



**TWO-MODE SUBHARMONIC GENERATING  
SYSTEM WITH COHERENT AND THERMAL  
LIGHT**

**By**

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**This Work is Dedicated**  
**to**  
**My parents whose loving memories still guide me**  
**through difficult times.**

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## Abstract

In this thesis, we have investigated the statistical and squeezing properties of the light produced by a two-mode subharmonic generating system in a cavity driven by two-mode coherent light and coupled to two-mode thermal reservoir via a single-port mirror. Employing the master equation for the system under consideration, we obtain c-number Langevin equations associated with the normal ordering and determined correlation properties of the noise forces. The solutions of the resulting differential equations are then used to calculate the mean and variance of the photon number of the cavity light. We have found that the variance of the photon number is greater than the mean of the photon number and hence the photon statistics of the cavity light mode is super-Poissonian. Moreover, applying the same solutions, we also obtained the antinormally-ordered characteristic function with the aid of which the Q function is determined. In addition, the Q function is used to calculate the photon number distribution for the cavity mode. Furthermore, we obtain the variance of the plus and minus quadrature of the two-mode cavity light produced by the two-mode subharmonic generating system. It is found that the two-mode driving coherent light has no effect on the squeezing properties of the cavity mode and the squeezing occurs in the plus quadrature. Finally, we have found that at steady state and at threshold, there is a 50% maximum squeezing of the two-mode cavity light when the cavity is coupled to vacuum reservoir.

## Introduction

Quantum optics is the subject that deals with optical phenomena that can only be explained by treating light as a stream of photons rather than as electromagnetic waves. In principle, the subject is as old as quantum theory itself, but in practice, it is a relatively new one, and has really only come to the fore during the last quarter of the twentieth century [1]. Quantum optics deals mainly with the quantum properties of the light generated by various optical systems such as lasers and with the effect of light on the dynamics of atoms. The quantum properties of light are largely determined by the state of the light mode and the well known quantum states of light are the number state, the chaotic state, the coherent state and the squeezed state [2].

A two-mode subharmonic generating system, consisting of a nonlinear crystal in a cavity pumped by a coherent light and coupled to a reservoir is a source of a two-mode squeezed light. When light interacts with nonlinear crystal it induces polarization. The amount of polarization of a material depends linearly or nonlinearly on the applied electric field [3]. The propagation of electromagnetic field in a nonlinear medium is affected by the dielectric polarization induced by the field. When electromagnetic radiation is incident on a nonlinear medium, a response with greater or less than the driving frequency can appear. The frequencies of the response in such a system are related to the driving frequency by an integer multiple and these are called harmonic responses [3]. But under certain conditions, a response with frequency less than the driving frequency can appear. The response with frequency less than the driving frequency is called subharmonics and the response with frequency greater than the driving frequency is called super harmonics. Some examples of nonlinear interactions are; second harmonic generation, sum frequency generation and subharmonic generation. Subharmonic generation [2 - 4] as well as second harmonic generation [2, 5, 6] is a typical process that leads

to the production of squeezing light. In second harmonic generation a photon of frequency  $\omega$  interacts with nonlinear material and is up converted into photon with twice the frequency of the initial photon. In sum frequency generation two photons of different frequencies are combined to provide a photon of frequency equal to the sum of the two frequencies.

There has been a considerable interest in the analysis of the quantum properties of the squeezed light generated by various quantum optical systems [2]. Squeezing is one of the nonclassical features of light that has attracted a great deal of interest [7 - 8]. In squeezed light the fluctuations in one quadrature is below the vacuum level at the expense of enhanced fluctuations in the conjugate quadrature, with the product of the uncertainty in the two quadratures satisfying the uncertainty relation. It is also worth mentioning that squeezed light has potential applications in the detection of weak signal and in low-noise communications [2, 10]. Hence it is vital to find new optical devices or to combine the existing ones to generate highly squeezed and bright light.

Subharmonic generator has been considered as an important source of squeezed light. It is one of the most interesting and well characterized optical device in quantum optics. In this device a pump photon interacts with a nonlinear crystal inside a cavity and is down-converted into two highly correlated photons [2]. If these photons have the same frequency the device is called a degenerate subharmonic generator, otherwise it is called a nondegenerate subharmonic generator [2, 11].

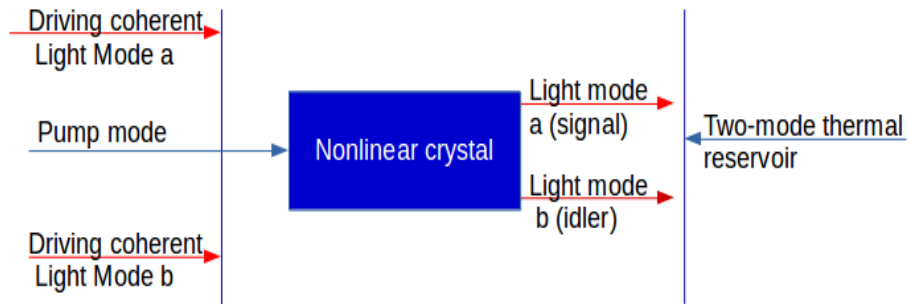
The main objective of this thesis is to study, employing c-number Langevin equation, the statistical and squeezing properties of the light produced by a two-mode subharmonic generating system in a cavity driven by two-mode coherent light and coupled to two-mode thermal reservoir via a single-port mirror.

## C-number Langevin Equations

In this section we develop the dynamics of the system, the master equation for the cavity mode under consideration. Applying this master equation, we determine c-number Langevin equations associated with the normal ordering. Finally, we can obtain the correlation properties of the noise forces and the solutions of the resulting c-number Langevin equations.

### 2.1 The master equation

We now seek to obtain the equation of evolution of the reduced density operator for the two-mode light produced by a nondegenerate subharmonic generator in a cavity driven by two-mode coherent light beams and coupled to two-mode thermal reservoir via a single-port mirror. In a nondegenerate subharmonic generator a



**Figure 2.1:** A nondegenerate subharmonic generator with the cavity modes driven by coherent light and coupled to thermal reservoir.

pump photon of frequency  $\omega$  is down converted into a highly correlated signal and idler photons of frequencies  $\omega_a$  and  $\omega_b$  such that  $\omega = \omega_a + \omega_b$  [2, 11].

The process of the parametric interaction can be described by the Hamiltonian

$$\hat{H} = ig(\hat{a}\hat{b}\hat{c}^\dagger - \hat{a}^\dagger\hat{b}^\dagger\hat{c}), \quad (2.1)$$

where  $\hat{a}$  and  $\hat{b}$  are the annihilation operators for the signal and idler light modes respectively,  $\hat{c}$  is the annihilation operator for the pump mode, and  $g$  is the coupling

constant. With the pump mode treated classically, the Hamiltonian given in Eq. (2.1) can be written as

$$\hat{H} = ig\beta(\hat{a}\hat{b} - \hat{a}^\dagger\hat{b}^\dagger), \quad (2.2)$$

or

$$\hat{H} = i\lambda(\hat{a}\hat{b} - \hat{a}^\dagger\hat{b}^\dagger), \quad (2.3)$$

where  $\lambda = g\beta$  is proportional to the amplitude of the pump mode, considered to be real and positive constant.

We consider the case in which the cavity modes are driven by two coherent light modes. With the driving modes treated classically, the interaction of the cavity modes with the driving modes can be described by the Hamiltonian

$$\hat{H} = i\varepsilon(\hat{a}^\dagger - \hat{a} + \hat{b}^\dagger - \hat{b}), \quad (2.4)$$

where,  $\varepsilon$  is proportional to the amplitude of the driving coherent light modes, considered to be real constant and the same for both modes.

The Hamiltonian describing the parametric interaction and the interaction of the cavity modes with the driving light modes is expressible, using Eqs. (2.3) and (2.4), as

$$\hat{H} = i\varepsilon(\hat{a}^\dagger - \hat{a} + \hat{b}^\dagger - \hat{b}) + i\lambda(\hat{a}\hat{b} - \hat{a}^\dagger\hat{b}^\dagger). \quad (2.5)$$

The master equation for two-mode cavity light driven by coherent light and coupled to two-mode thermal reservoir can be written as [2]

$$\begin{aligned} \frac{d\hat{\rho}}{dt} = & -i[\hat{H}, \hat{\rho}] + \frac{k}{2}(\bar{n} + 1)(2\hat{a}\hat{\rho}\hat{a}^\dagger - \hat{a}^\dagger\hat{a}\hat{\rho} - \hat{\rho}\hat{a}^\dagger\hat{a}) + \frac{k}{2}\bar{n}(2\hat{a}^\dagger\hat{\rho}\hat{a} - \hat{a}\hat{a}^\dagger\hat{\rho} - \hat{\rho}\hat{a}\hat{a}^\dagger) \\ & + \frac{k}{2}(\bar{n} + 1)(2\hat{b}\hat{\rho}\hat{b}^\dagger - \hat{b}^\dagger\hat{b}\hat{\rho} - \hat{\rho}\hat{b}^\dagger\hat{b}) + \frac{k}{2}\bar{n}(2\hat{b}^\dagger\hat{\rho}\hat{b} - \hat{b}\hat{b}^\dagger\hat{\rho} - \hat{\rho}\hat{b}\hat{b}^\dagger), \end{aligned} \quad (2.6)$$

where  $k$  is a cavity damping constant,  $\hat{\rho}$  is density operator and  $\bar{n}$  is the mean photon number of thermal reservoir.

Combining Eqs. (2.5) and (2.6), the equation of evolution of the density operator for the system under consideration is

$$\begin{aligned} \frac{d\hat{\rho}}{dt} = & \varepsilon(\hat{a}^\dagger\hat{\rho} + \hat{b}^\dagger\hat{\rho} - \hat{a}\hat{\rho} - \hat{b}\hat{\rho} - \hat{\rho}\hat{a}^\dagger - \hat{\rho}\hat{b}^\dagger + \hat{\rho}\hat{a} + \hat{\rho}\hat{b}) \\ & + \lambda(\hat{a}\hat{b}\hat{\rho} - \hat{a}^\dagger\hat{b}^\dagger\hat{\rho} - \hat{\rho}\hat{a}\hat{b} + \hat{\rho}\hat{a}^\dagger\hat{b}^\dagger) \\ & + \frac{k}{2}(\bar{n} + 1)(2\hat{a}\hat{\rho}\hat{a}^\dagger - \hat{a}^\dagger\hat{a}\hat{\rho} - \hat{\rho}\hat{a}^\dagger\hat{a}) + \frac{k}{2}\bar{n}(2\hat{a}^\dagger\hat{\rho}\hat{a} - \hat{a}\hat{a}^\dagger\hat{\rho} - \hat{\rho}\hat{a}\hat{a}^\dagger) \\ & + \frac{k}{2}(\bar{n} + 1)(2\hat{b}\hat{\rho}\hat{b}^\dagger - \hat{b}^\dagger\hat{b}\hat{\rho} - \hat{\rho}\hat{b}^\dagger\hat{b}) + \frac{k}{2}\bar{n}(2\hat{b}^\dagger\hat{\rho}\hat{b} - \hat{b}\hat{b}^\dagger\hat{\rho} - \hat{\rho}\hat{b}\hat{b}^\dagger). \end{aligned} \quad (2.7)$$

This represents the master equation for a two-mode subharmonic generator driven by coherent light and coupled to a two-mode thermal reservoir.

Now, we seek to obtain the c-number Langevin equations employing the master equation. The time evolution of the expectation value of an operator  $\hat{A}$  in the Schrödinger picture can be written as [2]

$$\frac{d}{dt}\langle\hat{A}\rangle = Tr\left(\frac{d\hat{\rho}}{dt}\hat{A}\right). \quad (2.8)$$

Now taking into account Eq. (2.7) along with (2.8), we have

$$\begin{aligned} \frac{d}{dt}\langle\hat{a}(t)\rangle &= \varepsilon Tr(\hat{a}^\dagger\hat{\rho}\hat{a} + \hat{b}^\dagger\hat{\rho}\hat{a} - \hat{a}\hat{\rho}\hat{a} - \hat{b}\hat{\rho}\hat{a} - \hat{\rho}\hat{a}^\dagger\hat{a} - \hat{\rho}\hat{b}^\dagger\hat{a} + \hat{\rho}\hat{a}^2 + \hat{\rho}\hat{b}\hat{a}) \\ &+ \lambda Tr(\hat{a}\hat{b}\hat{\rho}\hat{a} - \hat{a}^\dagger\hat{b}^\dagger\hat{\rho}\hat{a} - \hat{\rho}\hat{a}\hat{b}\hat{a} + \hat{\rho}\hat{a}^\dagger\hat{b}^\dagger\hat{a}) \\ &+ \frac{k}{2}(\bar{n} + 1)Tr(2\hat{a}\hat{\rho}\hat{a}^\dagger\hat{a} - \hat{a}^\dagger\hat{a}\hat{\rho}\hat{a} - \hat{\rho}\hat{a}^\dagger\hat{a}^2) \\ &+ \frac{k}{2}\bar{n}Tr(2\hat{a}^\dagger\hat{\rho}\hat{a}^2 - \hat{a}\hat{a}^\dagger\hat{\rho}\hat{a} - \hat{\rho}\hat{a}\hat{a}^\dagger\hat{a}) \\ &+ \frac{k}{2}(\bar{n} + 1)Tr(2\hat{b}\hat{\rho}\hat{b}^\dagger\hat{a} - \hat{b}^\dagger\hat{b}\hat{\rho}\hat{a} - \hat{\rho}\hat{b}^\dagger\hat{b}\hat{a}) \\ &+ \frac{k}{2}\bar{n}Tr(2\hat{b}^\dagger\hat{\rho}\hat{b}\hat{a} - \hat{b}\hat{b}^\dagger\hat{\rho}\hat{a} - \hat{\rho}\hat{b}\hat{b}^\dagger\hat{a}). \end{aligned} \quad (2.9)$$

or

$$\frac{d}{dt}\langle\hat{a}(t)\rangle = \varepsilon I_1 + \lambda I_2 + \frac{k}{2}(\bar{n} + 1)I_3 + \frac{k}{2}\bar{n}I_4 + \frac{k}{2}(\bar{n} + 1)I_5 + \frac{k}{2}\bar{n}I_6, \quad (2.10)$$

where

$$I_1 = Tr(\hat{a}^\dagger\hat{\rho}\hat{a} + \hat{b}^\dagger\hat{\rho}\hat{a} - \hat{a}\hat{\rho}\hat{a} - \hat{b}\hat{\rho}\hat{a} - \hat{\rho}\hat{a}^\dagger\hat{a} - \hat{\rho}\hat{b}^\dagger\hat{a} + \hat{\rho}\hat{a}^2 + \hat{\rho}\hat{b}\hat{a}),$$

$$I_2 = Tr(\hat{a}\hat{b}\hat{\rho}\hat{a} - \hat{a}^\dagger\hat{b}^\dagger\hat{\rho}\hat{a} - \hat{\rho}\hat{a}\hat{b}\hat{a} + \hat{\rho}\hat{a}^\dagger\hat{b}^\dagger\hat{a}),$$

$$I_3 = Tr(2\hat{a}\hat{\rho}\hat{a}^\dagger\hat{a} - \hat{a}^\dagger\hat{a}\hat{\rho}\hat{a} - \hat{\rho}\hat{a}^\dagger\hat{a}^2),$$

$$I_4 = Tr(2\hat{a}^\dagger\hat{\rho}\hat{a}^2 - \hat{a}\hat{a}^\dagger\hat{\rho}\hat{a} - \hat{\rho}\hat{a}\hat{a}^\dagger\hat{a}),$$

$$I_5 = Tr(2\hat{b}\hat{\rho}\hat{b}^\dagger\hat{a} - \hat{b}^\dagger\hat{b}\hat{\rho}\hat{a} - \hat{\rho}\hat{b}^\dagger\hat{b}\hat{a}),$$

and

$$I_6 = Tr(2\hat{b}^\dagger\hat{\rho}\hat{b}\hat{a} - \hat{b}\hat{b}^\dagger\hat{\rho}\hat{a} - \hat{\rho}\hat{b}\hat{b}^\dagger\hat{a}).$$

Applying the cyclic property of the trace operation together with the commutation relations

$$[\hat{a}, \hat{a}^\dagger] = 1, \quad (2.11)$$

$$[\hat{a}, \hat{b}^\dagger] = [\hat{a}, \hat{b}] = [\hat{a}^\dagger, \hat{b}^\dagger] = 0, \quad (2.12)$$

$$[\hat{a}^2, \hat{a}^\dagger] = 2\hat{a}, \quad (2.13)$$

$$[\hat{b}^2, \hat{b}^\dagger] = 2\hat{b}, \quad (2.14)$$

we evaluate the traces in Eq. (2.9).

$$I_1 = Tr(\hat{a}^\dagger\hat{\rho}\hat{a} + \hat{b}^\dagger\hat{\rho}\hat{a} - \hat{a}\hat{\rho}\hat{a} - \hat{b}\hat{\rho}\hat{a} - \hat{\rho}\hat{a}^\dagger\hat{a} - \hat{\rho}\hat{b}^\dagger\hat{a} + \hat{\rho}\hat{a}^2 + \hat{\rho}\hat{b}\hat{a})$$

$$\begin{aligned}
&= Tr(\hat{\rho}\hat{a}\hat{a}^\dagger + \hat{\rho}\hat{a}\hat{b}^\dagger - \hat{\rho}\hat{a}^2 - \hat{\rho}\hat{a}\hat{b} - \hat{\rho}\hat{a}^\dagger\hat{a} - \hat{\rho}\hat{b}^\dagger\hat{a} + \hat{\rho}\hat{a}^2 + \hat{\rho}\hat{b}\hat{a}) \\
&= Tr(\hat{\rho}[\hat{a}, \hat{a}^\dagger] + \hat{\rho}[\hat{a}, \hat{b}^\dagger] - \hat{\rho}[\hat{a}, \hat{b}]), \\
&= Tr(\hat{\rho}) = 1.
\end{aligned} \tag{2.15}$$

$$\begin{aligned}
I_2 &= Tr(\hat{a}\hat{b}\hat{\rho}\hat{a} - \hat{a}^\dagger\hat{b}^\dagger\hat{\rho}\hat{a} - \hat{\rho}\hat{a}\hat{b}\hat{a} + \hat{\rho}\hat{a}^\dagger\hat{b}^\dagger\hat{a}) \\
&= Tr(\hat{\rho}\hat{a}^2\hat{b} - \hat{\rho}\hat{a}\hat{a}^\dagger\hat{b}^\dagger - \hat{\rho}\hat{a}\hat{b}\hat{a} + \hat{\rho}\hat{a}^\dagger\hat{b}^\dagger\hat{a}) \\
&= Tr(-\hat{\rho}[\hat{a}, \hat{a}^\dagger]\hat{b}^\dagger) = Tr(-\hat{\rho}\hat{b}^\dagger),
\end{aligned} \tag{2.16}$$

or

$$I_2 = -\langle \hat{b}^\dagger \rangle. \tag{2.17}$$

In a similar manner, one can verify that

$$I_3 = -\langle \hat{a} \rangle, \tag{2.18}$$

$$I_4 = \langle \hat{a} \rangle, \tag{2.19}$$

$$I_5 = 0, \tag{2.20}$$

and

$$I_6 = 0. \tag{2.21}$$

Now substituting Eqs. (2.15), (2.17), (2.18), (2.19), (2.20) and (2.21) into (2.10), we have

$$\frac{d}{dt}\langle \hat{a}(t) \rangle = -\frac{k}{2}\langle \hat{a}(t) \rangle - \lambda\langle \hat{b}^\dagger(t) \rangle + \varepsilon. \tag{2.22}$$

Similarly

$$\frac{d}{dt}\langle \hat{a}^\dagger(t) \rangle = -\frac{k}{2}\langle \hat{a}^\dagger(t) \rangle - \lambda\langle \hat{b}(t) \rangle + \varepsilon. \tag{2.23}$$

Moreover, employing Eq. (2.7) along with Eq. (2.8), we see that

$$\begin{aligned}
\frac{d}{dt}\langle \hat{a}^2(t) \rangle &= \varepsilon Tr(\hat{a}^\dagger\hat{\rho}\hat{a}^2 + \hat{b}^\dagger\hat{\rho}\hat{a}^2 - \hat{a}\hat{\rho}\hat{a}^2 - \hat{b}\hat{\rho}\hat{a}^2 - \hat{\rho}\hat{a}^\dagger\hat{a}^2 - \hat{\rho}\hat{b}^\dagger\hat{a}^2 + \hat{\rho}\hat{a}^3 + \hat{\rho}\hat{b}\hat{a}^2) \\
&+ \lambda Tr(\hat{a}\hat{b}\hat{\rho}\hat{a}^2 - \hat{a}^\dagger\hat{b}^\dagger\hat{\rho}\hat{a}^2 - \hat{\rho}\hat{a}\hat{b}\hat{a}^2 + \hat{\rho}\hat{a}^\dagger\hat{b}^\dagger\hat{a}^2) \\
&+ \frac{k}{2}(\bar{n} + 1)Tr(2\hat{a}\hat{\rho}\hat{a}^\dagger\hat{a}^2 - \hat{a}^\dagger\hat{a}\hat{\rho}\hat{a}^2 - \hat{\rho}\hat{a}^\dagger\hat{a}^3) \\
&+ \frac{k}{2}\bar{n}Tr(2\hat{a}^\dagger\hat{\rho}\hat{a}^3 - \hat{a}\hat{a}^\dagger\hat{\rho}\hat{a}^2 - \hat{\rho}\hat{a}\hat{a}^\dagger\hat{a}^2) \\
&+ \frac{k}{2}(\bar{n} + 1)Tr(2\hat{b}\hat{\rho}\hat{b}^\dagger\hat{a}^2 - \hat{b}^\dagger\hat{b}\hat{\rho}\hat{a}^2 - \hat{\rho}\hat{b}^\dagger\hat{b}\hat{a}^2) \\
&+ \frac{k}{2}\bar{n}Tr(2\hat{b}^\dagger\hat{\rho}\hat{b}\hat{a}^2 - \hat{b}\hat{b}^\dagger\hat{\rho}\hat{a}^2 - \hat{\rho}\hat{b}\hat{b}^\dagger\hat{a}^2).
\end{aligned} \tag{2.24}$$

or

$$\frac{d}{dt}\langle \hat{a}^2(t) \rangle = \varepsilon T_1 + \lambda T_2 + \frac{k}{2}(\bar{n} + 1)T_3 + \frac{k}{2}\bar{n}T_4 + \frac{k}{2}(\bar{n} + 1)T_5 + \frac{k}{2}\bar{n}T_6, \tag{2.25}$$

where

$$T_1 = Tr(\hat{a}^\dagger \hat{\rho} \hat{a}^2 + \hat{b}^\dagger \hat{\rho} \hat{a}^2 - \hat{a} \hat{\rho} \hat{a}^2 - \hat{b} \hat{\rho} \hat{a}^2 - \hat{\rho} \hat{a}^\dagger \hat{a}^2 - \hat{\rho} \hat{b}^\dagger \hat{a}^2 + \hat{\rho} \hat{a}^3 + \hat{\rho} \hat{b} \hat{a}^2),$$

$$T_2 = Tr(\hat{a} \hat{b} \hat{\rho} \hat{a}^2 - \hat{a}^\dagger \hat{b}^\dagger \hat{\rho} \hat{a}^2 - \hat{\rho} \hat{a} \hat{b} \hat{a}^2 + \hat{\rho} \hat{a}^\dagger \hat{b}^\dagger \hat{a}^2),$$

$$T_3 = Tr(2\hat{a} \hat{\rho} \hat{a}^\dagger \hat{a}^2 - \hat{a}^\dagger \hat{a} \hat{\rho} \hat{a}^2 - \hat{\rho} \hat{a}^\dagger \hat{a}^3),$$

$$T_4 = Tr(2\hat{a}^\dagger \hat{\rho} \hat{a}^3 - \hat{a} \hat{a}^\dagger \hat{\rho} \hat{a}^2 - \hat{\rho} \hat{a} \hat{a}^\dagger \hat{a}^2),$$

$$T_5 = Tr(2\hat{b} \hat{\rho} \hat{b}^\dagger \hat{a}^2 - \hat{b}^\dagger \hat{b} \hat{\rho} \hat{a}^2 - \hat{\rho} \hat{b}^\dagger \hat{b} \hat{a}^2),$$

and

$$T_6 = Tr(2\hat{b}^\dagger \hat{\rho} \hat{b} \hat{a}^2 - \hat{b} \hat{b}^\dagger \hat{\rho} \hat{a}^2 - \hat{\rho} \hat{b} \hat{b}^\dagger \hat{a}^2).$$

We next evaluate the traces in Eq. (2.25). Applying the cyclic property of the trace operation, we obtain

$$\begin{aligned} T_1 &= Tr(\hat{\rho} \hat{a}^2 \hat{a}^\dagger + \hat{\rho} \hat{a}^2 \hat{b}^\dagger - \hat{\rho} \hat{a}^3 - \hat{\rho} \hat{a}^2 \hat{b} - \hat{\rho} \hat{a}^\dagger \hat{a}^2 - \hat{\rho} \hat{b}^\dagger \hat{a}^2 + \hat{\rho} \hat{a}^3 + \hat{\rho} \hat{b} \hat{a}^2). \\ &= Tr(\hat{\rho}[\hat{a}^2, \hat{a}^\dagger] + \hat{\rho}[\hat{a}^2, \hat{b}^\dagger] - \hat{\rho}[\hat{a}^2, \hat{b}]). \end{aligned} \quad (2.26)$$

We see that with the aid of Eqs. (2.10) - (2.12), we can rewrite Eq. (2.26) as

$$T_1 = Tr(2\hat{\rho} \hat{a}) = 2\langle \hat{a} \rangle. \quad (2.27)$$

Employing the cyclic property of trace operation, we have

$$\begin{aligned} T_2 &= Tr(\hat{a} \hat{b} \hat{\rho} \hat{a}^2 - \hat{a}^\dagger \hat{b}^\dagger \hat{\rho} \hat{a}^2 - \hat{\rho} \hat{a} \hat{b} \hat{a}^2 + \hat{\rho} \hat{a}^\dagger \hat{b}^\dagger \hat{a}^2). \\ &= Tr(\hat{\rho} \hat{a}^3 \hat{b} - \hat{\rho} \hat{a}^2 \hat{a}^\dagger \hat{b}^\dagger - \hat{\rho} \hat{a} \hat{b} \hat{a}^2 + \hat{\rho} \hat{a}^\dagger \hat{b}^\dagger \hat{a}^2) \\ &= Tr(\hat{\rho} \hat{a}[\hat{a}, \hat{b}] - \hat{\rho}[\hat{a}^2, \hat{a}^\dagger] \hat{b}^\dagger). \end{aligned} \quad (2.28)$$

In view of Eqs. (2.11) and (2.12), Eq. (2.28) reduces to

$$T_2 = -Tr(2\hat{\rho} \hat{a} \hat{b}^\dagger) = -2\langle \hat{a} \hat{b}^\dagger \rangle. \quad (2.29)$$

Moreover, applying the cyclic property of trace operation, we see that

$$\begin{aligned} T_3 &= Tr(2\hat{a} \hat{\rho} \hat{a}^\dagger \hat{a}^2 - \hat{a}^\dagger \hat{a} \hat{\rho} \hat{a}^2 - \hat{\rho} \hat{a}^\dagger \hat{a}^3). \\ &= Tr(2\hat{\rho} \hat{a}^\dagger \hat{a}^3 - \hat{\rho} \hat{a}^2 \hat{a}^\dagger \hat{a} - \hat{\rho} \hat{a}^\dagger \hat{a}^3) \\ &= Tr(\hat{\rho} \hat{a}^\dagger \hat{a}^3 - \hat{\rho} \hat{a}^2 \hat{a}^\dagger \hat{a}) \\ &= Tr(\hat{\rho}[\hat{a}^\dagger, \hat{a}^2] \hat{a}), \end{aligned}$$

or

$$T_3 = -Tr(2\hat{\rho} \hat{a}^2) = -2\langle \hat{a}^2 \rangle. \quad (2.30)$$

In a similar manner, one readily obtains

$$T_4 = 2\langle \hat{a}^2 \rangle. \quad (2.31)$$

$$T_5 = 0. \quad (2.32)$$

and

$$T_6 = 0. \quad (2.33)$$

Now substituting Eqs. (2.27), (2.29), (2.30), (2.31), (2.32) and (2.33) into (2.25), we get

$$\frac{d}{dt}\langle \hat{a}^2(t) \rangle = -k\langle \hat{a}^2(t) \rangle - 2\lambda\langle \hat{a}(t)\hat{b}^\dagger(t) \rangle + 2\varepsilon\langle \hat{a}(t) \rangle. \quad (2.34)$$

Furthermore, in view of Eqs. (2.7) and (2.8), we have

$$\begin{aligned} \frac{d}{dt}\langle \hat{a}^\dagger(t)\hat{a}(t) \rangle &= \varepsilon Tr(\hat{a}^\dagger\hat{\rho}\hat{a}^\dagger\hat{a} + \hat{b}^\dagger\hat{\rho}\hat{a}^\dagger\hat{a} - \hat{a}\hat{\rho}\hat{a}^\dagger\hat{a} - \hat{b}\hat{\rho}\hat{a}^\dagger\hat{a} - \hat{\rho}\hat{a}^{\dagger 2}\hat{a} - \hat{\rho}\hat{b}^\dagger\hat{a}^\dagger\hat{a} + \hat{\rho}\hat{a}\hat{a}^\dagger\hat{a} + \hat{\rho}\hat{b}\hat{a}^\dagger\hat{a}) \\ &+ \lambda Tr(\hat{a}\hat{b}\hat{\rho}\hat{a}^\dagger\hat{a} - \hat{a}^\dagger\hat{b}^\dagger\hat{\rho}\hat{a}^\dagger\hat{a} - \hat{\rho}\hat{a}\hat{b}\hat{a}^\dagger\hat{a} + \hat{\rho}\hat{a}^\dagger\hat{b}^\dagger\hat{a}^\dagger\hat{a}) \\ &+ \frac{k}{2}(\bar{n} + 1)Tr(2\hat{a}\hat{\rho}\hat{a}^{\dagger 2}\hat{a} - \hat{a}^\dagger\hat{a}\hat{\rho}\hat{a}^\dagger\hat{a} - \hat{\rho}\hat{a}^\dagger\hat{a}\hat{a}^\dagger\hat{a}) \\ &+ \frac{k}{2}\bar{n}Tr(2\hat{a}^\dagger\hat{\rho}\hat{a}\hat{a}^\dagger\hat{a} - \hat{a}\hat{a}^\dagger\hat{\rho}\hat{a}^\dagger\hat{a} - \hat{\rho}\hat{a}\hat{a}^{\dagger 2}\hat{a}) \\ &+ \frac{k}{2}(\bar{n} + 1)Tr(2\hat{b}\hat{\rho}\hat{b}^\dagger\hat{a}^\dagger\hat{a} - \hat{b}^\dagger\hat{b}\hat{\rho}\hat{a}^\dagger\hat{a} - \hat{\rho}\hat{b}^\dagger\hat{b}\hat{a}^\dagger\hat{a}) \\ &+ \frac{k}{2}\bar{n}Tr(2\hat{b}^\dagger\hat{\rho}\hat{b}\hat{a}^\dagger\hat{a} - \hat{b}\hat{b}^\dagger\hat{\rho}\hat{a}^\dagger\hat{a} - \hat{\rho}\hat{b}\hat{b}^\dagger\hat{a}^\dagger\hat{a}). \end{aligned} \quad (2.35)$$

or

$$\frac{d}{dt}\langle \hat{a}^\dagger(t)\hat{a}(t) \rangle = \varepsilon R_1 + \lambda R_2 + \frac{k}{2}(\bar{n} + 1)R_3 + \frac{k}{2}\bar{n}R_4 + \frac{k}{2}(\bar{n} + 1)R_5 + \frac{k}{2}\bar{n}R_6, \quad (2.36)$$

where

$$R_1 = Tr(\hat{a}^\dagger\hat{\rho}\hat{a}^\dagger\hat{a} + \hat{b}^\dagger\hat{\rho}\hat{a}^\dagger\hat{a} - \hat{a}\hat{\rho}\hat{a}^\dagger\hat{a} - \hat{b}\hat{\rho}\hat{a}^\dagger\hat{a} - \hat{\rho}\hat{a}^{\dagger 2}\hat{a} - \hat{\rho}\hat{b}^\dagger\hat{a}^\dagger\hat{a} + \hat{\rho}\hat{a}\hat{a}^\dagger\hat{a} + \hat{\rho}\hat{b}\hat{a}^\dagger\hat{a}),$$

$$R_2 = Tr(\hat{a}\hat{b}\hat{\rho}\hat{a}^\dagger\hat{a} - \hat{a}^\dagger\hat{b}^\dagger\hat{\rho}\hat{a}^\dagger\hat{a} - \hat{\rho}\hat{a}\hat{b}\hat{a}^\dagger\hat{a} + \hat{\rho}\hat{a}^\dagger\hat{b}^\dagger\hat{a}^\dagger\hat{a}),$$

$$R_3 = Tr(2\hat{a}\hat{\rho}\hat{a}^{\dagger 2}\hat{a} - \hat{a}^\dagger\hat{a}\hat{\rho}\hat{a}^\dagger\hat{a} - \hat{\rho}\hat{a}^\dagger\hat{a}\hat{a}^\dagger\hat{a}),$$

$$R_4 = Tr(2\hat{a}^\dagger\hat{\rho}\hat{a}\hat{a}^\dagger\hat{a} - \hat{a}\hat{a}^\dagger\hat{\rho}\hat{a}^\dagger\hat{a} - \hat{\rho}\hat{a}\hat{a}^{\dagger 2}\hat{a}),$$

$$R_5 = Tr(2\hat{b}\hat{\rho}\hat{b}^\dagger\hat{a}^\dagger\hat{a} - \hat{b}^\dagger\hat{b}\hat{\rho}\hat{a}^\dagger\hat{a} - \hat{\rho}\hat{b}^\dagger\hat{b}\hat{a}^\dagger\hat{a}),$$

and

$$R_6 = Tr(2\hat{b}^\dagger\hat{\rho}\hat{b}\hat{a}^\dagger\hat{a} - \hat{b}\hat{b}^\dagger\hat{\rho}\hat{a}^\dagger\hat{a} - \hat{\rho}\hat{b}\hat{b}^\dagger\hat{a}^\dagger\hat{a}).$$

We next evaluate the traces in Eq. (2.36)

$$R_1 = Tr(\hat{a}^\dagger\hat{\rho}\hat{a}^\dagger\hat{a} + \hat{b}^\dagger\hat{\rho}\hat{a}^\dagger\hat{a} - \hat{a}\hat{\rho}\hat{a}^\dagger\hat{a} - \hat{b}\hat{\rho}\hat{a}^\dagger\hat{a} - \hat{\rho}\hat{a}^{\dagger 2}\hat{a} - \hat{\rho}\hat{b}^\dagger\hat{a}^\dagger\hat{a} + \hat{\rho}\hat{a}\hat{a}^\dagger\hat{a} + \hat{\rho}\hat{b}\hat{a}^\dagger\hat{a})$$

$$\begin{aligned}
&= Tr(\hat{\rho}\hat{a}^\dagger\hat{a}\hat{a}^\dagger + \hat{\rho}\hat{a}^\dagger\hat{a}\hat{b}^\dagger - \hat{\rho}\hat{a}^\dagger\hat{a}^2 - \hat{\rho}\hat{a}^\dagger\hat{a}\hat{b} - \hat{\rho}\hat{a}^{\dagger 2}\hat{a} - \hat{\rho}\hat{b}^\dagger\hat{a}^\dagger\hat{a} + \hat{\rho}\hat{a}\hat{a}^\dagger\hat{a} + \hat{\rho}\hat{b}\hat{a}^\dagger\hat{a}) \\
&= Tr(\hat{\rho}\hat{a}^\dagger[\hat{a}, \hat{a}^\dagger] + \hat{\rho}[\hat{a}^\dagger, \hat{b}^\dagger]\hat{a} - \hat{\rho}[\hat{a}^\dagger, \hat{a}]\hat{a} - \hat{\rho}[\hat{a}^\dagger, \hat{b}]\hat{a}).
\end{aligned} \tag{2.37}$$

or

$$R_1 = Tr(\hat{\rho}\hat{a}^\dagger + \hat{\rho}\hat{a}) = \langle \hat{a}^\dagger \rangle + \langle \hat{a} \rangle. \tag{2.38}$$

$$\begin{aligned}
R_2 &= Tr(\hat{a}\hat{b}\hat{\rho}\hat{a}^\dagger\hat{a} - \hat{a}^\dagger\hat{b}^\dagger\hat{\rho}\hat{a}^\dagger\hat{a} - \hat{\rho}\hat{a}\hat{b}\hat{a}^\dagger\hat{a} + \hat{\rho}\hat{a}^\dagger\hat{b}^\dagger\hat{a}^\dagger\hat{a}) \\
&= Tr(\hat{\rho}\hat{a}^\dagger\hat{a}^2\hat{b} - \hat{\rho}\hat{a}^\dagger\hat{a}\hat{a}^\dagger\hat{b}^\dagger - \hat{\rho}\hat{a}\hat{b}\hat{a}^\dagger\hat{a} + \hat{\rho}\hat{a}^\dagger\hat{b}^\dagger\hat{a}^\dagger\hat{a}) \\
&= Tr(\hat{\rho}[\hat{a}^\dagger, \hat{a}]\hat{a}\hat{b} + \hat{\rho}\hat{a}^\dagger[\hat{a}^\dagger, \hat{a}]\hat{b}^\dagger),
\end{aligned} \tag{2.39}$$

or

$$R_2 = -\langle \hat{a}\hat{b} \rangle - \langle \hat{a}^\dagger\hat{b}^\dagger \rangle. \tag{2.40}$$

$$\begin{aligned}
R_3 &= Tr(2\hat{a}\hat{\rho}\hat{a}^{\dagger 2}\hat{a} - \hat{a}^\dagger\hat{a}\hat{\rho}\hat{a}^\dagger\hat{a} - \hat{\rho}\hat{a}^\dagger\hat{a}\hat{a}^\dagger\hat{a}) \\
&= Tr(2\hat{\rho}\hat{a}^{\dagger 2}\hat{a}^2 - \hat{\rho}\hat{a}^\dagger\hat{a}\hat{a}^\dagger\hat{a} - \hat{\rho}\hat{a}^\dagger\hat{a}\hat{a}^\dagger\hat{a}) \\
&= Tr(2\hat{\rho}\hat{a}^{\dagger 2}\hat{a}^2 - 2\hat{\rho}\hat{a}^\dagger\hat{a}\hat{a}^\dagger\hat{a}) \\
&= Tr(2\hat{\rho}\hat{a}^\dagger[\hat{a}^\dagger, \hat{a}]\hat{a})
\end{aligned}$$

or

$$R_3 = Tr(-2\hat{\rho}\hat{a}^\dagger\hat{a}) = -2\langle \hat{a}^\dagger\hat{a} \rangle. \tag{2.41}$$

Following similar procedure one can verify that

$$R_4 = 2\langle \hat{a}^\dagger\hat{a} \rangle + 2. \tag{2.42}$$

$$R_5 = 0. \tag{2.43}$$

and

$$R_6 = 0. \tag{2.44}$$

Substituting Eqs. (2.38), (2.40), (2.41), (2.42), (2.43) and (2.44) into (2.36), we obtain

$$\begin{aligned}
\frac{d}{dt}\langle \hat{a}^\dagger(t)\hat{a}(t) \rangle &= -k\langle \hat{a}^\dagger(t)\hat{a}(t) \rangle - \lambda(\langle \hat{a}(t)\hat{b}(t) \rangle + \langle \hat{a}^\dagger(t)\hat{b}^\dagger(t) \rangle) \\
&+ \varepsilon(\langle \hat{a}^\dagger(t) \rangle + \langle \hat{a}(t) \rangle) + k\bar{n}.
\end{aligned} \tag{2.45}$$

Furthermore, employing Eq. (2.7) along with Eq. (2.8), we see that

$$\begin{aligned}
\frac{d}{dt}\langle\hat{a}(t)\hat{a}^\dagger(t)\rangle &= \varepsilon Tr(\hat{a}^\dagger\hat{\rho}\hat{a}\hat{a}^\dagger + \hat{b}^\dagger\hat{\rho}\hat{a}\hat{a}^\dagger - \hat{a}\hat{\rho}\hat{a}\hat{a}^\dagger - \hat{b}\hat{\rho}\hat{a}\hat{a}^\dagger - \hat{\rho}\hat{a}^\dagger\hat{a}\hat{a}^\dagger - \hat{\rho}\hat{b}^\dagger\hat{a}\hat{a}^\dagger + \hat{\rho}\hat{a}^2\hat{a}^\dagger + \hat{\rho}\hat{b}\hat{a}\hat{a}^\dagger) \\
&+ \lambda Tr(\hat{a}\hat{b}\hat{\rho}\hat{a}\hat{a}^\dagger - \hat{a}^\dagger\hat{b}^\dagger\hat{\rho}\hat{a}\hat{a}^\dagger - \hat{\rho}\hat{a}\hat{b}\hat{a}\hat{a}^\dagger + \hat{\rho}\hat{a}^\dagger\hat{b}^\dagger\hat{a}\hat{a}^\dagger) \\
&+ \frac{k}{2}(\bar{n} + 1)Tr(2\hat{a}\hat{\rho}\hat{a}^\dagger\hat{a}\hat{a}^\dagger - \hat{a}^\dagger\hat{a}\hat{\rho}\hat{a}\hat{a}^\dagger - \hat{\rho}\hat{a}^\dagger\hat{a}^2\hat{a}^\dagger) \\
&+ \frac{k}{2}\bar{n}Tr(2\hat{a}^\dagger\hat{\rho}\hat{a}^2\hat{a}^\dagger - \hat{a}\hat{a}^\dagger\hat{\rho}\hat{a}\hat{a}^\dagger - \hat{\rho}\hat{a}\hat{a}^\dagger\hat{a}\hat{a}^\dagger) \\
&+ \frac{k}{2}(\bar{n} + 1)Tr(2\hat{b}\hat{\rho}\hat{b}^\dagger\hat{a}\hat{a}^\dagger - \hat{b}^\dagger\hat{b}\hat{\rho}\hat{a}\hat{a}^\dagger - \hat{\rho}\hat{b}^\dagger\hat{b}\hat{a}\hat{a}^\dagger) \\
&+ \frac{k}{2}\bar{n}Tr(2\hat{b}^\dagger\hat{\rho}\hat{b}\hat{a}\hat{a}^\dagger - \hat{b}\hat{b}^\dagger\hat{\rho}\hat{a}\hat{a}^\dagger - \hat{\rho}\hat{b}\hat{b}^\dagger\hat{a}\hat{a}^\dagger). \tag{2.46}
\end{aligned}$$

or

$$\frac{d}{dt}\langle(t)\hat{a}(t)\hat{a}^\dagger\rangle = \varepsilon N_1 + \lambda N_2 + \frac{k}{2}(\bar{n} + 1)N_3 + \frac{k}{2}\bar{n}N_4 + \frac{k}{2}(\bar{n} + 1)N_5 + \frac{k}{2}\bar{n}N_6, \tag{2.47}$$

where

$$N_1 = Tr(\hat{a}^\dagger\hat{\rho}\hat{a}\hat{a}^\dagger + \hat{b}^\dagger\hat{\rho}\hat{a}\hat{a}^\dagger - \hat{a}\hat{\rho}\hat{a}\hat{a}^\dagger - \hat{b}\hat{\rho}\hat{a}\hat{a}^\dagger - \hat{\rho}\hat{a}^\dagger\hat{a}\hat{a}^\dagger - \hat{\rho}\hat{b}^\dagger\hat{a}\hat{a}^\dagger + \hat{\rho}\hat{a}^2\hat{a}^\dagger + \hat{\rho}\hat{b}\hat{a}\hat{a}^\dagger),$$

$$N_2 = Tr(\hat{a}\hat{b}\hat{\rho}\hat{a}\hat{a}^\dagger - \hat{a}^\dagger\hat{b}^\dagger\hat{\rho}\hat{a}\hat{a}^\dagger - \hat{\rho}\hat{a}\hat{b}\hat{a}\hat{a}^\dagger + \hat{\rho}\hat{a}^\dagger\hat{b}^\dagger\hat{a}\hat{a}^\dagger),$$

$$N_3 = Tr(2\hat{a}\hat{\rho}\hat{a}^\dagger\hat{a}\hat{a}^\dagger - \hat{a}^\dagger\hat{a}\hat{\rho}\hat{a}\hat{a}^\dagger - \hat{\rho}\hat{a}^\dagger\hat{a}^2\hat{a}^\dagger),$$

$$N_4 = Tr(2\hat{a}^\dagger\hat{\rho}\hat{a}^2\hat{a}^\dagger - \hat{a}\hat{a}^\dagger\hat{\rho}\hat{a}\hat{a}^\dagger - \hat{\rho}\hat{a}\hat{a}^\dagger\hat{a}\hat{a}^\dagger),$$

$$N_5 = Tr(2\hat{b}\hat{\rho}\hat{b}^\dagger\hat{a}\hat{a}^\dagger - \hat{b}^\dagger\hat{b}\hat{\rho}\hat{a}\hat{a}^\dagger - \hat{\rho}\hat{b}^\dagger\hat{b}\hat{a}\hat{a}^\dagger),$$

and

$$N_6 = Tr(2\hat{b}^\dagger\hat{\rho}\hat{b}\hat{a}\hat{a}^\dagger - \hat{b}\hat{b}^\dagger\hat{\rho}\hat{a}\hat{a}^\dagger - \hat{\rho}\hat{b}\hat{b}^\dagger\hat{a}\hat{a}^\dagger).$$

Applying the cyclic property of the trace operation together with the commutation relation

$$\begin{aligned}
N_1 &= Tr(\hat{a}^\dagger\hat{\rho}\hat{a}\hat{a}^\dagger + \hat{b}^\dagger\hat{\rho}\hat{a}\hat{a}^\dagger - \hat{a}\hat{\rho}\hat{a}\hat{a}^\dagger - \hat{b}\hat{\rho}\hat{a}\hat{a}^\dagger - \hat{\rho}\hat{a}^\dagger\hat{a}\hat{a}^\dagger - \hat{\rho}\hat{b}^\dagger\hat{a}\hat{a}^\dagger + \hat{\rho}\hat{a}^2\hat{a}^\dagger + \hat{\rho}\hat{b}\hat{a}\hat{a}^\dagger), \\
&= Tr(\hat{\rho}\hat{a}\hat{a}^\dagger^2 + \hat{\rho}\hat{a}\hat{a}^\dagger\hat{b}^\dagger - \hat{\rho}\hat{a}\hat{a}^\dagger\hat{a} - \hat{\rho}\hat{a}\hat{a}^\dagger\hat{b} - \hat{\rho}\hat{a}^\dagger\hat{a}\hat{a}^\dagger - \hat{\rho}\hat{b}^\dagger\hat{a}\hat{a}^\dagger + \hat{\rho}\hat{a}^2\hat{a}^\dagger + \hat{\rho}\hat{b}\hat{a}\hat{a}^\dagger) \\
&= Tr(\hat{\rho}[\hat{a}, \hat{a}^\dagger]\hat{a}^\dagger - \hat{\rho}\hat{a}[\hat{a}^\dagger, \hat{a}]), \tag{2.48}
\end{aligned}$$

or

$$N_1 = \langle\hat{a}^\dagger\rangle + \langle\hat{a}\rangle. \tag{2.49}$$

In a similar manner, one can verify that

$$N_2 = -\langle\hat{a}\hat{b}\rangle - \langle\hat{a}^\dagger\hat{b}^\dagger\rangle. \tag{2.50}$$

$$N_3 = -2\langle\hat{a}\hat{a}^\dagger\rangle + 2. \tag{2.51}$$

$$N_4 = 2\langle \hat{a}\hat{a}^\dagger \rangle. \quad (2.52)$$

$$N_5 = 0. \quad (2.53)$$

and

$$N_6 = 0. \quad (2.54)$$

Now substitution of Eqs. (2.49) - (2.54) into (2.47) leads to

$$\begin{aligned} \frac{d}{dt}\langle \hat{a}(t)\hat{a}^\dagger(t) \rangle &= -k\langle \hat{a}(t)\hat{a}^\dagger(t) \rangle - \lambda(\langle \hat{a}(t)\hat{b}(t) \rangle + \langle \hat{a}^\dagger(t)\hat{b}^\dagger(t) \rangle) \\ &+ \varepsilon(\langle \hat{a}^\dagger(t) \rangle + \langle \hat{a}(t) \rangle) + k(\bar{n} + 1). \end{aligned} \quad (2.55)$$

Moreover, employing Eq. (2.7) along with Eq. (2.8), we see that

$$\begin{aligned} \frac{d}{dt}\langle \hat{a}(t)\hat{b}(t) \rangle &= \varepsilon Tr(\hat{a}^\dagger \hat{\rho} \hat{a} \hat{b} + \hat{b}^\dagger \hat{\rho} \hat{a} \hat{b} - \hat{a} \hat{\rho} \hat{a} \hat{b} - \hat{b} \hat{\rho} \hat{a} \hat{b} - \hat{\rho} \hat{a}^\dagger \hat{a} \hat{b} - \hat{\rho} \hat{b}^\dagger \hat{a} \hat{b} + \hat{\rho} \hat{a}^2 \hat{b} + \hat{\rho} \hat{b} \hat{a} \hat{b}) \\ &+ \lambda Tr(\hat{a} \hat{b} \hat{\rho} \hat{a} \hat{b} - \hat{a}^\dagger \hat{b}^\dagger \hat{\rho} \hat{a} \hat{b} - \hat{\rho} \hat{a} \hat{b} \hat{a} \hat{b} + \hat{\rho} \hat{a}^\dagger \hat{b}^\dagger \hat{a} \hat{b}) \\ &+ \frac{k}{2}(\bar{n} + 1) Tr(2\hat{a} \hat{\rho} \hat{a}^\dagger \hat{a} \hat{b} - \hat{a}^\dagger \hat{a} \hat{\rho} \hat{a} \hat{b} - \hat{\rho} \hat{a}^\dagger \hat{a}^2 \hat{b}) \\ &+ \frac{k}{2} \bar{n} Tr(2\hat{a}^\dagger \hat{\rho} \hat{a}^2 \hat{b} - \hat{a} \hat{a}^\dagger \hat{\rho} \hat{a} \hat{b} - \hat{\rho} \hat{a} \hat{a}^\dagger \hat{a} \hat{b}) \\ &+ \frac{k}{2}(\bar{n} + 1) Tr(2\hat{b} \hat{\rho} \hat{b}^\dagger \hat{a} \hat{b} - \hat{b}^\dagger \hat{b} \hat{\rho} \hat{a} \hat{b} - \hat{\rho} \hat{b}^\dagger \hat{b} \hat{a} \hat{b}) \\ &+ \frac{k}{2} \bar{n} Tr(2\hat{b}^\dagger \hat{\rho} \hat{b} \hat{a} \hat{b} - \hat{b} \hat{b}^\dagger \hat{\rho} \hat{a} \hat{b} - \hat{\rho} \hat{b} \hat{b}^\dagger \hat{a} \hat{b}). \end{aligned} \quad (2.56)$$

or

$$\frac{d}{dt}\langle \hat{a}(t)\hat{b}(t) \rangle = \varepsilon M_1 + \lambda M_2 + \frac{k}{2}(\bar{n} + 1)M_3 + \frac{k}{2} \bar{n} M_4 + \frac{k}{2}(\bar{n} + 1)M_5 + \frac{k}{2} \bar{n} M_6, \quad (2.57)$$

where

$$M_1 = Tr(\hat{a}^\dagger \hat{\rho} \hat{a} \hat{b} + \hat{b}^\dagger \hat{\rho} \hat{a} \hat{b} - \hat{a} \hat{\rho} \hat{a} \hat{b} - \hat{b} \hat{\rho} \hat{a} \hat{b} - \hat{\rho} \hat{a}^\dagger \hat{a} \hat{b} - \hat{\rho} \hat{b}^\dagger \hat{a} \hat{b} + \hat{\rho} \hat{a}^2 \hat{b} + \hat{\rho} \hat{b} \hat{a} \hat{b}),$$

$$M_2 = Tr(\hat{a} \hat{b} \hat{\rho} \hat{a} \hat{b} - \hat{a}^\dagger \hat{b}^\dagger \hat{\rho} \hat{a} \hat{b} - \hat{\rho} \hat{a} \hat{b} \hat{a} \hat{b} + \hat{\rho} \hat{a}^\dagger \hat{b}^\dagger \hat{a} \hat{b}),$$

$$M_3 = Tr(2\hat{a} \hat{\rho} \hat{a}^\dagger \hat{a} \hat{b} - \hat{a}^\dagger \hat{a} \hat{\rho} \hat{a} \hat{b} - \hat{\rho} \hat{a}^\dagger \hat{a}^2 \hat{b}),$$

$$M_4 = \frac{k}{2} \bar{n} Tr(2\hat{a}^\dagger \hat{\rho} \hat{a}^2 \hat{b} - \hat{a} \hat{a}^\dagger \hat{\rho} \hat{a} \hat{b} - \hat{\rho} \hat{a} \hat{a}^\dagger \hat{a} \hat{b}),$$

$$M_5 = Tr(2\hat{b} \hat{\rho} \hat{b}^\dagger \hat{a} \hat{b} - \hat{b}^\dagger \hat{b} \hat{\rho} \hat{a} \hat{b} - \hat{\rho} \hat{b}^\dagger \hat{b} \hat{a} \hat{b}),$$

and

$$M_6 = Tr(2\hat{b}^\dagger \hat{\rho} \hat{b} \hat{a} \hat{b} - \hat{b} \hat{b}^\dagger \hat{\rho} \hat{a} \hat{b} - \hat{\rho} \hat{b} \hat{b}^\dagger \hat{a} \hat{b}).$$

Now we evaluate the traces in Eq. (2.57) applying the trace's cyclic property and commutation relation given Eqs. (2.11) - (2.14)

$$M_1 = Tr(\hat{a}^\dagger \hat{\rho} \hat{a} \hat{b} + \hat{b}^\dagger \hat{\rho} \hat{a} \hat{b} - \hat{a} \hat{\rho} \hat{a} \hat{b} - \hat{b} \hat{\rho} \hat{a} \hat{b} - \hat{\rho} \hat{a}^\dagger \hat{a} \hat{b} - \hat{\rho} \hat{b}^\dagger \hat{a} \hat{b} + \hat{\rho} \hat{a}^2 \hat{b} + \hat{\rho} \hat{b} \hat{a} \hat{b})$$

$$\begin{aligned}
&= Tr(\hat{\rho}\hat{a}\hat{b}\hat{a}^\dagger + \hat{\rho}\hat{a}\hat{b}\hat{b}^\dagger - \hat{\rho}\hat{a}\hat{b}\hat{a} - \hat{\rho}\hat{a}\hat{b}^2 - \hat{\rho}\hat{a}^\dagger\hat{a}\hat{b} - \hat{\rho}\hat{b}^\dagger\hat{a}\hat{b} + \hat{\rho}\hat{a}^2\hat{b} + \hat{\rho}\hat{b}\hat{a}\hat{b}) \\
&= Tr(\hat{\rho}\hat{a}\hat{a}^\dagger\hat{b} - \hat{\rho}\hat{a}^\dagger\hat{a}\hat{b} + \hat{\rho}\hat{a}\hat{b}\hat{b}^\dagger - \hat{\rho}\hat{a}\hat{b}^\dagger\hat{b}) \\
&= Tr(\hat{\rho}[\hat{a}, \hat{a}^\dagger]\hat{b} + \hat{\rho}\hat{a}[\hat{b}, \hat{b}^\dagger]),
\end{aligned}$$

or

$$\begin{aligned}
M_1 &= Tr(\hat{\rho}\hat{b} + \hat{\rho}\hat{a}) = \langle \hat{b} \rangle + \langle \hat{a} \rangle. \tag{2.58} \\
M_2 &= Tr(\hat{a}\hat{b}\hat{\rho}\hat{a}\hat{b} - \hat{a}^\dagger\hat{b}^\dagger\hat{\rho}\hat{a}\hat{b} - \hat{\rho}\hat{a}\hat{b}\hat{a}\hat{b} + \hat{\rho}\hat{a}^\dagger\hat{b}^\dagger\hat{a}\hat{b}) \\
&= Tr(\hat{\rho}\hat{a}\hat{b}\hat{a}\hat{b} - \hat{\rho}\hat{a}\hat{b}\hat{a}^\dagger\hat{b}^\dagger - \hat{\rho}\hat{a}\hat{b}\hat{a}\hat{b} + \hat{\rho}\hat{a}^\dagger\hat{b}^\dagger\hat{a}\hat{b}) \\
&= Tr(-\hat{\rho}\hat{a}\hat{b}\hat{a}^\dagger\hat{b}^\dagger + \hat{\rho}\hat{a}^\dagger\hat{b}^\dagger\hat{a}\hat{b}) \\
&= Tr(\hat{\rho}\hat{a}^\dagger\hat{a}\hat{b}^\dagger\hat{b} - \hat{\rho}(\hat{a}^\dagger\hat{a} + 1)\hat{b}\hat{b}^\dagger) \\
&= Tr(\hat{\rho}\hat{a}^\dagger\hat{a}\hat{b}^\dagger\hat{b} - \hat{\rho}\hat{a}^\dagger\hat{a}\hat{b}\hat{b}^\dagger - \hat{\rho}\hat{b}\hat{b}^\dagger) \\
&= Tr(\hat{\rho}\hat{a}^\dagger\hat{a}[\hat{b}^\dagger, \hat{b}] - \hat{\rho}\hat{b}\hat{b}^\dagger), \\
&= Tr(-\hat{\rho}\hat{a}^\dagger\hat{a} - \hat{\rho}\hat{b}\hat{b}^\dagger) = Tr(-\hat{\rho}\hat{a}^\dagger\hat{a} - \hat{\rho}(\hat{b}^\dagger\hat{b} + 1))
\end{aligned}$$

or

$$\begin{aligned}
M_2 &= -\langle \hat{a}^\dagger\hat{a} \rangle - \langle \hat{b}^\dagger\hat{b} \rangle - 1. \tag{2.59} \\
M_3 &= Tr(2\hat{a}\hat{\rho}\hat{a}^\dagger\hat{a}\hat{b} - \hat{a}^\dagger\hat{a}\hat{\rho}\hat{a}\hat{b} - \hat{\rho}\hat{a}^\dagger\hat{a}^2\hat{b}) \\
&= Tr(2\hat{\rho}\hat{a}^\dagger\hat{a}\hat{b}\hat{a} - \hat{\rho}\hat{a}\hat{b}\hat{a}^\dagger\hat{a} - \hat{\rho}\hat{a}^\dagger\hat{a}^2\hat{b}) \\
&= Tr(\hat{\rho}\hat{a}^\dagger\hat{a}^2\hat{b} - \hat{\rho}\hat{a}\hat{a}^\dagger\hat{a}\hat{b}) \\
&= Tr(\hat{\rho}[\hat{a}^\dagger, \hat{a}]\hat{a}\hat{b}) = -Tr(\hat{\rho}\hat{a}\hat{b}) = -\langle \hat{a}\hat{b} \rangle. \tag{2.60}
\end{aligned}$$

Following a similar procedure, one can verify that

$$M_4 = \langle \hat{a}\hat{b} \rangle. \tag{2.61}$$

$$M_5 = -\langle \hat{a}\hat{b} \rangle. \tag{2.62}$$

and

$$M_6 = \langle \hat{a}\hat{b} \rangle. \tag{2.63}$$

Now substituting Eqs. (2.58) - (2.63) into (2.57), we get

$$\begin{aligned}
\frac{d}{dt}\langle \hat{a}(t)\hat{b}(t) \rangle &= -k\langle \hat{a}(t)\hat{b}(t) \rangle - \lambda\langle \hat{a}^\dagger(t)\hat{a}(t) \rangle - \lambda\langle \hat{b}^\dagger(t)\hat{b}(t) \rangle \\
&+ \varepsilon(\langle \hat{b}(t) \rangle + \langle \hat{a}(t) \rangle) - \lambda. \tag{2.64}
\end{aligned}$$

Following a similar procedure, one can readily obtain

$$\frac{d}{dt}\langle \hat{a}^\dagger(t)\hat{b}(t) \rangle = -k\langle \hat{a}^\dagger(t)\hat{b}(t) \rangle - \lambda(\langle \hat{b}^2(t) \rangle + \langle \hat{a}^{\dagger 2}(t) \rangle) + \varepsilon(\langle \hat{a}^\dagger(t) \rangle + \langle \hat{b}(t) \rangle), \tag{2.65}$$

$$\frac{d}{dt}\langle\hat{b}(t)\rangle = -\frac{k}{2}\langle\hat{b}(t)\rangle - \lambda\langle\hat{a}^\dagger(t)\rangle + \varepsilon, \quad (2.66)$$

$$\frac{d}{dt}\langle\hat{b}^\dagger(t)\rangle = -\frac{k}{2}\langle\hat{b}^\dagger(t)\rangle - \lambda\langle\hat{a}(t)\rangle + \varepsilon, \quad (2.67)$$

$$\frac{d}{dt}\langle\hat{b}^2(t)\rangle = -k\langle\hat{b}^2(t)\rangle - 2\lambda\langle\hat{b}(t)\hat{a}^\dagger(t)\rangle + 2\varepsilon\langle\hat{b}(t)\rangle, \quad (2.68)$$

$$\begin{aligned} \frac{d}{dt}\langle\hat{b}^\dagger(t)\hat{b}(t)\rangle &= -k\langle\hat{b}^\dagger(t)\hat{b}(t)\rangle - \lambda(\langle\hat{a}(t)\hat{b}(t)\rangle + \langle\hat{a}^\dagger(t)\hat{b}^\dagger(t)\rangle) \\ &+ \varepsilon(\langle\hat{b}^\dagger(t)\rangle + \langle\hat{b}(t)\rangle) + k\bar{n}, \end{aligned} \quad (2.69)$$

$$\begin{aligned} \frac{d}{dt}\langle\hat{b}(t)\hat{b}^\dagger(t)\rangle &= -k\langle\hat{b}(t)\hat{b}^\dagger(t)\rangle - \lambda(\langle\hat{a}(t)\hat{b}(t)\rangle + \langle\hat{a}^\dagger(t)\hat{b}^\dagger(t)\rangle) \\ &+ \varepsilon(\langle\hat{b}^\dagger(t)\rangle + \langle\hat{b}(t)\rangle) + k(\bar{n} + 1). \end{aligned} \quad (2.70)$$

Replacing  $\hat{a}$  by  $\alpha$ ,  $\hat{a}^\dagger$  by  $\alpha^*$ ,  $\hat{b}$  by  $\beta$  and  $\hat{b}^\dagger$  by  $\beta^*$ , the c-number equations corresponding to Eqs. (2.22), (2.23), (2.34), (2.45), (2.55), (2.64),(2.65), (2.66), (2.67), (2.68), (2.69) and (2.70), in the normal order can be written as

$$\frac{d}{dt}\langle\alpha(t)\rangle = -\frac{k}{2}\langle\alpha(t)\rangle - \lambda\langle\beta^*(t)\rangle + \varepsilon, \quad (2.71)$$

$$\frac{d}{dt}\langle\alpha^*(t)\rangle = -\frac{k}{2}\langle\alpha^*(t)\rangle - \lambda\langle\beta(t)\rangle + \varepsilon, \quad (2.72)$$

$$\frac{d}{dt}\langle\alpha^2(t)\rangle = -k\langle\alpha^2(t)\rangle - 2\lambda\langle\alpha(t)\beta^*(t)\rangle + 2\varepsilon\langle\alpha(t)\rangle, \quad (2.73)$$

$$\begin{aligned} \frac{d}{dt}\langle\alpha^*(t)\alpha(t)\rangle &= -k\langle\alpha^*(t)\alpha(t)\rangle - \lambda(\langle\alpha(t)\beta(t)\rangle + \langle\alpha^*(t)\beta^*(t)\rangle) \\ &+ \varepsilon(\langle\alpha^*(t)\rangle + \langle\alpha(t)\rangle) + k\bar{n}, \end{aligned} \quad (2.74)$$

$$\begin{aligned} \frac{d}{dt}\langle\alpha(t)\beta(t)\rangle &= -k\langle\alpha(t)\beta(t)\rangle - \lambda\langle\alpha^*(t)\alpha(t)\rangle - \lambda\langle\beta^*(t)\beta(t)\rangle \\ &+ \varepsilon(\langle\beta(t)\rangle + \langle\alpha(t)\rangle) - \lambda, \end{aligned} \quad (2.75)$$

$$\frac{d}{dt}\langle\alpha^*(t)\beta(t)\rangle = -k\langle\alpha^*(t)\beta(t)\rangle - \lambda(\langle\beta^2(t)\rangle + \langle\alpha^{*2}(t)\rangle) + \varepsilon(\langle\alpha^*(t)\rangle + \langle\beta(t)\rangle), \quad (2.76)$$

$$\frac{d}{dt}\langle\beta(t)\rangle = -\frac{k}{2}\langle\beta(t)\rangle - \lambda\langle\alpha^*(t)\rangle + \varepsilon, \quad (2.77)$$

$$\frac{d}{dt}\langle\beta^*(t)\rangle = -\frac{k}{2}\langle\beta^*(t)\rangle - \lambda\langle\alpha(t)\rangle + \varepsilon, \quad (2.78)$$

$$\frac{d}{dt}\langle\beta^2(t)\rangle = -k\langle\beta^2(t)\rangle - 2\lambda\langle\beta(t)\alpha^*(t)\rangle + 2\varepsilon\langle\beta(t)\rangle, \quad (2.79)$$

and

$$\begin{aligned} \frac{d}{dt}\langle\beta^*(t)\beta(t)\rangle &= -k\langle\beta^*(t)\beta(t)\rangle - \lambda(\langle\alpha(t)\beta(t)\rangle + \langle\alpha^*(t)\beta^*(t)\rangle) \\ &+ \varepsilon(\langle\beta^*(t)\rangle + \langle\beta(t)\rangle) + k\bar{n}. \end{aligned} \quad (2.80)$$

We claim that the equation of evolution of  $\alpha(t)$  (c-number Langevin equations) can be obtained from that of  $\langle\alpha(t)\rangle$ . This can be achieved by dropping the angular brackets in Eq. (2.71), and adding the noise forces  $f_\alpha(t)$ , so that the c-number Langevin equations for the cavity modes can be written as

$$\frac{d}{dt}\alpha(t) = -\frac{k}{2}\alpha(t) - \lambda\beta^*(t) + \varepsilon + f_\alpha(t). \quad (2.81)$$

Following similar procedure one can verify that

$$\frac{d}{dt}\alpha^*(t) = -\frac{k}{2}\alpha^*(t) - \lambda\beta(t) + \varepsilon + f_\alpha^*(t), \quad (2.82)$$

$$\frac{d}{dt}\beta(t) = -\frac{k}{2}\beta(t) - \lambda\alpha^*(t) + \varepsilon + f_\beta(t), \quad (2.83)$$

$$\frac{d}{dt}\beta^*(t) = -\frac{k}{2}\beta^*(t) - \lambda\alpha(t) + \varepsilon + f_\beta^*(t), \quad (2.84)$$

where  $f_\alpha(t)$  and  $f_\beta(t)$  are noise forces the properties of which remain to be determined.

Now we seek to determine the correlation properties of the noise forces  $f_\alpha(t)$  and  $f_\beta(t)$ . Upon comparing Eq. (2.71) and the expectation value of Eq. (2.81), we notice that

$$\langle f_\alpha(t) \rangle = 0. \quad (2.85)$$

In addition, comparing Eq. (2.77) and the expectation value of Eq. (2.83), we also see that

$$\langle f_\beta(t) \rangle = 0. \quad (2.86)$$

In the same manner, one can verify that

$$\langle f_\alpha^*(t) \rangle = \langle f_\beta^*(t) \rangle = 0. \quad (2.87)$$

Now substituting Eq. (2.81) into the relation

$$\frac{d}{dt}\langle\alpha^2(t)\rangle = \left\langle\alpha(t)\frac{d\alpha(t)}{dt}\right\rangle + \left\langle\frac{d\alpha(t)}{dt}\alpha(t)\right\rangle, \quad (2.88)$$

we obtain

$$\frac{d}{dt}\langle\alpha^2(t)\rangle = -k\langle\alpha^2(t)\rangle - 2\lambda\langle\alpha(t)\beta^*(t)\rangle + 2\varepsilon\langle\alpha(t)\rangle + \langle\alpha(t)f_\alpha(t)\rangle + \langle f_\alpha(t)\alpha(t)\rangle. \quad (2.89)$$

Comparison of Eqs. (2.73) and (2.89) indicates that

$$\langle\alpha(t)f_\alpha(t)\rangle + \langle f_\alpha(t)\alpha(t)\rangle = 0. \quad (2.90)$$

A formal solution of Eq. (2.81) can be written as

$$\alpha(t) = \alpha(0)e^{-\frac{kt}{2}} + \int_0^t e^{-\frac{k}{2}(t-t')} \left( f_\alpha(t') + \varepsilon - \lambda\beta^*(t') \right) dt'. \quad (2.91)$$

Multiplying both side Eq. (2.91) by  $f_\alpha(t)$  from the right side and taking the expectation value of the resulting expression, we have

$$\begin{aligned} \langle \alpha(t) f_\alpha(t) \rangle &= \langle \alpha(0) f_\alpha(t) \rangle e^{-\frac{kt}{2}} \\ &+ \int_0^t e^{-\frac{k}{2}(t-t')} \left( \langle f_\alpha(t') f_\alpha(t) \rangle + \varepsilon \langle f_\alpha(t) \rangle - \lambda \langle \beta^*(t') f_\alpha(t) \rangle \right) dt'. \end{aligned} \quad (2.92)$$

Since a noise force at some time  $t$  has no effect on system variables at time [2], we can write

$$\langle \alpha(0) f_\alpha(t) \rangle = \langle \alpha(0) \rangle \langle f_\alpha(t) \rangle, \quad (2.93)$$

$$\langle \beta^*(t') f_\alpha(t) \rangle = \langle \beta^*(t') \rangle \langle f_\alpha(t) \rangle, \quad (2.94)$$

so that taking into account Eq. (2.85), we see that

$$\langle \alpha(0) f_\alpha(t) \rangle = 0 \quad (2.95)$$

$$\langle \beta^*(t') f_\alpha(t) \rangle = 0 \quad (2.96)$$

Applying Eqs. (2.95) and (2.96) in (2.92), we obtain

$$\langle \alpha(t) f_\alpha(t) \rangle = \int_0^t e^{-\frac{k}{2}(t-t')} \langle f_\alpha(t') f_\alpha(t) \rangle dt'. \quad (2.97)$$

In a similar manner, we can verify that

$$\langle f_\alpha(t) \alpha(t) \rangle = \int_0^t e^{-\frac{k}{2}(t-t')} \langle f_\alpha(t) f_\alpha(t') \rangle dt'. \quad (2.98)$$

Now adding Eqs. (2.97) and (2.98), we get

$$\langle \alpha(t) f_\alpha(t) \rangle + \langle f_\alpha(t) \alpha(t) \rangle = \int_0^t e^{-\frac{k}{2}(t-t')} \left( \langle f_\alpha(t') f_\alpha(t) \rangle + \langle f_\alpha(t) f_\alpha(t') \rangle \right) dt'. \quad (2.99)$$

Comparison of Eqs. (2.90) and (2.99) shows that

$$\int_0^t e^{-\frac{k}{2}(t-t')} \left( \langle f_\alpha(t') f_\alpha(t) \rangle + \langle f_\alpha(t) f_\alpha(t') \rangle \right) dt' = 0, \quad (2.100)$$

and upon assuming

$$\langle f_\alpha(t') f_\alpha(t) \rangle = \langle f_\alpha(t) f_\alpha(t') \rangle, \quad (2.101)$$

we obtain

$$\int_0^t e^{-\frac{k}{2}(t-t')} \langle f_\alpha(t') f_\alpha(t) \rangle dt' = 0. \quad (2.102)$$

On the basis of the relation [2]

$$\int_0^t e^{-\frac{b}{2}(t-t')} \langle f^\dagger(t') f(t) \rangle dt' = M, \quad (2.103)$$

we assert that

$$\langle f^\dagger(t') f(t) \rangle = 2M \delta(t - t'). \quad (2.104)$$

where  $f(t)$  is noise force,  $b$  and  $M$  are constants. With the aid of this relation, Eq. (2.100), becomes

$$\langle f_\alpha(t')f_\alpha(t) \rangle = \langle f_\alpha(t)f_\alpha(t') \rangle = 0. \quad (2.105)$$

Moreover, employing Eqs. (2.81) and (2.82) in the relation

$$\frac{d}{dt}\langle \alpha^*(t)\alpha(t) \rangle = \left\langle \alpha^*(t)\frac{d}{dt}\alpha(t) \right\rangle + \left\langle \frac{d}{dt}\alpha^*(t)\alpha(t) \right\rangle, \quad (2.106)$$

one can easily establish that

$$\begin{aligned} \frac{d}{dt}\langle \alpha^*(t)\alpha(t) \rangle &= -k\langle \alpha^*(t)\alpha(t) \rangle - \lambda(\langle \alpha(t)\beta(t) \rangle + \langle \alpha^*(t)\beta^*(t) \rangle) \\ &+ \varepsilon(\langle \alpha^*(t) \rangle + \langle \alpha(t) \rangle) + \langle \alpha^*(t)f_\alpha(t) \rangle + \langle f_\alpha^*(t)\alpha(t) \rangle. \end{aligned} \quad (2.107)$$

Comparison of Eqs. (2.74) and (2.107) shows that

$$\langle \alpha^*(t)f_\alpha(t) \rangle + \langle f_\alpha^*(t)\alpha(t) \rangle = k\bar{n}. \quad (2.108)$$

A formal solution of Eq. (2.82) can be written as

$$\alpha^*(t) = \alpha^*(0)e^{-\frac{kt}{2}} + \int_0^t e^{-\frac{k}{2}(t-t')} \left( f_\alpha^*(t') + \varepsilon - \lambda\beta \right) dt'. \quad (2.109)$$

Multiplying both side of Eq. (2.109) by  $f_\alpha(t)$  from the right side and taking the expectation value of the resulting expression, we get

$$\begin{aligned} \langle \alpha^*(t)f_\alpha(t) \rangle &= \langle \alpha^*(0)f_\alpha(t) \rangle e^{-\frac{kt}{2}} \\ &+ \int_0^t e^{-\frac{k}{2}(t-t')} \left( \langle f_\alpha^*(t')f_\alpha(t) \rangle + \varepsilon\langle f_\alpha(t) \rangle - \lambda\langle \beta f_\alpha(t) \rangle \right) dt'. \end{aligned} \quad (2.110)$$

With the aid of the fact that a noise force at a certain time should not affect the system variables at earlier time, one obtains

$$\langle \alpha^*(t)f_\alpha(t) \rangle = \int_0^t e^{-\frac{k}{2}(t-t')} \langle f_\alpha^*(t')f_\alpha(t) \rangle dt'. \quad (2.111)$$

Multiplying both side of Eq. (2.91) from the left by  $f_\alpha^*(t)$  and taking the expectation value of the resulting expression and on account of the assertion that a noise operator at a certain time should not affect the system variables at earlier time, we have

$$\langle f_\alpha^*(t)\alpha(t) \rangle = \int_0^t e^{-\frac{k}{2}(t-t')} \langle f_\alpha^*(t)f_\alpha(t') \rangle dt'. \quad (2.112)$$

Adding Eqs. (2.111) and (2.112), we see that

$$\langle \alpha^*(t)f_\alpha(t) \rangle + \langle f_\alpha^*(t)\alpha(t) \rangle = \int_0^t e^{-\frac{k}{2}(t-t')} \left( \langle f_\alpha^*(t')f_\alpha(t) \rangle + \langle f_\alpha^*(t)f_\alpha(t') \rangle \right) dt'. \quad (2.113)$$

In view of Eq. (2.108), we can write Eq. (2.113) as

$$\int_0^t e^{-\frac{k}{2}(t-t')} \left( \langle f_\alpha^*(t') f_\alpha(t) \rangle + \langle f_\alpha^*(t) f_\alpha(t') \rangle \right) dt' = k\bar{n}. \quad (2.114)$$

Assuming that

$$\langle f_\alpha^*(t') f_\alpha(t) \rangle = \langle f_\alpha^*(t) f_\alpha(t') \rangle, \quad (2.115)$$

and taking into account Eqs. (2.103) and (2.104), one easily gets

$$\langle f_\alpha^*(t') f_\alpha(t) \rangle = \langle f_\alpha^*(t) f_\alpha(t') \rangle = k\bar{n}\delta(t-t') \quad (2.116)$$

Moreover, substituting eqs. (2.81) and (2.83) in the relation

$$\frac{d}{dt} \langle \alpha(t) \beta(t) \rangle = \left\langle \alpha(t) \frac{d}{dt} \beta(t) \right\rangle + \left\langle \frac{d\alpha(t)}{dt} \beta(t) \right\rangle, \quad (2.117)$$

we obtain

$$\begin{aligned} \frac{d}{dt} \langle \alpha(t) \beta(t) \rangle &= -k \langle \alpha(t) \beta(t) \rangle - \lambda \langle \alpha(t) \alpha^*(t) \rangle + \varepsilon \langle \alpha(t) \rangle + \varepsilon \beta(t) \\ &\quad - \lambda \langle \beta^*(t) \beta(t) \rangle + \langle \alpha(t) f_\beta(t) \rangle + \langle f_\alpha(t) \beta(t) \rangle. \end{aligned} \quad (2.118)$$

Upon comparing Eqs. (2.75) and (2.118), we see that

$$\langle \alpha(t) f_\beta(t) \rangle + \langle f_\alpha(t) \beta(t) \rangle = -\lambda. \quad (2.119)$$

Furthermore, multiplying both side of Eq. (2.91) by  $f_\beta(t)$  from the right and taking the expectation value of the resulting expression, we have

$$\begin{aligned} \langle \alpha(t) f_\beta(t) \rangle &= \langle \alpha(0) f_\beta(t) \rangle e^{-\frac{kt}{2}} \\ &\quad + \int_0^t e^{-\frac{k}{2}(t-t')} \left( \langle f_\alpha(t') f_\beta(t) \rangle + \varepsilon \langle f_\beta(t) \rangle - \lambda \langle \beta^* f_\beta(t) \rangle \right) dt'. \end{aligned} \quad (2.120)$$

On account of the assertion that a noise force at some time  $t$  should not affect system variables at earlier times, we can write

$$\langle \alpha(0) f_\beta(t) \rangle = \langle \alpha(0) \rangle \langle f_\beta(t) \rangle = 0, \quad (2.121)$$

and

$$\langle \beta^*(t) f_\beta(t) \rangle = \langle \beta^*(t) \rangle \langle f_\beta(t) \rangle = 0. \quad (2.122)$$

In view of these results, we can write Eq. (2.120) as

$$\langle \alpha(t) f_\beta(t) \rangle = \int_0^t e^{-\frac{k}{2}(t-t')} \langle f_\alpha(t') f_\beta(t) \rangle dt'. \quad (2.123)$$

It can also be verified in a similar manner

$$\langle f_\alpha(t) \beta(t) \rangle = \int_0^t e^{-\frac{k}{2}(t-t')} \langle f_\alpha(t) f_\beta(t') \rangle dt' \quad (2.124)$$

Adding Eqs. (2.123) and (2.124), we have

$$\langle \alpha(t)f_\beta(t) \rangle + \langle f_\alpha(t)\beta(t) \rangle = \int_0^t e^{-\frac{k}{2}(t-t')} \left( \langle f_\alpha(t')f_\beta(t) + f_\alpha(t)f_\beta(t') \rangle \right) dt'. \quad (2.125)$$

In view of Eqs. (2.119) and (2.125), we see that

$$\int_0^t e^{-\frac{k}{2}(t-t')} \left( \langle f_\alpha(t')f_\beta(t) + f_\alpha(t)f_\beta(t') \rangle \right) dt' = -\lambda. \quad (2.126)$$

Upon assuming

$$\langle f_\alpha(t')f_\beta(t) \rangle = \langle f_\alpha(t)f_\beta(t') \rangle, \quad (2.127)$$

and in view of Eqs. (2.103) and (2.104), we can put Eq. (2.126) in the form

$$\langle f_\alpha(t')f_\beta(t) \rangle = \langle f_\alpha(t)f_\beta(t') \rangle = -\lambda\delta(t-t'). \quad (2.128)$$

Following similar procedure, one can establish that

$$\langle \beta(t)f_\beta(t) \rangle + \langle f_\beta(t)\beta(t) \rangle = 0. \quad (2.129)$$

$$\langle \beta^*(t)f_\beta(t) \rangle + \langle f_\beta^*(t)\beta(t) \rangle = k\bar{n}. \quad (2.130)$$

Furthermore, it in similar manner

$$\langle f_\beta(t')f_\beta(t) \rangle = \langle f_\beta(t)f_\beta(t') \rangle = 0. \quad (2.131)$$

$$\langle f_\beta^*(t')f_\beta(t) \rangle = \langle f_\beta^*(t)f_\beta(t') \rangle = k\bar{n}\delta(t-t'). \quad (2.132)$$

In addition, we find

$$\langle f_\alpha^*(t')f_\beta^*(t) \rangle = \langle f_\alpha^*(t)f_\beta^*(t') \rangle = \langle f_\beta^*(t)f_\alpha^*(t') \rangle = \langle f_\beta^*(t')f_\alpha^*(t) \rangle = -\lambda\delta(t-t') \quad (2.133)$$

$$\langle f_\alpha(t')f_\beta^*(t) \rangle = \langle f_\alpha(t)f_\beta^*(t') \rangle = \langle f_\beta^*(t)f_\alpha(t') \rangle = \langle f_\beta^*(t')f_\alpha(t) \rangle = 0 \quad (2.134)$$

$$\langle f_\alpha^*(t')f_\alpha(t) \rangle = \langle f_\alpha^*(t)f_\alpha(t') \rangle = \langle f_\beta^*(t')f_\beta^*(t) \rangle = \langle f_\beta^*(t)f_\beta^*(t') \rangle = 0 \quad (2.135)$$

We would like to point out that Eqs. (2.105), (2.116), (2.128), (2.132), (2.133), (2.134) and (2.135) describe the Correlation properties of the noise forces  $f_\alpha(t)$  and  $f_\beta(t)$  associated with the normal ordering.

We then introduce new variable  $z_\pm(t) = \alpha(t) \pm \beta^*(t)$ . Adding Eq. (2.81), along with the complex conjugate of Eq. (2.83), we have

$$\frac{d}{dt} \left( \alpha(t) + \beta^*(t) \right) = \frac{-k}{2} \left( \alpha(t) + \beta^*(t) \right) - \lambda \left( \alpha(t) + \beta^*(t) \right) + 2\varepsilon + f_\alpha(t) + f_\beta^*(t), \quad (2.136)$$

or

$$\frac{d}{dt} z_+(t) = \frac{-k}{2} z_+(t) - \lambda z_+(t) + 2\varepsilon + f_\alpha(t) + f_\beta^*(t), \quad (2.137)$$

where

$$z_+(t) = \alpha(t) + \beta^*(t). \quad (2.138)$$

Furthermore, subtracting Eq. (2.81) from Eq. (2.83), we see that

$$\frac{d}{dt}(\alpha(t) - \beta^*(t)) = \frac{-k}{2}(\alpha(t) - \beta^*(t)) - \lambda(\alpha(t) - \beta^*(t)) + f_\alpha(t) - f_\beta^*(t), \quad (2.139)$$

or

$$\frac{d}{dt}z_-(t) = \frac{-k}{2}z_-(t) - \lambda z_-(t) + f_\alpha(t) - f_\beta^*(t), \quad (2.140)$$

where

$$z_-(t) = \alpha(t) - \beta^*(t). \quad (2.141)$$

Combining Eqs. (2.137) and (2.140), we get

$$\frac{d}{dt}z_\pm(t) = \frac{-1}{2}\lambda_\pm z_\pm(t) + \varepsilon \pm \varepsilon + f_\alpha(t) \pm f_\beta^*(t), \quad (2.142)$$

where

$$\lambda_\pm = k \pm 2\lambda, \quad (2.143)$$

and

$$z_\pm(t) = \alpha(t) \pm \beta^*(t). \quad (2.144)$$

In view of Eq. (2.142) together with Eq. (2.143), the solution of  $z_-(t)$  does not have a well-behaved solution for  $k < 2\lambda$ , we then identify  $k = 2\lambda$  as the threshold condition. For  $k > 2\lambda$ , the solution of Eq. (2.142) can be put in the form

$$z_\pm(t) = z_\pm(0)e^{-\lambda_\pm t/2} + \int_0^t e^{-\lambda_\pm(t-t')/2}[\varepsilon \pm \varepsilon + f_\alpha(t') \pm f_\beta^*(t')]dt'. \quad (2.145)$$

Now with the aid of Eqs. (2.144) and (2.145), we have

$$\alpha(t) + \beta^*(t) = (\alpha(0) + \beta^*(0))e^{-\lambda_+ t/2} + \int_0^t e^{-\lambda_+(t-t')/2}[2\varepsilon + f_\alpha(t') + f_\beta^*(t')]dt'. \quad (2.146)$$

$$\alpha(t) - \beta^*(t) = (\alpha(0) - \beta^*(0))e^{-\lambda_- t/2} + \int_0^t e^{-\lambda_-(t-t')/2}[f_\alpha(t') - f_\beta^*(t')]dt'. \quad (2.147)$$

Employing Eqs. (2.146) and (2.147), we obtain

$$\alpha(t) = E_+(t)\alpha(0) + E_-(t)\beta^*(0) + F_+(t) + F_-(t), \quad (2.148)$$

$$\beta(t) = E_+(t)\beta(0) + E_-(t)\alpha^*(0) + F_+^*(t) - F_-^*(t), \quad (2.149)$$

where

$$E_\pm(t) = \frac{1}{2}(e^{-\lambda_+ t/2} \pm e^{-\lambda_- t/2}), \quad (2.150)$$

and

$$F_\pm(t) = \frac{1}{2} \int_0^t e^{-\lambda_\pm(t-t')/2}[\varepsilon \pm \varepsilon + f_\alpha(t') \pm f_\beta^*(t')]dt'. \quad (2.151)$$

## Photon Statistics

In the previous sections, we have studied the solutions of the c-number Langevin equations for the nondegenerate subharmonic generator. The statistical properties of a light beam is described in terms of the mean photon number, the variance of the photon number, the power spectrum and the photon number distribution. The photon statistics of light modes based on the relation between the mean and variance of the photon number divided into three parts. Thus the photon statistics of a light mode for which  $(\Delta n)^2 = \bar{n}$  the light mode has Poissonian photon statistics (e.g. coherent light), the photon statistics of a light mode for which  $(\Delta n)^2 > \bar{n}$  is known as super-Poissonian (e.g chaotic light) and the photon statistics of a light mode for which  $(\Delta n)^2 < \bar{n}$  is called sub-Poissonian. We now wish to determine the mean photon number, the variance of the photon number, the power spectrum, the local mean photon number and the photon number distribution.

### 3.1 Single-mode photon statistics

In this section, we obtain the mean photon number for cavity mode  $a$  and cavity mode  $b$ , the variance of the photon number for the cavity mode  $a$  and cavity mode  $b$ , the power spectrum and the local mean photon number.

#### 3.1.1 The mean photon number for cavity mode $a$ and cavity mode $b$

For the system under consideration the mean number of photons for the cavity mode  $a$  can be represented by

$$\bar{n}_a(t) = \langle \hat{a}^\dagger(t) \hat{a}(t) \rangle. \quad (3.1)$$

The mean number of photons for the cavity mode  $a$  can also be expressed in terms of c-number variables associated with the normal ordering as

$$\bar{n}_a(t) = \langle \alpha^*(t) \alpha(t) \rangle. \quad (3.2)$$

Applying Eq. (2.148) along with the complex conjugate of (2.148), we readily obtain

$$\begin{aligned}
\bar{n}_a(t) &= \left\langle \left( E_+(t)\alpha^*(0) + E_-(t)\beta(0) + F_+^*(t) + F_-^*(t) \right) \left( E_+(t)\alpha(0) + E_-(t)\beta^*(0) + F_+(t) + F_-(t) \right) \right\rangle \\
&= E_+^2(t)\langle\alpha^*(0)\alpha(0)\rangle + E_+(t)E_-(t)\langle\alpha^*(0)\beta^*(0)\rangle + E_+(t)\langle\alpha^*(0)F_+(t)\rangle \\
&+ E_+(t)\langle\alpha^*(0)F_-(t)\rangle + E_-(t)E_+(t)\langle\beta(0)\alpha(0)\rangle + E_-^2(t)\langle\beta(0)\beta^*(0)\rangle \\
&+ E_-(t)\langle\beta(0)F_+(t)\rangle + E_-(t)\langle\beta(0)F_-(t)\rangle + E_+(t)\langle F_+^*(t)\alpha(0)\rangle + E_-(t)\langle F_+^*(t)\beta^*(0)\rangle \\
&+ \langle F_+^*(t)F_+(t)\rangle + \langle F_+^*(t)F_-(t)\rangle + E_+(t)\langle F_-^*(t)\alpha(0)\rangle + E_-(t)\langle F_-^*(t)\beta^*(0)\rangle \\
&+ \langle F_-^*(t)F_+(t)\rangle + \langle F_-^*(t)F_-(t)\rangle. \tag{3.3}
\end{aligned}$$

With the assertion that a noise force at a certain time should not affect the system variables at earlier time and the cavity radiation is initially in a two-mode vacuum state, we get

$$\bar{n}_a(t) = \langle F_+^*(t)F_+(t) \rangle + \langle F_+^*(t)F_-(t) \rangle + \langle F_-^*(t)F_+(t) \rangle + \langle F_-^*(t)F_-(t) \rangle, \tag{3.4}$$

where  $F_\pm(t)$  is defined by Eq. (2.151). On account of (2.151), we note that

$$\begin{aligned}
\langle F_+^*(t)F_+(t) \rangle &= \left\langle \left[ \frac{1}{2} \int_0^t e^{-\lambda_+(t-t')/2} (2\varepsilon + f_\alpha^*(t') + f_\beta(t')) dt' \right] \right. \\
&\quad \times \left. \left[ \frac{1}{2} \int_0^t e^{-\lambda_+(t-t'')/2} (2\varepsilon + f_\alpha(t'') + f_\beta^*(t'')) dt'' \right] \right\rangle \\
&= \frac{1}{4} \int_0^t e^{-\lambda_+(2t-t'-t'')/2} \left[ 4\varepsilon^2 + 2\varepsilon \left( \langle f_\alpha(t'') \rangle + \langle f_\beta^*(t'') \rangle + \langle f_\alpha^*(t') \rangle + \langle f_\beta(t') \rangle \right) \right. \\
&\quad \left. + \langle f_\alpha^*(t')f_\alpha(t'') \rangle + \langle f_\alpha^*(t')f_\beta^*(t'') \rangle + \langle f_\beta(t')f_\alpha(t'') \rangle + \langle f_\beta(t')f_\beta^*(t'') \rangle \right] dt' dt'', \tag{3.5}
\end{aligned}$$

so that application of Eqs. (2.85), (2.86), (2.97), (2.116), (2.128), (2.132) and (2.133) leads to

$$\begin{aligned}
\langle F_+^*(t)F_+(t) \rangle &= \varepsilon^2 \int_0^t e^{-\lambda_+(2t-t'-t'')/2} dt' dt'' \\
&+ \frac{k\bar{n} - \lambda}{2} \int_0^t e^{-\lambda_+(2t-t'-t'')/2} \delta(t' - t'') dt' dt''. \tag{3.6}
\end{aligned}$$

Now carrying out the integration, one easily gets

$$\langle F_+^*(t)F_+(t) \rangle = \frac{4\varepsilon^2}{\lambda_+^2} \left( 1 - 2e^{-\lambda_+t/2} + e^{-\lambda_+t} \right) + \frac{k\bar{n} - \lambda}{2\lambda_+} \left( 1 - e^{-\lambda_+t} \right). \tag{3.7}$$

Furthermore, applying Eq. (2.151), we have

$$\begin{aligned}
\langle F_-^*(t)F_-(t) \rangle &= \left\langle \left[ \frac{1}{2} \int_0^t e^{-\lambda_-(t-t')/2} (f_\alpha^*(t') - f_\beta(t')) dt' \right] \right. \\
&\quad \times \left. \left[ \frac{1}{2} \int_0^t e^{-\lambda_-(t-t'')/2} (f_\alpha(t'') - f_\beta^*(t'')) dt'' \right] \right\rangle
\end{aligned}$$

$$\begin{aligned}
&= \frac{1}{4} \int_0^t e^{-\lambda_-(2t-t'-t'')/2} \left[ \langle f_\alpha^*(t') f_\alpha(t'') \rangle - \langle f_\alpha^*(t') f_\beta^*(t'') \rangle \right. \\
&\quad \left. - \langle f_\beta(t') f_\alpha(t'') \rangle + \langle f_\beta(t') f_\beta^*(t'') \rangle \right] dt' dt'', \tag{3.8}
\end{aligned}$$

Now with the aid of Eqs. (2.116), (2.128), (2.132) and (2.133), one readily gets

$$\langle F_-^*(t) F_-(t) \rangle = \frac{k\bar{n} + \lambda}{2} \int_0^t e^{-\lambda_-(2t-t'-t'')/2} \delta(t' - t'') dt' dt''. \tag{3.9}$$

On performing the integration using the property of Dirac delta function, one can obtain

$$\langle F_-^*(t) F_-(t) \rangle = \frac{k\bar{n} + \lambda}{2\lambda_-} \left( 1 - e^{-\lambda_- t} \right). \tag{3.10}$$

In addition, applying Eq. (2.151), we find

$$\langle F_+^*(t) F_-(t) \rangle = 0, \tag{3.11}$$

and

$$\langle F_-^*(t) F_+(t) \rangle = 0. \tag{3.12}$$

Therefore, upon substituting Eqs. (3.7), (3.10), (3.11), (3.12) into (3.2), there follows

$$\begin{aligned}
\bar{n}_a(t) &= \frac{4\varepsilon^2}{\lambda_+^2} \left( 1 - 2e^{-\lambda_+ t/2} + e^{-\lambda_+ t} \right) + \frac{k\bar{n} - \lambda}{2\lambda_+} \left( 1 - e^{-\lambda_+ t} \right) \\
&\quad + \frac{k\bar{n} + \lambda}{2\lambda_-} \left( 1 - e^{-\lambda_- t} \right). \tag{3.13}
\end{aligned}$$

At steady state this expression goes over into

$$\bar{n}_a = \frac{4\varepsilon^2}{(k + 2\lambda)^2} + \frac{k^2 \bar{n}}{k^2 - 4\lambda^2} + \frac{2\lambda^2}{k^2 - 4\lambda^2}. \tag{3.14}$$

For  $\lambda = 0$  (absence of parametric interaction) Eq. (3.14) becomes

$$\bar{n}_a = \frac{4\varepsilon^2}{k^2} + \bar{n}. \tag{3.15}$$

This is the mean photon number due to the deriving coherent light and thermal reservoir.

Furthermore, in the absence of the parametric interaction ( $\lambda = 0$ ) and when the cavity is coupled to vacuum reservoir ( $\bar{n} = 0$ ) Eq. (3.14), we get

$$\bar{n}_a = \frac{4\varepsilon^2}{k^2}. \tag{3.16}$$

This represents the mean photon number due to the driving coherent light mode. Moreover, in the absence of the parametric interaction ( $\lambda = 0$ ) and the driving coherent light modes are absent ( $\varepsilon = 0$ ), Eq. (3.14) reduces to

$$\bar{n}_a = \bar{n}. \tag{3.17}$$

This is the mean photon number due to thermal reservoir.

Finally, in the absence of the driving coherent light modes ( $\varepsilon = 0$ ) and when the cavity is coupled to vacuum reservoir ( $\bar{n} = 0$ ), Eq. (3.14) takes the form

$$\bar{n}_a = \frac{2\lambda^2}{k^2 - 4\lambda^2}. \quad (3.18)$$

This represents the mean photon number produced by the nondegenerate subharmonic generator.

Following a procedure used to obtain the mean photon number for cavity mode  $b$  it can be verified that the mean photon number for the cavity mode  $b$  is expressible, at steady state, as

$$\bar{n}_b = \frac{4\varepsilon^2}{(k + 2\lambda)^2} + \frac{k^2\bar{n}}{k^2 - 4\lambda^2} + \frac{2\lambda^2}{k^2 - 4\lambda^2}. \quad (3.19)$$

For absence of parametric interaction ( $\lambda = 0$ ) Eq. (3.19) becomes

$$\bar{n}_b = \frac{4\varepsilon^2}{k^2} + \bar{n}. \quad (3.20)$$

This is the mean photon number for cavity mode  $b$  due to the driving coherent light and thermal reservoir.

Furthermore, in the absence of the parametric interaction ( $\lambda = 0$ ) and when the cavity is coupled to ordinary vacuum ( $\bar{n} = 0$ ) Eq. (3.19), we obtain

$$\bar{n}_b = \frac{4\varepsilon^2}{k^2}. \quad (3.21)$$

This describes the mean photon number due to the driving coherent light.

Moreover, in the absence of the parametric interaction ( $\lambda = 0$ ) and the driving coherent light modes are absent ( $\varepsilon = 0$ ), Eq. (3.19) reduces to

$$\bar{n}_b = \bar{n}. \quad (3.22)$$

This can be interpreted as the mean photon number due to thermal reservoir.

And also, in the absence of the driving coherent light modes ( $\varepsilon = 0$ ) and when the cavity is coupled to vacuum reservoir ( $\bar{n} = 0$ ), Eq. (3.19) takes the form

$$\bar{n}_b = \frac{2\lambda^2}{k^2 - 4\lambda^2}. \quad (3.23)$$

This describes the mean photon number of the cavity mode  $b$  photons produced by the nondegenerate subharmonic generator.

Hence the above result shows that the mean photon number for cavity mode  $a$  are the same as that of cavity mode  $b$ .

### 3.1.2 The cavity mode photon number variance

We now seek to obtain the variance of the photon number for the cavity mode  $a$  and the variance of the photon number for the cavity mode  $b$ . The variance of the photon number for the cavity mode  $a$  can be expressed in the normal order as

$$(\Delta n_a(t))^2 = \langle \hat{a}^{\dagger 2}(t) \hat{a}^2(t) \rangle + \langle \hat{a}^\dagger(t) \hat{a}(t) \rangle - \langle \hat{a}^\dagger(t) \hat{a}(t) \rangle^2. \quad (3.24)$$

The corresponding c-number equation is then

$$(\Delta n_a(t))^2 = \langle \alpha^{*2}(t) \alpha^2(t) \rangle + \langle \alpha^*(t) \alpha(t) \rangle - \langle \alpha^*(t) \alpha(t) \rangle^2. \quad (3.25)$$

Since the c-number Langevin equation for  $\alpha(t)$  has the form given by Eq. (2.81) inspection of this equation indicate that  $\alpha(t)$  is a Gaussian variable and noise force  $f(t)$  has the correlation properties obtained in chapter two. Moreover, on account of (2.148) along with the assumption that the cavity mode is initially in a vacuum state, we easily show that  $\langle \alpha(t) \rangle = 0$ . Hence  $\alpha(t)$  is a Gaussian variable with vanishing mean satisfy the relation [2]

$$\langle ABCD \rangle = \langle AB \rangle \langle CD \rangle + \langle AC \rangle \langle BD \rangle + \langle AD \rangle \langle BC \rangle, \quad (3.26)$$

one can write

$$\langle \alpha^{*2}(t) \alpha^2(t) \rangle = \langle \alpha^{*2}(t) \rangle \langle \alpha^2(t) \rangle + 2 \langle \alpha^*(t) \alpha(t) \rangle^2, \quad (3.27)$$

so that Eq. (3.25) can be expressed as

$$(\Delta n_a(t))^2 = \langle \alpha^{*2}(t) \rangle \langle \alpha^2(t) \rangle + \langle \alpha^*(t) \alpha(t) \rangle^2 + \langle \alpha^*(t) \alpha(t) \rangle, \quad (3.28)$$

or

$$(\Delta n_a(t))^2 = \langle \alpha^{*2}(t) \rangle \langle \alpha^2(t) \rangle + \bar{n}_a^2 + \bar{n}_a. \quad (3.29)$$

With the aid of Eqs. (2.148) and (2.151), one readily obtains

$$\langle \alpha^{*2}(t) \rangle = \langle \alpha^2(t) \rangle = \frac{4\varepsilon^2}{\lambda_+^2} \left( 1 - 2e^{-\lambda_+ t/2} + e^{-\lambda_+ t} \right). \quad (3.30)$$

And at steady state

$$\langle \alpha^{*2} \rangle_{ss} = \langle \alpha^2 \rangle_{ss} = \frac{4\varepsilon^2}{\lambda_+^2}. \quad (3.31)$$

Hence on account of Eqs. (3.29) together with (3.31), the variance of the photon number for the cavity mode  $a$  at steady state, can be expressed as

$$(\Delta n_a)_{ss}^2 = \frac{16\varepsilon^4}{\lambda_+^4} + \bar{n}_a^2 + \bar{n}_a. \quad (3.32)$$

We note that  $(\Delta n_a)_{ss}^2 > \bar{n}_a$  and hence the cavity radiation has super-Poissonian photon statistics.

Moreover, in the absence of the parametric interaction ( $\lambda = 0$ ) and the driving coherent light modes ( $\varepsilon = 0$ ) one can put Eq. (3.32), using Eq. (3.17), in the form

$$(\Delta n_a)_{ss}^2 = \bar{n}^2 + \bar{n}. \quad (3.33)$$

This represents the variance of the photon number due to thermal reservoir.

Finally, in the absence of the driving coherent light modes ( $\varepsilon = 0$ ) and when the cavity is coupled to ordinary vacuum ( $\bar{n} = 0$ ) and taking into account Eq. (3.18) along with Eq. (3.32), we note that

$$(\Delta n_a)_{ss}^2 = \frac{2k^2\lambda^2 - 4\lambda^4}{(k^2 - 4\lambda^2)^2}. \quad (3.34)$$

This describes the variance of the photon number of the cavity mode  $a$  photons produced by the nondegenerate subharmonic generator.

It can also be established following a similar procedure that, the variance of the photon number for cavity mode  $b$  can be expressed, at steady state, as

$$(\Delta n_b)_{ss}^2 = \frac{16\varepsilon^4}{\lambda_+^4} + \bar{n}_b^2 + \bar{n}_b. \quad (3.35)$$

We readily observe that  $(\Delta n_b)_{ss}^2 > \bar{n}_b$ , this indicates that the photon number statistics is super-Poissonian.

Moreover, for the absence of the parametric interaction ( $\lambda = 0$ ) and the driving coherent light modes are absent ( $\varepsilon = 0$ ) and using Eq. (3.22) one can put Eq. (3.35) in the form

$$(\Delta n_b)_{ss}^2 = \bar{n}^2 + \bar{n}. \quad (3.36)$$

This can be interpreted as the variance of the photon number due to thermal reservoir.

In addition, for  $\varepsilon = 0$  (the absence of the driving coherent light modes) and when the cavity is coupled to vacuum reservoir ( $\bar{n} = 0$ ) and taking into account Eq. (3.23) along with Eq. (3.35), we note that

$$(\Delta n_b)_{ss}^2 = \frac{2k^2\lambda^2 - 4\lambda^4}{(k^2 - 4\lambda^2)^2}. \quad (3.37)$$

This is represents the variance of the photon number of the cavity mode  $b$  photons produced by the parametric interaction.

Now the above result indicats that the variance of the photon number for cavity mode  $a$  are the same as that of cavity mode  $b$ .

### 3.1.3 Power spectrum

We wish here to determine the spectrum of mean photon number, usually known as the power spectrum for cavity mode  $a$ . The power spectrum of the cavity mode  $a$  with central frequency  $\omega_o$  is given by [12]

$$P(\omega) = \frac{1}{\pi} \text{Re} \int_0^\infty \langle \hat{a}^\dagger(t) \hat{a}(t + \tau) \rangle_{ss} e^{i(\omega - \omega_o)\tau} d\tau, \quad (3.38)$$

where ' $\text{Re}$ ' donotes the real part. The c-number equation corresponding to Eq. (3.38) one can write

$$P(\omega) = \frac{1}{\pi} \text{Re} \int_0^\infty \langle \alpha^*(t) \alpha(t + \tau) \rangle_{ss} e^{i(\omega - \omega_o)\tau} d\tau. \quad (3.39)$$

Now we seek to find the steady state expectation value  $\langle \alpha^*(t) \alpha(t + \tau) \rangle$  in Eq. (3.39). To this end, we realize that the solution of Eq. (2.137) can also be written as

$$z_+(t + \tau) = z_+(t) e^{-\lambda_+ \tau/2} + 2\varepsilon e^{-\lambda_+ \tau/2} \int_0^\tau e^{\lambda_+ \tau'/2} d\tau' + e^{-\lambda_+ \tau/2} \int_0^\tau e^{\lambda_+ \tau'/2} f_+(t + \tau') d\tau'. \quad (3.40)$$

With the aid of Eq. (2.138), we have

$$\begin{aligned} \alpha(t + \tau) + \beta^*(t + \tau) &= (\alpha(t) + \beta^*(t)) e^{-\lambda_+ \tau/2} + \frac{4\varepsilon}{\lambda_+} (1 - e^{-\lambda_+ \tau/2}) \\ &+ e^{-\lambda_+ \tau/2} \int_0^\tau e^{\lambda_+ \tau'/2} f_+(t + \tau') d\tau'. \end{aligned} \quad (3.41)$$

On the other hand, the solution of Eq. (2.140) is expressible as

$$z_-(t + \tau) = z_-(t) e^{-\lambda_- \tau/2} + e^{-\lambda_- \tau/2} \int_0^\tau e^{\lambda_- \tau'/2} f_-(t + \tau') d\tau'. \quad (3.42)$$

Now taking into account Eq. (2.141), we can write

$$\begin{aligned} \alpha(t + \tau) - \beta^*(t + \tau) &= (\alpha(t) - \beta^*(t)) e^{-\lambda_- \tau/2} \\ &+ e^{-\lambda_- \tau/2} \int_0^\tau e^{\lambda_- \tau'/2} f_-(t + \tau') d\tau'. \end{aligned} \quad (3.43)$$

Adding Eqs. (3.41) and (3.43), we see that

$$\begin{aligned} \alpha(t + \tau) &= \frac{1}{2} \alpha(t) (e^{-\lambda_+ \tau/2} + e^{-\lambda_- \tau/2}) + \frac{2\varepsilon}{\lambda_+} (1 - e^{-\lambda_+ \tau/2}) \\ &+ \frac{1}{2} \beta^*(t) (e^{-\lambda_+ \tau/2} - e^{-\lambda_- \tau/2}) \\ &+ \frac{1}{2} \int_0^\tau e^{-\lambda_+ (\tau - \tau')/2} f_+(t + \tau') d\tau' \\ &+ \frac{1}{2} \int_0^\tau e^{-\lambda_- (\tau - \tau')/2} f_-(t + \tau') d\tau'. \end{aligned} \quad (3.44)$$

Furthermore, taking the expectation value of both sides and on account of Eqs. (2.85) and (2.86), we obtain

$$\begin{aligned}\langle \alpha(t + \tau) \rangle &= \frac{1}{2} \langle \alpha(t) \rangle (e^{-\lambda + \tau/2} + e^{-\lambda - \tau/2}) + \frac{2\varepsilon}{\lambda_+} (1 - e^{-\lambda + \tau/2}) \\ &+ \frac{1}{2} \langle \beta^*(t) \rangle (e^{-\lambda + \tau/2} - e^{-\lambda - \tau/2}).\end{aligned}\quad (3.45)$$

Now application of the quantum regression theorem to Eq. (3.45) leads to

$$\begin{aligned}\langle \alpha^*(t) \alpha(t + \tau) \rangle &= \frac{1}{2} \langle \alpha^*(t) \alpha(t) \rangle (e^{-\lambda + \tau/2} + e^{-\lambda - \tau/2}) + \frac{2\varepsilon}{\lambda_+} \langle \alpha^*(t) \rangle (1 - e^{-\lambda + \tau/2}) \\ &+ \frac{1}{2} \langle \alpha^*(t) \beta^*(t) \rangle (e^{-\lambda + \tau/2} - e^{-\lambda - \tau/2}).\end{aligned}\quad (3.46)$$

Since  $\alpha(t)$  has a vanishing mean, we have

$$\begin{aligned}\langle \alpha^*(t) \alpha(t + \tau) \rangle_{ss} &= (\langle \alpha^*(t) \alpha(t) \rangle_{ss} + \langle \alpha^*(t) \beta^*(t) \rangle_{ss}) \frac{e^{-\lambda + \tau/2}}{2} \\ &+ (\langle \alpha^*(t) \alpha(t) \rangle_{ss} - \langle \alpha^*(t) \beta^*(t) \rangle_{ss}) \frac{e^{-\lambda - \tau/2}}{2}.\end{aligned}\quad (3.47)$$

From Eq. (3.14), we obtain

$$\langle \alpha^*(t) \alpha(t) \rangle_{ss} = \frac{4\varepsilon^2}{(k + 2\lambda)^2} + \frac{k^2 \bar{n}}{k^2 - 4\lambda^2} + \frac{2\lambda^2}{k^2 - 4\lambda^2},\quad (3.48)$$

and in similar manner

$$\langle \alpha^*(t) \beta^*(t) \rangle_{ss} = \frac{4\varepsilon^2}{(k + 2\lambda)^2} - \frac{2k\bar{n}\lambda}{k^2 - 4\lambda^2} - \frac{k\lambda}{k^2 - 4\lambda^2}.\quad (3.49)$$

Moreover, applying Eqs. (3.48) and (3.49), one easily gets

$$\langle \alpha^*(t) \alpha(t) \rangle_{ss} + \langle \alpha^*(t) \beta^*(t) \rangle_{ss} = 2\bar{n}_a - \frac{k\bar{n}}{k - 2\lambda} - \frac{\lambda}{k - 2\lambda},\quad (3.50)$$

and

$$\langle \alpha^*(t) \alpha(t) \rangle_{ss} - \langle \alpha^*(t) \beta^*(t) \rangle_{ss} = \frac{k\bar{n}}{k - 2\lambda} + \frac{\lambda}{k - 2\lambda}.\quad (3.51)$$

Now substituting Eqs. (3.50) and (3.51) into (3.47), we have

$$\begin{aligned}\langle \alpha^*(t) \alpha(t + \tau) \rangle_{ss} &= \left( 2\bar{n}_a - \frac{k\bar{n}}{k - 2\lambda} - \frac{\lambda}{k - 2\lambda} \right) \frac{e^{-\lambda + \tau/2}}{2} \\ &+ \left( \frac{k\bar{n}}{k - 2\lambda} + \frac{\lambda}{k - 2\lambda} \right) \frac{e^{-\lambda - \tau/2}}{2}.\end{aligned}\quad (3.52)$$

Furthermore, employing Eq. (3.52) in (3.39), we see that

$$\begin{aligned}p(\omega) &= \frac{1}{2\pi} \left( 2\bar{n}_a - \frac{k\bar{n}}{k - 2\lambda} - \frac{\lambda}{k - 2\lambda} \right) \text{Re} \int_0^\infty e^{-(\frac{\lambda_+}{2} - i(\omega - \omega_o))\tau} d\tau \\ &+ \frac{1}{2\pi} \left( \frac{k\bar{n}}{k - 2\lambda} + \frac{\lambda}{k - 2\lambda} \right) \text{Re} \int_0^\infty e^{-(\frac{\lambda_-}{2} - i(\omega - \omega_o))\tau} d\tau,\end{aligned}\quad (3.53)$$

and carrying out the integration, there follows

$$p(\omega) = \frac{1}{\pi} \left( 2\bar{n}_a - \frac{k\bar{n}}{k-2\lambda} - \frac{\lambda}{k-2\lambda} \right) \frac{k+2\lambda}{(k+2\lambda)^2 + 4(\omega - \omega_o)^2} + \frac{1}{\pi} \left( k\bar{n} + \lambda \right) \frac{1}{(k-2\lambda)^2 + 4(\omega - \omega_o)^2}, \quad (3.54)$$

or

$$p(\omega) = \frac{1}{\pi} \left[ \frac{2\bar{n}_a(k+2\lambda)}{(k+2\lambda)^2 + 4(\omega - \omega_o)^2} - \frac{(k\bar{n} + \lambda)(k+2\lambda)}{(k-2\lambda)(k+2\lambda)^2 + 4(\omega - \omega_o)^2} + \frac{k\bar{n} + \lambda}{(k-2\lambda)^2 + 4(\omega - \omega_o)^2} \right]. \quad (3.55)$$

This represents the power spectrum for cavity mode  $a$ .

### 3.1.4 The local mean photon number

We finally seek to calculate the local mean photon number of the cavity light. Upon integrating both sides of Eq. (3.38) over  $\omega$ , we readily obtain

$$\int_{-\infty}^{\infty} P(\omega) d\omega = \frac{1}{\pi} \text{Re} \int_0^{\infty} \langle \hat{a}^\dagger(t) \hat{a}(t+\tau) \rangle_{ss} e^{-i\omega_o\tau} d\tau \cdot \int_{-\infty}^{\infty} e^{i\omega\tau} d\omega, \quad (3.56)$$

so that using the fact that [10]

$$\int_{-\infty}^{\infty} e^{i\omega\tau} d\omega = 2\pi\delta(\tau) \quad (3.57)$$

we have

$$\int_{-\infty}^{\infty} P(\omega) d\omega = 2 \int_0^{\infty} \langle \hat{a}^\dagger(t) \hat{a}(t+\tau) \rangle_{ss} e^{-i\omega_o\tau} \delta(\tau) d\tau \quad (3.58)$$

$$= \int_{-\infty}^{\infty} \langle \hat{a}^\dagger(t) \hat{a}(t+\tau) \rangle_{ss} e^{-i\omega_o\tau} \delta(\tau) d\tau. \quad (3.59)$$

In view of the relation [10]

$$\int_{-\infty}^{\infty} f(x) \delta(x) dx = f(x)|_{x=0}, \quad (3.60)$$

we get

$$\int_{-\infty}^{\infty} P(\omega) d\omega = \bar{n}_a, \quad (3.61)$$

with  $\bar{n}_a(t) = \langle \hat{a}^\dagger(t) \hat{a}(t) \rangle$  being the steady state mean photon number of the cavity mode  $a$ .

On account of this relation, we observe that  $P(\omega)d\omega$  represents the steady state mean photon number of a cavity mode in the interval between  $\omega$  and  $\omega + d\omega$ .

We realize that the local mean photon number in the frequency interval between  $\omega' = -\mu$  and  $\omega' = \mu$  is expressible as

$$\bar{n}_{a\pm\mu} = \int_{-\mu}^{\mu} P(\omega') d\omega', \quad (3.62)$$

in which  $\omega' = \omega - \omega_o$ .

Finally, upon substituting Eq. (3.55) into Eq. (3.62), we get

$$\begin{aligned} \bar{n}_{a\pm\mu} &= \frac{2\bar{n}_a(k+2\lambda)}{4\pi} \int_{-\mu}^{\mu} \frac{1}{\left(\frac{k+2\lambda}{2}\right)^2 + \omega'^2} d\omega' - \frac{(k\bar{n} + \lambda)(k+2\lambda)}{4\pi(k-2\lambda)} \int_{-\mu}^{\mu} \frac{1}{\left(\frac{k+2\lambda}{2}\right)^2 + \omega'^2} d\omega' \\ &+ \frac{k\bar{n} + \lambda}{4\pi} \int_{-\mu}^{\mu} \frac{1}{\left(\frac{k-2\lambda}{2}\right)^2 + \omega'^2} d\omega', \end{aligned} \quad (3.63)$$

and carrying out the integration, applying the relation

$$\int_{-\mu}^{\mu} \frac{dy}{m^2 + y^2} = \frac{2}{m} \tan^{-1} \left( \frac{\mu}{m} \right), \quad (3.64)$$

we get

$$\bar{n}_{a\pm\mu} = \frac{2\bar{n}_a}{\pi} \tan^{-1} \left( \frac{2\mu}{k+2\lambda} \right) + \frac{k\bar{n} + \lambda}{\pi(k-2\lambda)} \left[ \tan^{-1} \left( \frac{2\mu}{k-2\lambda} \right) - \tan^{-1} \left( \frac{2\mu}{k+2\lambda} \right) \right], \quad (3.65)$$

or

$$\bar{n}_{a\pm\mu} = \bar{n}_a z_+(\mu) + \frac{k\bar{n} + \lambda}{k-2\lambda} z_-(\mu), \quad (3.66)$$

where

$$z_+(\mu) = \frac{2}{\pi} \tan^{-1} \left( \frac{2\mu}{k+2\lambda} \right), \quad (3.67)$$

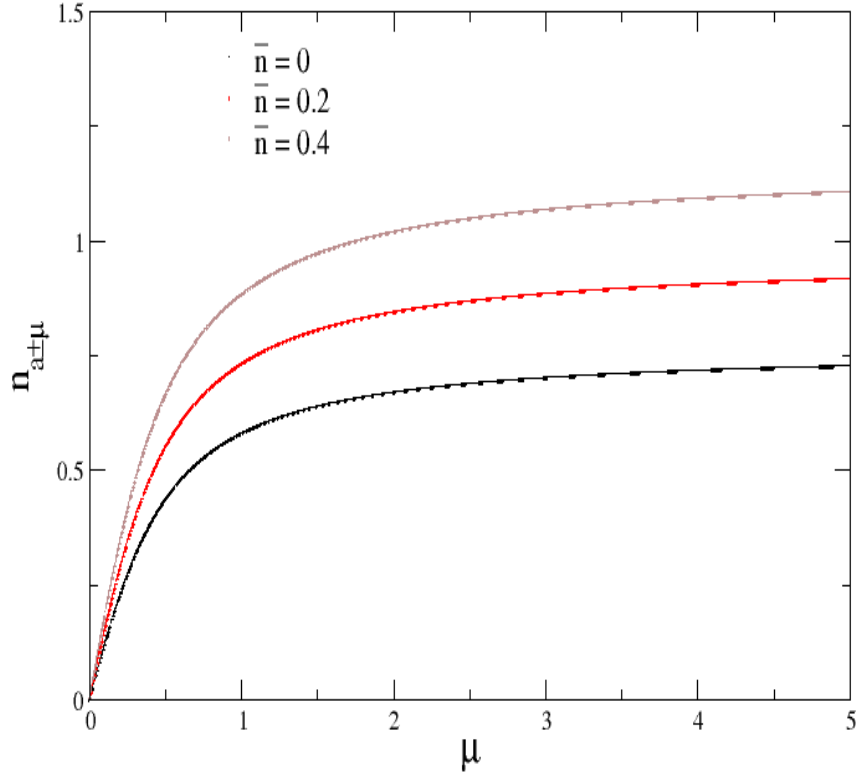
$$z_-(\mu) = \frac{1}{\pi} \left[ \tan^{-1} \left( \frac{2\mu}{k-2\lambda} \right) - \tan^{-1} \left( \frac{2\mu}{k+2\lambda} \right) \right]. \quad (3.68)$$

This describes the local mean photon number of the cavity mode  $a$ .

We indicate in the table below that as the values of  $\mu$  increases the local mean photon number approaches to the global mean photon number.

	$\bar{n} = 0$	$\bar{n} = 0.2$	$\bar{n} = 0.4$
$\mu$	$\bar{n}_{a\pm\mu}$	$\bar{n}_{a\pm\mu}$	$\bar{n}_{a\pm\mu}$
1	0.590	0.751	0.919
2	0.670	0.849	1.028
3	0.709	0.880	1.070
4	0.719	0.905	1.091
5	0.731	0.915	1.115
$\bar{n}_a$	0.765	0.965	1.165

**Table 3.1:** Values of  $\bar{n}_{a\pm\mu}$  and  $\bar{n}_a$  for different values of  $\mu$  and  $\bar{n}$ .



**Figure 3.1:** Plots of the local mean photon number [Eq. (3.65)] at steady state versus  $\mu$  for  $\lambda = 0$ ,  $\varepsilon = 0.35$ ,  $k = 0.8$ ,  $\bar{n} = 0$  (black curve),  $\bar{n} = 0.2$  (red curve),  $\bar{n} = 0.4$  (brown curve).

Moreover, inspection of the plots in Fig. 3.1 shows that the presence of thermal reservoir increases the local mean photon number of the cavity mode. In addition, one can also observe that the local mean photon number approaches the global mean photon number for a relatively small frequency interval.

## 3.2 Two-mode photon statistics

We now obtain the mean and the variance of the photon number for the the two-mode cavity light produced by nondegenerate subharmonic generating system.

### 3.2.1 The mean photon number for the the two-mode cavity light

The two-mode cavity light mean photon number can be expressed as [2]

$$\bar{n}_\gamma = \bar{n}_a + \bar{n}_b, \quad (3.69)$$

where  $\bar{n}_a$  and  $\bar{n}_b$  are the mean photon number for cavity mode  $a$  and cavity mode  $b$ .

We see that with the aid of Eqs. (3.14) and (3.19), we can write Eq. (3.69) as

$$\bar{n}_\gamma = \frac{8\varepsilon^2}{(k + 2\lambda)^2} + \frac{2k^2\bar{n}}{k^2 - 4\lambda^2} + \frac{4\lambda^2}{k^2 - 4\lambda^2}. \quad (3.70)$$

In addition, for  $\lambda = 0$  (absence of parametric interaction), we have

$$\bar{n}_\gamma = \frac{8\varepsilon^2}{k^2} + 2\bar{n}. \quad (3.71)$$

This is the mean photon number due to the two-mode deriving coherent light and two-mode thermal reservoir.

Furthermore, in the absence of the parametric interaction ( $\lambda = 0$ ) and when the cavity is coupled to ordinary vacuum ( $\bar{n} = 0$ ) Eq. (3.70), we get

$$\bar{n}_\gamma = \frac{8\varepsilon^2}{k^2}. \quad (3.72)$$

This represents the mean photon number due to the driving coherent light modes. Moreover, in the absence of the parametric interaction ( $\lambda = 0$ ) and the driving coherent light modes ( $\varepsilon = 0$ ), Eq. (3.70) reduces to

$$\bar{n}_\gamma = 2\bar{n}. \quad (3.73)$$

This is the mean photon number due to the two-mode thermal reservoir.

Lastly, in the absence of the driving coherent light modes ( $\varepsilon = 0$ ) and when the cavity is coupled to ordinary vacuum ( $\bar{n} = 0$ ), Eq. (3.70) takes the form

$$\bar{n}_\gamma = \frac{4\lambda^2}{k^2 - 4\lambda^2}. \quad (3.74)$$

This represents the steady-state mean photon number of the two-mode photons produced by the nondegenerate subharmonic generator.

### 3.2.2 The variance of the photon number for two-mode cavity light

The variance of the photon number for two-mode cavity light in the normal order defined as [2]

$$(\Delta n_\gamma)^2 = \langle \hat{c}^{\dagger 2} \hat{c}^2 \rangle + \langle \hat{c}^\dagger \hat{c} \rangle - \langle \hat{c}^\dagger \hat{c} \rangle^2, \quad (3.75)$$

where  $\hat{c} = \hat{a} + \hat{b}$  is annihilation operator for two-mode cavity light. The c-number equation corresponding to Eq. (3.75) is

$$(\Delta n_\gamma)^2 = \langle \gamma^{*2} \gamma^2 \rangle + \langle \gamma^* \gamma \rangle - \langle \gamma^* \gamma \rangle^2, \quad (3.76)$$

or

$$(\Delta n_\gamma)^2 = \langle \gamma^{*2} \gamma^2 \rangle + \bar{n}_\gamma - \bar{n}_\gamma^2, \quad (3.77)$$

where

$$\gamma = \alpha + \beta. \quad (3.78)$$

With the help of Eq. (3.78), one easily find

$$\langle \gamma^{*2} \gamma^2 \rangle = \langle (\alpha^* + \beta^*)^2 (\alpha + \beta)^2 \rangle \quad (3.79)$$

$$\begin{aligned} &= \langle \alpha^{*2} \alpha^2 \rangle + \langle \beta^{*2} \beta^2 \rangle + \langle \alpha^{*2} \beta^2 \rangle + 2\langle \alpha^{*2} \alpha \beta \rangle + \langle \alpha^2 \beta^{*2} \rangle + 2\langle \alpha \beta^{*2} \beta \rangle \\ &+ 2\langle \alpha^* \alpha^2 \beta^* \rangle + 2\langle \alpha^* \beta^* \beta^2 \rangle + 4\langle \alpha^* \alpha \beta^* \beta \rangle. \end{aligned} \quad (3.80)$$

Since  $\alpha(t)$  and  $\beta(t)$  are a Gaussian variable. Therefore the Gaussian variables with vanishing mean satisfy the relation from Eq. (3.26), thus on the basis of the relation, we have

$$\langle \alpha^{*2} \alpha^2 \rangle = \langle \alpha^{*2} \rangle \langle \alpha^2 \rangle + 2\langle \alpha^* \alpha \rangle^2, \quad (3.81)$$

on the basis of Eqs. (3.14) and (3.31), one can write

$$\langle \alpha^{*2} \alpha^2 \rangle_{ss} = \frac{16\varepsilon^4}{\lambda_+^4} + 2\bar{n}_a^2. \quad (3.82)$$

Furthermore, it can also established in a similar manner that

$$\langle \beta^{*2} \beta^2 \rangle_{ss} = \frac{16\varepsilon^4}{\lambda_+^4} + 2\bar{n}_b^2, \quad (3.83)$$

$$\langle \alpha^{*2} \beta^2 \rangle_{ss} = \frac{48\varepsilon^4}{\lambda_+^4}, \quad (3.84)$$

$$\langle \alpha^2 \beta^{*2} \rangle_{ss} = \frac{48\varepsilon^4}{\lambda_+^4}, \quad (3.85)$$

$$2\langle \alpha^{*2} \alpha \beta \rangle_{ss} = \frac{8\varepsilon^2}{\lambda_+^2} \left( \frac{4\varepsilon^2}{\lambda_+^2} + \frac{k\bar{n} - \lambda}{2\lambda_+} - \frac{k\bar{n} - \lambda}{2\lambda_-} \right) + \frac{16\varepsilon^2}{\lambda_+^2} \bar{n}_a, \quad (3.86)$$

$$2\langle \alpha \beta^{*2} \beta \rangle_{ss} = \frac{16\varepsilon^2}{\lambda_+^2} \bar{n}_b + \frac{8\varepsilon^2}{\lambda_+^2} \left( \frac{4\varepsilon^2}{\lambda_+^2} + \frac{k\bar{n} - \lambda}{2\lambda_+} - \frac{k\bar{n} - \lambda}{2\lambda_-} \right), \quad (3.87)$$

$$2\langle \alpha^* \alpha^2 \beta^* \rangle_{ss} = \frac{16\varepsilon^2}{\lambda_+^2} \bar{n}_a + \frac{8\varepsilon^2}{\lambda_+^2} \left( \frac{4\varepsilon^2}{\lambda_+^2} + \frac{k\bar{n} - \lambda}{2\lambda_+} - \frac{k\bar{n} - \lambda}{2\lambda_-} \right), \quad (3.88)$$

$$2\langle \alpha^* \beta^* \beta^2 \rangle_{ss} = \frac{8\varepsilon^2}{\lambda_+^2} \left( \frac{4\varepsilon^2}{\lambda_+^2} + \frac{k\bar{n} - \lambda}{2\lambda_+} - \frac{k\bar{n} - \lambda}{2\lambda_-} \right) + \frac{16\varepsilon^2}{\lambda_+^2} \bar{n}_b, \quad (3.89)$$

$$4\langle\alpha^*\alpha\beta^*\beta\rangle_{ss} = 4\bar{n}_a^2 + 4\left(\frac{4\varepsilon^2}{\lambda_+^2} + \frac{k\bar{n} - \lambda}{2\lambda_+} - \frac{k\bar{n} - \lambda}{2\lambda_-}\right)^2 + \frac{64\varepsilon^4}{\lambda_+^4}. \quad (3.90)$$

Now substituting Eqs. (3.82) - (3.90) into (3.80), we see that

$$\begin{aligned} \langle\gamma^{*2}\gamma^2\rangle_{ss} &= 8\bar{n}_a^2 + \frac{64\varepsilon^2}{\lambda_+^2}\bar{n}_a + \frac{192\varepsilon^4}{\lambda_+^4} + \frac{32\varepsilon^2}{\lambda_+^2}\left(\frac{4\varepsilon^2}{\lambda_+^2} + \frac{k\bar{n} - \lambda}{2\lambda_+} - \frac{k\bar{n} - \lambda}{2\lambda_-}\right) \\ &+ 4\left(\frac{4\varepsilon^2}{\lambda_+^2} + \frac{k\bar{n} - \lambda}{2\lambda_+} - \frac{k\bar{n} - \lambda}{2\lambda_-}\right)^2. \end{aligned} \quad (3.91)$$

Moreover, on account of Eq. (3.91), the variance of the photon number for two-mode cavity light given by (3.77) at steady state takes the form

$$\begin{aligned} (\Delta n_\gamma)_{ss}^2 &= 8\bar{n}_a^2 + \frac{64\varepsilon^2}{\lambda_+^2}\bar{n}_a + \frac{192\varepsilon^4}{\lambda_+^4} + \frac{32\varepsilon^2}{\lambda_+^2}\left(\frac{4\varepsilon^2}{\lambda_+^2} + \frac{k\bar{n} - \lambda}{2\lambda_+} - \frac{k\bar{n} - \lambda}{2\lambda_-}\right) \\ &+ 4\left(\frac{4\varepsilon^2}{\lambda_+^2} + \frac{k\bar{n} - \lambda}{2\lambda_+} - \frac{k\bar{n} - \lambda}{2\lambda_-}\right)^2 + \bar{n}_\gamma - \bar{n}_\gamma^2. \end{aligned} \quad (3.92)$$

Furthermore, in the absence of the parametric interaction ( $\lambda = 0$ ) and the driving coherent light modes are absent ( $\varepsilon = 0$ ) and using Eq. (3.17) and (3.73) one can put Eq. (3.92) as

$$(\Delta n_\gamma)_{ss}^2 = 4\bar{n}^2 + 2\bar{n}. \quad (3.93)$$

This represents the variance of the photon number due to the two-mode thermal reservoir.

Finally, in the absence of the driving coherent light modes ( $\varepsilon = 0$ ) and when the cavity is coupled to vacuum reservoir ( $\bar{n} = 0$ ) and taking into account Eqs. (3.18) and (3.74) along with Eq. (3.92), we note that

$$(\Delta n_\gamma)_{ss}^2 = \frac{8k^2\lambda^2}{(k^2 - 4\lambda^2)^2}. \quad (3.94)$$

This can be interpreted as the variance of the photon number of the two-mode cavity light photons produced by the nondegenerate subharmonic generator for parametric interaction.

We see that the above result tells us the variance of the photon number of the two-mode cavity light is greater than the mean photon number the two-mode cavity light and hence the light produced by nondegenerate subharmonic generator has super Poissonian photon statistics.

### 3.3 The Q function

We next seek to obtain the Q function for the two-mode cavity light produced by the system under consideration. To this end, the Q function for a two-mode cavity

light is expressible as [2]

$$Q(\alpha, \beta, t) = \frac{1}{\pi^4} \int \int d^2z d^2\eta \phi_a(z, \eta, t) \exp(z^* \alpha - z \alpha^* + \eta^* \beta - \eta \beta^*), \quad (3.95)$$

where the antinormally ordered characteristic function  $\phi_a(z^*, z, t)$  is defined in the Heisenberg picture by [13]

$$\phi_a(z, \eta, t) = Tr \left( \hat{\rho}(0) e^{-z^* \hat{a}(t)} e^{z \hat{a}^\dagger(t)} e^{-\eta^* \hat{b}(t)} e^{\eta \hat{b}^\dagger(t)} \right). \quad (3.96)$$

Using the identity [2]

$$e^{\hat{A}} e^{\hat{B}} = e^{\hat{B}} e^{\hat{A}} e^{[\hat{A}, \hat{B}]}, \quad (3.97)$$

the characteristic function defined by Eq. (3.96) can be expressed in terms of c-number variables associated with the normal ordering as

$$\phi_a(z, t) = \exp(-z^* z - \eta^* \eta) \langle \exp(z \alpha^*(t) - z^* \alpha(t) + \eta \beta^*(t) - \eta^* \beta(t)) \rangle. \quad (3.98)$$

We recall that  $\alpha(t)$  and  $\beta(t)$  are Gaussian variables with zero mean. One can put Eq. (3.98) in the form [14]

$$\phi_a(z, t) = \exp(-z^* z - \eta^* \eta) \exp\left(\frac{1}{2} \langle (z \alpha^*(t) - z^* \alpha(t) + \eta \beta^*(t) - \eta^* \beta(t))^2 \rangle\right), \quad (3.99)$$

or

$$\phi_a(z, t) = \exp(-z^* z - \eta^* \eta) \exp\left(\frac{1}{2} T_1\right), \quad (3.100)$$

where

$$T_1 = \langle (z \alpha^*(t) - z^* \alpha(t) + \eta \beta^*(t) - \eta^* \beta(t))^2 \rangle. \quad (3.101)$$

Now simplify Eq. (3.101), we have

$$\begin{aligned} T_1 &= -(\langle \alpha^*(t) \alpha(t) \rangle + \langle \alpha(t) \alpha^*(t) \rangle) z^* z - (\langle \beta^*(t) \beta(t) \rangle + \langle \beta(t) \beta^*(t) \rangle) \eta^* \eta \\ &+ 2 \langle \alpha^*(t) \beta^*(t) \rangle z \eta + 2 \langle \alpha(t) \beta(t) \rangle z^* \eta^* - 2 \langle \alpha^*(t) \beta(t) \rangle z \eta^* \\ &- 2 \langle \alpha(t) \beta^*(t) \rangle z^* \eta + \langle \alpha^{*2} \rangle z^2 + \langle \alpha^2 \rangle z^2 + \langle \beta^{*2} \rangle \eta^2 + \langle \beta^2 \rangle \eta^2. \end{aligned} \quad (3.102)$$

It can be readily verified that

$$\langle \alpha^{*2} \rangle_{ss} = \langle \beta^{*2} \rangle_{ss} = \langle \alpha^2 \rangle_{ss} = \langle \beta^2 \rangle_{ss} = \langle \alpha^*(t) \beta(t) \rangle_{ss} = \langle \alpha(t) \beta^*(t) \rangle_{ss} = \frac{4\varepsilon^2}{\lambda_+^2}, \quad (3.103)$$

$$\langle \alpha^*(t) \alpha(t) \rangle_{ss} = \langle \beta^*(t) \beta(t) \rangle_{ss} = \frac{4\varepsilon^2}{\lambda_+^2} + \frac{k\bar{n} - \lambda}{2\lambda_+} + \frac{k\bar{n} + \lambda}{2\lambda_-}, \quad (3.104)$$

$$\langle \alpha(t) \alpha^*(t) \rangle_{ss} = \langle \beta(t) \beta^*(t) \rangle_{ss} = \frac{4\varepsilon^2}{\lambda_+^2} + \frac{k\bar{n} - \lambda}{2\lambda_+} + \frac{k\bar{n} + \lambda}{2\lambda_-}, \quad (3.105)$$

$$\langle \alpha(t) \beta(t) \rangle_{ss} = \langle \alpha^*(t) \beta^*(t) \rangle_{ss} = \frac{4\varepsilon^2}{\lambda_+^2} + \frac{k\bar{n} - \lambda}{2\lambda_+} - \frac{k\bar{n} - \lambda}{2\lambda_-}. \quad (3.106)$$

Substituting Eqs. (3.103) - (3.106) into (3.102), we have

$$T_1 = -\left(\frac{8\varepsilon^2}{\lambda_+^2} + \frac{k\bar{n} - \lambda}{\lambda_+} + \frac{k\bar{n} + \lambda}{\lambda_-}\right)(z^*z + \eta^*\eta) + \left(\frac{8\varepsilon^2}{\lambda_+^2} + \frac{k\bar{n} - \lambda}{\lambda_+} - \frac{k\bar{n} + \lambda}{\lambda_-}\right)(z\eta + z^*\eta^*) - \frac{8\varepsilon^2}{\lambda_+^2}(z\eta^* + z^*\eta) + \frac{4\varepsilon^2}{\lambda_+^2}(z^2 + z^{*2} + \eta^2 + \eta^{*2}). \quad (3.107)$$

Now on the account of Eq. (3.107) expression (3.100) takes the form

$$\begin{aligned} \phi_a(z, \eta, t) = & \exp \left[ -\left(1 + \left(\frac{4\varepsilon^2}{\lambda_+^2} + \frac{k\bar{n} - \lambda}{\lambda_+} + \frac{k\bar{n} + \lambda}{\lambda_-}\right)(z^*z + \eta^*\eta)\right. \right. \\ & + \left.\left(\frac{4\varepsilon^2}{\lambda_+^2} + \frac{k\bar{n} - \lambda}{\lambda_+} - \frac{k\bar{n} + \lambda}{\lambda_-}\right)(z\eta + z^*\eta^*) \right. \\ & \left. \left. - \frac{4\varepsilon^2}{\lambda_+^2}(z\eta^* + z^*\eta) + \frac{2\varepsilon^2}{\lambda_+^2}(z^2 + z^{*2} + \eta^2 + \eta^{*2}) \right], \end{aligned} \quad (3.108)$$

or

$$\phi_a(z, \eta, t) = \exp \left[ -a(z^*z + \eta^*\eta) - b(z\eta + z^*\eta^*) - c(z\eta^* + z^*\eta) + d(z^2 + z^{*2} + \eta^2 + \eta^{*2}) \right], \quad (3.109)$$

where

$$a = 1 + \frac{4\varepsilon^2}{(k + 2\lambda)^2} + \frac{2k^2\bar{n}}{k^2 - 4\lambda^2} + \frac{4\lambda^2}{k^2 - 4\lambda^2}, \quad (3.110)$$

$$b = \frac{4k\bar{n}\lambda}{k^2 - 4\lambda^2} + \frac{2k\lambda}{k^2 - 4\lambda^2} - \frac{4\varepsilon^2}{(k + 2\lambda)^2}, \quad (3.111)$$

$$c = \frac{4\varepsilon^2}{(k + 2\lambda)^2}, \quad (3.112)$$

$$d = \frac{2\varepsilon^2}{(k + 2\lambda)^2}. \quad (3.113)$$

Now upon introducing Eq. (3.109) into (3.95), the Q function for the two-mode cavity light turns out to be

$$\begin{aligned} Q(\alpha, \beta, t) = & \frac{1}{\pi^4} \int d^2z d^2\eta \exp \left[ -a(z^*z + \eta^*\eta) - b(z\eta + z^*\eta^*) - c(z\eta^* + z^*\eta) \right. \\ & \left. + d(z^2 + z^{*2} + \eta^2 + \eta^{*2}) + z^*\alpha - z\alpha^* + \eta^*\beta - \eta\beta^* \right] \\ = & \frac{1}{\pi^2} \int \frac{d^2z}{\pi} \exp(-az^*z + dz^2 + dz^{*2} + z^*\alpha - z\alpha^*) \\ \times & \int \frac{d^2\eta}{\pi} \exp(-a\eta^*\eta - bz\eta - bz^*\eta^* \\ - & cz\eta^* - cz^*\eta + d\eta^2 + d\eta^{*2} + \eta^*\beta - \eta\beta^*). \end{aligned} \quad (3.114)$$

Furthermore, using the relation [2]

$$\begin{aligned} & \int \frac{d^2z}{\pi} \exp(-az^*z + bz + cz^* + Az^2 + Bz^{*2}) \\ = & \left[ \frac{1}{a^2 - 4AB} \right]^{\frac{1}{2}} \exp\left(\frac{abc + Ac^2 + Bb^2}{a^2 - 4AB}\right), \quad a > 0, \end{aligned} \quad (3.115)$$

and performing the integration over  $\eta$ , one readily obtains

$$\begin{aligned}
Q(\alpha, \beta, t) &= \frac{1}{\pi^2 \sqrt{a^2 - 4d^2}} \exp\left(\frac{-a\beta^* \beta + d\beta^2 + d\beta^{*2}}{a^a - 4d^2}\right) \\
&\times \int \frac{d^2 z}{\pi} \exp\left[-\left(\frac{a - ab^2 - ac^2 - 4dbc}{a^2 - 4d^2}\right) z^* z\right. \\
&- \left(\frac{\alpha^* + ab\beta - ac\beta^* + 2dc\beta - 2dc\beta^*}{a^2 - 4d^2}\right) z \\
&+ \left(\frac{\alpha - ac\beta + ab\beta^* - 2db\beta + 2dc\beta^*}{a^2 - 4d^2}\right) z^* \\
&\left. + \left(\frac{d + abc + dc^2 + db^2}{a^2 - 4d^2}\right) z^2 + \left(\frac{d + abc + dc^2 + db^2}{a^2 - 4d^2}\right) z^{*2}\right], \quad (3.116)
\end{aligned}$$

or

$$\begin{aligned}
Q(\alpha, \beta, t) &= \frac{1}{\pi^2 \sqrt{a^2 - 4d^2}} \exp\left(\frac{-a\beta^* \beta + d\beta^2 + d\beta^{*2}}{a^a - 4d^2}\right) \int \frac{d^2 z}{\pi} \exp\left[-P z^* z\right. \\
&- \left(\frac{\alpha^* + ab\beta - ac\beta^* + 2dc\beta - 2dc\beta^*}{a^2 - 4d^2}\right) z \\
&\left. + \left(\frac{\alpha - ac\beta + ab\beta^* - 2db\beta + 2dc\beta^*}{a^2 - 4d^2}\right) z^* + S z^2 + S z^{*2}\right], \quad (3.117)
\end{aligned}$$

where

$$P = \frac{a - ab^2 - ac^2 - 4dbc}{a^2 - 4d^2}, \quad (3.118)$$

$$S = \frac{d + abc + dc^2 + db^2}{a^2 - 4d^2}. \quad (3.119)$$

Furthermore, integrating over  $z$  the  $Q$  function can be written as

$$\begin{aligned}
Q(\alpha, \beta, t) &= \frac{1}{\pi^2 \sqrt{a^2 - 4d^2}} \exp\left(\frac{-a\beta^* \beta + d\beta^2 + d\beta^{*2}}{a^a - 4d^2}\right) \frac{1}{\sqrt{P^2 - 4S^2}} \\
&\times \exp\left[-\frac{p}{(a^2 - 4d^2)^2 (P^2 - 4S^2)} \alpha^* \alpha + \frac{S}{(a^2 - 4d^2)^2 (P^2 - 4S^2)} (\alpha^2 + \alpha^{*2})\right. \\
&- \frac{N}{(a^2 - 4d^2)^2 (P^2 - 4S^2)} \beta^* \beta - \frac{M}{(a^2 - 4d^2)^2 (P^2 - 4S^2)} (\alpha\beta + \alpha^* \beta^*) \\
&+ \frac{L}{(a^2 - 4d^2)^2 (P^2 - 4S^2)} (\alpha^* \beta + \alpha \beta^*) \\
&\left. + \frac{R}{(a^2 - 4d^2)^2 (P^2 - 4S^2)} (\beta^2 + \beta^{*2})\right], \quad (3.120)
\end{aligned}$$

where

$$\begin{aligned}
N &= a^2 b^2 P + a^2 c^2 P + 4d^2 c^2 P + 4d^2 b^2 P + 8dabcP \\
&+ 4a^2 bcS + 8dab^2 S + 8dac^2 S + 16d^2 bcS, \quad (3.121)
\end{aligned}$$

$$\begin{aligned}
R &= a^2 bcP + 2dab^2 P + 2dac^2 P + 4d^2 bcP + a^2 c^2 S \\
&+ 8dabcS + 4d^2 b^2 S + a^2 b^2 S + 4d^2 c^2 S, \quad (3.122)
\end{aligned}$$

$$M = abP + 2dcP + 2acS + 4dbS, \quad (3.123)$$

$$L = acP + 2dbP + 2abS + 4dcS. \quad (3.124)$$

Finally, the Q function for the two-modes turns out to be

$$\begin{aligned} Q(\alpha, \beta, t) = & \frac{1}{\pi^2 \sqrt{z}} \exp \left[ -u(\alpha^* \alpha + \beta^* \beta) + v(\alpha^2 + \alpha^{*2}) - w(\alpha\beta + \alpha^* \beta^*) \right. \\ & \left. + \mu(\alpha^* \beta + \alpha \beta^*) + \Psi(\beta^2 + \beta^{*2}) \right], \end{aligned} \quad (3.125)$$

where

$$z = (a^2 - 4d^2)(P^2 - 4S^2), \quad (3.126)$$

$$u = \frac{P}{z(a^2 - 4d^2)}, \quad (3.127)$$

$$v = \frac{S}{z(a^2 - 4d^2)}, \quad (3.128)$$

$$w = \frac{M}{z(a^2 - 4d^2)}, \quad (3.129)$$

$$\mu = \frac{L}{z(a^2 - 4d^2)}, \quad (3.130)$$

$$\Psi = \frac{dz + R}{z(a^2 - 4d^2)}. \quad (3.131)$$

### 3.3.1 The photon number distribution

We are now in a position to calculate the photon number distribution for the two-mode cavity light. The photon number distribution for observing  $m$  photons of cavity mode  $a$  and  $n$  photons of cavity mode  $b$  are expressible in terms of the Q function as [2]

$$P(m, n, t) = \frac{\pi^2}{m!n!} \frac{\partial^{2m}}{\partial \alpha^m \partial \alpha^{*m}} \frac{\partial^{2n}}{\partial \beta^n \partial \beta^{*n}} \left[ Q(\alpha, \beta, t) e^{\alpha\alpha^* + \beta\beta^*} \right]_{\alpha=\alpha^*=\beta=\beta^*=0}. \quad (3.132)$$

Now substituting Eq. (3.125) into (3.132), we have

$$\begin{aligned} P(m, n, t) = & \frac{1}{\sqrt{z} m! n!} \frac{\partial^{2m}}{\partial \alpha^m \partial \alpha^{*m}} \frac{\partial^{2n}}{\partial \beta^n \partial \beta^{*n}} \left[ \exp \left( -u(\alpha^* \alpha + \beta^* \beta) \right. \right. \\ & + v(\alpha^2 + \alpha^{*2}) - w(\alpha\beta + \alpha^* \beta^*) + \mu(\alpha^* \beta + \alpha \beta^*) \\ & \left. \left. + \Psi(\beta^2 + \beta^{*2}) + \alpha\alpha^* + \beta\beta^* \right) \right]_{\alpha=\alpha^*=\beta=\beta^*=0} \end{aligned} \quad (3.133)$$

$$\begin{aligned} = & \frac{1}{\sqrt{z} m! n!} \frac{\partial^{2m}}{\partial \alpha^m \partial \alpha^{*m}} \frac{\partial^{2n}}{\partial \beta^n \partial \beta^{*n}} \left[ \exp \left( (1-u)(\alpha^* \alpha + \beta^* \beta) + v(\alpha^2 + \alpha^{*2}) \right. \right. \\ & \left. \left. - w(\alpha\beta + \alpha^* \beta^*) + \mu(\alpha^* \beta + \alpha \beta^*) + \Psi(\beta^2 + \beta^{*2}) \right) \right]_{\alpha=\alpha^*=\beta=\beta^*=0} \end{aligned} \quad (3.134)$$

$$\begin{aligned}
&= \frac{1}{\sqrt{z}m!n!} \frac{\partial^{2m}}{\partial \alpha^m \partial \alpha^{*m}} \frac{\partial^{2n}}{\partial \beta^n \partial \beta^{*n}} \left[ \sum_{ijklpqrst} ((1-u)\alpha^*\alpha)^i ((1-u)\beta^*\beta)^j (v\alpha^2)^l (v\alpha^{*2})^k \right. \\
&\times \left. \frac{(-w\alpha\beta)^t (-w\alpha^*\beta^*)^o (\mu\alpha^*\beta)^p (\mu\alpha\beta^*)^q (\Psi\beta^2)^r (\Psi\beta^{*2})^s}{i!j!k!l!o!p!q!r!s!t!} \right]_{\alpha=\alpha^*=\beta=\beta^*=0}. \quad (3.135)
\end{aligned}$$

So that on carrying out the differentiation employing the relation  $\frac{\partial^n x^r}{\partial x^n} = \frac{r!}{(r-n)!} x^{r-n}$  [2] and applying the condition  $\alpha = \alpha^* = \beta = \beta^* = 0$ , there follows

$$\begin{aligned}
P(m, n, t) &= \frac{1}{\sqrt{z}m!n!} \sum_{ijklpqrst} \frac{(-1)^t (-1)^o (1-u)^{i+j} (v)^{k+l} (w)^{t+o} (\mu)^{p+q} (\Psi)^{r+s}}{i!j!k!l!o!p!q!r!s!t!} \\
&\times \frac{(i+2k+t+q)!}{(i+2k+t+q-m)!} \frac{(i+2l+o+p)!}{(i+2l+o+p-m)!} \frac{(j+t+p+2r)!}{(j+t+p+2r-n)!} \\
&\times \frac{(j+o+q+2s)!}{(j+o+q+2s-n)!} \delta_{i+2k+t+q,m} \delta_{i+2l+o+p,m} \\
&\times \delta_{j+t+p+2r,n} \delta_{j+o+q+2s,n}. \quad (3.136)
\end{aligned}$$

It is interesting to consider a special case in which the coherent driving light is absent ( $\varepsilon = 0$ ). Thus upon setting  $\varepsilon = 0$ , we get

$$v = \mu = \Psi = 0. \quad (3.137)$$

With the aid of Eq. (3.137) the photon number distribution reduces to

$$\begin{aligned}
P(m, n, t) &= \frac{(u'^2 - \omega'^2)}{m!n!} \sum_{ijto} \frac{(-1)^{t+o} (1-u')^{i+j} \omega'^{t+o}}{i!j!t!o!} \frac{(i+t)!}{(i+t-m)!} \\
&\times \frac{(i+o)!}{(i+o-m)!} \frac{(j+t)!}{(j+t-n)!} \frac{(j+o)!}{(j+o-n)!} \\
&\times \delta_{i+t,m} \delta_{i+o,m} \delta_{j+t,n} \delta_{j+o,n}. \quad (3.138)
\end{aligned}$$

Applying the properties of the Kronecker delta, we have

$$t = m - i, o = m - i, \quad (3.139)$$

and

$$t = n - j, o = n - j. \quad (3.140)$$

We note that

$$t = o = m - i = n - j. \quad (3.141)$$

Therefore for  $m = n$  and on account of Eqs. (3.141) along with the fact that a factorial is defined for nonnegative integers, the photon number distribution can be put in the form

$$P(n, n, t) = (u'^2 - \omega'^2) \sum_{i=0}^n \frac{n!^2 (1-u')^{2i} \omega'^{2(n-i)}}{i!^2 [(n-i)!]^2}, \quad (3.142)$$

where

$$u' = \frac{a'}{a'^2 - b'^2}, \quad (3.143)$$

$$\omega' = \frac{b'}{a'^2 - b'^2}, \quad (3.144)$$

$$a' = 1 + \frac{2k^2\bar{n}}{k^2 - 4\lambda^2} + \frac{4\lambda^2}{k^2 - 4\lambda^2}, \quad (3.145)$$

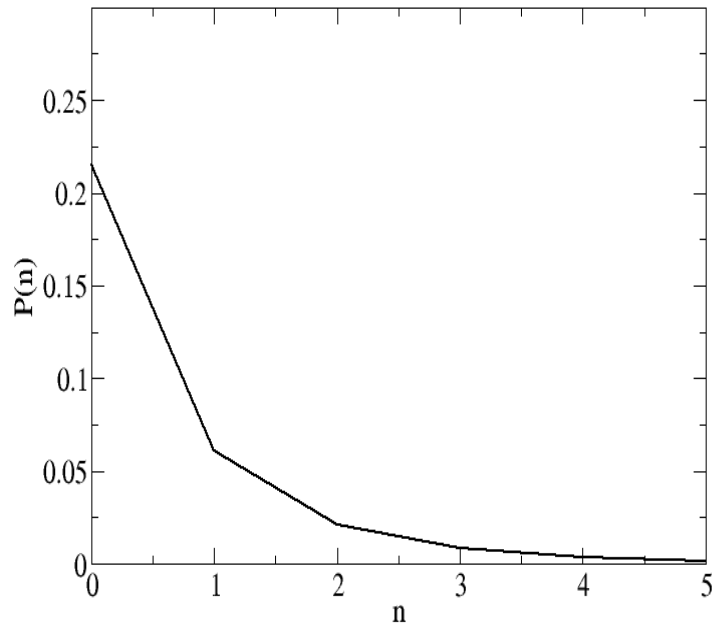
$$b' = \frac{4k\bar{n}\lambda}{k^2 - 4\lambda^2} + \frac{2k\lambda}{k^2 - 4\lambda^2}. \quad (3.146)$$

This describes the joint probability to observe equal number of cavity mode  $a$  and cavity mode  $b$  photons. From this result we note that there is a finite probability of finding equal number of photons in the two-modes.

In addition, in the absence of the parametric interaction ( $\lambda = 0$ ) and the driving coherent light modes ( $\varepsilon = 0$ ), Eq. (3.142) reduces to

$$P(n, n, t) = \frac{(2\bar{n})^{2n}}{(2\bar{n} + 1)^{2n+2}}. \quad (3.147)$$

This the photon number distribution for a two-mode chaotic light.



**Figure 3.2:** Plots of the photon number distribution [Eq. (3.142)] at steady state versus the photon number for  $\bar{n} = 0.5$ ,  $\lambda = 0.15$ ,  $\varepsilon = 0$  and  $k = 0.8$ .

As can be seen from Fig. 3.2 the steady state photon number distribution decreases with the photon number. Though the photons are generated in pairs in this quantum optical system, there is a finite probability to find odd number of photons inside the cavity. This is due to the damping of the cavity mode.

## Quadrature Variance and Quadrature Squeezing

Applying the solutions of the c-number Langevin equations along with the correlation properties of the noise forces, we calculate the quadrature variance and quadrature squeezing for the cavity light

### 4.1 Quadrature variance of the cavity mode

We now proceed to calculate the variance of the plus and minus quadrature of the the two-mode cavity light produced by the nondegenerate subharmonic generator. The squeezing properties of a two-mode cavity light is described by two quadrature operators [2]

$$\hat{c}_+ = \hat{c}^\dagger + \hat{c}, \quad (4.1)$$

$$\hat{c}_- = i(\hat{c}^\dagger - \hat{c}), \quad (4.2)$$

where

$$\hat{c} = \hat{a} + \hat{b}. \quad (4.3)$$

Using the commutation relation

$$[\hat{a}, \hat{a}^\dagger] = [\hat{b}, \hat{b}^\dagger] = 1, \quad (4.4)$$

we see that

$$[\hat{c}, \hat{c}^\dagger] = 2. \quad (4.5)$$

Now with the aid of Eqs. (4.1), (4.2), and (4.5), we readily find

$$\begin{aligned} [\hat{c}_+, \hat{c}_-] &= [\hat{c}^\dagger + \hat{c}, i(\hat{c}^\dagger - \hat{c})] \\ &= i[\hat{c}, \hat{c}^\dagger] + i[\hat{c}, \hat{c}^\dagger] \\ &= 4i. \end{aligned} \quad (4.6)$$

The operators  $\hat{c}_+$  and  $\hat{c}_-$  represents physical quantities called the plus and minus quadrature respectively.

A two-mode light is said to be in a squeezed state if either  $\Delta c_+ < \sqrt{2}$  and  $\Delta c_- > \sqrt{2}$  or  $\Delta c_+ > \sqrt{2}$  and  $\Delta c_- < \sqrt{2}$  such that  $\Delta c_+ \Delta c_- \geq 2$ . A squeezed state for which the equality holds is called a minimum uncertainty state. We then note that the noise in one quadrature of a squeezed state is below the vacuum state level at the expense of enhanced noise in the other quadrature.

The variance of the quadrature operators are defined by

$$(\Delta c_{\pm}(t))^2 = \langle \hat{c}_{\pm}^2(t) \rangle - \langle \hat{c}_{\pm}(t) \rangle^2. \quad (4.7)$$

Applying Eqs. (4.1) and (4.2), one can write (4.7) in the normal order as

$$(\Delta c_{\pm}(t))^2 = 2 \pm \langle \hat{c}^{\dagger 2}(t) + \hat{c}^2(t) \pm 2\hat{c}^{\dagger}(t)\hat{c}(t) \rangle \mp \langle \hat{c}^{\dagger}(t) \pm \hat{c}(t) \rangle^2. \quad (4.8)$$

This can be expressed in terms of c-number variables associated with the normal ordering as

$$(\Delta c_{\pm}(t))^2 = 2 \pm \langle \gamma^{*2}(t) + \gamma^2(t) \pm 2\gamma^*(t)\gamma(t) \rangle \mp \langle \gamma^*(t) \pm \gamma(t) \rangle^2, \quad (4.9)$$

where  $\gamma(t)$  is the c-number variable corresponding to the operator  $\hat{c}(t)$ . The c-number equation corresponding to Eq. (4.3) can be written as

$$\gamma = \alpha + \beta, \quad (4.10)$$

Introducing a new variable defined by

$$\gamma_{\pm}(t) = \gamma^*(t) \pm \gamma(t), \quad (4.11)$$

Eq. (4.9) can be rewritten as

$$(\Delta c_{\pm}(t))^2 = 2 \pm (\langle \gamma_{\pm}^2(t) \rangle - \langle \gamma_{\pm}(t) \rangle^2). \quad (4.12)$$

Now in view of Eq. (4.10), one can write

$$\frac{d}{dt}\gamma(t) = \frac{d}{dt}\alpha(t) + \frac{d}{dt}\beta(t) \quad (4.13)$$

Applying Eqs. (2.81) and (2.83) into (4.13), we have

$$\frac{d}{dt}\gamma(t) = \frac{-k}{2}\gamma(t) - \lambda\gamma^*(t) + 2\varepsilon + f_{\alpha}(t) + f_{\beta}(t), \quad (4.14)$$

where

$$\gamma(t) = \alpha(t) + \beta(t). \quad (4.15)$$

Taking the time derivatives of Eq. (4.11), we get

$$\frac{d}{dt}\gamma_{\pm}(t) = \frac{d}{dt}\gamma^*(t) \pm \frac{d}{dt}\gamma(t). \quad (4.16)$$

Furthermore, employing Eq. (4.14) and its complex conjugate along with (4.16), one can easily establish that

$$\frac{d}{dt}\gamma_{\pm}(t) = \frac{-1}{2}\lambda_{\pm}\gamma_{\pm}(t) + 2(\varepsilon \pm \varepsilon) + f_{\alpha}^*(t) \pm f_{\alpha}(t) + f_{\beta}^*(t) \pm f_{\beta}(t). \quad (4.17)$$

For  $2\lambda < k$ , the solution of Eq. (4.17) can be put in the form

$$\begin{aligned} \gamma_{\pm}(t) = & \gamma_{\pm}(0)e^{-\lambda_{\pm}t/2} + \int_0^t e^{-\lambda_{\pm}(t-t')/2} \left[ 2(\varepsilon \pm \varepsilon) \right. \\ & \left. + f_{\alpha}^*(t') \pm f_{\alpha}(t') + f_{\beta}^*(t') \pm f_{\beta}(t') \right] dt', \end{aligned} \quad (4.18)$$

so that taking the expectation value both side Eq. (4.18) and taking into account Eqs. (2.85), (2.86) and (2.87), we arrive at

$$\langle \gamma_{\pm}(t) \rangle = \langle \gamma_{\pm}(0) \rangle e^{-\lambda_{\pm}t/2} + \int_0^t e^{-\lambda_{\pm}(t-t')/2} 2(\varepsilon \pm \varepsilon) dt'. \quad (4.19)$$

Now carrying out the integration, one easily gets

$$\langle \gamma_{\pm}(t) \rangle = \langle \gamma_{\pm}(0) \rangle e^{-\lambda_{\pm}t/2} + \frac{4}{\lambda_{\pm}}(\varepsilon \pm \varepsilon)(1 - e^{-\lambda_{\pm}t/2}). \quad (4.20)$$

Moreover, applying the relation

$$\frac{d}{dt}\gamma_{\pm}^2(t) = \gamma_{\pm}(t)\frac{d\gamma_{\pm}(t)}{dt} + \frac{d\gamma_{\pm}(t)}{dt}\gamma_{\pm}(t), \quad (4.21)$$

along with Eq. (4.17), we have

$$\begin{aligned} \frac{d}{dt}\gamma_{\pm}^2(t) = & -\lambda_{\pm}\gamma_{\pm}^2(t) + 4(\varepsilon \pm \varepsilon)\gamma_{\pm}(t) + \gamma_{\pm}(t)f_{\alpha}^*(t) \pm \gamma_{\pm}(t)f_{\alpha}(t) \\ & + f_{\alpha}^*(t)\gamma_{\pm}(t) \pm f_{\alpha}(t)\gamma_{\pm}(t) + \gamma_{\pm}(t)f_{\beta}^*(t) \\ & \pm \gamma_{\pm}(t)f_{\beta}(t) + f_{\beta}^*(t)\gamma_{\pm}(t) \pm f_{\beta}(t)\gamma_{\pm}(t), \end{aligned} \quad (4.22)$$

or

$$\begin{aligned} \frac{d}{dt}\langle \gamma_{\pm}^2(t) \rangle = & -\lambda_{\pm}\langle \gamma_{\pm}^2(t) \rangle + 4(\varepsilon \pm \varepsilon)\langle \gamma_{\pm}(t) \rangle + \langle \gamma_{\pm}(t)f_{\alpha}^*(t) \rangle \pm \langle \gamma_{\pm}(t)f_{\alpha}(t) \rangle \\ & + \langle f_{\alpha}^*(t)\gamma_{\pm}(t) \rangle \pm \langle f_{\alpha}(t)\gamma_{\pm}(t) \rangle + \langle \gamma_{\pm}(t)f_{\beta}^*(t) \rangle \\ & \pm \langle \gamma_{\pm}(t)f_{\beta}(t) \rangle + \langle f_{\beta}^*(t)\gamma_{\pm}(t) \rangle \pm \langle f_{\beta}(t)\gamma_{\pm}(t) \rangle. \end{aligned} \quad (4.23)$$

Multiplying both side Eq. (4.18) by  $f_{\alpha}^*(t)$  from the right side and taking the expectation value of the resulting expression, we obtain

$$\begin{aligned} \langle \gamma_{\pm}(t)f_{\alpha}^*(t) \rangle = & \langle \gamma_{\pm}(0)f_{\alpha}^*(t) \rangle e^{-\lambda_{\pm}t/2} + 2(\varepsilon \pm \varepsilon) \int_0^t e^{-\lambda_{\pm}(t-t')/2} \langle f_{\alpha}^*(t) \rangle dt' \\ & + \int_0^t e^{-\lambda_{\pm}(t-t')/2} \left( \langle f_{\alpha}^*(t')f_{\alpha}^*(t) \rangle \pm \langle f_{\alpha}(t')f_{\alpha}^*(t) \rangle \right) dt' \\ & + \int_0^t e^{-\lambda_{\pm}(t-t')/2} \left( \langle f_{\beta}^*(t')f_{\alpha}^*(t) \rangle \pm \langle f_{\beta}(t')f_{\alpha}^*(t) \rangle \right) dt'. \end{aligned} \quad (4.24)$$

On account of the assertion that a noise force at time  $t$  should not affect system variables at earlier times, we note that

$$\langle \gamma_{\pm}(0) f_{\alpha}^*(t) \rangle = \langle \gamma_{\pm}(0) \rangle \langle f_{\alpha}^*(t) \rangle. \quad (4.25)$$

Now taking into account Eq. (2.87), we see that

$$\langle \gamma_{\pm}(0) f_{\alpha}^*(t) \rangle = 0. \quad (4.26)$$

Applying Eqs. (2.87), (2.116), (2.133), (2.134), (2.2.135) and (4.26) in (4.24), we get

$$\langle \gamma_{\pm}(t) f_{\alpha}^*(t) \rangle = \pm k\bar{n} \int_0^t e^{-\lambda_{\pm}(t-t')/2} \delta(t-t') dt' - \lambda \int_0^t e^{-\lambda_{\pm}(t-t')/2} \delta(t-t') dt'. \quad (4.27)$$

On the basis of the relation [2]

$$M \int_0^t e^{-b(t-t')/2} \delta(t-t') dt' = \frac{M}{2}, \quad (4.28)$$

one can write Eq. (4.27) as

$$\langle \gamma_{\pm}(t) f_{\alpha}^*(t) \rangle = -\frac{\lambda}{2} \pm \frac{k\bar{n}}{2}. \quad (4.29)$$

Furthermore, multiplying both side Eq. (4.18) by  $f_{\alpha}(t)$  from the right side and taking the expectation value of the resulting expression, we have

$$\begin{aligned} \langle \gamma_{\pm}(t) f_{\alpha}(t) \rangle &= \langle \gamma_{\pm}(0) f_{\alpha}(t) \rangle e^{-\lambda_{\pm}t/2} + 2(\varepsilon \pm \varepsilon) \int_0^t e^{-\lambda_{\pm}(t-t')/2} \langle f_{\alpha}(t) \rangle dt' \\ &+ \int_0^t e^{-\lambda_{\pm}(t-t')/2} \left( \langle f_{\alpha}^*(t') f_{\alpha}(t) \rangle \pm \langle f_{\alpha}(t') f_{\alpha}(t) \rangle \right) dt' \\ &+ \int_0^t e^{-\lambda_{\pm}(t-t')/2} \left( \langle f_{\beta}^*(t') f_{\alpha}(t) \rangle \pm \langle f_{\beta}(t') f_{\alpha}(t) \rangle \right) dt'. \end{aligned} \quad (4.30)$$

Employing Eqs. (2.85), (2.105), (2.116), (2.128), (2.134) and using the fact that a noise force at time  $t$  doesn't affect that cavity mode variables at earlier time, we get

$$\langle \gamma_{\pm}(t) f_{\alpha}(t) \rangle = k\bar{n} \int_0^t e^{-\lambda_{\pm}(t-t')/2} \delta(t-t') dt' \mp \lambda \int_0^t e^{-\lambda_{\pm}(t-t')/2} \delta(t-t') dt'. \quad (4.31)$$

On the basis of Eq. (4.28), one can write

$$\langle \gamma_{\pm}(t) f_{\alpha}(t) \rangle = \frac{k\bar{n}}{2} \mp \frac{\lambda}{2}. \quad (4.32)$$

Moreover, multiplying both side Eq. (4.18) by  $f_{\beta}(t)$  from the right side and taking the expectation value of the resulting expression, we have

$$\begin{aligned} \langle \gamma_{\pm}(t) f_{\beta}(t) \rangle &= \langle \gamma_{\pm}(0) f_{\beta}(t) \rangle e^{-\lambda_{\pm}t/2} + 2(\varepsilon \pm \varepsilon) \int_0^t e^{-\lambda_{\pm}(t-t')/2} \langle f_{\beta}(t) \rangle dt' \\ &+ \int_0^t e^{-\lambda_{\pm}(t-t')/2} \left( \langle f_{\alpha}^*(t') f_{\beta}(t) \rangle \pm \langle f_{\alpha}(t') f_{\beta}(t) \rangle \right) dt' \\ &+ \int_0^t e^{-\lambda_{\pm}(t-t')/2} \left( \langle f_{\beta}^*(t') f_{\beta}(t) \rangle \pm \langle f_{\beta}(t') f_{\beta}(t) \rangle \right) dt'. \end{aligned} \quad (4.33)$$

Substituting Eqs. (2.86), (2.128), (2.131), (2.132) and (2.134) into (4.33) with the assertion a noise force at time  $t$  should not affect system variables at earlier times, we find

$$\langle \gamma_{\pm}(t) f_{\beta}(t) \rangle = \mp \lambda \int_0^t e^{-\lambda_{\pm}(t-t')/2} \delta(t-t') dt' + k\bar{n} \int_0^t e^{-\lambda_{\pm}(t-t')/2} \delta(t-t') dt'. \quad (4.34)$$

In view of the relation of Eq. (4.28), we easily find

$$\langle \gamma_{\pm}(t) f_{\beta}(t) \rangle = \mp \frac{\lambda}{2} + \frac{k\bar{n}}{2}, \quad (4.35)$$

or

$$\langle \gamma_{\pm}(t) f_{\beta}(t) \rangle = \frac{k\bar{n}}{2} \mp \frac{\lambda}{2}. \quad (4.36)$$

It can also be established in a similar manner that

$$\langle \gamma_{\pm}(t) f_{\beta}^*(t) \rangle = -\frac{\lambda}{2} \pm \frac{k\bar{n}}{2}. \quad (4.37)$$

$$\langle f_{\alpha}^*(t) \gamma_{\pm}(t) \rangle = -\frac{\lambda}{2} \pm \frac{k\bar{n}}{2}. \quad (4.38)$$

$$\langle f_{\alpha}(t) \gamma_{\pm}(t) \rangle = \frac{k\bar{n}}{2} \mp \frac{\lambda}{2}. \quad (4.39)$$

$$\langle f_{\beta}^*(t) \gamma_{\pm}(t) \rangle = -\frac{\lambda}{2} \pm \frac{k\bar{n}}{2}. \quad (4.40)$$

$$\langle f_{\beta}(t) \gamma_{\pm}(t) \rangle = \frac{k\bar{n}}{2} \mp \frac{\lambda}{2}. \quad (4.41)$$

Now substituting Eqs. (4.29), (4.32), (4.36), (4.37), (4.38), (4.39), (4.40) and (4.41) into (4.23), we obtain

$$\frac{d}{dt} \langle \gamma_{\pm}^2(t) \rangle = -\lambda_{\pm} \langle \gamma_{\pm}^2(t) \rangle + 4(\varepsilon \pm \varepsilon) \langle \gamma_{\pm}(t) \rangle - 4\lambda \pm 4k\bar{n}. \quad (4.42)$$

This can be rewritten on account of (4.20) as

$$\begin{aligned} \frac{d}{dt} \langle \gamma_{\pm}^2(t) \rangle &= -\lambda_{\pm} \langle \gamma_{\pm}^2(t) \rangle + 4(\varepsilon \pm \varepsilon) \left[ \langle \gamma_{\pm}(0) \rangle e^{-\lambda_{\pm} t/2} \right. \\ &\quad \left. + \frac{4}{\lambda_{\pm}} (\varepsilon \pm \varepsilon) (1 - e^{-\lambda_{\pm} t/2}) \right] - 4\lambda \pm 4k\bar{n} \\ &= -\lambda_{\pm} \langle \gamma_{\pm}^2(t) \rangle + 4(\varepsilon \pm \varepsilon) \langle \gamma_{\pm}(0) \rangle e^{-\lambda_{\pm} t/2} \\ &\quad + \frac{16}{\lambda_{\pm}} (\varepsilon \pm \varepsilon)^2 (1 - e^{-\lambda_{\pm} t/2}) - 4\lambda \pm 4k\bar{n}. \end{aligned} \quad (4.43)$$

The solution of Eq. (4.43) can be put in the form

$$\begin{aligned} \langle \gamma_{\pm}^2(t) \rangle &= \langle \gamma_{\pm}^2(0) \rangle e^{-\lambda_{\pm} t} + 4(\varepsilon \pm \varepsilon) \langle \gamma_{\pm}(0) \rangle \int_0^t e^{-\lambda_{\pm}(t-t')/2} dt' \\ &\quad + \frac{16}{\lambda_{\pm}} (\varepsilon \pm \varepsilon)^2 \int_0^t e^{-\lambda_{\pm}(t-t')} dt' \\ &\quad - \frac{16}{\lambda_{\pm}} (\varepsilon \pm \varepsilon)^2 \int_0^t e^{-\lambda_{\pm}(t-t'/2)} dt' \\ &\quad + (-4\lambda \pm 4k\bar{n}) \int_0^t e^{-\lambda_{\pm}(t-t')} dt', \end{aligned} \quad (4.44)$$

so that on performing the integration, we get

$$\begin{aligned}
\langle \gamma_{\pm}^2(t) \rangle &= \langle \gamma_{\pm}^2(0) \rangle e^{-\lambda_{\pm} t} + \frac{8}{\lambda_{\pm}} (\varepsilon \pm \varepsilon) \langle \gamma_{\pm}(0) \rangle (e^{-\lambda_{\pm} t/2} - e^{-\lambda_{\pm} t}) \\
&+ \frac{16}{\lambda_{\pm}^2} (\varepsilon \pm \varepsilon)^2 (1 - e^{-\lambda_{\pm} t}) - \frac{32}{\lambda_{\pm}^2} (\varepsilon \pm \varepsilon)^2 (e^{-\lambda_{\pm} t/2} - e^{-\lambda_{\pm} t}) \\
&+ \left( \frac{-4\lambda \pm 4k\bar{n}}{\lambda_{\pm}} \right) (1 - e^{-\lambda_{\pm} t}) \\
&= \langle \gamma_{\pm}^2(0) \rangle e^{-\lambda_{\pm} t} + \frac{8}{\lambda_{\pm}} (\varepsilon \pm \varepsilon) \langle \gamma_{\pm}(0) \rangle (e^{-\lambda_{\pm} t/2} - e^{-\lambda_{\pm} t}) \\
&+ \frac{16}{\lambda_{\pm}^2} (\varepsilon \pm \varepsilon)^2 (1 - 2e^{-\lambda_{\pm} t/2} + e^{-\lambda_{\pm} t}) + \left( \frac{-4\lambda}{\lambda_{\pm}} \pm \frac{4k\bar{n}}{\lambda_{\pm}} \right) (1 - e^{-\lambda_{\pm} t}). \quad (4.45)
\end{aligned}$$

Moreover, on account of Eq. (4.20), we have

$$\begin{aligned}
\langle \gamma_{\pm}(t) \rangle^2 &= [\langle \gamma_{\pm}(0) \rangle e^{-\lambda_{\pm} t/2} + \frac{4}{\lambda_{\pm}} (\varepsilon \pm \varepsilon) (1 - e^{-\lambda_{\pm} t/2})] \\
&\times [\langle \gamma_{\pm}(0) \rangle e^{-\lambda_{\pm} t/2} + \frac{4}{\lambda_{\pm}} (\varepsilon \pm \varepsilon) (1 - e^{-\lambda_{\pm} t/2})] \\
&= \langle \gamma_{\pm}(0) \rangle^2 e^{-\lambda_{\pm} t} + \frac{8}{\lambda_{\pm}} (\varepsilon \pm \varepsilon) \langle \gamma_{\pm}(0) \rangle (e^{-\lambda_{\pm} t/2} - e^{-\lambda_{\pm} t}) \\
&+ \frac{16}{\lambda_{\pm}^2} (\varepsilon \pm \varepsilon)^2 (1 - 2e^{-\lambda_{\pm} t/2} + e^{-\lambda_{\pm} t}). \quad (4.46)
\end{aligned}$$

Finally, upon substituting Eqs. (4.45) and (4.46) into (4.12), the quadrature variances there follows

$$(\Delta c_{\pm}(t))^2 = 2 \pm \left[ \langle \gamma_{\pm}^2(0) \rangle e^{-\lambda_{\pm} t} - \langle \gamma_{\pm}(0) \rangle^2 e^{-\lambda_{\pm} t} + \left( \frac{-4\lambda}{\lambda_{\pm}} \pm \frac{4k\bar{n}}{\lambda_{\pm}} \right) (1 - e^{-\lambda_{\pm} t}) \right]. \quad (4.47)$$

At steady state, the quadrature variances become

$$(\Delta c_{\pm})^2 = 2 \pm \left( \frac{-4\lambda}{\lambda_{\pm}} \pm \frac{4k\bar{n}}{\lambda_{\pm}} \right). \quad (4.48)$$

Separatly the plus and minus quadrature variance can be written as

$$(\Delta c_{+})^2 = 2 + \frac{4(k\bar{n} - \lambda)}{k + 2\lambda}, \quad (4.49)$$

and

$$(\Delta c_{-})^2 = 2 + \frac{4(k\bar{n} + \lambda)}{k - 2\lambda}. \quad (4.50)$$

On taking into account Eqs. (4.49) and (4.50), we see that at steady state and at thershold

$$(\Delta c_{+})^2 = 1 + 2\bar{n}, \quad (4.51)$$

and

$$(\Delta c_-)^2 = \infty. \quad (4.52)$$

Moreover, when the cavity is coupled to vacuum reservoir ( $\bar{n} = 0$ ), Eqs. (4.49) and (4.50) reduces to

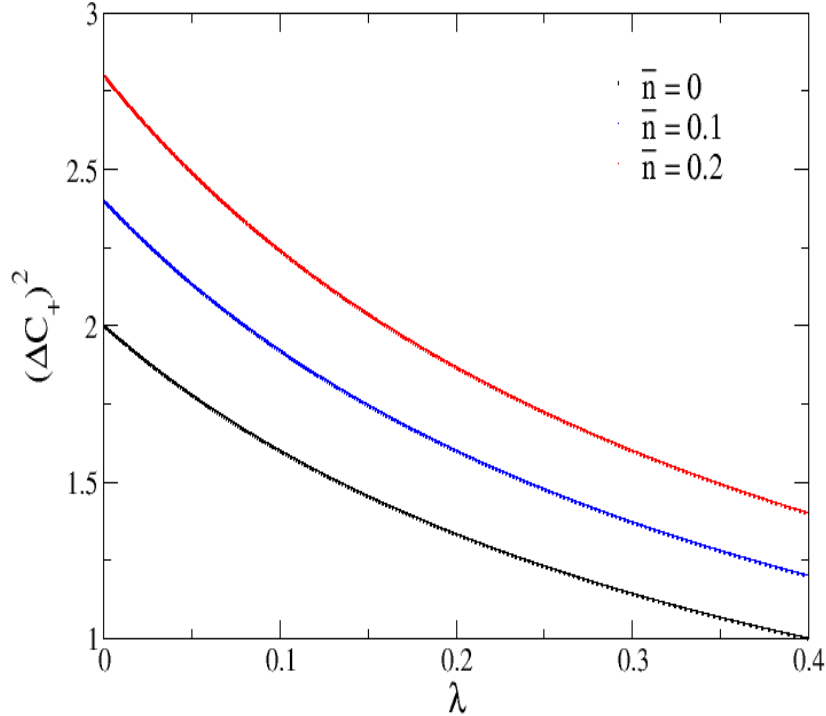
$$(\Delta c_+)^2 = \frac{2k}{k + 2\lambda}, \quad (4.53)$$

and

$$(\Delta c_-)^2 = \frac{2k}{k - 2\lambda}, \quad (4.54)$$

which is quadrature variance of the cavity light generated by two-mode subharmonic generating system coupled to a two-mode vacuum reservoir.

We observe from Eqs. (4.49) and (4.50) that the two-mode cavity light is in a squeezed state and the squeezing occurs in the plus quadrature. Moreover, the deriving coherent light has no effect on squeezing.



**Figure 4.1:** Plots of  $(\Delta c_+)^2$  [Eq. (4.49)] at steady state versus  $\lambda$  for  $k = 0.8$ ,  $\bar{n} = 0$  (black curve),  $\bar{n} = 0.1$  (blue curve),  $\bar{n} = 0.2$  (red curve)

It can be seen from the plots in Fig. 4.1 that generally the quadrature variance decreases as parametric interaction increases. However, the quadrature variance increases as the mean photon number of thermal reservoir increases.

## 4.2 The quadrature squeezing

In this section we seek to study the squeezing properties of the light generated by nondegenerate subharmonic generating system. To this end, we calculate the quadrature squeezing of the two-mode light relative to the quadrature variance of the two-mode cavity light in a vacuum state. That is [2]

$$S_+ = \frac{(\Delta c_+)_{vac}^2 - (\Delta c_+)^2}{(\Delta c_+)_{vac}^2}, \quad (4.55)$$

where  $(\Delta c_+)_{vac}^2$  is quadrature variance of the two-mode vacuum cavity light,  $S_+$  is plus quadrature squeezing and  $(\Delta c_+)^2$  is given by Eq. (4.49).

Now employing Eq. (4.49) the quadrature variance of the cavity light in a vacuum state can be expressed as

$$(\Delta c_+)_{vac}^2 = 2. \quad (4.56)$$

With the aid of Eqs. (4.49) and (4.56), we can put the quadrature squeezing as

$$S_+ = \frac{2\lambda - 2k\bar{n}}{k + 2\lambda}. \quad (4.57)$$

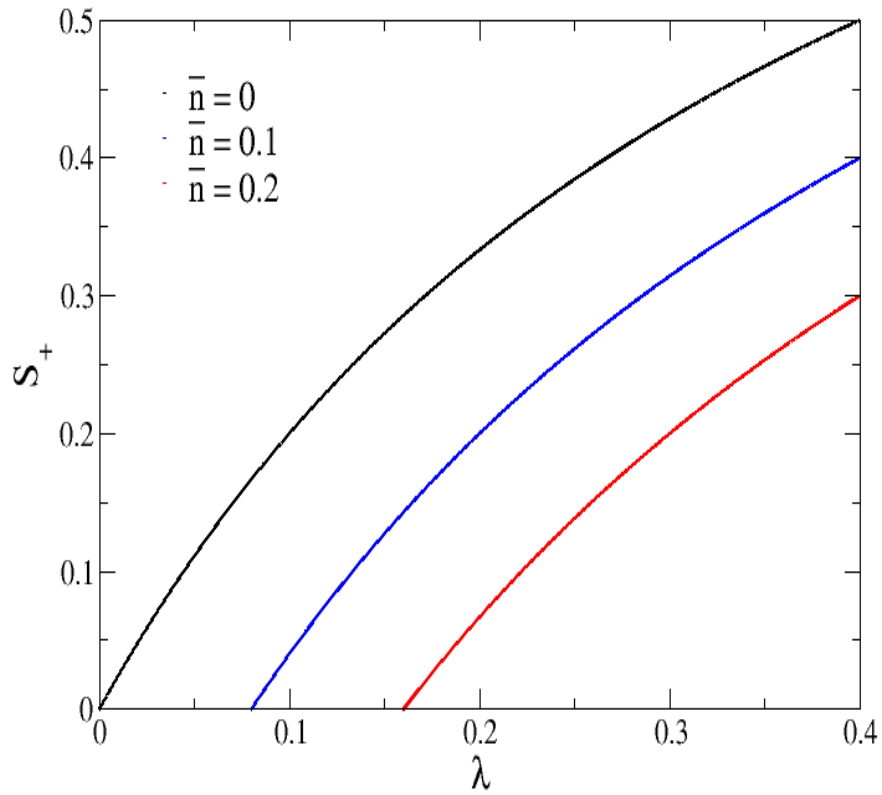
We see that at steady state and at thershold

$$S_+ = \frac{1}{2} - \bar{n}. \quad (4.58)$$

Furthermore, when the cavity is coupled to vacuum reservoir ( $\bar{n} = 0$ ), Eq. (4.57) reduces to

$$S_+ = \frac{2\lambda}{k + 2\lambda}. \quad (4.59)$$

We note that at steady state and at thershold, there is a 50 % squeezing of the two-mode cavity light when the cavity is coupled to vacuum reservoir. We would like to point out that this result is in complete agreement with that obtained in [2].



**Figure 4.2:** Plots of  $S_+$  [Eq. (4.57)] versus  $\lambda$  for  $k = 0.8$ ,  $\bar{n} = 0$  (black curve),  $\bar{n} = 0.1$  (blue curve),  $\bar{n} = 0.2$  (red curve)

We see from Fig. 4.2 that the quadrature squeezing increases with  $\lambda$ . Furthermore, the plot shows that the quadrature squeezing decreases as the mean photon number of the thermal reservoir increases.

## Conclusion

In this thesis we have studied the statistical and squeezing properties of the light generated by two-mode subharmonic generating system with a cavity light driven by two-mode coherent light beams and coupled to two-mode thermal reservoir via a single port-mirror. Applying the master equation for the system under consideration and following the procedure discussed in [2], we determined the c-number Langevin equations for the cavity mode variables associated with the normal-ordering. Using the solutions of these equations, we have calculated the mean and variance of the photon number and the local mean photon number for the cavity light. Our result shows that the mean photon number and the variance of the photon number for mode  $a$  are the same as that of mode  $b$ . We also note that the variance of the photon number is greater than the mean photon number of a cavity light and hence the light produced by nondegenerate subharmonic generator has super Poissonian photon statistics. In addition, we have found that the driving coherent light and the thermal reservoir enhance the mean and variance of the photon number for the cavity mode. Moreover, we have found that the local mean photon number approaches the global mean photon number for a relatively small frequency interval.

Furthermore, employing the solutions of the c-number Langevin equations, we obtained the antinormally-ordered characteristic function. And with the aid of this characteristic function, we have determined the Q-function for the two-mode cavity light. The resulting Q-function is then used to calculate the photon number distribution. From the results we have found, we note that the steady-state photon number distribution decreases with the photon number. And there is also a finite probability of finding even and odd number of photons in the two-mode cavity light.

Finally, we have determined the quadrature variance and quadrature squeezing of the cavity light. Our calculation of the quadrature variance of the two-mode

cavity light indicates that the cavity modes in a squeezed state and the squeezing occurs in the plus quadrature. In addition, we have shown that at steady state and at threshold, there is a maximum of 50% squeezing of the two-mode cavity light when the cavity is coupled to vacuum reservoir. We would like to point out that this result is in complete agreement with that obtained in [2]. We have also seen that the presence of the parametric interaction enhances the squeezing of the light generated by the systems under consideration, while the driving light has no effect on the squeezing. On the other hand, the presence of thermal reservoir decreases the quadrature squeezing of the cavity light.

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**DECLARATION**

ADDIS ABABA UNIVERSITY  
COLLEGE OF NATURAL AND COMPUTATIONAL SCIENCES  
DEPARTMENT OF PHYSICS

MSc Thesis

Two-Mode Subharmonic Generating System with Coherent and Thermal Light

Name of Candidate: Lemma Tirfie Zegbreal

I the under signed declare that the thesis is my original work and no part of it can be claimed as an intellectual property of anybody else except me and my advisors.

Signature: \_\_\_\_\_