LASER DYNAMICS WITH A SQUEEZED VACUUM

A Thesis Presented to the

School of Graduate Studies

Addis Ababa University

In Partial Fulfillment

of the Requirement for the Degree

of Master of Science in Physics

by

Tesfaye Kebede

January, 1999

ACKNOWLEDGMENT

I wish to express my deep gratitude to my advisor and instructor Dr. Fesseha Kassahun for his constant assistance, invaluable guidance and encouragement without any reservation during the whole period of my research work .

I would like to extend my appreciation to Deutscher Akademischer Austauschdienst (German Academic Exchange Service) DAAD for the financial support I received during my M. Sc. study.

Last but not least, I would like to acknowledge the Department of Physics and School of Graduate Studies, A.A.U. for all co-operation I had during my M. Sc. study.

Abstract

We present a detailed derivation of the c-number Langevin equations for a laser coupled to a squeezed vacuum reservoir from the corresponding operator Langevin equations. Employing these equations we investigate the effects of the squeezed vacuum on the photon statistics, the spectrum of intensity fluctuations, squeezing spectrum and the power spectrum for a laser operating below threshold. The same set of equations are also used to study the effects of the squeezed vacuum on the photon statistics, the power spectrum, and spectrum of intensity fluctuations for a laser operating above threshold. We find that the intensity fluctuations and laser linewidth depend on the relative phase between the squeezed vacuum and the laser.

CONTENTS

1. INTRODUCTION

The laser principle was first introduced by Schwalow and Townes and also by Prokorov [1,2] four years later after the discovery of the maser principle by Gordon, Zienger, and Townes in 1954 [3]. In their semiclassical treatment of atom-field interaction, Schwalow and Townes showed that a coherent radiation can be generated not only in the radio frequency range (maser) but also in the optical domain (laser). A laser consists of a set of atoms pumped to the upper level and interacting with a resonant radiation inside two parallel mirrors forming a cavity. One mirror has 100% reflectivity while the other mirror called port mirror is partially transmitting thereby giving rise to the output radiation. If the upper level is sufficiently populated, a condition called population inversion, the resonant radiation (cavity mode) gives rise to further stimulated emission. From the above discussion, it is clear that a laser theory should deal with three basic elements, an active medium, pumping to the upper level, and the radiation losses due to the cavity.

The quantum noise and photon statistics of a laser have been extensively investigated [4-9]. It has been found that the photon statistics of a laser operating well above threshold tends to be Poissonian [6]. Moreover, the intensity fluctuations of the output light of such a laser tends to have a flat spectrum at the shot-noise level [6]. In addition, the photon statistics of the radiation generated by a laser operating below threshold is identical to that of a thermal light [7,8].

A closer look at the quantum theory of a laser reveals that there are three dominant sources of noise which contribute to the laser output. These are pump fluctuations, spontaneous emission, and vacuum fluctuations entering the cavity through the port mirror. There has been several attempts to suppress one or more of these noise sources [10-14]. The replacement of the ordinary vacuum surrounding the port mirror by a squeezed vacuum would certainly modify the properties of the laser light.

The effects of a squeezed vacuum on the intensity fluctuations and laser linewidth have been investigated by several authors applying the Glauber-Sudarshan P-function or stochastic differential equations obtained from the pertinent Fokker-Planck equation [10-14]. It is predicted that the Schawlow-Townes limit of the laser linewidth can be reduced by as much as one-half of the linewidth of a laser coupled to an ordinary vacuum reservoir. In addition, it has been found that, due to the coupling of the laser with a squeezed vacuum, there is an increase in the intensity and intensity noise [13]. On the other hand, Marte *et al* [12] considered the effects of coupling the atoms of a laser to a squeezed vacuum reservoir and showed that the intensity fluctuations of the laser reduces and even sub-Poissonian statistics may result.

The Fokker-Planck approach involves sophisticated techniques to convert operator master equations into Fokker-Planck equations and via Ito calculus to the equivalent c-number Langevin equations [15]. Moreover, the derivation of the master equation by itself involves a great deal of mathematical manipulations.

The main objective of this thesis is to study the quantum noise and the photon statistics of a laser coupled to a squeezed vacuum reservoir applying c-number Langevin equations. We seek here to derive the c-number Langevin equations from the corresponding operator Langevin equations for the normal ordering applying the method discussed in reference [16]. From these set of c-number Langevin equations for atomic and radiation variables, we eliminate the atomic variables adiabatically and derive the Langevin equations for the radiation variables alone.

The resulting c-number Langevin equations are then used to study the photon statistics, spectrum of intensity fluctuations, squeezing spectrum, and power spectrum for a laser operating below threshold. Applying the same equations we also calculate the mean and variance of the photon number, the power spectrum, and the spectrum of intensity fluctuations for a laser operating above threshold.

2. LANGEVIN EQUATIONS

In this chapter we seek to derive the Langevin equations for a laser coupled to a squeezed vacuum reservoir. We shall follow the approach of Scully and Lamb [17], in which the pumping mechanism is modeled by the injection of a sequence of inverted atoms into the cavity. A laser coupled to a squeezed vacuum can be arranged as indicated in figure 1.

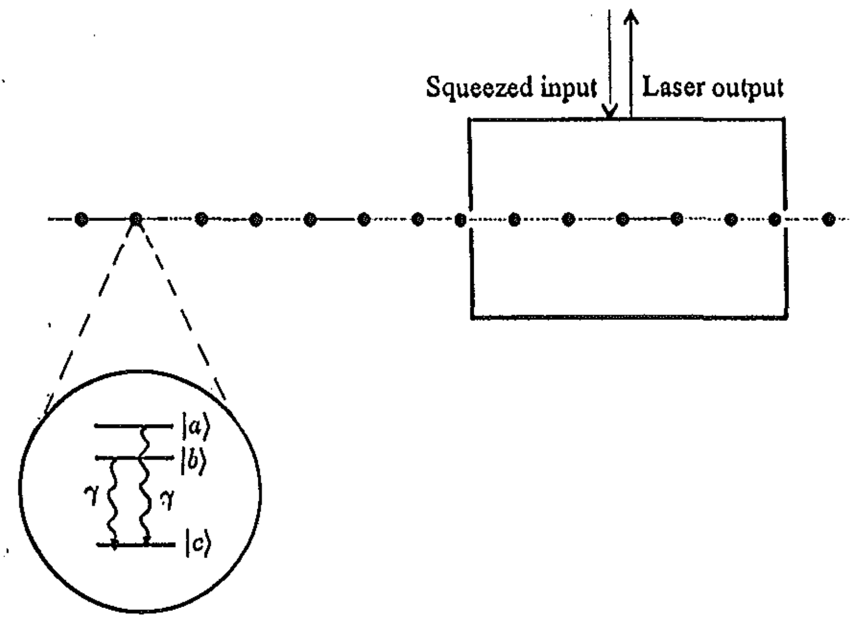

Fig. 1. A laser coupled to a squeezed vacuum

Atoms initially in level $|a\rangle$ are injected into the cavity at a rate r_a. The upper two levels $|a\rangle$ and $|b\rangle$ constitute the lasing levels and are at resonance with the cavity mode. The lowest atomic level $|c\rangle$ is a state to which atoms decay with a rate γ and the transition frequencies ω_{ac} and ω_{bc} are assumed to be far from resonance. A laser coupled to a squeezed vacuum reservoir is described in the interaction picture by the Hamiltonian

$$H = H_{AF} + H_{R_1F} + H_{R_2A}, \tag{2.1}$$

where H_{AF} is the atom-radiation interaction Hamiltonian, H_{R_1F} describes the interaction of the cavity mode with the squeezed vacuum reservoir and H_{R_2A} represents the interaction of the atoms with the ordinary vacuum reservoir. These Hamiltonians have the form

$$H_{AF} = i\hbar g \sum_j \Theta(t - t_j) \left(a^\dagger \sigma_-^j - \sigma_+^j a \right), \tag{2.2}$$

$$H_{R_1F} = i\hbar \sum_l g_l' \left(b_l^\dagger a \exp[-i(\omega_0 - \omega_l)t] - a^\dagger b_l \exp[i(\omega_0 - \omega_l)t] \right), \tag{2.3}$$

$$H_{R_2A} = i\hbar \sum_j \sum_{\vec{k}} g_{\vec{k}} \left(\sigma_{ac}^j c_{\vec{k}} \exp[i(\omega_{ac} - \omega_{\vec{k}})t] + \sigma_{bc}^j c_{\vec{k}} \exp[i(\omega_{bc} - \omega_{\vec{k}})t] + H.C. \right), \tag{2.4}$$

where a, b_l and $c_{\vec{k}}$ are the annihilation operators for the cavity mode, squeezed vacuum and ordinary vacuum reservoir modes, respectively. The unit step function $\Theta(t)$ is included to ensure that the j^{th} atom starts to interact at the injection time t_j. $\sigma_-^j = |b\rangle_{jj}\langle a|$ and $\sigma_+^j = |a\rangle_{jj}\langle b|$ are the lowering and the raising operators for the j^{th} atom, respectively. We assume the coupling constants g, g_l' and $g_{\vec{k}}$ to be real and

$$\sigma_{pq}^j = |p\rangle_{jj}\langle q|, \qquad (p, q = a, b, c) \tag{2.5}$$

2.1. Quantum Langevin Equations for the Cavity Mode

The operator a evolves in time according to

$$\dot{a} = \frac{1}{i\hbar}[a, H], \tag{2.6}$$

and using (2.1), we have

$$\dot{a} = \frac{1}{i\hbar}[a, H_{AF}] + \frac{1}{i\hbar}[a, H_{R_1F}] + \frac{1}{i\hbar}[a, H_{R_2A}]. \tag{2.7}$$

Since the cavity mode operator a commutes with the atomic operators σ_{pq}^j and the vacuum mode operators $c_{\vec{k}}$

$$[a, H_{R_2A}] = 0. \tag{2.8}$$

And using the commutation relation

$$[a, a^\dagger] = 1, \tag{2.9}$$

we see that

$$\frac{1}{i\hbar}[a, H_{AF}] = g \sum_j \Theta(t - t_j)\sigma_-^j, \tag{2.10a}$$

$$\frac{1}{i\hbar}[a, H_{R_1F}] = -\sum_l g_l' b_l \exp[i(\omega_0 - \omega_l)t]. \tag{2.10b}$$

Combining Eqs. (2.8), (2.10a), (2.10b) and (2.7), we have

$$\dot{a} = g \sum_j \Theta(t - t_j)\sigma_-^j - \sum_l g_l' b_l \exp\left[i(\omega_0 - \omega_l)t\right]. \tag{2.11}$$

In addition, applying the commutation relation

$$[b_l, b_{l'}^\dagger] = \delta_{ll'}, \tag{2.12}$$

one can easily show that

$$\dot{b}_l = g_l' a \exp\left[-i(\omega_0 - \omega_l)t\right]. \tag{2.13}$$

On formally integrating this, we obtain

$$b_l(t) = b_l(0) + g_l' \int_0^t dt'\, a(t') \exp\left[-i(\omega_0 - \omega_l)t'\right], \tag{2.14}$$

so that substitution of this into (2.11) leads to

$$\begin{aligned} \dot{a} = g \sum_j \Theta(t - t_j)\sigma_-^j \quad &- \quad \sum_l g_l' b_l(0) \exp\left[i(\omega_0 - \omega_l)t\right] \\ &- \quad \sum_l g_l'^2 \int_0^t dt'\, a(t') \exp\left[i(\omega_0 - \omega_l)(t - t')\right]. \end{aligned} \tag{2.15}$$

We now proceed to simplify the last term in Eq. (2.15). To this end, we let

$$\Gamma_0 = \sum_l g_l'^2 \int_0^t dt'\, a(t') \exp\left[i(\omega_0 - \omega_l)(t - t')\right]. \tag{2.16}$$

Assuming that the frequencies of the squeezed vacuum reservoir modes to be closely spaced, the summation can be replaced by an integration

$$\sum_l \longrightarrow \int_0^\infty \lambda(\omega)\, d\omega, \tag{2.17}$$

where $\lambda(\omega)$ is the density of reservoir modes. Hence Eq.(2.16) can be put in the form

$$\Gamma_0 = \int_0^t dt' a(t') \int_0^\infty d\omega \lambda(\omega) g'^2(\omega) \exp\left[i(\omega_0 - \omega)(t - t')\right]. \tag{2.18}$$

Furthermore, applying the method developed in Ref. [18] we set $\omega' = \omega - \omega_0$, and have

$$\Gamma_0 = \int_0^t dt' a(t') \int_{-\omega_0}^\infty d\omega' \lambda(\omega_0 + \omega') g'^2(\omega_0 + \omega') \exp\left[-i\omega'(t - t')\right]. \tag{2.19}$$

Since the exponential factor is a rapidly oscillating function of ω' except near $\omega' = 0$, one can replace $\lambda(\omega_0 + \omega')$ and $g'^2(\omega_0 + \omega')$ by $\lambda(\omega_0)$ and $g'^2(\omega_0)$ and extend the lower limit to $-\infty$. In view of this, we have

$$\Gamma_0 = \int_0^t dt' a(t')\lambda(\omega_0)g'^2(\omega_0) \int_{-\infty}^{\infty} d\omega' \exp\left[-i\omega'(t-t')\right], \tag{2.20}$$

so that on carrying out the integration, we find

$$\Gamma_0 = \frac{\mathcal{C}}{2}a(t), \tag{2.21}$$

where

$$\mathcal{C} = 2\pi\lambda(\omega_0)g'^2(\omega_0) \tag{2.22}$$

is the cavity decay rate. Hence the quantum Langevin equation for the cavity mode is

$$\dot{a} = -\frac{\mathcal{C}}{2}a(t) + g\sum_j \Theta(t-t_j)\sigma_-^j + F_c(t), \tag{2.23}$$

where

$$F_c(t) = -\sum_l g_l' b_l(0) \exp\left[i(\omega_0 - \omega_l)t\right] \tag{2.24}$$

is the noise operator associated with the squeezed vacuum reservoir.

For a squeezed vacuum reservoir [8]

$$\langle b_l \rangle = \langle b_l^\dagger \rangle = 0, \tag{2.25a}$$
$$\langle b_l^\dagger b_{l'} \rangle = N\delta_{ll'}, \tag{2.25b}$$
$$\langle b_l b_{l'}^\dagger \rangle = (N+1)\delta_{ll'}, \tag{2.25c}$$
$$\langle b_l b_{l'} \rangle = M\delta_{l',2l_0-l}, \tag{2.25d}$$
$$\langle b_l^\dagger b_{l'}^\dagger \rangle = M^*\delta_{l',2l_0-l}, \tag{2.25e}$$

where

$$N = \sinh^2(r), \tag{2.25f}$$
$$M = \cosh(r)\sinh(r)e^{i\phi} \tag{2.25g}$$

and r is a squeeze parameter, assumed to be real and positive and ϕ is the reference phase of the squeezed light. On account of (2.25a), we see that

$$\langle F_c(t) \rangle = \langle F_c^\dagger(t) \rangle = 0. \tag{2.26}$$

We now proceed to evaluate the correlation function $\langle F_C^\dagger(t)F_C(t')\rangle$. With the aid of (2.24), we have

$$\langle F_C^\dagger(t)F_C(t')\rangle = \sum_{l,l'} g_l' g_{l'}' \langle b_l^\dagger(0) b_{l'}(0)\rangle \exp\left[-i(\omega_0-\omega_l)t + i(\omega_0-\omega_{l'})t'\right]. \tag{2.27}$$

On introducing (2.25b) into (2.27), there follows

$$\begin{aligned}\langle F_C^\dagger(t)F_C(t')\rangle &= N\sum_l g_l'^2 \exp\left[-i(\omega_0-\omega_l)(t-t')\right] \\ &= N\int_0^\infty d\omega\lambda(\omega) g_l'^2(\omega)\exp\left[-i(\omega_0-\omega)(t-t')\right] \\ &= NC\,\delta(t-t')\end{aligned} \tag{2.28a}$$

Following a similar procedure, one can easily verify that

$$\langle F_C(t)F_C^\dagger(t')\rangle = C(N+1)\,\delta(t-t'), \tag{2.28b}$$

$$\langle F_C(t)F_C(t')\rangle = CM\,\delta(t-t'), \tag{2.28c}$$

$$\langle F_C^\dagger(t)F_C^\dagger(t')\rangle = CM^*\,\delta(t-t'). \tag{2.28d}$$

2.2. Quantum Langevin Equations for Atomic Operators

We next seek to obtain the quantum Langevin equations for the atomic operators. The lowering operator σ_-^j evolves in time according to

$$\dot{\sigma}_-^j = \frac{1}{i\hbar}[\sigma_-^j, H] \tag{2.29}$$

Applying Eq.(2.1), we see that

$$\dot{\sigma}_-^j = \frac{1}{i\hbar}[\sigma_-^j, H_{AF}] + \frac{1}{i\hbar}[\sigma_-^j, H_{R_1F}] + \frac{1}{i\hbar}[\sigma_-^j, H_{R_2A}]. \tag{2.30}$$

Since the atomic operators commute with the cavity mode operator a and the squeezed vacuum operator b_l, we have

$$[\sigma_-^j, H_{R_1F}] = 0. \tag{2.31}$$

Employing the identities

$$\sigma_-^j\sigma_+^{j'} = |b\rangle_{jj'}\langle b|\,\delta_{jj'}, \tag{2.32a}$$

$$\sigma_+^{j'}\sigma_-^{j} = |a\rangle_{j'j}\langle a|\,\delta_{jj'}, \tag{2.32b}$$

and the commutation relations

$$[\sigma_-^{j}, \sigma_{ac}^{j'}] = |b\rangle_{jj'}\langle c|\,\delta_{jj'}, \tag{2.33a}$$

$$[\sigma_-^{j}, \sigma_{bc}^{j'}] = [\sigma_-^{j}, \sigma_{ca}^{j'}] = 0, \tag{2.33b}$$

$$[\sigma_-^{j}, \sigma_{cb}^{j'}] = -|c\rangle_{j'j}\langle a|\,\delta_{jj'}, \tag{2.33c}$$

the first and third terms in Eq.(2.30) are found to be

$$\frac{1}{i\hbar}[\sigma_-^{j}, H_{AF}] = g\Theta(t-t_j)\left(\sigma_{aa}^{j} - \sigma_{bb}^{j}\right) a, \tag{2.34a}$$

$$\frac{1}{i\hbar}[\sigma_-^{j}, H_{R_2A}] = \sum_{\vec{k}} g_{\vec{k}}\left(\sigma_{bc}^{j} c_{\vec{k}} \exp\left[i(\omega_{ac}-\omega_{\vec{k}})t\right] + c_{\vec{k}}^{\dagger}\sigma_{ca}^{j}\exp\left[-i(\omega_{bc}-\omega_{\vec{k}})t\right]\right). \tag{2.34b}$$

Hence substituting Eqs.(2.31) and (2.34) into (2.30), we get

$$\begin{aligned}\dot{\sigma}_-^{j} &= g\Theta(t-t_j)\left(\sigma_{aa}^{j} - \sigma_{bb}^{j}\right) a \\ &\quad + \sum_{\vec{k}} g_{\vec{k}}\left(\sigma_{bc}^{j} c_{\vec{k}} \exp\left[i(\omega_{ac}-\omega_{\vec{k}})t\right] + c_{\vec{k}}^{\dagger}\sigma_{ca}^{j}\exp\left[-i(\omega_{bc}-\omega_{\vec{k}})t\right]\right).\end{aligned} \tag{2.35}$$

Following the same procedure and with the aid of the commutation relations

$$[\sigma_{aa}^{j}, \sigma_-^{j'}] = -|b\rangle_{j'j}\langle a|\,\delta_{jj'}, \tag{2.36a}$$

$$[\sigma_{aa}^{j}, \sigma_+^{j'}] = |a\rangle_{jj'}\langle b|\,\delta_{jj'}, \tag{2.36b}$$

$$[\sigma_{aa}^{j}, \sigma_{ac}^{j'}] = |a\rangle_{jj'}\langle c|\,\delta_{jj'}, \tag{2.36c}$$

$$[\sigma_{aa}^{j}, \sigma_{ca}^{j'}] = -|c\rangle_{j'j}\langle a|\,\delta_{jj'}, \tag{2.36d}$$

$$[\sigma_{aa}^{j}, \sigma_{bc}^{j'}] = [\sigma_{aa}^{j}, \sigma_{cb}^{j'}] = 0, \tag{2.36e}$$

and

$$[\sigma_{bb}^{j}, \sigma_-^{j'}] = |b\rangle_{jj'}\langle a|\,\delta_{jj'}, \tag{2.36f}$$

$$[\sigma_{bb}^{j}, \sigma_+^{j'}] = -|a\rangle_{jj'}\langle b|\,\delta_{jj'}, \tag{2.36g}$$

$$[\sigma_{bb}^{j}, \sigma_{bc}^{j'}] = |b\rangle_{jj'}\langle c|\,\delta_{jj'}, \tag{2.36h}$$

$$[\sigma_{bb}^{j}, \sigma_{cb}^{j'}] = -|c\rangle_{jj'}\langle b|\,\delta_{jj'}, \tag{2.36i}$$

$$[\sigma_{bb}^{j}, \sigma_{ac}^{j'}] = [\sigma_{bb}^{j}, \sigma_{ca}^{j'}] = 0, \tag{2.36j}$$

one can easily arrive at the following equations of evolution for the atomic operators

$$\begin{aligned}\dot{\sigma}_{aa}^{j} &= -g\Theta(t-t_j)\left(a^{\dagger}\sigma_{-}^{j} + \sigma_{+}^{j}a\right) \\ &\quad + \sum_{\vec{k}} g_{\vec{k}}\left(\sigma_{ac}^{j} c_{\vec{k}} \exp\left[i(\omega_{ac}-\omega_{\vec{k}})t\right] + c_{\vec{k}}^{\dagger}\sigma_{ca}^{j} \exp\left[-i(\omega_{ac}-\omega_{\vec{k}})t\right]\right).\end{aligned} \tag{2.37}$$

$$\begin{aligned}\dot{\sigma}_{bb}^{j} &= g\Theta(t-t_j)\left(a^{\dagger}\sigma_{-}^{j} + \sigma_{+}^{j}a\right) \\ &\quad + \sum_{\vec{k}} g_{\vec{k}}\left(\sigma_{bc}^{j} c_{\vec{k}} \exp\left[i(\omega_{bc}-\omega_{\vec{k}})t\right] + c_{\vec{k}}^{\dagger}\sigma_{cb}^{j} \exp\left[-i(\omega_{bc}-\omega_{\vec{k}})t\right]\right).\end{aligned} \tag{2.38}$$

Moreover, applying the commutation relation

$$[c_{\vec{k}}, c_{\vec{k}'}^{\dagger}] = \delta_{\vec{k}\vec{k}'}, \tag{2.39}$$

the equation of evolution for the operator $c_{\vec{k}}$ is found to be

$$\dot{c}_{\vec{k}} = -\sum_{j} g_{\vec{k}}\left(\sigma_{ca}^{j} \exp\left[-i(\omega_{ac}-\omega_{\vec{k}})t\right] + \sigma_{cb}^{j} \exp\left[-i(\omega_{bc}-\omega_{\vec{k}})t\right]\right). \tag{2.40a}$$

On formally integrating this equation we have

$$\begin{aligned}c_{\vec{k}}(t) = c_{\vec{k}}(0) - \int_0^t dt' \sum_{j} g_{\vec{k}}\Big(& \sigma_{ca}^{j}(t') \exp\left[-i(\omega_{ac}-\omega_{\vec{k}})t'\right] \\ & + \sigma_{cb}^{j}(t') \exp\left[-i(\omega_{bc}-\omega_{\vec{k}})t'\right]\Big).\end{aligned} \tag{2.40b}$$

Substitution of $c_{\vec{k}}(t)$ from (2.40b) and its complex conjugate into (2.35) leads to

$$\begin{aligned}\dot{\sigma}_{-}^{j} &= g\Theta(t-t_j)\left(\sigma_{aa}^{j} - \sigma_{bb}^{j}\right)a + \sum_{\vec{k}} g_{\vec{k}}\sigma_{bc}^{j}(t)c_{\vec{k}}(0)\exp\left[i(\omega_{ac}-\omega_{\vec{k}})t\right] \\ &\quad - \sum_{\vec{k}} g_{\vec{k}}^{2}\sigma_{bc}^{j}(t)\sum_{j'}\int_0^t dt'\sigma_{ca}^{j'}(t')\exp\left[i(\omega_{ac}-\omega_{\vec{k}})(t-t')\right] \\ &\quad - \sum_{\vec{k}} g_{\vec{k}}^{2}\sigma_{bc}^{j}(t)\sum_{j'}\int_0^t dt'\sigma_{cb}^{j'}(t')\exp\left[-i(\omega_{bc}-\omega_{\vec{k}})t' + i(\omega_{ac}-\omega_{\vec{k}})t\right]\end{aligned}$$

$$+\sum_{\vec{k}} g_{\vec{k}} c^{\dagger}_{\vec{k}}(0)\sigma^{j}_{ca}(t)\exp\left[-i(\omega_{bc}-\omega_{\vec{k}})t\right]$$
$$-\sum_{\vec{k}} g^{2}_{\vec{k}} \sum_{j'} \int_0^t dt' \sigma^{j'}_{bc}(t')\sigma^{j}_{ca}(t)\exp\left[-i(\omega_{bc}-\omega_{\vec{k}})(t-t')\right]$$
$$-\sum_{\vec{k}} g^{2}_{\vec{k}} \sum_{j'} \int_0^t dt' \sigma^{j'}_{ac}(t')\sigma^{j}_{ca}(t)\exp\left[-i(\omega_{bc}-\omega_{\vec{k}})t+i(\omega_{ac}-\omega_{\vec{k}})t'\right]. \tag{2.41}$$

We now let

$$\Gamma_1 = \sum_{\vec{k}} g^{2}_{\vec{k}}\sigma^{j}_{bc}(t) \sum_{j'} \int_0^t dt' \sigma^{j'}_{ca}(t')\exp\left[i(\omega_{ac}-\omega_{\vec{k}})(t-t')\right], \tag{2.42}$$

Where the coupling constant $g_{\vec{k}}$ is given by [8,18]

$$g^{2}_{\vec{k}} = \frac{\omega_{\vec{k}}}{2\hbar\epsilon_0 V} d^2\cos^2(\theta), \tag{2.43}$$

with d (assumed to be the same for each atom) being the electric-dipole matrix element.

Assuming that the $\omega_{\vec{k}}$ are closely spaced, the sum over $\vec{k}$ can be replaced by an integral over ω

$$\sum_{\vec{k}} \longrightarrow 2\frac{V}{(2\pi c)^3}\int_0^{2\pi} d\phi \int_0^{\pi} d\theta \sin(\theta) \int_0^{\infty} d\omega\ \omega^2, \tag{2.44}$$

where V is the cavity volume and the factor 2 is included to account for the two possible polarization of the vacuum modes. Now combination of (2.43) and (2.44) with (2.42) and setting $\omega' = \omega - \omega_{ac}$ leads to

$$\Gamma_1 = \frac{4d^2}{6\hbar\epsilon_0 c^3(2\pi)^2}\sigma^{j}_{bc}(t)\sum_{j'}\int_0^t dt'\sigma^{j'}_{ca}(t')\int_{-\omega_{ac}}^{\infty} d\omega'(\omega_{ac}+\omega')^3\exp\left[-i\omega'(t-t')\right]. \tag{2.45}$$

In addition, we assume that ω varies slowly around ω_{ac}. We can then replace $\omega_{ac}+\omega'$ in the above integral by ω_{ac} and extend the lower limit of the integration to infinity, one then readily obtains

$$\Gamma_1 = \frac{4d^2\omega^3_{ac}}{6\hbar\epsilon_0 c^3(2\pi)^2}\sigma^{j}_{bc}(t)\sum_{j'}\int_0^t dt'\sigma^{j'}_{ca}(t')\int_{-\infty}^{\infty} d\omega' \exp -i\omega'(t-t')]. \tag{2.46}$$

Upon carrying out the integration, we obtain

$$\Gamma_1 = \frac{d^2\omega^3_{ac}}{6\hbar\pi\epsilon_0 c^3}\sum_{j'}\sigma^{j}_{bc}\sigma^{j'}_{ca}. \tag{2.47}$$

Applying the identity

$$\sigma^{j}_{bc}\sigma^{j'}_{ca} = \sigma^{j}_{-}\delta_{jj'}, \tag{2.48}$$

we see that

$$\Gamma_1 = \frac{\gamma_{ac}}{2}\sigma_-^j, \tag{2.49a}$$

where

$$\gamma_{ac} = \frac{d^2\omega_{ac}^3}{3\hbar\pi\epsilon_0 c^3}. \tag{2.49b}$$

is the rate at which atoms decay from level $|a\rangle$ to level $|c\rangle$.

Denoting the sixth term in Eq.(2.41) by Γ_2 and following the same procedure, one can show that

$$\begin{aligned}\Gamma_2 &= \sum_{\vec{k}} g_{\vec{k}}^2 \sum_{j'} \int_0^t dt' \sigma_{bc}^{j'}(t')\sigma_{ca}^j(t)\exp\left[-i(\omega_{bc}-\omega_{\vec{k}})(t-t')\right] \\ &= \frac{\gamma_{bc}}{2}\sigma_-^j,\end{aligned} \tag{2.50a}$$

where

$$\gamma_{bc} = \frac{d^2\omega_{bc}^3}{3\hbar\pi\epsilon_0 c^3} \tag{2.50b}$$

is the rate at which atoms decay from level $|b\rangle$ to level $|c\rangle$. Assuming the same decay rate, i.e., $\gamma_{ac} = \gamma_{bc} = \gamma$

$$\Gamma_1 = \Gamma_2 = \frac{\gamma}{2}\sigma_-^j. \tag{2.51}$$

Since the exponential functions in the fourth and seventh terms in Eq.(2.41) are a rapidly oscillating functions of $\omega_{\vec{k}}$, their average value tends to vanish. Upon neglecting these terms, the quantum-Langevin equation for σ_-^j then reduces to

$$\dot{\sigma}_-^j = -\gamma\sigma_-^j + g\Theta(t-t_j)\left(\sigma_{aa}^j - \sigma_{bb}^j\right) a + F_{ba}^j(t), \tag{2.52}$$

where the noise operator associated with the j^{th} atom is given by

$$\begin{aligned}F_{ba}^j(t) &= \sigma_{bc}^j(t)\sum_{\vec{k}} g_{\vec{k}} c_{\vec{k}}(0)\exp\left[i(\omega_{ac}-\omega_{\vec{k}})t\right] \\ &\quad + \sigma_{ca}^j(t)\sum_{\vec{k}} g_{\vec{k}} c_{\vec{k}}^\dagger(0)\exp\left[i(\omega_{bc}-\omega_{\vec{k}})t\right].\end{aligned} \tag{2.53}$$

Substituting (2.40b) and its complex conjugate into Eqs.(2.37) and (2.38), and following a similar procedure, we obtain the following quantum-Langevin equations for $\sigma_{aa}^j(t)$ and $\sigma_{bb}^j(t)$

$$\dot{\sigma}_{aa}^j = -\gamma\sigma_{aa}^j - g\Theta(t-t_j)\left(a^\dagger\sigma_-^j + \sigma_+^j a\right) + F_{aa}^j(t), \tag{2.54}$$

$$\dot{\sigma}^j_{bb} = -\gamma\sigma^j_{bb} + g\Theta(t-t_j)\left(a^\dagger\sigma^j_- + \sigma^j_+ a\right) + F^j_{bb}(t), \tag{2.55}$$

where the noise operators are defined as

$$\begin{aligned} F^j_{aa}(t) &= \sigma^j_{ac}(t)\sum_{\vec{k}} g_{\vec{k}} c_{\vec{k}}(0)\exp\left[i(\omega_{ac}-\omega_{\vec{k}})t\right] \\ &+\sigma^j_{ca}(t)\sum_{\vec{k}} g_{\vec{k}} c^\dagger_{\vec{k}}(0)\exp\left[-i(\omega_{ac}-\omega_{\vec{k}})t\right], \end{aligned} \tag{2.56}$$

$$\begin{aligned} F^j_{bb}(t) &= \sigma^j_{bc}(t)\sum_{\vec{k}} g_{\vec{k}} c_{\vec{k}}(0)\exp\left[i(\omega_{bc}-\omega_{\vec{k}})t\right] \\ &+\sigma^j_{cb}(t)\sum_{\vec{k}} g_{\vec{k}} c^\dagger_{\vec{k}}(0)\exp\left[-i(\omega_{bc}-\omega_{\vec{k}})t\right]. \end{aligned} \tag{2.57}$$

For ordinary vacuum reservoir, we have

$$\langle c_{\vec{k}}(0)\rangle = \langle c^\dagger_{\vec{k}}(0)\rangle = 0, \tag{2.58a}$$

$$\langle c^\dagger_{\vec{k}}(0)c_{\vec{k}'}(0)\rangle = 0, \tag{2.58b}$$

$$\langle c_{\vec{k}}(0)c_{\vec{k}'}(0)\rangle = \langle c^\dagger_{\vec{k}}(0)c^\dagger_{\vec{k}'}(0)\rangle = 0, \tag{2.58c}$$

$$\langle c_{\vec{k}}(0)c^\dagger_{\vec{k}'}(0)\rangle = \delta_{\vec{k}\vec{k}'}. \tag{2.58d}$$

Applying (2.58a) it is easy to show that

$$\langle F^j_{ba}(t)\rangle = \langle F^j_{aa}(t)\rangle = \langle F^j_{bb}(t)\rangle = 0 \tag{2.59}$$

We next evaluate the non-vanishing correlation functions of the atomic noise operators. On account of (2.53), we have

$$\begin{aligned} \langle F^{j\dagger}_{ba}(t)F^j_{ba}(t')\rangle &= \langle\sigma^j_{cb}(t)\sigma^j_{bc}(t')\rangle\sum_{\vec{k},\vec{k}'} g_{\vec{k}}g_{\vec{k}'}\langle c^\dagger_{\vec{k}}(0)c_{\vec{k}'}(0)\rangle\exp\left[-i(\omega_{ac}-\omega_{\vec{k}})t - i(\omega_{ac}-\omega_{\vec{k}'})t'\right] \\ &+\langle\sigma^j_{cb}(t)\sigma^j_{ca}(t')\rangle\sum_{\vec{k},\vec{k}'} g_{\vec{k}}g_{\vec{k}'}\langle c^\dagger_{\vec{k}}(0)c^\dagger_{\vec{k}'}(0)\rangle\exp\left[-i(\omega_{ac}-\omega_{\vec{k}})t - i(\omega_{bc}-\omega_{\vec{k}'})t'\right] \\ &+\langle\sigma^j_{ac}(t)\sigma^j_{bc}(t')\rangle\sum_{\vec{k},\vec{k}'} g_{\vec{k}}g_{\vec{k}'}\langle c_{\vec{k}}(0)c_{\vec{k}'}(0)\rangle\exp\left[-i(\omega_{bc}-\omega_{\vec{k}})t + i(\omega_{ac}-\omega_{\vec{k}'})t'\right] \\ &+\langle\sigma^j_{ac}(t)\sigma^j_{ca}(t')\rangle\sum_{\vec{k},\vec{k}'} g_{\vec{k}}g_{\vec{k}'}\langle c_{\vec{k}}(0)c^\dagger_{\vec{k}'}(0)\rangle\exp\left[-i(\omega_{bc}-\omega_{\vec{k}})t - i(\omega_{bc}-\omega_{\vec{k}'})t'\right]. \end{aligned} \tag{2.60}$$

Applying (2.58b-2.58d) and the fact that

$$\sum_{\vec{k}} g_{\vec{k}}^2 \exp\left[-i(\omega_{bc} - \omega_{\vec{k}})(t - t')\right] = \gamma\, \delta(t - t'), \tag{2.61}$$

Eq.(2.60) read

$$\langle F_{ba}^{j\dagger}(t) F_{ba}^{j}(t') \rangle = \gamma \langle \sigma_{aa}^{j}(t) \rangle \delta(t - t'). \tag{2.62}$$

Following a similar procedure the remaining non-vanishing correlation functions are found to be

$$\langle F_{aa}^{j}(t) F_{aa}^{j}(t') \rangle = \gamma \langle \sigma_{aa}^{j}(t) \rangle \delta(t - t'), \tag{2.63}$$

$$\langle F_{bb}^{j}(t) F_{bb}^{j}(t') \rangle = \gamma \langle \sigma_{bb}^{j}(t) \rangle \delta(t - t'), \tag{2.64}$$

$$\langle F_{ba}^{j}(t) F_{aa}^{j}(t') \rangle = \gamma \langle \sigma_{-}^{j}(t) \rangle \delta(t - t'), \tag{2.65}$$

$$\langle F_{ba}^{j\dagger}(t) F_{bb}^{j}(t') \rangle = \gamma \langle \sigma_{+}^{j}(t) \rangle \delta(t - t'). \tag{2.66}$$

2.3. Quantum Langevin Equations for Macroscopic Atomic Operators

In the preceding section we derived the quantum-Langevin equations for individual atomic operators. We next change the operators for individual atoms to operators which describe the macroscopic atomic properties. This proves to be necessary for the approximation techniques employed later in this chapter. While the individual atomic operators are very sensitive to an adiabatic approximation, the averaged, macroscopic quantities can be treated by such a technique. Therefore we define the following operators:

$$M(t) = \sum_{j} \Theta(t - t_j)\, \sigma_{-}^{j}(t), \tag{2.67}$$

$$N_a(t) = \sum_{j} \Theta(t - t_j)\, \sigma_{aa}^{j}(t), \tag{2.68}$$

$$N_b(t) = \sum_{j} \Theta(t - t_j)\, \sigma_{bb}^{j}(t). \tag{2.69}$$

The operator M represents the macroscopic atomic polarization, and the operators N_a and N_b specify the number of atoms in the two excited atomic levels $|a\rangle$ and $|b\rangle$ respectively. With this definitions Eq.(2.23) for the cavity mode simplifies to

$$\dot{a} = -\frac{\mathcal{C}}{2} a + gM + F_{\mathcal{C}}(t). \tag{2.70}$$

The Langevin equations for the macroscopic atomic operators can be found by differentiating Eqs (2.67), (2.68) and (2.69) with respect to time and substituting Eqs (2.52), (2.54), (2.55) respectively. For example, for the operator N_a we obtain,

$$\dot{N}_a = \sum_j \left[\sigma^j_{aa}(t)\dot{\Theta}(t-t_j) + \Theta(t-t_j)\dot{\sigma}^j_{aa}(t)\right]. \tag{2.71}$$

Substituting Eq.(2.54) into (2.71)and applying the following identity

$$\dot{\Theta}(t-t_j) = \delta(t-t_j), \tag{2.72a}$$

$$\Theta(t-t_j)\Theta(t-t_j) = \Theta(t-t_j), \tag{2.72b}$$

we get

$$\dot{N}_a = \sum_j \delta(t-t_j)\sigma^j_{aa}(t_j) - \gamma N_a - g\left(a^\dagger M + M^\dagger a\right) + \sum_j \Theta(t-t_j)F^j_{aa}(t). \tag{2.73}$$

The first term on the right-hand side of Eq.(2.73) corresponds to the injection of atoms into the cavity. To see this clearly let us evaluate the expectation value of this term;

$$\left\langle \sum_j \delta(t-t_j)\sigma^j_{aa}(t_j) \right\rangle = \left\langle \sum_j \delta(t-t_j)\langle\sigma^j_{aa}(t_j)\rangle \right\rangle_s . \tag{2.74}$$

The index s on the brackets in (2.74) indicates that we still have to perform the statistical average over the injection times. Since atoms are initially in level $|a\rangle$ we see that

$$\langle\sigma^j_{aa}(t_j)\rangle = 1, \tag{2.75a}$$

$$\langle\sigma^j_{bb}(t_j)\rangle = 0, \tag{2.75b}$$

and hence

$$\left\langle \sum_j \delta(t-t_j)\sigma^j_{aa}(t_j) \right\rangle = \left\langle \sum_j \delta(t-t_j) \right\rangle_s . \tag{2.76}$$

If we assume a time independent atomic injection rate r_a, this average can be evaluated as

$$\begin{aligned}\left\langle \sum_j \delta(t-t_j) \right\rangle_s &= r_a \int_{-\infty}^{\infty} dt_j\, \delta(t-t_j) \\ &= r_a. \end{aligned} \tag{2.77}$$

In order to obtain the correct Langevin equation in which the reservoir average of the noise operator is zero, we add and subtract the expectation value of the first term of Eq.(2.73), and obtain

$$\dot{N}_a = r_a - \gamma N_a - g\left(a^\dagger M + M^\dagger a\right) + F_a(t), \tag{2.78}$$

where

$$F_a(t) = \sum_j \Theta(t-t_j)F_{aa}^j(t) + \sum_j \delta(t-t_j)\sigma_{aa}^j(t_j) - r_a \tag{2.79}$$

is the total noise operator for the atomic operator N_a. It is easy to verify that

$$\langle F_a(t)\rangle = 0. \tag{2.80}$$

In a similar way we can derive the quantum-Langevin equations for the remaining atomic operators.

$$\dot{N}_b = -\gamma N_b + g\left(a^\dagger M + M^\dagger a\right) + F_b(t), \tag{2.81}$$

$$\dot{M} = -\gamma M + g\left(N_a - N_b\right)a + F_M(t), \tag{2.82}$$

where the noise operators are given by

$$F_b(t) = \sum_j \Theta(t-t_j)F_{bb}^j(t) + \sum_j \delta(t-t_j)\sigma_{bb}^j(t_j), \tag{2.83}$$

$$F_M(t) = \sum_j \Theta(t-t_j)F_{ba}^j(t) + \sum_j \delta(t-t_j)\sigma_-^j(t_j). \tag{2.84}$$

Since we have assumed the atoms to be initially in level $|a\rangle$, there is no injection term in Eqs.(2.81) and (2.82). Employing the identities

$$\langle\sigma_{bb}^j(t_j)\rangle = 0, \tag{2.85a}$$

$$\langle\sigma_-^j(t_j)\rangle = 0, \tag{2.85b}$$

there follows

$$\langle F_b(t)\rangle = 0, \tag{2.86}$$

$$\langle F_M(t)\rangle = 0. \tag{2.87}$$

Next, we seek to evaluate the non-vanishing correlation functions of the atomic noise operators. Using Eq.(2.79), we have

$$
\begin{aligned}
\langle F_a(t)F_a(t')\rangle \;=\;& \left\langle \sum_{i,j} \Theta(t-t_j)\Theta(t'-t_i)\langle F^j_{aa}(t)F^i_{aa}(t')\rangle \right\rangle_s \\
&+\left\langle \sum_{i,j} \Theta(t-t_j)\,\delta(t'-t_i)\langle F^j_{aa}(t)\sigma^i_{aa}(t_i)\rangle \right\rangle_s \\
&-2r_a\left\langle \sum_{j} \Theta(t-t_j)\langle F^j_{aa}(t)\rangle \right\rangle_s \\
&+\left\langle \sum_{i,j} \delta(t-t_j)\Theta(t'-t_i)\langle \sigma^j_{aa}(t_j)F^i_{aa}(t')\rangle \right\rangle_s \\
&+\left\langle \sum_{i,j} \delta(t-t_j)\delta(t'-t_i)\langle \sigma^j_{aa}(t_j)\sigma^i_{aa}(t_i)\rangle \right\rangle_s \\
&-2r_a\left\langle \sum_{j} \delta(t-t_j)\langle \sigma^j_{aa}(t_j)\rangle \right\rangle_s + r_a^2.
\end{aligned}
\tag{2.88}
$$

The statistical average over the injection times denoted by s is separated from the quantum mechanical expectation value. Since individual atoms are completely independent of each other, we have the following:

$$
\text{for } i \neq j \qquad \begin{aligned} \langle \sigma^j_{aa}(t_j)\sigma^i_{aa}(t_i)\rangle &= \langle \sigma^j_{aa}(t_j)\rangle\langle \sigma^i_{aa}(t_i)\rangle \\ &= 1, \end{aligned} \tag{2.89}
$$

$$
\text{for } i = j \qquad \begin{aligned} \langle \sigma^j_{aa}(t_j)\sigma^i_{aa}(t_i)\rangle &= \langle \sigma^j_{aa}(t_j)\rangle \\ &= 1, \end{aligned} \tag{2.90}
$$

and also,

$$
\langle F^j_{aa}(t)\sigma^i_{aa}(t_i)\rangle = \langle \sigma^j_{aa}(t_j)F^i_{aa}(t')\rangle = 0. \tag{2.91}
$$

Using (2.89-91) and (2.63), Eq.(2.88) can be written as

$$
\begin{aligned}
\langle F_a(t)F_a(t')\rangle \;=\;& \gamma\left\langle \sum_{j} \Theta(t-t_j)\Theta(t'-t_j)\langle \sigma^j_{aa}(t)\rangle \right\rangle_s \delta(t-t') + \left\langle \sum_{j} \delta(t-t_j)\delta(t'-t_j) \right\rangle_s \\
&+\left\langle \sum_{\substack{i,j\\ i\neq j}} \delta(t-t_j)\delta(t'-t_i) \right\rangle_s - 2r_a\left\langle \sum_{j} \delta(t-t_j) \right\rangle_s + r_a^2.
\end{aligned}
\tag{2.92}
$$

We see that

$$\left\langle \sum_j \delta(t-t_j)\delta(t'-t_j) \right\rangle_s = \left\langle \sum_j \delta(t-t_j) \right\rangle_s \delta(t-t')$$

$$= r_a \delta(t-t'), \tag{2.93}$$

$$\left\langle \sum_{\substack{i,j \\ i\neq j}} \delta(t-t_j)\delta(t'-t_i) \right\rangle_s = \left\langle \sum_j \delta(t-t_j) \right\rangle_s \left\langle \sum_j \delta(t'-t_i) \right\rangle_s$$

$$= r_a^2, \tag{2.94}$$

and

$$\Theta(t-t_j)\Theta(t'-t_j) = \Theta(t-t_j)\delta(t-t'). \tag{2.95}$$

Hence, Eq.(2.92) has the form

$$\langle F_a(t)F_a(t')\rangle = (\gamma\langle N_a\rangle + r_a)\delta(t-t'). \tag{2.96}$$

Applying the definition of $F_b(t)$, we see that

$$\begin{aligned}
\langle F_b(t)F_b(t')\rangle \;=\; & \left\langle \sum_{i,j} \Theta(t-t_i)\Theta(t'-t_j)\langle F^i_{bb}(t)F^j_{bb}(t')\rangle \right\rangle_s \\
& + \left\langle \sum_{i,j} \Theta(t-t_i)\,\delta(t'-t_j)\langle F^i_{bb}(t)\sigma^j_{bb}(t_j)\rangle \right\rangle_s \\
& + \left\langle \sum_{i,j} \delta(t-t_i)\Theta(t'-t_j)\langle \sigma^i_{bb}(t_i)F^j_{bb}(t')\rangle \right\rangle_s \\
& + \left\langle \sum_{i,j} \delta(t-t_i)\delta(t'-t_j)\langle \sigma^i_{bb}(t_i)\sigma^j_{bb}(t_j)\rangle \right\rangle_s .
\end{aligned} \tag{2.97}$$

The last term of (2.97) can be written as

$$\begin{aligned}
\left\langle \sum_{i,j} \delta(t-t_i)\delta(t'-t_j)\langle \sigma^i_{bb}(t_i)\sigma^j_{bb}(t_j)\rangle \right\rangle_s \;=\; & \left\langle \sum_{i} \delta(t-t_i)\delta(t'-t_i)\langle \sigma^i_{bb}(t_i)\rangle^2 \right\rangle_s \\
& + \left\langle \sum_{\substack{i,j \\ i\neq j}} \delta(t-t_i)\delta(t'-t_j)\langle \sigma^i_{bb}(t_i)\rangle\langle \sigma^j_{bb}(t_j)\rangle \right\rangle_s \\
=\; & 0.
\end{aligned} \tag{2.98}$$

Combining (2.64), (2.98) and the following relation

$$\langle \sigma^i_{bb}(t_i) F^j_{bb}(t') \rangle = \langle F^i_{bb}(t) \sigma^j_{bb}(t_j) \rangle = 0, \tag{2.99}$$

with (2.97), we obtain

$$\langle F_b(t) F_b(t') \rangle = \gamma \langle N_b \rangle \, \delta(t - t'). \tag{2.100}$$

Again from the definition of $F_b(t)$ and $F_M(t)$ we see that

$$\begin{aligned} \langle F_b(t) F_M(t') \rangle &= \left\langle \sum_{i,j} \Theta(t - t_i) \Theta(t' - t_j) \langle F^i_{bb}(t) F^j_{ba}(t') \rangle \right\rangle_s \\ &+ \left\langle \sum_{i,j} \Theta(t - t_i) \, \delta(t' - t_j) \langle F^i_{bb}(t) \sigma^j_-(t_j) \rangle \right\rangle_s \\ &+ \left\langle \sum_{i,j} \delta(t - t_i) \Theta(t' - t_j) \langle \sigma^i_{bb}(t_i) F^j_{ba}(t') \rangle \right\rangle_s \\ &+ \left\langle \sum_{i,j} \delta(t - t_i) \delta(t' - t_j) \langle \sigma^i_{bb}(t_i) \sigma^j_-(t_j) \rangle \right\rangle_s . \end{aligned} \tag{2.101}$$

Since individual atoms are completely independent and are initially in the upper level, it can be easily verified that,

$$\langle \sigma^i_{bb}(t_i) \sigma^j_-(t_j) \rangle = 0, \tag{2.102a}$$

$$\langle \sigma^i_{bb}(t_i) F^j_{ba}(t') \rangle = 0, \tag{2.102b}$$

$$\langle F^i_{bb}(t) \sigma^j_-(t_j) \rangle = 0. \tag{2.102c}$$

So that on substituting Eqs.(2.102) and the complex conjugate of (2.66) into (2.101) we get

$$\langle F_b(t) F_M(t') \rangle = \gamma \langle M \rangle \, \delta(t - t'). \tag{2.103}$$

Similarly the remaining non-vanishing correlation functions for the noise operators are found to be

$$\langle F_a(t) F_M(t') \rangle = \gamma \langle M^\dagger \rangle \, \delta(t - t'), \tag{2.104}$$

$$\langle F^\dagger_M(t) F_M(t') \rangle = (\gamma \langle N_a \rangle + r_a) \, \delta(t - t'). \tag{2.105}$$

2.4. c-Number Langevin Equations

In this section we seek to obtain the c-number equations corresponding to the operator equations (2.70), (2.78), (2.81), and(2.82). In order to obtain a unique relationship between operator and c-number Langevin equations, we have to define a certain ordering of operators. We choose here the normal ordering $a^\dagger, M^\dagger, N_a, N_b, M, a$, with the corresponding c-numbers $\alpha^*, m^*, n_a, n_b, m, \alpha$, respectively.

Equations (2.70), (2.78), (2.81), and (2.82) are already in the chosen order so that in accordance with the discussion given in reference [16] the c-number equations corresponding to Eqs(2.70), (2.78), (2.81), and (2.82) can be written as

$$\dot{\alpha} = -\frac{\mathcal{C}}{2}\alpha + gm + f_{\mathcal{C}}(t) + \eta_0(t), \tag{2.106}$$

$$\dot{m} = -\gamma m + g(n_a - n_b)\alpha + f_m(t) + \eta_1(t), \tag{2.107}$$

$$\dot{n}_a = r_a - \gamma n_a - g(\alpha^* m + m^*\alpha) + f_a(t) + \eta_2(t), \tag{2.108}$$

$$\dot{n}_b = -\gamma n_b + g(\alpha^* m + m^*\alpha) + f_b(t) + \eta_3(t), \tag{2.109}$$

where the functions $f_i(t)$ are the noise forces associated with the corresponding noise operators. And the $\eta_j(t)$ are independent noise forces the properties of which remain to be determined [16].

It is easy to see that the two time correlation functions for the noise forces $f_i(t)$ are the same as the normally ordered correlation functions of the corresponding noise operators, so that the non-vanishing correlations of interest are

$$\langle f_{\mathcal{C}}^*(t) f_{\mathcal{C}}(t')\rangle = \mathcal{C}N\delta(t-t'), \tag{2.110a}$$

$$\langle f_{\mathcal{C}}(t) f_{\mathcal{C}}(t')\rangle = \mathcal{C}M\delta(t-t'), \tag{2.110b}$$

$$\langle f_{\mathcal{C}}^*(t) f_{\mathcal{C}}^*(t')\rangle = \mathcal{C}M^*\delta(t-t'), \tag{2.110c}$$

$$\langle f_a(t) f_a(t')\rangle = (\gamma\langle n_a\rangle + r_a)\delta(t-t'), \tag{2.110d}$$

$$\langle f_b(t) f_b(t')\rangle = \gamma\langle n_b\rangle\delta(t-t'), \tag{2.110e}$$

$$\langle f_b(t) f_m(t')\rangle = \gamma\langle m\rangle\delta(t-t'), \tag{2.110f}$$

$$\langle f_a(t) f_m(t')\rangle = \gamma\langle m^*\rangle\delta(t-t'), \tag{2.110g}$$

$$\langle f_m^*(t)f_m(t')\rangle = (\gamma\langle n_a\rangle + r_a)\delta(t-t'). \tag{2.110h}$$

We now proceed to determine the properties of the noise forces $\eta_j(t)$ by imposing the requirement that the c-number equations of evolution for the first and the second moments have identical form as the corresponding operator equations. We note that the expectation values of Eqs.(2.70), (2.78), (2.81), and (2.82) and Eqs.(2.106), (2.107), (2.108), and (2.109) will have identical form if

$$\langle \eta_j(t)\rangle = 0. \tag{2.111}$$

Employing (2.70) it can be established that

$$\begin{aligned}\frac{d}{dt}\langle a(t)a(t)\rangle &= \langle \dot{a}(t)a(t)\rangle + \langle a(t)\dot{a}(t)\rangle \\ &= -\mathcal{C}\langle a(t)a(t)\rangle + 2g\langle a(t)M(t)\rangle \\ &\quad+\langle F_{\mathcal{C}}(t)a(t)\rangle + \langle a(t)F_{\mathcal{C}}(t)\rangle,\end{aligned} \tag{2.112}$$

and with the aid of (2.106) the corresponding c-number equation is

$$\begin{aligned}\frac{d}{dt}\langle \alpha(t)\alpha(t)\rangle &= -\mathcal{C}\langle \alpha(t)\alpha(t)\rangle + 2g\langle m(t)\alpha(t)\rangle \\ &\quad+2\langle \alpha(t)f_{\mathcal{C}}(t)\rangle + 2\langle \alpha(t)\eta_0(t)\rangle.\end{aligned} \tag{2.113}$$

Equations (2.112) and (2.113) are identical if

$$\langle \alpha(t)\eta_0(t)\rangle = 0. \tag{2.114}$$

Formally integrating (2.106) we have

$$\alpha(t) = \alpha(0) - \frac{\mathcal{C}}{2}\int_0^t \alpha(t')dt' + g\int_0^t m(t')dt' + \int_0^t f_{\mathcal{C}}(t')dt' + \int_0^t \eta_0(t')dt', \tag{2.115}$$

so that the left-hand side of Eq.(2.114) can be written as

$$\begin{aligned}\langle \alpha(t)\eta_0(t)\rangle &= \langle \alpha(0)\eta_0(t)\rangle - \frac{\mathcal{C}}{2}\int_0^t \langle \alpha(t')\eta_0(t)\rangle dt' + g\int_0^t \langle m(t')\eta_0(t)\rangle dt' \\ &\quad+\int_0^t \langle f_{\mathcal{C}}(t')\eta_0(t)\rangle dt' + \int_0^t \langle \eta_0(t')\eta_0(t)\rangle dt'.\end{aligned} \tag{2.116}$$

Assuming that the system variables at earlier time are not affected by the noise force at a later time, we have

$$\langle \alpha(0)\eta_0(t)\rangle = \langle \alpha(t')\eta_0(t)\rangle = \langle m(t')\eta_0(t)\rangle = 0. \tag{2.117a}$$

Noting that $f_C(t)$ and $\eta_0(t)$ are independent noise forces we see that

$$\langle f_C(t')\eta_0(t)\rangle = \langle f_C(t')\rangle\langle\eta_0(t)\rangle = 0. \tag{2.117b}$$

Hence Eq.(2.116) takes the form

$$\langle\alpha(t)\eta_0(t)\rangle = \int_0^t \langle\eta_0(t')\eta_0(t)\rangle dt'. \tag{2.118}$$

Comparing (2.114) with (2.118) we see that [16]

$$\langle\eta_0(t')\eta_0(t)\rangle = 0. \tag{2.119a}$$

By a similar argument one can easily show that

$$\langle\eta_0^*(t')\eta_0^*(t)\rangle = 0, \tag{2.119b}$$

$$\langle\eta_0^*(t')\eta_0(t)\rangle = 0. \tag{2.119c}$$

Again applying (2.78) we have

$$\frac{d}{dt}\langle N_a(t)N_a(t)\rangle = 2r_a\langle N_a\rangle + -2\gamma\langle N_aN_a\rangle - g\left[\langle a^\dagger MN_a\rangle + \langle M^\dagger aN_a\rangle\right]$$

$$-g\left[\langle N_a a^\dagger M\rangle + \langle N_aM^\dagger a\rangle\right] + \langle F_aN_a\rangle + \langle N_aF_a\rangle. \tag{2.120a}$$

The first term in the first square brackets and the second term in the second square brackets are not in the chosen order. Therefore we have to bring N_a to the left of M in the first case and $M^\dagger$ to the left of N_a in the second case. To do this we first evaluate the commutation relation $[M, N_a]$. With the aid of Eqs.(2.67) and (2.68) this commutation relation can be written as

$$\begin{aligned}
[M, N_a] &= M\,N_a - N_a\,M \\
&= \sum_{i,j}\Theta(t-t_i)\Theta(t-t_j)\sigma_-^i(t)\sigma_{aa}^j(t) \\
&\quad -\sum_{i,j}\Theta(t-t_i)\Theta(t-t_j)\sigma_{aa}^j(t)\sigma_-^i(t) \\
&= \sum_{i,j}\Theta(t-t_i)\Theta(t-t_j)[\sigma_-^i(t), \sigma_{aa}^j(t)].
\end{aligned}$$

We note that

$$[\sigma_-^i(t), \sigma_{aa}^j(t)] = \sigma_-^j(t)\,\delta_{ij},$$

and hence we get

$$[M, N_a] = \sum_j \Theta(t - t_j)\sigma_-^j(t) = M.$$

Applying this commutation relation we can rewrite (2.120a) in the normal order as

$$\frac{d}{dt}\langle N_a(t)N_a(t)\rangle = 2r_a\langle N_a\rangle - 2\gamma\langle N_a N_a\rangle - 2g\left[\langle a^\dagger N_a M\rangle + \langle M^\dagger N_a a\rangle\right]$$

$$-g\left[\langle a^\dagger M\rangle + \langle M^\dagger a\rangle\right] + \langle F_a N_a\rangle + \langle N_a F_a\rangle. \tag{2.120b}$$

We now use Eq.(2.108) to obtain the corresponding c-number equation

$$\frac{d}{dt}\langle n_a(t)n_a(t)\rangle = 2r_a\langle n_a\rangle - 2\gamma\langle n_a n_a\rangle - 2g\left[\langle \alpha^* m n_a\rangle + \langle m^*\alpha n_a\rangle\right] + 2\langle n_a\eta_2\rangle + 2\langle n_a f_a\rangle. \tag{2.121}$$

Equations (2.120b) and (2.121) are identical if

$$2\langle n_a(t)\eta_2(t)\rangle = -g(\langle\alpha^* m\rangle + \langle m^*\alpha\rangle). \tag{2.122}$$

On formally integrating (2.108) we find

$$n_a(t) = n_a(0) + r_a t - \gamma\int_0^t n_a(t')dt' - g\int_0^t (\alpha^*(t')m(t') + m^*(t')\alpha(t'))dt'$$

$$+\int_0^t f_a(t')dt' + \int_0^t \eta_2(t')dt', \tag{2.123}$$

in view of this one readily obtains

$$\langle n_a(t)\eta_2(t)\rangle = \int_0^t \langle\eta_2(t)\eta_2(t')\rangle dt'. \tag{2.124}$$

On comparing (2.122) and (2.124) we see that [16]

$$\langle\eta_2(t)\eta_2(t')\rangle = -g(\langle\alpha^* m\rangle + \langle m^*\alpha\rangle)\delta(t - t'). \tag{2.125}$$

In a similar manner, the remaining non-vanishing correlation functions are found to be

$$\langle\eta_1(t)\eta_1(t')\rangle = 2g\langle m\alpha\rangle\delta(t - t'), \tag{2.126a}$$

$$\langle\eta_2(t')\eta_3(t)\rangle = g(\langle\alpha^* m\rangle + \langle m^*\alpha\rangle)\delta(t - t'), \tag{2.126b}$$

$$\langle\eta_3(t)\eta_3(t')\rangle = -g(\langle\alpha^* m\rangle + \langle m^*\alpha\rangle)\delta(t - t'). \tag{2.126c}$$

2.5. Adiabatic Elimination of Atomic Variables

The cavity decay rate $\mathcal{C}$ is much smaller than the atomic decay rate γ, so that the evolution of the atomic variables happens on a much shorter time scale than the cavity mode [4,8]. Under this condition we can adiabatically eliminate the atomic variables m, n_a, and n_b and derive an equation for α alone. Thus we first set the time derivative of m in Eq.(2.107) equal to zero and obtain the adiabatic value for the atomic polarization

$$m = \frac{g}{\gamma}(n_a - n_b)\alpha + \frac{1}{\gamma}(f_m + \eta_1). \tag{2.127}$$

Substituting this result into the equations for α, n_a and n_b yields

$$\dot{\alpha} = -\frac{\mathcal{C}}{2}\alpha + \frac{g^2}{\gamma}(n_a - n_b)\alpha + \frac{g}{\gamma}(f_m + \eta_1) + f_{\mathcal{C}} + \eta_0, \tag{2.128a}$$

$$\dot{n}_a = r_a - \gamma n_a - \frac{2g^2}{\gamma}(n_a - n_b)\alpha^*\alpha - \frac{g}{\gamma}(f_m^*\alpha + \alpha^* f_m) - \frac{g}{\gamma}(\eta_1^*\alpha + \alpha^*\eta_1) + f_a + \eta_2, \tag{2.128b}$$

$$\dot{n}_b = -\gamma n_b + \frac{2g^2}{\gamma}(n_a - n_b)\alpha^*\alpha + \frac{g}{\gamma}(f_m^*\alpha + \alpha^* f_m) + \frac{g}{\gamma}(\eta_1^*\alpha + \alpha^*\eta_1) + f_b + \eta_3. \tag{2.128c}$$

We next adiabatically eliminate the population variables n_a and n_b by setting their time derivative equal to zero. Solving the resulting set of two coupled linear equations, we obtain

$$n_a = \frac{1}{\gamma(1 + \frac{4g^2}{\gamma^2}I)}\left[r_a\left(1 + \frac{2g^2}{\gamma^2}I\right) + \left(1 + \frac{2g^2}{\gamma^2}I\right)\zeta_a + \frac{2g^2}{\gamma^2}I\,\zeta_b\right], \tag{2.129a}$$

$$n_b = \frac{1}{\gamma(1 + \frac{4g^2}{\gamma^2}I)}\left[r_a\frac{2g^2}{\gamma^2}I + \left(1 + \frac{2g^2}{\gamma^2}I\right)\zeta_b + \frac{2g^2}{\gamma^2}I\,\zeta_a\right], \tag{2.129b}$$

in which I is the intensity $|\alpha|^2$ of the radiation field and the noise functions ζ_a and ζ_b are defined by

$$\zeta_a = f_a + \eta_2 - \frac{g}{\gamma}(f_m^*\alpha + \alpha^* f_m) - \frac{g}{\gamma}(\eta_1^*\alpha + \alpha^*\eta_1), \tag{2.130a}$$

$$\zeta_b = f_b + \eta_3 + \frac{g}{\gamma}(f_m^*\alpha + \alpha^* f_m) + \frac{g}{\gamma}(\eta_1^*\alpha + \alpha^*\eta_1). \tag{2.130b}$$

Substituting Eq.(2.129a) and (2.129b) into (2.128a), we obtain an equation of evolution for α alone

$$\dot{\alpha} = -\frac{\mathcal{C}}{2}\alpha + \frac{\mathcal{A}\alpha}{2(1 + \frac{\mathcal{B}}{\mathcal{A}}I)} + f_\alpha, \tag{2.131}$$

where $\mathcal{A}$ and $\mathcal{B}$ are the gain and the saturation coefficients for the laser and are given by

$$\mathcal{A} = \frac{2r_a g^2}{\gamma^2}, \tag{2.132a}$$

$$\mathcal{B} = \frac{4g^2}{\gamma^2}\mathcal{A}, \tag{2.132b}$$

and the noise force f_α is given by

$$f_\alpha = f_C + \eta_0 + \frac{g}{\gamma}(f_m + \eta_1) + \frac{g^2\alpha}{\gamma^2(1+\frac{\mathcal{B}}{\mathcal{A}}I)}(\zeta_a - \zeta_b). \tag{2.133}$$

Employing this expression for f_α and the definition for ζ_a and ζ_b one can easily show that

$$\langle f_\alpha(t)\rangle = 0 \tag{2.134}$$

Applying (2.110), (2.130a), (2.130b) and the adiabatic values (2.127), (2.129a), (2.129b), the two time correlation function for the noise force f_α is found to be

$$\langle f_\alpha(t)f_\alpha(t')\rangle = \left\langle \mathcal{C}M - \frac{\mathcal{B}\alpha^2}{4\left(1+\frac{\mathcal{B}}{\mathcal{A}}I\right)^2}\left(3+\frac{\mathcal{B}}{\mathcal{A}}I\right)\right\rangle \delta(t-t'), \tag{2.135a}$$

$$\langle f_\alpha^*(t)f_\alpha^*(t')\rangle = \left\langle \mathcal{C}M^* - \frac{\mathcal{B}\alpha^{*2}}{4\left(1+\frac{\mathcal{B}}{\mathcal{A}}I\right)^2}\left(3+\frac{\mathcal{B}}{\mathcal{A}}I\right)\right\rangle \delta(t-t'), \tag{2.135b}$$

$$\langle f_\alpha^*(t)f_\alpha(t')\rangle = \left\langle \mathcal{C}N + \frac{\mathcal{A}}{\left(1+\frac{\mathcal{B}}{\mathcal{A}}I\right)^2}\left[1+\frac{\mathcal{B}}{4\mathcal{A}}I\left(3+\frac{\mathcal{B}}{\mathcal{A}}I\right)\right]\right\rangle \delta(t-t'). \tag{2.135c}$$

We use these results to determine the the photon statistics, the power spectrum and the quadrature fluctuations.

3. BELOW THRESHOLD LASER DYNAMICS

In this chapter we wish to consider a limiting case of laser operation. In the linear approximation where $\mathcal{B} = 0$, Eq.(2.131) reduces to

$$\dot{\alpha} = -\frac{1}{2}(\mathcal{C} - \mathcal{A})\alpha + f_\alpha(t), \tag{3.1}$$

in which the noise force $f_\alpha(t)$ in this approximation is given (cf. 2.133) by

$$f_\alpha(t) = f_\mathcal{C}(t) + \eta_0(t) + \frac{g}{\gamma}(f_m(t) + \eta_1(t)), \tag{3.2}$$

and characterized by

$$\langle f_\alpha(t)\rangle = 0, \tag{3.3a}$$

$$\langle f_\alpha(t) f_\alpha(t')\rangle = \mathcal{C}M\delta(t - t'), \tag{3.3b}$$

$$\langle f_\alpha^*(t) f_\alpha^*(t')\rangle = \mathcal{C}M^*\delta(t - t'), \tag{3.3c}$$

$$\langle f_\alpha^*(t) f_\alpha(t')\rangle = (\mathcal{C}N + \mathcal{A})\delta(t - t'). \tag{3.3d}$$

Since no well-behaved solution of Eq.(3.1) exists for $\mathcal{A} > \mathcal{C}$, we interpret $\mathcal{A} = \mathcal{C}$ as the threshold condition. Hence, for a laser coupled to a squeezed vacuum reservoir and operating below threshold, i.e., $\mathcal{A} < \mathcal{C}$, the solution of Eq.(3.1) can be written as

$$\alpha(t) = c(t)\alpha(0) + R(t), \tag{3.4}$$

where

$$c(t) = \exp\left[-\frac{1}{2}(\mathcal{C} - \mathcal{A})t\right], \tag{3.5a}$$

$$R(t) = c(t)\int_0^t dt' f_\alpha(t')\, c(-t'). \tag{3.5b}$$

3.1. Photon Statistics

In this section we seek to calculate, applying the Q-function, the mean and variance of the photon number as well as the photon number distribution of a laser coupled to a squeezed vacuum and operating below threshold. We obtain the Q-function via the antinormally ordered characteristic function.

3.1.1. The Q-function

The Q-function for a laser coupled to a squeezed vacuum reservoir and operating below threshold is expressible as

$$Q(\alpha^*, \alpha, t) = \frac{1}{\pi^2} \int d^2z \Phi(z^*, z, t) \exp\left[z^*\alpha - z\alpha*\right], \tag{3.6}$$

where the antinormally ordered characteristic function in the Heisenberg picture is expressed as [16]

$$\Phi(z^*, z, t) = Tr\left\{\rho(0) \exp\left[-z^* a(t)\right] \exp\left[z a^\dagger(t)\right]\right\}, \tag{3.7}$$

here $\rho(0)$ is the density operator for the cavity mode and the reservoir at the initial time.

We carry out our analysis using c-number variables corresponding to the normal ordering. To this end, applying the identity

$$e^A e^B = e^B e^A e^{[A,B]}, \tag{3.8}$$

one can put expression (3.7) in the normal order

$$\Phi(z^*, z, t) = e^{-z^* z} \left\langle \exp\left[z a^\dagger(t)\right] \exp\left[-z^* a(t)\right]\right\rangle. \tag{3.9}$$

Hence the characteristic function can be expressed as

$$\Phi(z^*, z, t) = e^{-z^* z} \left\langle \exp\left[z\alpha^*(t) - z^*\alpha(t)\right]\right\rangle. \tag{3.10}$$

Using (3.4) and its complex conjugate, we can rewrite (3.10) as

$$\Phi(z^*, z, t) = e^{-z^* z} \left\langle \exp\left[z c(t)\alpha^*(0) - z^* c(t)\alpha(0) + zR^*(t) - z^* R(t)\right]\right\rangle. \tag{3.11}$$

The radiation variables at the initial time and the noise forces at a later time are uncorrelated. In accordance with the discussion given in Ref. [16] we can write (3.11) in the form

$$\Phi(z^*, z, t) = e^{-z^* z} \left\langle \exp\left[z c(t)\alpha^*(0) - z^* c(t)\alpha(0)\right]\right\rangle$$

$$\times \left\langle \exp\left[zR^*(t) - z^* R(t)\right]\right\rangle. \tag{3.12}$$

We consider the case for which the cavity mode is initially in a vacuum state. It then turns out that

$$\langle\alpha(0)\rangle = 0 \tag{3.13a}$$

and hence

$$\left\langle \exp\left[zc(t)\alpha^*(0) - z^*c(t)\alpha(0)\right]\right\rangle = 1. \tag{3.13b}$$

Consequently,

$$\Phi(z^*, z, t) = e^{-z^*z}\left\langle \exp\left[zR^*(t) - z^*R(t)\right]\right\rangle. \tag{3.14}$$

Since the expression in the square brackets is a random Gaussian variable, i.e., has a vanishing mean, we can express (3.14) in the form [19]

$$\Phi(z^*, z, t) = e^{-z^*z}\exp\left[\frac{1}{2}\langle\chi^2(t)\rangle\right], \tag{3.15a}$$

where

$$\chi(t) = zR^*(t) - z^*R(t). \tag{3.15b}$$

We observe that

$$\langle\chi^2(t)\rangle = z^2\langle R^{*2}(t)\rangle - 2zz^*\langle R^*(t)R(t)\rangle + z^{*2}\langle R^2(t)\rangle. \tag{3.16}$$

With the aid of (3.5b), we have

$$\langle R^{*2}(t)\rangle = c^2(t)\int_0^t dt'\int_0^t dt''\langle f_\alpha^*(t')f_\alpha^*(t'')\rangle c(-t')c(-t''), \tag{3.17}$$

so that substituting (3.3c) into (3.17), we find

$$\begin{aligned}\langle R^{*2}(t)\rangle &= \mathcal{C}M^*c^2(t)\int_0^t dt'c^2(-t')\\ &= \frac{\mathcal{C}M^*}{\mathcal{C}-\mathcal{A}}(1-c^2(t)).\end{aligned} \tag{3.18a}$$

Similarly

$$\langle R^2(t)\rangle = \frac{\mathcal{C}M}{\mathcal{C}-\mathcal{A}}(1-c^2(t)), \tag{3.18b}$$

$$\langle R^*(t)R(t)\rangle = \frac{\mathcal{C}N+\mathcal{A}}{\mathcal{C}-\mathcal{A}}(1-c^2(t)). \tag{3.18c}$$

Now combination of (3.18) with (3.16) yields

$$\begin{aligned}\langle\chi^2(t)\rangle = {}&\frac{\mathcal{C}M}{\mathcal{C}-\mathcal{A}}(1-c^2(t))(z^2+z^{*2})\\ &-\frac{2(\mathcal{C}N+\mathcal{A})}{\mathcal{C}-\mathcal{A}}(1-c^2(t))zz^*,\end{aligned} \tag{3.19}$$

in which we have assumed $M = M^*$. Hence the characteristic function takes the form

$$\Phi(z^*, z, t) = \exp\left[-L(t)zz^* + H(t)(z^2+z^{*2})\right], \tag{3.20}$$

where

$$H(t) = \frac{\mathcal{C}M}{2(\mathcal{C}-\mathcal{A})}\left(1 - c^2(t)\right), \tag{3.21a}$$

$$L(t) = \frac{\mathcal{C}(N+1)}{\mathcal{C}-\mathcal{A}} - \left(\frac{\mathcal{C}N+\mathcal{A}}{\mathcal{C}-\mathcal{A}}\right)c^2(t). \tag{3.21b}$$

Finally, substituting (3.20) into (3.6) and carrying out the integration, the Q-function for a laser coupled to a squeezed vacuum reservoir and operating below threshold is found to be

$$Q(\alpha^*, \alpha, t) = \frac{1}{\pi S(t)} \exp\left[-L'(t)\alpha\alpha^* + H'(t)(\alpha^2 + \alpha^{*2})\right], \tag{3.22}$$

where

$$S^2(t) = L^2(t) - 4H^2(t), \tag{3.23a}$$

$$L'(t) = \frac{L(t)}{S^2(t)}, \tag{3.23b}$$

$$H'(t) = \frac{H(t)}{S^2(t)}. \tag{3.23c}$$

3.1.2. The Mean Photon Number and Variance

We now proceed to calculate using the Q-function the mean, variance and the photon number distribution for the cavity mode under consideration.

A. *TheMeanPhotonNumber*

The mean photon number is expressible as

$$\langle n\rangle = Tr\left\{\rho a^\dagger a\right\} \tag{3.24a}.$$

Applying the commutation relation

$$[a, a^\dagger] = 1 \tag{3.24b}$$

along with the fact that

$$Tr\{\rho\} = 1,$$

Eq.(3.24a) can be put in the form

$$\langle n\rangle = Tr\left\{\rho a a^\dagger\right\} - 1. \tag{3.25}$$

Employing the identity operator

$$I = \int \frac{d^2\alpha}{\pi} |\alpha\rangle\langle\alpha|,$$

we get

$$\langle n \rangle = \int d^2\alpha \frac{1}{\pi} \langle\alpha|\rho|\alpha\rangle \alpha\alpha^* - 1$$

or

$$\langle n \rangle = \int d^2\alpha \; Q(\alpha^*, \alpha, t)\alpha\alpha^* - 1, \tag{3.26}$$

so that on applying the Q-function (3.22), there follows

$$\begin{aligned} \langle n \rangle &= \frac{1}{S(t)} \int \frac{d^2\alpha}{\pi} \alpha\alpha^* \exp\left[-L'(t)\alpha\alpha^* + H'(t)(\alpha^2 + \alpha^{*2})\right] - 1 \\ &= \frac{1}{S(t)} \left(-\frac{\partial}{\partial L'}\right) \int \frac{d^2\alpha}{\pi} \exp\left[-L'(t)\alpha\alpha^* + H'(t)(\alpha^2 + \alpha^{*2})\right] - 1. \end{aligned} \tag{3.27}$$

Finally, carrying out the integration and the differentiation, the mean photon number is found to be

$$\begin{aligned} \langle n \rangle &= L(t) - 1 \\ &= \frac{\mathcal{C}N + \mathcal{A}}{\mathcal{C} - \mathcal{A}} (1 - \exp\left[-(\mathcal{C} - \mathcal{A})t\right]). \end{aligned} \tag{3.28}$$

At steady state, this result reduces to

$$\langle n \rangle = \frac{\mathcal{C}N + \mathcal{A}}{\mathcal{C} - \mathcal{A}}. \tag{3.29}$$

If $\mathcal{A} = 0$ i.e., there are no atoms in the cavity, the mean photon number in the cavity at steady state is N, which is the same as the mean photon number of the reservoir. For $N = M = 0$, i.e., a laser coupled to an ordinary vacuum reservoir, we have [6]

$$\langle n \rangle = \frac{\mathcal{A}}{\mathcal{C} - \mathcal{A}}. \tag{3.30}$$

B. *Variance*

The variance of the photon number is expressible as

$$\Delta n^2 = \langle (a^\dagger a)^2 \rangle - \langle a^\dagger a \rangle^2. \tag{3.31}$$

On account of the commutation relation (3.24b), this expression takes the form

$$\Delta n^2 = \langle a^2 a^{\dagger 2} \rangle - \langle a^\dagger a \rangle^2 - 3\langle a^\dagger a \rangle - 2. \tag{3.32}$$

Following a similar procedure used in arriving at (3.26), the first term in (3.32) can be expressed in terms of the Q-function as

$$\langle a^2 a^{\dagger 2}\rangle = \int d^2\alpha Q(\alpha^*, \alpha, t)\alpha^2\alpha^{*2}, \tag{3.33}$$

so that inview of (3.22), we have

$$\begin{aligned}\langle a^2 a^{\dagger 2}\rangle &= \frac{1}{S(t)}\int \frac{d^2\alpha}{\pi}\alpha^2\alpha^{*2}\exp\left[-L'(t)\alpha^*\alpha + H'(t)(\alpha^2+\alpha^{*2}\right] \\ &= \frac{1}{S(t)}\left(\frac{\partial^2}{\partial L'^2}\right)\int\frac{d^2\alpha}{\pi}\exp\left[-L'(t)\alpha^*\alpha + H'(t)(\alpha^2+\alpha^{*2})\right].\end{aligned} \tag{3.34}$$

Upon carrying out the integration, we find

$$\langle a^2 a^{\dagger 2}\rangle = \frac{1}{S(t)}\left(\frac{\partial^2}{\partial L'^2}\right)\left[\frac{1}{L'^2 - 4H'^2}\right]^{\frac{1}{2}}, \tag{3.35}$$

so that performing the differentiation along with (3.23), we get

$$\langle a^2 a^{\dagger 2}\rangle = 2L^2(t) - 4H^2(t). \tag{3.36}$$

Hence substituting (3.36) and (3.28) into (3.32), the variance of the photon number is found to be

$$\Delta n^2 = \langle n\rangle(\langle n\rangle + 1) - 4H^2. \tag{3.37}$$

At steady state one can easily show that

$$\Delta n^2 > \langle n\rangle.$$

This shows that the photon statistics of the radiation is supper-Poissonian. For $M = N = 0$ Eq.(3.37) reduces to

$$\Delta n^2 = \langle n\rangle(\langle n\rangle + 1), \tag{3.38}$$

where $\langle n\rangle$ is given by (3.30). We observe that the radiation inside the cavity is in a chaotic state.

3.1.3. The Photon Number Distribution

The photon number distribution $P(n, t)$ at any time can be expressed in terms of the Q-function as [20]

$$P(n,t) = \frac{\pi}{n!}\frac{\partial^{2n}}{\partial\alpha^n\partial\alpha^{*n}}\left[Q(\alpha^*, \alpha, t)\exp\left[\alpha\alpha^*\right]\right]_{\alpha=\alpha^*=0}. \tag{3.39}$$

Therefore applying the Q-function (3.22) we have

$$P(n,t) = \frac{1}{S(t)n!}\frac{\partial^{2n}}{\partial\alpha^n\partial\alpha^{*n}}\Big[\exp\left[b(t)\alpha\alpha^* + H'(t)(\alpha^2+\alpha^{*2})\right]\Big]_{\alpha=\alpha^*=0}, \tag{3.40}$$

where

$$b(t) = 1 - L'(t). \tag{3.41}$$

Expanding the exponential function in power series, we have

$$P(n,t) = \frac{1}{S(t)n!}\sum_{i,j,k}\frac{b^i H'^{(j+k)}}{i!j!k!}\frac{\partial^{2n}}{\partial\alpha^n\partial\alpha^{*n}}\left[\alpha^{i+2j}\alpha^{*i+2k}\right]_{\alpha=\alpha^*=0} \tag{3.42}$$

and in view of the fact that

$$\frac{\partial^n}{\partial\alpha^n}\alpha^p = \frac{p!}{(p-n)!}\alpha^{p-n}, \tag{3.43}$$

we find

$$P(n,t) = \frac{1}{S(t)n!}\sum_{i,j,k}\left[\frac{b^i H'^{(j+k)}(i+2k)!(i+2j)!\alpha^{i+2j-n}\alpha^{*i+2k-n}}{i!j!k!(i+2k-n)!(i+2j-n)!}\right]_{\alpha=\alpha^*=0}. \tag{3.44}$$

Applying the condition $\alpha = \alpha^* = 0$, we get

$$P(n,t) = \frac{1}{S(t)n!}\sum_{i,j,k}\left[\frac{b^i H'^{(j+k)}(i+2k)!(i+2j)!}{i!j!k!(i+2k-n)!(i+2j-n)!}\delta_{i+2j,n}\delta_{i+2k,n}\right], \tag{3.45}$$

so that on account of the results

$$j = k,$$
$$j = \frac{n-i}{2},$$

the photon number distribution can be expressed as

$$P(n,t) = \frac{n!}{S(t)}\sum_{i=0}^{n}\frac{b^i(t)H'^{n-i}(t)}{i!\left[(\frac{n-i}{2})!\right]^2}. \tag{3.46}$$

For $N = M = 0$, we obtain

$$b = \frac{\mathcal{A}}{\mathcal{C}},$$
$$S = \frac{\mathcal{C}}{\mathcal{C}-\mathcal{A}},$$
$$H' = 0.$$

We thus note that

$$H'^{n-i} = \delta_{n,i} \tag{3.47}$$

Hence combining this results with (3.46), we obtain the photon number distribution of a laser coupled to an ordinary vacuum reservoir at steady state:

$$P(n) = \left(1 - \frac{\mathcal{A}}{\mathcal{C}}\right)\left(\frac{\mathcal{A}}{\mathcal{C}}\right)^{n}. \tag{3.48}$$

In terms of the mean photon number

$$P(n) = \frac{\langle n\rangle^{n}}{(\langle n\rangle + 1)^{n+1}}, \tag{3.49}$$

which is the photon number distribution of a chaotic light.

One can also easily establish that for $\mathcal{A} = 0$ Eq.(3.46) at steady state reduces to the photon number distribution of a squeezed vacuum reservoir [20]

$$P(n) = \frac{(-1)^{n}}{2^{n} n!}\frac{\tanh^{n} r}{\cosh r} H_{n}^{2}(0), \tag{3.50}$$

where $H_n(0)$ is a Hermiete polynomial.

3.2. Spectrum of Intensity Fluctuations

Next we wish to calculate the intensity fluctuations spectrum of the output radiation. The spectrum of intensity fluctuations is expressible as

$$S(\omega) = \int_{-\infty}^{\infty} d\tau \left\langle I^{out}(t+\tau), I^{out}(t)\right\rangle_{ss} \exp[i\omega\tau], \tag{3.51}$$

where

$$\left\langle I^{out}(t+\tau), I^{out}(t)\right\rangle = \left\langle I^{out}(t+\tau) I^{out}(t)\right\rangle - \left\langle I^{out}(t+\tau)\right\rangle \left\langle I^{out}(t)\right\rangle, \tag{3.52a}$$

$$I^{out}(t) = a^{\dagger}_{out}(t) a_{out}(t). \tag{3.52b}$$

Since (3.52b) is in normal order, the corresponding c-number expression is

$$I^{out}(t) = \alpha^{*}_{out}(t)\alpha_{out}(t). \tag{3.53}$$

The output variable $\alpha_{out}(t)$ is expressible in terms of the interacavity variable $\alpha(t)$ and the input variable $\alpha_{in}(t)$ as

$$\alpha_{out}(t) = \sqrt{\mathcal{C}}\ \alpha(t) - \alpha_{in}(t), \tag{3.54}$$

where the input variable is given by

$$\alpha_{in} = \frac{1}{\sqrt{\mathcal{C}}}\ f_{\mathcal{C}}(t). \tag{3.55}$$

Employing (3.4)one can write Eq.(3.54) as

$$\alpha_{out} = \sqrt{\mathcal{C}}\ c(t)\alpha(0) + \sqrt{\mathcal{C}}\ R(t) - \alpha_{in}(t), \tag{3.56}$$

with the aid of which we get

$$\langle I^{out}(t)\rangle = \mathcal{C}c^2(t)\langle\alpha^*(0)\alpha(0)\rangle + \mathcal{C}\langle R^*(t)R(t)\rangle - \sqrt{\mathcal{C}}\langle R^*(t)\alpha_{in}(t)\rangle$$
$$-\sqrt{\mathcal{C}}\langle\alpha_{in}^*(t)R(t)\rangle + \langle\alpha_{in}^*(t)\alpha_{in}(t)\rangle. \tag{3.57}$$

For a cavity mode initially in a vacuum state, we see that

$$\langle\alpha^*(0)\alpha(0)\rangle = \langle\alpha^*(0)\rangle\langle\alpha(0)\rangle = 0. \tag{3.58}$$

Moreover, applying the definition of R(t), the second term in (3.57) can be written as

$$\begin{aligned}\langle R^*(t)R(t)\rangle &= c^2(t)\int_0^t dt'\int_0^t dt''\langle f_\alpha^*(t')f_\alpha(t'')\rangle c(-t')c(-t'')\\ &= (\mathcal{C}N+\mathcal{A})c^2(t)\int_0^t dt'c^2(-t')\\ &= \frac{\mathcal{C}N+\mathcal{A}}{\mathcal{C}-\mathcal{A}}\left(1-c^2(t)\right).\end{aligned} \tag{3.59}$$

Now using (3.5b) and (3.55), we obtain

$$\sqrt{\mathcal{C}}\langle R^*(t)\alpha_{in}(t)\rangle = c(t)\int_0^t\langle f_\alpha^*(t')f_\mathcal{C}(t)\rangle c(-t')dt', \tag{3.60}$$

so that on account of (3.2) we have

$$\sqrt{\mathcal{C}}\langle R^*(t)\alpha_{in}(t)\rangle = c(t)\int_0^t\langle f_\mathcal{C}^*(t')f_\mathcal{C}(t)\rangle c(-t')dt'. \tag{3.61}$$

Hence substituting (2.110a) into (3.61) and carrying out the integration, we find

$$\sqrt{\mathcal{C}}\langle R^*(t)\alpha_{in}(t)\rangle = \frac{\mathcal{C}N}{2}. \tag{3.62}$$

Similarly

$$\sqrt{\mathcal{C}}\langle\alpha_{in}^*(t)R(t)\rangle = \frac{\mathcal{C}N}{2}. \tag{3.63a}$$

We note that the mean photon number of the reservoir is

$$\langle\alpha_{in}^*(t)\alpha_{in}(t)\rangle = N. \tag{3.63b}$$

Hence combination of (3.58), (3.59), (3.62) and (3.63) with (3.57) yields

$$\begin{aligned}\langle I^{out}(t)\rangle &= \frac{\mathcal{C}(\mathcal{C}N+\mathcal{A})}{\mathcal{C}-\mathcal{A}}(1-c^2(t)) + (1-\mathcal{C})N \\ &= \mathcal{C}\langle n\rangle + (1-\mathcal{C})N. \end{aligned} \tag{3.64}$$

Upon replacing t by $t+\tau$ we get

$$\begin{aligned}\langle I^{out}(t+\tau)\rangle &= \frac{\mathcal{C}(\mathcal{C}N+\mathcal{A})}{\mathcal{C}-\mathcal{A}}(1-c^2(t+\tau)) + (1-\mathcal{C})N \\ &= \mathcal{C}\langle n\rangle_{t+\tau} + (1-\mathcal{C})N, \end{aligned} \tag{3.65}$$

where $\langle n\rangle_{t+\tau}$ is the mean photon number of the the radiation inside the cavity at time $t+\tau$ given by (3.28). At steady state

$$\langle I^{out}(t)\rangle_{ss} = \langle I^{out}(t+\tau)\rangle_{ss}. \tag{3.66}$$

With the aid of (3.52b), the first term in Eq.(3.52a) can be written as

$$\langle I^{out}(t+\tau)I^{out}(t)\rangle = \langle a^{\dagger}_{out}(t+\tau)a_{out}(t+\tau)a^{\dagger}_{out}(t)a_{out}(t)\rangle. \tag{3.67}$$

On account of the commutation relation

$$[a_{out}(t+\tau), a^{\dagger}_{out}(t)] = \delta(\tau), \tag{3.68}$$

expression (3.67) can be put in normal order as

$$\begin{aligned}\langle I^{out}(t+\tau)I^{out}(t)\rangle &= \langle a^{\dagger}_{out}(t+\tau)a_{out}(t)\rangle\delta(\tau) \\ &+\langle a^{\dagger}_{out}(t+\tau)a^{\dagger}_{out}(t)a_{out}(t+\tau)a_{out}(t)\rangle, \end{aligned} \tag{3.69}$$

so that the corresponding c-number expression is

$$\begin{aligned}\langle I^{out}(t+\tau)I^{out}(t)\rangle &= \langle \alpha^{*}_{out}(t+\tau)\alpha_{out}(t)\rangle\delta(\tau) \\ &+\langle \alpha^{*}_{out}(t+\tau)\alpha^{*}_{out}(t)\alpha_{out}(t+\tau)\alpha_{out}(t)\rangle. \end{aligned} \tag{3.70}$$

With the help of (3.56) one can write

$$\begin{aligned}\langle \alpha^{*}_{out}(t+\tau)\alpha_{out}(t)\rangle &= \mathcal{C}\langle R^{*}(t+\tau)R(t)\rangle - \sqrt{\mathcal{C}}\langle R^{*}(t+\tau)\alpha_{in}(t)\rangle \\ &-\sqrt{\mathcal{C}}\langle \alpha^{*}_{in}(t+\tau)R(t)\rangle + \langle \alpha^{*}_{in}(t+\tau)\alpha_{in}(t)\rangle, \end{aligned} \tag{3.71}$$

Now employing (3.5b) we see that

$$\begin{aligned}\langle R^*(t+\tau)R(t)\rangle &= c(t+\tau)c(t)\int_0^{t+\tau}dt'\int_0^t dt''\langle f_\alpha^*(t')f_\alpha(t'')\rangle c(-t')c(-t'')\\ &= (\mathcal{C}N+\mathcal{A})c(t+\tau)c(t)\int_0^t dt'c^2(-t')\\ &= \frac{\mathcal{C}N+\mathcal{A}}{\mathcal{C}-\mathcal{A}}\left(1-c^2(t)\right)c(\tau).\end{aligned} \tag{3.72}$$

And using the definition (3.55) the second term in Eq.(3.71) can be evaluated

$$\begin{aligned}\sqrt{\mathcal{C}}\langle R^*(t+\tau)\alpha_{in}(t)\rangle &= \langle R^*(t+\tau)f_{\mathcal{C}}(t)\rangle\\ &= c(t+\tau)\int_0^{t+\tau}dt'\langle f_\alpha^*(t')f_{\mathcal{C}}(t)\rangle c(-t')\\ &= c(t+\tau)\int_0^{t+\tau}dt'\langle f_{\mathcal{C}}^*(t')f_{\mathcal{C}}(t)\rangle c(-t')\\ &= \mathcal{C}Nc(\tau).\end{aligned} \tag{3.73}$$

Similarly one finds

$$\sqrt{\mathcal{C}}\langle\alpha_{in}^*(t+\tau)R(t)\rangle = \mathcal{C}Nc(t)\int_0^t dt'\delta(t'-(t+\tau)),$$

in this expression t' can not exceed t, but the Dirac delta function is zero except at $t'=t+\tau$. Hence

$$\sqrt{\mathcal{C}}\langle\alpha_{in}^*(t+\tau)R(t)\rangle = 0, \tag{3.74a}$$

$$\langle\alpha_{in}^*(t+\tau)\alpha_{in}(t)\rangle = N\delta(\tau). \tag{3.74b}$$

Finally substituting (3.72), (3.73) and (3.74) into (3.71) we get

$$\langle\alpha_{out}^*(t+\tau)\alpha_{out}(t)\rangle = N\delta(\tau)+\frac{\mathcal{C}(\mathcal{C}N+\mathcal{A})}{\mathcal{C}-\mathcal{A}}\left(1-c^2(t)\right)c(\tau)-\mathcal{C}Nc(\tau). \tag{3.75}$$

For a cavity mode initially in a vacuum state, we have

$$\langle\alpha_{out}(t)\rangle = 0, \tag{3.76}$$

so that inview of this we can express the second term in (3.70) as [20]

$$\begin{aligned}\langle\alpha_{out}^*(t+\tau)\alpha_{out}^*(t)\alpha_{out}(t+\tau)\alpha_{out}(t)\rangle &= \langle\alpha_{out}^*(t+\tau)\alpha_{out}^*(t)\rangle\langle\alpha_{out}(t+\tau)\alpha_{out}(t)\rangle\\ &+\langle\alpha_{out}^*(t+\tau)\alpha_{out}(t+\tau)\rangle\langle\alpha_{out}^*(t)\alpha_{out}(t)\rangle\\ &+\langle\alpha_{out}^*(t+\tau)\alpha_{out}(t)\rangle\langle\alpha_{out}^*(t)\alpha_{out}(t+\tau)\rangle.\end{aligned} \tag{3.77}$$

Applying (3.56), we see that

$$\langle \alpha^*_{out}(t+\tau)\alpha^*_{out}(t)\rangle = \mathcal{C}\langle R^*(t+\tau)R^*(t)\rangle - \sqrt{\mathcal{C}}\langle R^*(t+\tau)\alpha^*_{in}(t)\rangle$$

$$-\sqrt{\mathcal{C}}\langle \alpha^*_{in}(t+\tau)R^*(t)\rangle + \langle \alpha^*_{in}(t+\tau)\alpha^*_{in}(t)\rangle, \qquad (3.78)$$

so that the first term in this equation can be written as

$$\begin{aligned}
\langle R^*(t+\tau)R^*(t)\rangle &= c(t+\tau)c(t)\int_0^{t+\tau} dt' \int_0^t dt'' \langle f^*_\alpha(t')f^*_\alpha(t'')\rangle c(-t')c(-t'') \\
&= \mathcal{C}M^* c(t+\tau)c(t)\int_0^t dt' c^2(-t') \\
&= \frac{\mathcal{C}M^*}{\mathcal{C}-\mathcal{A}}\left(1-c^2(t)\right)c(\tau).
\end{aligned}$$

$$(3.79a)$$

Similarly

$$\sqrt{\mathcal{C}}\langle \alpha^*_{in}(t+\tau)R^*(t)\rangle = 0, \qquad (3.79b)$$

$$\langle \alpha^*_{in}(t+\tau)\alpha^*_{in}(t)\rangle = M^*\delta(\tau), \qquad (3.79c)$$

$$\sqrt{\mathcal{C}}\langle R^*(t+\tau)\alpha^*_{in}(t)\rangle = \mathcal{C}M^*c(\tau), \qquad (3.79d)$$

upon substituting (3.79a-79d) into (3.78) we find

$$\langle \alpha^*_{out}(t+\tau)\alpha^*_{out}(t)\rangle = M^*\delta(\tau) + \frac{\mathcal{C}^2M^*}{\mathcal{C}-\mathcal{A}}\left(1-c^2(t)\right)c(\tau) - \mathcal{C}M^*c(\tau). \qquad (3.80)$$

Taking the complex conjugate of (3.80) and (3.75), we get

$$\langle \alpha_{out}(t+\tau)\alpha_{out}(t)\rangle = M\delta(\tau) + \frac{\mathcal{C}^2M}{\mathcal{C}-\mathcal{A}}\left(1-c^2(t)\right)c(\tau) - \mathcal{C}M c(\tau), \qquad (3.81)$$

$$\langle \alpha^*_{out}(t)\alpha_{out}(t+\tau)\rangle = N\delta(\tau) + \frac{\mathcal{C}(\mathcal{C}N+\mathcal{A})}{\mathcal{C}-\mathcal{A}}\left(1-c^2(t)\right)c(\tau) - \mathcal{C}Nc(\tau). \qquad (3.82)$$

Introducing (3.75), (3.80), (3.81) and (3.82) into (3.77) at steady state we obtain

$$\begin{aligned}
\langle \alpha^*_{out}(t+\tau)\alpha^*_{out}(t)\alpha_{out}(t+\tau)\alpha_{out}(t)\rangle &= \langle I^{out}(t)\rangle^2_{ss} + \left(\frac{\mathcal{C}^2\mathcal{A}^2(M^2+(N+1)^2)}{(\mathcal{C}-\mathcal{A})^2}\right)c^2(\tau) \\
&+ \left(\frac{2\mathcal{C}\mathcal{A}(M^2+N(N+1))}{\mathcal{C}-\mathcal{A}}\right)c(\tau)\delta(\tau) \\
&+ \left(M^2+N^2\right)\delta(\tau),
\end{aligned}$$

$$(3.83)$$

where we have assumed $M = M^*$. Finally, substituting (3.83) and (3.75) into (3.70) we get at steady state

$$\begin{aligned}\langle I^{out}(t+\tau)I^{out}(t)\rangle &= \langle I^{out}(t)\rangle^2 + \left(\frac{\mathcal{C}^2\mathcal{A}^2(M^2+(N+1)^2)}{(\mathcal{C}-\mathcal{A})^2}\right)c^2(\tau)\\ &+ \left(\frac{\mathcal{C}\mathcal{A}(2M^2+(2N+1)(N+1))}{\mathcal{C}-\mathcal{A}}\right)c(\tau)\delta(\tau)\\ &+ (M^2+N(N+1)))\,\delta(\tau),\end{aligned} \tag{3.84}$$

so that combination of (3.66) and (3.84), with (3.52a) at steady state yields

$$\begin{aligned}\langle I^{out}(t+\tau), I^{out}(t)\rangle_{ss} &= +\left(\frac{\mathcal{C}^2\mathcal{A}^2(M^2+(N+1)^2)}{(\mathcal{C}-\mathcal{A})^2}\right)c^2(\tau)\\ &+ \left(\frac{\mathcal{C}\mathcal{A}(2M^2+(2N+1)(N+1))}{\mathcal{C}-\mathcal{A}}\right)c(\tau)\delta(\tau)\\ &+ (M^2+N(N+1))\,\delta(\tau).\end{aligned} \tag{3.85}$$

Inserting this expression into (3.51) and carrying out the integration for the second and third terms we obtain

$$S(\omega) = a + b\int_{-\infty}^{\infty} d\tau \exp\left[(i\omega - (\mathcal{C}-\mathcal{A}))\tau\right], \tag{3.86}$$

where

$$a = M^2 + N(N+1) + \frac{\mathcal{C}\mathcal{A}(2M^2+(N+1)(2N+1))}{\mathcal{C}-\mathcal{A}}, \tag{3.87a}$$

$$b = \frac{\mathcal{C}^2\mathcal{A}^2(M^2+(N+1)^2)}{(\mathcal{C}-\mathcal{A})^2}. \tag{3.87b}$$

The remaining integration can be evaluated applying the stationarity property and is found to be

$$\int_{-\infty}^{\infty} d\tau \exp\left[(i\omega - (\mathcal{C}-\mathcal{A}))\tau\right] = \frac{2(\mathcal{C}-\mathcal{A})}{\omega^2+(\mathcal{C}-\mathcal{A})^2}, \tag{3.87c}$$

and hence the intensity fluctuations spectrum take the form

$$S(\omega) = a + \frac{2b(\mathcal{C}-\mathcal{A})}{\omega^2+(\mathcal{C}-\mathcal{A})^2}. \tag{3.88}$$

We observe that the spectrum of intensity fluctuations is a Lorentzian with a width of $2(\mathcal{C}-\mathcal{A})$. For $N = M = 0$ we see that

$$a = \frac{\mathcal{C}\mathcal{A}}{\mathcal{C}-\mathcal{A}},$$

$$b = \frac{\mathcal{C}^2\mathcal{A}^2}{(\mathcal{C}-\mathcal{A})^2},$$

so that (3.88) reduces to

$$S(\omega) = \frac{\mathcal{C}\mathcal{A}}{\mathcal{C}-\mathcal{A}} + \frac{2\mathcal{C}^2\mathcal{A}^2/(\mathcal{C}-\mathcal{A})}{\omega^2+(\mathcal{C}-\mathcal{A})^2}. \tag{3.89}$$

This shows that the spectrum of intensity fluctuations for a laser coupled to an ordinary vacuum reservoir is a Lorentzian with the same width of $2(\mathcal{C}-\mathcal{A})$. Hence the effect the squeezed vacuum reservoir is to increase the peak of the spectrum. It has no effect on the width of the spectrum.

3.3. Squeezing Spectrum

The squeezing spectrum of the output radiation is expressible as

$$S_i(\omega) = \int_{-\infty}^{\infty} d\tau \left\langle a_i^{out}(t+\tau), a_i^{out}(t)\right\rangle_{ss} \exp[i\omega\tau], \tag{3.90}$$

where the a_i^{out} (for i=1,2) are the quadrature operators for the output radiation defined by

$$a_1^{out}(t) = a_{out}(t) + a_{out}^{\dagger}(t), \tag{3.91a}$$

$$a_2^{out}(t) = i[a_{out}^{\dagger}(t) - a_{out}(t)]. \tag{3.91b}$$

We now proceed to calculate the squeezing spectrum for the first quadrature operator of the output radiation. We note that for normal order

$$\langle a_1^{out}(t+\tau), a_1^{out}(t)\rangle = \delta(\tau) + \langle \alpha_1^{out}(t+\tau), \alpha_1^{out}(t)\rangle, \tag{3.92}$$

where

$$\alpha_1^{out}(t) = \alpha_{out}(t) + \alpha_{out}^{*}(t). \tag{3.93}$$

Applying the relation (3.52a) we see that

$$\langle \alpha_1^{out}(t+\tau), \alpha_1^{out}(t)\rangle = \langle \alpha_1^{out}(t+\tau)\alpha_1^{out}(t)\rangle - \langle \alpha_1^{out}(t+\tau)\rangle\langle \alpha_1^{out}(t)\rangle. \tag{3.94}$$

With the aid of (3.93) and (3.56) one can easily establish that

$$\langle \alpha_1^{out}(t+\tau)\rangle = \langle \alpha_1^{out}(t)\rangle = 0, \tag{3.95}$$

and also the first term in (3.94) can be written as

$$\langle \alpha_1^{out}(t+\tau)\alpha_1^{out}(t)\rangle = \langle \alpha_{out}(t+\tau)\alpha_{out}(t)\rangle + \langle \alpha_{out}(t+\tau)\alpha_{out}^*(t)\rangle$$

$$+\langle \alpha_{out}^*(t+\tau)\alpha_{out}(t)\rangle + \langle \alpha_{out}^*(t+\tau)\alpha_{out}^*(t)\rangle. \tag{3.96}$$

Each of these terms has been calculated in the preceding section and are given by Eqs.(3.81), (3.82), and (3.80). Introducing these terms into (3.96) at steady state we find

$$\langle \alpha_1^{out}(t+\tau)\alpha_1^{out}(t)\rangle = 2(M+N)\delta(\tau) + \frac{\mathcal{CA}(2(N+M)+2)}{\mathcal{C}-\mathcal{A}}c(\tau), \tag{3.97}$$

so that substituting this into (3.92) we get

$$\langle a_1^{out}(t+\tau), a_1^{out}(t)\rangle_{ss} = [2(M+N)+1]\delta(\tau) + \frac{\mathcal{CA}(2(N+M)+2)}{\mathcal{C}-\mathcal{A}}c(\tau). \tag{3.98}$$

Upon introducing this into (3.90) and carrying out the integration, we find

$$S_1(\omega) = e^{2r} + \frac{\mathcal{CA}(e^{2r}+1)}{\omega^2 + \left(\frac{\mathcal{C}-\mathcal{A}}{2}\right)^2}. \tag{3.99}$$

Similarly for $i = 2$

$$S_2(\omega) = e^{-2r} + \frac{\mathcal{CA}(e^{-2r}+1)}{\omega^2 + \left(\frac{\mathcal{C}-\mathcal{A}}{2}\right)^2}. \tag{3.100}$$

In both cases the squeezing spectrum is a Lorentzian with a width of $(\mathcal{C}-\mathcal{A})$. The effect of the squeezed vacuum reservoir is to increase the peak of the spectrum for the first quadrature and to decrease the peak of the spectrum for the second quadrature, it has no effect on the width of the spectrum. For sufficiently large r, squeezing will occur in the second quadrature of the output radiation.

We now proceed to calculate the quadrature fluctuations for the cavity mode.

$$\Delta a_1^2 = \langle a_1^2(t)\rangle - \langle a_1(t)\rangle^2, \tag{3.101}$$

so that for normal order of the operators we have

$$\Delta a_1^2 = \langle \alpha^2(t)\rangle - \langle \alpha(t)\rangle^2 + \langle \alpha^{*2}(t)\rangle - \langle \alpha^*(t)\rangle^2$$

$$+2(\langle \alpha^*(t)\alpha(t)\rangle - \langle \alpha^*(t)\rangle\langle \alpha(t)\rangle) + 1. \tag{3.102}$$

Applying (3.4), one can easily show that

$$\langle \alpha(t)\rangle = \langle \alpha^*(t)\rangle = 0. \tag{3.103}$$

In addition, in view of (3.4) and (3.18), we see that

$$\begin{aligned}(\langle\alpha^{*2}(t)\rangle)^* = \langle\alpha^2(t)\rangle &= \langle R(t)R(t)\rangle \\ &= \frac{\mathcal{C}M}{\mathcal{C}-\mathcal{A}}\left(1-c^2(t)\right),\end{aligned} \tag{3.104}$$

and

$$\begin{aligned}\langle\alpha^*(t)\alpha(t)\rangle &= \langle R^*(t)R(t)\rangle \\ &= \frac{\mathcal{C}N+\mathcal{A}}{\mathcal{C}-\mathcal{A}}\left(1-c^2(t)\right).\end{aligned} \tag{3.105}$$

Combination of (3.103), (3.104),and (3.105) with (3.102) at steady state leads to ($M = M^*$)

$$\Delta a_1^2 = \frac{\mathcal{C}e^{2r}+\mathcal{A}}{\mathcal{C}-\mathcal{A}}, \tag{3.106}$$

Analogously

$$\Delta a_2^2 = \frac{\mathcal{C}e^{-2r}+\mathcal{A}}{\mathcal{C}-\mathcal{A}}, \tag{3.107}$$

If we choose the squeeze parameter r to be

$$r > \frac{1}{2}\ln\left(\frac{\mathcal{C}}{\mathcal{C}-2\mathcal{A}}\right), \tag{3.108}$$

for $\mathcal{C} > 2\mathcal{A}$, we see that

$$\Delta a_1^2 > 1,$$

$$\Delta a_2^2 < 1,$$

which implies that the squeezing occures in the second quadrature.

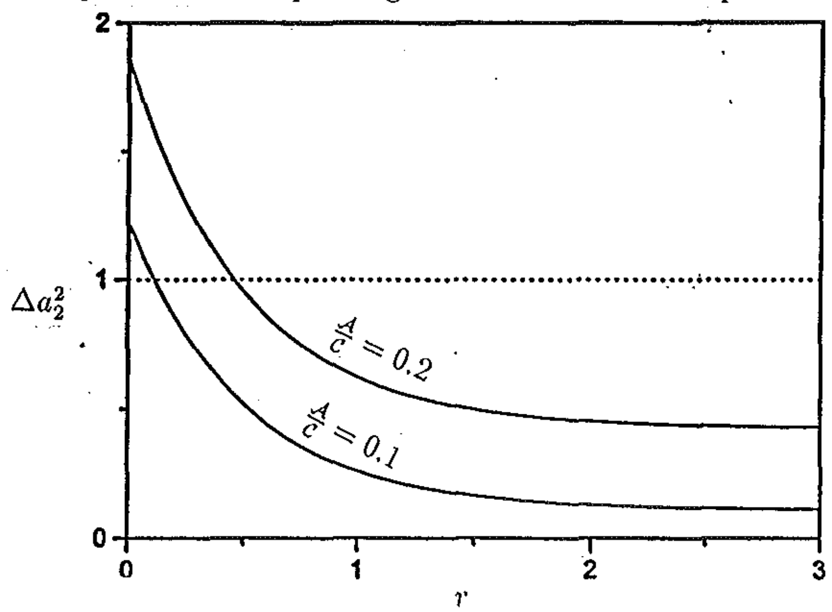

Fig. 2. Plots of the quadrature variance Δa_2^2 versus r.

3.4. Power Spectrum

Now we seek to calculate the power spectrum of the cavity mode from which we can find the laser linewidth. It is expressible as

$$S(\omega) = \int_{-\infty}^{\infty} d\tau \langle a^{\dagger}(t) a(t+\tau)\rangle_{ss} \exp[i\omega\tau]. \tag{3.109a}$$

Since we have a normal ordering of operators the corresponding expression interms of c-number variables is then

$$S(\omega) = \int_{-\infty}^{\infty} d\tau \langle \alpha^{*}(t) \alpha(t+\tau)\rangle_{ss} \exp[i\omega\tau]. \tag{3.109b}$$

Using (3.4) we see that

$$\begin{aligned}
\langle \alpha^{*}(t)\alpha(t+\tau)\rangle &= c(t+\tau)c(t)\int_{0}^{t} dt' \int_{0}^{t+\tau} dt'' \langle f_{\alpha}^{*}(t') f_{\alpha}(t'')\rangle c(-t')c(-t'') \\
&= (\mathcal{C}N + \mathcal{A}) c(t+\tau)c(t)\int_{0}^{t} dt' c^{2}(-t') \\
&= \frac{\mathcal{C}N + \mathcal{A}}{\mathcal{C} - \mathcal{A}} \left(1 - c^{2}(t)\right) c(\tau),
\end{aligned} \tag{3.110}$$

and thus at steady state this reduces

$$\langle \alpha^{*}(t)\alpha(t+\tau)\rangle_{ss} = \frac{\mathcal{C}N + \mathcal{A}}{\mathcal{C} - \mathcal{A}} \exp[-1/2(\mathcal{C} - \mathcal{A})\tau]. \tag{3.111}$$

Introducing (3.111) into (3.109b) and carrying out the integration yields

$$S(\omega) = \frac{\mathcal{C}N + \mathcal{A}}{\omega^{2} + (\frac{\mathcal{C}-\mathcal{A}}{2})^{2}}. \tag{3.112a}$$

One can also easily establish, employing Eq.(3.75), that the power spectrum for the output radiation to be

$$S^{out}(\omega) = N + \frac{\mathcal{C}\mathcal{A}(N+1)}{\omega^{2} + (\frac{\mathcal{C}-\mathcal{A}}{2})^{2}}. \tag{3.112b}$$

We observe that the power spectrum in both cases is a Lorentzian with a width of $(\mathcal{C} - \mathcal{A})$, which is independent of the squeeze parameter r. Again in this case the effect of the squeezed vacuum reservoir is to increase the peak of the spectrum only.

4. ABOVE THRESHOLD LASER DYNAMICS

In the preceding chapter we neglected the nonlinear saturation effects and discussed a restricted laser operation in which the gain coefficient is less than the cavity decay rate. In this chapter we consider a laser operating above threshold, i.e., $\mathcal{A} > \mathcal{C}$, with all the nonlinear effects included. To this end, we can express $\alpha(t)$ interms of intensity and phase variables as

$$\alpha = \sqrt{I} e^{i\theta}. \tag{4.1}$$

On differentiating with respect to time and comparing the real and imaginary parts with Eq.(2.131), one can easily arrive at the following equation of evolution for the variable I and θ:

$$\dot{I} = -\mathcal{C} I + \frac{\mathcal{A} I}{(1 + \frac{\mathcal{B}}{\mathcal{A}} I)} + f_I(t), \tag{4.2a}$$

$$\dot{\theta} = f_\theta(t), \tag{4.2b}$$

where the noise forces are given by

$$f_I(t) = \alpha^* f_\alpha + \alpha f_\alpha^*, \tag{4.3a}$$

$$f_\theta(t) = \frac{i}{2I}(\alpha f_\alpha^* - \alpha^* f_\alpha). \tag{4.3b}$$

We next proceed to evaluate the mean of the noise forces. Taking into account (4.3a), we note that

$$\langle f_I(t) \rangle = \langle \alpha^*(t) f_\alpha(t) \rangle + \langle \alpha(t) f_\alpha^*(t) \rangle. \tag{4.4}$$

On formally integrating Eq.(2.131), we find

$$\alpha(t) = \alpha(0) - \frac{\mathcal{C}}{2} \int_0^t \alpha(t')dt' + \frac{1}{2} \int_0^t \frac{\mathcal{A}\alpha(t')}{1 + \frac{\mathcal{B}}{\mathcal{A}} I(t')} dt' + \int_0^t f_\alpha(t')dt', \tag{4.5}$$

with the aid of which we can write the first term in (4.4) as

$$\langle \alpha^*(t) f_\alpha(t) \rangle = \langle \alpha^*(0) f_\alpha(t) \rangle - \frac{\mathcal{C}}{2} \int_0^t \langle \alpha^*(t') f_\alpha(t) \rangle dt' + \frac{1}{2} \int_0^t \left\langle \frac{\mathcal{A}\alpha^*(t') f_\alpha(t)}{1 + \frac{\mathcal{B}}{\mathcal{A}} I(t')} \right\rangle dt'$$
$$+ \int_0^t \langle f_\alpha^*(t') f_\alpha(t) \rangle dt'. \tag{4.6}$$

Since the radiation variables at the earlier times and the noise force at a later time are uncorrelated, all terms vanish except the last term in (4.6). Hence in view of (2.135c), we have

$$\langle \alpha^*(t) f_\alpha(t) \rangle = \frac{1}{2}\left\langle \mathcal{C}N + \frac{\mathcal{A}}{(1+\frac{\mathcal{B}}{\mathcal{A}}I)^2}\left[1 + \frac{\mathcal{B}}{4\mathcal{A}}I\left(3 + \frac{\mathcal{B}}{\mathcal{A}}I\right)\right]\right\rangle. \tag{4.7a}$$

Similarly,

$$\langle \alpha(t) f^*_\alpha(t) \rangle = \frac{1}{2}\left\langle \mathcal{C}N + \frac{\mathcal{A}}{(1+\frac{\mathcal{B}}{\mathcal{A}}I)^2}\left[1 + \frac{\mathcal{B}}{4\mathcal{A}}I\left(3 + \frac{\mathcal{B}}{\mathcal{A}}I\right)\right]\right\rangle. \tag{4.7b}$$

Hence the mean of the noise force f_I is

$$\langle f_I(t) \rangle = \left\langle \mathcal{C}N + \frac{\mathcal{A}}{(1+\frac{\mathcal{B}}{\mathcal{A}}I)^2}\left[1 + \frac{\mathcal{B}}{4\mathcal{A}}I\left(3 + \frac{\mathcal{B}}{\mathcal{A}}I\right)\right]\right\rangle. \tag{4.8}$$

Taking the expectation value of (4.2a), we see that

$$\langle \dot{I} \rangle = -\mathcal{C}\langle I \rangle + \left\langle \frac{\mathcal{A}I}{1+\frac{\mathcal{B}}{\mathcal{A}}I} \right\rangle + \langle f_I(t) \rangle$$

Consider

$$\left\langle \frac{\mathcal{A}I}{1+\frac{\mathcal{B}}{\mathcal{A}}I} \right\rangle + \langle f_I(t) \rangle = \mathcal{C}N + \left\langle \frac{\mathcal{A}I}{1+\frac{\mathcal{B}}{\mathcal{A}}I} + \frac{\mathcal{A}}{(1+\frac{\mathcal{B}}{\mathcal{A}}I)^2}\left[1 + \frac{\mathcal{B}}{4\mathcal{A}}I\left(3 + \frac{\mathcal{B}}{\mathcal{A}}I\right)\right]\right\rangle.$$

For a laser operating well above threshold [4,8]

$$\mathcal{A} \gg \mathcal{C}, \quad I \gg \frac{\mathcal{A}}{\mathcal{B}} \gg 1. \tag{4.9}$$

In view of this we can rewrite the above expression as

$$\left\langle \frac{\mathcal{A}I}{1+\frac{\mathcal{B}}{\mathcal{A}}I} \right\rangle + \langle f_I(t) \rangle = \mathcal{C}N + \left\langle \frac{\mathcal{A}I}{1+\frac{\mathcal{B}}{\mathcal{A}}I}(1+\mathcal{B}/4\mathcal{A}) \right\rangle$$

$$\cong \mathcal{C}N + \left\langle \frac{\mathcal{A}I}{1+\frac{\mathcal{B}}{\mathcal{A}}I} \right\rangle.$$

Hence, on comparing the left and the right hand side, we see that

$$\langle f_I(t) \rangle = \mathcal{C}N. \tag{4.10}$$

Including the contribution of (4.10) in the drift term of (4.2a), we may write the langevin equation for the variable I as,

$$\dot{I} = -\mathcal{C}(I-N) + \frac{\mathcal{A}I}{(1+\frac{\mathcal{B}}{\mathcal{A}}I)} + f_I'(t), \tag{4.11}$$

where

$$f_I'(t) = f_I(t) - \mathcal{C}N, \tag{4.12}$$

so that

$$\langle f_I'(t)\rangle = 0, \tag{4.13a}$$

$$\langle f_I'(t)f_I'(t')\rangle = \langle f_I(t)f_I(t')\rangle - \mathcal{C}^2N^2. \tag{4.13b}$$

Similarly, applying the definition of $f_\theta(t)$ and Eq.(4.7), one can easily show that

$$\langle f_\theta(t)\rangle = 0. \tag{4.14}$$

With the aid of (4.3a) and suppressing the time dependence of α and α^*, the two time correlation function for $f_I(t)$ is

$$\langle f_I(t)f_I(t')\rangle = \langle \alpha^{*2}f_\alpha(t)f_\alpha(t')\rangle + 2\langle \alpha\alpha^* f_\alpha^*(t)f_\alpha(t')\rangle + \langle \alpha^2 f_\alpha^*(t)f_\alpha^*(t')\rangle.$$

Assuming that the radiation variables can be decoupled from the noise forces,we have

$$\langle f_I(t)f_I(t')\rangle = \langle \alpha^{*2}\rangle\langle f_\alpha(t)f_\alpha(t')\rangle + 2\langle \alpha\alpha^*\rangle\langle f_\alpha^*(t)f_\alpha(t')\rangle + \langle \alpha^2\rangle\langle f_\alpha^*(t)f_\alpha^*(t')\rangle.$$

Substituting Eq.(2.135) for the noise correlation functions and Eq.(4.1) into the above equation, we obtain

$$\langle f_I(t)f_I(t')\rangle = \left\langle 2I\mathcal{C}(\sinh^2 r + \sinh r\cosh r\cos(\phi - 2\theta)) + \frac{2\mathcal{A}I}{(1+\frac{\mathcal{B}}{\mathcal{A}}I)^2}\right\rangle \delta(t-t'), \tag{4.15}$$

where we have substituted Eq.(2.25g) for the value of M and M^*. If we choose the phase of the squeezed vacuum to be

$$\phi = 2\theta, \tag{4.16}$$

Eq.(4.15) reduces to

$$\langle f_I(t)f_I(t')\rangle = \left\langle I\mathcal{C}(e^{2r}-1) + \frac{2\mathcal{A}I}{(1+\frac{\mathcal{B}}{\mathcal{A}}I)^2}\right\rangle \delta(t-t'), \tag{4.17a}$$

and hence

$$\langle f_I'(t)f_I'(t')\rangle = \left\langle I\mathcal{C}(e^{2r}-1) + \frac{2\mathcal{A}I}{(1+\frac{\mathcal{B}}{\mathcal{A}}I)^2}\right\rangle \delta(t-t') - \mathcal{C}^2N^2. \tag{4.17b}$$

On the other hand, if we choose

$$\phi - 2\theta = \pi/2, \tag{4.18}$$

we get

$$\langle f_I'(t)f_I'(t')\rangle = \left\langle 2I\mathcal{C}N + \frac{2\mathcal{A}I}{(1+\frac{\mathcal{B}}{\mathcal{A}}I)^2}\right\rangle \delta(t-t') - \mathcal{C}^2N^2. \tag{4.19}$$

Similarly, for $f_\theta(t)$ we find

$$\langle f_\theta(t)f_\theta(t')\rangle = \left\langle \frac{1}{4I}\left[2\mathcal{C}(\sinh^2 r - \sinh r\cosh r\cos(\phi - 2\theta)) + \frac{2\mathcal{A}}{(1+\frac{\mathcal{B}}{\mathcal{A}}I)^2} + \frac{\mathcal{B}I(3+\frac{\mathcal{B}}{\mathcal{A}}I)}{(1+\frac{\mathcal{B}}{\mathcal{A}}I)^2}\right]\right\rangle \delta(t-t').$$

If we let $X = 1 + \frac{\mathcal{B}}{\mathcal{A}}I$, the above expression can be written as

$$\langle f_\theta(t)f_\theta(t')\rangle = \left\langle \frac{1}{4I}\left[2\mathcal{C}(\sinh^2 r - \sinh r\cosh r\cos(\phi - 2\theta)) + \mathcal{A}(1+1/X)\right]\right\rangle \delta(t-t').$$

Choosing the phase of the squeezed vacuum reservoir given by (4.16), the above expression has the form

$$\langle f_\theta(t)f_\theta(t')\rangle = \left\langle \frac{\mathcal{A}(1+1/X) + \mathcal{C}(e^{-2r}-1)}{4I}\right\rangle \delta(t-t'). \tag{4.20a}$$

On the other hand, if we make the choice (4.18), we obtain

$$\langle f_\theta(t)f_\theta(t')\rangle = \left\langle \frac{\mathcal{A}(1+1/X) + 2\mathcal{C}N}{4I}\right\rangle \delta(t-t'). \tag{4.20b}$$

4.1. The Mean Photon Number and Variance

A. *The Mean Photon Number*

Since $I = \alpha^*\alpha$, where α and α^* are the c-number variables corresponding to the operators a and $a^\dagger$ for the normal ordering, one can write

$$\langle n\rangle = \langle a^\dagger a\rangle = \langle I\rangle. \tag{4.21}$$

It follows from Eq.(4.11) that at steady state

$$-\mathcal{C}(\langle I\rangle + N) + \left\langle \frac{\mathcal{A}I}{1+\frac{\mathcal{B}}{\mathcal{A}}I}\right\rangle = 0, \tag{4.22}$$

where we have used Eq.(4.13a). If we make the assumption

$$\langle I^{-1}\rangle = \langle I\rangle^{-1},$$

we obtain the steady state mean photon number of a laser coupled to a squeezed vacuum reservoir and operating above threshold:

$$\langle n\rangle = I_0 = \frac{1}{2}\left[N + \frac{\mathcal{A}}{\mathcal{B}}\left(\frac{\mathcal{A}-\mathcal{C}}{\mathcal{C}}\right)\right] + \sqrt{\frac{1}{4}\left[N + \frac{\mathcal{A}}{\mathcal{B}}\left(\frac{\mathcal{A}-\mathcal{C}}{\mathcal{C}}\right)\right]^2 + \frac{N\mathcal{A}}{\mathcal{B}}}. \tag{4.23}$$

In view of (4.9) we can neglect the last term in the square root and obtain

$$I_0 = N + \frac{\mathcal{A}}{\mathcal{B}}\left(\frac{\mathcal{A}-\mathcal{C}}{\mathcal{C}}\right). \tag{4.24}$$

Setting $N = 0$, we get the steady state mean photon number for a laser coupled to an ordinary vacuum reservoir [8]

$$I_0 = \frac{\mathcal{A}}{\mathcal{B}}\left(\frac{\mathcal{A}-\mathcal{C}}{\mathcal{C}}\right). \tag{4.25}$$

B. *Variance*

The variance of the photon number for the cavity mode is given by

$$\begin{aligned}\Delta n^2 &= \langle a^\dagger a a^\dagger a\rangle - \langle a^\dagger a\rangle^2 \\ &= \langle a^\dagger a^\dagger a a\rangle + \langle a^\dagger a\rangle - \langle a^\dagger a\rangle^2 \\ &= \langle I^2\rangle - \langle I\rangle^2 + \langle I\rangle \\ &= \Delta I^2 + \langle I\rangle.\end{aligned} \tag{4.26}$$

To determine the fluctuations in the quantity $I(t)$, we first linearize $I(t)$ around its steady state value. Defining

$$\Delta I(t) = I(t) - I_0, \tag{4.27}$$

and substituting for $I(t) = I_0 + \Delta I(t)$ into Eq.(4.11), we have

$$\frac{d}{dt}(\Delta I) = -\mathcal{C}(I_0 + \Delta I) + \mathcal{C}N + \frac{\mathcal{A}(I_0+\Delta I)}{1+\frac{\mathcal{B}}{\mathcal{A}}(I_0+\Delta I)} + f_I'(t). \tag{4.28}$$

Expanding the third term in power series around $\Delta I = 0$, and neglecting second and higher order terms, we obtain

$$\begin{aligned}\frac{d}{dt}(\Delta I) &= -\mathcal{C}(I_0 + \Delta I) + \mathcal{C}N + \frac{\mathcal{A}(I_0+\Delta I)}{1+\frac{\mathcal{B}}{\mathcal{A}}I_0} - \frac{\mathcal{B}I_0\Delta I}{(1+\frac{\mathcal{B}}{\mathcal{A}}I_0)^2} + f_I' \\ &\cong -\mathcal{C}I_0 + \mathcal{C}N + \frac{\mathcal{A}^2}{\mathcal{B}} - \left(\mathcal{C} - \frac{\mathcal{A}}{(1+\frac{\mathcal{B}}{\mathcal{A}}I_0)^2}\right)\Delta I + f_I'\end{aligned} \tag{4.29}$$

In view of (4.9) and (4.24), we see that

$$\mathcal{C}I_0 \approx \mathcal{C}N + \mathcal{A}^2/\mathcal{B},$$

and thus

$$\frac{d}{dt}(\Delta I) = -\kappa \Delta I + f_I', \tag{4.30a}$$

where

$$\kappa = \mathcal{C} - \frac{\mathcal{A}}{(1+\frac{\mathcal{B}}{\mathcal{A}}I)^2}. \tag{4.30b}$$

Therefore

$$\begin{aligned}\frac{d}{dt}(\Delta I)^2 &= 2\langle \Delta I \frac{d}{dt}(\Delta I)\rangle \\ &= -2\kappa(\Delta I)^2 + 2\langle \Delta I(t) f_I'(t)\rangle. \end{aligned}\tag{4.31}$$

The steady state solution of this equation is then

$$(\Delta I)^2 = \frac{1}{\kappa}\langle \Delta I(t) f_I'(t)\rangle_{ss}. \tag{4.32}$$

On formally integrating Eq.(4.30), we find

$$\Delta I(t) = \Delta I(0)e^{-\kappa t} + e^{-\kappa t}\int_0^t dt' f_I'(t')e^{\kappa t'}, \tag{4.33}$$

and hence

$$\langle \Delta I(t) f_I'(t)\rangle = \langle \Delta I(0) f_I'(t)\rangle e^{-\kappa t} + e^{-\kappa t}\int_0^t dt' \langle f_I'(t') f_I'(t)\rangle e^{\kappa t'}, \tag{4.34}$$

Applying (4.17b) for $\phi = 2\theta$, we obtain at steady state

$$\langle \Delta I(t) f_I'(t)\rangle_{ss} = \frac{I_0 \mathcal{C}}{2}(e^{2r}-1) + \frac{\mathcal{A}I_0}{(1+\frac{\mathcal{B}}{\mathcal{A}}I_0)^2} - \frac{\mathcal{C}^2 N^2}{\kappa}, \tag{4.35}$$

so that combination of this with (4.32) leads to

$$(\Delta I)^2 = \frac{I_0 \mathcal{C}}{2\kappa}(e^{2r}-1) + \frac{\mathcal{A}I_0}{\kappa(1+\frac{\mathcal{B}}{\mathcal{A}}I_0)^2} - \frac{\mathcal{C}^2 N^2}{\kappa^2}. \tag{4.36}$$

In view of (4.9) we see that

$$\kappa \approx \mathcal{C},$$

and

$$\frac{\mathcal{A}I_0}{\kappa(1+\frac{\mathcal{B}}{\mathcal{A}}I_0)^2} \approx \frac{\mathcal{A}}{\mathcal{B}}.$$

Hence

$$(\Delta I)^2 = \frac{I_0}{2}(e^{2r}-1) - N^2 + \frac{\mathcal{A}}{\mathcal{B}}. \tag{4.37}$$

Finally substituting this into (4.26), the variance of the photon number at steady state is found to be

$$\Delta n^2 = \frac{I_0}{2}(e^{2r}+1) - N^2 + \frac{\mathcal{A}}{\mathcal{B}}. \tag{4.38a}$$

If we apply (4.19) for $\phi - 2\theta = \pi/2$, in Eq.(4.34) we obtain

$$(\Delta I)^2 = I_0 N - N^2 + \frac{\mathcal{A}}{\mathcal{B}},$$

and the corresponding variance of the photon number is then

$$\Delta n^2 = I_0(N+1) - N^2 + \frac{\mathcal{A}}{\mathcal{B}}. \tag{4.38b}$$

For $r = 0$, both Eq.(4.38a) and (4.38b) reduces to the variance of the photon number for a laser coupled to an ordinary vacuum reservoir [6,8], given by

$$\Delta n^2 = I_0 + \frac{\mathcal{A}}{\mathcal{B}}. \tag{4.39}$$

We observe that, the variance of the photon number increases when the laser is coupled to a squeezed vacuum reservoir as compared to the variance of a laser coupled to an ordinary vacuum reservoir.

4.2. Power Spectrum

We now wish to calculate the power spectrum of the cavity mode and hence the laser linewidth. We recall that the expression for the power spectrum interms of c-number variables is given by

$$S(\omega) = \int_{-\infty}^{\infty} d\tau \langle \alpha^*(t+\tau)\alpha(t)\rangle_{ss} \exp[i\omega\tau]. \tag{4.40}$$

In the region well-above threshold, the fluctuations in I becomes relatively small. This stabilization in I permits us to replace I in Eq.(4.1) by its steady state value (4.21b), and write

$$\alpha(t) = \sqrt{I_0} e^{i\theta(t)}, \tag{4.41}$$

so that

$$\langle \alpha^*(t+\tau)\alpha(t)\rangle = I_0 \langle \exp[-i(\theta(t+\tau) - \theta(t))]\rangle. \tag{4.42}$$

On formally integrating Eq.(4.2b) we find

$$\theta(t+\tau) - \theta(t) = \int_t^{t+\tau} f_\theta(t')dt'. \tag{4.43}$$

Since $\langle[\theta(t+\tau)-\theta(t)]\rangle = 0$, one can express Eq.(4.42) as (cf. 3.15a)

$$\langle\alpha^*(t+\tau)\alpha(t)\rangle = I_0 \exp\left[-\frac{1}{2}\langle(\theta(t+\tau)-\theta(t))^2\rangle\right]. \tag{4.44}$$

Applying Eq.(4.43), we find

$$\langle(\theta(t+\tau)-\theta(t))^2\rangle = \int_t^{t+\tau} dt' \int_t^{t+\tau} dt'' \langle f_\theta(t')f_\theta(t'')\rangle, \tag{4.45a}$$

We recall that for $\phi - 2\theta = 0$,

$$\langle f_\theta(t)f_\theta(t')\rangle = \left\langle\frac{\mathcal{A}(1+1/X)+\mathcal{C}(e^{-2r}-1)}{4I}\right\rangle \delta(t-t'), \tag{4.45b}$$

where

$$\mathcal{A}(1+1/X) = \mathcal{A}\left(1+\frac{1}{1+\frac{\mathcal{B}}{\mathcal{A}}I}\right).$$

Replacing I by its steady state value (4.24), we see that

$$\mathcal{A}(1+1/X) = \mathcal{A}\left(1+\frac{1}{\frac{\mathcal{B}}{\mathcal{A}}N+\frac{\mathcal{A}}{\mathcal{C}}}\right).$$

In view of (4.9) and for N not sufficiently large, we can neglect $\frac{\mathcal{B}}{\mathcal{A}}N$ as compared to $\frac{\mathcal{A}}{\mathcal{C}}$ and have

$$\mathcal{A}(1+1/X) = \mathcal{A}+\mathcal{C}. \tag{4.45c}$$

Combining this with Eq.(4.45b), we get

$$\langle f_\theta(t)f_\theta(t')\rangle = \left\langle\frac{\mathcal{A}+\mathcal{C}e^{-2r}}{4I}\right\rangle \delta(t-t'). \tag{4.45d}$$

Hence, substituting this into Eq.(4.45a) and carrying out the integration, we obtain

$$\langle(\theta(t+\tau)-\theta(t))^2\rangle = \left\langle\frac{\mathcal{A}+\mathcal{C}e^{-2r}}{4I}\right\rangle \tau. \tag{4.46}$$

With this Eq.(4.44) becomes at steady state

$$\langle\alpha^*(t+\tau)\alpha(t)\rangle_{ss} = I_0 \exp\left[-\frac{1}{2}\left(\frac{\mathcal{A}+\mathcal{C}e^{-2r}}{4I_0}\right)\tau\right]. \tag{4.47}$$

Substituting this into Eq.(4.40) and carrying out the integration, the power spectrum in this case has the form

$$S(\omega) = \frac{2I_0D_1}{\omega^2+D_1^2}, \tag{4.48}$$

where

$$2D_1 = \frac{\mathcal{A}+\mathcal{C}e^{-2r}}{4I_0}. \tag{4.49}$$

The power spectrum is a Lorentzian with a width (laser linewidth) of $2D_1$. For $r = 0$, the width has the form

$$2D_1 = \frac{\mathcal{A}+\mathcal{C}}{4I_0}, \tag{4.50}$$

which is the linewidth of a laser coupled to an ordinary vacuum reservoir [8]. We see that the effect of the squeezed vacuum reservoir is to decrease the laser linewidth.

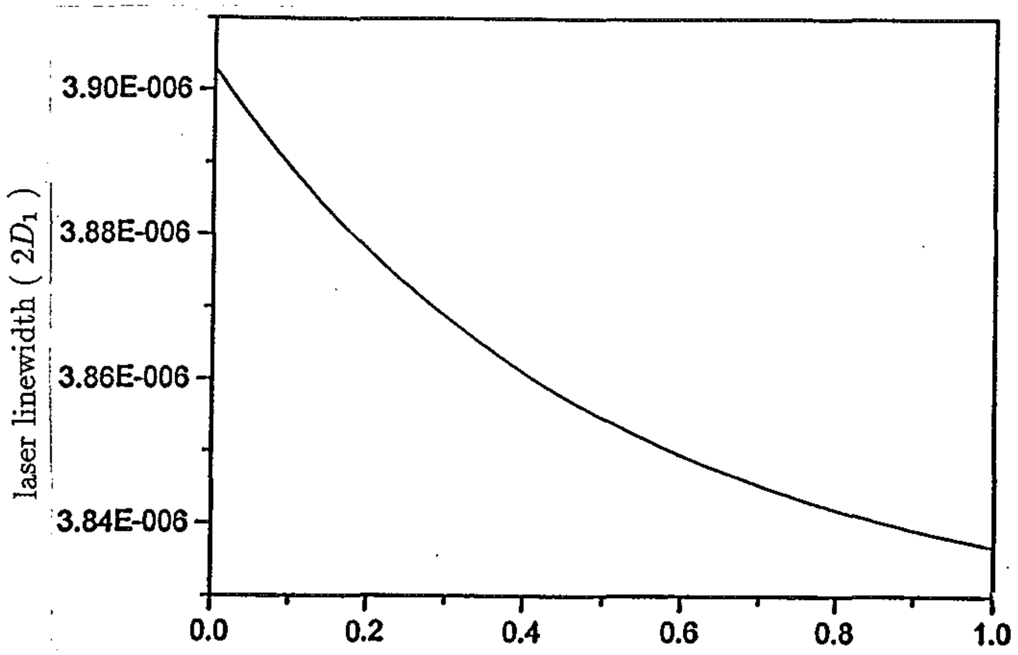

Fig. 3. Plot of the laser linewidth ($2D_1$) versus r for $\mathcal{A}/\mathcal{B} = 1600$, $\mathcal{A} = 1.2$,$\mathcal{C} = 0.02\mathcal{A}$

Similarily, using Eq.(4.45c) into (4.20b) for $\phi - 2\theta = \pi/2$, we obtain

$$\langle f_\theta(t) f_\theta(t')\rangle = \left\langle \frac{\mathcal{A}+\mathcal{C}(2N+1)}{4I} \right\rangle \delta(t-t').$$

And hence using this correlation function, one can easily show that

$$\langle \alpha^*(t+\tau)\alpha(t)\rangle_{ss} = I_0 \exp\left[-\frac{1}{2}\left(\frac{\mathcal{A}+\mathcal{C}(2N+1)}{4I_0}\right)\tau\right]. \tag{4.51}$$

The corresponding power spectrum is then

$$S'(\omega) = \frac{2I_0 D_2}{\omega^2 + D_2^2}, \tag{4.52}$$

where the width of the spectrum is

$$2D_2 = \frac{\mathcal{A}+\mathcal{C}(2N+1)}{4I_0} \tag{4.53}$$

In this case, when $\phi - 2\theta = \pi/2$, the width of the spectrum increases.

4.3. Spectrum of Intensity Fluctuations

The spectrum of intensity fluctuations for the cavity mode is expressible as

$$S(\omega) = \int_{-\infty}^{\infty} d\tau \langle \hat{I}(t+\tau), \hat{I}(t)\rangle_{ss} e^{i\omega\tau}. \tag{4.54}$$

One can write, for the normal ordering of operators, that

$$\begin{aligned}\langle \hat{I}(t+\tau), \hat{I}(t)\rangle_{ss} &= \langle a^\dagger(t+\tau)a(t)\rangle_{ss}\delta(\tau) \\ &+\langle I(t+\tau)I(t)\rangle_{ss} - \langle I(t+\tau)\rangle_{ss}\langle I(t)\rangle_{ss}\end{aligned} \tag{4.55}$$

where $I = \alpha\alpha^*$ is the c-number equivalent of the operator $\hat{I} = a^\dagger a$ for the normal ordering. Combining (4.27) and (4.33), we have

$$I(t) = I_0 + \Delta I(0)e^{-\kappa t} + e^{-\kappa t}\int_0^t dt' f_I'(t')e^{\kappa t'}, \tag{4.56a}$$

such that

$$\langle I(t)\rangle_{ss} = I_0. \tag{4.56b}$$

With the aid of (4.56a) one can easily show that

$$\langle I(t+\tau)I(t)\rangle_{ss} = I_0^2 - \frac{C^2N^2}{\kappa^2} + \frac{d}{2\kappa}e^{-\kappa\tau}, \tag{4.57}$$

where

$$d = \begin{cases} I_0 C(e^{2r}-1) + \frac{2AI_0}{(1+\frac{B}{A}I_0)^2}, & \text{for } \phi = 2\theta \\ 2I_0CN + \frac{2AI_0}{(1+\frac{B}{A}I_0)^2}, & \text{for } \phi - 2\theta = \pi/2 \end{cases} \tag{4.58}$$

Substituting (4.57) and (4.56b) into (4.55), we have

$$\langle \hat{I}(t+\tau), \hat{I}(t)\rangle_{ss} = \langle a^\dagger(t+\tau)a(t)\rangle_{ss}\delta(\tau) - \frac{C^2N^2}{\kappa^2} + \frac{d}{2\kappa}e^{-\kappa\tau}. \tag{4.59}$$

Introducing this into (4.54) and carrying out the integration, we get

$$S(\omega) = I_0 - \frac{C^2N^2}{\kappa^2}\delta(\omega) + \frac{d}{\omega^2+\kappa^2}. \tag{4.60a}$$

For $\omega \neq 0$

$$S(\omega) = I_0 + \frac{d}{\omega^2+\kappa^2}. \tag{4.60b}$$

The spectrum of intensity fluctuations is a Lorenzian with a half width of κ, where κ is given by

$$\kappa = \mathcal{C} - \frac{\mathcal{A}}{(\frac{\mathcal{B}}{\mathcal{A}}N + \frac{\mathcal{A}}{\mathcal{C}})^2} \tag{4.61}$$

For $r = 0$, the spectrum has the form [22,23]

$$S'(\omega) = I_0 + \frac{d'}{\omega^2 + \kappa'^2}, \tag{4.62}$$

where the half width of the spectrum in this case is given by

$$\kappa' = \mathcal{C} - \frac{\mathcal{C}^2}{\mathcal{A}}, \tag{4.63}$$

and d' is obtained from Eq.(4.58) by setting $r = 0$, and I_0 is given by (4.25). Comparing (4.61) and (4.63), we observe that the width of the spectrum increases due to the coupling of the laser with a squeezed vacuum reservoir.

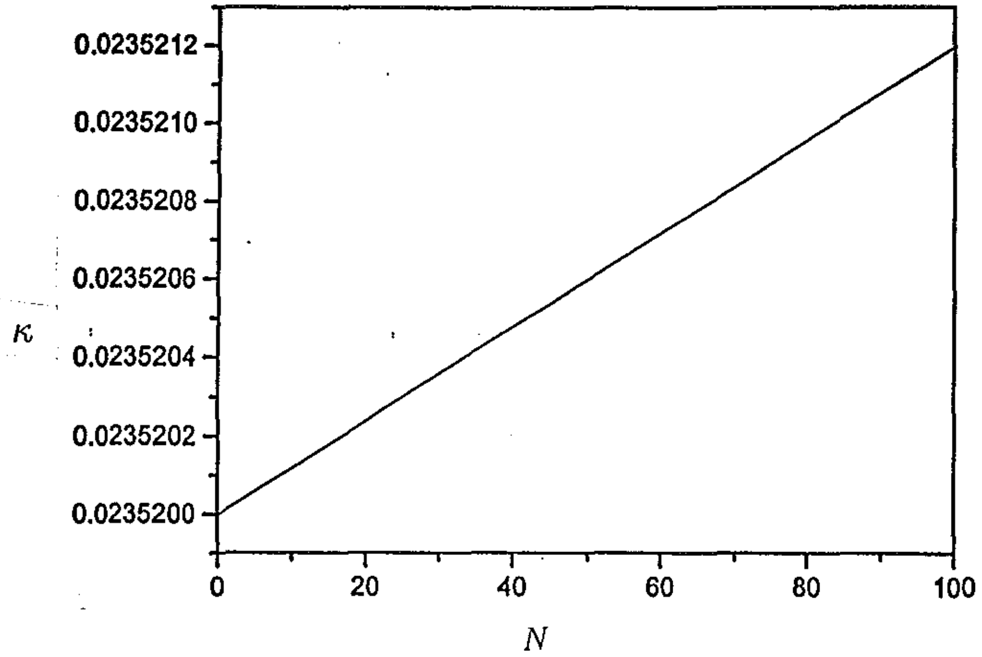

Fig. 4. Plot of the half width of the spectrum of intensity fluctuations (κ) versus N for $\mathcal{A}/\mathcal{B} = 1600$, $\mathcal{A} = 1.2$, $\mathcal{C} = 0.02\mathcal{A}$

5. CONCLUSION

We have obtained the c-number Langevin equations for the atomic and radiation variables from the corresponding operator Langevin equations applying a new method discussed in reference [16]. Furthermore, applying the principle of adiabatic elimination, we have derived the equation of evolution for the cavity mode variables.

With the aid of these equations, we have determined the antinormally ordered characteristic function defined in the Heisenberg picture which in turn is used to calculate the Q-function for a laser operating below threshold. The Q-function has been used to calculate the mean and variance of the photon number, and the photon number distribution. We have seen that the effects of the squeezed vacuum are to increase the mean photon number and to decrease the variance of the photon number. We also found that the photon statistics of the radiation is supper-Poissonian. We have calculated the spectrum of intensity fluctuations and the power spectrum for the same regime of operation and found in both cases that the effect of the squeezed vacuum is to increase the peak of the spectrum with no effect on the width of the spectrum. Moreover, the squeezed vacuum increases the peak of the squeezing spectrum for the first quadrature and decreases the peak of the spectrum for the second quadrature of the output radiation with no effect on the width of the spectrum. In addition, for sufficiently large squeeze parameter r, the cavity mode will be in a squeezed state.

We have also investigated the properties of a laser coupled to a squeezed vacuum operating above threshold. We have found that the intensity fluctuations and laser linewidth depend on the instantaneous relative phase between the squeezed vacuum and the laser. When the squeezed vacuum and the laser are in phase, the intensity fluctuations increases and the laser linewidth decreases, and when they are out of phase by $\pi/2$, both the intensity fluctuations and laser linewidth increase. Moreover, the width of the spectrum of intensity fluctuations for a laser operating above threshold increases and the peak of the spectrum depends on the relative phase difference.

We strongly believe that the method we have used in determining the c-number Langevin equations provides a convenient means of obtaining stochastic differential equations. In addition, the method employed to determine the characteristic function defined in the Heisnberg picture provides a significant mathematical simplicity.

REFERENCES

[1] A. M. Prokorov,Sov. Phys. JETP 34, 1658 (1958).

[2] A. L. Schawlow and C. H. Townes, Phys. Rev. 112, 40 (1958).

[3] J. P. Gordon, H. J. Zeiger, and C. H. Townes, Phys. Rev. 95, 282 (1954).

[4] W. H. Louisell, Quantum Statistical Properties of Radiation (Wiley, New York, 1973).

[5] Y. Yamamato, S. Machida, and O. Nilson, Phys. Rev. A 34, 4025 (1980).

[6] D. F. Walls and G. J. Milburn, Quantum optics (Springer-Verlag, Berlin, 1994).

[7] C. W. Gardiner, Quantum Noise (Springer–Verlag, Berlin, 1994).

[8] M. O. Scully and M. S. Zubairy, Quantum Optics (Cambridge, United Kingdom, 1997).

[9] M. O. Scully, Phys. Rev. Lett. 55, 2802 (1985).

[10] J. Gea–Banacloche, Phys. Rev. Lett. 59, 543 (1987).

[11] Monika A. M. Marte *et al*, Phys. Rev. A 37, 1235 (1988).

[12] Monika A. M. Marte *et al*, Phys. Rev. A 38, 3577 (1988).

[13] C. Ginzel *et al*, Phys. Rev. A 42, 4164 (1990).

[14] P. R. Rice *et al*, Phys. Rev. A 50, 4176 (1994).

[15] M. J. Collet and C. W. Gardiner, Phys. Rev. A 31, 3761 (1985).

[16] K. Fesseha, Output Analysis of a Degenerate Parametric Oscillator Coupled to a Squeezed Reservoir (to be published).

[17] M. O. Scully, W. E. Lamb, Phys. Rev. 159, 208 (1967).

[18] K. Fesseha, Fundamentals of Quantum Optics (to be published).

[19] W. Chow, W. Koch, and M. Sargent III, Semiconductor-Laser Physics (Springer-Verlag, Berlin, 1994).

[20] B. Daniel and K. Fesseha, Opt. Commun. 151, 384 (1998).

[21] A. S. Lane *et al*, Phys. Rev. A 38, 788 (1988).

[22] E. Jakman and R. Loudon, J. Phys. A: Math. Gen. 24 (1991).

[23] E. R. Pike and S. Sarkar, The Quantum Theory of Radiation, (Claredon Prees London, 1995).