MONOPOLE SYNCHROTRON RADIATION

AND

ELECTRON-MONOPOLE SCATTERING

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The power radiated per harmonic into a unit solid angle by a monopole in uniform circular motion is computed. Then the polarization of monopole synchrotron radiation is investigated, and found to have certain distinct features compared with that of electron synchrotron radiation. This, we hope, may serve as a theoretical background in the search for monopoles. Finally, the energy transferred from a scattered monopole to an electron initially at rest is determined.

1. INTRODUCTION

The notion of a monopole was introduced by Dirac [1] in 1931 from symmetry consideration of the field equations of electrodynamics. The requirement that these equations be symmetrical [2] with respect to charge as well as current densities, in addition to the symmetry between electric and magnetic fields inherent in them, leads to postulating the existence of a magnetic charge or a monopole. Such a particle in a state of rest would be a source of magnetic field. In the presence of magnetic charge and current densities, ρ_g and $\vec{J}_g$, in addition to electric densities, ρ_e and $\vec{J}_e$, Maxwell's equations would then assume the form

$$\text{curl } \vec{H} - \frac{1}{c}\frac{\partial \vec{E}}{\partial t} = \frac{4\pi}{c}\vec{J}_e , \quad \text{div } \vec{E} = 4\pi\rho_e ,$$

$$\text{curl } \vec{E} + \frac{1}{c}\frac{\partial \vec{H}}{\partial t} = -\frac{4\pi}{c}\vec{J}_g , \quad \text{div } \vec{H} = 4\pi\rho_g .$$

However, the modified Maxwell's equations give rise to singularities [3] of the electromagnetic potential. The line along which the electromagnetic potential fails to be analytic is called the Dirac string. The Dirac string can be moved arbitrarily by a suitable gauge transformation, and hence is not physically observable [4]. More recently, alternative formulations have originated in which the Dirac singularities are avoided. A typical example is the formulation given by Wu and Young in which space is divided into overlapping regions, on each of which a smooth vector potential is defined [5].

A quantum mechanical consideration of the motion of an electric chargee in the field of a stationary monopole of strength g, that takes into account the arbitrariness in the location of the Dirac string and single - valuedness of the wave function, leads to the Dirac quantization condition (See for a derivation of this appendix A)

$$\frac{eg}{\hbar c} = \frac{n}{2}$$

where n is an integer. Thus the mere existence of a single monopole in the universe would offer a beautiful explanation of the observed discrete nature of electric charge.

Although the notion of a monopole is over half a century old, so far there is no experimental evidence for the existence of monopoles. Nevertheless, the quantization of electric charge is one of the most fundamental and stri~~c~~king features of the physical world, and there seems to be no explanation for it apart from the theory of monopoles [6]. This then provides some ground for believing in the existence of monopoles [7]. In light of this, <u>the main objective of this project is to propose a method of identifying monopoles based on the polarization of their synchrotron radiation. Another objective being the determination of the energy transferred from a scattered monopole to an electron initially at rest.</u>

In order to realize the aforementioned objectives, applying the variational principle and postulating the equation of motion of a monopole in an electromagnetic field, Maxwell's equations for monopole fields are obtained. And comparison with the ordinary Maxwell's equations provides a means of obtaining the first set of equations from the second, or vice versa. Besides, the motion of a monopole in a uniform electric field is found to be along a circle of constant radius. Next, starting from the monopole field equations, integral expressions for the scalar and vector potentials are derived, from which the electric and magnetic fields produced by a monopole in uniform circular motion are calculated.

Furthermore, the power radiated per harmonic into a unit solid angle is computed. Then the polarization of monopole synchrotron radiation is investigated, and found to have certain distinct features compared with that of electron synchrotron radiation. This, we hope, may serve as a theoretical background in the search for monopoles. The radiation emitted by a relativistic charge in uniform circular motion is called synchrotron radiation, because historically such radiation was first observed in an electron synchrotron [3]. In such an accelerator, a charge moves continuously in a circle of constant radius since any energy lost due to radiation is immediately compensated.

Analysis of the motion of a monopole in the field of an electron (or vice versa) shows that, in the center-of-mass reference frame, the trajectory of the monopole (or the electron) is along the surface of a cone. With the aid of this, the energy transferred from a scattered monopole to an electron initially at rest is determined. In our discussion of the motion of an electron and a monopole in the field of each other, we have assumed the monopole to be stable like the electron, the stability of which is inferred from observation.

Finally, we point out that wherever the symbol h appears in this manuscript, it must be read as $\hbar$.

2. MONOPOLE FIELD EQUATIONS

In order to obtain Maxwell's equations for the fields produced by monopoles, we shall first seek a tensor whose elements are the components of monopole electric and magnetic fields. To this end, we write the action integral for a monopole in an electromagnetic field, by analogy with that of an electric charge [9], in the form

$$S = \int_a^b \left(-\, mcds - \frac{g}{c} A_i dx^i\right)$$

where m and g are the rest mass and magnetic charge of the monopole, and $A_i = (\phi, -\vec{A})$ is the four - potential of the external field. The first term in the integrand characterizes a free monopole and the second one describes the interaction of the monopole with the external field. Applying the variational principle,

$$\delta S = \delta\int_a^b \left(-\, mcds - \frac{g}{c} A_i dx^i\right) = 0$$

or

$$\int_a^b \left(-\, mc\delta ds - \frac{g}{c} A_i d\delta x^i - \frac{g}{c} \delta A_i dx^i\right) = 0$$

But $\delta ds^2 = 2ds\ \delta ds$ implies $\delta ds = \dfrac{\delta ds^2}{2ds}$

Then employing the relation $ds = (dx_i dx^i)^{\frac{1}{2}}$,

$$\delta ds = \frac{1}{2ds}\,\delta(dx_i dx^i) = \frac{1}{2ds}\,(dx_i d\delta x^i + d\delta x_i dx^i)$$

or, taking into account that

$dx^i d\delta x_i = dx_i d\delta x^i$, we have $\delta ds = \dfrac{dx_i d\delta x^i}{ds}$

so that $$\int_a^b \left(-\frac{mc\,dx_i\,d\delta x^i}{ds} - \frac{g}{c} A_i d\delta x^i - \frac{g}{c}\,\delta A_i dx^i\right) = 0 .$$

After integrating by parts the first two terms in the integrand, one obtains

$$\int_a^b \left(\frac{mc\,d^2x_i\,\delta x^i}{ds} + \frac{g}{c}\,dA_i\,\delta x^i - \frac{g}{c}\,\delta A_i dx^i\right) - \left[\left(\frac{mc\,dx_i}{ds} + \frac{g}{c} A_i\right)\delta x^i\right]_a^b = 0.$$

Since the variations of the coordinates at the fixed points vanish, the quantity in square brackets is zero. Moreover, when the expressions

$$dA_i = \frac{\partial A_i}{\partial x^k}\,dx^k \text{ and } \delta A_i = \frac{\partial A_i}{\partial x^k}\,\delta x^k$$

are substituted, there results the equation

$$\int_a^b \left(\frac{mc\,d^2x_i\,\delta x^i}{ds} + \frac{g}{c}\frac{\partial A_i}{\partial x^k}\,dx^k\delta x^i - \frac{g}{c}\frac{\partial A_i}{\partial x^k}\,dx^i\delta x^k\right) = 0. \tag{2.1}$$

Using the fact that $ds = cd\tau$, we write

$$\frac{d^2x_i}{ds} = \frac{d^2x_i}{cd\tau} = du_i = \left(\frac{du_i}{d\tau}\right)d\tau , \tag{2.2}$$

where u_i $\left(= \frac{dx_i}{cd\tau}\right)$ are the components of the four - velocity and τ is the proper time of the monopole. Now combining (2.1) with Equation (2.2) we find

$$\int_a^b \left(mc\,\frac{du_i}{d\tau}\,\delta x^i d\tau + \frac{g}{c}\frac{\partial A_i}{\partial x^k}\,dx^k\delta d^i - \frac{g}{c}\cdot\frac{\partial A_i}{\partial x^k}\,dk^i\delta x^k\right) = 0.$$

Furthermore, interchanging in the third term the indices i and k, as they are summed over in accordance with the Einstein summation convention, and setting in the second and third terms $dx^k = cu^k d\tau$ leads to

$$\int \left(mc \frac{du_i}{d\tau} + g \frac{\partial A_i}{\partial x^k} u^k - g \frac{\partial A_k}{\partial x^i} u^k\right) \delta x^i d\tau = 0$$

or

$$\int_a^b \left[mc \frac{du_i}{d\tau} - g \left(\frac{\partial A_k}{\partial x^i} - \frac{\partial A_i}{\partial x^k}\right) u^k\right] \delta x^i d\tau \quad = 0 \ .$$

The variations δx^i are quite arbitrary, and hence the above equation holds true provided that

$$mc \frac{du_i}{d\tau} = g \left(\frac{\partial A_k}{\partial x^i} - \frac{\partial A_i}{\partial x^k}\right) u^k$$

or

$$\frac{dp^i}{d\tau} = g \left(\frac{\partial A^k}{\partial x_i} - \frac{\partial A^i}{\partial x_k}\right) u_k$$

where p^i $(= mcu^i)$ are the components of the four - momentum of the monopole. We now introduce an antisymmetric field tensor defined by

$$F^{ik} = \frac{\partial A^k}{\partial x_i} - \frac{\partial A^i}{\partial x_k} \ .$$

Thus

$$\frac{dp^i}{d\tau} = gF^{ik} u_k . \tag{2.3}$$

These are the equations of motion of the monopole in four-dimentional notation.

Next setting $i = 0$ and $k = 0,1,2,3$ in Equation (2.3) and noting that $p^o = \frac{\varepsilon}{c}$, we have

$$\frac{1}{c} \frac{d\varepsilon}{d\tau} = g \ (F^{oo} u_o + F^{o1} u_1 + F^{o2} u_2 + F^{o3} u_3).$$

Then recalling that $u_i = \frac{dx_i}{cd\tau}$ in which $x_i = (ct, -\vec{x})$ and

$$d\tau = dt\ (1 - \frac{v^2}{c^2})^{\frac{1}{2}} = \frac{dt}{\gamma} \text{ yields}$$

$$\frac{\gamma}{c}\frac{d\varepsilon}{dt} = g\ [F^{oo}\gamma\frac{d(ct)}{cdt} - F^{o1}\gamma\frac{dx_1}{cdt} - F^{o2}\gamma\frac{dx_2}{cdt} - F^{o3}\gamma\frac{dx_3}{cdt}]$$

from which follows

$$\frac{1}{c}\frac{d\varepsilon}{dt} = gF^{oo} - \frac{g}{c}\ (F^{o1}v_x + F^{o2}v_y + F^{o3}v_z). \tag{2.4a}$$

In a similar fashion, (for i = 1,2,3 and k = o,1,2,3) one can obtain the following three equations:

$$\frac{dp_x}{dt} = gF^{1o} - \frac{g}{c}\ (F^{11}v_x + F^{12}v_y + F^{13}v_z) \tag{2.4b}$$

$$\frac{dp_y}{dt} = gF^{2o} - \frac{g}{c}\ (F^{21}v_x + F^{22}v_y + F^{23}v_z) \tag{2.4c}$$

$$\frac{dp_z}{dt} = gF^{3o} - \frac{g}{c}\ (F^{31}v_x + F^{32}v_y + F^{33}v_z) \tag{2.4d}$$

In order to determine the components of the monopole field tensor F^{ik}, we postulate that the equations of motion of the monopole in three dimentional notation be

$$\frac{d\varepsilon}{dt} = g\vec{v}\cdot\vec{H}$$

and

$$\frac{d\vec{p}}{dt} = g\vec{H} - \frac{g}{c}\ \vec{v} \times \vec{E} \tag{2.5}$$

or, expressed in terms of components

$$\frac{d\varepsilon}{dt} = g(v_x H_x + v_y H_y + v_z H_z) \tag{2.6a}$$

$$\frac{dp_x}{dt} = gH_x - g/_c(E_zV_y - E_yV_z) \quad (2.6b)$$

$$\frac{dp_y}{dt} = gH_y - g/_c(E_xV_z - E_zV_x) \quad (2.6c)$$

$$\frac{dp_z}{dt} = gH_z - g/_c(E_yV_x - E_xV_y) \quad (2.6d)$$

Since both sets of equations (2.4) and (2.6) describe the same motion of the monopole, their corresponding equations must be exactly the same. As a consequence, we finally obtain the following matrix for the monopole field tensor

$$F^{ik} = \begin{pmatrix} 0 & -H_x & -H_y & -H_z \\ H_x & 0 & E_z & -E_y \\ H_y & -E_z & 0 & E_x \\ H_z & E_y & -E_x & 0 \end{pmatrix} \quad (2.7)$$

in which the indices i,k (= 0,1,2,3) label the rows and columns respectively. The complete action integral for a system of a monopole and an electromagnetic field has the form [10]

$$S = \int(- mcds - \frac{g}{c} A_k dX^k - \frac{1}{16\pi c} F_{ik}F^{ik} d^4x),$$

where the last term in the integrand represents the external field in the absence of charges, the other two terms having the same significance mentioned before. Here A_i and F_{ik} are the four-potential and field tensor of the external field plus the field produced by the monopole, and therefore are functions of the coordinates and velocity of the monopole. Replacing the point magnetic charge g by a continuous charge distribution of density ρ ($g \to \rho dv$) transforms the action integral into

$$S = \int (- mcds - \rho/_c \, A_k dX^k dv - \frac{1}{16\Pi c} F_{ik} F^{ik} d^4x).$$

Then employing the relation $j^k = \rho \frac{dx^k}{dt}$ and recalling that $dx^o = cdt$, one obtains

$$S = \int (- mcds - \frac{j^k A_k d^4x}{c^2} - \frac{1}{16\Pi c} F_{ik} F^{ik} d^4x). \tag{2.8}$$

The equation of motion of a particle in an external field is found by assuming the field to be given and varying the trajectory of the particle. But to obtain field equations, we assume the motion of the charge distribution to be given and vary only the potentials which serve as the "coordinates" of the system. Accordingly, the variation of the first term in Equation (2.8) is zero, and hence Equation (2.8) assumes the form

$$\delta S = - \frac{1}{c} \int (\frac{1}{c} j^k \delta A_k + \frac{1}{16\Pi} F_{ik} \delta F^{ik} + \frac{1}{16\Pi} F^{ik} \delta F_{ik}) d^4x .$$

Furthermore, substitution of $F_{ik} \delta F^{ik} = F^{ik} \delta F_{ik}$ and application of the variational principle leads to

$$- \frac{1}{c} \int (\frac{1}{c} j^k \delta A_k + \frac{1}{8\Pi} F^{ik} \delta F_{ik}) d^4x = 0$$

and noting that $\delta F_{ik} = \frac{\partial \delta A_k}{\partial x^i} - \frac{\partial \delta A_i}{\partial x^k}$, we have

$$- \frac{1}{c} \int (\frac{1}{c} j^k \delta A_k + \frac{1}{8\Pi} F^{ik} \frac{\partial}{\partial x^i} \delta A_k - \frac{1}{8\Pi} F^{ik} \frac{\partial}{\partial x^k} \delta A_i) d^4x = 0.$$

Next, interchanging in the third term the indices i and k (for the same reason given previously) and then replacing $-F^{ki}$ by F^{ik} yields

$$-\frac{1}{c}\int\left(\frac{1}{c}\,j^k\delta A_k + \frac{1}{4\Pi}\,F^{ik}\,\frac{\partial}{\partial x^i}\,\delta A_k\right)d^4x = 0$$

or, using the fact that

$$F^{ik}\,\frac{\partial\delta A_k}{\partial x^i} = \frac{\partial}{\partial x^i}\,(F^{ik}\delta A_k) - \frac{\partial F^{ik}}{\partial x^i}\,\delta A_k\,,$$

one finds

$$-\frac{1}{c}\int\left(\frac{1}{c}\,j^k\delta A_k - \frac{1}{4\Pi}\,\frac{\partial F^{ik}}{\partial x^i}\,\delta A_k\right)d^4x - \frac{1}{4\Pi c}\int\frac{\partial}{\partial x^i}(F^{ik}\delta A_k)\,d^4x = 0\,. \tag{2.9}$$

According to Gauss' theorem

$$\int\frac{\partial}{\partial x^i}\,(F^{ik}\delta A_k)\,d^4x = \oint F^{ik}\delta A_k\,ds_k\,.$$

This integral is zero in view of the requirement that the variations δA_k must vanish at the limits of integration. Consequently, Equation (2.9) goes over into

$$\int\left(\frac{1}{c}\,j^k - \frac{1}{4\Pi}\,\frac{\partial F^{ik}}{\partial x^i}\right)\delta A_k\,d^4x = 0\,.$$

Since the variations δA_k are arbitrary, the quantity in brackets must be zero. As a result, we obtain finally

$$\frac{\partial F^{ik}}{\partial x^i} = \frac{4\Pi}{c}\,j^k\,. \tag{2.10}$$

As will be shown pretty soon, this equation is equivalent to a pair of monopole field equations in three-dimensional form. To this end, setting $k = 1,2,3$ and for each case $i = 0,1,2,3$ we have the following set of equations:

$$\frac{\partial F^{o1}}{\partial x^o} + \frac{\partial F^{11}}{\partial x^1} + \frac{\partial F^{21}}{\partial x^2} + \frac{\partial F^{31}}{\partial x^3} = \frac{4\Pi}{c} j^1$$

$$\frac{\partial F^{o2}}{\partial x^o} + \frac{\partial F^{12}}{\partial x^1} + \frac{\partial F^{22}}{\partial x^2} + \frac{\partial F^{32}}{\partial x^3} = \frac{4\Pi}{c} j^2$$

$$\frac{\partial F^{o3}}{\partial x^o} + \frac{\partial F^{13}}{\partial x^1} + \frac{\partial F^{23}}{\partial x^2} + \frac{\partial F^{33}}{\partial x^3} = \frac{4\Pi}{c} j^3$$

Then substituting the values for the components of the monopole field tensor F^{ik} from Equation (2.7) and recalling that $\frac{\partial}{\partial x^o} = \frac{\partial}{c\partial t}$, one finds

$$-\frac{\partial H_x}{c\partial t} - \left(\frac{\partial E_z}{\partial y} - \frac{\partial E_y}{\partial z}\right) = \frac{4\Pi}{c} j_x$$

$$-\frac{\partial H_y}{c\partial t} - \left(\frac{\partial E_x}{\partial x} - \frac{\partial E_z}{\partial y}\right) = \frac{4\Pi}{c} j_y$$

$$-\frac{\partial H_z}{c\partial t} - \left(\frac{\partial E_y}{\partial x} - \frac{\partial E_x}{\partial y}\right) = \frac{4\Pi}{c} j_z$$

These three equations can be combined in a single vector equation:

$$\text{curl}\ \vec{E} + \frac{1}{c}\frac{\partial \vec{H}}{\partial t} = -\frac{4\Pi}{c}\vec{J} \tag{2.11}$$

where $\vec{J} = (j_x, j_y, j_z)$ is the current density vector.

Finally, for k = o and i = o,1,2,3 we have from Equation (2.10) that

$$\frac{\partial F^{oo}}{c\partial t} + \frac{\partial F^{o1}}{\partial x} + \frac{\partial F^{o2}}{\partial y} + \frac{\partial F^{o3}}{\partial z} = \frac{4\Pi}{c} j^o .$$

On taking into account Equation (2.7) and the fact that $j^o = c\rho$, the above equation becomes:

$$\frac{\partial H_x}{\partial x} + \frac{\partial H_y}{\partial y} + \frac{\partial H_z}{\partial z} = 4\pi\rho$$

or $$\operatorname{div} \vec{H} = 4\pi\rho. \tag{2.12}$$

Equations (2.11) and (2.12) are two of the four field equations we are looking for.

The remaining pair of field equations are obtained employing the definition of the monopole field tensor

$$F^{ik} = \frac{\partial A^k}{\partial X_i} - \frac{\partial A^i}{\partial X_k}. \tag{2.13a}$$

One can also write:

$$F^{k\ell} = \frac{\partial A^\ell}{\partial X_k} - \frac{\partial A^k}{\partial X_\ell} \tag{2.13b}$$

$$F^{\ell i} = \frac{\partial A^i}{\partial X_\ell} - \frac{\partial A^\ell}{\partial X_i} \tag{2.13c}$$

Now differentiating Equations (2.13a), (2.13b) and (2.13c) with respect to X_ℓ , X_i, and X_k gives

$$\frac{\partial F^{ik}}{\partial X_\ell} = \frac{\partial^2 A^k}{\partial X_\ell \partial X_i} - \frac{\partial^2 A^i}{\partial X_\ell \partial X_k}$$

$$\frac{\partial F^{k\ell}}{\partial X_i} = \frac{\partial^2 A^\ell}{\partial X_i \partial X_k} - \frac{\partial^2 A^k}{\partial X_i \partial X_\ell}$$

$$\frac{\partial F^{\ell i}}{\partial X_k} = \frac{\partial^2 A^i}{\partial X_k \partial X_\ell} - \frac{\partial^2 A^\ell}{\partial X_k \partial X_i}$$

and adding these three equations yields

$$\frac{\partial F^{ik}}{\partial X_\ell} + \frac{\partial F^{k\ell}}{\partial X_i} + \frac{\partial F^{\ell i}}{\partial X_k} = 0. \tag{2.14}$$

This expression is a tensor of the third rank, which is antisymmetric in all the three indices. The only nonzero components are those for which $i \neq k \neq \ell$.

Next, substituting the sequence of integers (2,0,3), (3,0,1), and (1,0,2) for the values of the indices (i,k,ℓ) in Equation (2.14), we obtain the following equations:

$$\frac{\partial F^{20}}{\partial X_3} + \frac{\partial F^{03}}{\partial X_2} + \frac{\partial F^{32}}{\partial X_0} = 0$$

$$\frac{\partial F^{30}}{\partial X_1} + \frac{\partial F^{01}}{\partial X_3} + \frac{\partial F^{13}}{\partial X_0} = 0$$

$$\frac{\partial F^{10}}{\partial X_2} + \frac{\partial F^{02}}{\partial X_1} + \frac{\partial F^{21}}{\partial X_0} = 0$$

and on taking into account Equation (2.7),

$$\left(\frac{\partial H_z}{\partial y} - \frac{\partial H_y}{\partial z}\right) - \frac{\partial E_x}{c\partial t} = 0$$

$$\left(\frac{\partial H_x}{\partial z} - \frac{\partial H_z}{\partial x}\right) - \frac{\partial E_y}{c\partial t} = 0$$

$$\left(\frac{\partial H_y}{\partial x} - \frac{\partial H_x}{\partial y}\right) - \frac{\partial E_z}{c\partial t} = 0$$

These three equations can be combined in a single vector equation:

$$\text{curl } \vec{H} - \frac{1}{c}\frac{\partial \vec{E}}{\partial t} = 0$$

Finally, putting $i = 1$, $k = 2$, and $\ell = 3$ in Equation (2.14),

$$\frac{\partial F^{12}}{\partial X_3} + \frac{\partial F^{23}}{\partial X_1} + \frac{\partial F^{31}}{\partial X_2} = 0$$

and substituting the values for the components of the field tensor F^{ik} from Equation (2.7) gives

$$-\frac{\partial E_z}{\partial z} - \frac{\partial E_x}{\partial x} - \frac{\partial E_y}{\partial y} = 0.$$

But this equation is exactly

$$-\operatorname{div} \vec{E} = 0$$

or $$\operatorname{div} \vec{E} = 0 \,. \tag{2.16}$$

Now we rewrite here Equations (2.11), (2.12), (2.15) and (2.16):

$$\operatorname{curl} \vec{E} + \frac{1}{c}\frac{\partial \vec{H}}{\partial t} = -\frac{4\Pi}{c}\vec{J}$$

$$\operatorname{div} \vec{H} = 4\Pi\rho$$

$$\operatorname{curl} \vec{H} - \frac{1}{c}\frac{\partial \vec{E}}{\partial t} = 0$$

$$\operatorname{div} \vec{E} = 0$$

These are Maxwell's equations for the fields produced by magnetic charge and current densities, ρ and $\vec{J}$. For comparison, we also write here the ordinary Maxwell's equations:

$$\operatorname{curl} \vec{H} - \frac{1}{c}\frac{\partial \vec{E}}{\partial t} = \frac{4\Pi}{c}\vec{J}$$

$$\operatorname{div} \vec{E} = 4\Pi\rho$$

$$\operatorname{curl} \vec{E} + \frac{1}{c}\frac{\partial \vec{H}}{\partial t} = 0$$

$$\operatorname{div} \vec{H} = 0$$

where ρ and $\vec{J}$ are the electric charge and current densities. One can easily see that the first set of equations can be obtained from the second using the following transformation:

$$\vec{H}_e \to \vec{E}_g,\ \vec{E}_e \to -\vec{H}_g,\ \rho_e \to -\rho_g,\ \vec{J}_e \to -\vec{J}_g$$

3. MOTION OF A MONOPOLE IN A UNIFORM ELECTRIC FIELD

We now consider the motion of a monopole in a uniform electric field directed along the z-axis. According to (2.5), the equation of motion of the monopole is

$$d\vec{p}/dt = - g/c\ \vec{v} \times \vec{E}. \tag{3.1}$$

Then taking the dot product of both sides of (3.1) with the velocity vector $\vec{v}$, and using the identity $\vec{a}\cdot(\vec{a}\times\vec{b}) = 0$ leads to

$$\vec{v}\cdot d\vec{p}/dt = 0 .$$

Multiplying both sides of this equation by the relativistic mass of the monopole m, we have

$$m\vec{v}\cdot d\vec{p}/dt = 0$$

or

$$\tfrac{1}{2}\ d\vec{p}^{\,2}/dt = 0$$

from which it follows that the total energy of the monopole

$$\varepsilon = (p^2c^2 + m_o^2c^4)^{\frac{1}{2}}$$

is a constant of motion. Of course this conclusion holds true provided that there exists a means by which any energy lost due to radiation is immediately compensated, as in synchrotron accelerators. With the expression for the momentum $\vec{p} = \varepsilon\vec{v}/c^2$ substituted into (3.1), the equation of motion thus takes the form

$$\varepsilon\, d\vec{v}/dt = -\, gc\, \vec{v} x \vec{E}$$

or, expressed in terms of components

$$\mathring{v}_x = -\omega v_y,\ \mathring{v}_y = \omega v_x,\ \text{and}\ \mathring{v}_z = 0\ , \tag{3.2a}$$

where

$$\omega = gcE/\varepsilon\ . \tag{3.2b}$$

Furthermore, multiplying the first equation of (3.2a) by i and adding the result to the second yields

$$\frac{d}{dt}(v_y + iv_x) = -\, i\omega(v_y + iv_x)$$

so that

$$v_y + iv_x = ae^{-i\omega t}\ , \tag{3.3}$$

where a is a complex constant. This can be written in the form

$$a = v_o e^{-i\alpha} \tag{3.4}$$

in which $\vec{v}_o = (v_{ox}, v_{oy}, o)$ is the initial velocity and α is the initial phase. Combining Equation (3.3) with (3.4) then gives

$$v_y + iv_x = v_o e^{-i(\omega t + \alpha)}$$

or, applying Euler's formula

$$v_y + iv_x = v_o \cos(\omega t + \alpha) - iv_o \sin(\omega t + \alpha)\,.$$

Now equating real and imaginary parts, one obtains

$$v_x = -\, v_o \sin(\omega t + \alpha)\ \text{and}\ v_y = v_o \cos(\omega t + \alpha)$$

and integration of these two equations with respect to the variable t yields

$$x = x_o + \frac{v_o}{\omega} \cos (\omega t + \alpha) , \tag{3.5a}$$

$$y = y_o + \frac{v_o}{\omega} \sin (\omega t + \alpha). \tag{3.5b}$$

In addition, noting that the z-component of the initial velocity is zero, we have

$$z = z_o , \tag{3.5c}$$

where $\vec{r}_o = (x_o, y_o, z_o)$ is the initial position of the monopole. Hence, inspection of Equation (3.5) shows that a monopole in a uniform electric field moves along a circle of radius $R = v_o/\omega$, in a plane perpendicular to the field. We see that ω is the angular frequency of the monopole and with the aid of Equation (3.2b), the radius can be written as

$$R = \varepsilon v_o/gcE.$$

4. POTENTIALS OF FIELDS PRODUCED BY MOVING MONOPOLES

We shall now derive the equations determining the scalar and vector potentials of fields produced by arbitrarily moving monopoles. To this end, we write once more the monopole field equations:

$$\text{curl } \vec{E} + \frac{1}{c}\frac{\partial \vec{H}}{\partial t} = -\frac{4\Pi}{c}\vec{J} \tag{4.1a}$$

$$\text{div } \vec{H} = 4\Pi\rho \tag{4.1b}$$

$$\text{curl } \vec{H} - \frac{1}{c}\frac{\partial \vec{E}}{\partial t} = 0 \tag{4.1c}$$

$$\text{div } \vec{E} = 0 \tag{4.1d}$$

As a consequence of (4.1d), one can express the electric field $\vec{E}$ in terms of the curl of a vector potential $\vec{A}$:

$$\vec{E} = -\text{ curl } \vec{A} \tag{4.2}$$

when this is substituted into (4.1c), there results the equation

$$\nabla x(\vec{H} + \frac{1}{c}\frac{\partial \vec{A}}{\partial t}) = 0$$

from which it follows that the quantity in brackets can be written as the gradient of a scalar function ϕ :

$$\vec{H} + \frac{1}{c}\frac{\partial \vec{A}}{\partial t} = -\nabla\phi$$

or

$$\vec{H} = -\nabla\phi - \frac{1}{c}\frac{\partial \vec{A}}{\partial t} \tag{4.3}$$

Substitution of (4.2) and (4.3) into Equation (4.1a) yields

$$\nabla^2 \vec{A} - \frac{1}{c^2}\frac{\partial^2 \vec{A}}{\partial t^2} - \nabla(\nabla\cdot\vec{A} + \frac{1}{c}\frac{\partial\phi}{\partial t} = -\frac{4\Pi}{c} J \tag{4.4}$$

on employing the identity

$$\nabla \times \nabla \times \vec{A} = \nabla(\nabla\cdot\vec{A}) - \nabla^2\vec{A} \cdot$$

Finally, imposing the Lorentz condition

$$\nabla\cdot\vec{A} + \frac{1}{c}\frac{\partial\phi}{\partial t} = 0$$

reduces Equation (4.4) to

$$\nabla^2\vec{A} - \frac{1}{c^2}\frac{\partial^2\vec{A}}{\partial t^2} = -\frac{4\Pi}{c}\vec{J}. \tag{4.5}$$

Furthermore, combining Equation (4.1b) with (4.3) one finds

$$\nabla^2\phi + \frac{1}{c}\frac{\partial}{\partial t}\nabla\cdot\vec{A} = -4\Pi\rho .$$

According to the Lorentz condition

$$\nabla\cdot\vec{A} = -\frac{1}{c}\frac{\partial\phi}{\partial t} \cdot$$

Therefore

$$\nabla^2\phi - \frac{1}{c^2}\frac{\partial^2\phi}{\partial t^2} = -4\Pi\rho . \tag{4.6}$$

The solution of (4.5) or (4.6) is the sum of the general solution of the homogeneous part and a particular solution of the inhomogeneous equation. But the solutions of the homogeneous parts represent potentials due to other sources. Hence the potentials of the fields produced by the magnetic charge and current densities under consideration must be particular solutions of (4.5) and (4.6). Then to obtain a solution of Equation (4.6), we express the magnetic charge density ρ in terms of the δ-function:

$$\rho(x,y,z,t) = \int d^4x'\delta(x-x')\ \delta(y-y')\ \delta(z-z')\ \delta(t-t')\ \rho(x',y',z',t'),$$

where $$d^4x' = dx'dy'dz'dt'.$$

On substituting the Fourier integral representation of the δ-functions

$$\delta(x-x') = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{ik(x-x')}dk_1,$$

$$\delta(t-t') = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{i\kappa(t-t')}d\kappa,$$

and similar expressions for $\delta(y-y')$, and $\delta(z-z')$ we obtain

$$\phi(x,y,z,t) = \frac{1}{(2\pi)^4}\int d^4x'\int_{-\infty}^{\infty} d^4k\ e^{i\vec{k}\cdot\vec{R} + i\kappa T}\rho(x',y',z',t') \qquad (4.7)$$

in which $\vec{k} = (k_1,k_2,k_3)$, $\vec{R} = [(x-x'),\ (y-y'),\ (z-z')]$,

$T = t-t'$, and $d^4k = dk_1dk_2dk_3d\kappa$.

Now with the aid of (4.7), Equation (4.6) takes the form

$$\nabla^2\phi - \frac{1}{c^2}\frac{\partial^2\phi}{\partial t^2} = -\frac{4\pi}{(2\pi)^4}\int d^4x'\int_{-\infty}^{\infty} d^4k\ e^{i\vec{k}\cdot\vec{R} + i\kappa T}\rho(x',y',z',t'). \qquad (4.8)$$

It is easy to check that

$$\phi(x,y,z,t) = -\frac{4\pi}{(2\pi)^4}\int d^4x'\int_{-\infty}^{\infty} \frac{d^4k\ e^{i\vec{k}\cdot\vec{R}+i\kappa T}\rho(x',y',z',t')}{(ik_1)^2+(ik_2)^2+(ik_3)^2-(\frac{i\kappa}{c^2})^2}$$

is a solution of Equation (4.8). Since

$$(ik_1)^2+(ik_2)^2+(ik_3)^2-(\frac{i\kappa}{cc})^2 = \frac{\kappa^2}{c^2} - k^2,$$

$$\phi(x,y,z,t) = -\frac{4\pi c^2}{(2\pi)^4}\int d^4k'\int_{-\infty}^{\infty} \frac{d^4x\ e^{i\vec{k}\cdot\vec{R}+i\kappa T}\rho(x',y',z',t')}{\kappa^2 - c^2k^2}. \qquad (4.9)$$

We next apply the residue theorem to evaluate the integral

$$\int_{-\infty}^{\infty} \frac{e^{i\kappa T} d\kappa}{\kappa^2 - c^2 k^2} .$$

The function $$f(z) = \frac{e^{izT}}{z^2 - c^2 k^2}$$

has two simple poles at $z = -ck$ and $z = ck$ where $z = \kappa + iy$.

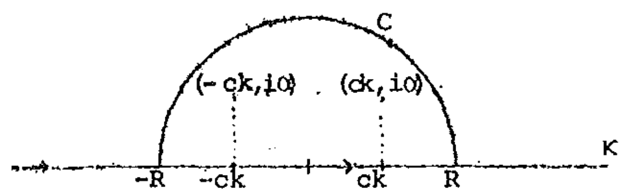

Fig. 1 Contour of Integration in the Calculation of the Scalar Potential

Since we cannot integrate through singular points, we move the poles an infinitesimal distance α off the κ-axis as indicated in Figure 1. Then by the residue theorem

$$\int_{-R}^{R} \frac{e^{i\kappa T} d\kappa}{\kappa^2 - c^2 k^2} + \int_C \frac{e^{izT} dz}{z^2 - c^2 k^2} = 2\pi i \left[\text{Res } f(-ck) + \text{Res } f(ck)\right].$$

But $$\text{Res } f(ck) = \lim_{\substack{z \to ck \\ \alpha \to 0}} \left[\frac{(z-ck-i\alpha) e^{izT}}{(z-ck-i\alpha)(z+ck-i\alpha)} \right]$$

$$= \frac{e^{ickT}}{2ck} .$$

Similarly $$\text{Res } f(-ck) = -\frac{e^{-ickT}}{2ck} .$$

Consequently $$\int_{-R}^{R} \frac{e^{i\kappa T} d\kappa}{\kappa^2 - c^2 k^2} + \int_C \frac{e^{izT} dz}{z^2 - c^2 k^2} = \frac{\pi i}{ck} \left(e^{ickT} - e^{-ickT}\right). \qquad (\cdot.10)$$

Since $$\lim_{R \to \infty} \int_C \frac{e^{izT}}{z^2 - c^2 k^2} = 0$$

and
$$\lim_{R\to\infty} \int_{-R}^{R} \frac{e^{i\kappa T} d}{\kappa^2 - c^2 k^2} = \int_{-\infty}^{\infty} \frac{e^{i\kappa T} d}{\kappa^2 - c^2 k^2} ,$$

Equation (4.10) reduces to

$$\int_{-\infty}^{\infty} \frac{e^{i\kappa T} d\kappa}{\kappa^2 - c^2 k^2} = \frac{\pi i}{ck} \left(e^{ickT} - e^{-ickT}\right).$$

Combination of this with Equation (4.9) then yields

$$\phi(x,y,z,t) = \frac{(-4\pi)(i\pi c)}{(2\pi)^4} \int d^4k' \int_{-\infty}^{\infty} \frac{d^3k\left(e^{ickT} - e^{-ickT}\right)}{k} e^{i\vec{k}\cdot\vec{R}} \rho(x',y',z',t') \tag{4.11}$$

where $d^3k = dk_1 dk_2 dk_3$.

We now wish to carry out the integration with respect to the variable k. To this end, we introduce a spherical coordinate system (k,θ,ϕ). In this system, $d^3k = k^2 \sin\theta \, dk \, d\theta \, d\phi$ where $o < \theta < \pi$, $o < \phi < 2\pi$ and $o < k < \infty$.

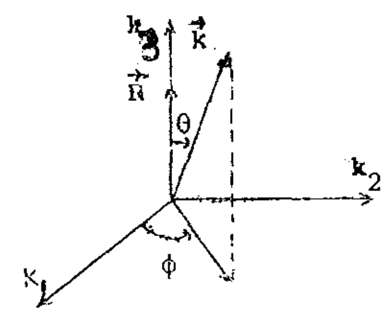

Fig. 2 A Spherical Coordinate System in k-space

If we take vector $\vec{R}$ parallel to the k_3-axis, then

$$\vec{k}\cdot\vec{R} = kR \cos\theta .$$

Therefore

$$\int \frac{d^3k \; d^{ickT + i\vec{k}\cdot\vec{R}}}{k} = \int \frac{dk \; d^{ickT + ikR\cos\theta}}{k} k^2 \sin\theta \, d\theta \, d\phi$$

$$= \int_0^{2\pi} d\phi \int_0^{\infty} dk \left[e^{ickT} \int_0^{\pi} e^{ikR \cos\theta} k \sin\theta \, d\theta\right]$$

$$= -\frac{2\pi}{iR} \int_0^{\infty} dk \left[e^{ickT} \left(e^{ikR \cos\theta}\right)\Big|_{\theta=0}^{\theta=\pi}\right]$$

$$= -\frac{2\pi}{iR} \int_0^{\infty} dk \left[e^{ickT} \left(e^{-ikR} - e^{ikR}\right)\right]$$

$$= -\frac{2\pi}{iR} \int_0^{\infty} dk \left[e^{ik(cT-R)} - \int_0^{\infty} dk \, e^{ik(cT+R)}\right]$$

In a similar fashion one obtains

$$\int_0^{\infty} \frac{d^3k \, e^{-ickT + i\vec{k}\cdot\vec{R}}}{k} = -\frac{2\pi}{iR} \left[\int_0^{\infty} dk \, e^{ik(-cT+R)} - \int_0^{\infty} dk \, e^{-ik(cT+R)}\right].$$

Hence

$$\int_0^{\infty} \frac{d^3k}{k} \left(e^{ickT} - e^{-ickT}\right) e^{i\vec{k}\cdot\vec{R}} = -\frac{2}{iR} \left[\int_0^{\infty} dk \, e^{ik(cT-R)} - \int_0^{\infty} dk \, e^{ik(cT+R)} + \int_0^{\infty} dk \, e^{-ik(cT+R)} - \int_0^{\infty} dk \, e^{ik(-cT+R)}\right].$$

Changing the variable of integration k to -k in the last two integrals, we have

$$\int_0^{\infty} \frac{d^3k}{k} \left(e^{ickT} - e^{-ickT}\right) e^{i\vec{k}\cdot\vec{R}} = -\frac{2\pi}{iR} \left[\int_0^{\infty} dk \, e^{ik(cT-R)} - \int_0^{\infty} dk \, e^{ik(cT+R)} - \int_{-\infty}^{0} dk \, e^{ik(cT+R)} + \int_{-\infty}^{0} dk \, e^{ik(cT-R)}\right]$$

$$= -\frac{2\pi}{iR} \left[\int_{-\infty}^{\infty} dk \, e^{ik(cT-R)} - \int_{-\infty}^{\infty} dk \, e^{ik(cT+R)}\right].$$

When the expressions

$$\int_{-\infty}^{\infty} dk\ e^{ik(cT-R)} = 2\pi\delta(cT-R) \text{ and } \int_{-\infty}^{\infty} dk\ e^{ik(cT+R)} = 2\pi\delta(cT+R)$$

are substituted, there results the equation

$$\int_{0}^{\infty} \frac{dk}{k}(e^{ickT}-e^{-ickT})\ e^{i\vec{k}\cdot\vec{R}} = -\frac{(2\pi)^2}{iR}[\delta(cT-R) - \delta(cT+R)].$$

Thus combining this with Equation (4.11) yields

$$\phi(x,y,z,t) = c\int\frac{d^3x'}{R}[\delta(cT-R) - \delta(cT+R)]\rho(x',y',z',t')dt'.$$

Furthermore, setting $c(t-t') - R = 0$, we find $t' = t-R/c$.

Then we can write

$$\delta(cT-R) = \frac{\delta(t'-t+R/c)}{\left|\frac{d}{dt'}(ct-ct'-R)\right|_{t'=t-R/c}}$$

Since t and t' are independent variables and R does not depend on t',

$$\frac{dt}{dt'} = \frac{dR}{dt'} = 0\ .$$

Therefore

$$\delta(cT-R) = \frac{\delta(t'-t+R/c)}{c}.$$

The radiation emitted at the retarded time t' is observed at a later time t due to the finite speed of propagation of the radiation. Then

$$(cT+R) > 0$$

and hence

$$\delta(cT+R) = 0\ . \tag{4.14}$$

Now employing (4.13) and (4.14) in Equation (4.12), one obtains

$$\phi(\vec{r},t) = \int\frac{dv'}{R}\delta(t'-t+R/c)\ \rho(\vec{r}',t')dt',$$

where

$$dv' = dx'_1\, dx'_2\, dx'_3 \text{ and } R = |\vec{r}-\vec{r}'|.$$

On integrating with respect to the variable t',

$$\phi(\vec{r},t) = \int \frac{dv'\rho(\vec{r}',t-R/_c)}{|\vec{r}-\vec{r}'(t-R/_c)|} .$$

The magnetic charge density can be expressed in terms of the δ-function as

$$\rho(\vec{r}',t-R/_c) = \int\rho(\vec{r}',\tau)\ \delta(\tau-t+R/_c)\,d\tau.$$

Also

$$\rho(\vec{r}',\tau) = g\delta[\vec{r}'-\vec{z}(\tau)]$$

in which $\vec{z}$ is the position vector of a point monopole g. It then follows from the above three equations that

$$\phi(\vec{r},t) = g\int dv' \int \frac{\delta[\vec{r}'-\vec{z}(\tau)]\delta(\tau-t+R/_c)\,d\tau}{|\vec{r}-\vec{r}'|} .$$

Finally, after performing the integration with respect to the variable v', the expression for the scalar potential of the field produced by a point monopole g assumes the form

$$\phi(\vec{r},t) = g\int \frac{\delta(\tau-t+R/_c)\,d\tau}{R} ,$$

where

$$R = |\vec{r}-\vec{z}(\tau)|.$$

Similarly we have for the vector potential

$$\vec{A}(\vec{r},t) = g/_c \int \frac{\vec{v}(\tau)\ \delta(\tau-t+R/_c)\,d\tau}{R} .$$

5. INTENSITY OF RADIATION BY A MONOPOLE IN UNIFORM CIRCULAR MOTION

Here we want to consider the intensity of radiation emitted by a monopole in uniform circular motion. The most convenient starting point in this case is the vector potential:

$$\vec{A}(\vec{r},t) = {}^{g}\!/_{c} \int \frac{\vec{v}(\tau)\ \delta(\tau - t + {}^{R}\!/_{c})\,d\tau}{|\vec{r}(t) - \vec{z}(\tau)|} \tag{5.1}$$

where $\vec{R} = \vec{r}(t) - \vec{z}(\tau)$ is the vector from the location of the monopole $\vec{z}(\tau)$ to the observation point $\vec{r}\ (t)$.

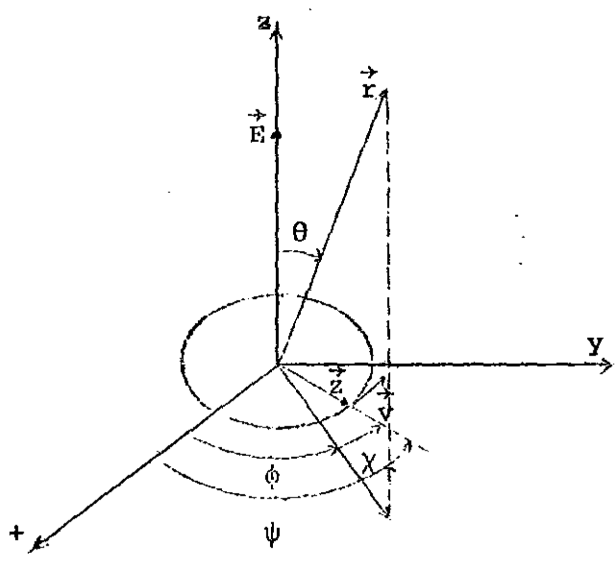

Fig. 3 A Monopole in Uniform Circular Motion

The arrangement of Figure 3 corresponds to the motion of a monopole of positive magnetic charge along a circle of radius R_o in the xy-plane, in a uniform electric field $\vec{E}$ directed along the positive z-axis. According to this figure

$$\vec{r} = (r \sin\theta \cos\phi\ ,\ r \sin\theta \sin\phi,\ r \cos\theta) \tag{5.2a}$$

and $$\vec{z} = (R_o \cos\psi,\ R_o \sin\psi,\ 0) \tag{5.2b}$$

on assuming the initial position of the monopole to be $(R_o, 0)$.

At large distance from the system of the monopole $(r >> R_o)$, the dipole approximation of R is given by

$$R = |\vec{r}-\vec{z}| = r - \vec{z}\circ\nabla r = r - \frac{\vec{r}\cdot\vec{r}}{r}. \tag{5.3}$$

Now employing Equation (5.2), one obtains

$$\vec{z}\circ\vec{r} = R_o\ r \sin\theta\,(\cos\psi\cos\phi + \sin\psi\sin\phi)$$

or, noting that $\cos\psi\cos\phi + \sin\psi\sin\phi = \cos\ (\psi-\phi)$,

$$\vec{z}\circ\vec{r} = R_o\ r \sin\theta\cos\ \chi \tag{5.4a}$$

in which $$\chi = \psi-\phi\ . \tag{5.4b}$$

Thus substitution of (5.4a) into Equation (5.3) yields

$$R = r - R_o\ \sin\theta\cos\ \chi \tag{5.5}$$

from which follows

$$\frac{1}{R} = \frac{1}{r-R_o\ \sin\theta\cos\ \chi} = \frac{1}{r}\,(1 - R_o\ \sin\theta\cos\chi/r)^{-1}$$

and on applying the Binomial approximation

$$\frac{1}{R} = \frac{1}{r} + R_o \sin\theta\cos\ \chi/r^2\ .$$

If we keep only terms of the first order, then

$$\frac{1}{R} = \frac{1}{r}\ .$$

Finally, substituting this expression and (5.5) into (5.1) gives

$$\vec{A}(\vec{r},t) = {}^{g}/_{cr} \int \vec{v}(\tau)\ \delta(\tau-t+{}^{r}/_{c} - R_o \sin\theta \cos\chi/_{c})\ d\tau\ . \tag{5.6}$$

Since the motion of the monopole under consideration is periodic, we expand the δ-function in a Fourier series

$$\delta(\tau') = \frac{1}{\sqrt{T}} \sum_{n=-\infty}^{\infty} c_n e^{in\omega\tau'}, \tag{5.7}$$

where the functions $\frac{1}{\sqrt{T}}$ exp $in\omega\tau'$ are orthonormal in the interval $[-{}^{T}/_{2}, {}^{T}/_{2}]$ (See for a proof of this appendix B), ω and T are the angular frequency and period of the monopole, n is an integer, and

$$\tau' = \tau - t + {}^{r}/_{c} - R_o \sin\theta \cos\chi/_{c}. \tag{5.8}$$

The coefficient of expansion c_n is determined by

$$c_n = \frac{1}{\sqrt{T}} \int_{-T/2}^{T/2} e^{-in\omega\tau'}\ \delta(\tau')d\tau'\ .$$

Then using the property of the δ-function

$$\int_{-x_o}^{x_o} f(x)\ \delta(x)dx = f(o),$$

we obtain

$$c_n = \frac{1}{\sqrt{T}}\ . \tag{5.9}$$

Now with the aid of (5.7) along with (5.8), and (5.9), Equation (5.6) takes the form

$$\vec{A}(\vec{r},t) = {}^{g}/_{crT} \sum_{n=-\infty}^{\infty} \int \vec{v}(\tau) e^{in\omega\tau'}\ d\tau. \tag{5.10}$$

Next, we wish to put the quantity "$in\omega\tau'$" in a form convenient for further calculation. Taking into account expression (5.8), we write

$$in\omega\tau' = in\omega(\tau - t + r/_c - R_o \sin\theta\cos\chi/_c)$$

$$= in(\omega\tau - \omega t + \omega r/_c - \omega R_o \sin\theta\cos\chi/_c)$$

$$= in(\omega\tau - \phi + \pi/_2) \quad -(\omega t - \frac{\omega r}{c} - \phi + \pi/_2) - \omega R_o \sin\theta\cos\chi/_c$$

or

$$in\omega\tau' = in(\alpha - \gamma - \beta \sin\theta\cos\chi)$$

in which

$$\alpha = \omega\tau - \phi + \pi/_2 ,$$

$$\gamma = \omega t - \frac{\omega r}{c} - \phi + \pi/_2 ,$$

$$\beta = \omega R_o /_c .$$

It follows from Equation (5.12) that

$$d\tau = \frac{d\alpha}{\omega} . \tag{5.13}$$

In addition, noting that $\psi = \omega\tau$ Equation (5.12) becomes

$$\alpha = \psi - \phi + \pi/_2$$

and then using expression (5.4b)

$$\alpha = \chi + \pi/_2$$

or

$$\chi = \alpha - \pi/_2 .$$

Thus

$$\cos\chi = \sin\alpha . \tag{5.14}$$

Finally, combining Equation (5.10) with (5.11), (5.13), and (5.14), we obtain the following expression for the vector potential:

$$\vec{A}(\vec{r},t) = q/_{cr\omega T} \sum_{n=-\infty}^{\infty} \int_{-\pi}^{\pi} \vec{v}\, e^{in(\alpha-\gamma-\beta\sin\theta\sin\alpha)} d\alpha .$$

On introducing the notation

$$\vec{A}(n) = g/cr \frac{1}{2\pi} \int_{-\pi}^{\pi} \vec{v}\, e^{in(\alpha-\beta \sin\theta \sin\alpha)} d\alpha, \tag{5.15}$$

the vector potential can be put in the form

$$\vec{A}(\vec{r},t) = \sum_{n=-\infty}^{\infty} \vec{A}(n)\, e^{-in\gamma}. \tag{5.16}$$

It is easy to see from Equation (5.15) that the spherical components of $\vec{A}(n)$ can be written as

$$A_r(n) = \frac{g}{cr}\frac{1}{2\pi} \int_{-\pi}^{\pi} v_r\, e^{in(\alpha-\beta\sin\theta\sin\alpha)} d\alpha \tag{5.17a}$$

$$A_\theta(n) = \frac{g}{cr}\frac{1}{2\pi} \int_{-\pi}^{\pi} v_\theta\, e^{in(\alpha-\beta\sin\theta\sin\alpha)} d\alpha$$

and

$$A_\phi(n) = \frac{g}{cr}\frac{1}{2\pi} \int_{-\pi}^{\pi} v_\phi\, e^{in(\alpha-\beta\sin\theta\sin\alpha)} d\alpha. \tag{5.17b}$$

Noting that $\vec{E} = -\operatorname{curl} \vec{A}$ and employing the expression for the curl of vectr $\vec{A}$ in spherical coordinates,

$$\vec{E} = -\frac{1}{r\sin\theta}\left[\frac{\partial}{\partial\theta}(\sin\theta\, A_\phi) - \frac{\partial A_\theta}{\partial\phi}\right]\vec{e}_r - \left[\frac{1}{r\sin\theta}\frac{\partial A_r}{\partial\phi} - \frac{1}{r}\frac{\partial}{\partial r}(rA_\phi)\right]\vec{e}_\theta$$

$$- \left[\frac{1}{r}\frac{\partial}{\partial r}(rA_\theta) - \frac{1}{r}\frac{\partial A_r}{\partial\theta}\right]\vec{e}_\phi$$

But as shown in appendix C, the radial component of the electric field is zero. Besides, according to Equation (5.17a), the terms $\left(\frac{1}{r\sin\theta}\frac{\partial A_r}{\partial\phi}\right)$ and $\left(\frac{1}{r}\frac{\partial A_r}{\partial\theta}\right)$ are each proportional to $\frac{1}{r^2}$. Thus if we neglect terms of order higher than one, the expression for the electric field reduces to

$$\vec{E} = \frac{1}{r}\frac{\partial}{\partial r}(rA_\phi)\,\vec{e}_\theta - \frac{1}{r}\frac{\partial}{\partial r}(rA_\theta)\,\vec{e}_\phi\,. \tag{5.18}$$

In order to determine the quantities A_θ and A_ϕ , using Equation (5.2b) we write

$$\vec{v}(\tau) = \frac{d\vec{z}}{d\tau} = R_o \sin\psi \frac{d\psi}{d\tau} \vec{i} + R_o \cos\psi \frac{d\psi}{d\tau} \vec{j}$$

or, since $\frac{d\psi}{d\tau} = \omega$ and $v = R_o \omega$,

$$\vec{v} = - v \sin\psi \, \vec{i} + v \cos\psi \, \vec{j}.$$

Then recalling that $\vec{e}_\theta = \cos\theta\cos\phi\,\vec{i} + \cos\theta\sin\phi\,\vec{j} - \sin\theta\,\vec{k}$ and $\psi - \phi = \alpha - \pi/2$, we have

$$v_\theta = \vec{v}\cdot\vec{e}_\theta = - v \cos\theta\,(\sin\psi\cos\phi - \cos\psi\sin\phi) = - v \cos\theta \sin(\alpha - \pi/2)$$

or

$$v_\theta = v \cos\theta\cos\alpha .$$

Substitution of this into (5.17b) yields

$$A_\theta(n) = \frac{g\, v\cos\theta}{cr} \frac{1}{2\pi} \int_{-\pi}^{\pi} \cos\alpha\, e^{in(\alpha - \beta\sin\theta\sin\alpha)} d\alpha.$$

Now setting $\beta = v/c$ and multiplying both numerator and denominator by $\sin\theta$, one finds

$$A_\theta(n) = \frac{g \operatorname{cotan}\theta}{r} \frac{1}{2\pi} \int_{-\pi}^{\pi} \beta \sin\theta \cos\alpha\, e^{in(\alpha - \beta\sin\theta\sin\alpha)} d\alpha.$$

Furthermore, since

$$\frac{de^{-in\beta\sin\theta\sin\alpha}}{-in} = \beta\sin\theta\cos\alpha\, e^{-in\beta\sin\theta\cos\alpha} d\alpha,$$

$$A_\theta(n) = \frac{g \operatorname{cotan}\theta}{r} \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{e^{in\alpha}}{-in} de^{-in\beta\sin\theta\sin\alpha}$$

and integration by parts leads to

$$A_\theta(n) = \frac{g \text{ cotan } \theta}{r} \left\{ \frac{1}{2\pi} \left[\frac{e^{in\alpha}}{-in} e^{-in\,\beta \sin\theta \sin\alpha} \right]_{\alpha=-\pi}^{\alpha=\pi} + \frac{1}{2\pi} \int_{\pi}^{\pi} e^{in(\alpha - \beta \sin\theta \sin\alpha)} d\alpha.\right.$$

But $\quad \frac{1}{-2\pi in} e^{in\alpha} \Big|_{\alpha=-\pi}^{\alpha=\pi} = \frac{1}{-n\pi} \left(\frac{e^{in\pi} - e^{-in\pi}}{2i}\right) = \frac{\sin n\pi}{-n\pi}$.

Hence, noting that $\lim\limits_{n\to o} \frac{\sin n\pi}{-n\pi} = -1$,

$$\frac{1}{-2\pi in} e^{in\alpha} \Big|_{\alpha=-\pi}^{\alpha=\pi} = \begin{cases} -1 & \text{if } n = o \\ o & \text{if } n \neq o \end{cases}$$

In addition, for $n = o$

$$\frac{1}{2\pi} \int_{-\pi}^{\pi} e^{in(\alpha - \beta \sin\theta \sin\alpha)} d\alpha = 1.$$

Consequently, for any integer different from zero, the expression for $A_\theta(n)$ takes the form

$$A_\theta(n) = \frac{g \text{ cotan } \theta}{r} \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{in(\alpha - \beta \sin\theta \sin\alpha)} d\alpha$$

or, in terms of the Bessel functions defined by

$$J_n(n\beta \sin\theta) = \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{in(\alpha - \beta \sin\theta \sin\alpha)} d\alpha,$$

$$A_\theta(n) = \frac{g \text{ cotan } \theta}{r} J_n(n\beta \sin\theta). \tag{5.19}$$

According to Equation (5.16),

$$A_\theta = \sum_{n=-\infty}^{\infty} A_\theta(n) e^{-in\gamma}$$

and therefore with the aid of (5.19)

$$A_\theta = \frac{g \operatorname{cotan} \theta}{r} \sum_{n=-\infty}^{\infty} J_n(n\beta \sin\theta)\, e^{-in\gamma}$$

or, on replacing n by -n

$$A_\theta = \frac{g \operatorname{cotan} \theta}{r} \sum_{n=-\infty}^{\infty} J_{-n}(-n\beta \sin\theta) e^{in\gamma}.$$

Since

$$J_{-n}(-n\beta \sin\theta) = J_n(n\beta\sin\theta), \tag{5.20}$$

finally we have

$$A_\theta = \frac{g \operatorname{cotan} \theta}{r} \sum_{n=-\infty}^{\infty} J_n(n\beta \sin\theta) e^{in\gamma}. \tag{5.21}$$

Next, using the fact that $\vec{e}\phi = -\sin\phi\,\vec{i} + \cos\phi\,\vec{j}$ along with $\psi - \phi = \alpha - \pi/2$, we write

$$v_\phi = \vec{v}\cdot\vec{e}_\phi = v(\sin\psi\sin\phi + \cos\psi\cos\phi) = v\cos(\psi-\phi) = v\cos(\alpha-\pi/2)$$

or

$$v_\phi = v\sin\alpha.$$

When this expression is substituted into (5.17c), there results the equation

$$A_\phi(n) = \frac{gv}{cr}\frac{1}{2\pi}\int_{-\pi}^{\pi} \sin\alpha\; e^{in(\alpha-\beta\sin\theta\sin\alpha)}\,d\alpha.$$

On taking into account that

$$\frac{d}{d(n\beta\sin\theta)}\frac{1}{2\pi}\int_{-\pi}^{\pi} e^{in(\alpha-\beta\sin\theta\sin\alpha)}\,d\alpha = J_n'(n\beta\sin\theta)$$

$$= -\frac{i}{2\pi}\int_{-\pi}^{\pi}\sin\alpha\, e^{in(\alpha-\beta\sin\theta\sin\alpha)}\,d\alpha,$$

Equation (5.22) goes over into

$$A_\phi(n) = \frac{ig\beta}{r} J_n'(n\beta\sin\theta)$$

in which $\beta = v/_c$.

Thus
$$A_\phi = \sum_{n=-\infty}^{\infty} A_\phi(n) e^{-in\gamma} = \frac{ig\beta}{r} \sum_{n=-\infty}^{\infty} J'_n\,(n\,\beta \sin\,\theta) e^{-in\gamma}.$$

Finally, on replacing n by -n and noting that

$$J'_{-n}(-n\,\beta \sin\,\theta) = -J'_n\,(n\,\beta \sin\,\theta), \tag{5.23}$$

$$A_\phi = \frac{-ig\beta}{r} \sum_{n=-\infty}^{\infty} J'_n\,(n\,\beta \sin\,\theta) e^{in\gamma}\,. \tag{5.24}$$

we shall now obtain explicit expressions for the angular components of the electric field in terms of the Bessel functions or their first derivatives. In view of Equation (5.18),

$$E_\Theta = \frac{1}{r} \frac{\partial}{\partial r}\,(rA_\phi). \tag{5.25}$$

But with the help of (5.24)

$$\frac{\partial}{\partial r}\,(rA_\phi) = \frac{\partial}{\partial r}\,[-\,ig\,\beta \sum_{n=-\infty}^{\infty} J'_n(n\,\beta \sin\,\theta) e^{in\gamma}\,]$$

$$= -ig\beta \sum_{n=-\infty}^{\infty} J'_n(n\,\beta \sin\,\theta)\,\frac{d}{dr}\,e^{in\gamma},$$

where
$$\gamma = \omega t - \frac{\omega r}{c} - \phi + \pi/_2$$

from which follows
$$\frac{d\gamma}{dr} = -\frac{\omega}{c} = -\frac{\beta}{R_o}\,. \tag{5.26}$$

Therefore

$$\frac{\partial}{\partial r}\,(rA_\phi) = \frac{-g\beta^2}{R_o} \sum_{n=-\infty}^{\infty} n\,J'_n(n\,\beta \sin\,\theta) e^{in\gamma}$$

and hence

$$E_\theta = -\frac{g\beta^2}{R_o r}\sum_{n=-\infty}^{\infty} nJ'_n\,(n\,\beta\sin\,\theta)e^{in\gamma}. \tag{5.27}$$

From Equation (5.18), we see that

$$E_\phi = -\frac{1}{r}\,\frac{\partial}{\partial r}\,(rA_\theta).$$

Then combining this with (5.21), one obtains

$$E_\phi = \frac{-g\,\text{cotan}\,\theta}{r}\sum_{n=-\infty}^{\infty} J_n(n\,\beta\,\sin\,\theta\,)\,\frac{d}{dr}\,e^{in\gamma}$$

or, on employing Equation (5.26)

$$E_\phi = \frac{ig\beta}{R_o r}\,\text{cotan}\,\theta\sum_{n=-\infty}^{\infty} nJ_n(n\,\beta\,\sin\,\theta)e^{in\gamma}. \tag{5.28}$$

We are now in a position to calculate the intensity of radiation emitted by a monopole in uniform circular motion into a unit solid angle. The energy flux density is given by the poynting vector

$$\vec{S} = \frac{c}{4\pi}\,\vec{E}\times\vec{H}\,. \tag{5.29}$$

Thus the amount of energy passing per unit time through an element of spherical surface dA with center at the origin and radius r can be written as

$$dW = \vec{S}\cdot d\vec{A}\,, \tag{5.30}$$

where $d\vec{A} = r^2 d\Omega\,\vec{n}$ with $d\Omega = \sin\theta\,d\theta\,d\phi$ and $\vec{n} = \frac{\vec{r}}{r}$. (5.31)

Applying (5.29) and the first equation of (5.31) in (5.30), one finds

$$\frac{dW}{d\Omega} = \frac{c\,r^2}{4\pi}\,[\,\vec{E}\times\vec{H}\,]\cdot\vec{n}\,.$$

It is shown in appendix C that $\vec{H} = \vec{n} x \vec{E}$.

Then

$$\frac{dW}{d\Omega} = \frac{cr^2}{4\pi} [\vec{E} x (\vec{n} x \vec{E})] \cdot \vec{n}$$

and using the identity $\vec{a} x (\vec{b} x \vec{c}) = \vec{b}(\vec{a}.\vec{c}) - \vec{c}(\vec{a}.\vec{b})$, we obtain

$$\frac{dW^2}{d\Omega} = \frac{cr^2}{4\pi} [\vec{n}(\vec{E}.\vec{E}) - \vec{E}(\vec{E}.\vec{n})] \cdot \vec{n}$$

Since the electric field $\vec{E}$ is normal to the unit vector $\vec{n}$ along which the radiation propagates, $\vec{E}.\vec{n} = 0$.

Consequently
$$\frac{dW}{d\Omega} = \frac{cr^2}{4\pi} |\vec{E}|^2$$

or, expressed in terms of the spherical components of the electric field $\vec{E}$

$$\frac{dW}{d\Omega} = \frac{cr^2}{4\pi} E_{\theta} E^*_{\theta} + E_{\phi} E^*_{\phi}) .$$

Moreover, employing (5.25) and (5.26) along with their complex conjugates,

$$\frac{dW}{d\Omega} = \frac{cr^2}{4\pi} \quad \frac{g^2\beta^4}{R_o^2 r^2} \sum_{n=-\infty}^{\infty} nn' J'_{n'} (n' \beta \sin \theta) J'_n (n \beta \sin \theta)$$

$$+ \frac{g^2\beta^2}{R_o^2 r^2} \cot an^2 \theta \sum_{n=-\infty}^{\infty} nn' J_n (n' \beta \sin \theta) J_n (n \beta \sin \theta) e^{i(n-n')\gamma} \qquad (5.32)$$

we next compute the average intensity of the radiation over the period T:

$$<\frac{dW}{d\Omega}> = \frac{1}{T} \int_{-T/2}^{T/2} \frac{dW}{d\Omega} dt .$$

Combining this with Equation (5.32), we have

$$\left\langle\frac{dW}{d\Omega}\right\rangle = \Big[\frac{cg^2\beta^4}{4\pi R_o^2}\sum_{n=-\infty}^{\infty} nn'J'_{n'}(n'\beta\sin\theta)J'_n(n\beta\sin\theta)$$

$$+\frac{cg^2\beta^2\cot an^2\theta}{4\pi R_o^2}\sum_{n=-\infty}^{\infty} nn'J_{n'}(n'\beta\sin\theta)J_n(n\beta\sin\theta)\Big]\frac{1}{T}\int_{-T/2}^{T/2} e^{i(n-n')\gamma}dt\,, \tag{5.33}$$

where $\gamma = \omega t + \gamma_o$ with $\gamma_o = \pi/2 - \phi - \frac{\omega r}{c}$.

Then changing the variable of integration t to γ and noting that $dt = \frac{d\gamma}{\omega}$,

$$\frac{1}{T}\int_{-T/2}^{T/2} e^{i(n-n')\gamma}dt = \frac{1}{T\omega}\int_{-\frac{\omega T}{2}+\gamma_o}^{\frac{\omega T}{2}+\gamma_o} e^{i(n-n')\gamma}d\gamma$$

$$= \frac{e^{i(n-n')\gamma}}{2\pi i(n-n')}\Big|_{\gamma=\gamma_o-\pi}^{\gamma=\pi+\gamma_o}$$

$$= \frac{e^{i(n-n')\gamma_o}}{\pi(n-n')}\Big[\frac{e^{i\pi(n-n')}-e^{-i\pi(n-n')}}{2i}\Big]$$

$$= e^{i(n-n')\gamma_o}\Big[\frac{\sin\pi(n-n')}{\pi(n-n')}\Big]$$

Since $\lim_{n\to n'}\frac{\sin\pi(n-n')}{\pi(n-n')} = 1$ and for $n\neq n'$ $\frac{\sin\pi(n-n')}{\pi(n-n')} = 0$, (5.34)

$$\frac{1}{T}\int_{-T/2}^{T/2} e^{i(n-n')\gamma}dt = \delta_{nn'}\,.$$

Thus substitution of this into Equation (5.33) yields

$$\left\langle\frac{dW}{d\Omega}\right\rangle = \frac{cg^2\beta^2}{4\pi R_o^2}\Big[\sum_{n=-\infty}^{\infty} n^2\beta^2 J'^2_n(n\beta\sin\theta) + \cot an^2\theta\sum_{n=-\infty}^{\infty} n^2 J_n^2(n\beta\sin\theta)\Big]$$

$$= \frac{cg^2\beta^2}{4\pi R_o^2}\Big[\sum_{n=-\infty}^{-1} n^2\beta^2 J_n'^2(n\beta\sin\theta) + \sum_{n=1}^{\infty} n^2\beta^2 J_n'^2(n\beta\sin\theta)$$

$$+ \sum_{n=-\infty}^{-1}\cot^2\theta\, n^2 J_n^2(n\beta\sin\theta) + \sum_{n=1}^{\infty}\cot^2\theta\, n^2 J_n^2(n\beta\sin\theta)\Big].$$

Furthermore, replacing n by -n in the first and third terms and then applying (5.20) and (5.23), one finds

$$\left\langle\frac{dW}{d\Omega}\right\rangle = \frac{cg^2\beta^2}{2\pi R_o^2}\Big\{\sum_{n=1}^{\infty} n^2\big[\beta^2 J_n'^2(n\beta\sin\theta) + \cot^2\theta\, J_n^2(n\beta\sin\theta)\big]\Big\},$$

from which we finally obtain the expression for the intensity of radiation emitted per harmonic into a unit solid angle

$$\frac{dp_n}{d\Omega} = \frac{cg^2\beta^2 n^2}{2\pi R_o^2}\big[\beta^2 J_n'^2(n\beta\sin\theta) + \cot^2\theta\, J_n^2(n\beta\sin\theta)\big]. \tag{5.35a}$$

A similar expression was first derived by Schott for the intensity of radiation emitted by an electric charge in uniform circular motion [11]:

$$\frac{dp_n}{d\Omega} = \frac{ce^2\beta^2 n^2}{2\pi R_o^2}\big[\beta^2 J_n'^2(n\beta\sin\theta) + \cot^2\theta\, J_n^2(n\beta\sin\theta)\big]. \tag{5.35b}$$

6. POLARIZATION OF MONOPOLE SYNCHROTRON RADIATION

According to Equations (5.27) and (5.28), the monochromatic component of the electric field can be written in the form

$$\vec{E}_n = (C_\theta \vec{e}_\theta + i\, C_\phi \vec{e}_\phi)\, e^{in\gamma} , \tag{6.1a}$$

where

$$C_\theta = -g\beta^2 n J_n'/R_o r \text{ and } C_\phi = g\beta \cot\!an\,\theta\, n J_n/R_o r \tag{6.1b}$$

In order to investigate the polarization of monopole synchrotron radiation in some detail, we express $\vec{E}_n$ as

$$\vec{E}_n = (E_f \vec{a}_f + F_g \vec{a}_g)\, e^{in\gamma}$$

in which

$$\vec{a}_f = \cos z\, \vec{\beta}_2 + e^{i\eta} \sin z\, \vec{\beta}_3 , \tag{6.2a}$$

$$\vec{a}_g = \sin z\, \vec{\beta}_2 - e^{i\eta} \cos z\, \vec{\beta}_3 , \tag{6.2b}$$

and

$$\vec{a}_f \cdot \vec{a}_f^* = \vec{a}_g \cdot \vec{a}_g^* = 1$$

$$\vec{a}_f \cdot \vec{a}_g = \vec{a}_g \cdot \vec{a}_f^* = 0$$

In addition, the unit vectors $\vec{\beta}_2$ and $\vec{\beta}_3$ are required to satisfy the condition

$$\vec{\beta}_2 \cdot \vec{\beta}_3 = 0 \text{ and } \vec{\beta}_2 \times \vec{\beta}_3 = \frac{\vec{r}}{r} . \tag{6.3}$$

The parameters z and η specify different states of polarization. And comparison of $\vec{e}_\phi \times (-\vec{e}_\theta) = \frac{\vec{r}}{r}$ with the second equation of (6.3) shows that

$$\vec{\beta}_2 \,||\, \vec{e}_\phi \text{ and } \vec{\beta}_3 \,||\, (-\vec{e}_\theta) .$$

a) We now consider the state of polarization when $z = o$ and $\eta = \pi$. Accordingly, it follows from equation (6.2) that

$$\vec{a}_f = \vec{\beta}_2 \text{ and } \vec{a}_g = \vec{\beta}_3$$

on employing the fact that $e^{i\pi} = -1$.

Thus

$$\vec{E}_{nf} = [(C_\theta \vec{e}_\theta + iC_\phi \vec{e}_\phi) . \vec{\beta}_2] \vec{\beta}_2 e^{in\gamma}$$

or, noting that $\vec{e}_\theta \circ \vec{\beta}_2 = o$ and $\vec{e}_\phi \circ \vec{\beta}_2 = 1$,

$$\vec{E}_{nf} = iC_\phi \vec{\beta}_2 e^{in\gamma} . \tag{6.4a}$$

Also

$$\vec{E}_{ng} = [(C_\theta \vec{e}_\theta + iC_\phi \vec{e}_\phi) \cdot \vec{\beta}_3] \vec{\beta}_3 e^{in\gamma}$$

and since

$$\vec{e}_\theta \cdot \vec{\beta}_3 = -1 \text{ and } \vec{e}_\phi \cdot \vec{\beta}_3 = 0 ,$$

$$\vec{E}_{ng} = - C_\theta \vec{\beta}_3 e^{in\gamma} . \tag{6.4b}$$

Equation (6.4) indicates that $\vec{E}_{nf}$ and $\vec{E}_{ng}$ are linearly polarized along $\vec{\beta}_2$ and $\vec{\beta}_3$ respectively. In view of Equations (5.35a) and (6.1b), the intensities with the f- and g- polarization are then given by

$$\frac{dp_{nf}}{d\Omega} = cg^2 \beta^2 \text{cotan}^2 \theta \; n^2 J_n^2 / 2\pi R_o^2 \tag{6.5a}$$

and

$$\frac{dp_{ng}}{d\Omega} = cg^2 \beta^4 \; n^2 J'^2_n / 2\pi R_o^2 . \tag{6.5b}$$

As shown in appendix C, $H_\theta = - E_\phi$ and $H_\phi = E_\theta$. Hence in accordance with Equation (6.1a), the monochromatic component of the magnetic field can be written in the form

$$\vec{H}_n = (-iC_\phi \vec{e}_\theta + C_\theta \vec{e}_\phi) e^{in\gamma}.$$

Now application of the transformation $\vec{H}_n^g \to -\vec{E}_n^e$ and $g \to -e$ to the above equation along with (6.1b) yields the following expression for the monochromatic component of the electric field produced by an electron in uniform circular motion:

$$\vec{E}'_n = (iC'_\phi \vec{e}_\theta + C'_\theta \vec{e}_\phi) e^{in\gamma}; \qquad (6.6)$$

where

$$C'_\theta = -e\beta^2 n J'_n / R_o r \text{ and } C'_\phi = -e\beta \cot\!an\,\theta\, n J_n / R_1 r.$$

Here we have introduced a prime sign in order to distinguish quantities that belong to the electron from that of the monopole. With the aid of Equations (6.4) and (6.6), one can show that $\vec{E}'_{nf}$ and $\vec{E}'_{ng}$ are linearly polarized, as in monopole synchrotron radiation, along $\vec{\beta}_2$ and $\vec{\beta}_3$. But the intensities with the f- and g- polarization are given by

$$\frac{dp'_{nf}}{d\Omega} = c e^2 \beta^4 n^2 J'^2_n / 2\pi R_o^2 \qquad (6.7a)$$

and

$$\frac{dp'_{ng}}{d\Omega} = c e^2 \beta^2 \cot\!an^2\theta\; n^2 J_n^2 / 2\pi R_o^2 . \qquad (6.7b)$$

Since for a relativistic electron θ is of the order of $(\pi/2 \pm \sqrt{1-\beta^2})$ [12], $\cos\theta \simeq \mp \sin\sqrt{1-\beta^2}$ and $\sin\theta \simeq \cos\sqrt{1-\beta^2}$.

Then

$$\cot\!an^2\theta \simeq \frac{\sin^2\sqrt{1-\beta^2}}{\cos^2\sqrt{1-\beta^2}} << 1 .$$

It follows from this

$$\beta^2 J_n'^2 > \cot\!an^2\theta J_n^2 \qquad (6.8)$$

and hence in view of Equation (6.7)

$$\frac{dp'_{nf}}{d\Omega} > \frac{dp'_{ng}}{d\Omega} . \qquad (6.9a)$$

However, in the case of monopole synchrotron radiation, one can easily see from (6.5) and (6.8) that

$$\frac{dp_{ng}}{d\Omega} > \frac{dp_{nf}}{d\Omega} . \qquad (6.9b)$$

b) We next consider the polarization state when $z = \pi/4$ and $\eta = \pi$.

As a result we obtain from Equation (6.2) that

$$\vec{a}_f = \frac{\sqrt{2}}{2}(\vec{\beta}_2-\vec{\beta}_3) \text{ and } \vec{a}_g = \frac{\sqrt{2}}{2}(\vec{\beta}_2+\vec{\beta}_3) .$$

Thus

$$\vec{E}_{nf} = [(C_\theta\vec{e}_\theta + iC_\phi\vec{e}_\phi)\cdot\frac{\sqrt{2}}{2}(\vec{\beta}_2-\vec{\beta}_3)]\frac{\sqrt{2}}{2}(\vec{\beta}_2-\vec{\beta}_3)e^{in\gamma}$$

or

$$\vec{E}_{nf} = \frac{1}{2}[C_\theta + iC_\phi](\vec{\beta}_2-\vec{\beta}_3)e^{in\gamma}$$

This shows that $\vec{E}_{nf}$ is linearly polarized along the vector making an angle of -45° with $\vec{\beta}_2$.

And

$$\vec{E}_{ng} = [(C_\theta\vec{e}_\theta + iC_\phi\vec{e}_\phi)\cdot\frac{\sqrt{2}}{2}(\vec{\beta}_2+\vec{\beta}_3)]\frac{\sqrt{2}}{2}(\vec{\beta}_2+\vec{\beta}_3)e^{in\gamma}$$

or

$$\vec{E}_{ng} = \frac{1}{2}[iC_\phi - C_\theta](\vec{\beta}_2+\vec{\beta}_3)e^{in\gamma}$$

which indicates that $\vec{E}_{ng}$ is linearly polarized along the vector making an angle of 45° with $\vec{\beta}_2$. In this case, the intensities with the f- and g-polarization are equal:

$$\frac{dp_{nf}}{d\Omega} = \frac{dp_{ng}}{d\Omega} = \frac{cr^2}{4\pi}\,(C_\theta^{\,2} + C_\phi^{\,2})$$

or

$$\frac{dp_{nf}}{d\Omega} = \frac{dp_{ng}}{d\Omega} = \frac{cg^2\beta^2 n^2}{4\pi R_o^2}\,(\beta^2 J_n'^{\,2} + \cot an^2\theta\, J_n^2)\,.$$

In a similar manner, one can show that $\vec{E}'_{nf}$ and $\vec{E}'_{ng}$ have the same polarization as $\vec{E}_{nf}$ and $\vec{E}_{ng}$, and

$$\frac{dp'_{nf}}{d\Omega} = \frac{dp'_{ng}}{d\Omega} = \frac{ce^2\beta^2 n^2}{4\pi R_o^2}\,(\beta^2 J_n'^{\,2} + \cot an^2\theta\, J_n^2)\,.$$

c) Finally we consider the case when $Z = \pi/4$ and $\eta = \pi/2$. Then, on taking into account the fact that $e^{i\pi/2} = i$, one obtains from Equation (6.2) that

$$\vec{a}_f = \frac{\sqrt{2}}{2}\,(\vec{\beta}_2 + i\vec{\beta}_3) \text{ and } \vec{a}_g = \frac{\sqrt{2}}{2}\,(\vec{\beta}_2 - i\vec{\beta}_3)\,.$$

Consequently

$$\vec{E}_{nf} = [(C_\theta\vec{e}_\theta + iC_\phi\vec{e}_\phi)\cdot\vec{a}_f^{*}]\,\frac{\sqrt{2}}{2}\,(\vec{\beta}_2 + i\vec{\beta}_3)\,e^{in\gamma}$$

or, noting that $\vec{a}_g^{*} = \frac{\sqrt{2}}{2}\,(\vec{\beta}_2 - i\vec{\beta}_3)$, we have

$$\vec{E}_{nf} = \frac{i}{2}\,[C_\theta + C_\phi]\,(\vec{\beta}_2 + i\vec{\beta}_3)\,e^{in\gamma} \tag{6.10a}$$

and

$$\vec{E}_{ng} = \frac{i}{2}\,[C_\phi - C_\theta]\,(\vec{\beta}_2 - i\vec{\beta}_3)\,e^{in\gamma}. \tag{6.10b}$$

It follows from Equation (6.10) along with $\gamma = t - \frac{\omega r}{c} - \phi + \pi/2$ that $\vec{E}_{nf}$ and $\vec{E}_{ng}$ are respectively right and left circularly polarized. In a similar fashion, one can show that $\vec{E}'_{nf}$ and $\vec{E}'_{ng}$ have the same polarization as $\vec{E}_{nf}$ and $\vec{E}_{ng}$. The intensities with the f- and g- polarization are now given by

$$\frac{dp_{nf}}{d\Omega} = \frac{cr^2}{4\pi} (C_\theta^2 + 2C_\theta C_\phi + C_\phi^2)$$

and

$$\frac{dp_{ng}}{d\Omega} = \frac{cr^2}{4} (C_\theta^2 - 2C_\theta C_\phi + C_\phi^2)$$

or, using Equation (6.1b)

$$\frac{dp_{nf}}{d\Omega} = \frac{cg^2\beta^2 n^2}{4\pi R_o^2} (\beta^2 J_n'^2 - 2\beta \text{ cotan}\theta\ J_n' J_n + \text{cotan}^2\theta J_n^2)$$

and

$$\frac{dp_{ng}}{d\Omega} = \frac{cg^2\beta^2 n^2}{4\pi R_o^2} (\beta^2 J_n'^2 - 2\beta \text{ cotan}\theta\ J_n' J_n + \text{cotan}^2\theta\ J_n^2)$$

Morever, in the case of electron synchrotron radiation

$$\frac{dp'_{nf}}{d\Omega} = \frac{ce^2\beta^2 n^2}{4\pi R_o^2} (\beta^2 J_n'^2 - 2\beta \text{ cotan}\theta\ J_n' J_n + \text{cotan}^2\theta J_n^2)$$

and

$$\frac{dp'_{ng}}{d\Omega} = \frac{ce^2\beta^2 n^2}{4\pi R_o^2} (\beta^2 J_n'^2 + 2\beta \text{ cotan}\theta\ J_n' J_n + \text{cotan}^2\theta J_n^2).$$

It is easy to see from the above four Equations that

$$\frac{dp_{ng}}{d\Omega} > \frac{dp_{nf}}{d\Omega} \text{ and } \frac{dp'_{ng}}{d\Omega} > \frac{dp'_{nf}}{d\Omega}. \qquad (6.11)$$

7. ENERGY TRANSFER IN ELECTRON-MONOPOLE SCATTERING

We now want to determine the energy transferred from a fast monopole to an electron initially at rest. The procedure we follow here is to obtain first the trajectory of an electron in the field of a monopole, and then the square of the velocity at a time long after the scattering is over. Obviously, using this, we are finally able to find the energy transferred to the electron.

The equation of motion of an electron in the field produced by a monopole is

$$\frac{d\vec{p}_1}{dt} = e\vec{E}_2 + \frac{e}{c}\,\vec{v}_1 x \vec{H}_2 \tag{7.1a}$$

and that of a monopole in the field of an electron is

$$\frac{d\vec{p}_2}{dt} = g\vec{H}_1 - g/_c\;\vec{v}_2 x\,\vec{E}_1, \tag{7.1b}$$

where the subscripts (1) and (2) label quantities that belong to the electron and the monopole respectively. We write here once more Equations (4.2) and (4.3):

$$\vec{E}_2 = -\nabla x\vec{A}_2, \tag{7.2a}$$

$$\vec{H}_2 = -\nabla\phi_2 - \frac{1}{c}\frac{\partial\vec{A}_2}{\partial t}. \tag{7.2b}$$

And as consequences of the ordinary Maxwell's equations, we have

$$\vec{E}_1 = -\nabla\phi_1 - \frac{1}{c}\;\frac{\partial\vec{A}_1}{\partial t}, \tag{7.2c}$$

and

$$\vec{H}_1 = \nabla x\,\vec{A}_1. \tag{7.2d}$$

The potentials of the fields produced by an electron and a monopole have the form

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The equation of motion of an electron in the field produced by a monopole is

$$\frac{d\vec{p}_1}{dt} = e\vec{E}_2 + \frac{e}{c}\,\vec{v}_1 \times \vec{H}_2 \tag{7.1a}$$

and that of a monopole in the field of an electron is

$$\frac{d\vec{p}_2}{dt} = g\vec{H}_1 - g/_c\;\vec{v}_2 \times \vec{E}_1, \tag{7.1b}$$

where the subscripts (1) and (2) label quantities that belong to the electron and the monopole respectively. We write here once more Equations (4.2) and (4.3):

$$\vec{E}_2 = -\nabla\times\vec{A}_2, \tag{7.2a}$$

$$\vec{H}_2 = -\nabla\phi_2 - \frac{1}{c}\frac{\partial\vec{A}_2}{\partial t}\,. \tag{7.2b}$$

And as consequences of the ordinary Maxwell's equations, we have

$$\vec{E}_1 = -\nabla\phi_1 - \frac{1}{c}\,\frac{\partial\vec{A}_1}{\partial t}\,, \tag{7.2c}$$

and

$$\vec{H}_1 = \nabla\times\vec{A}_1\,. \tag{7.2d}$$

The potentials of the fields produced by an electron and a monopole have the form

$$\phi_i = e_i \int \frac{\delta(\tau_i - t + R_{i/c})}{R_i} d\tau_i \tag{7.3a}$$

and

$$\vec{A}_i = \frac{e_i}{c} \int \frac{\vec{v}_i(\tau_i)\ \delta(\tau_i - t + \frac{R_i}{c})}{R_i} d\tau_i \ , \tag{7.3b}$$

where . $$i = 1,2,\ e_1 = e,\ e_2 = g,\ \vec{R}_1 = \vec{r}_2(t) - \vec{r}_1(\tau_1),$$

$$\vec{R}_2 = \vec{r}_1(t) - \vec{r}_2(\tau_2),\ \text{and}\ \vec{v}_i = \frac{d\vec{r}_i}{d\tau_i}.$$

Expanding the δ-function in powers of R/c and neglecting all terms except the first gives

$$\delta(\tau_i - t + \frac{R_i}{c}) = \delta(\tau_i - t).$$

Substitution of this expression into (7.3) leads to

$$\phi_i = e_i \int \frac{\delta(\tau_i - t)}{R_i} d\tau_i = \frac{e_i}{R_i} \tag{7.4a}$$

and

$$\vec{A}_i = \frac{e_i}{c} \int \frac{\vec{v}_i(\tau_i)\ \delta(\tau_i - t)}{R_i} d\tau_i = \frac{e_i \vec{v}_i(t)}{cR_i} \ . \tag{7.4b}$$

Now combining Equation (7.2) with (7.4) and dropping terms of the order of

$$\frac{\partial}{\partial t} (\frac{\vec{v}_i}{R_i}) \ ,$$

one finds

$$\vec{E}_2 = - g/_c\, \nabla \times (\frac{\vec{v}_2}{R_2}) \ , \tag{7.5a}$$

$$\vec{H}_2 = - g\nabla(\frac{1}{R_2}) \ , \tag{7.5b}$$

$$\vec{E}_1 = - e\nabla(\frac{1}{R_1}) \ , \tag{7.6a}$$

and

$$\vec{H}_1 = \frac{e}{c}\ \nabla \times \left(\frac{\vec{v}_1}{R_1}\right) \ . \tag{7.6b}$$

Putting Equations (7.5) and (7.6) into (7.1a) and (7.1b) respectively, and using the identity $\nabla \times (\phi\vec{A}) = \phi\nabla \times \vec{A} - \vec{A} \times \nabla\phi$ yields

$$\frac{d\vec{p}_1}{dt} = \frac{-eg}{cR_2}\ \nabla \times \vec{v}_2 + \frac{eg}{c}(\vec{v}_2 - \vec{v}_1) \times \nabla\left(\frac{1}{R_2}\right) \tag{7.7a}$$

and

$$\frac{d\vec{p}_2}{dt} = \frac{eg}{c}\,\frac{1}{R_1}\ \nabla \times \vec{v}_1 - \frac{eg}{c}(\vec{v}_1 - \vec{v}_2) \times \nabla\left(\frac{1}{R_1}\right). \tag{7.7b}$$

If we keep only terms of the order of v/c , Equation (7.7) reduces to

$$\frac{d\vec{p}_1}{dt} = \frac{eg}{c}\ \mathring{\vec{R}}_1 \times \nabla\left(\frac{1}{R_2}\right) \ ,$$

$$\frac{d\vec{p}_2}{dt} = -\ \frac{eg}{c}\ \mathring{\vec{R}}_2 \times \nabla\left(\frac{1}{R_1}\right) \ ,$$

where
$$\mathring{\vec{R}}_1 = \mathring{\vec{r}}_2 - \mathring{\vec{r}}_1 \text{ and } \mathring{\vec{R}}_2 = \mathring{\vec{r}}_1 - \mathring{\vec{r}}_2 \ .$$

For $v \ll c$, the time derivative of $\vec{p}_1$ and $\vec{p}_2$ can be written as

$$\mathring{\vec{p}}_1 = m_1\ddot{\vec{r}}_1 \text{ and } \mathring{\vec{p}}_2 = m_2\ddot{\vec{r}}_2 \ ,$$

where m_1 and m_2 are the rest mass of the electron and the monopole.

Consequently

$$m_1\ddot{\vec{r}}_1 = \frac{eg}{c}\ \mathring{\vec{R}}_1 \times \nabla\left(\frac{1}{R_2}\right),$$

$$m_2\ddot{\vec{r}}_2 = \frac{-eg}{c}\ \mathring{\vec{R}}_2 \times \nabla\left(\frac{1}{R_1}\right).$$

Since
$$\nabla\left(\frac{1}{R_1}\right) = -\frac{\vec{R}_1}{R_1^3} \text{ and } \nabla\left(\frac{1}{R_2}\right) = -\frac{\vec{R}_2}{R_2^3} \ ,$$

finally the equations of motion assume the form

$$m_1\ddot{\vec{r}}_1 = \frac{eg}{c}\frac{\mathring{\vec{R}}\times\vec{R}}{R^3}\,, \tag{7.8a}$$

$$m_2\ddot{\vec{r}}_2 = \frac{-eg}{c}\frac{\mathring{\vec{R}}\times\vec{R}}{R^3}\,, \tag{7.8b}$$

where $\qquad \vec{R} = \vec{R}_1 = -\vec{R}_2 = \vec{r}_2 - \vec{r}_1.$

Next, adding Equations (7.8a) and (7.8b) we have

$$m_1\ddot{\vec{r}}_1 + m_2\ddot{\vec{r}}_2 = \frac{d}{dt}(m_1\mathring{\vec{r}}_1 + m_2\mathring{\vec{r}}_2) = 0 \tag{7.9}$$

so that $\vec{p}_1 + \vec{p}_2 = m_1\vec{v}_{o1} + m_2\vec{v}_{o2} = \vec{p}_o$ is constant. Thus the total energy of the system of an electron and a monopole is conserved. The position of the center-of-mass is given by

$$\vec{R}_{c.m.} = \frac{m_1\vec{r}_1 + m_2\vec{r}_2}{m_1+m_2}\,.$$

Then

$$\frac{d\vec{R}_{c.m}}{dt} = \frac{m_1\mathring{\vec{r}}_1 + m_2\mathring{\vec{r}}_2}{m_1+m_2} = \frac{\vec{p}_o}{m_1+m_2}\,. \tag{7.10}$$

Equation (7.10) shows that the center-of-mass moves with a uniform velocity relative to the laboratory system. In the center-of-mass system,

$$m_1\vec{r}_1 + m_2\vec{r}_2 = 0$$

or

$$\vec{r}_2 = -\frac{m_1\vec{r}_1}{m_2}\,.$$

Recalling that $\qquad \vec{r}_2 = \vec{R} + \vec{r}_1$, we have

$$\vec{r}_1 = -\frac{\mu}{m_1}\vec{R}\,, \tag{7.11a}$$

and similarly $$\vec{r}_2 = \frac{\mu}{m_2}\vec{R}\ , \tag{7.11b}$$

where $\mu\ (=\frac{m_1 m_2}{m_1+m_2})$ is the reduced mass of the system.

Now combining Equation (7.11a) with (7.8a), or Equation (7.11b) with (7.8b) gives

$$\ddot{\vec{R}} = -\frac{eg}{\mu c R^3}\,\dot{\vec{R}}x\vec{R} \tag{7.12}$$

Taking the cross product of both sides of this equation with vector $\vec{R}$, and using the identity $\vec{a}x(\vec{b}x\vec{c}) = \vec{b}(\vec{a}.\vec{c})-\vec{c}(\vec{a}.\vec{b})$ one finds

$$\vec{R}x\ddot{\vec{R}} + \frac{eg\,\dot{\vec{R}}}{\mu c R} - \frac{eg\,(\vec{R}.\dot{\vec{R}})\vec{R}}{\mu c R^3} = 0$$

or

$$\frac{d}{dt}\ (\vec{R}x\dot{\vec{R}} + \frac{eg}{\mu c R}\vec{R}) = 0.$$

It then follows from this that the quantity in brackets is an integral of motion. Denoting this quantity by vector $\vec{C}$, we have

$$\vec{C} = \vec{R}x\dot{\vec{R}} + \frac{eg}{\mu c R}\vec{R}. \tag{7.13}$$

Vector $\vec{C}$ is determined from the following set of initial conditions in the laboratory system at time $t = -t_\infty$, long before the scattering takes place:

$$\begin{aligned} \vec{r}^{\,o}_{1L} &= (o,o,b_1), & \dot{\vec{r}}^{\,o}_{1L} &= (o,o,o) \\ \vec{r}^{\,o}_{2L} &= (o,-R_{-\infty}, -b_2) & \dot{\vec{r}}^{\,o}_{2L} &= (o,v_2,o). \end{aligned} \tag{7.14}$$

At the initial time $(t=-t_\infty)$, the monopole has a velocity of v_2 and is located, with impact parameter $b=b_1+b_2$ far from the electron.

Furthermore, noting that $\vec{R}^{o} = \vec{r}^{\,o}_{2L} - \vec{r}^{\,o}_{1L} = \vec{r}^{\,o}_{2} - \vec{r}^{\,o}_{1}$ and taking into account Equation (7.14), vectors $\vec{R}^{o}$ and $\dot{\vec{R}}^{o}$ can be expressed in the laboratory or center-of-mass system as

$$\vec{R}^{o} = (o, -R_{-\infty}, -b) \text{ and } \dot{\vec{R}}^{o} = (o, v_2, o). \tag{7.15}$$

Thus employing Equations (7.13) and (7.15) we obtain

$$\vec{C} = [bv_2, -eg\,R_{-\infty}/\mu c\,(R^2_{-\infty} + b^2)^{\frac{1}{2}}, -egb/\mu c\,(R^2_{-\infty} + b^2)^{\frac{1}{2}}]$$

or, on taking into account that $R_{-\infty} \gg b$

$$\vec{C} = (bv_2, -\frac{eg}{\mu c}, o). \tag{7.16}$$

Finally, taking the dot product of both sides of Equation (7.13) with vector $\vec{R}$ and recalling that $\vec{a}.(\vec{a}x\vec{b}) = o$ yields

$$\vec{C}.\vec{R} = \frac{egR}{\mu c} \tag{7.17a}$$

or

$$\cos\theta = \frac{eg}{\mu cC}, \tag{7.17b}$$

where θ is the angle between vectors $\vec{C}$ and $\vec{R}$ and the magnitude of vector $\vec{C}$ in view of Equation (7.16) is given by

$$C = (b^2v_2^2 + \frac{e^2g^2}{\mu^2c^2})^{\frac{1}{2}}. \tag{7.18}$$

It then follows from Equations (.17b) and (7.18) that in the center-of-mass system, the trajectory of the electron, or for that matter of the monopole, is along the surface of a cone the axis of which is directed along vector $\vec{C}$, and whose vertex angle is

$$\alpha = 2\theta = 2\arccos\left(\frac{eg}{\mu cC}\right)$$

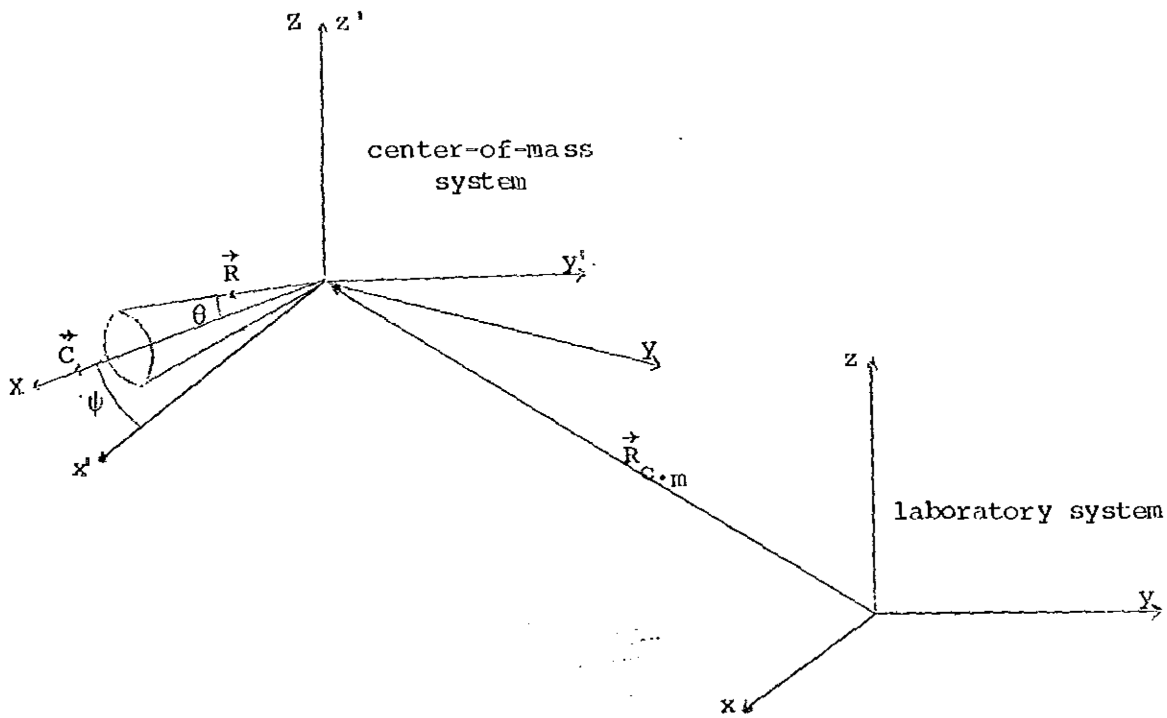

Fig. 4 Center-of-mass and Laboratory Systems

A clockwise rotation about the z'-axis through angle ψ takes the x'- and y'-axes into the X- and Y-axes. Noting that $o<\psi<\pi/2$ and taking into account Equation (7.16)

$$\cos\psi = \frac{bv_2}{C} \text{ and } \sin\psi = \frac{eg}{\mu cC}. \tag{7.19}$$

in which $eg > o$.

Thus it follows from the second equation of (7.19) and Equation (7.17b) that

$$\sin\psi = \cos\theta .$$

Also, according to Equation (7.18)

$$b^2 v_2^2 = C^2 - \frac{e^2 g^2}{\mu^2 c^2}$$

and combining this with Equation (7.17b) gives

$$b^2 v_2^2 = C^2 \sin^2\theta \tag{7.20a}$$

or

$$\sin\theta = \frac{bv_2}{C} . \tag{7.20b}$$

Therefore

$$\cos\psi = \sin\theta .$$

The matrix of transformation in terms of $\sin\theta$ and $\cos\theta$ is then given by

$$\alpha_{ik} = \begin{vmatrix} \sin\theta & -\cos\theta & o \\ \cos\theta & \sin\theta & o \\ o & o & 1 \end{vmatrix}$$

In the x'y'x'- system the velocity vector $\vec{v}_2 = (o, v_2, o)$. With the aid of this and the transformation matrix, $\vec{v}_2$ can be written in the XYZ-system as

$$\vec{v}_2 = (-v_2\cos\theta , v_2\sin\theta , o) . \tag{7.21}$$

In addition, vector $\vec{R}$ can be expressed in the form

$$\vec{R} = (R\cos\Theta , R\sin\Theta\cos\phi , R\sin\Theta\sin\phi) , \tag{7.22}$$

where ϕ is the angle between the projection of $\vec{R}$ on the YZ-plane and the Y-axis.

Hence

$$\mathring{\vec{R}} = (\mathring{R}\cos\theta , \mathring{R}\sin\theta\cos\phi - R\sin\theta\sin\phi\mathring{\phi} , \mathring{R}\sin\theta\sin\phi + R\sin\theta\cos\phi\mathring{\phi}) . \tag{7.23}$$

from which follows

$$\overset{\circ}{\vec{R}}{}^2 = \overset{\circ}{R}{}^2 + R^2\sin^2\theta\overset{\circ}{\phi}{}^2. \tag{7.24}$$

Furthermore, applying Equations (7.22) and (7.23) we find

$$\vec{R}x\overset{\circ}{\vec{R}} = (R^2\sin^2\theta\overset{\circ}{\phi}\ ,\ -R^2\sin\theta\cos\theta\cos\phi\overset{\circ}{\phi}\ ,\ -R^2\sin\theta\cos\theta\sin\phi\overset{\circ}{\phi})$$

and taking the dot product of $\vec{R}x\overset{\circ}{\vec{R}}$ with vector $\vec{C} = (C,o,o)$ yields

$$\vec{C}\cdot(\vec{R}x\overset{\circ}{\vec{R}}) = CR^2\sin^2\theta\overset{\circ}{\phi}\ . \tag{7.25}$$

Besides, in view of Equation (7.13)

$$\vec{R}x\overset{\circ}{\vec{R}} = \vec{C} - \frac{eg}{\mu cR}\vec{R}$$

so that

$$\vec{C}\cdot(\vec{R}x\overset{\circ}{\vec{R}}) = C^2\frac{eg}{\mu cR}\vec{C}\cdot\vec{R}$$

or, on employing Equation (7.17)

$$\vec{C}\cdot(\vec{R}x\overset{\circ}{\vec{R}}) = C^2\sin^2\theta.$$

It then follows from this and Equation (7.24) that

$$R^2\overset{\circ}{\phi} = C\ . \tag{7.26}$$

Next, the dot product of both sides of Equation (7.12) with $\overset{\circ}{\vec{R}}$ leads to

$$\overset{\circ}{\vec{R}}.\overset{\circ\circ}{\vec{R}} = o$$

or

$$\frac{1}{2}\frac{d}{dt}(\overset{\circ}{\vec{R}}{}^2) = o,$$

which shows that the quantity in brackets is a constant of motion, and in accordance with the second equation of (7.15)

$$\overset{\circ}{\vec{R}}{}^2 = \overset{\circ}{\vec{R}}{}_o^2 = v_2^2$$

so that Equation (7.24) now becomes

$$v_2^2 = \mathring{R}^2 + R^2\sin^2\theta\mathring{\phi}^2.$$

Then multiplying throughout by R^2, we have

$$R^2v_2^2 = R^2\mathring{R}^2 + R^4\sin^2\theta\mathring{\phi}^2 ,$$

from which follows

$$\frac{1}{4}\left(\frac{dR^2}{dt}\right)^2 = R^2v_2^2 - R^4\sin^2\theta\mathring{\phi}^2$$

or

$$\frac{1}{4v_2^2}\left(\frac{dR^2}{dt}\right)^2 = R^2 - \frac{R^4\sin^2\theta\mathring{\phi}^2}{v_2^2}.$$

Using Equations (7.20a) and (7.26), one finally obtains

$$\frac{1}{4v_2^2}\left(\frac{dR^2}{dt}\right)^2 = R^2 - b^2 .$$

We now proceed to find the solution of this equation along with the initial condition $R^o = (R_{-\infty}^2 + b^2)^{\frac{1}{2}}$. To this end, we write

$$\int_{R^o}^{R} \frac{dR^2}{(R^2 - b^2)^{\frac{1}{2}}} = \pm v_2 \int_{-t_\infty}^{t} dt.$$

Then

$$(R^2 - b^2)^{\frac{1}{2}}\Big|_{R^o}^{R} = \pm v_2 t\Big|_{-t_\infty}^{t}$$

or

$$R^2 = [R_{-\infty} \pm v_2(t + t_\infty)]^2 + b^2. \qquad (7.27)$$

We choose the time scale in such a way that the separation between the electron and the monopole has the minimum value b at time $t = o$. As a result, we obtain from Equation (7.27) that

$$R_{-\infty} \pm v_2 t_\infty = 0$$

or

$$R_{-\infty} = \mp v_2 t_\infty \; . \tag{7.28}$$

Since $R_{-\infty} > 0$, we must take the plus sign in Equation (7.28), or the minus sign in (7.27).

Hence $$R^2 = [R_{-\infty} - v_2(t + t_\infty)]^2 + b^2.$$

When the expression $R_{-\infty} = v_2 t_\infty$ is substituted into this, there results the equation

$$R^2 = v_2^2 t^2 + b^2 \tag{7.29}$$

which is valid for any time t. We immediately see that

$$\lim_{t \to t_\infty} R^2 = v_2^2 t_\infty^2 + b^2 = R_{-\infty}^2 + b^2$$

as it should be.

Also, $$\lim_{t \to -t_\infty} R^2 = \lim_{t \to t_\infty} (v_2^2 t^2 + b^2) = R_{-\infty}^2 + b^2. \tag{7.30}$$

In view of Equation (7.29), the time derivative of R is given by

$$\mathring{R} = \frac{v_2^2 t}{(v_2^2 t^2 + b^2)^{\frac{1}{2}}}$$

and on taking into account that $v_2 t_\infty \gg b$

$$\lim_{t \to t_\infty} \mathring{R} = \frac{v_2^2 t_\infty}{(v_2^2 t_\infty^2 + b^2)^{\frac{1}{2}}} = \frac{v_2}{\left(1 + \frac{b^2}{v^2 t_\infty^2}\right)^{\frac{1}{2}}} = v_2 \; . \tag{7.31}$$

We now want to obtain an expression for ϕ as a function of the time t. We see from Equations (7.26) and (7.29) that

$$\phi(t) = C \int_{t-\infty}^{t} \frac{dt}{v_2^2 t^2 + b^2} + \phi_o ,$$

where ϕ_o is the angle between the projection of $\vec{R}^o$ on the YZ-plane and the Y-axis. Since

$$\frac{dt}{v_2^2 t^2 + b^2} = \frac{1}{bv_2} \frac{d(v_2 t/b)}{[(v_2 t/b)^2 + 1)]},$$

the expression for ϕ can then be written as

$$\phi(t) = \frac{C}{bv_2} \int_{-t\infty}^{t} \frac{d(v_2 t/b)}{(v_2 t/b)^2 + 1} + \phi_o = \frac{C}{bv_2} \arctan \left(\frac{v_2 t}{b}\right)\Big|_{-t_\infty}^{t} + \phi_o$$

or

$$\phi(t) = \frac{C}{bv_2} \arctan \left(\frac{v_2 t}{b}\right) - \frac{C}{bv_2} \arctan \left(\frac{-v_2 t_\infty}{b}\right) + \phi_o. \tag{7.32}$$

Applying the first equation of (7.15) and the transformation matrix, we find the following expression for $\vec{R}^o$ in the XYZ-system:

$$\vec{R}^o = (R_{-\infty} \cos\theta , - R_{-\infty} \sin\theta , - b)$$

Consequently

$$\cos\phi_o = \frac{-R_{-\infty} \sin\theta}{(R_{-\infty}^2 \sin^2\theta + b^2)^{\frac{1}{2}}} = \frac{-1}{(1+b^2/R_{-\infty}^2 \sin^2\theta)^{\frac{1}{2}}}$$

or

$$\cos\phi_o = -1 \text{ implies } \phi_o = \pi. \tag{7.33}$$

Moreover, $\arctan \left(\frac{-v_2 t_\infty}{b}\right) \simeq \arctan(-\infty)$.

It then follows from this $\arctan\left(\frac{-v_2 t_\infty}{b}\right) = \frac{3\pi}{2}$. (7.34)

Putting Equations (7.33) and (7.34) into (7.32), one finally obtains

$$\phi(t) = \frac{C}{bv_2} \arctan\left(\frac{v_2 t}{b}\right) - \frac{C}{bv_2}\left(\frac{3\pi}{2}\right) + \pi$$

and noting that $\arctan\left(\frac{v_2 t_\infty}{b}\right) \simeq \arctan(\infty) = \frac{\pi}{2}$,

$$\phi_\infty = \lim_{t \to t_\infty} \phi(t) = \pi - \frac{C\pi}{bv_2} .$$

Therefore $$\cos \phi_\infty = - \cos\left(\frac{C\pi}{bv_2}\right). \qquad (7.35)$$

In order to determine the square of the velocity of the electron at time $t = t_\infty$, long after the scattering is over, we write

$$\vec{r}_{1L} = \vec{R}_{c.m} + \vec{r}_1$$

or, using Equation (7.11a)

$$\vec{r}_{1L} = \vec{R}_{c.m} - \frac{\mu}{m_1} \vec{R} .$$

Thus

$$\mathring{\vec{r}}_{1L} = \mathring{\vec{R}}_{c.m} - \frac{\mu}{m_1} \mathring{\vec{R}}$$

and substituting Equation (7.10) into this, we have

$$\mathring{\vec{r}}_{1L} = \frac{\mu}{m_1} (\vec{v}_2 - \mathring{\vec{R}}),$$

from which follows

$$\mathring{\vec{r}}^{\,2}_{1L} = \frac{\mu^2}{m_1^2} (v_2^2 - 2\vec{v}_2 \cdot \mathring{\vec{R}} + v_2^2) \qquad (7.36)$$

on recalling that $\mathring{\vec{R}}^2 = v_2^2$.

But employing (7.21) along with (7.23), one finds

$$\vec{v}_2 \cdot \mathring{\vec{R}} = - v_2 \mathring{R} \cos^2\theta + v_2 \mathring{R} \sin^2\theta \cos\phi - v_2 R \sin^2\theta \sin\phi \mathring{\phi}$$

and on taking into account Equations (7.26), (7.30), (7.31), and (7.35)

$$\lim_{t \to t_\infty} \vec{v}_2 \cdot \vec{R} = v_2^2 (-\cos^2\theta - \sin^2\theta \cos \frac{C\pi}{bv_2}) - \frac{Cv_2 \sin^2\theta \sin\phi_\infty}{(R_{-\infty}^2 + b^2)^{\frac{1}{2}}}$$

or, noting that the last term is zero

$$\lim_{t \to t_\infty} \vec{v}_2 \cdot \mathring{\vec{R}} = v_2^2 (- \cos^2\theta - \sin^2\theta \cos \frac{C\pi}{bv_2}) .$$

Finally, combining this with Equation (7.36) we obtain

$$\lim_{t \to t_\infty} \mathring{\vec{r}}_{1L}^{\,2} = \frac{2\mu^2 v_2^2}{m_1^2} (1 + \cos^2\theta + \sin^2\theta \cos \frac{C\pi}{bv_2}) .$$

Whence the energy transferred to the electron from the scattered monopole is

$$\Delta\varepsilon = \frac{1}{2} m_1 (\lim_{t \to t_\infty} \mathring{\vec{r}}_{1L}^{\,2}) = \frac{\mu^2 v_2^2}{m_1} (1 + \cos^2\theta + \sin^2\theta \cos \frac{C\pi}{bv_2}) .$$

8. DISCUSSION

Although symmetry consideration of Maxwell's Equations has served as a theoretical background for postulating the existence of monopoles, the strong point for sustaining the belief in the existence of monopoles, despite the negative experimental result up to now, is the ability of the monopole theory to account quantization of electric charge. Then either the confirmation of the existence of monopoles, or the formulation of a theory without the notion of a monopole that can explain charge quantization, seems to be inevitable.

It is found in the first case considered in Chapter 6 that the intensity with the g-polarization is stronger than that with the f-polarization for monopole synchrotron radiation. However, just quite the opposite holds true for electron synchrotron radiation. This then provides a means by which one can experimentally distinguish monopole synchrotron radiation from that of the electron. It is, of course, possible to come across another radiation in which the β_3-component is larger than the β_2-component. But the results of the second and third cases for monopole synchrotron radiation can supplement the conclusion drawn with the aid of the result obtained in the first case. Moreover, although in the second case the intensities with the f- and g-polarization are equal both for monopole and electron synchrotron radiation, in view of Equation (5.35) and the Dirac quantization condition (A14), the intensity of the monopole radiation is $(h^2c^2/4e^4)$ times stronger than that of the electron if both are moving along circles of the same radii. Obviously, a monopole produced in a collision process can easily be identified by the spiral or parabolic track the monopole would leave when moving in a uniform electric or magnetic field.

The result of the discussion of energy transfer in electron-monopole scattering may be used in the calculation of the energy loss per unit distance by a monopole traversing a metal, due to the interaction of the monopole with conduction electrons.

Appendix A The Dirac Quantization Condition

The Schroedinger equation for an electron in the field of a stationary monopole is

$$\frac{h^2}{2m}(-i\nabla - \frac{e}{hc}\vec{A})^2\psi = E\psi, \tag{A1}$$

where $\vec{A}$ is the vector potential of the field. A change in the location of the Dirac string gives rise to a gauge transformation of the vector potential:

$$\vec{A} \to \vec{A}' = \vec{A} + g\,\mathrm{grad}\,\chi, \tag{A2}$$

where χ is an arbitrary function of the coordinates and time. We suppose that under transformation (A2) the wave function takes the form

$$\psi \to \psi' = e^{ieg\chi/hc}\psi. \tag{A3}$$

Then

$$(-i\nabla - \frac{e}{hc}\vec{A}')\psi' = (-i\nabla - \frac{e}{hc}\vec{A} - \frac{eg}{hc}\,\mathrm{grad}\,\chi)\,e^{ieg\chi/hc}\psi$$

$$= -i\nabla\,(e^{ieg\chi/hc}\psi) - \frac{e}{hc}\vec{A}\,e^{ieg\chi/hc}\psi - \frac{eg}{hc}\,\mathrm{grad}\,\chi\,e^{ieg\chi/hc}\psi$$

$$= \frac{eg}{hc}\mathrm{grad}\,\chi\,e^{ieg\chi/hc}\psi - ie^{ieg\chi/hc}\nabla\psi - \frac{e}{hc}\vec{A}\,e^{ieg\chi/hc}\psi$$

$$- \frac{eg}{hc}\,\mathrm{grad}\,\chi\,e^{ieg\chi/hc}\psi$$

$$= e^{ieg\chi/hc}(-i\nabla - \frac{e}{hc}\vec{A})\psi \tag{A4}$$

Introducing the notation

$$\vec{D} = -i\nabla - \frac{e}{hc}\vec{A}$$

and $$\vec{D}' = -i\nabla - \frac{e}{hc}\vec{A}',$$

one can write Equation (A4) as

$$\vec{D}'\psi' = e^{ieg\chi/hc}D\psi \tag{5A}$$

or $$\vec{D}'\psi' = e^{ieg\chi/hc}\Psi$$

in which $$\Psi = D\psi . \tag{A6}$$

According to Equation (A2)

$$\Psi' = e^{ieg\chi/hc}\Psi$$

so that $$\vec{D}'\psi' = \Psi'$$

It then follows from this

$$\vec{D}'(\vec{D}'\psi') = \vec{D}'\Psi'$$

and with the aid of Equations (A5) and (A6), we have

$$\vec{D}'(\vec{D}'\psi') = e^{ieg\chi/hc}\vec{D}\Psi = e^{ieg\chi/hc}D(\vec{D}\psi)$$

or

$$\vec{D}'^2\psi' = e^{ieg\chi/hc}\,\vec{D}^2\psi . \tag{A7}$$

Since the Schroedinger equation is gauge invariant,

$$\frac{h}{2m}\vec{D}'^2\psi' = E\psi'.$$

When (A3) and (A7) are substituted into this, there results the equation

$$\frac{h}{2m}\vec{D}^2\psi = E\psi$$

which is exactly the same as (A1), thereby conforming the validity of assumption (A3).

The vector potential of the field produced by a stationary monopole at the origin of a coordinate system may be expressed in the form

$$\vec{A}(\vec{r}) = \frac{g}{r}\,\frac{\vec{n} \times \vec{r}}{r - \vec{n} \cdot \vec{r}} \tag{A8}$$

where g is the magnetic charge of the monopole and $\vec{n}$ is a unit vector with arbitrary direction. It is evident that $\vec{A}$ is singular along the (+ ive) n-axis. According to Equation (A2), one can write

$$\vec{A}(\vec{n}_2) = \vec{A}(\vec{n}_1) + g \,\text{grad}\,\chi$$

with

$$\vec{n}_2 = (o,o,1) \text{ and } \vec{n}_1) = o,o,-1).$$

Thus

$$\text{grad}\,\chi = \frac{1}{g}[\vec{A}(\vec{n}_2) - \vec{A}(\vec{n}_1)]$$

from which follows

$$\S\, \text{grad}\,\chi \cdot d\vec{r} = \frac{1}{g}\S[\vec{A}(\vec{n}_2) - \vec{A}(\vec{n}_1)] \cdot d\vec{r}$$

or

$$\Delta\chi = \frac{1}{g}\,[\S\vec{A}(\vec{n}_2) \cdot d\vec{r} - \S\,\vec{A}(\vec{n}_1) \cdot d\vec{r}]. \tag{A9}$$

In view of Equation (A8)

$$\vec{A}\ (\vec{n}_2) = \frac{g}{r}\,\frac{\vec{n}_2 \times \vec{r}}{r - \vec{n}_2 \cdot \vec{r}}\,.$$

If vector $\vec{r}$ is taken to be along the xy-plane, then

$$\vec{r} = (r \cos\phi,\ r \sin\phi,\ o)$$

where ϕ is the angle between the x-axis and vector $\vec{r}$.

Hence

$$\vec{n}_2 \times \vec{r} = -r \sin\phi\,\vec{i} + r \cos\phi\,\vec{j}$$

In addition, $\vec{n}_2 \cdot \vec{r} = o$ as the angle between the two vectors is $\pi/2$.

Consequently

$$\vec{A}(\vec{n}_2) = \frac{g}{r^2}(-r \sin\phi\, \vec{i} + r \cos\phi\, \vec{j})$$

so that

$$\oint \vec{A}(\vec{n}_2)\cdot d\vec{r} = \frac{g}{r^2}\int_o^{2\pi}(-r \sin\phi\, \vec{i} + r\cos\phi\, \vec{j})\cdot(-r\sin\phi\, d\phi\vec{i} + r\cos\phi\, d\phi\vec{j})$$

$$= g\int_o^{2\pi} d\phi$$

$$= 2\pi g\,. \qquad \text{(A10)}$$

Similarly

$$\oint \vec{A}(\vec{n}_1)\cdot d\vec{r} = -2\pi g$$

Now combination of Equations (A9), (A10), and (A11) yields

$$\Delta\chi = 4\pi \qquad \text{(A12)}$$

Furthermore,

$$\psi' = e^{ieg\,\chi/hc}\psi$$

$$= e^{ieg\,(\chi+\Delta\chi)/hc}\psi\,.$$

$$= e^{ieg\chi/hc}\, e^{ieg\,\Delta\chi/hc}\psi\,.$$

The requirement that ψ' be single-valued leads to

$$e^{ieg\Delta\chi/hc} = 1.$$

Applying Euler's formula and then equating real and imaginary parts, one obtains

$$\cos\ (eg\,\Delta\chi/hc) = 1$$

from which follows

$$eg\Delta\chi/hc = 2\pi n \tag{A13}$$

where n is an integer.

Finally, when expression (A12) is substituted into (A13), there results the Dirac quantization condition

$$eg/hc = n/2 \,. \tag{A14}$$

Appendix B|| The Orthonormal Functions $1/\sqrt{T}\exp in\omega\tau'$

The functions $\phi_n(\tau') = \frac{1}{\sqrt{T}} e^{in\omega\tau'}$ are orthonormal in the interval $[-T/2, T/2]$ if

$$\int_{-T/2}^{T/2} \phi_{n'}^*(\tau')\,\phi_n(\tau')d\tau' = \delta nn'$$

with $\omega T = 2\pi$.

Then

$$\int_{-T/2}^{T/2} \phi_n^*(\tau')\phi_n(\tau')d\tau' = \frac{1}{T}\int_{-T/2}^{T/2} e^{i(n-n')\omega\tau'}d\tau'$$

$$= \frac{e^{i(n-n')\omega\tau'}}{i(n-n')\omega T}\Bigg|_{-T/2}^{T/2}$$

$$= \frac{1}{(n-n')\pi}\left[\frac{e^{i(n-n')\pi}-e^{-i(n-n')\pi}}{2i}\right]$$

$$= \frac{\sin(n-n')\pi}{(n-n')\pi}$$

On employing Equation (5.34) we obtain

$$\int_{-T/2}^{T/2} \phi_n^*(\tau')\,\phi_n(\tau')d\tau' = \delta nn'$$

showing that the functions $\phi_n(\tau')$ are orthonormal.

Appendix C || Relations Between the Electric and Magnetic Fields

In the wave zone, the monopole field equations take the form

$$\text{curl } \vec{E} = -\frac{1}{c}\frac{\partial \vec{H}}{\partial t}, \tag{C1}$$

$$\text{div } \vec{H} = 0, \tag{C2}$$

$$\text{curl } \vec{H} = \frac{1}{c}\frac{\partial \vec{E}}{\partial t}, \tag{C3}$$

$$\text{div } \vec{E} = 0. \tag{C4}$$

Taking the curl of both sides of Equation (C1), we have

$$\nabla \times \nabla \times \vec{E} = -\frac{1}{c}\frac{\partial}{\partial t} \nabla \times \vec{H}.$$

Combining this with Equation (C3) then gives

$$\nabla \times \nabla \times \vec{E} = -\frac{1}{c^2}\frac{\partial^2 \vec{E}}{\partial t^2}$$

so that on employing the identity $\nabla \times \nabla \times \vec{E} = \nabla(\nabla \cdot \vec{E}) - \nabla^2 \vec{E}$ along with Equation (C4)

$$\nabla^2 \vec{E} - \frac{1}{c^2}\frac{\partial^2 \vec{E}}{\partial t^2} = 0. \tag{C5}$$

Similarly, one can show that

$$\nabla^2 \vec{H} - \frac{1}{c^2}\frac{\partial^2 H}{\partial t^2}. \tag{C6}$$

It can easily be verified that

$$\vec{E}(\vec{r},t) = \vec{E}_0 e^{i\vec{k}\cdot\vec{r} - i\omega t} \tag{C7}$$

and

$$\vec{H}(\vec{r},t) = \vec{H}_0 e^{i\vec{k}\cdot\vec{r} - i\omega t} \tag{C8}$$

are respectively solutions of (C5) and (C6) with $\omega = ck$. Equation (C7) can be written in the form

$$\vec{E} = (E_{ox}\vec{i} + E_{oy}\vec{j} + E_{oz}\vec{k})e^{ik_1x + ik_2y + ik_3z - i\omega t}.$$

Thus

$$\nabla\times\vec{E} = \left(\frac{\partial E_z}{\partial} - \frac{\partial E_y}{\partial z}\right)\vec{i} + \left(\frac{\partial E_x}{\partial z} - \frac{\partial E_z}{\partial x}\right)\vec{j} + \left(\frac{\partial E_y}{\partial x} - \frac{\partial E_x}{\partial y}\right)\vec{k}$$

$$= i[(E_{oz}k_2 - E_{oy}k_3)\vec{i} + (E_{ox}k_3 - E_{oz}k_1)\vec{j} + (E_{oy}k_1 - E_{ox}k_2)\vec{k}]\, e^{i\vec{k}\cdot\vec{r} - i\omega t}$$

$$= i\vec{k}\times\vec{E}_o e^{i\vec{k}\cdot\vec{r} - i\omega t} \qquad \text{(C9)}$$

Now combination of Equation (C1) with (C9) yields

$$i\vec{k}\times\vec{E}_o e^{i\vec{k}\cdot\vec{r} - i\omega t} = -\frac{1}{c}\frac{\partial\vec{H}}{\partial t}. \qquad \text{(C10)}$$

But using Equation (C8) one obtains

$$\frac{\partial\vec{H}}{\partial t} = -i\omega\vec{H}_o e^{i\vec{k}\cdot\vec{r} - i\omega t}.$$

Therefore Equation (C10) goes over into

$$i\vec{k}\times\vec{E}_o e^{i\vec{k}\cdot\vec{r} - i\omega t} = ik\vec{H}_o e^{i\vec{k}\cdot\vec{r} - i\omega t}$$

from which follows on taking into account Equations (C7) and (C8) that

$$\vec{H} = \frac{\vec{k}}{k} \times \vec{E}.$$

Since in the wave zone $\vec{k}/k \simeq \vec{r}/r$, the expression for the relation between the electric and magnetic fields takes the form

$$\vec{H} = \frac{\vec{r}}{r} \times \vec{E}. \tag{C11}$$

We shall next obtain a relation between the spherical components of the electric and magnetic fields. To this end, we write in accordance with Equation (C11)

$$H_r\vec{e}_r + H_\theta\vec{e}_\theta + H_\phi\vec{e}_\phi = \vec{e}_r \times (E_r\vec{e}_r + E_\theta\vec{e}_\theta + E_\phi\vec{e}_\phi).$$

On taking into account that

$$\vec{e}_r \times \vec{e}_r = o, \quad \vec{e}_r \times \vec{e}_\theta = \vec{e}_\phi, \text{ and } \vec{e}_r \times \vec{e}_\phi = -\vec{e}_\theta$$

we find

$$H_r = o, \quad H_\theta = -E_\phi, \text{ and } H_\phi = E_\theta.$$

It follows from Equation (C11) that

$$\vec{E} = \vec{H} \times \frac{\vec{r}}{r}.$$

Then with the aid of this, one can also show that

$$E_r = o, \quad E_\theta = H_\phi, \text{ and } E_\phi = -H_\theta.$$

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