Addis Ababa University

Office of Graduate Program

Faculty of Computer and Mathematical Science

Department of Mathematics

Graduate Project Report

On

Dyck path Combinatorics

Submitted to the Department of Mathematics in Partial fulfillment of the requirements for Master's degree of Science in Mathematics

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Declaration

I declare that this project has been composed by me and that no part of the project has formed the basis for the award of any Degree, Diploma, Associate ship, Fellowship or any other similar title to me.

Abinet kefeni

Signature ____________________Date___________________________

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(i)Summary of the project

This project is all about combinatorial enumeration of Dyck paths and generalization which is intimately enumerated by Catalan numbers, Fine numbers, Motzkin numbers, bijection, and generating function.

The project also gives a unified presentation and history of the previous results in the literature that mention Dyck paths.

Among the settings that mention this project, section one looks about Historical background of Dyck path. In section two it is intended to count the number of Dyck paths according to several parameters, such as number of length, number of peaks, number of valleys, number of double rises, number of returns to the x-axis, the odd length of the downward path ending on the x-axis, and also number of Dyck paths that have no peaks at height two, number of Dyck paths whose first down step is followed by another down step, number of Dyck paths that have n-1 up steps, number of Dyck paths that have n-1 peaks and containing neither long slopes nor lonely peaks, number of Dyck paths with no hills, number of Dyck paths whose initial peak is at height k, whose first peak has even height and whose first peak has odd height etc.

And finally the last section, presents the Dyck path statistics using both Ordinary and Bivariate generating functions.

(ii) Acknowledgements

In preparing this project, I would like to express my appreciation to my advisors **Doctor Seyoum Getu** and **Professor Melkamu zeleke** for their patience in repeatedly reading the draft manuscript of this project and for making constructive comments and suggestions from which I have immeasurably benefited.

Second, I would like to extend grateful acknowledgement to my wife **Azeb.W** in particular for their blessed support and much appreciated willingness to shoulder social responsibility on my behalf.

Last but not least, I am also indebted for many friends and to all those who helped me one way or another for the accomplishment of my study.

(iii)Definition of key terms

- **Catalan numbers** are often defined by $C_n = \frac{(2n)!}{(n+1)n!} = \frac{1}{n+1}\binom{2n}{n}, n \geq 0$.The sequence of Catalan number begins with 1, 1, 2, 5, 14, 42, … .The generating function is $C(x) = \frac{1-\sqrt{1-4x}}{2x}$.
- **Lattice point** is a point on the Cartesian plane with integral coordinates.
- **Dyck paths(mountain ranges)** are lattice paths starting and ending on the horizontal axis using steps $(1,1)$ *and* $(1,-1)$, and never going below the x-axis.
- **Height** on Dyck path is a vertical displacement from the x-axis.
- **Slope** is any $(1,1)$ *step or* $(1,-1)$ *step on* a Dyck paths.
- **Long slope** is a slope of length more than two $(1,1)$ steps or more than two $(1,-1)$ steps on Dyck paths.
- **Peak** at height k on a Dyck path is a lattice point on the path with coordinate y=k that is preceded by a (1, 1) step and immediately followed by a (1,-1) step.
- **lonely peak** is a peak if both the two slopes joined at a peak have length more than one step
- **Hill** in a Dyck path is a pair of consecutive steps (1, 1) and (1,-1), giving a peak of height 1.
- **Valley** at height k on Dyck path is a point with coordinate y=k that is preceded by down-step and immediately followed by up-step.
- **Fine numbers** begin with the sequence 1, 0, 1, 2, 6, 18, 57, … and the generating function is $F(z) = \frac{1}{z}\frac{1-\sqrt{1-4z}}{3-\sqrt{1-4z}}$.
- **Motzkin numbers** begin with the sequence $\{M_n\}_{n=0}^{\infty} = 1, 1, 2, 4, 9, 21, 51,\ldots$ where $M_n = \frac{1}{n+1}\sum_{i=0}^{n}\binom{n+1}{i}\binom{n+1-i}{i+1} = \sum_{i=0}^{n}\binom{n}{2i}C_i$ and the generating function can be given as $M(x) = \frac{1-x-\sqrt{1-2x-3x^2}}{2x^2}$, $x \neq 0$ is the closed form ([5],p.238), (appendix (6))
- **Standard Young Tableau** (SYT) is a Ferrier's shape on n boxes in which each box contains one of the elements of $[n] = \{1, 2, \ldots, n\}$ so that all boxes contain different numbers and the rows and columns increase going down and going to the right.
- **Semi-length** of Dyck path is total number of up steps or total number of down Steps.

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Section One

1.1.Historical background of Dyck paths.

Dyck path of length 2n is a lattice path from (0, 0) to (0, 2n) in the plane integer consisting of steps $(1,1)$ *and* $(1,-1)$ and that never passes below the x-axis. **Walther Franz Anton von Dyck** (1856–1934) was born in Munich, Germany. The *Dyck language* in the theory of formal languages is named after him. He studied mathematics at the Munich Polytechnikum under Alexander Brill and Felix Klien. In 1879, he received his doctorate and became an assistant to Klien. He joined the faculty at the University of Leipzig in 1882. Two years later, he was appointed professor at the Munich Polytechnikum, where he served as director, rector, and second chairman. Dyck made significant contributions to group theory, function theory, topology, and algebraic characteristic theory. He is credited with the modern definition of a mathematical group. Dyck was also the editor of Kepler's works [13].

1.2. Introduction.

Dyck paths are a lattice paths starting and ending on the horizontal axis using steps $(1,1) and\ (1,-1)$, and which never goes below the x-axis. A peak at height k on a Dyck path is a point on the path with coordinate $y = k$ that is preceded by a $(1,1)$ step and immediately followed by a $(1,-1)$ step. Valley at height k on a Dyck path is a point on the path with coordinate $y = k$ that is preceded by a $(1,-1)$ step and immediately followed by a $(1,1)$ step [12].

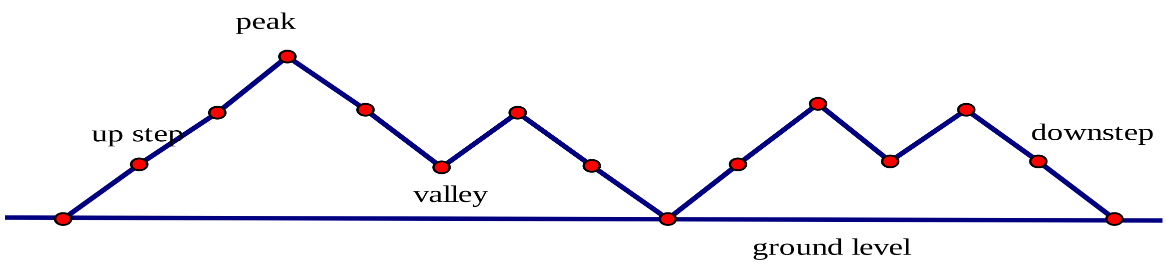


Fig.1.1: A Dyck 7-path with 2 components, 2DUDs, and height 3

The size (or semi length) of a Dyck path is its number of up steps and a Dyck path of size n is a Dyck n-path. The empty Dyck path (of size 0) is denoted by ε. A return down step is one that returns the path to ground level. A primitive Dyck path is one with exactly one return (necessarily at the end). (Note that the empty Dyck path ε is not primitive.) Its returns split a nonempty Dyck path into one or more primitive Dyck paths, called its components. We use D_n to denote the number of the set of Dyck paths of size n [1].

Dyck paths that are mostly studied in this project are lattice paths starting and ending on the horizontal axis using steps $(1,1) and\ (1,-1)$, and never goes below the x-axis. Such type of Dyck paths starting at (0, 0) and ending at (2n, 0) counted by the most important numbers, (that counts more than 66 objects) [11], [13] is called Catalan numbers $, C_n = \frac{1}{n+1}\binom{2n}{n}$,

$i.e.\ 1, 1, 2, 5, 14, 42 \ldots$ whose generating function is $D(x) = \frac{1-\sqrt{1-4x}}{2x}$ (see Appendix (3))

In this project paper Dyck paths having different character that are counted by Fine numbers and Motzkin numbers are also presented using generating function and Bijection method. In the second section we introduce recurrence relation and generating function of Dyck path. At the end, in section three we will discuss Dyck paths statistics using ordinary generating functions and Bivariate generating functions.

Section two

2.1. Recurrence relation and generating function of Dyck path.

There is a Bijection between the number of Dyck paths and the number of lattice paths such that in the $n \times n$ square consisting of only north and east steps and which do not pass below the line $y = x$ (or the main diagonal) in the grid. It starts at $(0,0)$ and ends at (n, n). By necessity, the first step must be north and the final step must be east. By rotating 45^0 each lattice paths in to clockwise direction, or by changing each north steps in the lattice path into north-east steps and each south steps in to south-east we obtain the corresponding Dyck paths [11], [13]. The lattice paths for $1 \leq n \leq 4$ are given below.

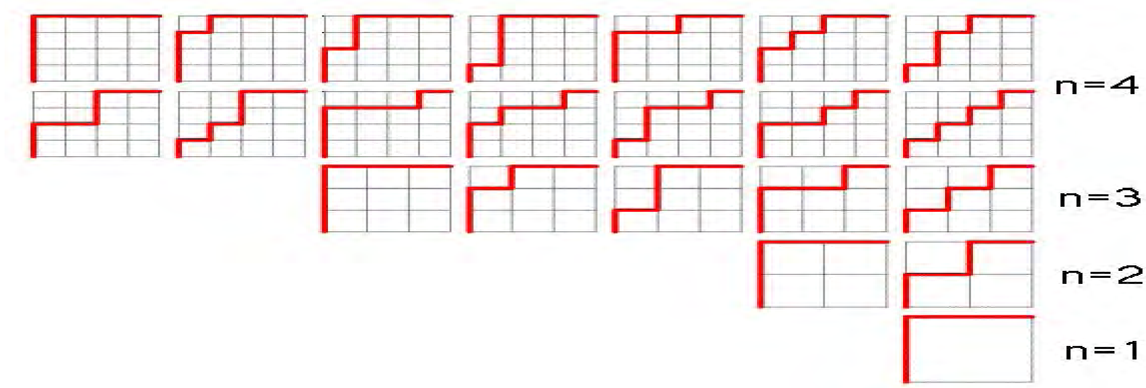


Fig.2.1: Lattice path for $1 \leq n \leq 4$

Let us enumerate the lattice paths that are Dyck paths.

(i)There are a total of $\binom{2n}{n}$ paths from (0,0) to (n, n) with north and east steps in nxn grid. We choose n steps in the north direction and the remainder must travel east.

(ii)Now consider paths which cross below the line $y = x$ and let us call it the first step that crosses below the diagonal is 'bad' step. Take the position of the path after the first 'bad' step and interchange the north and east steps which is equivalent to reflecting this portion of the path in the line $y = x - 1$ as in the figure below.

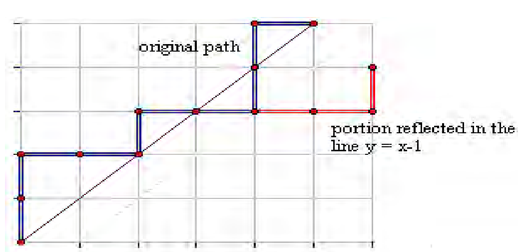


Fig.2.2: Reflecting a lattice path

Now we have a situation where, for the $2n$ steps, $n+1$ are in one direction and $n-1$ are in the opposite direction. Infact, this reflection maps the point (x, y) to the point $(x', y') = (y+1, x-1)$ as shown geometrically below.

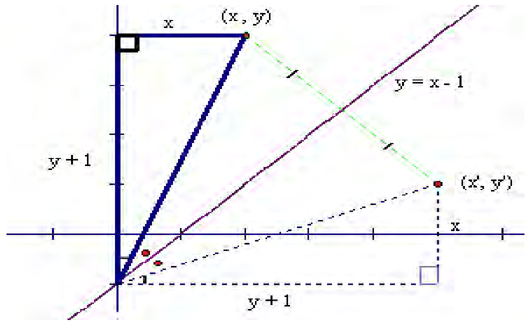


Fig.2.3: Reflecting lattice path on the line $y = x - 1$

The overall effect then is to take $(n, n) \longrightarrow (n+1, n-1)$ or to create $(n+1)X(n-1)$ grid. On this grid there are $\binom{2n}{n-1}$ ways of choosing a path. Since each path that crosses the diagonal can be uniquely transformed like this there is a one-to-one correspondence.

So the total number of such paths is given by $\binom{2n}{n-1}$. Thus the total number of lattice paths, i.e. those which lie entirely above or touch but don't cross the line $y = x$, will be equal to the total number of paths (= $\binom{2n}{n}$) minus the number of paths which cross below the line $y = x$, (=$\binom{2n}{n-1}$)

i.e. $\binom{2n}{n} - \binom{2n}{n-1} = \frac{(2n)!}{n!n!} - \frac{(2n)!}{(n-1)!(n+1)!} = \frac{(2n)!}{n!(n-1)!}\left(\frac{1}{n} - \frac{1}{n+1}\right)$

$$= \frac{(2n)!}{n!(n-1)!}\left(\frac{n+1-n}{n(n+1)}\right) = \frac{1}{n+1}\frac{(2n)!}{n!n!} = \frac{1}{n+1}\binom{2n}{n}.$$

$$= \frac{1}{n+1}\binom{2n}{n}$$

So that the total number of lattice paths, i.e. those which lie entirely above or touch but don't cross the line $y = x$, will be equal to the Catalan number. So the total number of Dyck paths will be equal to the Catalan number.

$$D_n = \frac{1}{n+1}\binom{2n}{n}$$

In the following section we observe a detailed explanation by forming recurrence relations.

2.1.1. Recurrence relation of Dyck paths.

Commonly known that sequence of Catalan numbers 1, 1, 2, 5, 14, 42 . . . are defined by $C_n = \frac{1}{n+1}\binom{2n}{n}$ [11], [13], [see Appendix (3)], we have established that this sequence counts the number of Dyck paths D_n.

Now we want to find the recurrence relation for the number of Dyck paths D_n by providing a combinatorial explanation.

Table.2.1: lists of the first five numbers of Dyck paths.

NOTE: This corresponds to the Catalan numbers for $0 \leq n \leq 5$

n		D_n
0	D_0(by definition)	1
1	D_0D_0=(1)(1)=1	1
2	$D_0D_1+D_1D_0$ =(1)(1)+ (1)(1)=2	2
3	$D_0D_2+D_1D_1+D_2D_0$=(1)(2)+ (1)(1)+ (2)(1)=5	5
4	$D_0D_3+D_1D_2+D_2D_1+D_3D_0$=(1)(5)+ (1)(2)+ (2)(1)+(5)(1)=14	14
5	$D_0D_4+D_1D_3+D_2D_2+D_3D_1+D_4D_0$=(1)(14)+ (1)(5)+ (2)(2)+(5)(1)+ (14)(1)=42	42

Now let us see the combinatorial explanation of the following figure as it is the core point of recurrence relation D_n. In this figure, it is clear that there are n steps NE and n step SE. We will call this a path of length 2n. For example, D_5 is composed of paths of length 10 which have 5 steps NE and 5 steps SE. If the path starts at (0,0), it must come back down to the x-axis at some point. We will consider k to represent the SE step in which the path first reaches the x-axis.

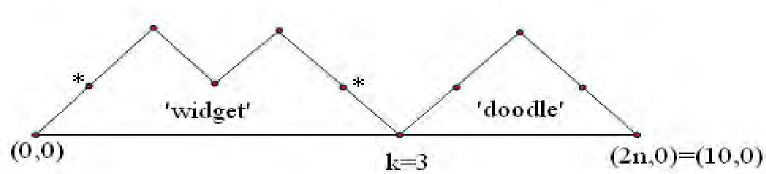


Fig. 2.4: Dyck path of length 10

NOTE: we will use the term 'widget' to denote the Dyck path from (0, 0) to k, and the term 'doodle' to indicate the remainder of the Dyck path beyond k.

In the above figure, it first reaches the x-axis on the third SE step, so k=3. That means that there is a path of length 4 (=2 SE steps) between the asterisks in the above figure. We know that the first step on the path must be in NE direction and the last segment must be SE since there is only one way to do these, so in the widget we need only consider the path between the asterisks. For any Dyck path, the length of the portion between the asterisks will be k-1. Beyond k, there are n-k NE and n-k SE steps, for a path of length n-k in the doodle. By the multiplication principle, the number of paths for each k will be equal to $D_{k-1}D_{n-k}$.It is equal to the number of pairs$(x, y)$where x is a widget of size k-1 and y is a doodle of size n-k such that the size of widget plus the size of the doodle is equal to n-1.These correspond to the combinations in the centre column of the above table. For example in the row where n=5 in the above table, the pairs (0,4), (1,3), (2,2)(3,1), (4,0) all sum to four. However, the path can first reach the axis either when k=1 or when k=2, or …, or when k=n since the k^{th} SE step can occur anywhere from the first to the n^{th} SE step. Since these cases are disjoint, we apply the additional principle to combine them. So we consider (set of paths for k=1) + (set of paths for k=2) +… + (set of paths for k=n). These will partition the total number of paths.

The above discussion establishes the following proposition.

Proposition 2.1: If we consider D_0=1 to be the empty path (since there is one way to do this), then the total number of Dyck paths, from (0,0) to $(2n, 0)$ is given by:

$$D_n = \sum_{k=1}^{n} D_{k-1}\, D_{n-k} \qquad were\ D_0 = 1$$

2.1.2. Generating functions of Dyck paths.

Now we use the result of the previous section to find algebraically $D(x)$ is the generating function of the infinite sequence$\{D_n\}_{n=0}^{\infty} = 1, 1, 2, 5, 14, 42, \ldots$.

Now to find the explicit form of $D(x)$ from previous section

$$\mathrm{D_n} = \sum_{\mathrm{k}=1}^{\mathrm{n}} \mathrm{D_{k-1}}\, \mathrm{D_{n-k}} \qquad \mathrm{were\ D_0} = 1$$

Let

$$D(x) = \sum_{n=0}^{\infty} D_n\, x^n$$

Generating function for D_n.

For example $D(x) = 1 + x + 2x^2 + 5x^3 + 14x^5 + \cdots$ is the generating function of the above infinite sequence.

$$D(x) = D_0x^0 + D_1x^1 + D_2x^2 + D_3x^3 + \cdots$$

$$\begin{aligned}(D(x))^2 &= (D_0x^0 + D_1x^1 + D_2x^2 + D_3x^3 + \cdots)(D_0x^0 + D_1x^1 + D_2x^2 + D_3x^3 + \cdots) \\ &= D_0D_0x^0 + (D_0D_1 + D_1D_0)x^1 + (D_0D_2 + D_1D_1 + D_2D_0)x^2 + \cdots \\ &\quad + (D_0D_n + D_1D_{n-1} + \cdots + D_nD_0)x^n + \cdots \\ &= D_1x^0 + D_2x^1 + D_3x^2 + D_4x^3 + \cdots \\ &= \frac{D(x)-D_0}{x}\end{aligned}$$

$$x(D(x))^2 = D(x) - 1$$

$$x(D(x))^2 - D(x) - 1 = 0$$

Solving this using the quadratic formula, we obtain

$$D(x) = \frac{1 \pm \sqrt{1-4x}}{2x}$$

Noting that using Mathematica syntax expansion

$$D(x) = \frac{1 + \sqrt{1-4x}}{2x}$$

Has negative term, and given that our sequence D_n has only positive terms, we reject this root.

Therefore, the generating function can be written in the form:

$$D(x) = \frac{1-\sqrt{1-4x}}{2x} = \sum_{n=0} D_n\, x^n$$

Or

x→0

$D(x) = \frac{1+\sqrt{1-4x}}{2x} \to \frac{1}{0}$

But as x→0

$D(x) = \frac{1-\sqrt{1-4x}}{2x} \to \frac{0}{0}$ indetrmenet form

(An alternative way)

Let $D(x) = \sum_{n=0}^{\infty} D_n\, x^n$

be a generating function for the infinite sequence D_n. Using the recurrence relation it is evident that

$$(D(x))^2 = \sum_{n=0}^{\infty}(\sum_{k=0}^{n} D_{n-k}\, D_K)x^n = \sum_{n=0}^{\infty} D_{n+1}x^n \,.$$

Now we shall show that $(D(x))^2 = \sum_{n=0}^{\infty} D_{n+1}x^n$.

Let $F(x) = \frac{d}{dx}[xD(x)] = D(x) + xD'(x) = 1 + 2x + 6x^2 + 20x^3 + \cdots$

Then $F(x) = \sum_{n=0}^{\infty}\binom{2n}{n}x^n = \frac{1}{\sqrt{1-4x}}$

Therefore

$$xD(x) = \int_0^x F(t)dt = \int_0^x \frac{dt}{\sqrt{1-4t}} = \frac{1}{2}\left(1-\sqrt{1-4x}\right)$$

Hence

$$(D(x))^2 = \frac{1-2\sqrt{1-4x}+(1-4x)}{4x^2}$$

$$= \frac{1-\sqrt{1-4x}}{2x^2} - \frac{1}{x} = \frac{D(x)-1}{x}$$

$$= \frac{1}{x}\sum_{n=0}^{\infty} D_n x^n = \sum_{n=0}^{\infty} D_{n+1}x^n$$

$$\therefore (D(x))^2 = \sum_{n=0}^{\infty} D_{n+1}x^n.$$

Now, if we multiply both sides of this equation by x and then rearrange the terms, we develop a quadratic equation in D(x).

$$x(D(x))^2 = x\sum_{n=0}^{\infty} D_{n+1}x^n = \sum_{n=0}^{\infty} D_{n+1}x^{n+1} = \sum_{n=1}^{\infty} D_n x^n = \sum_{n=0}^{\infty} D_n x^n - 1$$

$$\Rightarrow x(D(x))^2 - D(x) + 1 = 0$$

Solving this using the quadratic formula, we obtain

$$D(x) = \frac{1 \pm \sqrt{1-4x}}{2x}$$

Noting that using Mathematica syntax expansion

$$D(x) = \frac{1+\sqrt{1-4x}}{2x}$$

has negative term, and given that our sequence D_n has only positive terms, we reject this root.

Therefore, the generating function can be written in the form:

$$D(x) = \frac{1-\sqrt{1-4x}}{2x} = \sum_{n=0} D_n\, x^n$$

Or

x→0

$D(x) = \frac{1+\sqrt{1-4x}}{2x} \rightarrow \frac{1}{0}$

But as x→0

$D(x) = \frac{1-\sqrt{1-4x}}{2x} \rightarrow \frac{0}{0}$ Indetrmenet form

The above results lead to the following theorem:

Theorem 2.1: The generating function for number of Dyck paths (D_n) length 2n is

$$D(x) = \frac{1-\sqrt{1-4x}}{2x} = \sum_{n=0}^{\infty} D_n\, x^n$$

2.2. Enumerating the number of Dyck path according to different parameters.

In this part we count the number of Dyck paths according to several parameters, such as length, number of peaks, number of double rises, number of returns to the x-axis etc.

Theorem:2.2;

(a) The number of Dyck paths that move from the origin to the lattice point $(2n + 2, 0)$ on the Cartesian plane such that:

• It travel from the lattice point $(0, 0)$ to $(2n + 2, 0)$ *with steps* $(1, 1)$ or $(1, -1)$ and never crossing the x-axis.

• The length of the downward path are odd are counted by Catalan numbers

$$D_n = \frac{1}{n+1}\binom{2n}{n} \; for\; n \geq 0.$$

(b) The number of Dyck paths from the origin to the lattice point *(*$2n$ +2,0*)* on the Cartesian plane such that:

• It travel from the lattice point *(0, 0)* to *(*$2n$ + 2, 0*) with steps (*1, 1*)* or *(*1, − 1*)* and never crossing the *x*-axis.

• Whose first down-step followed by another down-step are counted by Catalan numbers

$$D_n = \frac{1}{n+1}\binom{2n}{n} \; for\; n \geq 0.$$

(c) Enumerating the number of incomplete Dyck paths that:

• It travel from the lattice point *(0, 0) with steps (*1, 1*)* or *(*1, − 1*)* and never crossing the *x*-axis.

• That has n-1 up steps are counted by Catalan numbers

$$D_n = \frac{1}{n+1}\binom{2n}{n} \; for\; n \geq 0.$$

(d) The number of Dyck paths that:

• It travel from the lattice point *(0, 0) with steps (*1, 1*)* or *(*1, − 1*)* and never crossing the *x*-axis.

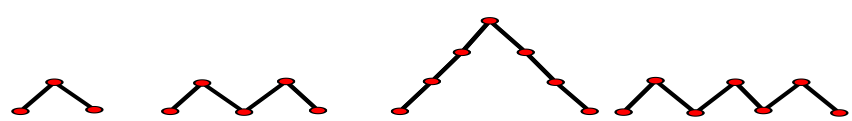
• Do not contain three consecutive up steps or three consecutive down steps.

• That has n-1 peaks are counted by Catalan numbers

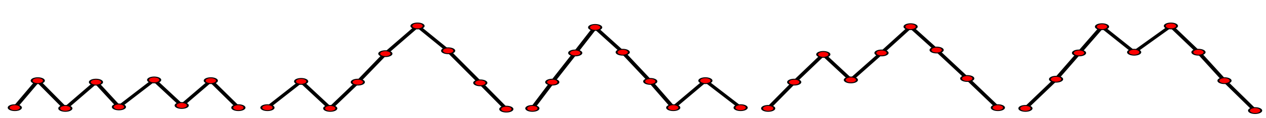
$$D_n = \frac{1}{n+1}\binom{2n}{n} \; for\; n \geq 0.$$

Illustration of [(a). (b). (c). (d)]

(a) The various Dyck paths for $0 \leq n \leq 3$ are



D_0=1 D_1=1 D_2=2

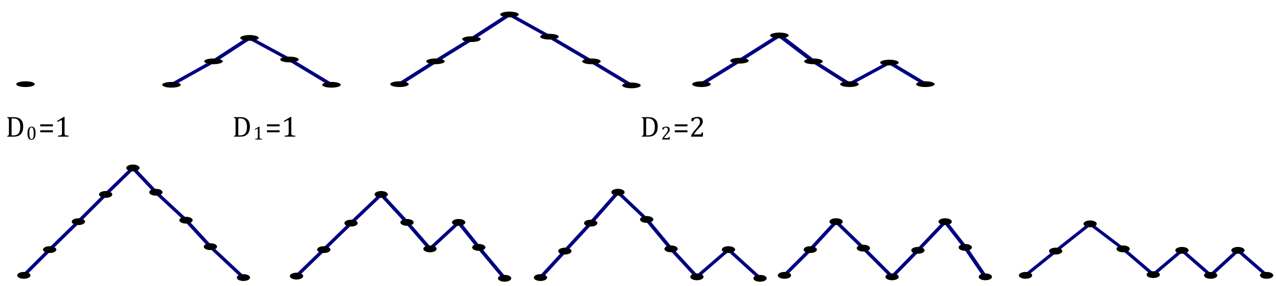


D_3=5

Fig. 2.5: Dyck path having a length of 2n+2 and length of the Downward path are odd.

In each case D_0=1, D_1=1, D_2=2, D_3=5, D_4=14, D_5=42 …

(b) The various Dyck paths for $0 \leq n \leq 3$.



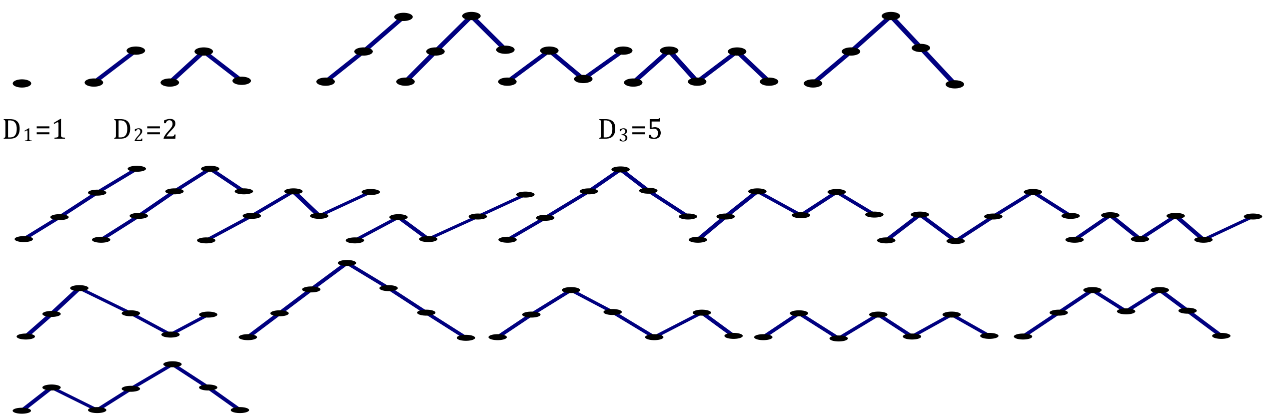
$D_0=1$ $D_1=1$ $D_2=2$

$D_3=5$

Fig. 2.6: Dyck path having a length of 2n+2 and whose first Down step followed by another down Step.

In each case $D_0=1$, $D_1=1$, $D_2=2$, $D_3=5$, $D_4=14$, $D_5=42$ …

(c) The various incomplete Dyck paths for $1 \leq n \leq 4$ are.



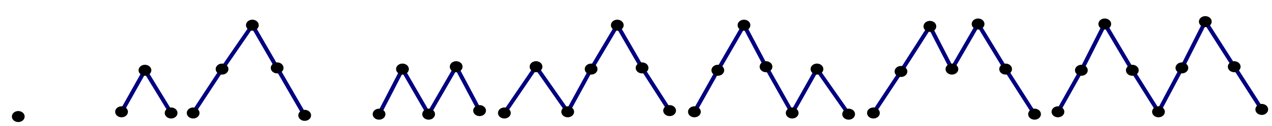
$D_1=1$ $D_2=2$ $D_3=5$

$D_4=14$

Fig.2.7: Incomplete Dyck path having n-1 up step

In each case $D_1=1$, $D_2=2$, $D_3=5$, $D_4=14$, $D_5=42$ …

(d) The various paths for $1 \leq n \leq 3$ are.



$D_1=1$ $D_2=2$ $D_3=5$

Fig. 2.8: Dyck path having n-1 peaks and do not contain three consecutive up steps or three consecutive down steps.

Proof: [(a), (b), (c), (d)]

In each case $D_1=1, D_2=2, D_3=5, D_4=14, D_5=42$ ...

We observe that

$$\frac{D_2}{D_1}=\frac{2}{1}=\frac{6}{3}=\frac{4.2-2}{2+1}$$

$$\frac{D_3}{D_2}=\frac{5}{2}=\frac{10}{4}=\frac{4.3-2}{3+1}$$

$$\frac{D_4}{D_3}=\frac{14}{5}=\frac{14}{5}=\frac{4.4-2}{4+1}$$

$$\frac{D_5}{D_4}=\frac{42}{14}=\frac{18}{6}=\frac{4.5-2}{5+1}$$

.
.
.

Noticing a clear pattern that

$$D_n=\frac{4n-2}{n+1}D_{n-1}\,, n\geq 1$$

When n=1, this yield$D_1=D_0$ but $D_1=1$ so we can define $D_0=1$.

An explicit formula for D_n are as follows

$$D_n=\frac{4n-2}{n+1}D_{n-1}=\frac{1}{n+1}(4n-2)D_{n-1}=\frac{1}{n+1}.\frac{2(2n-1)}{n}\binom{2n-2}{n-1}$$

$$=\frac{1}{n+1}\frac{(2n)(2n-1)(2n-2)!}{n(n-1)!\,n(n-1)}=\frac{1}{n+1}\frac{(2n)!}{n!\,n!}$$

$$=\frac{(2n)!}{(n+1)!n!}=\frac{1}{n+1}\binom{2n}{n}$$

So that for general case the number of the n[th] Dyck path of [(a), (b), (c), (d)] are obtain by

$$D_n=\frac{1}{n+1}\binom{2n}{n}\ for\ n\geq 0.$$

Theorem: 2.3;

(a)The number of Dyck paths that move from the origin to the lattice point $(2n, 0)$ on the Cartesian plane such that:

• With steps $(1, 1)$ or $(1, -1)$ and never crossing the x-axis.

• Containing neither long slopes nor lonely peaks.

• Having n-peaks are counted by Motzkin numbers $M_n=\frac{1}{n+1}\sum_{i=0}^{n}\binom{n+1}{i}\binom{n+1-i}{i+1}$

A peak of a Dyck path is said to be a lonely peak if both the two slopes joined at a peak have length more than one step. Here a peak is a point at which an Up-step is immediately followed by the Down-step. A long slope is a slope of length more than 2-steps.

Proof:

The various Dyck paths for 0≤ n ≤ 3 are.

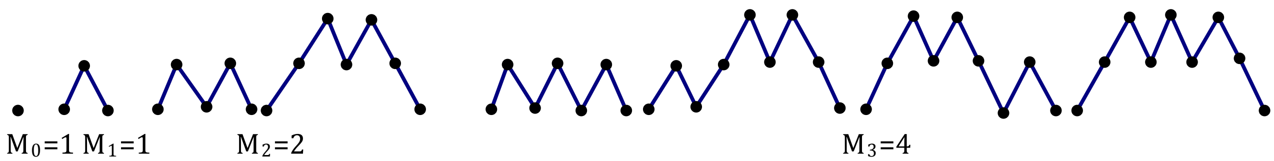


Fig.2.9: Dyck path having neither long slopes nor lonely peaks and having, n-peaks.

In each case M_0=1, M_1=1, M_2=2, M_3=4, M_4=9, M_5=21 …

Let, M_n = # of Dyck paths with n peaks those containing neither long slopes nor lonely peaks and using the steps (1, 1), (1,-1) and never falling below the x- axis. The generating function is from the sequence $\{M_n\}_{n=0}^{\infty} = 1, 1, 2, 4, 9, 21, 51\ldots$ can be given as

$M(x) = \sum_{n=0}^{\infty} \mathrm{M_n}\, x^n$.

$$M(x) = 1 + x + 2x^2 + 4x^3 + 9x^4 + \cdots$$

Then it has the functional equation,

$$\Longrightarrow x^2M^2(x) + xM(x) - M(x) + 1 = 0$$
$$\Longrightarrow x^2M^2(x) + (x-1)M(x) + 1 = 0$$

Using the quadratic formula,

$$M(x) = \frac{1 - \mathrm{x} - \sqrt{1 - 2x - 3x^2}}{2x^2}\ , \mathrm{x} \neq 0$$

is the closed form.

Using Mathematica syntax for the expansion of $\sqrt{1 - 2x - 3x^2}$ or using binomial expansion, $\frac{1-\mathrm{x}-\sqrt{1-2x-3x^2}}{2x^2}$ is exactly equal to the Generating function of Motzkin numbers, but

$\frac{1-\mathrm{x}+\sqrt{1-2x-3x^2}}{2x^2}$ is not. As we have seen in the generating function of Dyck paths with n peaks those containing neither long slopes nor lonely peaks have a functional equation

$M(x) = 1 + xM(x) + x^2\, M^2(x)$.

Let $z = xM(x)$

$\Longrightarrow M(x) = 1 + z + z^2 \qquad\qquad \Longrightarrow z = x(1 + z + z^2)$

By Combinatorial interpretation, using Lagrange's Inversion Formula (LIF),

Let $f(z) \ = \ 1 + z + z^2$

Then $M_n \ = [z^{n+1}]f(z)^{n+1}$

$= \frac{1}{n+1}[z^n](1 + z + z^2)^{n+1} \Longrightarrow \mathrm{M_n} = \frac{1}{n+1} \sum_{i=0}^{n} \binom{n+1}{i} \binom{n+1-i}{i+1}$

So that we conclude that number of Dyck paths with n peaks those containing neither long slopes nor lonely peaks and using the steps (1, 1), (1,-1) and never falling below the x- axis are enumerated by Motzkin numbers[see Appendix (6)].

Proposition 2.2

(a)The generating function for Dyck paths of length 2n that moves from the origin to the lattice point *(*2n , 0*)* on the Cartesian plane such that:

•with steps *(*1, 1*)* or *(*1, − 1*)*, and never crossing the *x*-axis.

• Whose initial peak is at height k is $z^k C^k$.

Proof: Every Dyck path whose initial peak is at height k has a unique factorization of the form k up steps and k Dyck paths each between the consecutive down steps. And let each up- step be denoted by z.

Fig.2.10: Dyck path whose initial peak is at height k having unique factorization of the form k up steps and k Dyck paths each between the consecutive down steps

Thus by the multiplication principle the appropriate generating function is $z^k C^k$.

Proposition 2.3:

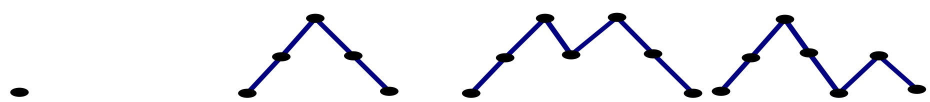
(a)The generating function of Dyck paths of length 2n that moves from the origin to the lattice point $(2n, 0)$ on the Cartesian plane such that:

•using steps $(1, 1)$ or $(1, -1)$, and never crossing the x-axis.

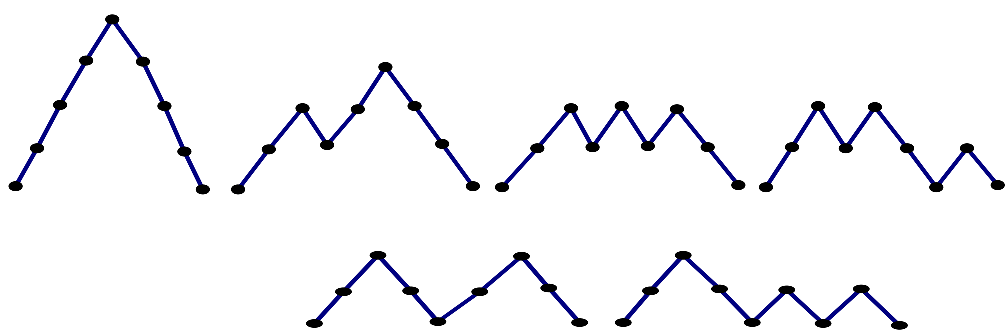
• Whose first peak has even height is $F(\mathrm{z}) = \frac{1}{1-z^2C^2}$

Proof:

The various Dyck paths for $0 \le n \le 4$ are



$F_0 = 1$ $F_1 = 0$ $F_2 = 1$ $F_3 = 2$



$F_4 = 6$

Fig. 2.11: Dyck paths whose first peak has even height.

Let, $F_n = \#$ of Dyck paths whose first peak has even height gives a sequence

$\{F_n\}_{n=0}^{\infty} = 1, 0, 1, 2, 6, 18, 57, \ldots$ We can find its generating function as follows:

Let $F(z) = \sum_{n=0}^{\infty} F_n\, z^n$ be the generating function for $\{F_n\}_{n\geq 0}$.

From Proposition 2.2 the generating functions for Dyck paths of length 2n whose initial peak is at height k is $z^k C^k$.

If $$k = 0 \rightarrow z^0C^0 = (zc)^0 = 1$$

$$k = 2 \ \rightarrow \ z^2C^2 = (zc)^2$$
$$k = 4 \ \rightarrow \ z^4C^4 = (zc)^4$$
$$k = 6 \ \rightarrow \ z^6C^6 = (zc)^6$$
$$\vdots$$

They are disjoint so they sum up to

$$\Leftrightarrow F(z) = (zc)^o + (zc)^2 + (zc)^4 + (zc)^6 + \cdots$$
$$= 1 + z^2C^2 + z^4C^4 + \cdots$$
$$= 1 + (z^2C^2)^1 + (z^2C^2)^2 + (z^2C^2)^3 + \cdots$$
$$= \frac{1}{1-z^2C^2}$$

This is the generating function of Fine numbers. [see Appendix (4)].Therefore the number of Dyck paths whose first peak has even height are counted by Fine numbers.

<u>Proposition.2.4</u>:(a)

The generating function of Dyck paths of length 2n that moves from the origin to the lattice point $(2n\ ,0)$ on the Cartesian plane such that:

- using steps $(1, 1)$ or $(1, -1)$, and never crossing the x-axis.
- Whose first peak has odd heights is zCF.

<u>Proof</u>:

From preposition 2.2, the generating functions for Dyck paths (mountain ranges) of length 2n whose initial peak is at height $k\ is\ z^kC^k$.

If
$$k = 1 \ \rightarrow \ z^1C^1 = (zc)^1 = zc$$
$$k = 3 \ \rightarrow \ z^3C^3 = (zc)^3$$
$$k = 5 \ \rightarrow \ z^5C^5 = (zc)^5$$
$$k = 7 \ \rightarrow \ z^7C^7 = (zc)^7$$
$$\vdots$$

They are disjoint so they sum up to

$$\Leftrightarrow D(z) = (zc)^1 + (zc)^3 + (zc)^5 + (zc)^7 + \cdots$$

$$= z^1C^1 + z^3C^3 + z^5C^5 + z^7C^7 + \cdots$$

$$= zC(1 + z^2C^2 + z^4C^4 + z^6C^6 + \cdots)$$

$$= zC(1 + (z^2C^2)^1 + (z^2C^2)^2 + (z^2C^2)^3 + \cdots)$$

$$= ZC\left(\frac{1}{1-z^2C^2}\right)$$

$$= zCF$$

<u>Theorems 2.4</u>:

(a)The number of Dyck paths of length 2n that move from the origin to the lattice point *(2n* , 0*)* on the Cartesian plane such that:

• using steps *(*1, 1*)* or *(*1, − 1*)*, and never crossing the *x*-axis.

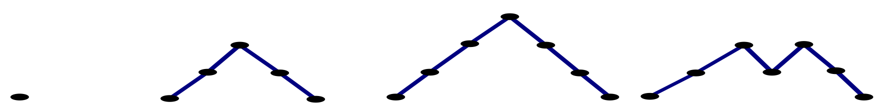
•With no hills. i.e no peaks at height one is the Fine number F_n for $n \geq 0$. [see Appendix (4)].

<u>Proof</u>: From section 2.1 the generating function of Dyck paths of length $2n$ is

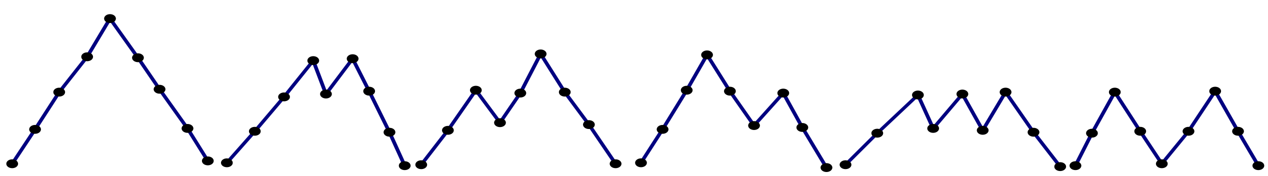
$$C = \frac{1-\sqrt{1-4z}}{2z}$$

A hill in a Dyck path is a pair of consecutive steps giving a peak of at height 1.

The various possible paths for $0 \leq n \leq 4$ are.



F_0=1 F_1=0 F_2=1 F_3=2



F_4=6

Fig. 2.12:Dyck paths with no hills.

F_n = # Of Dyck paths with no hills using the steps (1, 1), (1,-1), and never falling below the x-axis. For the general case $\{F_n\}_{n=0}^{\infty}$=F_0, F_1, F_2, F_3, F_4, F_5, F_6...= *1, 0, 1, 2, 6, 18, 57*, ... so that this shows the fine numbers [see Appendix (4)] and enumerates the number of Dyck paths with no hills. Let $F(z) = \sum_{n=0}^{\infty} F_n z^n$ be the generating function for $\{F_n\}_{n\geq 0}$.We can find its generating function as follows:

Take any hill free Dyck path of length $2n$.

Case-1: The path is an empty path that is counted by 1(since there is one way to do this).

Case-2: Non-empty hill free Dyck path.

Here we have n up steps and n down steps, so we can mark each up step with a z. Since the path is non-empty, it must start with an up step. The section of the path between the first up step and the first down step returning to the horizontal axis must be a non trivial Dyck path, so that we do not have a hill.

Once the path returns to the horizontal axis, it must again avoid hills, so the generating function for this part of the path is F.

More formally, the path has a unique factorization of the form (empty, up step, non-trivial Dyck path, down step, hill free path).

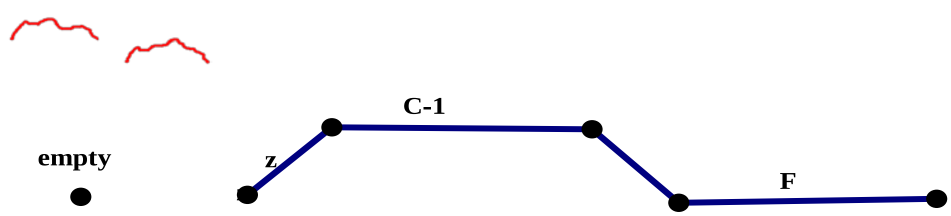


Fig. 2.13: unique factorization of Dyck path the form (up step, non-trivial Dyck path, down step, hill free path).

Then $F = 1 + z(C-1)F = 1 + z(zC^2)F$

$$\Leftrightarrow F(1 - z^2C^2) = 1$$

$$\Leftrightarrow F = \frac{1}{1-z^2C^2} = \frac{1}{(1-zC)(1+zC)} = \frac{C}{1+zC} \quad by\ identity\ C = 1 + zC^2 = \frac{1}{1-zC}$$

$$\Leftrightarrow F = \frac{\frac{1-\sqrt{1-4z}}{2z}}{1+z\left(\frac{1-\sqrt{1-4z}}{2z}\right)} = \frac{1}{z}\frac{1-\sqrt{1-4z}}{3-\sqrt{1-4z}}$$

Thus $F = \frac{1}{1-z^2C^2} = \frac{C}{1+zC} = \frac{1}{z}\frac{1-\sqrt{1-4z}}{3-\sqrt{1-4z}}$

Let $G_K(x)$ be the generating function for the number of Dyck paths of length 2n with no peaks at height k with k≥2. It is known that from section 2.2 theorems 2.4, $G_1(x)$ is the generating function of Dyck paths of length 2n with no hill. In this part, we want to derive the recurrence

$$G_k(x) = \frac{1}{1 - x\mathrm{G}_{\mathrm{K}-1}(x)}, \mathrm{K} \geq 2, were, G_1(x) = \frac{2}{1 + 2x + \sqrt{1-4x}}$$

Theorems 2.5: Let $G_m(x) = \sum_{\mathrm{m}\geq 0} \mathrm{g(m,n)}x^{\mathrm{n}}$ be the generating function for dyck paths of length 2n with no peaks at height m, m≥1. Then $G_m(x) = \frac{1}{1-x\mathrm{G}_{\mathrm{m}-1}(x)}$; $\mathrm{m} \geq 1$ is the recurrence relation for Dyck paths of length 2n with no peaks at height m with m≥1.

Proof: The set of all Dyck paths of length 2n, n≥0, with no peaks at height m consists of the trivial path (the origin) and paths with general form shown in the diagram.

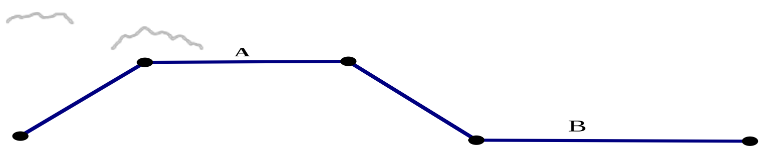


Fig 2.14: The set of all Dyck paths of length 2n, n≥0, with no peaks at height m consists of the trivial path (the origin) and paths with general form.

It starts with a north-east step followed by a segment labeled A which represents any Dyck path of length 2k, 0≤ k ≤ n-1, with no peaks at height m-1. A is followed by a south-east step followed by a segment labeled B which represents any Dyck path of length 2n-2-2k with no peaks at he*ight m. Therefore*

$$g(m,0) = 1, g(m,n) = \sum_{k=0}^{n-1} g(m-1,k)g(m,n-1-k) = [x^{n-1}]\{G_{m-1}(x)G_m(x)\}.$$

$$i.e.\, g(m,0) = 1, g(m,n) = [x^n]\{G_{m-1}(x)G_m(x)\}; n \geq 1,$$

Where $[x^n]$ denotes "coefficient of x^n in". That is,

$$G_m(x) = 1 + x\mathrm{G}_{\mathrm{m}-1}(x)\mathrm{G}_{\mathrm{m}}(x)$$

$$G_m(x) - x\mathrm{G}_{\mathrm{m}-1}(x)\mathrm{G}_{\mathrm{m}}(x) = 1$$

$$G_m(x)(1 - +x\mathrm{G}_{\mathrm{m}-1}(x)) = 1$$

$$G_m(x) = \frac{1}{1-x\mathrm{G}_{\mathrm{m}-1}(x)} ; \mathrm{m} \geq 2$$

Theorems 2.6: The generating function for Dyck paths of length 2n starting and ending on the x-axis with no peaks at height k, where k≥2 is given by

$$\cfrac{1}{1-\cfrac{x}{1-\cfrac{x}{1-\cfrac{x}{1-\cfrac{x}{\ddots\, 1-\cfrac{x}{1-\cfrac{x}{1-x^2c^2(x)}}}}}}}$$

The continued fraction contains exactly k levels.

Proof: From theorem 2.5 we have that

$$G_k(x) = \frac{1}{1 - x\mathrm{G}_{\mathrm{k}-1}(x)} \text{ ; k} \geq 2$$

Substitute at $G_{k-1}(x) = \frac{1}{1-x\mathrm{G}_{\mathrm{k}-2}(x)}$

$$= \frac{1}{1 - x\left(\frac{1}{1 - x\mathrm{G}_{\mathrm{k}-2}(x)}\right)} = \cfrac{1}{1 - \cfrac{x}{1 - x\mathrm{G}_{\mathrm{k}-2}(x)}}$$

Continue this process up to k-1 steps

$$= \cfrac{1}{1 - \cfrac{x}{1 - \cfrac{x}{x\mathrm{G}_{\mathrm{k}-3}(x)}}}$$

$$= \cfrac{1}{1-\cfrac{x}{1-\cfrac{x}{\ddots\, 1-\cfrac{x}{1-\cfrac{x}{1-x\mathrm{G}_1}}}}} \qquad \text{---------------------------}(*)$$

So from the above (Theorem 2.4) G_1 is the set of Dyck paths without hill enumerated by fine numbers and the generating function is $G_1(x) = \frac{1}{1-x^2C^2(x)}$

Substitute this into (*) We obtain that

$$= \cfrac{1}{1-\cfrac{x}{1-\cfrac{x}{\ddots\, 1-\cfrac{x}{1-\cfrac{x}{1-\cfrac{x}{1-x^2\mathrm{C}^2(x)}}}}}}$$

After this we can also prove theorem 2.4 by this formula:

Proof: with the notation of the above theorem 2.3, we will prove that

$$G_1(x) = \sum_{n=0}^{\infty} g(1,n)x^n = \frac{1}{1 - x^2C^2}$$

Obviously, g(1,0)=1 and g(1,1)=0.

For n≥2, a Dyck path of length 2n with no peaks at height 1 has the form of the (Fig 2.14) in the proof of Theorem 2.5 with A any Dyck path of length 2k, 1≤k≤n-1, and B a Dyck path of length 2n-2k-2 with no peaks at height 1.Therefore, for n≥2, we have

$$g(1,n) = \sum_{k=1}^{n-1} c_k\, g(1, n-k-1) = [x^{n-1}]\{c(x)G_1(x)\} - g(1, n-1)$$

$$= [x^n]\{xc(x)G_1(x)\} - g(1, n-1)$$

Therefore

$$G_1(x) = 1 + \sum_{n\geq 2} g(1,n)x^n = 1 + xc(x)G_1(x) - x - xG_1(x) + x$$

$$= 1 + xG_1(x)(c(x) - 1) = 1 + xG_1(x)xc^2(x)$$

That is

$$G_1(x) = \frac{1}{1 - x^2C^2}$$

Theorem 2.7: The number of Dyck paths of length 2n +2 with no peaks at height 2 is the Catalan numberC_{n-1}, for $n \geq 0$.

Proof: From theorem 2.5 $\quad G_2(x) = \frac{1}{1 - xG_1(x)}$

$$= \frac{1}{1 - x\frac{C(x)}{1 + xC(x)}}$$

$$=1+ xC(x)$$

$$= 1 + x(1 + x + 2x^2 + 5x^3 + \cdots)$$

$$= 1 + x + x^2 + 2x^3 + 5x^4 + \cdots$$

Therefore the coefficient of x^n in 1+ $x\mathrm{C}(x)$ is $C_{n-1} = \frac{1}{n}\binom{2n-2}{n-1}$,for $n \geq 0$.

A point on the Dyck path is called a valley at height k if it is a point with coordinate y=k that is preceded by down-step and immediately followed by up-step. In this part we find an explicit expression for the generating function for the number of Dyck paths starting at (0,0) and ending at (2n,0) with exactly r valley at height 0. This allows us to express this function via Chebyshev polynomial of the second kind and the generating function for the Catalan number.

From the above $C_n = \frac{1}{(1+n)}\binom{2n}{n}$ is the nth Catalan number and $c(x) = \frac{1-\sqrt{1-4x}}{2x}$ is the generating function of Catalan numbers.

Chebyshev polynomials of the second kind are defined by $u_r(cos\theta) = \frac{\sin(r+1)\theta}{\sin\theta}, for\ r \geq 0.$ Evidently, $u_r(x)$ is a polynomial of degree r in x with integer coefficients when $cos\theta = x$. Some of them are $u_0(x) = 1, u_1(x) = 2x,\ u_2(x) = 4x^2 - 1, u_3(x) = 8x^3 - 4x, \ldots$

This Chebyshev polynomials were invented for the needs of approximation theory, but are also widely used in various other branches of mathematics, including algebra, Combinatorics, Number theory, and lattice paths.

For $k \geq 0$ we define $R_k(x)$ by

$$R_K(x) = \frac{U_{K-1}(\frac{1}{2\sqrt{x}})}{\sqrt{x}U_K(\frac{1}{2\sqrt{x}})}$$

For example, $R_0(X) = 0, R_1(X) = 1, R_2(X) = \frac{1}{(1-x)}$. it is easy to see that for any k, $R_K(x)$is a rational function in x.

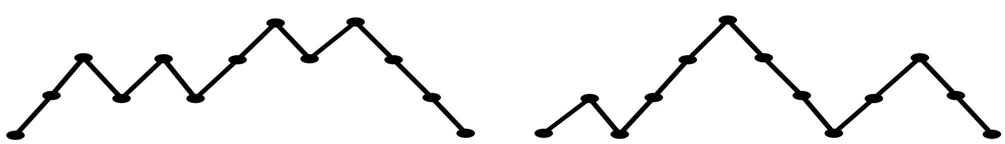


Fig 2.15: Two Dyck paths having length 12

For example, in the above Figure presents two Dyck paths; the path on the left has two valleys at height 1 and one valley at height 2, and the path on the right has only two valleys at height 0. The number of all Dyck paths starting at (0, 0) and ending at (2n, 0) with exactly r valleys at height k we denote by $valley_k^r(n)$. the corresponding generating function is denoted by $valley_k^r(x)$.

In this part we enumerate the valley of the Dyck path by finding explicit formulas for the generating functions $valley_0^r(x)$;$r \geq 0$.

Corollary 2.1: For all r≥0

$$valley_0^r(x) = \delta_{r,0} + x^{r+1}C^{r+1}(x)$$

Proof: From ([12], Theorem 2.4, P.7), were

$$valley_K^r(x) = \delta_{r,0}R_{k+1}(x) + \frac{x^r c^{r+1}(x)}{U_{k+1}^2\left(\frac{1}{2\sqrt{x}}\right)(1 - x(R_{k+1}(x) - 1)C(x))^{r+1}}$$

For k=0

$$valley_0^r(x) = \delta_{r,0}R_1(x) + \frac{x^r c^{r+1}(x)}{U_1^2\left(\frac{1}{2\sqrt{x}}\right)(1 - x(R_1(x) - 1)C(x))^{r+1}}$$

From $\quad R_1(x) = \frac{U_0(\frac{1}{2\sqrt{x}})}{\sqrt{x}U_1(\frac{1}{2\sqrt{x}})}$

$$= \delta_{r,0}\left(\frac{u_0\left(\frac{1}{2\sqrt{x}}\right)}{\sqrt{x}u_1\left(\frac{1}{2\sqrt{x}}\right)}\right) + \frac{x^r c^{r+1}(x)}{u_1^2\left(\frac{1}{2\sqrt{x}}\right)\left(1 - x\left(\left(\frac{u_0\left(\frac{1}{2\sqrt{x}}\right)}{\sqrt{x}u_1\left(\frac{1}{2\sqrt{x}}\right)}\right) - 1\right)c(x)\right)^{r+1}}$$

From Chebyshev polynomials of the second kind $u_r(cos\theta) = \frac{\sin(r+1)\theta}{\sin\theta}, for\ r \geq 0$, and $u_r(x)$ is a polynomial of degree r in x with integer coefficients when $cos\theta = x$. But here $cos\theta = \frac{1}{2\sqrt{x}}$

$u_0(cos\theta) = \frac{\sin(0+1)\theta}{\sin\theta} = \frac{\sin\theta}{\sin\theta} = 1$ ……………………………………………………………(*)

$u_1(cos\theta) = u_1\left(\frac{1}{2\sqrt{x}}\right) = \frac{\sin(1+1)\theta}{\sin\theta} = \frac{\sin 2\theta}{\sin\theta} = \frac{2\sin\theta\cos\theta}{\sin\theta} = 2\cos\theta = \frac{2}{2\sqrt{x}} = \frac{1}{\sqrt{x}}$ ……………………………(**)

From (*) and (**)

$$= \delta_{r,0}\left(\frac{1}{\sqrt{x}(\frac{1}{\sqrt{x}})}\right) + \frac{x^{r}c^{r+1}(x)}{\left(\frac{1}{\sqrt{x}}\right)^{2}\left(1 - x\left(\left(\frac{1}{\sqrt{x}\left(\frac{1}{\sqrt{x}}\right)}\right) - 1\right)c(x)\right)^{r+1}}$$

$$= \delta_{r,0} + \frac{x^{r}c^{r+1}(x)}{\frac{1}{x}\,(1 - x(1-1)c(x))^{r+1}}$$

$$= \delta_{r,0} + \frac{x^{r}c^{r+1}(x)}{x^{-1}}$$

$$= \delta_{r,0} + x^{r+1}c^{r+1}(x)$$

$\therefore \quad valley_0^r(x) = \delta_{r,0} + x^{r+1}C^{r+1}(x)$

From this we can to conclude that $[x^n]\{\delta_{r,0} + x^{r+1}c^{r+1}(x)\} = \frac{r+1}{n}\binom{2n-r-1}{n+1}$.In other words, the number of Dyck paths starting at (0, 0) and ending at (2n, 0) with exactly r valleys at height 0 is given by $\frac{r+1}{n}\binom{2n-r-1}{n+1}$

2.3 Some Combinatorics Bijections with Dyck paths

The idea of **Bijections,** or **Bijective** proofs, is used very often in counting arguments. When we want to enumerate elements of a set A, we can instead prove that a Bijection f: A→B exists with some set B, whose number of elements we know. Usually this is done by first defining a function f, then showing this f is indeed a Bijection from A into B. Once that is done, we can conclude that |A|= |B|. Sometimes we do not need the actual number of elements in the sets, just the fact that the two sets have the same number of elements. In that case, the method of Bijections can save us the actual counting. Hence in combinatory, Bijective method is a proof technique that finds a Bijective function $f: A \to B$ between two sets A and B, thus proving that they have the same number of elements, $|A| = |B|$. One place the technique is useful is where we wish to know the size of A, but can find no direct way of counting its elements. Then establishing a Bijection from A to some more easily countable B solves the problem. Another useful feature of the technique is that the nature of the Bijection itself often provides powerful insights into each or both of the sets is Bijective.

(A) Bijection between correctly parenthesized sequences and Dyck paths.

Interestingly, Dyck path and correctly parenthesized expressions are essentially the same number. In other words, there is a Bijection between the two. And counted by Catalan numbers. This Bijection can be exhibited by replacing an up step with a left parenthesis and a down step with a right parenthesis as in the following figure. Clearly this procedure is reversible.

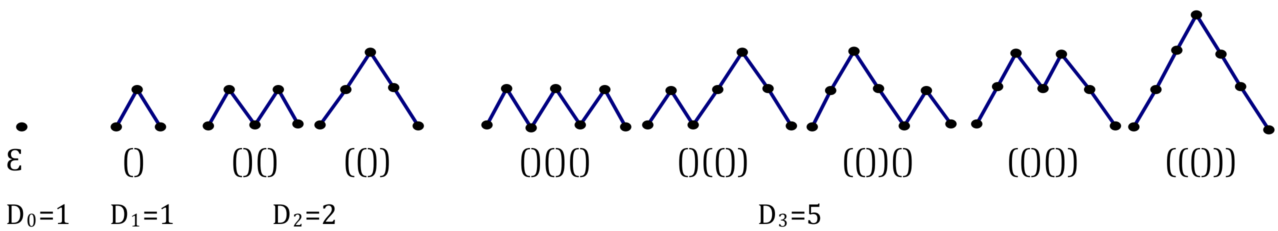


Fig. 2.16: Dyck path corresponding to correctly parenthesized sequences.

(b) Bijection between two Catalan families of Dyck paths.

From section 2.1 and section 2.2 the set of all Dyck paths of length 2n and the set of all Dyck paths of length 2n +2 that have no peaks at height 2 are counted by Catalan numbers. Because of this we call it Catalan families of Dyck paths.

Let α be the set of all Dyck paths of length 2n and let β be the set of all Dyck paths of length 2n+2 with no peaks at height 2. We define a Bijection between α and β as follows. First, starting with a Dyck path P from α, we obtain a Dyck path $\dot{P}$ from β using the following steps.

(1) Attach a Dyck path of length 2 to the left of P to produce $\dot{P}$.

(2) Let $\dot{S}$ be a maximal sub-Dyck path of $\dot{P}$ with $\dot{S}$ having no peaks at height 1. To each Such $\dot{S}$ add a north-east step at the beginning and a south-east step at the end to produce sub-Dyck path $\tilde{S}$. This step produces a Dyck path $\tilde{P}$.

(3) From $\tilde{P}$ eliminate each Dyck path of length 2 that is to the immediate left of each$\tilde{S}$. We now have a unique element $\widehat{P}$ of β.

To obtain P from $\widehat{P}$, we reverse the steps as follows:

(1) Let $\hat{S}$ be a sub-Dyck path of $\widehat{P}$ between two consecutive points on the x-axis with $\hat{S}$ having no peaks at height 1. To each $\hat{S}$ add a Dyck path of length 2 immediately to the left. This step produces a Dyck path $\tilde{P}$.

(2) Let $\tilde{S}$ be a maximal sub-Dyck path of $\tilde{P}$. From each such $\widetilde{S}$ remove the left-most north-east step and the right-most south-east step to produce a sub-Dyck path $\dot{S}$. This step produces a Dyck path $\dot{P}$ of length 2n+2.

(3) From $\dot{P}$, remove the left-most Dyck path of length 2 to produce P:

For example, we obtain a Dyck path of length 18 with no peaks at height 2 starting with a Dyck path of length 16 as follows:

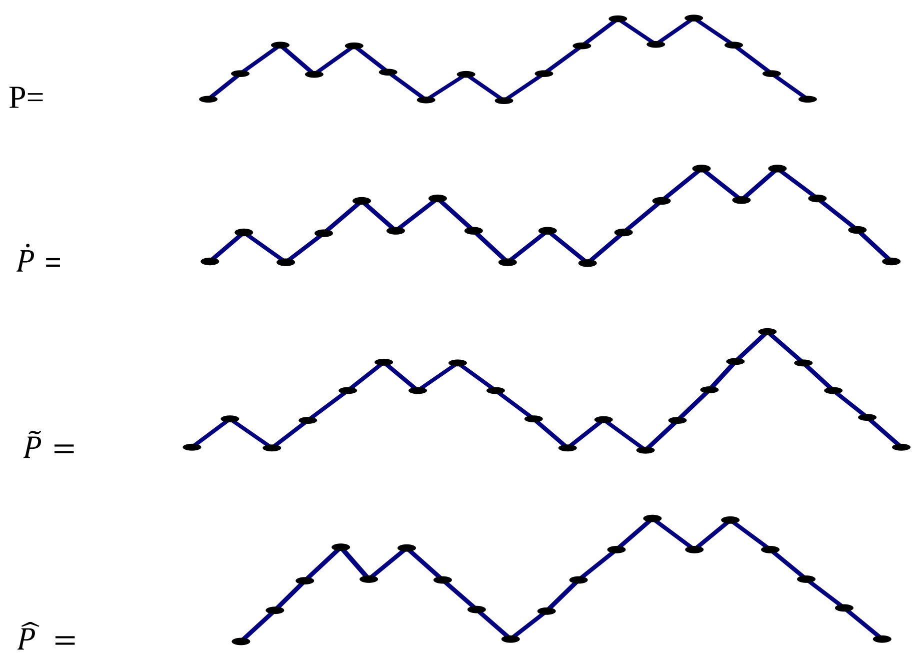
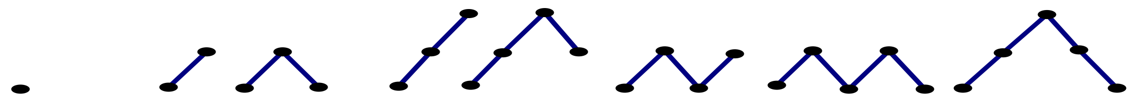


Fig. 2.17: Dyck paths corresponding to two Catalan families Dyck paths

(c) Bijection between the set of incomplete Dyck paths and the set Dyck path.

The set of incomplete Dyck paths that start from the lattice point *(0, 0)*, and using steps $(1, 1)$ or $(1, -1)$, and never crossing the *x*-axis having n-1 up steps and the set of Dyck path has

Bijection relation. This Bijection can be exhibited directly as follows: At the end of each incomplete Dyck paths in the figure add an up step and then enough down steps to reach the horizontal axis. For example consider the paths in figure 2.18 from incompletes Dyck path adding an up step at the end of each path and then enough down steps yields the Dyck paths in figure 2.19.



$D_1=1$ $D_2=2$ $D_3=5$

Fig.2.18: Incompletes Dyck paths.

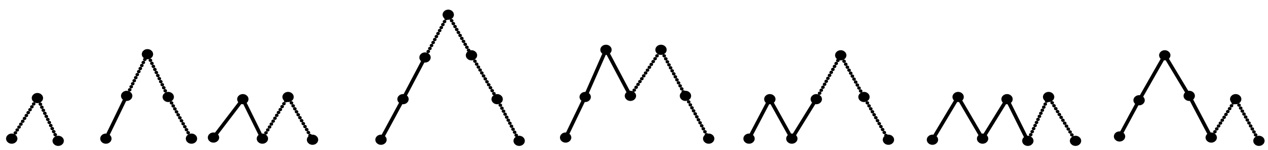


Fig.2.19:Dyck paths corresponding to Incompletes Dyck paths

Clearly, this algorithm is reversible: From the end of a Dyck paths (with n up steps and n down steps), delete the last sequence of down steps and the up step immediately preceding it. This results in an incomplete path with n-1 up steps. For example, consider the Dyck paths in figure 2.20(C).Deleting the portions in dote yields the incomplete path in figure 2.20(D).

Figure(C)

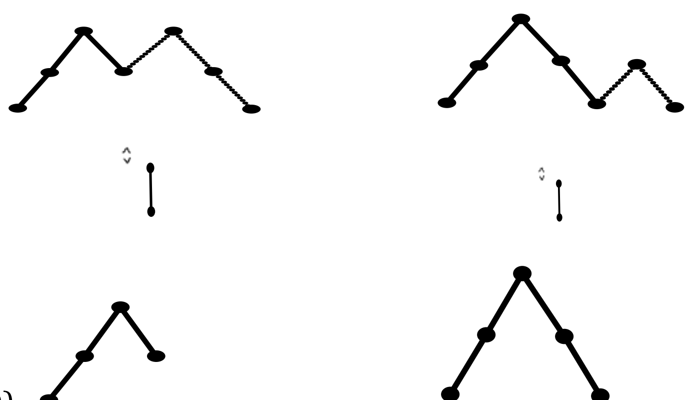


Figure (D)

Fig.2.20: Incompletes Dyck paths corresponding to Dyck paths.

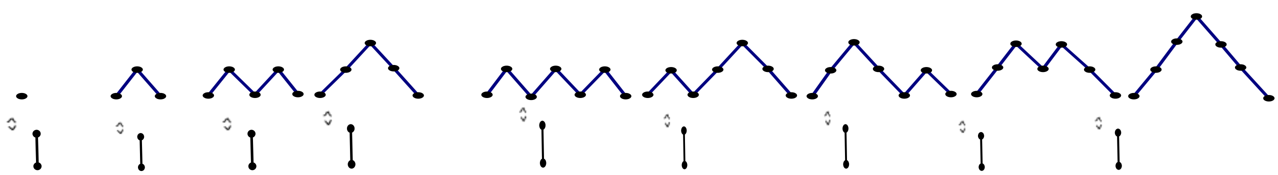
(d) Bijection between the set of sequences of $n\,(1's)$ and $n\,(-1's)$ such that every partial sum is non-negative integers and the set of Dyck paths of length $2n$.

We present a simple Bijection between this set of sequences and a well known Catalan enumerated set of Dyck paths of length $2n$.

i.e. $1 \longmapsto up\ step$

$-1 \longmapsto down\ step$

The condition that every partial sum is non-negative in the sequence implies, we cannot have more number of $-1's$ than $1's$ when reading from left to right and finally we will have equal number of $1's$ and $-1's$ which directly corresponds to the property of Dyck path.



ε 1-1 1-11-1 11-1-1 1-11-11-1 1-111-1-1 11-1-11-1 11-11-1-1 111-1-1-1

$D_0=1$ $D_1=1$ $D_2=2$ $D_3=5$

Fig. 2.21: Dyck path corresponding to the set of sequences of n $1's$ and n $-1's$ such that every partial sum is non-negative integers.

(e) Bijection between the set of standard young tableau of the shape (n, n) and the set of Dyck paths without hill of length $2n$.

<u>Definition</u> : A standard Young Tableau (SYT) is a Ferrer's shape on n boxes in which each box contains one of the elements of $[2n] = \{1, 2, \dots, 2n\}$ so that all boxes contain different numbers and the rows and columns increase going down and going to the right. So Standard Young Tableaux (SYT) of shape (n, n) where there is no column of consecutive integers.

<u>Example</u>: We have six such standard young tableau of shape (4,4).

These are:

1	2	3	4
5	6	7	8

1	2	4	5
3	6	7	8

1	2	4	6
3	5	7	8

1	2	5	6
3	4	7	8

1	2	3	6
4	5	7	8

1	2	3	5
4	6	7	8

Fig.2.22: six standard young tableau of shape (4,4).

There is a Bijection f between the set of sequences of n $(1's)$ and n $(-1's)$ such that every partial sum is non-negative integers and the set of Dyck paths. And there is also a Bijection g between the set of Standard Young Tableau of the shape (n, n) and the set of sequences of n $1's$ and n $-1's$ such that every partial sum is non-negative integers.

It is as follows:

Given a Standard Young Tableau T of the shape (n, n),

Define the sequence $a_1\ a_2 a_3 \ldots \ldots a_{2n}$ by

$$a_i = \begin{cases} 1 \quad if \quad a_i \quad appears\ in\ row\ 1\ of\ T \\ -1 \quad if \quad a_i\ appears\ in\ row\ 2\ of\ T \end{cases}$$

Example: Standard Young Tableau T of the shape $(4, 4)$, has sequence $a_1 = 1,\ a_2 = 2, a_3 = 3 \ldots a_8 = 8$

So, now, the composition fog of the two bijections is the bijection between the set of standard young tableau of the shape (n, n) and the set of Dyck paths of length $2n$.

Example:

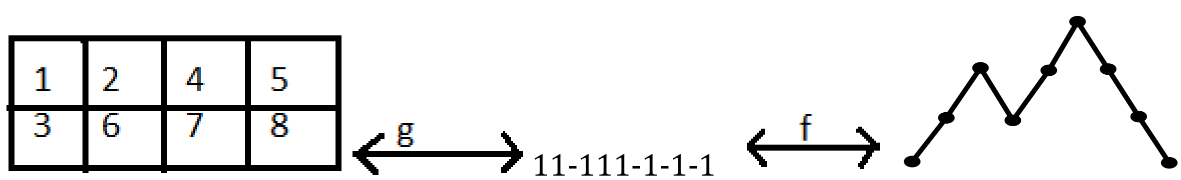


Fig.2.23: composition fog of the two bijections.

(f) Bijection between the set of Dyck paths of length $2n + 2$ and the set of all path pairs of length n (APP(n)) .

Definition-1: The set of all path pairs (APP) is the set of all pairs of paths such that;

i) Both paths are composed of unit east and north steps.

ii) Both paths start at (0,0) and have a common end point.

iii) The upper path never goes strictly below the lower path.

Example: Let $APP(n) = The\ set\ of\ all\ path\ pairs\ of\ length\ n.$

For $n = 2$, $APP(2) = 5$ these are;

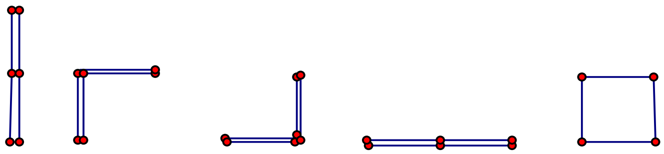


Fig.2.24 $APP(2) = The\ set\ of\ all\ path\ pairs\ of\ length\ 2$

We can encode a path pair as follows;

A= step apart, upper path goes north, lower path east;

T = step together, upper path goes east, lower path north;

E = both steps east;

N = both steps north.

Example: $ANTEATEN$ is the code for the path pair below:

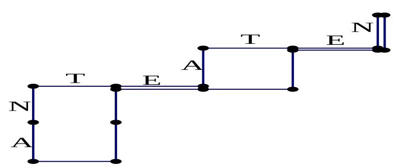


Fig 2.25: $ANTEATEN$ code path pair.

Remark: Reading from the left we must never have more $T's$ than $A's$ and at the end the number of $T's$ is the same as the number of $A's$.

Definition-2: By a joint step of a path pair we mean a pair of superposed steps (one from each path of the pair).

Example: In the pair of paths encoded $ANTEATEN$ both $E's$ and the last $N's$ represent joint steps.

Proof: a Bijection between the set of Dyck paths of length $2n + 2$ and $APP(n)$.As before let $u\ and\ d$ denote the steps $(1,1) and$ $(1, -1)$ respectively. Then we convert a path pair to a Dyck path by converting $A \longmapsto uu, T \mapsto dd, E \mapsto du, and\ N \mapsto ud$ and adding a u to the start and a d to the end of the resulting word. Consider the following path pair $ATEAT$ of length 5. The above operation gives us $uuddduuudd$ which is a Dyck path of length 12.

Fig 2.26: Bijection between the set of Dyck paths of length $2n + 2$ and $APP(n)$.

Thus $APP(n)$ is the Catalan enumerated set where $|APP(n)| = C_{n+1}$.

(g) Bijection between the set of all Dyck paths of length $2n$ without hill and the set of all path pairs of length n (APP(n)) with no joint steps .

<u>**Proof**</u>**:** We consider the Bijection between $APP(n)$ and the set of Dyck paths of length $2n + 2$, defined in the proof of above.

It is easy to see that, in this Bijection, to joint E steps there correspond valleys at level zero and to joint N steps there correspond peaks at level two.

<u>**Example**</u>: Consider the following:

Now consider a path pair of length n with no joint steps.

ATAT

Dyck path of length 10 with no peaks at level 2

Removing the first and the last steps, we obtain a Dyck path of length 8 with no hills.

Fig 2.27: Bijection between the set of Dyck paths of length $2n$ without hill and $APP(n)$.

Thus, to a path pair of length n with no joint steps there correspond elevated Dyck path (i.e. with the exception of the end points, they stay strictly above the horizontal axis) of length $2n + 2$, with no peaks at level 2.

Removing the first and the last step of this Dyck path, we obtain a Dyck path of length $2n$ with no hills.

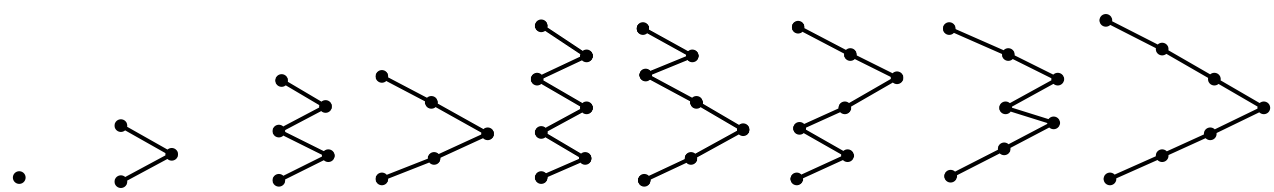
Remark: Having no valleys at level 0 and no peaks at level 2, guarantees that the paths to stay above the horizontal axis and to be hill free while removing the first and the last step. Therefore, the set of all path pairs of length n with no joint steps is the Fine number enumerated set.

(h) Bijection between the set of all Dyck paths of length $2n$ and the set of all particle movement that start at the origin moves n units to the right and n units to the left along the x-axis, one unit at time, after moving 2n units, the particle returns to the origin .

In the set of all particles that start from origin moves n units to the right and n units to the left along the x-axis the number of moves to the left does not exceed the number of moves to the right.

Look at the various possible moves for $0 \leq n \leq 3$.

The figures are distorted to avoid over lapping line segments. The right moves by up stroke and left moves by down stroke.



$D_0=1$ $D_1=1$ $D_2=2$ $D_3=5$

Fig 2.28: Various possible move of particle for $0 \leq n \leq 3$.

These figures clearly resemble the mountain ranges in the figure with Dyck paths graph, especially if the mountains are rotated through 90^0 degree in the counterclockwise direction. Consequently, switching an up stroke with a unit line segment in the positive direction and a down stroke with a unit line segment in the negative direction establishes a one-to-one correspondence (Bijection) between the set of all Dyck paths and the set of all particles that start from origin move n units to the right and n units to the left along the x-axis the number of moves to the left does not exceed the number of moves to the right. Both are counted by the Catalan numbers.

Section three

3.1Dyck path statistics using Ordinary generating functions

We can find so money statistics for Dyck path relative to different parametric. In this section we will see behavior of certain statistics.

Definition: A return, denoted by X_R, (to the horizontal axis) consists of a non-trivial path, a point on the horizontal axis, and another path (possibly trivial).

First we consider the statistic number of returns.

Remark: We are assuming that all paths are equally likely to be chosen.

(i)From the definition, we can see that, the generating function for the total number of returns of all Dyck paths of a given length is

$$(C-1)C$$

$$\Leftrightarrow (C-1)C = C^2 - C = \frac{C-1}{z} - C = \frac{C}{z} - \frac{1}{z} - C$$

$$\Leftrightarrow [z^n](C-1)C = [z^n]\left(\frac{C-1}{z} - C\right)$$

$$= [z^n]\left(\frac{C-1}{z}\right) - [z^n]C$$

$$= [z^{n+1}](C-1) - [z^n]C$$

$$= C_{n+1} - C_n$$

To find the expected number of returns, we divide the total number of returns by the total number of paths. $\Leftrightarrow \mathrm{E}(X_R) = \frac{C_{n+1}-C_n}{C_n} = \frac{\frac{1}{n+2}\binom{2n+2}{n+1} - \frac{1}{n+1}\binom{2n}{n}}{\frac{1}{n+1}\binom{2n}{n}}$

$$= \frac{\frac{1}{n+2}\frac{(2n+2)!}{(n+1)!(n+1)!} - \frac{1}{n+1}\frac{(2n)!}{n!n!}}{\frac{1}{n+1}\frac{(2n)!}{n!n!}}$$

$$= \frac{(2n+2)(2n+1)}{(n+2)(n+1)} - 1$$

$$= \frac{3n(n+1)}{(n+2)(n+1)} = \frac{3n}{n+2}$$

$\Leftrightarrow \mathrm{E}(X_R) = \frac{3n}{n+2} \longrightarrow 3$

Next, we consider the statistic height of the first peak, denoted X_H.

(i)The generating function for Dyck paths whose initial peak is at height k is $z^k C^k$ (Preposition 2.2)

Claim: The generating function for the sum of heights of the first peaks at height k over all Dyck paths of length $2n$ is $kz^k C^k$.

Proof: Since the height of each such Dyck path is k and the total number of such paths is $[z^n](zC)^k$, the appropriate generating function for the sum of heights of paths of the first peaks at height k over all Dyck paths of length $2n$ is $kz^k C^k$.

Then the generating function for the sum of the heights of the first peaks of all Dyck paths of length $2n$ is just taking the sum over all possible k.

$$\begin{aligned}\sum_{k=1}^{\infty} kz^k C^k &= zC + 2z^2C^2 + 3z^3C^3 + \cdots \\ &= zC(1 + 2zC + 3z^2C^2 + \cdots) \\ &= zC\frac{d\left(\frac{1}{1-zC}\right)}{dzc} \\ &= \frac{zC}{(1-zC)^2} = (zC)C^2 = (zC^2)C \\ &= (C-1)C = C^2 - C\end{aligned}$$

To find the expected number of the heights of the first peaks of all Dyck paths of length $2n$ we divide the total number of the heights of the first peaks of all Dyck paths of length $2n$ by the total number of paths. $\Leftrightarrow \mathrm{E}(X_H) = \frac{C_{n+1} - C_n}{C_n} = \frac{C_{n+1}}{C_n} - 1$

$$= \frac{(2n+2)(2n+1)}{(n+2)(n+1)} - 1$$

$$= \frac{3n(n+1)}{(n+2)(n+1)} = \frac{3n}{n+2}$$

This is the same as the result obtained at the statistic 'number of returns" and therefore,

$$E(X_H) = \frac{3n}{n+2} \longrightarrow 3$$

Dyck path statistics using Ordinary generating function is summarized in the following table.

Table-3.1 :Dyck paths statistics using Ordinary generating function, total number of returns, and the sum of the heights of the first peaks.

Statistics	Generating function	Closed formula for the expaected number	Asymptotic value.
Total number of returns of Dyck paths of length $2n$.	$\frac{C-1}{z} - C$	$\frac{C_{n+1}-C_n}{C_n} = \frac{3n}{n+2}$	3
The sum of the heights of the first peaks of all Dyck paths of length $2n$.	$C^2 - C$	$\frac{C_{n+1}-C_n}{C_n} = \frac{3n}{n+2}$	3

3.2 Dyck path statistics using Bivariate generating functions

We find the Bivariate generating function for the sum of the heights of the first peaks over all Dyck path of length $2n$.Mark each up step by z and each up step before the first peak by t.

For Dyck paths denote the Bivariate generating function by Δ.

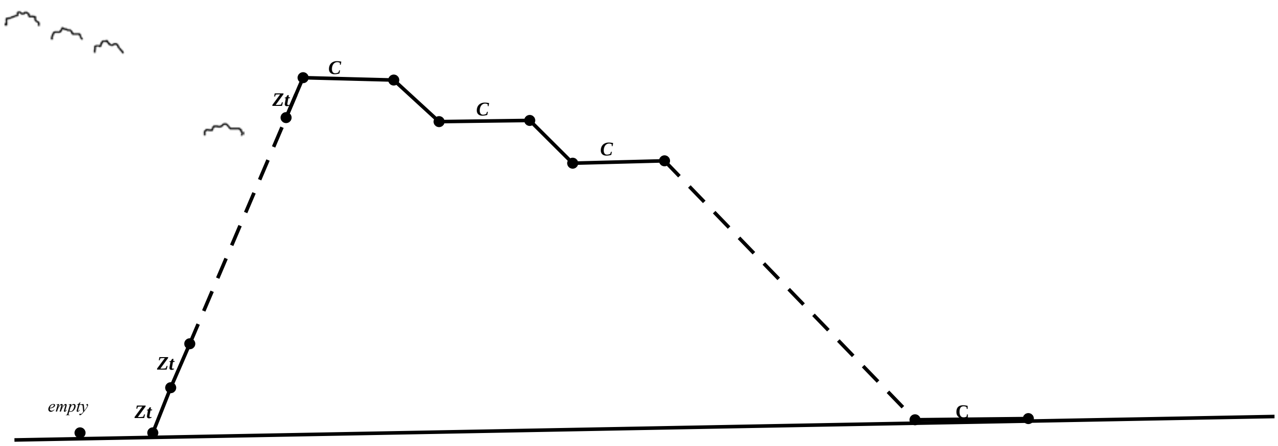


Fig 3.1: Decompositions of Dyck path Mark each up step by z and each up step befor the first peak by t.

From preposition2.2 the generating function of Dyck paths whose first height at k is $z^k c^k$ but here Mark each up step by z and each up step befor the first peak by t. So that the generating function is $t^k z^k c^k$

If
$$k = 0 \ \rightarrow t^0 z^0 C^0 = (tzc)^0 = 1$$
$$k = 1 \ \rightarrow \ t^1 z^1 C^1 = (tzc)^1$$
$$k = 2 \ \rightarrow \ t^2 z^2 C^2 = (tzc)^2$$
$$k = 3 \ \rightarrow \ t^3 z^3 C^3 = (tzc)^3$$
$$\vdots$$

By sum up $\Delta(t, z) = \Delta = 1 + (tzc)^1 + (tzc)^2 + (tzc)^3 + \cdots$

Then we obtain the Bivariate generating function

$$\Delta(t, z) = \Delta = \frac{1}{1-ztC}$$

Differentiating this with respect to $t. and\ setting\ t = 1$,

$$\frac{d\left(\frac{1}{1-ztC}\right)}{dt}$$
$$= \frac{zC}{(1-ztC)^2}$$

$And\ set\ t = 1$

$$= (zC)C^2 = (zC^2)C$$

$$= (C-1)C = C^2 - C$$

we obtain again the same expressions as above. Now we consider the statistic number of peaks, denoted by X_p.

We use the method of Bivariate generating function.

i) Let $\lambda(t,z)$ be the bivariate generating function for Dyck paths according to up-step (marked by z) and number of peaks (marked by t).

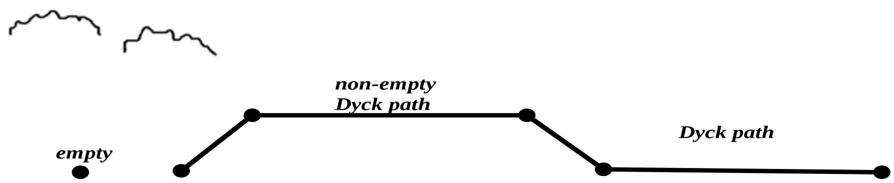


Fig 3.2: Decomposition of Dyck path according to up-step(marked by z) and number of peaks(marked by t)

Hence $\lambda(t,z) = \lambda = 1 + z(\lambda - 1 + t)\lambda$

<u>Note</u>: $\lambda(1,z) = C$

By implicit differentiation we have,

$$\frac{\partial\lambda}{\partial t} = \frac{\partial}{\partial t}(1 + z\lambda^2 - z\lambda + zt\lambda)$$

$$= 2z\lambda\frac{\partial\lambda}{\partial t} - z\frac{\partial\lambda}{\partial t} + z\lambda + zt\frac{\partial\lambda}{\partial t} = \frac{\partial\lambda}{\partial t}(2z\lambda - z + zt) + z\lambda$$

$$\Leftrightarrow \frac{\partial\lambda}{\partial t}(1 - 2z\lambda + z - zt) = z\lambda$$

$$\Leftrightarrow \frac{\partial\lambda}{\partial t} = \frac{z\lambda}{1-2z\lambda+z-zt}$$

$$\Leftrightarrow \left(\frac{\partial\lambda}{\partial t}\right)_{t=1} = \frac{z\lambda(1,z)}{1-2z\lambda(1,z)} = \frac{zC}{1-2zC}$$

$= \frac{zC}{1-(1-\sqrt{1-4z})} = \frac{zC}{\sqrt{1-4z}} = zBC = \frac{B-1}{2}$, were, B is generating function of central Binomial coefficient[appendix(5)]

Thus $E(X_p) = \frac{1}{C_n}[z^n]\left(\frac{\partial\lambda}{\partial t}\right)_{t=1} = \frac{\frac{1}{2}\binom{2n}{n}}{C_n} = \frac{\frac{1}{2}\binom{2n}{n}}{\frac{1}{n+1}\binom{2n}{n}} = \frac{n+1}{2}$

Dyck path statistics using Bivariate generating function is summarized in the following table.

<u>Table</u>-3.2 Dyck path statistics using Bivariate generating function according to its semi-length and the number of peaks.

Statistics	**Generating function**	**Closed formula for $E(X_p)$**
Dyck path according to its semi-length and the number of peaks.	$\frac{B-1}{2}$	$\frac{n+1}{2}$

Conclusion

There is little doubt that the best known and most studied area of Combinatorics is Dyck paths. When the set is associated with a Catalan numbers, Fine numbers, and Motzkin numbers. In an attempt to count the number of Dyck paths according to several parameters, such as number of length, number of peaks, number of valleys, number of double rises, number of returns to the x-axis, the odd length of the downward path ending on the x-axis, and also number of Dyck paths that have no peaks at height two, number of Dyck paths whose first down step is followed by another down step, number of Dyck paths that have n-1 up steps, number of Dyck paths that have n-1 peaks and containing neither long slopes nor lonely peaks, number of Dyck paths with no hills, number of Dyck paths whose initial peak is at height k, whose first peak has even height and whose first peak has odd height etc. has become a subject of great interest.

As further study of this topic, we could explore the generating function of Dyck paths with respect to different parameters and restrictions to enumerate it. For example, we could explore the generating function of Dyck paths whose initial peak is at height k. Another possible topic could be Number of peaks before and after the first return. Emeric Deutsch Dyck path Enumeration provides guidance for further exploration of these ideas.

For the future we will need to study Dyck paths in-depth to enumerate it. We conclude this project by remembering that it needs further research.

Appendix

(1) An infinite lower triangular matrix A is called a Riordan array if its Bivariate generating function $G(t,z)$ is of the form$G(t,z) = \frac{g(z)}{(1-th(z))}$.

We denote $A = (g(z),\ h(z))$.

(2) The Lagrange inversion theorem is used in subsection 2.2, under the following specialized form.

Assume that a generating function $A(z)$ satisfies the functional equation

$$A(z) = 1 + z\mathrm{H}\big(\mathrm{A(z)}\big),$$

Where $\mathrm{H}(\lambda)$ is a polynomial in λ. Then $A(z) = 1 + z\mathrm{H}\big(\mathrm{A(z)}\big)$, has a unique solution $\mathrm{A(z)}$ and if $G(\lambda)$ is a polynomial in λ, then

$$[z^n]G\big(A(z)\big) = \frac{1}{n}[\lambda^{n-1}]G'(1+\lambda)\big(\mathrm{H}(1+\lambda)\big)^n \quad (n \geq 1).$$

(3) We define the Catalan function $C = C(x)$ implicitly by

$$zC^2 - C + 1 = 0, \quad C(0) = 1$$

The coefficient C_n of x^n in the Maclaurin series of $C(z)$ is known as the n^{th} Catalan number.

From $zC^2 - C + 1 = 0, \quad C(0) = 1$ we obtain that

$$C_0 = 1$$

$$C_n = C_0C_{n-1} + C_1C_{n-2} + \cdots + C_{n-2}C_1 + C_{n-1}C_{0,} \ (n \geq 1).$$

This shows, in particular, that the Catalan numbers are positive integers.

From

$$zC^2 - C + 1 = 0, \quad C(0) = 1$$

We obtain the explicit expression of the Catalan function,

$$C(z) = \frac{1 - \sqrt{1 - 4z}}{2z},$$

From where one can derive that

$$C_n = \frac{1}{n+1}\binom{2n}{n}, n \geq 0$$

(it is sequence M1459 in[10]). The Catalan numbers are probably the most frequently occurring combinatorial numbers after the binomial coefficients (for example, in R.Stanley's [11] there will be given at least 65 combinatorial structures that are counted by the Catalan numbers.

(4) We define the Fine function$F = F(z)$

$$F(z) = \frac{1}{z}\frac{1 - \sqrt{1 - 4z}}{3 - \sqrt{1 - 4z}}$$

The coefficient $F_0 = 1, F_1 = 0, F_2 = 1, F_3 = 2, F_4 = 6, F_5 = 18, F_6 = 57, F_7 = 186, F_8 = 622$ (it is sequence M1459 in[10]).

From Catalan function

$$C(z) = \frac{1 - \sqrt{1 - 4z}}{2z}$$

And

$$F(z) = \frac{1}{z}\frac{1 - \sqrt{1 - 4z}}{3 - \sqrt{1 - 4z}}$$

Eliminating the square root, we find

$$F = \frac{C}{1 + zC}$$

Multiplying both the numerator and the denominator of the right-hand side of

$$F = \frac{C}{1 + zC}$$

by $1 - zC$ and making use of $zC^2 - C + 1 = 0$, we obtain

$$F = \frac{1}{1 - z^2C^2}$$

From

$$F = \frac{C}{1 + zC}$$

One can derive easily

$$2F + zF = 1 + c$$

This implies the recurrence relation

$$2\mathrm{F}_n + \mathrm{F}_{\mathrm{n}-1} = \mathrm{C}_n, (\mathrm{n} \geq 2)$$

(5) The central binomial coefficient **CBC**s $\binom{2\mathrm{n}}{n}$are centrally located in even-numbered rows in Pascal's triangle,
as Figure shows.

[1]

1 1

1 [2] 1

1 3 3 1

1 4 [6] 4 1

1 5 10 10 5 1

1 6 15 [20] 15 6 1

Pascal's Triangle

The generating function is

$\mathrm{B} = \sum_{n=0}^{\infty}\binom{2n}{n} x^n = \frac{1}{\sqrt{1-4x}}$

(6) Motzkin numbers has the sequence

$\{M_n\}_{n=0}^{\infty} = 1, 1, 2, 4, 9, 21, 51, \dots$, The generating function can be given as

$M(x) = \sum_{n=0}^{\infty} \mathrm{M_n}\, x^n$.

$$M(x) = 1 + x + 2x^2 + 4x^3 + 9x^4 + \cdots$$

Then it has the functional equation,

$$\Longrightarrow x^2M^2(x) + xM(x) - M(x) + 1 = 0$$

Using the quadratic formula,

$M(x) = \frac{1-\mathrm{x}-\sqrt{1-2x-3x^2}}{2x^2}$, x ≠ 0 is the closed form.

Using Mathematica for the expansion of $\sqrt{1-2x-3x^2}$ or using binomial expansion, $\frac{1-\mathrm{x}-\sqrt{1-2x-3x^2}}{2x^2}$ is exactly equal to $M(x)$, but $\frac{1-\mathrm{x}+\sqrt{1-2x-3x^2}}{2x^2}$ is not equal to $M(x)$.

$\mathrm{n^{th}}$ Motzkin number $\mathrm{M_n}$ using the generating function

As we have seen in in the above the generating function of Motzkin numbers has a functional equation $M(x) = 1 + xM(x) + x^2\, M^2(x)$.

Let $z = xM(x)$

$\Longrightarrow M(x) = 1 + z + z^2$

$\Longrightarrow z = x(1 + z + z^2)$

By Combinatorial interpretation, using Lagrange's Inversion Formula (LIF),

Let $f(z) = 1 + z + z^2$

Then $M_n = [z^{n+1}]f(z)^{n+1}$

$= \frac{1}{n+1}[z^n](1 + z + z^2)^{n+1}$

$\Longrightarrow M_n = \frac{1}{n+1}\sum_{i=0}^{n}\binom{n+1}{i}\binom{n+1-i}{i+1}$

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