflections.

Summing rotations and reflections, we find that a regular even n-gon has 2n symmetries.

In a regular odd n-gon there are also n-rotations and n-reflections summing rotations and reflections we have 2n symmetries.

As a result each symmetry acts as a permutation on the corners. A symmetry of Ω followed by another, that is the composition of two symmetries is again a symmetry, and the motion that leaves every thing fixed is the identify symmetry. Hence we conclude that the symmetries of Ω acts as a permutation group G_C on its corners, as a permutation group G_E on its edges. Thus a set of permutations which results by considering all the symmetries of a figure is automatically a corner symmetry group. For example;

- The rotations about the center of a square through the angles 0°, 90°, 180° and 270° acting on the corners of the square give the four permutations.

$$\rho_4^0 = i = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 2 & 3 & 4 \end{pmatrix} \quad , \quad \rho_4^2 = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 4 & 1 & 2 \end{pmatrix}$$

$$\rho_4 = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 4 & 1 \end{pmatrix} \quad , \quad \rho_4^3 = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 1 & 2 & 3 \end{pmatrix}$$

- The reflections of corners through the lines joining opposite corners or through the lines joining midpoints of opposite sides, acting on the corners of the square give the four permutations.

$$T_1 = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 4 & 3 & 2 \end{pmatrix} \quad , \quad T_2 = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 2 & 1 & 4 \end{pmatrix},$$

$$T_3 = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 1 & 4 & 3 \end{pmatrix} \quad , \quad T_4 = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 3 & 2 & 1 \end{pmatrix}$$

Thus the corner symmetry group of a square is

$$G_C = \{ \rho_4^0 = i , \rho_4 , \rho_4^2 , \rho_4^3 , T_1 , T_2 , T_3 , T_4 \}$$

To compute that the edge symmetry group G_E of a square

Let the edges be labeled by a, b, c and d and the corners by 1, 2, 3 and 4. Replacing 1 by a, 2 by b, 3 by c and 4 by d .

For example, in ρ_4^2 we get $\begin{pmatrix} a & b & c & d \\ c & d & a & b \end{pmatrix}$ and in T_2 we get $\begin{pmatrix} a & b & c & d \\ c & b & a & d \end{pmatrix}$.Thus the permutations of the edge forms a group called G_E.

In a similar way we can obtain the symmetry group of a regular n - gon for $n \geq 3$.

Besides the n- rotations $\rho_n^0 = i , \rho_n , \rho_n^2 , \ldots, \rho_n^{n-1}$ we have n - reflections $T_1, T_2, \ldots, T_n$.

The result group $D_n = \{ \rho_n^0 = i , \rho_n , \rho_n^2 , \ldots , \rho_n^{n-1} , T_1 , T_2 , \ldots, T_n \}$ of 2n permutations of {1,2, 3, 4, . . ., n} is an instance of a dihedral group of order 2n.

Example 2.1: (The dihedral group of order 12).

Consider the regular hexagon with its vertices labeled 1, 2, 3, 4, 5 and 6. Its corner symmetry group D_6 contains 6 rotations and 6 reflections.

The 6 rotations are

$$\rho_5^0 = i = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 1 & 2 & 3 & 4 & 5 & 6 \end{pmatrix}, \quad \rho_5 = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 2 & 3 & 4 & 5 & 6 & 1 \end{pmatrix} , \quad \rho_5^2 = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 3 & 4 & 5 & 6 & 1 & 2 \end{pmatrix}$$

$$\rho_5^3 = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 4 & 5 & 6 & 1 & 2 & 3 \end{pmatrix}, \quad \rho_5^4 = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 5 & 6 & 1 & 2 & 3 & 4 \end{pmatrix} , \quad \rho_6^5 = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 6 & 1 & 2 & 3 & 4 & 5 \end{pmatrix}$$

The 6 reflections are

$$T_1 = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 1 & 6 & 5 & 4 & 3 & 2 \end{pmatrix}, \quad T_2 = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 3 & 2 & 1 & 6 & 5 & 4 \end{pmatrix}, \quad T_3 = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 5 & 4 & 3 & 2 & 1 & 6 \end{pmatrix}$$

$$T_4 = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 2 & 1 & 6 & 5 & 4 & 3 \end{pmatrix}, \quad T_5 = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 4 & 3 & 2 & 1 & 6 & 5 \end{pmatrix}, \quad T_6 = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 6 & 5 & 4 & 3 & 2 & 1 \end{pmatrix}$$

Suppose we have a group G of permutations of a set X where X = {1, 2, 3, . . ., n}

- Let C be a collection of colorings of X (a coloring of X is an assignment of a color to each element of X)
- Let c be a coloring of X in which the colors of 1,2,3, . . .,n are denoted by c(1) , c(2), . . ., c(n), respectively.
- Let $f = \begin{pmatrix} 1 & 2 & 3 \ldots k \ldots n \\ i_1 & i_2 & i_3 \ldots i_k \ldots i_n \end{pmatrix}$ be a permutation in G.

Then f * c is defined to be the coloring in which i_k has the color c(k), that is, (f * c) (i_k) = c(k) or using the inverse of f, (f * c) (k) = c(f^{-1}(k))

In words, since f moves k to i_k, the color of k namely c(k) moves to f(k) = i_k and becomes the color of i_k.

The set C of colorings is required to have the property that for all f in G and all c in C, f * c is also in C. This implies that f permutes the colorings in C and thus G acts as a permutation group on the set C of colorings. Hence f * c denotes the coloring in C in to which c is sent by f.

The basic relation ship that holds between the two operations o (composition of permutations in G) and * (action of permutations in G on colorings in C) is

(gof) * (c) = g * (f * c) --------------------------- (1)

-The left hand side of (1) is the coloring in which the color of k moves to (gof) (k)

-The right side is the coloring in which the color of k moves to f(k) and then moves to g(f(k)). Since (gof) (k) = g(f(k), we have (gof) * (c) = g * (f * c)

Definition: Let G be a group of permutation acting on a set $X = \{1, 2, 3, \ldots, n\}$ of the first positive integers. Let C be a collection of colorings of X such that for all f in G and all c in C, the coloring $f * c$ of X is also in C.

Thus G acts on C in the sense that it takes colorings in C to colorings in C. Let c_1 and c_2 be two colorings in C, we define c_1 to be equivalent to (under the action of G) c_2 denoted by $c_1 \sim c_2$ provided there is a permutation f in G such that $f * c_1 = c_2$. **Two colorings are inequivalent provided they ore not equivalent.**

From the definition we have

a. $c \sim c$ for each coloring c ; (because $i * c = c$)

b. If $c_1 \sim c_2$ then $c_2 \sim c_1$

(If $f * c_1 = c_2$ for some f in G then $f^{-1} * c_2 = c_1$)

c. if $c_1 \sim c_2$ and $c_2 \sim c_3$ then $c_1 \sim c_3$ (if $f * c_1 = c_2$ and $g * c_2 = c_3$ then

$(gof) * c_1 = g * (f * c_1) = g * c_2 = c_3$)

Remark: Equivalence partitions the colorings of C in to parts with two colorings being in the same part if and only if they are equivalent.

3. Burnside's Theorem

In this section we derive and apply Burnside's formula for counting the number of inequivalent colorings of a set X under the action of a group of permutations of X. Let G be a group of permutations of X and let C be a set of colorings of X such that G acts on C. This means f * c is in C for all f in G and all c in C, and each f in G permutes the colorings in C. It is possible that for an appropriate choice of f and c, we have

$$f * c = c$$

If we allow either f to vary over all permutations in G or all c to vary over all colorings in C, then we get the following.

- G(c) = { f : f in G , f * c = c } the set of all permutations in G which fix the coloring c, and
- C(f) = { c: c in C, f * c = c} the set of all colorings in C which are fixed by f.

The set G(c) of all permutations that fix the coloring c is called the stabilizer of c.

Theorem 3:1 For each coloring c, the stabilizer G(c) of c is a permutation group. Moreover for any permutations f and g in G, g * c = f * c if and only if $f^{-1} \circ g$ is in G(c).

Proof: let f and g both fix c then (gof)(c) = g(f(c)) = g(c) = c

$\Rightarrow$ gof is in G(c)

Therefore G(c) is closed under composition.

Since the identify i fixes every coloring, i fixes c thus i is in G(c)

Let f fixes c, since $(f^{-1} \circ f) * (c) = f^{-1} * (f * c)$ we have $i * c = f^{-1} * c$

$$\Rightarrow c = f^{-1} * c$$

$$\Rightarrow f^{-1} \text{ is in } G(c)$$

Hence G(c) is closed under inverse.

Therefore G(c) is a permutation group.

Suppose that $f * c = g * c$

By the relation $(g o f) * (c) = g * (f * c)$

We get $(f^{-1} o\, g) * c = f^{-1} * (g * c)$

$$= f^{-1} * (f * c)$$

$$= (f^{-1} o\, f) * c$$

$$= i * c$$

$$= c$$

Therefore $f^{-1} o\, g$ fixes c and hence $f^{-1} o\, g$ is in G(c)

Conversely suppose that $f^{-1} o\, g$ is in G(c)

$$\Rightarrow (f^{-1} o\, g) * c = c$$

$$\Rightarrow (f^{-1} o\, g) * c = i * c$$

$$\Rightarrow (f^{-1} o\, g) * c = (f^{-1} o\, f) * c$$

$$\Rightarrow f^{-1} * (g * c) = f^{-1} * (f * c)$$

$$\Rightarrow g * c = f * c.$$

Hence $g * c = f * c$ if and only if $f^{-1} o\, g$ is in G(c)

Corollary 3:1 Let c be a coloring in C then the number $|\{ f * c : f \text{ in } G \}|$ of colorings that are equivalent to c equals the number

$$\frac{|G|}{|G(c)|}$$

obtained by dividing the number of permutations in G by the number of permutations in the stabilizer of c.

Proof: let f and g be permutations in G such that g satisfies

$$g * c = f * c.$$

By theorem 3:1 , $f^{-1} \circ g$ is in G(c)

$$\Rightarrow \quad f^{-1} \circ g = h \text{ for some h in G(c)}$$

$$\Rightarrow \quad f \circ (f^{-1} \circ g) = f \circ h$$

$$\Rightarrow \quad g = f \circ h$$

$\because$ g is a permutation in { f o h: h is in G(c) }

If $f \circ h = f \circ h_1$ then $h = h_1$ by cancellation low

$\Rightarrow$ g is uniquely determined.

Hence the number of permutations in the set { f o h: h is in G(c) } equals the number $| G(c) |$.

Thus for each permutation f there are exactly $| G(c) |$ permutations that have the same effect on c as f.

Since there are $| G |$ permutations altogether, the number $| \{ f * c : f \text{ in } G \} |$ of colorings equivalent to c equals

$$\frac{| G |}{| G(c) |}$$

The following theorem of Burnside gives a formula for counting the number of inequivalent colorings.

Theorem 3:2 Let G be a group of permutations of X and let C be a set of colorings of X such that f * c is in C for all f in G and all c in C. Then the number N (G , C) of inequivalent colorings in C is given by

$$N(G, C) = \frac{1}{| G |} \sum_{f \text{ in } G} | C(f) |$$

Proof: The proof is depending on counting the number of pairs (f , c) such that f fixes c, that is such that f * c = c, in two different ways and then equating counts. One we to count is to consider each f in G and compute the number of colorings that f fixes and then add up all quantities. Since C(f) is the set of colorings which are fixed by f, counting in this way we get $\sum_{f \text{ in } G} | \, C(f) \, |$ ----------- (1)

Another way two count is to consider each c in C and compute the number of permutations f such that f * c = c and then add up all the quantities. Since the set of all f such that f * c = c is the stabilizer G(c) of c, each c contributes $| \, G(c) \, |$ to the sum

By corollary 3:1 , $| \, G(c) \, | = \dfrac{| \, G \, |}{\textit{the number of colorings equivalent to c}}$ ------------- (2)

Hence counting in the way we get

$$\sum_{c \text{ in } C} \frac{| \, G \, |}{\textit{the number of colorings equivalent to c}} \quad \text{------------------------------(3)}$$

But the sum in (3) can be simplified if we group the colorings by equivalence classes. Two colorings in the same equivalence class contribute the same amount (2) to this sum and hence the total contribution of every equivalence class is $| \, G \, |$. Since the number of equivalence classes is the number $N(G, C)$ of ineqivalent colorings, thus (3) equals $N(G, C) \times | \, G \, |$ ------------------------------- (4)

Equating (1) and (4) we get

$$\sum_{f \text{ in } G} | \, C(f) \, | \quad = \quad N(G, C) \times | \, G \, |$$

4. Pólya's Counting Formula

Using Burnside's Theorem to count the number of inequivalent colorings in the presence of a permutation group G acting on set C of colorings is dependent on being able to compute the number $|C(f)|$ of colorings in C fixed by a permutation f in G. This computation can be facilitated by consideration of the **cyclic structure of a permutation.**

Let $f = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ 3 & 1 & 9 & 7 & 6 & 5 & 4 & 8 & 2 \end{pmatrix}$ be permutation of $\{1,2,3,4,5,6,7,8,9\}$.Then the following are partitions of the permutation in to directed cycles

$$1 \to 3 \to 9 \to 2 \to 1 \ , \quad 4 \to 7 \to 4 \ , \quad 5 \to 6 \to 5 \ , \quad 8 \to 8$$

Let us write

[1 3 9 2] for the permutation of {1, 2,. . ., 9} which sends 1 to 3, 3 to 9, 9 to 2, 2 to 1, and fixes the remaining integers. Thus

$$[1\ 3\ 9\ 2] = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ 3 & 1 & 9 & 4 & 5 & 6 & 7 & 8 & 2 \end{pmatrix}$$

We call such a permutation, in which certain of the elements are permuted in a cycle and the remaining elements, if any, are fixed, a cycle permutation more briefly a cycle. If the number of elements in the cycle is k, then we call it is a k cycle. Thus [1 6 3 5] is a 4 cycle. The other directed cycles of the permutation

$f = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ 3 & 1 & 9 & 7 & 6 & 5 & 4 & 8 & 2 \end{pmatrix}$ are [4 7] , [5 6] and [8]

Since each integer in the permutation f occurs in exactly one of the cycles in the factorization, it is easy to check that the partition of the permutation f in to cycles corresponds to a factorization (with respect to the composition $\circ$) of f in to permutation cycles. Thus we can write f interms of its cycle factorization as

$$f = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ 3 & 1 & 9 & 7 & 6 & 5 & 4 & 8 & 2 \end{pmatrix} = [1\ 3\ 9\ 2] \circ [4\ 7] \circ [5\ 6] \circ [8]$$

In the factorization of the above permutation:

- It doesn't matter in which order we write the cycles because each element occurs in exactly one cycle.
- The 1-cycle [8] is just the identity permutation and thus could be omitted in the factorization with out affecting its validity. But for counting problem it is useful to include all 1-cycles.

Let f be any permutation of the set X. Then with respect to the operation of composition, f has a factorization.

$f = (i_1, i_2, \ldots, i_p) \circ (j_1, j_2, \ldots, j_q) \circ, \ldots, \circ (m_1, m_2, \ldots, m_r)$ ----------------------- (1)

in to cycles where each integer in X occurs in exactly one of the cycles.

We call (1) the cycle factorization of f. The cycle factorization of f is unique apart from the order in which the cycles appear and this order is arbitrary.

Example 4.1: Determine the cycle factorization of each permutation in the dihedral group D_6 of order 12 (the corner symmetry group of a regular hexagon).The permutations in D_6 were computed in page 6 and 7.The cycle factorization of each permutation in D_6 is given by the table below.

D_6	Cycle factorization
ρ_6^0	$[1]o[2]o[3]o[4]o[5]o[6]$
ρ_6	$[1\ 2\ 3\ 4\ 5\ 6]$
ρ_6^2	$[1\ 3\ 5]o[2\ 4\ 6]$
ρ_6^3	$[1\ 4]o[2\ 5]o[3\ 6]$
ρ_6^4	$[1\ 5\ 3]o[2\ 4\ 6]$
ρ_6^5	$[1\ 6\ 5\ 4\ 3\ 2]$
T_1	$[1]o[2\ 6]o[3\ 5]o[4]$
T_2	$[1\ 3]o[2]o[4\ 6]o[5]$
T_3	$[1\ 5]o[24]o[3]o[6]$
T_4	$[1\ 2]o[3\ 6]o[4\ 5]$
T_5	$[1\ 4]o[2\ 3]o[5\ 6]$
T_6	$[1\ 6]o[2\ 5]o[3\ 4]$

Notice that in the cycle factorization of dihedral group of D_6 of order 12, for the

- identity permutation i, all cycles are 1-cycles, this shows that identity permutation fixes all elements.
- Rotations ρ_6 and ρ_6^5 one 6-cycle occur since the rotations are about 60° and 300° respectively.
- Rotations ρ_6^2 and ρ_6^4 two 3-cycles occur since each of the rotations are about 120° and 240° respectively.
- reflections T_1, T_2 and T_3 two 1-cycles and two 2-cycles occur since each of these reflections is about a line joining two opposite corners of the hexagon and these corners are thus fixed.

- reflections T_4, T_5 and T_6 and for the rotation ρ_6^3 three 2-cycles occur since the reflections are about the line joining the mid point of opposite sides and the rotation is about 180°

The reflections in the corner symmetry group of a regular n-gon with

- n-even behave similarly, that is half of them have two 1- cycle and half of them have two 2-cycles
- n-odd each reflection has one 1-cycle since each such reflection is about a line joining a corner to the mid point of the opposite sides and hence only the one corner is fixed and also each reflection has $\frac{n-1}{2}$ 2-cycles

Let us examine the importance of cycle decomposition in counting inequivalent colorings using the following examples.

Example 4.2: Let f be the permutation of $X = \{1,2,\ldots,7\}$ defined by

$$f = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 3 & 2 & 7 & 4 & 6 & 5 & 1 \end{pmatrix}$$

The cycle factorization of f is f = [1 3 7] o [2] o [4] o [5 6].

Suppose that we color the elements of X with the colors; red, white, blue, green and yellow, and C be the set of all such colorings. How many colorings in C are left fixed by f?

Solution: Let c be a coloring such that f * c = c.

First consider the 3 cycle [1 3 7]. This 3 cycle moves the color of

1 to 3 , 3 to 7 , 7 to 1

Since the coloring c is fixed by f, we see that color of 1 = color of 3 = color of 7 = color of 1. This means that 1, 3, and 7 have the same color and 5 and 6 also have the same color and no restriction placed on 2 and 4 since they belong to 1- cycle. Pick any one of the five colors; red, white, blue green and yellow, for { 1 3 7 } (5 choices) for {5 6} (5 choices), for { 2 } (5 choices) and for {4} (5 choices). Thus $5^4 = 625$ colorings are fixed by f. That is $|C(f)| = 625$

Note that the exponent 4 in the answer is **the number of cycles of f**, in its cycle factorization and the answer is independent of the sizes of the cycles.

Theorem 4:1 Let f be a permutation of a set X. Suppose we have k colors available with which to color the elements of X. Let C be the set of all colorings of X. then the number $|C(f)|$ of colorings of C that are fixed by f equals $k^{\#f}$ where #f is the number of cycles in the cycle factorization of a permutation f.

Proof: Let c is in C such that c is fixed by f.

Let $f = [\, i_1\, i_2 \dots i_p\,] \circ [\, j_1\, j_2 \dots j_q\,] \circ \dots \circ [\, m_1\, m_2 \dots m_r\,]$ be the cycle factorization of f .

Let k = the number of colors available to color.

Since f fixes the coloring c

Color of i_1 = color of i_2 = … = color of i_p

Color of j_1 = color of j_2 = … = color of j_q

. . .

. . .

. . .

Color of m_1 = color of m_2 = … = color of m_r

Pick any one of the k colors

For $\{ i_1, i_2, \dots, i_p \}$ k choices

$\{ j_1, j_2, \dots, j_q \}$ k choice

. . .

. . .

. . .

$\{ m_1, m_2, \ldots, m_r \}$ k choices

Thus a total of $k^{\text{number of cycles}} = k^{\#f}$ colorings are fixed by f.

Hence $| C(f) | = k^{\#f}$

Example 4.3: How many inequivalent ways are there to color the corners of a regular heptagon with the colors; red, white and blue?

Solution: Let C = the set of all $3^7 = 2187$ colorings of the corners of heptagon with colors; red, white and blue. The corner symmetry group of a regular heptagon is the dihedral group D_7. For f in D_7, the following table shows cycle factorization of f, #f and $| C(f) |$

f in D_7	Cycle factorization	#f	$\vert C(f) \vert$
ρ_7^0	[1]0[2]0[3]0[4[5]*o*[6]*o*[7]	7	$3^7 = 2187$
ρ_7	[1 2 3 4 5 6 7]	1	$3^1 = 3$
ρ_7^2	[1 3 5 7 2 4 61]	1	3
ρ_7^3	[1 4 7 3 6 2]	1	3
ρ_7^4	[1 5 2 6 3 7 4]	1	3
ρ_7^5	[1 6 4 2 7 5 3]	1	3
ρ_7^6	[1 7 6 5 4 3 2]	1	3
T_1	[1]*o*[2 7]*o*[3 6]*o*[45]	4	$3^4 = 81$
T_2	[2]*o*[13]*o*[4 7]*o*[56]	4	81
T_3	[3]*o*[1 5]*o*[2 4]*o*[6 7]	4	81
T_4	[4]*o*[1 7]*o*[2 6]*o*[3 5]	4	81
T_5	[5]*o*[12]*o*[3 7]*o*[4 6]	4	81
T_6	[6]*o*[1 4]*o*[2 3]*o*[5 7]	4	81
T_7	[7]*o*[1 6]*o*[2 5]*o*[3 4]	4	81

$$\text{Hence } N(D_7, C) = \frac{1}{| D_7 |} \sum_{f \text{ in } D_7} | C(f) |$$

$$= \frac{1}{14} [2187 + 6(3) + 7(81)]$$

$$= \frac{1}{14}(2187 + 18 + 567)$$

$$= \frac{1}{14}(2772)$$

$$= 198$$

Theorem 3:2 and Theorem 4:1 gives as a method to compute the number of inequivalent colorings in the presence of a group G of permutations of a set X. This method requires that we be able to compute the cycle factorization (or at least the number of cycles in the cycle factorization) of each permutation in G. In order to be able to compute the number of inequivalent colorings for more general sets C of colorings, we introduce a generating function for the number of permutations in G whose cycle factorizations have the same number of cycles of each size.

4.1 Generating function for number of permutation

Let f be a permutation of X where X has n-elements. Suppose that the cycle factorization of f has e_1 1-cycles, e_2 2-cycle,… and e_n n-cycles. Since each element of X occurs in exactly one cycle in the cycle factorization of f, the numbers $e_1, e_2, \ldots, e_n$ are non negative integers satisfying

$$1e_1 + 2e_2 + \ldots + ne_n = n$$

We call the n - tuple $(e_1, e_2, \ldots, e_n)$ the type of the permutation f and write

$$\text{type}(f) = (e_1, e_2, \ldots, e_n)$$

Note that the number of cycles in the cycle factorization of f is

$$\#f = e_1 + e_2 + \ldots + e_n.$$

Different permutations may have the same type since the type of a permutation depends only on the size of the cycles in its cycle factorization and not on which elements are in which cycles. Since we now want to distinguish permutations only by type, we introduce n - indeterminates $z_1, z_2, \dots, z_n$ where z_k is to correspond to a k-cycle (k =1, 2, … , n).

To each permutation f with type(f) = ($e_1, e_2, \dots, e_n$), we associate the monomial of f (cycle structure term).

Definition: For f in G the monomial (cycle structure term) of f is a product of the variables z_i, with one copy of z_i for each cycle of length i in f (including 1-cycles). That is

$$\text{mon}(f) = z_1^{e_1} z_2^{e_2} \dots z_n^{e_n}$$

The total degree of mon(f) is the number #f of cycles in the cycle factorization of f. Let G be a group of permutation of X, summing these monomials for each f in G we get the generating function for the number of permutations.

$$\sum_{f \text{ in } G} mon(f) = \sum_{f \text{ in } G} z_1^{e_1} z_2^{e_2} z_3^{e_3} \dots z_n^{e_n} \quad \text{------------------} \quad (1)$$

If we combine like terms in (1), the coefficient of $z_1^{e_1} z_2^{e_2} \dots z_n^{e_n}$ equals the number of permutations in G of type ($e_1, e_2, \dots, e_n$)

4.2 Cycle index and pattern inventory

The cycle structure of permutations played an important role in counting problems. Our goal in this section is to find the Cycle index and pattern inventory for the number of different ways to k-colors of a set X acted on by a group G. The cycle index polynomial P_G of a permutation group G is the sum of the monomials (cycle structure terms) of f in G, divided by the order of G:

Thus the cycle index $P_G(z_1, z_2, \dots, z_n) = \dfrac{1}{|G|} \sum_{f \text{ in } G} z_1^{e_1} z_2^{e_2} z_3^{e_3} \dots z_n^{e_n}$

Definition: The pattern inventory of the k-colorings of a set X acted on by a group G is the generating function for the number of inequivalent k-colorings of X. That is, the pattern inventory is a polynomial I($u_1, u_2, \ldots, u_k$) (representing colors 1 to k) such that the coefficient on $u_1^{p1}\, u_2^{p2} \cdots u_k^{pk}$ is the number of colorings of X with p_1 elements of color u_1, p_2 elements of color $u_2, \ldots, p_k$ elements of color u_k etc. Note that each term must satisfy $p_1 + p_2 + \cdots + p_k = |X|$.

Example 4.4: Determine the cycle index of the dihedral group D_6

Solution : Using the cycle factorization of each permutation in D_6 on page 15, we give the type of each permutation and its associated monomial in the table below.

D_6	Cycle factorization	$\neq f$	type	monomial
$\rho_6^0 = i$	$[1]o[2]o[3]o[4]o[5]o[6]$	6	$(6,0,0,0,0,0)$	z_1^6
ρ_6	$[1\ 2\ 3\ 4\ 5\ 6]$	1	$(0,0,0,0,0,1)$	z_6
ρ_6^2	$[1\ 3\ 5]o[2\ 4\ 6]$	2	$(0,0,2,0,0,0)$	z_3^2
ρ_6^3	$[1\ 4]o[2\ 5]o[3\ 6]$	3	$(0,3,0,0,0,0)$	z_2^3
ρ_6^4	$[1\ 5\ 3]o[2\ 4\ 6]$	2	$(0,0,2,0,0,0)$	z_3^2
ρ_6^5	$[1\ 6\ 5\ 4\ 3\ 2]$	1	$(0,0,0,0,0,1)$	z_6
T_1	$[1]o[2\ 6]o[3\ 5]o[4]$	4	$(2,2,0,0,0,0)$	$z_1^2\, z_2^2$
T_2	$[1\ 3]o[2]o[4\ 6]o[5]$	4	$(2,2,0,0,0,0)$	$z_1^2\, z_2^2$
T_3	$[1\ 5]o[24]o[3]o[6]$	4	$(2,2,0,0,0,0)$	$z_1^2\, z_2^2$
T_4	$[1\ 2]o[3\ 6]o[4\ 5]$	3	$(0,3,0,0,0,0)$	z_2^3
T_5	$[1\ 4]o[2\ 3]o[5\ 6]$	3	$(0,3,0,0,0,0)$	z_2^3
T_6	$[1\ 6]o[2\ 5]o[3\ 4]$	3	$(0,3,0,0,0,0)$	z_2^3

The cycle index of D_6 = P_{D_6} ($z_1, z_2, z_3, z_4, z_5, z_6$)

$$= \frac{1}{12}\,(z_1^6 + 4z_2^3 + 2z_3^2 + 3z_1^2\, z_2^2 + 2z_6)$$

We can now determine the number of inequivalent colorings among all the colorings of a set X using a specified set of colors provided we know the cycle index of the group G of permutations of X.

Theorem 4:2: let X be a set of n - elements and suppose we have a set of k colors available with which to color the elements of X. Let C be the set of all k^n colorings of X. Let G be a group of permutations of X. Then the number of inequivalent colorings is the number.

$N(G, C) = P_G(k, k, \dots, k)$ obtained by substituting $z_i = k$ $(i = 1,2,\dots, n)$ in the cycle index of G.

Proof:- The cycle index of G is the average of the sum of the monomials associated with the permutation f in G. That is

$$P_G(z_1,z_2,\dots,z_n) = \frac{1}{|G|}\sum z_1^{e_1} z_2^{e_2} \dots z_n^{e_n}$$

By Theorem 4:1 , the number of colorings in C which are fixed by f equals

$$k^{\#f} = k^{e_1+e_2+\dots+e_n} = k^{e_1} k^{e_2} \dots k^{e_n}$$

where $(e_1, e_2, \dots, e_n) = \text{type}(f)$

By theorem 3:2 , the number of inequivalent colorings is

$$N(G, C) = \frac{1}{|G|} \sum_{f \text{ in } G} k^{e_1} k^{e_2} \dots k^{e_n}$$

$$= P_G(k, k, \dots, k)$$

Example 4.5: What is the number of inequivalent ways to color the corners of a regular hexagon if we are given a set of k colors.

Solution: The cycle index of the dihedral group D_6 is

$$P_{D_6}\ (z_1 ,z_2, z_3, z_4, z_5, z_6) = \frac{1}{12}\ (z_1^6 + 4 z_2^3 + 2 z_3^2 + 3 z_1^2 z_2^2 + 2 z_6)$$

By theorem 4: 2, the number of inequivalent colorings

$$P_{D_6}\ (k, k, k, k, k, k) = \frac{k^6 + 4k^3 + 2k^2 + 3k^4 + 2k}{12}$$

$$\text{If } k = 2,\quad P_{D_6}\ (2,2,2,2\ 2,2) = \frac{2^6 + 4.2^3 + 2.2^2 + 3 .2^4 + 2.2}{12} = 13$$

$$\text{If } k = 6\ ,\quad P_{D_6}\ (6,6,6,6,6,6) = \frac{6^6 + 4.6^3 + 2.6^2 + 3 .6^4 + 2.6}{12} = 4291$$

The formula in Theorem 4:2 gives a satisfactory way to count the number of inequivalent colorings in C provided C is the set of all colorings possible with k given colors. However the formula requires one to know the number of permutations of each type in the group G of permutations, and so can be difficult to apply. But it is as simple as one could expect given the fact that G can be any permutation group on the set X of objects being colored.

In Theorem 3:2 the only restriction on C is that G acts as a permutation group on C, that is, each permutation f in G takes a coloring c of C to another coloring f * c of C. Under these more general circumstances we have same formal way to determine the inequivalent colorings, and we show how the cycle index of G can be used to determine the number of inequivalent colorings where the number of **times each color is used is specified.**

Let C be the set of all colorings of X in which the number of elements in X of each color have been specified. **For each permutation f of X and each coloring c in C, the number of times a particular color appears in C is the same as the number of times that color appears in f * c.**

Put another way, permuting the objects in X along with their colors doesn't change the number of colors of each kind.

This means that any group G of permutations of X acts as a permutation group on such a set of colorings C.

Example 4.6: How many inequivalent colorings are there for the corners of a regular 6-gon in which four corners are colored red and two are colored blue?

Solution: Let C be the set of all colorings of the corners of a 6-gon with four corners colored red and two colored blue.

$\Rightarrow$ the number of colorings in C = C(6,4) = C(6,2) = 15 .

The cycle factorization of each permutation in G along with the number of colorings in C fixed by the permutation in D_6 is given by

D_6	cycle factorization	Number of fixed colorings
ρ_6^0	$[1]o[2]o[3]o[4]o[5]o[6]$	15
ρ_6	$[1\ 2\ 3\ 4\ 5\ 6]$	0
ρ_6^2	$[1\ 3\ 5]o[2\ 4\ 6]$	0
ρ_6^3	$[1\ 4]o[2\ 5]o[3\ 6]$	3
ρ_6^4	$[1\ 5\ 3]o[2\ 4\ 6]$	0
ρ_6^5	$[1\ 6\ 5\ 4\ 3\ 2]$	0
T_1	$[1]o[2\ 6]o[3\ 5]o[4]$	3
T_2	$[1\ 3]o[2]o[4\ 6]o[5]$	3
T_3	$[1\ 5]o[2\ 4]o[3]o[6]$	3
T_4	$[1\ 2]o[3\ 6]o[4\ 5]$	3
T_5	$[1\ 4]o[2\ 3]o[5\ 6]$	3
T_6	$[1\ 6]o[2\ 5]o[3\ 4]$	3

The reason that none of the rotations different from the identity and ρ_6^3 fixes any coloring is that for ρ_6 and ρ_6^5 to fix a coloring, all colors in the coloring must be the same (and so we don't have four red and two blue colors as specified). For ρ_6^2 and ρ_6^4 to fix a coloring, three of the colors in the coloring must be the same (and we don't have four red and two blue colors as specified). Each reflection fixes three colorings in C. This is because for the 6-gon, three of the reflections have type (2,2,0,0.0) and the other three reflections have type (0,3,0,0,0,0). In order to have two blue corners in a fixed coloring, we must color blue the corners in one of the two 2-cycles or two of 1-cycles for those with type (2,2,0,0,0,0) and we must color blue in one of the three 2-cycles for those with type (0,3,0,0,0,0) in the factorization. Thus by Theorem 3:2

$$N(G,D_6) = \frac{15 + 7(3)}{12} = 3$$

In order to apply Burnside's Theorem to determine the number of inequivalent colorings when the number of occurrences of each color is specified, we must be able to determine the number of such colorings fixed by a permutation

Let f be a permutation of the set X and suppose that

$$\text{Type}(f) = (e_1, e_2, \ldots, e_n) \text{ and}$$

$$\text{Mon}(f) = z_1^{e_1} z_2^{e_2} \ldots z_n^{e_n}$$

Thus f has e_1 1-cycles, e_2 2-cycles,..., e_n n-cycle in the cycle factorization.

Suppose we have only two colors red and blue, Let $C_{p,q}$ denotes, the set of all colorings of X with p element colored red and $q = n - p$ elements colored blue. **A coloring in $C_{p,q}$ is fixed by f if and only if for each cycle in the cycle factorization of f all elements have the same color.**

Thus to determine the number of colorings in $C_{p,q}$ fixed by f, we can think of assigning colors to a cycle in such a way that the number of elements that get assigned the color red is p (and hence the number assigned the color blue is $q = n - p$).

Suppose that t_1 of the 1-cycles get assigned red, t_2 of the 2-cycles get red,. . . and t_n of the n-cycles get red, we have

$$P = t_1 1 + t_2 2 + \ldots + t_n n$$

Hence the number $\left| C_{p,q}(f) \right|$ of colorings in $C_{p,q}$ which are fixed by f equals the number of the solutions of

$P = t_1\,1 + t_2\,2 + \ldots + t_n\,n$ in integers $t_1, t_2, \ldots, t_n$ **satisfying $0 \le t_i \le e_i$ (i =1,…, n)**

Now consider the color red as a variable r and the color blue as a variable b which we can manipulate algebraically in the usual way. Then the number of solution of $P = t_1\,1 + t_2\,2 + \ldots + t_n\,n$, satisfying $0 \le t_i \le e_i$ (i = 1, 2,…, n) is the coefficient of $r^p\, b^q$ in the expansion of

$(r+b)^{e_1}\,(r^2+b^2)^{e_2}\,(r^3+b^3)^{e_3} \ldots (r^n+b^n)^{e_n}$ obtained by making the substitutions $z_1 = r + b$, $z_2 = r^2 + b^2, \ldots, z_n = r^n + b^n$ in the monomial of f

Thus the two variables:

cycle index is $P_G(r + b, r^2 + b^2, \ldots\ r^n + b^n)$ and

Generating function is $\sum (r+b)^{e_1} + (r^2+b^2)^{e_2} \ldots (r^n+b^n)^{e_n}$

Hence the number of inequivalent colorings in $C_{p,q}$ equals the coefficient of $r^p\, b^q$ in the expansion

$$P_G(r+b, r^2+b^2, \ldots, r^n+b^n) = \frac{1}{|G|} \sum (r+b)^{e_1} + (r^2+b^2)^{e_2} \ldots (r^n+b^n)^{e_n}$$

4.3. Pólya's Theorem

The next theorem commonly called Pólya's Theorem generalizes the way how we determine the number of inequivalent colorings for k number of colors. The motivation, derivation and application of the theorem has been the primary purpose of this seminar report.

Theorem 4:3 (Pólya's Theorem)

Let X be a set of elements. Let G be a group of permutation of X. Let (u_1,u_2,… u_k) be a set of k colors and let C be any set of colorings of X with the property that G acts as a permutation group on C, then the generating function for the number of inequivalent colorings of C according to the number of colors of each kind is the expression

$$P_G\,(u_1 + u_2 + ... + u_k\,,\, u_1^2 + u_2^2 + ... + u_k^2\,, ...,\, u_1^n + u_2^n \;+ ... + u_k^n\,) \text{ ----------------(1)}$$

obtained from the cycle index P_G (z_1, z_2, … z_k) by making the substitution $z_j = u_1^j + u_2^j + ... + u_k^j)$ (j = 1, 2, 3, … , n)

In other words the coefficient of $u_1^{p_1}\ u_2^{p_2} ...\ u_k^{p_k}$ in (1) equals the number of inequivalent colorings in C **with p_i elements of X colored u_i (i = 1, 2, …, k)** Substituting u_i = 1 for i = 1,2,…,k in (1) we get the sum of its coefficients and hence the total number of inequivalent colorings of X with k available colors. Since the substitution yields P_G (k, k, …, k) it follows that Theorem 4:3 (Pólya's Theorem) is a refinement of Theorem 4:2. Theorem 4:3 contains more detailed information than Theorem 4:2 which coincides with Theorem 4:2 if each u_i is replaced with 1

Example 4.7: Determine the generating function for inequivalent coloring of the corners of a rectangle which is not a square with k colors.

Solution: The corner symmetry group G_C of a rectangle contains, the identity, a rotation ρ by 180° about the center of the rectangle and two reflections $T_1\, and\, T_2$ about the lines joining the midpoints of opposite sides. That is,

$G_C = \{i, \rho, T_1, T_2\}$

If we label the corners of a rectangle by 1, 2, 3 and 4, then the cycle factorization, the types and the monomials of the permutations in G_C can be given as follows

f in G$_C$	cycle factorization	type	monomial
i	$[1]o[2]o[3]o[4]$	$(4,0,0,0)$	z_1^4
ρ	$[1\ 3]o[2\ 4]$	$(0,2,0,0)$	z_2^2
T_1	$[1\ 2]o[3\ 4]$	$(0,2,0,0)$	z_2^2
T_2	$[1 4]o[2\ 3]$	$(0,2,0,0)$	z_2^2

Generating function for number of permutations = $z_1^4 + 3\ z_2^2$

Generating function with k colors is $\frac{1}{4}(\ k^4 + 3k^2)$

Cycle index $p(z_1,z_2.z_3.z_4) = \frac{1}{4}\ (z_1^4 + 3\ z_2^2)$

Hence there are $\frac{1}{4}(k^4 + 3k^2)$ different ways to color the corners of a rectangle with k colors.

Suppose we have two colors; red r and blue b

Generating function for two colorings $= \frac{1}{4}(r + b)^4\ +\ 3(r^2 + b^2)^2$

$= r^4 + r^3b + 3r^2b^2 + rb^3 + b^4$ (pattern Inventory)

Thus there are

- one inequivalent coloring with all red
- one inequivalent coloring with all blue
- one inequivalent coloring with 3 corners red and 1 corner blue
- one inequivalent coloring with 3 corners blue and 1 corner red
- three inequivalent colorings with 2 corners red and 2 corners blue

The total number of inequivalent colorings, the sum of the coefficient is 7

Example 4.8: Benzene is a chemical with the formula C_6H_6. The six carbon atoms are arranged in a ring and all are equivalent. How many possible structures are there if 2-Chlorine and 2-Bromines are added to make $C_6H_2Cl_2Br_2$

Solution: The problem is similar to that of coloring the corners of a regular hexagon since the carbon atoms form a regular hexagon shape.

The index P_{D_6} of D_6 is given by

$$P_{D_6}(z_1, z_2, z_3, z_4, z_5, z_6) = \frac{1}{12}(z_1^6 + 3z_1^2 z_2^2 + 4z_2^3 + 2z_3^2 + 2z_6)$$

Determine the coefficient of $h^2c^2b^2$ in

$$\frac{1}{12}[(h+c+b)^6 + 3(h+c+b)^2(h^2+c^2+b^2)^2 + 4(h^2+c^2+b^2)^3$$
$$+2(h^3+c^3+b^3)^2 + 2(h^6+c^6+b^6)]$$

Thus coefficient of $h^2c^2b^2$ in :-

- $(h+c+b)^6$ is $\frac{6!}{2!2!2!} = 6 \times 5 \times 3 = 90$
- $3(h+c+b)^2(h^2+c^2+b^2)^2 =$

  $$3\left(\frac{2!}{2!0!0!}\cdot\frac{2!}{0!1!1!} + \frac{2!}{0!2!0!}\,\frac{2!}{1!0!1!} + \frac{2!}{0!0!2!}\,\frac{2!}{1!1!0!}\right) = 18$$

- $4(h^2+c^2+b^2)^3 = 4\left(\frac{3!}{1!1!1!}\right) = 24$
- $2(h^3+c^3+b^3)^2 = 0$
- $2(h^6+c^6+b^6) = 0$

Therefore coefficient of $h^2c^2b^2 = \frac{1}{12}(90+18+24) = 11$

Hence there are 11 possible structures.

Example 4.9: Let n be a prime number then each of ρ_n, $\rho_n^2, \ldots \rho_n^{n-1}$ is an n cycle.

Proof: If ρ_n^i (i = 1,2,…,n-1) contains m cycles then by symmetry the cycle factorization of ρ_n^i contains only m cycles.

$\Rightarrow$ m is the factor of n

$\Rightarrow$ m = 1 or m = n since n is prime

$\Rightarrow$ m = n otherwise ρ_n^i is identity for all i = 1,2,…,n-1

$\Rightarrow$ ρ_n^i is an n-cycle.

Hence each of ρ_n, $\rho_n^2, \ldots \rho_n^{n-1}$ is an n-cycle.

4:5 Beads on a Necklace

Count the number of ways to arrange beads on a necklace, where there are k-different colors of beads and n-total beads arranged on the necklace. With a necklace, we can obviously rotate it around, so if we label the beads in order as a, b, c, d then for a tiny necklace with only four beads, the pattern "red, red, blue, blue" is clearly the same as "red, blue, blue, red". etc. also, since the necklace is just made of beads, we can turn it over, so if there were four colors, although we can't rotate "red, green, yellow, blue" into "blue, yellow, green, red", we can flip over the necklace and make those two colorings identical. Let us solve this problem in the special case where k = 2 and n = 4, that is with 4 beads and 2 colors, red and blue. These are rrrr, bbbb, bbrr, brbr, rrrb, bbbr, a total of 6 solutions.

Now let us apply Pólya's method

If the bead position are called a, b, c and d, here are the permutations that map the necklace in to itself (a)(b)(c)(d), (adcb) , (ac) (bd) , (abcd) , (ad)(bc) , (ac)(b)(d), (ab)(cd) and (bd)(a)(c).

We can write down the cycle index: $P = \frac{1}{8}(z_1^4 + 2z_4 + 3z_2^2 + 2z_1^2 z_2)$

Since there are two colors; red ad blue, substitute as before

$$P = \frac{1}{8}[(r+b)^4 + 2(r+b)^2(r^2+b^2) + 3(r^2+b^2)^2 + 2(r^4+b^4)]$$

$$= r^4 + r^3 b + 2r^2 b^2 + r b^3 + b^4$$

The sum of the coefficients is 6 implies these are 6 ways to arrange the beads. Clearly with a small numbers of beads and colors, it is probably easier just to do a brute-force enumeration. But if the number of beads or colors gets large, Pólya's method become more and more attractive.

Let n be a prime number, consider a necklace that can be made from beads of k different colors.

Since we have n beads and k colors we have k^n ways to color all the beads. If we rotate the necklace we have $i, \rho_n, \rho_n^2, \ldots, \rho_n^{n-1}$ and if we turn it over, we get $T_1, T_2, \ldots, T_n$ permutations that map the necklace in to it self

The number of colorings fixed by the identity is $|C(i)| = k^n$

Since n is prime each of ρ_n^i (i = 1, 2,…, n-1) is an n-cycle then the number of colorings fixed by each ρ_n^i (i = 1,2,…, n-1) are equal. That is

$$|C(\rho_n)| = |C(\rho_n^2| = \ldots = |C(\rho_n^{n-1})| = k$$

Since n is prime each of T_i, $i = 1, 2, \ldots, n$ has one 1-cycle and $\frac{n-1}{2}$ 2-cycles

$$\Rightarrow |C(T_1)| = |C(T_1)| = \ldots = |C(T_n)| = k^{1+\frac{n-1}{2}} = k^{\frac{n+1}{2}}$$

Therefore number of different necklaces is equal to $\frac{1}{2n}\sum_f |C(f)|$. That is

$$\frac{1}{2n}\sum_f |C(f)| = \frac{1}{2n}[k^n + (n-1)k + n(k^{\frac{n+1}{2}})]$$

For example, look at a necklace with 17 beads in it. Using the above formula for n = 17 will show that the cycle index polynomial we need is

$$P = \frac{z_1^{17} + 16\, z_{17} + 17\, z_1\, z_2^8}{34}$$

Let us try to solve this with 4 colors; red r, blue b, yellow y and green g, of beads to obtain

$$P = \frac{(r+b+y+g)^{17} + 16(r^{17}+b^{17}+y^{17}+g^{17}) + 17(r+b+y+g)(r^2+b^2+y^2+g^2)^8}{34}$$

Substituting r = b = y = g = 1 yields 505421344 solutions.

If we have a really strong stomach, we can multiply out the expression for p and get the breakdown for various color combinations.

To illustrate with our example above to count the number of necklaces with 2 red, 4 blue, 3 yellow and 8 green beads, we simply look for coefficients of terms like $r^2b^4y^3g^8$.

Using the formula for multinomial coefficients, the coefficient of $r^2b^4y^3g^8$ in:

- $(r+b+y+g)^{17} = \frac{17!}{2!4!3!8!} = 30630600$
- $16(r^{17}+b^{17}+y^{17}+g^{17}) = 0$
- $17(r+b+y+g)(r^2+b^2+y^2+g^2)^8 = 17\ \frac{8!}{1!2!1!4!} = 14280$

Therefore, there are $\frac{30630600 + 14280}{34} = 901320$ way to make such necklace.

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