Applications of Fixed Point Method to Semilinear Elliptic Equations

BY

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Approval

This thesis has been examined and approved as meeting the requirements for the partial fulfillment of Master of Science in Mathematics.

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Declaration

I, Chernet Zewdie, with student ID GSK|0096|06, hereby declare that this thesis entitled "**Applications of Fixed Point Method to Semilinear Elliptic Equations**" has been compiled and organized by myself under the supervision of Dr. Mengistu Goa and that it has never been submitted for completion of graduate qualification at any higher learning institution. Any work done by others has been acknowledged and referenced accordingly.

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List of Mathematical Notations

Notation	Meaning
Ω	An open subset of R^N(Omega)
$\mathbb{C}$	The set of all complex numbers
$\mathbb{R}$	The set of all Real numbers
Δu	Laplacian of u
Φ	Phi
δ	Delta
$\mathbb{N}$	The set of all natural numbers
T	Operator
$\mathcal{L}^p(\Omega)$	Space of p^{th}-power lebesgue integrable functions on Ω
$\mathbb{R}^n$	Euclidean n-space
$\nabla u = \left(\frac{\partial u}{\partial x_1}, \frac{\partial u}{\partial x_2}, \dots, \frac{\partial u}{\partial x_n}\right)$	Gradient of the function u
$C_c^\infty(\Omega)$	The class of all infinitely differentiable Functions on Ω with compact support.
$W^{m,p}(\Omega)$	The sobolev space of functions whose distributional Derivatives up to m^{th} order belongs to$\mathcal{L}^p(\Omega)$.
$D^\alpha{}_\varphi$	The derivative of the function φ of order α.
∂B^n	The boundary of B^n i.e $\partial B^n = \overline{B^n}/_{B^n}$
$H^m\ (\Omega)$	Sobolev space on Ω
$\boldsymbol{\mu}$	Mu
$\overline{B(0\,;1)}$	Closed ball of radius 1 centered at 0 .
$\forall$	For all
$B(x\,,r)$	The open ball of radius r $>$ 0 centered at the point $x \in \mathbb{R}^n$

Abstract

In the study of differential equations there are two fundamental questions: Is there a solution? And what is it? One of the most elegant was to prove that an equation has a solution is to pose it as fixed point problem, that is, to find a function f such that x is a solution if and only if $f(x) = x$. Results from fixed point theory can then be employed to show that f has a fixed point. However, the results of fixed point theory are often non-constructive: they guarantee that a fixed point exists but do not help in finding the fixed point. Thus these methods tend to answer the first questions, but not the second. One such result is Schauder's fixed point theorem. This theorem is broadly applicable in proving the existence of solutions to differential equations.

In this thesis we present a selection of fixed point theorems with applications in semilinear elliptic equation. We begin with the Banach fixed point theorem. Then , prove in succession the fixed point theorems of Brouwer, Schauder and Schaeffer after which we conclude with applications for semilinear elliptic equation.

Introduction

Fixed point theory is a fascinating subject ,with an enormous number of applications in various fields of mathematics.

The goal of this thesis is to introduce some basic methods to establish existence of solutions to non-linear equations in infinite-dimensional spaces, such as semilinear elliptic equations.

Fixed point theorems guarantee the existence of a fixed point under appropriate conditions on the map f and the set X. Over the content of this thesis we present major fixed point theorems and prove some fundamental results in partial differential equations by reducing semilinear elliptic problems to fixed point problems. We use knowledge of basic real and functional analysis. Familiarity with results from topology is helpful.

The thesis is divided into four chapters, each chapter containing several sections.

In the first chapter there are three sections which review some basic notations and results: the notations of metric space and normed vector space, Hilbert space, Sobolev spaces, operators between normed space and monotone operators.

In chapter two, we present important fixed point theorems illustrating the conditions for existence and uniqueness of solutions for semilinear partial elliptic equation. such as Banach's fixed point theorem, Brouwer fixed point theorem, Schauder’s fixed point theorem and Schaeffer’s fixed point theorem.

In chapter three we discuss applications in semilinear elliptic equation including partial differential equations, some results in functional analysis and solution method for semilinear elliptic equations by using fixed point theorem.

The last chapter is focused on summarization and conclusion of the thesis.

Chapter One

Preliminary Concepts

1.1 Metric Space and Normed Vector Space

In this chapter we introduce certain definitions and results concerning about metric space and normed vector spaces, Hilbert space, Sobolev spaces, operators between normed spaces and monotone operators.

Definition 1.1.1. Let X be a non-empty set. A function $d: X \times X \to [0,\infty)$ is said to be a metric if and only if

a) $d(x, y) = 0$ if and only if $x = y$ for all $x, y \in X$;

b) $d(x, y) = d(y, x)$ for all $x, y \in X$;

c) $d(x, y) \leq d(x, z) + d(z, y)$ for all $x, y, z \in X$.

Then the pair (X, d) is called metric space.

Definition1.1.2. Let X be a vector space over the field $\mathbb{R}(or\ \mathbb{C})$. A function

$\| \cdot \|: X \times X \to [0,\infty)$ is said to be a norm if and only if for each $x, y \in X$ and $\alpha \in \mathbb{R}(or\ \mathbb{C})$,

a) $\| x \| = 0$ if and only if $x = 0$;

b) $\| \alpha x \| = |\alpha| \| x \|$;

c) $\| x + y \| \leq \| x \| + \| y \|$.

Then the pair $(X; \| \cdot \|)$ is called normed vector space. Any normed vector space is a metric space when we use the norm induced metric $d(x, y) = \|x - y\|$.

Difinition 1.1.3. A sequence $\{x_n\}$ in a metric space (X, d) is said to be a Cauchy sequence provided for each $\varepsilon > 0$, there is an index N for which if n , m $\geq N$, then $d(x_n, x_m) < \varepsilon$. In other words , in a normed space, a Cauchy sequence $\{ x_n\}$ is one such that $\forall \varepsilon > 0$ there is an index N such that $n, m \geq N$ implies $\|x_n - x_m\| < \varepsilon$.

Definition 1.1.4. Let X be a vector space over the field $\mathbb{R}$. A function $\langle \cdot, \cdot \rangle: X \times X \to \mathbb{R}$ is said to be an inner product if and only if for each $x, y, z \in X$ and $\alpha, \beta \in \mathbb{R}$,

a) $\langle x, x \rangle \geq 0$, and $\langle x, x \rangle = 0$ if and only if $x = 0$;

b) $\langle \alpha x + \beta y, z \rangle = \alpha \langle x, z \rangle + \beta \langle y, z \rangle$;

c) $\langle x, y \rangle = \langle y, x \rangle$.

A pre-Hilbert space (or an inner product space) is a vector space X with the inner product defined on X. An inner product on X defines a norm on X given by $\| x \| = \sqrt{\langle x, x \rangle}$ and a metric on X given by

$$d(x , y) = \| x - y \| = \sqrt{\langle x - y, x - y \rangle}$$

Hence inner product spaces are normed vector spaces.

A metric space X is said to be complete provided every Cauchy sequence in X converges to a point in X. A normed vector space that is complete in a meteric induced by the norm is called Banach space.

A vector space with an inner product that is a Banach space with respect to the induced norm is called a Hilbert space.

Example 1.1.1. $\mathbb{R}^n$ *and* $\mathbb{C}^n$ with the inner product taken to be the standared dot product , $\langle x, y \rangle = \sum_{i=1}^{n} x_i \overline{y_i}$ are Hilbert space.

1.2 Sobolev Spaces

Spaces of weakly differentiable functions, so called Sobolev spaces, play an important role in analysis . Sobolev and Lebesegue spaces are important examples of Hilbert spaces.

Defination 1.2.1. Let $\Omega \subset \mathbb{R}^n$ be an open set, $m \in \mathbb{N}, 1 \leq p \leq \infty$. The Sobolev space $W^{m,p}(\Omega)$ is defined by

$$W^{m,p}(\Omega) = \{ u \in \mathcal{L}^p(\Omega) | D^\alpha u \in \mathcal{L}^p(\Omega), \qquad \text{for all } |\alpha| \leq m \}$$

The space $W^{m,p}(\Omega)$ is a normed space (V. Burenkov,1998). Contained in $\mathcal{L}^p(\Omega)$ with the norm

$$\| u \|_{W^{m,p}(\Omega)} = \begin{cases} \left(\sum_{|\alpha| \leq m} \| D^\alpha u \|_{\mathcal{L}^p(\Omega)}^p \right)^{\frac{1}{p}} & if \quad 1 \leq p < \infty \\ \max_{|\alpha| \leq m} \| D^\alpha u \|_{\mathcal{L}^p(\Omega)}^p & if \quad p = \infty \end{cases}$$

In particular, if $p = 2$, we write $H^m(\Omega)$ instead of $W^{m,2}(\Omega)$. The corresponding norm $\| u \|_{W^{m,p}(\Omega)}$ be written as $\| u \|_{H^m(\Omega)}$ and it is generated by the inner product

$$(u , v)_{H^m(\Omega)} = \sum_{|\alpha| \leq m} \int_\Omega D^\alpha u \, D^\alpha v, for\ all\ u , v \in H^m(\Omega)$$

When there is no confusion we shall often write $W^{1,p}$ instead of $W^{1,p}(\Omega)$. The space $W^{1,p}$ is equipped with the norm

$$\|u\|_{W^{1,p}} = (\|u\|^p{}_{\mathcal{L}^p} + \|\nabla u\|^p{}_{\mathcal{L}^p})^{\frac{1}{p}}$$

for $1 < p < \infty$.

The space $H^1(\Omega)$ is equipped with the scalar product

$$(u\,,v)_{H^1(\Omega)} = (u\,,v)_{\mathcal{L}^2(\Omega)} + \sum_{i=1}^{N}\left(\frac{\partial u}{\partial x_i},\frac{\partial v}{\partial x_i}\right)_{\mathcal{L}^2(\Omega)} = \int_\Omega uv + \sum_{i=1}^{N}\frac{\partial u}{\partial x_i}\frac{\partial v}{\partial x_i}\,.$$

The associated norm

$$\|u\|_{H^1} = \left(\|u\|^2_{\mathcal{L}^2(\Omega)} + \sum_{i=1}^{N}\left\|\frac{\partial u}{\partial x_i}\right\|^2_{L^2(\Omega)} \right)^{\frac{1}{2}}$$

is equivalent to the $W^{1,2}$ norm.

Remark 1.2.1. In the definition of $W^{1,p}$ we could equally well have used $C_c^\infty(\Omega)$ as set of test functions φ (instead of $C_c^1(\Omega)$) .

Remark 1.2.2. It is clear that if $u \in C^1(\Omega) \cap \mathcal{L}^p(\Omega)$ and $if\ \frac{\partial u}{\partial x_i} \in \mathcal{L}^p(\Omega)$ for all i=1,2,... ,N (here $\frac{\partial u}{\partial x_i}$ means the usual partial derivatives of u) , then $u \in w^{1,p}(\Omega)$. Furthermore , the usual partial derivatives coincide with the partial derivatives in the $W^{1,p}$ sense, so the notation is consistent. In particular , if Ω is bounded , then $c^1(\overline{\Omega}) \subset W^{1,p}(\Omega)$ for all $1 \leq P \leq \infty$. Conversely, one can show that if $u \in W^{1,p}(\Omega)$ for some $1 \leq P \leq \infty$ and $if\ \frac{\partial u}{\partial x_i} \in C(\Omega)$ for all i =1,2,. .. , N (here $\frac{\partial u}{\partial x_i}$ means the partial derivative in the $W^{1,p}$ sense) ,then $u \in C^1(\Omega)$; more precisely, there exists a function $\bar{u} \in c^1(\Omega)$ such that $u = \overline{u}$ a.e .

Remark 1.2. 3. Let $\{u_n\}$ be a sequence in $W^{1,p}$ such that $u_n \to u$ in $\mathcal{L}^p$ and (∇u_n) converges to some limit in $(\mathcal{L}^p)^N$. Then $u \in W^{1,p}$ and $\|u_n - u\|_{W^{1,p}} \to 0$. When $1 < P \leq \infty$ it suffices to know that $u^n \to u$ in $\mathcal{L}^P$ and that (∇u^n) is bounded in $(\mathcal{L}^p)^N$ to conclude that $u \in W^{1,p}$.

Lemma 1.2.1 (poincare's inequality).[1 corollary 9.19. H.Brezis, 2010, p 290.]

Suppose that $1 \leq P < \infty$ and Ω is a bounded Lipschitz open set.Then there exists a constant C(depending on Ω and P) such that

$$\|u\|_{W_0^{1,p}(\Omega)} \leq c\,\|\nabla u\|_{\mathcal{L}^2(\Omega)}, \qquad \forall u \in {W_0}^{1,p}(\Omega).$$

In Particullar, the expression $\|\nabla u\|_{\mathcal{L}^p}$ is a norm on $W_0^{1,p}(\Omega)$, and it is equivalent to the norm $\|u\|_{W^{1,P}}$; on $H_0^1(\Omega)$ the expression $\sum_{i=1}^{N}\int_\Omega \frac{\partial u}{\partial x_i}\cdot\frac{\partial v}{\partial x_i}$ is a scalar product that induces the norm $\|\nabla u\|_{\mathcal{L}^2}$ and it is equivalent to the norm $\|u\|_{H^1}$.

Remark 1.2.4. Poincare's inequality remains true if Ω has finite measure and also if Ω has a bounded projection on some axis.

Proposition 1.2.1[Prop. 8.1: H.Brezis, 2010,p.203].The space $W^{1,p}$ is a Banach space $for\ 1 \leq P \leq \infty$. It is reflexive for $1 < P < \infty$ and separable for $1 \leq p < \infty$. The space H^1 is a separable Hilbert Space.

1. 3 Operators between Normed Spaces

Let X and Y be normed Spaces. We define an operator T to be a mapping from a domian $D(T) \subset X$ in to Y, and write $T: D(T) \rightarrow Y$. We denote the action of an operator T on an element $x \in D(T)$ by $T(x)$. The Kernel (or null Space) and the range (or image) of T are defined by

$$Ker(T) := \{x \in D(T) \mid T(x) = 0\}.$$

$$Im\ (T) = \{y \in Y \mid T(x) = y\ for\ x \in D(T)\}.$$

Moreover, if T is linear, ker(T) and Im(T) form subspaces of X and Y respectively.

Definition 1.3.1. Let $T: D(T) \rightarrow Y$ be an operator between normed spaces X and Y, where $D(T) \subset X$. We say that

$(\boldsymbol{i})$.T is linear if $T(\alpha x_1 + \beta\, x_2) = \alpha T x_1 + \beta\, T x_2\ \ for\ all\ x_1, x_2 \in D(T)$ and for all $\alpha, \beta \in \mathbb{R}$.

$(\boldsymbol{ii})$. T is bounded if there exists a constant M > 0 such that $\|T_x\|_Y \leq M\|X\|_X$.

The smallest such M (if it exists) is called the operator norm of T, denoted $\|T\|_{\mathcal{L}(x,y)}$. That is

$$\|T\|_{\mathcal{L}(x,y)} := \sup_{x\in D(T)} \frac{\|T_x\|_Y}{\|x\|_X}, \text{ x}\neq 0.$$

(iii). T is injective if $T(x_1) = T(x_2)$ implies that $x_1 = x_2, \forall x_1, x_{2\,\in D(T)}$.

$(\boldsymbol{iV})$. T is surjective if for all $y \in Y$ there exists X ∈ D(T) such that $T(x) = y$.

$(\boldsymbol{V})$. Let $Y = \mathbb{R}$, then T is said to be corecive if $\|x\|_X \rightarrow \infty$ implies $T(x) \rightarrow \infty$.

Definition 1.3.2 (**Continuity of operators)**. Let T be an operator between normed spaces X and Y. **T** is said to be

a) Continuous if it is continuous at each $x \in X$, $t$hat is, if for all $\varepsilon > 0$ there exists a $\delta > o$ such that $\|T(x_1) - T(x_2)\|_Y < \varepsilon$ whenever $\|x_1 - x_2\|_X < \delta \quad \forall\, x_1, x_2 \in$ X.

b) Hölder continuous with exponent α if there exists $\alpha \in (0,1]$ and C > 0 such that

$$\|T(x_1) - T(x_2)\|_Y \leq c\|x_1 - x_2\|^{\alpha}{}_X, \forall x_1, x_2 \in \text{X}.$$

c) Lipschitz continuous if it is Hölder continuous with exponent α =1, that is, there exists $L > 0$ such that

$$\|T(x_1) - T(x_2)\|_Y \leq L\|x_1 - x_2\|_X, \forall x_1, x_2 \in X.$$

If $L < 1$, we call T a contraction. Then , it clear that the following chain of implications holds.

$$T \text{ is Lipschitz continuous } \Longrightarrow T \text{ is Hölder continuous } \Longrightarrow T \text{ is continuous}.$$

Definition 1.3.3. Let X and Y be normed Spaces. A bilinear form a on $X \times Y$ is a mapping $a: X \times Y \to \mathbb{R}$ linear with respect to each arguments. That is , for $x, x_1, x_2 \in X \quad y, y_1, y_2 \in Y$ and c , d $\in \mathbb{R}$,We have

$$a(cx_1 + dx_2, y) = ca(x_1, y) + da(x_2, y),$$

$$a(x, cy_1 + dy_2) = ca(x, y_1) + da(x, y_2).$$

If there exists a real number M > 0 such that for all $x \in X$ and $y \in Y$, we have $\|a(x, y)\| \leq M\|x\|_Y\|y\|_Y$

then a is said to be bounded. The smallest such M is called the norm of a, and is denoted $\|a\|_{\mathcal{L}}(X \times Y, \mathbb{R})$.

Definition 1.3.4. Let H be a real Hilbert space. An operator $T: H \to H$ satisfying $\lim_{\|u\|_{H \to \infty}} \|T(u)\|_H = \infty$ is called weakly coercive.

Definition 1.3.5. Let H be a real Hilbert Space and let $T: H \to H$ be an operator

a) T is said to be a monotone operator if for any u ,v $\in$ H

$$(u - v, T(u) - T(v))_H \geq 0 \text{ ---- (1.6.1)}$$

b) T is strictly monotone if for any $u, v \in H$ and u≠v the strict inequality holds in (1.6.1).

c) **T** is called strongly monotone if there exists $c > 0$ Such that $(u - v, T(u) - T(v))_H \geq C\|u - v\|^2{}_H \, for\ any\ u, v \in H.$

Remark 1.3.1. It is clear that a strongly monotone operator is strictly monotone and , therefore, monotone. Also every strongly monotone operator is weakly coercive. Indeed **T** is Strongly monotone implies

$$\left(u\,,T(u)-T(0)\right)_H \geq\; c\|u\|^2{}_H \quad --- \; (1.6.2)$$

The Schwartz inequality yields

$$(u\,,T(u)-T(0))_H \;\leq \|u\|_H\,[\|T(u)\|_H \;+\|T(0)\|_H\,] \;--- (1.6.3)$$

putting (1.6.2) and (1.6.3) together we get

$$\|T(u)\|_H \;\geq c\|u\|_H - \|T(0)\|_H$$

and the weak coercivity follows.The following theorem is a basic assertion of this section .

Theorem 1.3.1. $[\,4.page.209]$. Let H be a real Hilbert space and $T\colon H \to H$ be continuous, monotone and weakly coercive. Then $T(H) = H$

Propostion 1.3.1. Let H be a real Hilbert Space and $S\colon H \to H$ be a continuous and Strongly monotone operator. Then **S**(H) = H.

Now, first we prove that **S**(H) is closed.

Lemma 1.3.1. $proof[see\ 4.page.211]$ Let D be a closed set in $H\,,S\colon D \to H$ be a continuous and strongly monotone operator. Then $S(D)$ is a closed set in H.

Lemma 1.3.2. $proof[see\ 4.page.214-216]$ Let $D \subset H$ be an open subset,$S\colon D \to H$ be continuous and strongly monotone. Then **S**(D) is an open subset of H.

Chapter Two

Fixed Point Theorems

This section will discuss about fixed point theorems such as: Banach's fixed point theorem, Brouwer's theorems, Schauder's fixed point theorem and Schaeffer's fixed point theorems . Fixed point theorems concern maps f of a set X in to itself that under certain conditions, admit a fixed point that is a point $x \in X$ such that $f(x) = x$.

Definition 2.1. Let (X, d) be a metric space and $f : M \subset X \to X$ be a map. A solution of $f(x) = x$ is called a fixed point of f.

2.1 Banach's Fixed Point Theroem

Theorem 2.1.1 (Banach Fixed Point Theorem). Let (X , d) be a complete metric Space and $M \subseteq X$ be non- empty and closed. If a map $T: M \subset X \to M$ a contraction .Then the equation $T(x) = x$ has a unique solution (T has a unique fixed point x ∈ M).

Proof. Note that X is complete, and $M \subset X$ is non- empty and closed, therefore (M, d) is also a complete metric space. So it is sufficient to consider the case $M = X$

<u>Existence:</u> Let T be contraction. Then there exists a constant k ∈ [0 , 1) such that

$$d\big(T(x), T(y)\big) \leq \; kd(x , y), for\ any\ x, y\ in\ X.$$

Since the sequence $\{x_n \}$ define by, $x_{n+1} = T(x_n)$ for all $n \in \mathbb{N}$,

$$d(x_2 , x_1) = d(T(x_1) , T(x_0)) \leq kd(x_1 , x_0)$$

for some $k \in [0 , 1)$. Continuing inductively, we have

$$d(x_{n+1} , x_n) = d\big(T(x_n), T(x_{n-1})\big) \leq k^n d(x_1 , x_0).$$

Thus for $n < m$, we have

$$d(x_n , x_m) \leq d(x_n , x_{n+1}) + \cdots + d(x_{m-1} , d_m)$$

$$\leq (k^n + k^{n+1} + \cdots + k^{m-1}) d(x_1 , x_0)$$

$$\leq \frac{k^n}{1-k} d(x_1 , x_0),$$

where we have made use of the triangle inequality and the properties of sums.

Since $0 \leq K < 1$, we find $\frac{k^n}{1-k} \to 0$ as n → ∞.

Hence $\{x^n\}$ is a cauchy sequence, and since x is complete it has a limit x ∈ X.

Since contraction maps are continuous , so it follows that

$$T(x) = lim_{n\to\infty} T(x_n) = lim_{n\to\infty} x_{n+1} = x, as\ desired.$$

uniquenss: To see that the fixed point $x \in X$ is unique , suppose that there is $x' \neq x$ in X such that x' also a fixed point.Then , $d(T(x) , T(x')) = d(x , x')$ since $T(x) = x$ and $T(x') = x'$. Since T is contractive $d(f(x) , f(x')) < d(x , x')$, which is a contradiction. Therefore the fixed point $x \in X$ is unique .

2.2 Brouwer Fixed Point Theorem

Notation. Denote the unit ball in ℝ^n by $\boldsymbol{B^n} := \boldsymbol{B}\overline{(\mathbf{0}\, ; \mathbf{1})} = \{x \in \mathbb{R}^n : |x| \leq 1\}$ and the unit sphere (the boundary of the unit ball) by

$$S^{n-1} \coloneqq \{x \in \mathbb{R}^n : |x| = 1\} = \partial B^n$$

Definition 2.2.1. Let **A** be a subset of a topological space X. A retraction is a map r:X⟶ A such that $r(x) = x$ for all $x \in A$. If there exists a retraction from $X\ to\ A$, we say A is a retract of X .

The contraction mapping therorem only required few pre-requisties on the space $\boldsymbol{X}$ but strong conditions on the map **T**. For Brouwer's (and later Leray-Schauder's) fixed point theorem we only need continuity of **T** but strong conditions on the space X .We will also loose uniqueness of a fixed point and (x_n) , defined via $x_{n+1} = f(x_n)$ does not necessarily converge.

Example 2.2.1. Let $X = [0 , 1)$ and $f(x) = \frac{x+1}{2}$. $There$ is no fixed point . Here f is continuous and X is connected , $but\ X\ is\ not\ compact$.

Example 2.2.2. Let $X = \left[0, \frac{1}{3}\right] \cup \left[\frac{2}{3}, 1\right]$ and $f(x) = \frac{1}{2}$. There is no fixed point . Here f is continuous and X is compact , but X is not connected .

Example 2.2.3. f : $[0 , 1] \to \mathbb{R}$, where $f(x) = x^3 + x - 1$, $f(x) = 0$ has a unique solution.

Note: $f(0) = -1$, $f(1) = 1$ and $f'(x) = 3x^2+1$ which implies $1 \leq f'(x) \leq 4$.Set $g(x) = x - \frac{1}{4}f(x)$. Then $g'(x) = 1 - \frac{1}{4}f'(x)$. Which implies $0 \leq g'(x) \leq 1 - \frac{1}{4} = \frac{3}{4}$

$g(0) = \frac{1}{4}$ and $g(1) = 1 - \frac{1}{4} = \frac{3}{4}$ complet. So , $g : [0,1] \to \left[\frac{1}{4}, \frac{3}{4}\right] \subset [0,1]$. Here $[0,1]$ is a closed subset then compact.

By Mean value theorem we have $|g(x) - g(y)| = |g'(a)(x - y)|$

$= |g'(a)||x - y|$

$= \frac{3}{4}|x - y|$. So , g is a contraction. Hence g has a unique fixed point in $[0,1]$.

Lemma 2.2.1(No Retraction Theorem). There is no continuous retraction $\boldsymbol{r}: B^n \to S^{n-1}$. Intuitively, it is not difficult to see why this Lemma holds. If we fix every point on the surface of the sphere, there is no function that continuously ‟ makes room” for every mapped point from the interior of the sphere.

Theorem 2.2.1. Let $K \subset R^n$ closed, convex, *bounded* and $\boldsymbol{T} : B^n \to B^n$ be a continuous function. Then , **T** has a fixed point $x \in B^n$.

Proof. Suppose there exists a map $\boldsymbol{T} : B^n \to B^n$with no fixed points. Construct the map

$\boldsymbol{r}: B^n \to S^{n-1}$ by extending a ray along the path from $\boldsymbol{x}$ to $T(x)$ and defining $\boldsymbol{r}(x)$ to be the intersection of the ray with the sphere S^{n-1} (see figure1).

The map $\boldsymbol{r}$ is well-defined since $x \neq T(x)$ for any $x \in B^n$, and continuous since **T** is continuous . Moreover, $r(x) = x$ for all $x \in S^{n-1}$, so **r** is a retraction from B^n to s^{n-1}. But this contradicts Lemma 2.2.1 above , which says that no such retraction exists. Hence, **T** must have a fixed point.

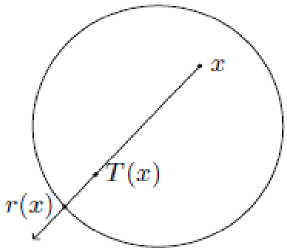

figure 1. The map r for n = 2

Note that it does not follow from Brouwer fixed point theorem that the fixed point is unique. Consider for instance the identity operator on a compact set K in $\mathbb{R}^n$ for which every $x \in K$ is a fixed point.

Remark 2.2.1. Brouwer's fixed point theorem is false in infinite dimensional spaces .The reason behind this is that a closed ball is not compact. Note also that in an infinite dimensional space, a continuous function may well be unbounded on a closed and bounded sets (and this fact makes the study of nonlinear equations much more difficult).

2.3 Schauder's Fixed Point Theorem

Definition 2.3.1. Let X be a normed vector space and $F = (x_1, x_2, \dots, x_n)$ a finite sub set of X. Then $conv(F)$, the convex hull of F, is defined by

$$Conv(F) = \left\{ \sum_{j=1}^{n} t_j x_j : \sum_{j=1}^{n} t_j = 1, \qquad t_j \geq 0 \right\}.$$

Lemma 2.3.1 (Schauder projection Lemma). Let K be a compact sub set of a normed vector space X , with metric **d** induced by the norm $\|.\|$. Given $\varepsilon > 0$, there exists a finite subset $F \subseteq X$ and a map $P: K \to conv(F)$ such that $d(P(x), x) < \varepsilon$ for all $x \in K$. This map is called the schauder projection.

Proof. Take a finite ε-net for the compact set K to obtain a set $F = \{x_1, \dots, x_n\}$ for i=1, ... ,n, define functions ϕ_i:K→ $\mathbb{R}$ by

$$\phi_i(\mathrm{x}) = \begin{cases} \varepsilon - d(x, x_i) \; if \; x \in B(x_{i,\varepsilon}) \\ 0 \qquad\qquad otherwise \end{cases}$$

We see that ϕ_i is strictly positive on $B(x_{i\,;\,\varepsilon})$ and vanishes elsewhere.

Therefore $\sum_{i=1}^{n} \phi_i(x) > 0$ for all $x \in K$. We define the schauder projection

$$P : K \to conv(F) \; by \; P(x) = \sum_{i=1}^{n} \frac{\phi_{i(x)}}{\phi(x)} x_i \; where \; \phi(x) = \sum_{i=1}^{n} \phi_i(x).$$

The map P is continuous since all ϕ_i are . Moreover,

$$d(P(x), x) = \left\| \sum_{i=1}^{n} \frac{\phi_{i(x)}}{\phi(x)} x_i - \sum_{i=1}^{n} \frac{\phi_i(x)}{\phi(x)} x \right\| = \left\| \sum_{i=1}^{n} \frac{\phi_{i(x)}}{\phi(x)} (x_i - x \right\|$$

$$\leq \sum_{i=1}^{n} \frac{\phi_i(x)}{\phi(x)} \|x_i - x\| < \sum_{i=1}^{n} \frac{\phi_i(x)}{\phi(x)} \varepsilon = \varepsilon$$

because ϕ_i(x) = 0 if $\|x_i - x\| \geq \varepsilon$.

Definition 2.3.3.(convex sets). A Set $K \subset \mathbb{R}^n$ is said to be convex if either K= $\emptyset$ or, whenever we take two points in K, the segment that connects them is entirely contained in K. i.e. $\lambda x_1 + (1-\lambda)x_2 \in$ K $\forall_{\lambda \in [0,1]}, \forall x_1, x_2 \in K$.

Example 2.3.1. $\boldsymbol{\mathcal{L}}$ **:** $\mathbb{R} \to \mathbb{R}^{\boldsymbol{n}}$ be linear and $\alpha \in R$. Then the sets

$\{x \in \mathbb{R}^n \mid \ell(x) < \alpha\}$, $\{x \in \mathbb{R}^n \mid \ell(x) \leq \alpha\}$,

$\{x \in \mathbb{R}^n \mid \ell(x) \geq \alpha\}$, $\{x \in \mathbb{R}^n \mid \ell(x) > \alpha\}$ are convex.

Theorem 2.3.1 (Schauder's Fixed Point Theorem). Let X be a Banach space . Let M ⊆ **X** be a non-empty convex and closed. If $T : M \to M$ is compact, then T has a fixed point.

Proof. Let **K** denote the closure of $T(M)$ which, by hypothesis, is compact. For each natural number n, Let F_n be a finite $\frac{1}{n}$-net for K and let $P_n: K \to conv(F_n)$ be the corresponding Schauder projection. The convexity of M implies that conv(F_n) ⊆ K ; define

$$Tn;\ conv(F_n) \to conv(F_n)\ by\ T_n := (P_N\ 0T) \mid conv(F_n)\,.$$

This implies that T_n has fixed points because for K be a non-empty, compact and convex subset of R^n, every continuous map $T: K \to K$ has a fixed point. For each $n \in \mathbb{N}$, we choose one such fixed point of T_n and call it x_n. Since K is compact $\{x_n\}$ has a convergent subsequence, which we denote $\{x_{n'}\}$. This sequence converges to some $x \in K$ as $n' \to \infty$, which we claim is the desired fixed point.

From lemma 2.3.1 above we obtain

$$d(T(x), x_{n'}) \leq d(T(x), T(x_{n'})) + d(T(x_{n'}), T_{n'}(x_{n'})) \to 0\ as\ n' \to \infty.$$

Since **T** is continuous and

$$d(T(x_{n'}), T_{n'}(x_{n'})) = d(T(x_{n'}), x_{n'}) < \frac{1}{n'}\,.$$

Thus $\{x_{n'}\}$ converges to both x and $T(x)$. Limits are unique, so $T(x) = x$, as desired .

In Practice, it is often awkward to apply Schauder's fixed point therorem as one needs to find an appropriate set M- and such a set is rarely obvious. This gives rise to an alternative formulation, known as Schaeffer's fixed point therorem, in which we do not have to identify an explict convex, compact set.

Therorem 2.3.2(Schaeffer's Fixed Point Theorem). Let **X** be a Banach space and T:X → X be continuous and compact mapping. If the set $\{x \in X: x = \lambda T(x)$ for some λ ∈ [0,1]} is bounded, then **T** has a fixed point.

Proof. By hypothesis, we can choose a constant M so large that ||x|| < M if x = λT(x) for some $\lambda \in [0,1]$. Define a retraction

$$r: X \to B(0\,;\,M)\ by\ \mathbf{r}(x) = \begin{cases} x & if\ \|x\| \leq M \\ (M/\|x\|)_x & if\ \|x\| > M \end{cases}$$

and observe that the composition $(\boldsymbol{r}\ oT): B\ (o\,;\,M) \to B(0\,;\,M)$ is compact since T is compact.

Let K denote the closed convex hull of $(\boldsymbol{r}oT)\ (B(o;M))$. The set K is covex by definition, and the compactness of $\boldsymbol{roT}$ implies K is compact.

By Schauder's fixed point theorem, there exists a fixed point $x \in K$ of the restriction $(\boldsymbol{r}oT)|K: K \to K$. We claim that x is also a fixed point of T. To show this, it is sufficient to porve that $T(x) \in K$. Suppose not. Then

$$\|T(x)\| > M\ and\ \ x = r(T(x)) = \frac{M}{\|T(x)\|}T(x) \quad \ldots\ldots\ldots\ldots (2.3.1)\,.$$

Which implies $\|x\| = \|M/\|T(x)\|\ T(x)\| = M$. On the other hand , M / $\|T(x)\|$ ∈ (0 , 1) , so our choice of M and (2.3.1) also imply $\|x\|$ < M , a contradiction. Therefore ,T has fixed point.

ChapterThree

Application of Semilinear Elliptic Equations

3.1 Partial Differential Equation

Most real world physical systems including gas dynamics, fluid mechanics, elasticity, relativity, ecology, neurology, thermodynamics, and many more are modeled by non-linear partial differential equations. Non-linear differential equations can take on a number of different forms. In this thesis we will treat semilinear elliptic equations.

A Partial differential equation (PDE) for a function $u(x, y, \ldots)$ is a relation of the form

$$f(x\ , y, \ldots, u, u_x, u_y, \ldots, u_{xx}\ , u_{xy}, \ldots)\ =\ 0\ ,$$

Where f is a given function of the independent variables $x\ , y, \ldots,$ and of the unknown function u and of a finite or infinite number of its derivatives. We call u a solution of f if after substitution of $u(x, y\ , \ldots)$ and its partial derivatives , is satisfied identically in $x\ , y, \ldots$ in some region Ω in the space of these independent variables. A differential operator $\mathcal{L}$ is said to be a linear if for any two element u and v in S and $\alpha, \beta \in \mathbb{R}(or\ \mathbb{C})$, we have

$$\mathcal{L}(\alpha u\ +\ \beta v) =\ \alpha\mathcal{L}(u) +\ \beta\mathcal{L}(v)$$

Partial differential equations can be classified based on linearity:

1. Linear partial differential equations 2. Nonlinear partial differential equations

Nonlinear partial differential equations are also classified as:

1) **Semilinear Partial Differential Equations:** Semilinear partial differential equations represent the simplest type of non-linearity. They are linear in all terms except those of $zero^{th}$ order and the coefficients of the highest order derivatives depend only on the independent variables.

Example 3.1.1. $U_t - k\Delta U -\ e^{\frac{-c}{u}}\ =\ 0$ (reaction diffusion equation) with Δ denoting the laplacian ($e.g\ , \Delta \equiv \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\ in\ 2 - D$ Cartesian coordinates); k is a constant diffusion coefficient and **c** is a constant ‟reaction energy.”

2) **Quasilinear Partial Differential Equations**: Partial differential equations are said to be quasilinear if it is linear in the nth-order derivatives (highest derivative).

   $\boldsymbol{Example}\ \mathbf{3.1.2}.\quad u_t + uu_x + u_{xxx} = 0\ \ (Dispersive\ Wave\ Equations).$

3) **Fully Nonlinear Partial Differential Equations:** The fully nonlinear, case is characterized by nonlinearities even in the highest derivatives of the differential operator.

   $\boldsymbol{Example}\ \mathbf{3.1.3}.\ \boldsymbol{U^2}_{\boldsymbol{xy}} - U_{xx}U_{yy} = F(x,y)\quad (More\ Amper\ Equation).$

4) **Almost linear:** If it is nonlinear only with the dependent variable.

   Example 3.1.4. $U_{tt} - U_{xx} + U^3 = 0$ (Wave Equation with Interaction).

The classification of second order partial differential equations depends on the form of the leading part of the equation consisting of the second order terms. So, for simplicity of the notation , we combine the lower order terms and write the equation in the following form.

$$A(x,y)\frac{\partial^2 u}{\partial x^2} + B(x,y)\frac{\partial^2 u}{\partial x\,\partial y} + C(x,y)\frac{\partial^2 u}{\partial y^2} = \Phi(x,y,u,\frac{\partial u}{\partial x},\frac{\partial u}{\partial y}).$$

As we shall see, there are fundamentally three types of partial differential equations. These are:-

a) **Elliptic Partial Differential Equations:** Elliptic equations are typically associated with steady-state behavior. Assume,

$Au_{xx} + Bu_{xy} + Cu_{yy} + F(x,y,u,u_x,u_y) = 0$ be second order partial differential equations, if the discriminate $B^2 - 4AC < 0$, then the type of partial differential equation is elliptic.

Example 3.1.5. The two dimensional Laplace equation $u_{xx} + u_{yy} = 0$ is an elliptic partial differential equation.

b) **Parabolic Partial Differential Equation:** If the discriminate $B^2 - 4AC = 0$, then the type of partial differential equation is parabolic**.**

   Example 3.1.6. $\boldsymbol{a\,\frac{\partial^2 u}{\partial x^2} - \frac{\partial u}{\partial t} = 0}$ is a parabolic partial differential equation**.**

c) **Hyperbolic Partial Differentia Equation:** If the discriminate $B^2 - 4AC > 0$, then the type of partial differential equation is hyperbolic**.**

   Example 3.1.7. $u_{xx} - u_{yy} = 0$ is a Hyperbolic partial differential equation.

3.2 Some Results in Functional Analysis

Before beginning our study of semilinear eliptic partial differential equation , we first need some prerequiste facts .

Definition 3.2.1(Continuous embedding). Let X and Y be normed spaces. We say that X is continuously embedded on Y if $X \subset Y$ and $\exists\, c > 0$ such that $\|u\|_Y \leq c\|u\|_X$ where $u \in X$.

Definition 3.2.2 (Compact embedding). Let X and Y be Banach spaces and $X \subset Y$. We say that X is compactly embedded in Y, written $X \subset\subset Y$, $provided$ X is continuously embedded in Y and that each bounded sequence in X has a convergent subsequence in Y.

Corollary 3.2.1. Let $\mu \geq 0$ .Then the map $g \mapsto (-\Delta + \mu I)^{-1} g = v$ is

i. Continuous as a map from $\mathcal{L}^2(\Omega)$ to $H_0^1(\boldsymbol{\Omega})$, i.e,

$$\|V\|_{H_0^1(\Omega)} \leq C(\Omega)\|g\|_{\mathcal{L}^2(\Omega)} ,$$

where **C** is a constant dependent on **Ω**.

ii, Compact as a map from $\mathcal{L}^2(\Omega)$ to $\mathcal{L}^2(\Omega)$.

Proof. The first part is due to the fact that $\mathcal{L}^2(\Omega)$ is continuously embedded in $H^{-1}(\Omega)$.

The second part follows

$$(-\Delta + \mu \mathrm{I})^{-1} : \mathcal{L}^2(\Omega) \to \mathcal{L}^2(\Omega)$$

can be viewed as compostion of the continuous map $(-\Delta + \mu \mathrm{I})^{-1} : \mathcal{L}^2(\Omega) \to H_0^1(\Omega)$

and the compact embedding $H_0^1(\Omega) \to \mathcal{L}^2(\Omega)$ and as the compstions of a compact linear operator and a continuous linear operator is again compact.

Corollary 3.2.2. Given $f \in C(R)$ Such that $|f(t)| \leq a(1 + |t|)$ where a > 0 ,the map

$u \mapsto f(u)$ is Continuous from $\mathcal{L}^2(\Omega)$ to $\mathcal{L}^2(\Omega)$.

Lemma 3.2.3 (**Poincare' inequality for Zero mean**). Let $\boldsymbol{\Omega} \subset \mathbf{R}^{\mathbf{n}}$ be open, bounded and connectd with Lipschitz boundary. Let $1 \leq P \leq \infty$. Then there exists a constant $C(\Omega, P)$ such that

$$\|u - u_n\|_{\mathcal{L}^p}(\Omega) \leq c(\Omega, p)\|\nabla u\|_{\mathcal{L}^p}(\Omega) \qquad \forall u \in W^{1,p}(\Omega) \quad \ldots.. (3.2.1)$$

Where $u_\Omega := \frac{1}{\Omega}\int_\Omega u dx$. In Particular for $1 \le P < \infty$ there exist constants $C_1(P)$ such that

$$\int_{B_r \backslash B_{r/2}} \left| u - u_{B_r \backslash B_{r/2}} \right|^p \text{dx} \le c_1(p) r^p \int_{B_r \backslash B_{r/2}} |\nabla u|^p \text{ dx} \quad \text{- - --------(3.2.2)}$$

$$\int_{B_r} \left| u - u_{B_r} \right|^p \text{dx} \le c_1(p) r^p \int_{B_r} |\nabla u|^p \text{ dx} \qquad \text{-------- (3.2.3)}$$

Remark 3.2.1 . This can be combined with the sobolev embedding theorem.

For $p \le n$, $\frac{1}{q} \le \frac{1}{p} - \frac{1}{n}$ and $q \neq \infty$ we thus obtain the sobolev- poincare' inequality

$$\|u - u_n\|_{\mathcal{L}^q}(\Omega) \le c(\Omega, p)\|\nabla u\|_{\mathcal{L}^p}(\Omega) \qquad \forall_u \in W^{1,p}(\Omega) \quad \text{--------- (3.2.4)}$$

Notation: We write u_{x_0} , r as a shorthand for $U_B(x_0, r)$. - - - - (3.2.5)

Proof. We may assume that $u_n = 0$. Assume that the estimate (3.2.1) does not hold . Then for $k \in N$ there exist $u_k \in w^{1,p}(\Omega)$ such that

$$\|u_k\|_{\mathcal{L}^p} \ge K\|\nabla u_k\|_{\mathcal{L}^p} , \quad \int_\Omega u_k \, dx = 0 \quad \text{-- - - ---- (3.2.6)}$$

Set $v_k = u_k / \|u_k\|\mathcal{L}^{\mathrm{P}}$ .Then,

$$\|v_k\|_{\mathcal{L}^p} = 1 , \|\nabla v_k\|_{\mathcal{L}\boldsymbol{P}} \to 0, \int_\Omega v_k \, dx = 0 \text{ -------------- (3.2.7)}$$

In particular the functions v_k form a bounded set in $W^{1,p}(\Omega)$. Hence by the compact sobolev theorem there exists a subsequence V_{k_j} such that $V_{k_j} \longrightarrow$ V in $\mathcal{L}^p(\Omega)$ as j $\longrightarrow \infty$. Since we also know that $\nabla_{v_{k_j}} \to 0$ in $\mathcal{L}^p(\Omega ; R^n)$ it follows that $V_{k_j} \longrightarrow v$ in $W^{1,p}(\Omega)$ and $\nabla V = 0$. Thus we get $\|V\|_{\mathcal{L}^p} = 1$,

$$\nabla V = 0 \quad \int_\Omega V dx = 0 \text{ ------- (3.2.8)}$$

Since Ω is connected the condition $\nabla V = 0$ implies that V is constant in Ω.

The condition $\int_\Omega v dx = 0$ then implies that $V = 0$. This contradicts the fact that

$\|V\|_{\mathcal{L}^p} = 1$. This contradiction finishes the proof of (3.2.1).

Thus (3.2.2) and (3.2.3) hold for $r = 1$. To obtain the result for general $\boldsymbol{r}$ it suffices to apply the result for $r = 1$ the rescaled function $u_r(z) = u(x_0 + r_z)$.

Corollary 3.2.4. A norm can be defined on $H_0^1(\Omega)$ by

$$\|u\|_{H_0^1(\Omega)} = \|\nabla u\|_{\mathcal{L}^2(\Omega)}\text{, for } u \in H_0^1(\Omega).$$

This norm is equivalent to the standard norm on $H^1(\Omega)$ as described proposition 3.2.1 below.

Proposition 3.2.1. $H^1(\Omega)$ is a Hilbert space with inner product

$$< f , g > = \int_\Omega \nabla f . \nabla g \, dx + \int_\Omega f g \, dx.$$

Proposition 3.2.2. The Sobolev Space $H_0^1(\Omega)$ embeds continuously in to $\mathcal{L}^2(\Omega)$.

3.3 Solution Method for Semilinear Elliptic Equations by using Fixed Point Theorem

In this section we study semilinear elliptic partial differential equations of the form

$$\begin{cases} -\Delta u = f(u) & in\ \Omega \\ U = O & on\ \partial\Omega \end{cases} \quad - - - - (3.3.1)$$

where $\Omega \subseteq \mathbb{R}^n$ is open, bounded and smooth and $f : \mathbb{R} \to \mathbb{R}$ is a given function.

Theorem 3.3.1. Let $\Omega \subseteq \mathbb{R}^n$ be an open, bounded and smooth domain and $f \in C(\mathbb{R})$ be a given bounded function. The boundary value problem (3.3.1) has a weak solution $u \in H_0^1(\Omega)$,

i.e the following formulation holds $\int_\Omega \nabla u . \nabla \phi dx = \int_\Omega f(u) \phi dx$.

Sketch of proof. Define a map $T: \mathcal{L}^2(\Omega) \to \mathcal{L}^2(\Omega)$ by $u \longmapsto (-\Delta)^{-1}(f(u))$ .Our strategy is to show that T satisfies the hypotheses of Schauder's fixed point theorem , which will then yield the desired weak solution.

Step 1. T is continuous. corollary 3.2.2 shows that $u \mapsto f(u)$ is continuous from $\mathcal{L}^2(\Omega)$ to itself ; corollary 3.2.1 shows that $(-\Delta)^{-1}$ is continuous from $\mathcal{L}^2(\Omega)$ in to $H_0^1(\Omega)$, which is continuously embedded in $\mathcal{L}^2(\Omega)$ by proposition 3.2.2 .

Step 2. Find a closed non-empty bounded convex set M such that $T: M \to M$. Given $u \in \mathcal{L}^2(\Omega)$, T(u) satisfies

$$\int_\Omega \nabla T(u) \nabla T(u)\, dx = \int_\Omega f(u)\, T(u)\, dx \le a|\Omega| \|T(u)\|_{\mathcal{L}^2(\Omega)}. \quad \text{---------}(3.3.2)$$

by the cauchy-schwarz inequality. Using poincare's inequality then gives

$$\|T(u)\|^2_{\mathcal{L}^2(\Omega)} \leq C\|\nabla T(u)\|^2_{\mathcal{L}^2(\Omega)} \leq \mathrm{a}|\Omega|C\|T(u)\|_{\mathcal{L}^2(\Omega)}$$

for some constant C. Set $r = a|\Omega|C$ and choose $\mathrm{M} := \{u \in \mathcal{L}^2(\Omega) : \|u\|\mathcal{L}^2(\Omega) \leq r\}$.

Hence $T: M \to M$.

Step 3. T is compact. Using Poincare's inequality on the right-hand side of (3.2.2) ,we obtain $\|\nabla T(u)\|^2_{\mathcal{L}^2(\Omega)} \leq K\|\nabla T(u)\|^2_{\mathcal{L}^2(\Omega)}$ for some constant K.

Thus $\|\nabla Tu\|_{\mathcal{L}^2(\Omega)} \leq K$, which implies T_u is bounded in $H^1(\Omega)$ by corollary 3.2.4, and since the embedding of $H^1(\Omega)$ in to $\mathcal{L}^2(\Omega)$ is compact, T is compact.

Apply Schauder's fixed point theorem to conclude T has a fixed Point $u \in M$. By our choice of solution operator, this u lies in the sobolev space $H_0^1\,(\Omega)$.

Example 3.3.1. consider the problem

$$\begin{cases} -\Delta u + u = f & on\ \ I = (0,1) \\ u(0) = 0\,, & u'(1) = 0 \end{cases} \quad \text{-----------------------(3.3.3)}$$

If u is a classical solution of (3.3.3) we have

$\int_I \nabla \mathrm{u} \nabla \mathrm{v} + \int_I uv = \int_I fv$,$\forall \in H^1(I)$ with

$$V(0) = 0$$

The appropriate space to work in is

$$H = \{v \in H^1(I)\ (I);\ V(0) = 0\}$$

Equipped with the H^1 scalar product.

Chapter Four

Summary and Conclusion

4.1 Summary

In this thesis we introduce some basic methods to establish the existence of solutions to semilinear elliptic equations. Systems of nonlinear equations arises in different areas of applied sciences such as engineering, physics, medicines, chemistry and other sciences .

Solving systems of semilinear elliptic equation is much more difficult when compared with solving the linear part and finding solutions of these problems is one of the most important parts of the studies in functional analysis. Because of this reason a number of Mathematicians developed different fixed point methods to solve problems of systems of semilinear elliptic equations . In this thesis we present several fixed point theorems and prove some fundamental results in partial differential equations by reducing semilinear problems to fixed point problems . Some important theorems for solving system of semilinear elliptic equations are Banach fixed point theorem , Brouwer fixed point theorem , Schauder fixed point theorem and Schaeffer fixed point theorem .

4.2 Conclusion

A fixed point of a function $\boldsymbol{f} : X \to X$ is a point $x_0 \in X$ such that $f(x_0) = x_0$.

In this thesis we have presented some of the most basic material needed for a study of solving semilinear partial differential equation. Some of this are: Metric space and normed spaces, inner product spaces,operators between normed spaces, Sobolev spaces and different fixed point theorems.

Spaces of weakly differentiable functions, so called Sobolev spaces, play an important role in analysis.

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