RATE AND CROSS SECTION FOR ELECTRON-POSITRON PAIR CREATION BY: (I) RELATIVISTIC MUON AND (II) PROTON IMPACT ON INTENSE LASER BEAMS

By

Lijalem Kelemu

SUBMITTED IN PARTIAL FULFILLMENT OF THE
REQUIREMENTS FOR THE DEGREE OF
MASTER OF SCIENCE IN PHYSICS
AT
ADDIS ABABA UNIVERSITY
ADDIS ABABA, ETHIOPIA
JUNE 2013

ADDIS ABABA UNIVERSITY
DEPARTMENT OF
PHYSICS

Supervisor: ______________________
Dr. S.Bhatnagar

Examiners: ______________________

____

ADDIS ABABA UNIVERSITY

Date: **June 2013**

Author: **Lijalem Kelemu**

Title: **RATE AND CROSS SECTION FOR ELECTRON-POSITRON PAIR CREATION BY: (I) RELATIVISTIC MUON AND (II) PROTON IMPACT ON INTENSE LASER BEAMS**

Department: **Physics**

Degree: **M.Sc.** Convocation: **June** Year: **2013**

Signature of Author

Table of Contents

Table of Contents **iv**

List of Figures **v**

Acknowledgements **vi**

Abstract **vii**

1 Introduction **1**

2 Rate and cross section for e^-e^+ pair creation by relativistic muon impact on intense laser beams **3**

2.1 Total pair production rate 3

2.2 Total cross section 16

3 The rate and cross section for e^+e^- pair creation by proton impact on intense laser beams **34**

3.1 Partial rate corresponding to the process $p + r\omega \rightarrow p' + e^+ + e^-$ taking Proton at rest 38

3.2 Total cross section of the process $p + r\omega \rightarrow p' + e^+ + e^-$taking proton at rest 39

4 Conclusion **49**

Appendix **50**

Bibliography **51**

List of Figures

2.1 Decomposition of the Feynman graph in to perturbation series for the process $\mu + r\omega \longrightarrow \mu' + e^{-}e^{+}$. 4

3.1 Feynman diagram of the considered process in the lowest order in. The dashed line represents the virtual photon propagated between the projectile and the electron positron vertices, and the double lines stand for the exact lepton wave functions in the laser field (Volkov states). 34

3.2 Feynman diagram on scattering of $e^{-}e^{+}$ at an external field (x) 35

Acknowledgements

First and foremost I thank the almighty God, my strength and power. What I return for You without Your glory? I would like to express my sincere thanks to my advisor and instructor Dr. S. Bhatnagar for his guidance, assistance, and contribution of valuable suggestions. His beautiful lectures has made a deep impression on me. I wish to thank Arba Minch University for its financial support. Last but not least, I thank my family and friends for their encouragement and support.

Abstract

We first study the rate and cross section for electron-positron pair creation by relativistic muon impact on intense laser beams through the process $\mu + r\omega \longrightarrow \mu' + e^+ + e^-$. We then study the rate and cross section for electron-positron pair creation by proton impact on intense laser beam through the process $p + r\omega \longrightarrow p' + e^+ + e^-$ in the proton rest frame. The Feynman diagrams the pair creation by relativistic muon and proton impact on a circularly polarized high frequency laser beam is evaluated with in the framework of laser-dressed quantum electrodynamics employing relativistic Volkov states.

Chapter 1

Introduction

The modification of the perturbative QED in strong electromagnetic fields was initiated soon after the invention of the laser, leading to the theory of strong-field QED, also known as laser dressed-QED. The creation of electron-positron pairs in very strong laser fields was investigated by theoreticians already in the 1960s and 1970s.

In recent years, interest in nonlinear pair creation processes via multiphoton absorption from external laser fields [1, 2, 3] has raised.

Here, we consider e^+e^- pair production by the collision of charged particles with laser fields. There are two possible ways in which the creation of the pairs can take place.

1) Bethe- Heitler type: where the pair is created by a virtual photon from the Coulomb field of the projectile particle and real photons (r) from the laser field. Due to the number of photon originally treated the linear case of pair production by a single high-energy photon (r=1) [4], and nonlinear which involves the absorption of $r > 1$ laser photons and thus depends on the laser intensity in a nonlinear way. In the case of a muon projectile, the symbolic equation for this process is $\mu + r\omega \rightarrow \mu' + e^-e^+$ with the laser frequency ω.

2) Breit-Wheeler [5]: nonlinear type where the pair creation takes place by the collision of the laser beam with a real photon.

We first worked out the rate and cross section for electron-positron pair creation by relativistic muon on intense laser beam throughout the process $\mu + r\omega \rightarrow \mu' + e^+ + e^-$.

The calculation of cross section for this process also done rigorously. However in [6], the calculation of cross section for this process was not done.

We then calculated the rate and cross section for electron-positron pair creation by proton impact on intense laser which is similar to earlier process, through out the process $p+r\omega \longrightarrow p'+e^{+}+e^{-}$ treated proton as an effective Dirac particle. However since proton is quite massive ($m_p = 1840 m_e$) than electron and positron. One can approximate the proton to be at rest before and after scattering and the thus consider this process to be taking place on the Coulomb field of the heavy particle(proton). One can thus treat this process in first order in perturbative theory. Similar to muon projectile the calculation of cross section for this process also done rigorously. However in [6], the calculation of cross section for this process was not done.

This thesis is organized as follows: chapter 2 deals electron-positron pair creation by relativistic muon impact on intense laser beams, the rate of production and total cross section are calculated. In chapter 3 also deals electron-positron pair creation by proton impact on intense laser beams, rate of production and total cross section also calculated when proton is at rest frame before and after scattering. Finally we conclude the work in chapter 4.

Chapter 2

Rate and cross section for e^-e^+ pair creation by relativistic muon impact on intense laser beams

2.1 Total pair production rate

A quantity of interest to be calculated for this process is the rate of production R. Physically, the rate of production is the probability per unit time for the produced electron-positron pairs per projectile muon for a laser beam of infinite length. First we begin finding the amplitude (S_{fi}) from the perturbation series in the Fig. 2.1. In our case we use from Fig. 2.1 the first two rows on the right-hand side show the four leading graphs for one-photon processes. From 2^{nd} order perturbation theory the amplitude (S_{fi}) of the process is given by

$$S_{fi} = \frac{\alpha_f}{i} \iint d^4x d^4y \bar{\psi}_{p_- s_-}(x) \gamma^\mu \psi_{p_+ s_+}(x) D_{\mu\nu}(x-y) \bar{\Psi}_{P' S'}(y) \gamma^\nu \Psi_{PS}(y). \tag{2.1.1}$$

Here, x and y the space-time coordinates of the produced pair and the scattering particle respectively.

$D_{\mu\nu}(x-y)$ is the virtual photon propagator between the two vertices.

$\bar{\Psi}_{P' S'}(y) \gamma^\nu \Psi_{PS}(y)$ is the muon vertex, and

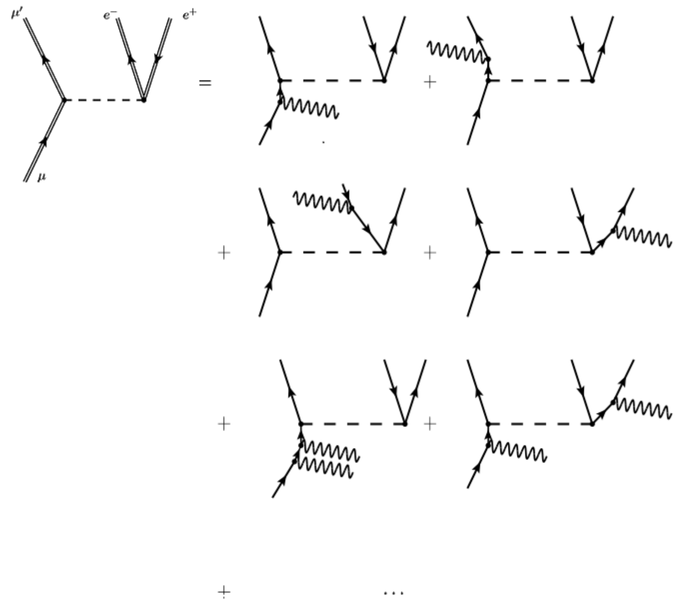

Figure 2.1: Decomposition of the Feynman graph in to perturbation series for the process $\mu + r\omega \longrightarrow \mu' + e^- e^+$.

$\bar{\psi}_{p_-s_-}(x)\gamma^\mu\psi_{p_+s_+}(x)$ the electron-positron vertex with the Dirac matrices γ^μ. ψ and Ψ are the Volkov states [7] of the respective particles which solve the Dirac equation for $spin^{1/2}$ particle in an electromagnetic field,

$$(\not{\partial} - e\not{A} - m)\psi = 0,$$

with the laser potential $A^\mu = (0, A)$ in the radiation gauge. We consider a circularly polarized laser field, so that the four potential reads $A^\mu(\eta) = a_1 cos(\eta) + a_2 sin(\eta)$ with the laser phase $\eta = kx$, where k is the wave vector of the laser field. The four-vectors $a_{1/2}$ read $a_1 = (0, a, 0, 0)$ and $a_2 = (0, 0, a, 0)$, where a being the amplitude of the laser potential. Volkov wave function of electron with the kinetic momentum p_- and the spin projection s_- thus can be written as

$$\psi_{p_-s_-}(x) = \sqrt{\frac{m}{Vq_-^0}}\left(1 - \frac{e\not{k}\not{A}}{2(k.p_-)}\right) u_{p_-s_-}e^{iS_-}, \tag{2.1.2}$$

with the action

$$S_- = -(q_-x) + \frac{e(a_1.p_-)}{e(k.p_-)}sin(\eta) - \frac{e(a_2.p_-)}{(k.p_-)}cos(\eta),$$

and effective momentum (q_-) of the electron in the laser field [8],

$$q_- = p_- + \frac{e^2 a^2}{2(k.p_-)} k^\mu.$$

Thus, we can write

$$\bar{\psi}_{p_- s_-}(x) = \sqrt{\frac{m}{V q_-^0}}\ \bar{u}_{p_- s_-} \left(1 - \frac{e \not{A} \not{k}}{2(k.p_-)}\right) e^{-iS_-}.$$

The Volkov wave function of positron with the kinetic momentum p_+ and the spin projection s_+ thus can be written as

$$\psi_{p_+ s_+}(x) = \sqrt{\frac{m}{V q_+^0}} \left(1 + \frac{e \not{k} \not{A}}{2(k.p_+)}\right)\ v_{p_+ s_+} e^{iS_+}, \tag{2.1.3}$$

with the action

$$S_+ = (q_+ x) + \frac{e(a_1.p_+)}{e(k.p_+)} sin(\eta) - \frac{e(a_2.p_+)}{(k.p_+)} cos(\eta),$$

and effective momentum (q_+) of the positron the laser field,

$$q_+ = p_+ + \frac{e^2 a^2}{2(k.p_+)} k^\mu.$$

Here, $m = m_e$ is the electron mass, V is normalization volume, u_{p,s_-} and $v_{p_+ s_+}$ are free Dirac spinors of electron and positron respectively. The corresponding effective mass is $m_{(*)} = m(1 + \zeta^2)$ with the laser intensity parameter reads $\zeta = \frac{ea}{m_e c^2}$.

Replacement of the mass m(mass of e^- or e^+) by the projectile mass M(mass of muon), the kinetical momentum p_- by P or $P^{'}$ for the incoming or scattered muon respectively, and the effective momentum q_- by Q or $Q^{'}$ yields the Volkov states for the initial and scattered muon. The effective muon mass is obtained by $M_* = M(1 + \Xi^2)$, and the corresponding intensity parameter reads $\Xi = \frac{ea}{M} = \zeta/M$. We concentrate our study on the multiphoton regime $\zeta \ll 1$ where the intensity of the laser field is comparatively low and the photon energy is high. Therefore, we neglect vacuum polarization effects which become important only at near critical laser intensities $I \gtrsim 10^{29}\ W/cm^2$ [9,10] and employ a free photon propagator for the description of the virtual photon which is propagated between the two vertices:

$$D_{\mu\nu}(x-y) = 4\pi \int \frac{d^4}{(2\pi)^2} \frac{e^{iq(x-y)}}{q^2} g^{\mu\nu}, \tag{2.1.4}$$

with the integration variable q describing the momentum of the virtual photon. Then one can write the Volkov wave function of incoming muon as

$$\Psi_{PS}(y) = \sqrt{\frac{M}{VQ^0}} \left(1 - \frac{e\not{k}\not{A}}{2(k.P)}\right) u_{PS} e^{iS}, \tag{2.1.5}$$

with the action

$$S = -(Qy) + \frac{e(a_1.P)}{(k.P)} sin(\eta) - \frac{e(a_2.P)}{(k.P)} cos(\eta), \tag{2.1.6}$$

and effective momentum of incoming muon is

$$Q = P + \frac{e^2 a^2}{2(k.P)} k^{\mu}.$$

For scattered muon we can write the corresponding Volkov wave function as

$$\Psi_{P'S'}(y) = \sqrt{\frac{M}{VQ^{0'}}} \left(1 - \frac{e\not{k}\not{A}}{2(k.P')}\right) u_{P's'} e^{iS'}, \tag{2.1.7}$$

with the action

$$S' = -(Q' y) + \frac{e(a_1.P')}{(k.P')} sin(\eta) - \frac{e(a_2.P')}{(k.P')} cos(\eta), \tag{2.1.8}$$

and effective momentum of scattered muon is

$$Q' = P' + \frac{e^2 a^2}{2(k.p')} k^{\mu}.$$

Thus, we can write

$$\bar{\Psi}_{P'S'}(y) = \sqrt{\frac{M}{VQ^{0'}}} \;\; \bar{u}_{P's'} \left(1 - \frac{e\not{A}\not{k}}{(2k.P')}\right) e^{-iS'}, \tag{2.1.9}$$

This implies insert the wave function of electron-positron and incoming muon with scattered muon in Eq. (2.1.1), we can write the amplitude as

$$S_{fi} = -i\alpha_f \iiint d^4q d^4x d^4y \sqrt{\frac{m^2}{V^2 q_+^0 q_-^0}} \;\; \bar{u}_{p_-s_-} \left(1 - \frac{e\not{A}\not{k}}{2(k.p_-)}\right) e^{-iS_-} \gamma^{\mu} \left(1 + \frac{e\not{k}\not{A}}{2(k.p_+)}\right) v_{p_+s_+} e^{iS_+}$$
$$\frac{1}{(2\pi)^4} \frac{e^{iq(x-y)}}{q^2} g^{\mu\nu} \sqrt{\frac{M^2}{V^2 Q^0 Q^{0'}}} \;\; \bar{u}_{P'S'} \left(1 - \frac{e\not{A}\not{k}}{2(k.P')}\right) e^{-iS'} \gamma^{\nu} \left(1 - \frac{e\not{k}\not{A}}{2(k.P)}\right) u_{PS} e^{iS}.$$

$$= -i\frac{\alpha_f mM}{(2\pi)^4 V^2}\sqrt{\frac{1}{q_+^0 q_-^0 Q^0 Q'^0}}\iiint d^4q d^4x d^4y \;\; \bar{u}_{p_-s_-}\left(1-\frac{e\not{A}\not{k}}{2(k.p_-)}\right)\gamma^\mu\left(1+\frac{e\not{k}\not{A}}{2(k.p_+)}\right)$$
$$v_{p_+s_+}e^{i(S_++qx-S_-)}\bar{u}_{P'S'}\left(1-\frac{e\not{A}\not{k}}{2(k.P')}\right)\gamma^\nu\left(1-\frac{e\not{k}\not{A}}{2(k.P)}\right)u_{PS}e^{i(S-qy-S')}g^{\mu\nu}. \quad (2.1.10)$$

To simplify the work first find out the integral in terms of the co-ordinate x of e^+e^-, in this case $\eta = kx$. From Eq. (2.1.10)

$$\int d^4x\bar{u}_{p_-s_-}\left(1-\frac{e\not{A}\not{k}}{2(k.p_-)}\right)\gamma^\mu\left(1+\frac{e\not{k}\not{A}}{2(k.p_+)}\right)]v_{p_+s_+}e^{i(S_++qx-S_-)}$$
$$= \int d^4x\bar{u}_{p_-s_-}\left(\gamma^\mu + \frac{e\gamma^\mu\not{k}\not{A}}{2(k.p_+)} - \frac{e\not{A}\not{k}\gamma^\mu}{2(k.p_-)} - \frac{e^2\not{A}\not{k}\gamma^\mu}{2(k.p_-)}\frac{\not{k}\not{A}}{2(k.p_+)}\right)v_{p_+s_+}e^{i(S_++qx-S_-)}. \quad (2.1.11)$$

For mathematical simplicity, the laser field is taken to be a monochromatic plane wave of circular polarization with four-potential $A_\mu(\eta) = a_1cos(\eta) + a_2sin(\eta)$ must satisfying the following condition

$a_1.a_2 = 0, a_1^2 = a_2^2 = 2a^2.$

From this the terms $e^2\not{A}\not{k}\gamma^\mu\not{k}\not{A}$ in Eq. (2.1.11) becomes

$e^2\not{A}\not{k}\gamma^\mu\not{k}\not{A} = e^2\not{A}\not{A}\gamma^\mu\not{k}\not{k}$

$=e^2A^\mu.A^\mu k^\mu\not{k}.$

But

$$A^\mu.A^\mu = a_1^2cos^2(\eta) + a_2^2sin^2(\eta) + a_1^2.a_2^2cos(\eta)sin(\eta)$$
$$= 2a^2(sin^2(\eta) + cos^2(\eta)) = 2a^2. \quad (2.1.12)$$

Therefore, $e^2\not{A}\not{k}\gamma^\mu\not{k}\not{A} = 2e^2a^2k^\mu\not{k}.$

This implies RHS of Eq. (2.1.12) becomes

$$\int d^4x\bar{u}_{p_-s_-}\left(\gamma^\mu + \frac{e\gamma^\mu\not{k}[\not{a}_1cos(kx)+\not{a}_2sin(kx)]}{2(k.p_+)} - \frac{e[\not{a}_1cos(kx)+\not{a}_2sin(kx)]\not{k}\gamma^\mu}{2(k.p_-)}\right.$$
$$\left. - \frac{e^2a^2k^\mu\not{k}}{2(k.p_+)(k.p_-)}\right)v_{p_+s_+}e^{i(S_++qx-S_-)}$$

$$= \int d^4x\bar{u}_{p_-s_-}\left[\left(\gamma^\mu - \frac{e^2a^2k^\mu\not{k}}{2(k.p_+)(k.p_-)}\right) + \frac{e}{2}\left(\frac{\gamma^\mu\not{k}\not{a}_1}{(k.p_+)} - \frac{\not{a}_1\not{k}\gamma^\mu}{(k.p_-)}\right)cos(kx)\right.$$
$$\left. + \frac{e}{2}\left(\frac{\gamma^\mu\not{k}\not{a}_2}{(k.p_+)} - \frac{e\not{a}_2\not{k}\gamma^\mu}{(k.p_-)}\right)sin(kx)\right]e^{i(S_++qx-S_-)}v_{p_+s_+} \quad (2.1.13)$$

Substituting each action in to exponential function of the terms in Eq. (2.1.13)

$$e^{i(S_{+}+qx-S_{-})}=e^{i(q+q_{+}+q_{-}).x}e^{-i\left(\frac{ea_1p_-}{(k.p_-)}-\frac{ea_1p_+}{(k.p_+)}\right)sin(kx)+i\left(\frac{ea_2p_-}{(k.p_-)}-\frac{ea_2p_+}{(k.p_+)}\right)cos(kx)}.$$

But

$$e^{-i\left(\frac{ea_1p_-}{(kp_-)}-\frac{ea_1p_+}{(kp_+)}\right)sin(kx)+i\left(\frac{ea_2p_-}{(k.p_-)}-\frac{ea_2p_+}{(k.p_+)}\right)cos(kx)}=e^{-i\alpha sin(kx-\eta_o)}.$$

Therefore

$$e^{i(S_{+}+qx-S_{-})}=e^{i(q+q_{+}+q_{-}).x}e^{-i\alpha sin(kx-\eta_o)}.$$

Then Eq. (2.1.13) which is the RHS of Eq. (2.1.11) can be written as

$$\int d^4x\bar{u}_{p_-s_-}\left[\left(\gamma^\mu-\frac{e^2a^2k^\mu\not{k}}{2(k.p_+)(k.p_-)}\right)+\frac{e}{2}\left(\frac{\gamma^\mu\not{k}\not{a}_1}{(k.p_+)}-\frac{\not{a}_1\not{k}\gamma^\mu}{(k.p_-)}\right)cos(kx)+\right.$$
$$\left.\frac{e}{2}\left(\frac{\gamma^\mu\not{k}\not{a}_2}{(k.p_+)}-\frac{e\not{a}_2\not{k}\gamma^\mu}{(k.p_-)}\right)sin(kx)\right]e^{-i\alpha sin(kx-\eta_o)}e^{(q+q_++q_-).x}\upsilon_{p_+s_+}. \tag{2.1.14}$$

From cylindrical Bessel function expansion of trigonometric functions in the appendix. Let as consider

$$B_n=J_n(\bar{\alpha})e^{in\eta_o},$$

$$C_n=\frac{1}{2}\left(e^{(n+1)\eta_o}J_{n+1}(\bar{\alpha})+e^{(n-1)\eta_o}J_{n-1}(\bar{\alpha})\right),$$

$$D_n=\frac{1}{2i}\left(e^{(n+1)\eta_o}J_{n+1}(\bar{\alpha})-e^{(n-1)\eta_o}J_{n-1}(\bar{\alpha})\right).$$

The functions $J_i(\bar{\alpha})$ are the regular cylindrical Bessel functions [11] of integer order. Their argument is $\bar{\alpha}=\sqrt{\alpha_1^2+\alpha_2^2}$ and the angle η_o is given by $cos\eta_o=\alpha_1/\alpha$, $sin\eta_o=\alpha_2/\alpha$ with

$$\alpha_j=\frac{e(a_j.p_-)}{(k.p_-)}-\frac{e(a_j.p_+)}{(k.p_+)}, j=1,2.$$

From these equation

$$\int d^4x\bar{u}_{p_-s_-}\left(\gamma^\mu+\frac{e\gamma^\mu\not{k}\not{A}}{2(k.p_+)}-\frac{e\not{A}\not{k}\gamma^\mu}{2(k.p_-)}-\frac{e^2\not{A}\not{k}\gamma^\mu}{2(k.p_-)}\frac{\not{k}\not{A}}{2(k.p_+)}\right)\upsilon_{p_+s_+}e^{i(S_{+}+qx-S_{-})}.$$

$$=\sum_{n=-\infty}^{\infty}\int d^4x\bar{u}_{p_-s_-}\left[\left(\gamma^\mu-\frac{e^2a^2k^\mu\not{k}}{2(k.p_+)(k.p_-)}\right)B_n+\frac{e}{2}\left(\frac{\gamma^\mu\not{k}\not{a}_1}{(k.p_+)}-\frac{\not{a}_1\not{k}\gamma^\mu}{(k.p_-)}\right)C_n\right.$$
$$\left.+\frac{e}{2}\left(\frac{\gamma^\mu\not{k}\not{a}_2}{(k.p_+)}-\frac{e\not{a}_2\not{k}\gamma^\mu}{(k.p_-)}\right)D_n\right]e^{(q+q_++q_--nk).x}\upsilon_{p_+s_+}. \tag{2.1.15}$$

But one can find

$$\int d^4xe^{(q+q_++q_--nk).x} = (2\pi)^4\delta^4(q+q_++q_--nk). \tag{2.1.16}$$

This implies

$$\int d^4x\bar{u}_{p_-s_-}\left(\gamma^\mu + \frac{e\gamma^\mu \not{k}\not{A}}{2(k.p_+)} - \frac{e\not{A}\not{k}\gamma^\mu}{2(k.p_-)} - \frac{e^2\not{A}\not{k}\gamma^\mu}{2(k.p_-)}\frac{\not{k}\not{A}}{2(k.p_+)}\right)v_{p_+s_+}e^{i(S_++qx-S_-)}$$

$$= \sum_{n=-\infty}^{\infty}\bar{u}_{p_-s_-}\left[\left(\gamma^\mu - \frac{e^2a^2k^\mu\not{k}}{2(k.p_+)(k.p_-)}\right)B_n + \frac{e}{2}\left(\frac{\gamma^\mu\not{k}\not{a}_1}{(k.p_+)} - \frac{\not{a}_1\not{k}\gamma^\mu}{(k.p_-)}\right)C_n + \frac{e}{2}\left(\frac{\gamma^\mu\not{k}\not{a}_2}{(k.p_+)} - \frac{e\not{a}_2\not{k}\gamma^\mu}{(k.p_-)}\right)D_n\right]v_{p_+s_+}(2\pi)^4\delta^4(q+q_++q_--nk)$$

$$= \sum_{n=-\infty}^{\infty}(2\pi)^4M_\mu(e^-e^+|n)\delta^4(q+q_++q_--nk), \tag{2.1.17}$$

where the electronic spinor matrix product $M(e^+e|n)$ can be written as

$$M_\mu(e^-e^+|n) = \bar{u}_{p_-s_-}\left[\left(\gamma^\mu - \frac{e^2a^2k_\mu\not{k}}{2(k.p_+)(k.p_-)}\right)B_n + \frac{e}{2}\left(\frac{\gamma^\mu\not{k}\not{a}_1}{(k.p_+)} - \frac{\not{a}_1\not{k}\gamma^\mu}{(k.p_-)}\right)C_n + \frac{e}{2}\left(\frac{\gamma^\mu\not{k}\not{a}_2}{(k.p_+)} - \frac{e\not{a}_2\not{k}\gamma^\mu}{(k.p_-)}\right)D_n\right]v_{p_+s_+}. \tag{2.1.18}$$

By the same method from the above find out the integral in terms of the co-ordinate y of $\mu\mu'$. In this case where $\eta = ky$ and

$A_\nu(\eta) = a_1cos(\eta) + a_2sin(\eta).$

From Eq. (2.1.10) we get

$$\int d^4y\bar{u}_{P'S'}\left(1 - \frac{e\not{k}\not{A}}{2(k.P')}\right)\gamma^\nu\left(1 - \frac{e\not{A}\not{k}}{2(k.P)}\right)]u_{PS}e^{i(S-S'-qy)}$$

$$= \int d^4y\bar{u}_{P'S'}\left(\gamma^\nu + \frac{e^2\not{A}\not{k}\gamma^\nu}{2(k.P')}\frac{\not{k}\not{A}}{2(k.P)} - \frac{e\gamma^\nu\not{A}\not{k}}{2(k.P)} - \frac{e\not{k}\not{A}\gamma^\nu}{2(k.P')}\right)u_{PS}e^{i(S-S'-qy)}. \tag{2.1.19}$$

But the terms $e^2\not{A}\not{k}\gamma^\nu\not{k}\not{A}$ in the above equation

$$e^2\not{A}\not{k}\gamma^\nu\not{k}\not{A} = e^2A^\nu.A^\nu k^\nu\not{k}$$
$$= 2e^2a^2k^\nu\not{k}.$$

From this the RHS Eq. (2.1.19) become

$$\int d^4y\bar{u}_{P'S'}\left[\gamma^\nu + \frac{e^2a^2k^\nu \not{k}}{2(k.P')(k.P)} - \frac{e\gamma^\nu \not{k}[\not{a}_1cos(\eta) + a_2sin(\eta)]}{2(k.P)} - \frac{e\not{k}(\not{a}_1cos(\eta) + \not{a}_2sin(\eta))\not{k}\gamma^\nu}{(2k.P')}\right]u_{PS}e^{i(S-S'-qy)}$$

$$= \int d^4y\bar{u}_{P'S'}\left[\left(\gamma^\mu + \frac{e^2a^2k^\nu\not{k}}{2(k.P)(k.P')}\right) - \frac{e}{2}\left(\frac{\gamma^\nu\not{k}\not{a}_1}{(k.P)} + \frac{\not{a}_1\not{k}\gamma^\nu}{(k.P')}\right)cos(kx) - \frac{e}{2}\left(\frac{\gamma^\nu\not{k}\not{a}_2}{(k.P)} + \frac{e\not{a}_2\not{k}\gamma^\nu}{(k.P')}\right)sin(kx)\right]e^{i(S-S^{'}-qy)}u_{ps}. \quad (2.1.20)$$

Substituting each action in the exponential function the term in Eq. (2.1.20)

$$e^{i(S-S'-qy)} = e^{(q+Q-Q').ye^{-i\left(\frac{e(a_1.P')}{(k.P')} - \frac{e(a_1P)}{(k.P)}\right)sin(ky)+i\left(\frac{e(a_2.P')}{(k.P')} - \frac{ea_2P}{(k.P)}\right)cos(k)}}.$$

But

$$e^{-i\left(\frac{e(a_1.P')}{(k.P')} - \frac{e(a_1.P)}{(k.P)}\right)sin(ky)+i\left(\frac{e(a_2.P')}{(k.P')} - \frac{e(a_2.P)}{(k.P)}\right)cos(k)]} = e^{-i\beta sin(ky-k_o)}.$$

Therefore

$$e^{i(S-S'-qy)} = e^{(q+Q-Q').ye^{-i\beta sin(ky-k_o)}}.$$

Then Eq. (2.1.20) which is the RHS of Eq. (2.1.19) can be written as

$$\int d^4y\bar{u}_{p's'}\left[\left(\gamma^\mu + \frac{e^2a^2k^\nu\not{k}}{2(k.P)(k.P')}\right) - \frac{e}{2}\left(\frac{\gamma^\nu\not{k}\not{a}_1}{(k.P)} + \frac{\not{a}_1\not{k}\gamma^\nu}{(k.P')}\right)cos(kx) - \frac{e}{2}\left(\frac{\gamma^\nu\not{k}\not{a}_2}{(k.P)} + \frac{e\not{a}_2\not{k}\gamma^\nu}{(k.P')}\right)sin(kx)\right]u_{PS}e^{(q+Q-Q'-).ye^{-i\beta sin(ky-k_o)}}. \quad (2.1.21)$$

From cylindrical Bessel function expansion of trigonometric functions in the appendix. Let as consider

$$F_N = J_N(\bar{\beta})e^{iN\eta_o},$$

$$G_N = \frac{1}{2}\left(e^{(N+1)\eta_o}J_{N+1}(\bar{\beta}) + e^{(N-1)k_o}J_{N-1}(\bar{\beta})\right),$$

$$H_N = \frac{1}{2i}\left(e^{(N+1)k_o}J_{N+1}(\bar{\beta}) - e^{(N-1)k_o}J_{N-1}(\bar{\beta})\right).$$

The argument of the Bessel functions $J_N(\bar{\beta})$ for this vertex reads

$$\bar{\beta} = \sqrt{\beta_1^2 + \beta_2^2},$$

with

$$\beta_j = \frac{e(a_j.P')}{(k.P')} - \frac{e(a_j.P)}{(k.P)}, j = 1, 2.$$

Making use of the angle k_0 defined by $cosk_o = \beta_1/\bar{\beta}$, and $sink_o = \beta_2/\bar{\beta}$

From these equation

$$\int d^4y \bar{u}_{P'S'} \left(\gamma^\nu + \frac{e^2 \not{A} \not{k} \gamma^\nu}{2(k.P')} \frac{\not{k} \not{A}}{2(k.P)} - \frac{e \gamma^\nu \not{A} \not{k}}{2(k.P)} - \frac{e \not{k} \not{A} \gamma^\nu}{2(k.P')} \right) u_{PS} e^{i(S-S'-qy)}$$

$$= \sum_{N=-\infty}^{\infty} \int d^4y \bar{u}_{P'S'} \left[\left(\gamma^\mu + \frac{e^2 a^2 k^\nu \not{k}}{2(k.P)(k.P')} \right) D_N - \frac{e}{2} \left(\frac{\gamma^\nu \not{k} \not{a}_1}{(k.P)} + \frac{\not{a}_1 \not{k} \gamma^\nu}{(k.P')} \right) F_N - \frac{e}{2} \left(\frac{\gamma^\nu \not{k} \not{a}_2}{(k.P)} + \frac{e \not{a}_2 \not{k} \gamma^\nu}{(k.P')} \right) H_N \right] e^{-i(q+Q+kN-Q')y} u_{PS}. \quad (2.1.22)$$

But one can fined

$$\int d^4y e^{-i(q+Q+kN-Q')y} = (2\pi)^4 \delta^4(q + Q + kN - Q'). \quad (2.1.23)$$

Therefor

$$\int d^4y \bar{u}_{P'S'} \left(\gamma^\nu + \frac{e^2 \not{A} \not{k} \gamma^\nu}{2(k.P')} \frac{\not{k} \not{A}}{2(k.P)} - \frac{e \gamma^\nu \not{A} \not{k}}{2(k.P)} - \frac{e \not{k} \not{A} \gamma^\nu}{2(k.P)'} \right) u_{PS} e^{i(S-S'-qy)}$$

$$\begin{aligned}
&= \sum_{N=-\infty}^{\infty} \int d^4y \bar{u}_{P'S'} \left[\left(\gamma^\mu + \frac{e^2 a^2 k^\nu \not{k}}{2(k.P)(k.P')} \right) D_N - \frac{e}{2} \left(\frac{\gamma^\nu \not{k} \not{a}_1}{(k.P)} + \frac{\not{a}_1 \not{k} \gamma^\nu}{(k.P')} \right) F_N \right. \\
&\qquad \left. - \frac{e}{2} \left(\frac{\gamma^\nu \not{k} \not{a}_2}{(k.P)} + \frac{e \not{a}_2 \not{k} \gamma^\nu}{(k.P')} \right) H_N \right] u_{PS} (2\pi)^4 \delta^4(q + Q + kN - Q') \\
&= \sum_{N=-\infty}^{\infty} M_\nu(\mu\mu'|N)(2\pi)^4 \delta^4(q + Q + kN - Q') \\
&= \sum_{N=-\infty}^{\infty} M_\mu(\mu\mu'|N)(2\pi)^4 \delta^4(q + Q + kN - Q'), \quad (2.1.24)
\end{aligned}$$

The corresponding muonic spinor-matrix product $M_\mu(\mu\mu'|N)$ is

$$M_\mu(\mu\mu'|N) = \bar{u}_{P'S'} \left[\left(\gamma^\mu + \frac{2e^2 a^2 k^\mu \not{k}}{2(k.P)(k.P')} \right) F_N - \frac{e}{2} \left(\frac{\gamma^\mu \not{k} \not{a}_1}{(k.P')} + \frac{\not{a}_1 \not{k} \gamma^\mu}{(k.P)} \right) G_N - \frac{e}{2} \left(\frac{\gamma^\mu \not{k} \not{a}_2}{(k.P')} + \frac{e \not{a}_2 \not{k} \gamma^\mu}{(k.P)} \right) H_N \right] u_{PS}. \quad (2.1.25)$$

Combinations of Eq. (2.1.17) and Eq. (2.1.24) insert in to Eq. (2.1.10) the amplitude become

$$S_{fi} = -i\frac{\alpha_f mM}{(2\pi)^4 V^2\sqrt{q_+^0 q_-^0 Q^0 Q'^0}}\int\frac{d^4q}{q^2}\sum_{n,N=-\infty}^{\infty} M_\mu(e^-e^+|n)M_\mu(\mu\mu'|N)$$
$$(2\pi)^4\delta^4(q+q_+ + q_- - nk)(2\pi)^4\delta^4(q+Q+kN-Q'). \qquad (2.1.26)$$

From the properties of Dirac delta function

$$\int d^4x\delta^4(x-a)\delta^4(x-a) = \delta^4(a-b). \qquad (2.1.27)$$

This implies Eq. (2.1.26) becomes

$$S_{fi} = \frac{(2\pi)^4\alpha_f mM}{iV^2\sqrt{q_+^0 q_-^0 Q^0 Q'^0}}\sum_{n,N=-\infty}^{\infty} M_\mu(e^-e^+|n)M_\mu(\mu\mu'|N)\frac{\delta^4(q+q_+ + q_- - nk - Q - Nk + Q')}{(q_+ + q_- - nk)^2}. \qquad (2.1.28)$$

The expressions by the energy-momentum conserving δ-function at the first vertex integer number n and in the second vertex integer number N in Eq. (2.1.28) counts the laser photons that are emitted (if $n > 0$) or absorbed (if $n < 0$) by the electron and positron. Similarly, N is the number of laser photons emitted (if $N < 0$) or absorbed (if $N > 0$) by the muons. But we can separate the amplitude in to a sum of partial amplitude $s_{fi}^{(r)}$ for one particular photon order r=n+N it gives N=r-n. From this we can write the partial amplitude for the process

$$S_{fi}^{(r)} = N'\sum_{n=-\infty}^{\infty} M_\mu(e^-e^+|n)M_\mu(\mu\mu'|r-n)\frac{\delta^4(q+q_+ + q_- - nk - Q - Nk + Q')}{(q_+ + q_- - nk)^2}, \qquad (2.1.29)$$

where

$$N' = \frac{(2\pi)^4\alpha_f mM}{iV^2\sqrt{q_+^0 q_-^0 Q^0 Q'^0}}.$$

The total transition amplitude and its squared can be written as

$$S_{fi} = \sum_{r\geq r_o} s_{fi}^r, \qquad \left|s_{fi}^{(r)}\right|^2 = \sum_{r\geq r_o} |s_{fi}^{(r)}|^2,$$

where r^o is the minimal number for which the threshold relation $r\omega \geqslant 2m_ec^2(1-\frac{m_e}{M})$ is fulfilled. Because of the delta function in Eq. (2.1.29), there is no double sum over r, r' so that the summation in Eq. (2.1.30) can be pulled out. The square of the partial amplitude $S_{fi}^{(r)}$ is

$$\left|S_{fi}^{(r)}\right|^2 = |N'|^2 \left|\sum_{n=-\infty}^{\infty} M_\mu(e^-e^+|n) M_\mu(\mu\mu'|r-n)\right|^2 \frac{\left(\delta^4(q+q_++q_--nk-Q-Nk+Q')\right)^2}{(q_++q_--nk)^2(q_++q_--n'k)^2}. \tag{2.1.30}$$

It is possible to show that

$$\left(\delta^4(q+q_++q_--nk-Q-Nk+Q')\right)^2 = \frac{TV}{(2\pi)^4}\delta^4(q+q_++q_--nk-Q-Nk+Q'),$$

where T is time factor and the volume factor V which is the same as the normalization of Volkov states. Therefore

$$\left|S_{fi}^{(r)}\right|^2 = |N'|^2 \left|\sum_{n=-\infty}^{\infty} M_\mu(e^-e^+|n) M_\mu(\mu\mu'|r-n)\right|^2 \frac{TV\delta^4(q+q_++q_--nk-Q-Nk+Q')}{(2\pi)^4(q_++q_--nk)^2(q_++q_--n'k)^2}, \tag{2.1.31}$$

where the invariant spinor-matrix product

$$\left|\sum_{n=-\infty}^{\infty} M_\mu(e^-e^+|n) M_\mu(\mu\mu'|r-n)\right|^2 = \sum_{n,n'=-\infty}^{\infty} M_\mu(e^-e^+|n) M_\mu(\mu\mu'|r-n) M_\nu^+(e^-e^+|n) M_\nu^+,(\mu\mu'|r-n).$$

This implies

$$\left|S_{fi}^{(r)}\right|^2 = |N'|^2 \sum_{n,n'=-\infty}^{\infty} M_\mu(e^-e^+|n) M_\mu(\mu\mu'|r-n) M_\nu^+(e^-e^+|n) M_\nu^+(\mu\mu'|r-n) \frac{TV\delta^4(q+q_++q_--nk-Q-Nk+Q')}{(2\pi)^4(q_++q_--nk)^2(q_++q_--n'k)^2}. \tag{2.1.32}$$

The total pair production rate is obtained from (2.1.32) by averaging over the possible initial spin states summing over the final spin states integrating over the final momenta and dividing the result by a unit time T:

$$R = \int \frac{d^3Q'}{(2\pi)^3} \int \frac{d^3q_+}{(2\pi)^3} \int \frac{d^3q_-}{(2\pi)^3} \frac{1}{2} \sum_{ss'} \sum_{s_+s_-} \frac{\left|S_{fi}^{(r)}\right|^2}{T}. \tag{2.1.33}$$

First evaluate with the completeness relation for the components of the Dirac spinors $u_{p_- s_-}$ and $v_{p_+ s_-}$, we can perform the spin summation by introducing the matrices

$$ {}_r\Gamma^n_\mu = \left(\gamma^\mu - \frac{e^2 a^2 k^\mu \not{k}}{2(k.p_+)(k.p_-)}\right) B_n + \frac{e}{2}\left[\left(\frac{\gamma^\mu \not{k} \not{a}_1}{(k.p_+)} - \frac{\not{a}_1 \not{k} \gamma^\mu}{(k.p_-)}\right) C_n + \left(\frac{\gamma^\mu \not{k} \not{a}_2}{(k.p_+)} - \frac{\not{a}_2 \not{k} \gamma^\mu}{(k.p_-)}\right) D_n\right] \tag{2.1.34} $$

for the vertex of the produced pair and

$$ {}_r\Delta^n_\mu = \left(\gamma^\mu + \frac{e^2 a^2 k^\mu \not{k}}{2(k.P)(k.P')}\right) F_{r-n} - \frac{e}{2}\left[\left(\frac{\gamma^\mu \not{k} \not{a}_1}{(k.P)} + \frac{\not{a}_1 \not{k} \gamma^\mu}{(k.P')}\right) G_{r-n} - \left(\frac{\gamma^\mu \not{k} \not{a}_2}{(k.P)} + \frac{e \not{a}_2 \not{k} \gamma^\mu}{(k.P')}\right) H_{r-n}\right] \tag{2.1.35} $$

for the projectile vertex.

So that the spinor matrix

$$ M_\mu(e^- e^+|n) = \bar{u}_{p_- s_-}\ {}_r\Gamma^n_\mu\ v_{p_+ s_+},\ \ and $$

$$ M_\mu(\mu\mu'|r-n) = \bar{u}_{p' s'}\ {}_r\Delta^n_\mu u_{ps}. $$

This implies

$$ M^+_\nu(e^- e^+|n') = (\bar{u}_{p_- s_-}\ {}_r\Gamma^{n'}_\nu\ v_{p_+ s_+})^+ $$

$$ = v^+_{p_+ s_+}\ \Gamma^{n'+}_\nu\ \bar{u}^+_{p_- s_-} \quad but\ \ \bar{u}_{p_- s_-} = u^+_{p_- s_-}\gamma_4 $$

$$ = v^+_{p_+ s_+}\gamma_4\gamma_4\ \Gamma^{n'+}_\nu\ \gamma_4 u_{p_- s_-}\ \ where\ \ \gamma_4\gamma_4 = \gamma_4^2 = I\ \ and\ \ \gamma_4 = \gamma_4^+ $$

$$ = \bar{v}_{p_+ s_+}\bar{\Gamma}^{n'}_\nu u_{p_- s_-}, \qquad where\ \ \bar{\Gamma}^{n'}_\nu = \gamma_4\ \Gamma^{n'+}_\nu\ \gamma_4. $$

By the same method

$$ M^+_\nu(\mu,\mu'|(r-n') = \bar{u}_{ps}\ {}_r\bar{\Delta}^{n'}{}_\nu u_{p's'}, \qquad where\ \ {}_r\bar{\Delta}^{n'}{}_\nu = \gamma_{4r}\Delta^{n'+}_\nu\gamma_4. $$

Insert Eq. (2.1.34) and Eq. (2.1.35) in to the summation of spin and we set

$$ \sum_{ss'}\sum_{s_+ s_-} M_\mu(e^- e^+|n) M_\nu(\mu,\mu'|r-n) M^+_\mu(e^- e^+|n') M^+_\nu(\mu,\mu'|r-n') $$

$$ = \sum_{ss'}\sum_{s_+ s_-} \bar{u}_{p_- s_-}\ {}_r\Gamma^n_\mu v_{p_+ s_+}\quad \bar{u}_{p' s}\ {}_r\Delta^n_\mu u_{ps}\quad \bar{v}_{p_+ s_+ r}\bar{\Gamma}^{n'}_\nu u_{p_- s_-}\ \bar{u}_{ps}\ {}_r\bar{\Delta}^{n'}{}_\nu u_{p's'} $$

$$= \sum_{ss'} \sum_{s_+s_-} \bar{u}_{p_-s_-} \; {}_r\Gamma^n_\mu v_{p_+s_+} \bar{v}_{p_+s_+} \; {}_r\bar{\Gamma}^{n'}_\nu u_{p_-s_-} \bar{u}_{p's'} \; {}_r\Delta^n_\mu u_{ps} \bar{u}_{ps} \; {}_r\bar{\Delta}^{n'}{}_\nu u_{p's'}$$

$$= \sum_{ss'} \sum_{s_+s_-} {}_r\Gamma^n_\mu \quad v_{p_+s_+} \bar{v}_{p_+s_+} \quad {}_r\bar{\Gamma}^{n'}_\nu \quad u_{p_-s_-} \bar{u}_{p_-s_-} \quad {}_r\Delta^n_\mu \quad u_{ps} \bar{u}_{ps} \; {}_r \quad \bar{\Delta}^{n'}{}_\nu \quad u_{p's'} \bar{u}_{p's'}$$

$$= \sum_{s_+s_-} {}_r\Gamma^n_\mu \quad v_{p_+s_+} \bar{v}_{p_+s_+} \quad {}_r\bar{\Gamma}^{n'}_\nu \quad u_{p_-s_-} \bar{u}_{p_-s_-} \quad \sum_{ss'} {}_r\Delta^n_\mu \quad u_{ps} \bar{u}_{ps} \; {}_r \quad \bar{\Delta}^{n'}{}_\nu \quad u_{p's'} \bar{u}_{p's'}. \tag{2.1.36}$$

From the summation of Dirac spinors of spin given in [12]

$$\sum_s u_{p_-s_-} \bar{u}_{p_-s_-} = \frac{\not{p} + m}{2m}, and \tag{2.1.37}$$

$$\sum_s v_{p_+s_+} \bar{v}_{p_+s_+} = \frac{\not{p} - m}{2m} \tag{2.1.38}$$

From these summation and Eq. (2.1.36)

$$\sum_{ss'} \sum_{s_+s_-} M_\mu(e^-e^+|n) M_\nu(\mu, \mu'|r-n) M^+_\mu(e^-e^+|n') M^+_\nu(\mu, \mu'|r-n')$$

$$= Tr\left[{}_r\Gamma^n_\mu \left(\frac{\not{p}_+ - m}{2m}\right) {}_r\bar{\Gamma}^{n'}_\nu \left(\frac{\not{p}_- + m}{2m}\right)\right] \; Tr\left[{}_r\Delta^n_\mu \left(\frac{\not{P} + M}{2M}\right) {}_r \; \bar{\Delta}^{n'}{}_\nu \left(\frac{\not{P}' + M}{2M}\right)\right]$$

$$= T^{nn'}_r. \tag{2.1.39}$$

Therefore

$$R = \int \frac{d^3Q'}{(2\pi)^3} \int \frac{d^3q_+}{(2\pi)^3} \int \frac{d^3q_-}{(2\pi)^3} \frac{1}{2} |N'|^2 \sum_{nn'=-\infty}^{\infty} \frac{T^{nn'}_r}{T} \frac{TV\delta^4(q + q_+ + q_- - nk - Q - Nk + Q')}{(2\pi)^4 (q_+ + q_- - nk)^2 (q_+ + q_- - n'k)^2}. \tag{2.1.40}$$

2.2 Total cross section

The scattering cross section can be defined as the total pair production rate dividing by the velocity of the incoming particle per unit volume. From the previous section total rate of production given in Eq. (2.1.40), so that

$$\frac{d\sigma}{d\Omega_f} = \frac{dR}{\frac{|v_i|}{V}}. \tag{2.2.1}$$

This implies

$$\sigma = \int \frac{d^3Q'}{(2\pi)^3} \int \frac{d^3q_+}{(2\pi)^3} \int \frac{d^3q_-}{(2\pi)^3} \int d\Omega_f \frac{1}{2} \frac{|N'|^2}{\frac{|v_i|}{V}} \sum_{nn'=-\infty}^{\infty} T_r^{nn'} \frac{V\delta^4(q+q_+ + q_- - nk - Q - Nk + Q')}{(2\pi)^4(q_+ + q_- - nk)^2(q_+ + q_- - n'k)^2}. \tag{2.2.2}$$

When we calculated the cross section we must confronted with the task of evaluating traces of special combinations of γ matrices using standard trace technology. To simplify the calculation first find the trace of product of gamma matrix in Eq. (2.1.39)

$$Tr\left[{}_r\Gamma_\mu^n \left(\frac{\not{p}_+ - m}{2m}\right) {}_r\bar{\Gamma}_\nu^{n'} \left(\frac{\not{p}_- + m}{2m}\right)\right] = \frac{1}{4m^2} Tr\left[{}_r\Gamma_\mu^n (\not{p}_+ - m) {}_r\bar{\Gamma}_\nu^{n'} (\not{p}_- + m)\right]$$

$$= \frac{1}{4m^2} Tr\left[{}_r\Gamma^n{}_\mu \not{p}_+ {}_r\bar{\Gamma}_\nu^{n'} \not{p}_- +_r \Gamma_\mu^n \not{p}_+ {}_r\bar{\Gamma}_\nu^{n'} m +_r \Gamma_\mu^n {}_r\bar{\Gamma}_\nu^{n'} \not{p}_- m - m_r^2 \Gamma_\mu^n {}_r\bar{\Gamma}_\nu^{n'}\right]$$

$$= \frac{1}{4m^2}\left[Tr\left[{}_r\Gamma_\mu^n \not{p}_+ {}_r\bar{\Gamma}_\nu^{n'} \not{p}_-\right] + mTr\left[{}_r\Gamma_\mu^n \not{p}_+ {}_r\bar{\Gamma}_\nu^{n'}\right] + mTr\left[{}_r\Gamma_\mu^n {}_r\bar{\Gamma}_\nu^{n'} \not{p}_-\right] - m^2 Tr\left[{}_r\Gamma_\mu^n {}_r\bar{\Gamma}_\nu^{n'}\right]\right]. \tag{2.2.3}$$

Due to trace identities involving γ matrices the trace of an odd number of γ matrices vanishes. From the above Eq. (2.2.3) the terms which vanishes are

$$mTr\left[{}_r\Gamma^n{}_\mu \not{p}_+ {}_r\bar{\Gamma}_\nu^{n'}\right] = mTr\left[{}_r\Gamma^n{}_{\mu r}\bar{\Gamma}_\nu^{n'} \not{p}_-\right] = 0.$$

This implies

$$Tr\left[\Gamma_\mu^n \left(\frac{\not{p} - m}{2m}\right) {}_r\bar{\Gamma}_\nu^{n'} \left(\frac{\not{p}_- + m}{2m}\right)\right] = \frac{1}{4m^2}\left[Tr[{}_r\Gamma_\mu^n \not{p}_+ {}_r\bar{\Gamma}_\nu^{n'} \not{p}_-] - m^2 Tr[{}_r\Gamma_\mu^n {}_r\bar{\Gamma}_\nu^{n'}]\right]. \tag{2.2.4}$$

First find out

$$Tr[{}_r\Gamma^n_\mu \not{p}_+ {}_r\bar{\Gamma}^{n'}_\nu \not{p}_-]. \tag{2.2.5}$$

Substituting the vertex of the produced pair from Eq. (2.1.34) in to Eq. (2.2.5)

$$Tr[{}_r\Gamma^n_\mu \not{p}_+ {}_r\bar{\Gamma}^{n'}_\nu \not{p}_-]$$

$$\begin{aligned}
&= Tr\Bigg[\Bigg\{\left(\gamma^\mu - \frac{e^2a^2k_\mu \not{k}}{2(k.p_+)(k.p_-)}\right)B_n + \frac{e}{2}\Bigg[\left(\frac{\gamma^\mu \not{k}\not{a}_1}{(k.p_+)} - \frac{\not{a}_1\not{k}\gamma^\mu}{(k.p_-)}\right)C_n + \left(\frac{\gamma^\mu \not{k}\not{a}_2}{(k.p_+)} - \frac{\not{a}_2\not{k}\gamma^\mu}{(k.p_-)}\right)D_n\Bigg]\Bigg\}\not{p}_+ \\
&\Bigg\{\left(-\gamma^\nu + \frac{e^2a^2k_\nu \not{k}'}{2(k.p_+)(k.p_-)}\right)B_n^+ + \frac{e}{2}\Bigg[\left(-\frac{\not{a}_1'\not{k}'\gamma^\nu}{(k.p_+)} + \frac{\gamma^\nu\not{k}'\not{a}_1'}{(k.p_-)}\right)C_n^+ + \left(-\frac{\not{a}_2'\not{k}'\gamma^\nu}{(k.p_+)} + \frac{\gamma^\nu\not{k}'\not{a}_2'}{(k.p_-)}\right)D_n^+\Bigg]\Bigg\}\not{p}_-\Bigg] \\
&= Tr\Bigg[\Bigg\{-\gamma^\mu\not{p}_+\gamma^\nu\not{p}_- + (ea)^2\frac{k^\nu\gamma^\mu\not{p}_+\not{k}'\not{p}_- + k^\mu\not{k}\not{p}_+\gamma^\nu\not{p}_-}{2(k.p_+)(k.p_-)} - (ea)^4k^\mu k^\nu\frac{\not{k}\not{p}_+\not{k}'\not{p}_-}{4((k.p_+)(k.p_-))^2}\Bigg\}\left|B_n\right|^2 \\
&+\frac{e}{2}\Bigg\{\frac{-\gamma^\mu\not{p}_+\not{a}_1'\not{k}'\gamma^\nu\not{p}_-}{(k.p_+)} + \frac{\gamma^\mu\not{p}_+\gamma^\nu\not{k}'\not{a}_1'\not{p}_-}{(k.p_-)} + \frac{(ea)^2k^\mu}{2(k.p_+)(k.p_-)}\left(\frac{\not{k}\not{p}_+\not{a}_1'\not{k}'\gamma^\nu\not{p}_-}{(k.p_+)} - \frac{\not{k}\not{p}_+\gamma^\nu\not{k}'\not{a}_1'\not{p}_-}{(k.p_-)}\right)\Bigg\}B_nC_n^+ \\
&+\frac{e}{2}\Bigg\{\frac{-\gamma^\mu\not{p}_+\not{a}_2'\not{k}'\gamma^\nu\not{p}_-}{(k.p_+)} + \frac{\gamma^\mu\not{p}_+\gamma^\nu\not{k}'\not{a}_2'\not{p}_-}{(k.p_-)} + \frac{(ea)^2k^\mu}{2(k.p_+)(k.p_-)}\left(\frac{\not{k}\not{p}_+\not{a}_2'\not{k}'\gamma^\nu\not{p}_-}{(k.p_+)} - \frac{\not{k}\not{p}_+\gamma^\nu\not{k}'\not{a}_2'\not{p}_-}{(k.p_-)}\right)\Bigg\}B_nD_n^+ \\
&+\frac{e}{2}\Bigg\{\frac{-\gamma^\mu\not{k}\not{a}_1\not{p}_+\gamma^\nu\not{p}_-}{(k.p_+)} + \frac{\not{a}_1\not{k}\gamma^\mu\not{p}_+\gamma^\nu\not{p}_-}{(k.p_-)} + \frac{(ea)^2k^\nu}{2(k.p_+)(k.p_-)}\left(\frac{\gamma^\mu\not{k}\not{a}_1\not{p}_+\not{k}'\not{p}_-}{(k.p_+)} - \frac{\not{a}_1\not{k}\gamma^\mu\not{p}_+\not{k}'\not{p}_-}{(k.p_-)}\right)\Bigg\}C_nB_n^+ \\
&+\frac{e^2}{4}\Bigg\{\frac{-\gamma^\mu\not{k}\not{a}_1\not{p}_+\not{a}_1'\not{k}'\gamma^\nu\not{p}_-}{(k.p_+)^2} + \frac{\gamma^\mu\not{k}\not{a}_1\not{p}_+\gamma^\nu\not{k}'\not{a}_1'\not{p}_-}{(k.p_+)(k.p_-)} + \frac{\not{a}_1\not{k}\gamma^\mu p_+\not{a}_1'\not{k}'\gamma^\nu\not{p}_-}{(k.p_+)(k.p_-)} - \frac{\not{a}_1\not{k}\gamma^\mu p_+\gamma^\nu\not{k}'\not{a}_1'\not{p}_-}{(k.p_-)^2}\Bigg\}\left|C_n\right|^2 \\
&+\frac{e^2}{4}\Bigg\{\frac{-\gamma^\mu\not{k}\not{a}_1\not{p}_+\not{a}_2'\not{k}'\gamma^\nu\not{p}_-}{(k.p_+)^2} + \frac{\gamma^\mu\not{k}\not{a}_1\not{p}_+\gamma^\nu\not{k}'\not{a}_2'\not{p}_-}{(k.p_+)(k.p_-)} + \frac{\not{a}_1\not{k}\gamma^\mu p_+\not{a}_2'\not{k}'\gamma^\nu\not{p}_-}{(k.p_+)(k.p_-)} - \frac{\not{a}_1\not{k}\gamma^\mu p_+\gamma^\nu\not{k}'\not{a}_2'\not{p}_-}{(k.p_-)^2}\Bigg\}C_nD_n^+ \\
&+\frac{e}{2}\Bigg\{\frac{-\gamma^\mu\not{k}\not{a}_2\not{p}_+\gamma^\nu\not{p}_-}{(k.p_+)} + \frac{\not{a}_2\not{k}\gamma^\mu\not{p}_+\gamma^\nu\not{p}_-}{(k.p_-)} + \frac{(ea)^2k^\nu}{2(k.p_+)(k.p_-)}\left(\frac{\gamma^\mu\not{k}\not{a}_2\not{p}_+\not{k}'\not{p}_-}{(k.p_+)} - \frac{\not{a}_2\not{k}\gamma^\mu\not{p}_+\not{k}'\not{p}_-}{(k.p_-)}\right)\Bigg\}D_nB_n^+ \\
&+\frac{e^2}{4}\Bigg\{\frac{-\gamma^\mu\not{k}\not{a}_2\not{p}_+\not{a}_1'\not{k}'\gamma^\nu\not{p}_-}{(k.p_+)^2} + \frac{\gamma^\mu\not{k}\not{a}_2\not{p}_+\gamma^\nu\not{k}'\not{a}_1'\not{p}_-}{(k.p_+)(k.p_-)} + \frac{\not{a}_2\not{k}\gamma^\mu p_+\not{a}_1'\not{k}'\gamma^\nu\not{p}_-}{(k.p_+)(k.p_-)} - \frac{\not{a}_2\not{k}\gamma^\mu p_+\gamma^\nu\not{k}'\not{a}_1'\not{p}_-}{(k.p_-)^2}\Bigg\}D_nC_n^+ \\
&+\frac{e^2}{4}\Bigg\{\frac{-\gamma^\mu\not{k}\not{a}_2\not{p}_+\not{a}_2'\not{k}'\gamma^\nu\not{p}_-}{(k.p_+)^2} + \frac{\gamma^\mu\not{k}\not{a}_2\not{p}_+\gamma^\nu\not{k}'\not{a}_2'\not{p}_-}{(k.p_+)(k.p_-)} + \frac{\not{a}_2\not{k}\gamma^\mu p_+\not{a}_2'\not{k}'\gamma^\nu\not{p}_-}{(k.p_+)(k.p_-)} - \frac{\not{a}_2\not{k}\gamma^\mu p_+\gamma^\nu\not{k}'\not{a}_2'\not{p}_-}{(k.p_-)^2}\Bigg\}\left|D_n\right|^2\Bigg].
\end{aligned} \tag{2.2.6}$$

Let as evaluate each trace in Eq. (2.2.6) using trace technology

$$Tr\Bigg[-\gamma^\mu\not{p}_+\gamma^\nu\not{p}_- + (ea)^2\frac{k^\nu\gamma^\mu\not{p}_+\not{k}'\not{p}_- + k^\mu\not{k}\not{p}_+\gamma^\nu\not{p}_-}{2(k.p_+)(k.p_-)}$$

$$-(ea)^4k^\mu k^\nu\frac{\not{k}\not{p}_+\not{k}'\not{p}_-}{4[(k.p_+)(k.p_-)]^2}\Bigg]\left|B_n\right|^2 = l_n\left|B_n\right|^2,$$

$$
\begin{aligned}
&\frac{e}{2}\ Tr\Bigg[\frac{-\gamma^\mu \not p_+ \not a_1 \not k \gamma^\nu \not p_-}{(k.p_+)} + \frac{\gamma^\mu \not p_+ \gamma^\nu \not k \not a_1 \not p_-}{(k.p_-)} \\
&+ \frac{(ea)^2 k^\mu}{2(k.p_+)(k.p_-)}\left(\frac{\not k \not p_+ \not a_1 \not k \gamma^\nu \not p_-}{(k.p_+)} - \frac{\not k \not p_+ \gamma^\nu \not k \not a_1 \not p_-}{(k.p_-)}\right)\Bigg] B_n C_n^+ = l_n^+ B_n C_n^+, \\
&\frac{e}{2}\ Tr\Bigg[\frac{-\gamma^\mu \not p_+ \not a_2 \not k \gamma^\nu \not p_-}{(k.p_+)} + \frac{\gamma^\mu \not p_+ \gamma^\nu \not k \not a_2 \not p_-}{(k.p_-)} \\
&+ \frac{(ea)^2 k^\mu}{2(k.p_+)(k.p_-)}\left(\frac{\not k \not p_+ \not a_2 \not k \gamma^\nu \not p_-}{(k.p_+)} - \frac{\not k \not p_+ \gamma^\nu \not k \not a_2 \not p_-}{(k.p_-)}\right)\Bigg] B_n D_n^+ = m_n B_n D_n^+, \\
&\frac{e}{2}\ Tr\Bigg[\frac{-\gamma^\mu \not k \not a_1 \not p_+ \gamma^\nu \not p_-}{(k.p_+)} + \frac{\not a_1 \not k \gamma^\mu \not p_+ \gamma^\nu \not p_-}{(k.p_-)} \\
&\quad + \frac{(ea)^2 k^\nu}{2(k.p_+)(k.p_-)}\left(\frac{\gamma^\mu \not k \not a_1 \not p_+ \not k \not p_-}{(k.p_+)} - \frac{\not a_1 \not k \gamma^\mu \not p_+ \not k \not p_-}{(k.p_-)}\right)\Bigg] C_n B_n^+ = m_n^+ C_n B_n^+, \\
&\frac{e^2}{4}\ Tr\Bigg[\frac{-\gamma^\mu \not k \not a_1 \not p_+ \not a_1 \not k \gamma^\nu \not p_-}{(k.p_+)^2} + \frac{\gamma^\mu \not k \not a_1 \not p_+ \gamma^\nu \not k \not a_1 \not p_-}{(k.p_+)(k.p_-)} + \frac{\not a_1 \not k \gamma^\mu p_+ \not a_1 \not k \gamma^\nu \not p_-}{(k.p_+)(k.p_-)} \\
&\qquad - \frac{\not a_1 \not k \gamma^\mu p_+ \gamma^\nu \not k \not a_1 \not p_-}{(k.p_-)^2}\Bigg] \big|C_n\big|^2 = p_n \big|C_n\big|^2, \\
&\frac{e^2}{4}\ Tr\Bigg[\frac{-\gamma^\mu \not k \not a_1 \not p_+ \not a_2 \not k \gamma^\nu \not p_-}{(k.p_+)^2} + \frac{\gamma^\mu \not k \not a_1 \not p_+ \gamma^\nu \not k \not a_2 \not p_-}{(k.p_+)(k.p_-)} + \frac{\not a_1 \not k \gamma^\mu p_+ \not a_2 \not k \gamma^\nu \not p_-}{(k.p_+)(k.p_-)} \\
&\qquad - \frac{\not a_1 \not k \gamma^\mu p_+ \gamma^\nu \not k \not a_2 \not p_-}{(k.p_-)^2}\Bigg] C_n D_n^+ = p_n^+ C_n D_n^+, \\
&\frac{e}{2}\ Tr\Bigg[\frac{-\gamma^\mu \not k \not a_2 \not p_+ \gamma^\nu \not p_-}{(k.p_+)} + \frac{\not a_2 \not k \gamma^\mu \not p_+ \gamma^\nu \not p_-}{(k.p_-)} \\
&\quad + \frac{(ea)^2 k^\nu}{2(k.p_+)(k.p_-)}\left(\frac{\gamma^\mu \not k \not a_2 \not p_+ \not k \not p_-}{(k.p_+)} - \frac{\not a_2 \not k \gamma^\mu \not p_+ \not k \not p_-}{(k.p_-)}\right)\Bigg] D_n B_n^+ = s_n D_n B_n^+, \\
&\frac{e^2}{4}\ \Bigg[\frac{-\gamma^\mu \not k \not a_2 \not p_+ \not a_1 \not k \gamma^\nu \not p_-}{(k.p_+)^2} + \frac{\gamma^\mu \not k \not a_2 \not p_+ \gamma^\nu \not k \not a_1 \not p_-}{(k.p_+)(k.p_-)} + \frac{\not a_2 \not k \gamma^\mu p_+ \not a_1 \not k \gamma^\nu \not p_-}{(k.p_+)(k.p_-)} \\
&\qquad - \frac{\not a_2 \not k \gamma^\mu p_+ \gamma^\nu \not k \not a_1 \not p_-}{(k.p_-)^2}\Bigg] D_n C_n^+ = s_n^+ D_n C_n^+, and \\
&\frac{e^2}{4}\ \Bigg[\frac{-\gamma^\mu \not k \not a_2 \not p_+ \not a_2 \not k \gamma^\nu \not p_-}{(k.p_+)^2} + \frac{\gamma^\mu \not k \not a_2 \not p_+ \gamma^\nu \not k \not a_2 \not p_-}{(k.p_+)(k.p_-)} + \frac{\not a_2 \not k \gamma^\mu p_+ \not a_2 \not k \gamma^\nu \not p_-}{(k.p_+)(k.p_-)} \\
&\qquad - \frac{\not a_2 \not k \gamma^\mu p_+ \gamma^\nu \not k \not a_2 \not p_-}{(k.p_-)^2}\Bigg] \big|D_n\big|^2.
\end{aligned}
$$

Where the abbreviations

$$
\begin{aligned}
l_n = 4\Bigg[&(p_+.p_-) - 2p_+p_- + (ea)^2 k \frac{p_+(k.p_-) - k(p_+.p_-) + p_-(k.p_+)}{(k.p_+)(k.p_-)} \\
&- (ea)^4 k^2 \frac{2(k.p_+)(k.p_-) - k^2(p_+.p_-)}{4[(k.p_+)(k.p_-)]^2}\Bigg],
\end{aligned}
$$

$$l_n^+ = 2e\Bigg[\Bigg\{ -(k.a_1)[2p_+p_- + (p_-.p_+)] + (p_+.a_1)[2kp_-)(p_-.k)] + (p_+.k)[-2p_-a_1]$$
$$-(a_1.p_-)\Bigg\}\frac{1}{(k.p_+)} + \Bigg\{(a_1.p_-)[2kp_+ - (k.p_+)] + (k.p_-)[(p_+.a_1) - 2p_+a_1]$$
$$+(k.a_1)[(p_+.p_-) - 2p_+p_-]\Bigg\}\frac{1}{(k.p_-)}$$
$$+(ea)^2k\Bigg\{2(k.p_+)[k(p_-.a_1) - a_-(k.p_-)] + 2p_+(k.a_1)(k.p_-) + k^2[(p_+.a_1)p_-$$
$$-p_+(p_-.a_1) + a_1(p_+.p_-)]\Bigg\}\frac{1}{(k.p_+)^2(k.p_-)} - (ea)^2k\Bigg\{2(p_+.k)[p_-(k.a_1) - (p_-.k)]$$
$$+k(p_+.a_1)[2(p_-.k) - kp_-] + k(p_+.p_-)[ka_1 - 2(k.a_1)] + k^2p_-(p_+.a_1)\Bigg\}\frac{1}{2(k.p_+)(k.p_-)^2}\Bigg],$$

$$m_n = 2e\Bigg[\Bigg\{ -(k.a_2)[2p_+p_- - (p_-.p_+)] - (p_+.a_2)[2kp_-) - (p_-.k)] - (p_+.k)[-2p_-a_2$$
$$+(a_2.p_-)]\Bigg\}\frac{1}{(k.p_+)} + \Bigg\{(a_2.p_-)[2kp_+ - (k.p_+)] + (k.p_-)[(p_+.a_2) - 2_+a_1] + (k.a_2)[(p_+.p_-)$$
$$-2p_+p_-]\Bigg\}\frac{1}{(k.p_-)} + (ea)^2k\Bigg\{2(k.p_+)[k(p_-.a_2) - a_-(k.p_-)] + 2p_+(k.a_2)(k.p_-)$$
$$+k^2[(p_+.a_2)p_- - p_+(p_-.a_2) + a_2(p_+.p_-)]\Bigg\}\frac{1}{(k.p_+)^2(.kp_-)}$$
$$-(ea)^2k\Bigg\{2(p_+.k)[p_-(k.a_2) - (p_-.k)] + k(p_+.a_2)[2(p_-.k) - kp_-]$$
$$+k(p_+.p_-)[ka_2 - 2(k.a_2)] + k^2p_-(p_+.a_2)\Bigg\}\frac{1}{2(k.p_+)(k.p_-)^2}\Bigg],$$

$$m_n^+ = 2e\Bigg[\Bigg\{ -(p_+.a_1)[2kp_- - (k.p_-)] - (k.p_-)[(p_+.a_1) - 2p_+a_1]$$
$$-(k.a_1)[2p_+p_- - (p_+.p_-)]\Bigg\}\frac{1}{(k.p_+)} + \Bigg\{(a_1.k)[2p_+p_- - (p_+p_-)] - 2a_1p_+(k.p_-)$$
$$+kp_-(a_1.p_+) + (a_1.p_-)[2kp_+ - (k.p_+)]\Bigg\}\frac{1}{(k.p_-)} + (ea)^2k\Bigg\{2k(a_1.p_-)(k.p_+)$$
$$-2p_+(a_1.k)(k.p_-) - 2(a_1.k)(p_+.p_-) - 2a_1(k.p_+)(k.p_-)\Bigg\}\frac{1}{2(k.p_+)^2(kp_-)}$$
$$-(ea)^2k\Bigg\{2k(a_1.k)(p_-.p_+) + 2p_-(a_1.k)(k.p_+) + 2(k.p_-)[k(a_1.p_+) - a_1(k.p_+)]$$
$$+k^2[a_1(p_-.p_+) - p_-(a_1.p_+) + p_+(a_1.p_-)]\Bigg\}\frac{1}{2(k.p_+)(k.p_-)^2}\Bigg],$$

$$p_n = e^2\Big[-\Big\{2k^2(a_1.p_+)[a_1p_- + (a_1.p_-)] + 2a_1^2(k.p_+)[(k.p_-) - kp_-] + (a_1k)^2p_+p_-$$
$$-a_1^2kp_+(k.p_-) + k^2a_1p_+(a_1.p_-) + (a_1.k)(k.p_+)[6p_-k - 4(k.p_-k)] + ka_1^2(k.p_+k)$$
$$-k^2a_1(a_1.p_+)\Big\}\frac{1}{(k.p_+)^2} - \Big\{4(a_1.p_-)(a_1.k)[2p_+k - (k.p_+)] + k^2(a_1.p_-)[2(a_1.p_+)$$
$$-a_1p_+] + 2a_1^2[(k.p_-)(-2kp_+ + (k.p_+)) - k^2(p_+p_- + (p_+p_-))]\Big\}\frac{1}{(k.p_-)^2}$$
$$+\Big\{4k^2(a_1.p_+)(a_1.p_-) - 3ka_1(a_1.p_-)(k.p_+) + 4ka_1^2p_+(k.p_-) + 3ka_1^2p_-(k.p_+)$$
$$-5a_1p_-(k.p_+)(a_1.k) - 7a_1k(k.p_-)(a_1.p_+) + 5a_1k(a_1.k)(p_+.p_-) - 3a_1^2k^2(p_+.p_-)$$
$$-8a_1p_+(a_1.k)(k.p_-) + 7p_+p_-(a_1.k)^2 + 3p_+a_1k^2(a_1.p_-) - 4k^2a_1^2p_+p_-$$
$$+6(k.p_+)(a_1.k)(k.p_-) - k^2(a_1.p_-)(a_1.p_+) + 6a_1^2(k.p_-)(k.p_+)3p_-a_1k^2(a_1.p_+)$$
$$-4p_-a_1(k.p_+)(a_1.k) + 6p_-k(a_1.p_+)(a_1.k) - (a_1.k)^2(p_+.p_+) - 2kp_-(a_1.p_+)(a_1.k)$$
$$+2(k.p_-)(a_1.p_+)(a_1.k) + 2k^2p_-a_1(a_1.p_+) - 2a_1^2kp_+(k.p_- - 2kp_+(a_1.k)(a_1.p_-)\Big\}\frac{1}{(k.p_+)(k.p_-)}\Big],$$

$$p_n^+ = e^2\Big[-\Big\{2(a_1.p_+)(k.a_2)\big[2kp_- - (k.p_-)\big] + 2(a_2.p_+)(k.a_1)\big[2kp_- - (k.p_-)\big]$$
$$+k^2(a_1.a_2)[2p_-p_+ - (p_-.p_+)] + k^2(a_2.p_-)[(a_1.p_+) - 2a_1p_+] + k^2(a_1.p_+)\big[-2a_2p_-$$
$$-(a_2.p_-)\big] - k^2a_1a_2(p_-.p_+)\Big\}\frac{1}{(k.p_+)^2} - \Big\{(a_1.a_2)\big[-2kp_+(k.p_-) + k^2\big(2p_+p_- - (p_+.p_-)\big)$$
$$-2kp_+(k.p_-) + 2(k.p_+)(k.p_-)\big] + (a_1.k)\big(2kp_+(p_-.a_2) - 2(k.p_+)(p_-.a_2)\big) - a_1a_2k^2(p_+.p_-)$$
$$+k^2(a_1.p_+)(a_2.p_-)\Big\}\frac{1}{(k.p_-)^2} + \Big\{(k.p_+)\big[-4a_1p_-(k.a_2) + 2(a_1.p_-)(k.a_2)$$
$$-5a_2p_-(k.a_1) + 2(a_2.p_-(k.a_1) + a_1a_2(k.p_-) - 4a_1k(a_2.p_-) - 2a_2k(a_1.p_-)\big]$$
$$+2k^2\big[a_2p_+(a_1.p_-) - 2a_2a_1(p_-.p_+) + a_2p_-(a_1.p_+) - (a_1.p_+)(a_2.p_-) - (a_2.a_1)(p_-.p_+)$$
$$-2(a_2.a_1)p_-p_+ + (a_1.p_-)(a_2.p_+)\big] + a_1k\big[kp_-(a_2.p_+) + (k.a_2)(p_-.p_+) - 4(a_2.p_+)(p_-.k)\big]$$
$$+2(k.p_-)\big[-2a_2p_+(k.a_1) + (a_2.p_+)(k.a_1) - (a_2.a_1)(k.p_+) - a_1p_+(k.a_2)$$
$$+(a_1.p_+)(k.a_2) - p_+k(a_2.a_1)\big] + 2(a_2.k)\big[3p_+p_-(a_1.k) - 2(p_+p_-)(a_1.k) + (p_+.a_1)p_-k\big]$$
$$+3k\big[-a_2(a_1.p_+)(p_-.k) + p_-(a_2.a_1)(k.p_+)\big]\Big\}\frac{1}{(k.p_+)(k.p_-)}\Big],$$

$$s_n = 2e\Bigg[\Bigg\{-(p_+.a_2)[2kp_- - (k.p_-)] - (k.p)[(p_+.a_2) - 2p_+a_2] - (k.a_2)[2p_+p_-$$
$$-(p_+.p_-)]\Bigg\}\frac{1}{(k.p_+} + \Bigg\{(a_2.k)[2p_+p_- - (p_+p_-)] - 2a_2p_+(k.p_-) + kp_-(a_2.p_+)$$
$$+(a_2.p_-)[2kp_+ - (k.p_+)]\Bigg\}\frac{1}{(k.p_-)} + (ea)^2k\Bigg\{2k(a_2.p_-)(k.p_+) - 2p_+(a_2.k)(k.p_-)$$
$$-2(a_2.k)(p_+.p_-) - 2a_2(k.p_+)(k.p_-)\Bigg\}\frac{1}{2(k.p_+)^2(k.p_-)]} - (ea)^2k\Bigg\{2k(a_2.k)(p_-.p_+)$$
$$+2p_-(a_2.k)(k.p_+) + 2(k.p_-)[k(a_2.p_+) - a_2(k.p_+)] + k^2[a_2(p_-.p_+) - p_-(a_2.p_+)$$
$$+p_+(a_2.p_-)]\Bigg\}\frac{1}{2(k.p_+)(k.p_-)^2}\Bigg],$$

$$s_n^+ = e^2\Bigg[-\Bigg\{2(a_2.p_+)(k.a_1)\big[2kp_- - (k.p_-)\big] + 2(a_1.p_+)(k.a_2)\big[2kp_- - (k.p_-)\big]$$
$$+k^2(a_2.a_1)[2p_-p_+ - (p_-.p_+)] + k^2(a_1.p_-)[(a_2.p_+) - 2a_2p_+] + k^2(a_2.p_+)\big[-2a_1p_-$$
$$-(a_1.p_-)\big] - k^2a_2a_2(p_-.p_+)\Bigg\}\frac{1}{(k.p_+)^2} - \Bigg\{(a_1.a_2)\big[-2kp_+(k.p_-) + k^2\big(2p_+p_- - (p_+.p_-)\big)$$
$$-2kp_+(k.p_-) + 2(k.p_+)(k.p_-)\big] + (a_2.k)\big(2kp_+(p_-.a_1) - 2(k.p_+)(p_-.a_1)\big)$$
$$-a_1a_2k^2(P_+.p_-) + k^2(a_2.p_+)(a_2.p_-)\Bigg\}\frac{1}{(k.p_-)^2} + \Bigg\{(k.p_+)\big[-4a_2p_-(k.a_1)$$
$$+2(a_2.p_-)(k.a_1) - 5a_2p_-(k.a_2) + 2(a_1.p_-(k.a_2) + a_1a_2(k.p_-) - 4a_2k(a_1.p_-)$$
$$-2a_1k(a_2.p_-)\big] + 2k^2\big[a_1p_+(a_2.p_-) - 2a_1a_2(p_-.p_+) + a_1p_-(a_2.p_+) - (a_2.p_+)(a_1.p_-)$$
$$-(a_2.a_1)(p_-.p_+) - 2(a_2.a_1)p_-p_+ + (a_2.p_-)(a_1.p_+)\big] + a_1k\big[kp_-(a_1.p_+) + (k.a_1)(p_-.p_+)$$
$$-4(a_1.p_+)(p_-.k)\big] + 2(k.p_-)\big[-2a_1p_+(k.a_2) + (a_1.p_+)(k.a_2) - (a_1.a_2)(k.p_+)$$
$$-a_2p_+(k.a_1) + (a_2.p_+)(k.a_1) - p_+k(a_1.a_2)\big] + 2(a_1.k)\big[3p_+p_-(a_2.k) - 2(p_+p_-)(a_2.k)$$
$$+(p_+.a_2)p_-k\big] + 3k\big[-a_1(a_1.p_+)(p_-.k) + p_-(a_2.a_1)(k.p_+)\big]\Bigg\}\frac{1}{(k.p_+)(k.p_-)}\Bigg], and$$

$$t_n = e^2\Big[-\Big\{2k^2(a_2.p_+)[a_2 p_- + (a_2.p_-)] + 2a_2^2(k.p_+)[(k.p_-) - kp_-] + (a_2 k)^2 p_+ p_-$$
$$-a_2^2 kp_+(k.p_-) + k^2 a_2 p_+(a_2.p_-) + (a_2.k)(k.p_+)[6p_- k - 4(k.p_- k)] + ka_2^2(k.p_+ k)$$
$$-k^2 a_2(a_2.p_+)\Big\}\frac{1}{(k.p_+)^2} - \Big(4(a_2.p_-)(a_2.k)[2p_+ k - (k.p_+)] + k^2(a_2.p_-)[2(a_2.p_+) - a_2 p_+]$$
$$+2a_2^2[(k.p_-)(-2kp_+ + (k.p_+)) - k^2(p_+ p_- + (p_+ p_-))]\Big)\frac{1}{(k.p_-)^2} + \Big\{4k^2(a_2.p_+)(a_2.p_-)$$
$$-3ka_2(a_2.p_-)(k.p_+) + 4ka_2^2 p_+(k.p_-) + 3ka_2^2 p_-(k.p_+) - 5a_1 p_-(k.p_+)(a_1.k)$$
$$-7a_2 k(k.p_-)(a_2.p_+) + 5a_2 k(a_2.k)(p_+.p_-) - 3a_2^2 k^2(p_+.p_-) - 8a_2 p_+$$
$$(a_2.k)(k.p_-) + 7p_+ p_-(a_2.k)^2 + 3p_+ a_2 k^2(a_2.p_-) - 4k^2 a_2^2 p_+ p_- + 6(k.p_+)(a_2.k)(k.p_-)$$
$$-k^2(a_2.p_-)(a_2.p_+) + 6a_2^2(k.p_-)(k.p_+)3p_- a_2 k^2(a_2.p_+) - 4p_- a_2(k.p_+)(a_2.k)$$
$$+6p_- k(a_2.p_+)(a_2.k) - (a_2.k)^2(p_+.p_+) - 2kp_-(a_2.p_+)(a_2.k) + 2(k.p_-)(a_2.p_+)(a_2.k)$$
$$+2k^2 p_- a_2(a_2.p_+) - 2a_2^2 kp_+(k.p_- - 2kp_+(a_2.k)(a_2.p_-)\Big\}\frac{1}{(k.p_+)(k.p_-)}\Big]$$

Therefor

$$Tr[{}_r\mathit{\Gamma}^n_\mu \not{p}_+ {}_r\bar{\mathit{\Gamma}}^{n'}_\nu \not{p}_-] = l_n|B_n|^2 + l_n^+ B_n C_n^+ + m_n B_n D_n^+ + m_n^+ C_n B_n^+ + p_n|C_n|^2$$
$$+ p_n^+ C_n D_n^+ + s_n D_n B_n^+ + s_n^+ D_n C_n^+ + t_n|D_n|^2. \tag{2.2.7}$$

Secondly evaluate

$$Tr[{}_r\mathit{\Gamma}^n_\mu {}_r\bar{\mathit{\Gamma}}^{n'}_\nu]. \tag{2.2.8}$$

Substituting the vertex of the produced pair from Eq. (2.1.34) in to Eq. (2.2.8)

$$Tr\Big[{}_r\mathit{\Gamma}^n_\mu {}_r\bar{\mathit{\Gamma}}^{n'}_\nu\Big]$$
$$= Tr\Big[\Big\{\Big(\gamma^\mu - \frac{e^2 a^2 k_\mu \not{k}}{2(k.p_+)(k.p_-)}\Big)B_n + \frac{e}{2}\Big[\Big(\frac{\gamma^\mu \not{k}\not{a}_1}{(k.p_+)} - \frac{\not{a}_1 \not{k}\gamma^\mu}{k.p_-}\Big)C_n + \Big(\frac{\gamma^\mu \not{k}\not{a}_2}{(k.p_+)} - \frac{\not{a}_2 \not{k}\gamma^\mu}{(k.p_-)}\Big)D_n\Big]\Big\}$$
$$\Big\{\Big(-\gamma^\nu + \frac{e^2 a^2 k_\nu \not{k}'}{2(k.p_+)(k.p_-)}\Big)B_n^+ + \frac{e}{2}\Big[\Big(-\frac{\not{a}_1' \not{k}'\gamma^\nu}{(k.p_+)} + \frac{\gamma^\nu \not{k}'\not{a}_1'}{(k.p_-)}\Big)C_n^+ + \Big(-\frac{\not{a}_2' \not{k}'\gamma^\nu}{(k.p_+)} + \frac{\gamma^\nu \not{k}'\not{a}_2'}{(k.p_-)}\Big)D_n^+\Big]\Big\}\Big]$$
$$= Tr\Big[\Big\{-\gamma^\mu\gamma^\nu + (ea)^2\frac{k^\nu\gamma^\mu\not{k}' + k^\mu\not{k}\gamma^\nu}{2(k.p_+)(k.p_-)} - (ea)^4 k^\mu k^\nu\frac{\not{k}\not{k}'}{4[(k.p_+)(k.p_-)]^2}\Big\}|B_n|^2$$
$$+ \frac{e}{2}\Big\{\frac{-\gamma^\mu \not{a}_1' \not{k}'\gamma^\nu}{(k.p_+)} + \frac{\gamma^\mu\gamma^\nu \not{k}'\not{a}_1' \not{p}_-}{(k.p_-)} + \frac{(ea)^2 k^\mu}{2(k.p_+)(k.p_-)}\Big(\frac{\not{k}\not{a}_1'\not{k}'\gamma^\nu}{(k.p_+)} - \frac{\not{k}\gamma^\nu\not{k}'\not{a}_1'}{(k.p_-)}\Big)\Big\}B_n C_n^+$$

$$+\frac{e}{2}\Bigg\{\frac{-\gamma^\mu \not{a}'_2 \not{k}' \gamma^\nu}{(k.p_+)}+\frac{\gamma^\mu \gamma^\nu \not{k}' \not{a}'_2}{(k.p_-)}+\frac{(ea)^2 k^\mu}{2(k.p_+)(k.p_-)}\left(\frac{\not{k} \not{a}'_2 \not{k}' \gamma^\nu}{(k.p_+)}-\frac{\not{k} \gamma^\nu \not{k}' \not{a}'_2}{(k.p_-)}\right)\Bigg\}B_n D_n^+$$

$$+\frac{e}{2}\Bigg\{\frac{-\gamma^\mu \not{k} \not{a}_1 \gamma^\nu}{(k.p_+)}+\frac{\not{a}_1 \not{k} \gamma^\mu \gamma^\nu}{(k.p_-)}+\frac{(ea)^2 k^\nu}{2(k.p_+)(k.p_-)}\left(\frac{\gamma^\mu \not{k} \not{a}_1 \not{k}'}{(k.p_+)}-\frac{\not{a}_1 \not{k} \gamma^\mu \not{k}'}{k.p_-}\right)\Bigg\}C_n B_n^+$$

$$+\frac{e^2}{4}\Bigg\{\frac{-\gamma^\mu \not{k} \not{a}_1 \not{a}'_1 \not{k}' \gamma^\nu}{(k.p_+)^2}+\frac{\gamma^\mu \not{k} \not{a}_1 \gamma^\nu \not{k}' \not{a}'_1}{(k.p_+)(k.p_-)}+\frac{\not{a}_1 \not{k} \gamma^\mu \not{a}'_1 \not{k}' \gamma^\nu}{(k.p_+)(k.p_-)}-\frac{\not{a}_1 \not{k} \gamma^\mu \gamma^\nu \not{k}' \not{a}'_1}{(k.p_-)^2}\Bigg\}\big|C_n\big|^2$$

$$+\frac{e^2}{4}\Bigg\{\frac{-\gamma^\mu \not{k} \not{a}_1 \not{a}'_2 \not{k}' \gamma^\nu}{(k.p_+)^2}+\frac{\gamma^\mu \not{k} \not{a}_1 \gamma^\nu \not{k}' \not{a}'_2}{(k.p_+)(k.p_-)}+\frac{\not{a}_1 \not{k} \gamma^\mu \not{a}'_2 \not{k}' \gamma^\nu}{(k.p_+)(k.p_-)}-\frac{\not{a}_1 \not{k} \gamma^\mu \gamma^\nu \not{k}' \not{a}'_2}{(k.p_-)^2}\Bigg\}C_n D_n^+$$

$$+\frac{e}{2}\Bigg\{\frac{-\gamma^\mu \not{k} \not{a}_2 \gamma^\nu}{(k.p_+)}+\frac{\not{a}_2 \not{k} \gamma^\mu \gamma^\nu}{(k.p_-)}+\frac{(ea)^2 k^\nu}{2(k.p_+)(k.p_-)}\left(\frac{\gamma^\mu \not{k} \not{a}_2 \not{k}'}{(k.p_+)}-\frac{\not{a}_2 \not{k} \gamma^\mu \not{k}'}{(k.p_-)}\right)\Bigg\}D_n B_n^+$$

$$+\frac{e^2}{4}\Bigg\{\frac{-\gamma^\mu \not{k} \not{a}_2 \not{a}'_1 \not{k}' \gamma^\nu}{(k.p_+)^2}+\frac{\gamma^\mu \not{k} \not{a}_2 \gamma^\nu \not{k}' \not{a}'_1}{(k.p_+)(k.p_-)}+\frac{\not{a}_2 \not{k} \gamma^\mu \not{a}'_1 \not{k}' \gamma^\nu}{(k.p_+)(k.p_-)}-\frac{\not{a}_2 \not{k} \gamma^\mu \gamma^\nu \not{k}' \not{a}'_1}{(k.p_-)^2}\Bigg\}D_n C_n^+$$

$$+\frac{e^2}{4}\Bigg\{\frac{-\gamma^\mu \not{k} \not{a}_2 \not{a}'_2 \not{k}' \gamma^\nu}{(k.p_+)^2}+\frac{\gamma^\mu \not{k} \not{a}_2 \gamma^\nu \not{k}' \not{a}'_2}{(k.p_+)(k.p_-)}+\frac{\not{a}_2 \not{k} \gamma^\mu \not{a}'_2 \not{k}' \gamma^\nu}{(k.p_+)(k.p_-)}-\frac{\not{a}_2 \not{k} \gamma^\mu \gamma^\nu \not{k}' \not{a}'_2}{(k.p_-)^2}\Bigg\}\big|D_n\big|^2\Bigg] \qquad (2.2.9)$$

Let as evaluate each trace in Eq. (2.2.9) using trace technology

$$Tr\Bigg[-\gamma^\mu\gamma^\nu+(ea)^2\frac{k^\nu\gamma^\mu\not{k}'+k^\mu\not{k}\gamma^\nu}{2(k.p_+)(k.p_-)}-(ea)^4k^\mu k^\nu\frac{\not{k}\not{k}'}{4[(k.p_+)(k.p_-)]^2}\Bigg]\big|B_n\big|^2=x_n\big|B_n\big|^2,$$

$$\frac{e}{2}\;Tr\Bigg[\frac{-\gamma^\mu \not{a}'_1 \not{k}' \gamma^\nu}{(k.p_+)}+\frac{\gamma^\mu \gamma^\nu \not{k}' \not{a}'_1}{(k.p_-)}+\frac{(ea)^2 k^\mu}{2(k.p_+)(k.p_-)}\left(\frac{\not{k} \not{a}'_1 \not{k}' \gamma^\nu}{(k.p_+)}-\frac{\not{k} \gamma^\nu \not{k}' \not{a}'_1}{(k.p_-)}\right)\Bigg]B_n C_n^+=y_n B_n C_n^+,$$

$$\frac{e}{2}\;Tr\Bigg[\frac{-\gamma^\mu \not{a}'_2 \not{k}' \gamma^\nu}{(k.p_+)}+\frac{\gamma^\mu \gamma^\nu \not{k}' \not{a}'_2}{(k.p_-)}+\frac{(ea)^2 k^\mu}{2(k.p_+)(k.p_-)}\left(\frac{\not{k} \not{a}'_2 \not{k}' \gamma^\nu}{(k.p_+)}-\frac{\not{k} \gamma^\nu \not{k}' \not{a}'_2}{(k.p_-)}\right)\Bigg]B_n D_n^+=y_n^+ B_n D_n^+,$$

$$\frac{e}{2}\;Tr\Bigg[\frac{-\gamma^\mu \not{k} \not{a}_1 \gamma^\nu}{(k.p_+)}+\frac{\not{a}_1 \not{k} \gamma^\mu \gamma^\nu}{(k.p_-)}+\frac{(ea)^2 k^\nu}{2(k.p_+)(k.p_-)}\left(\frac{\gamma^\mu \not{k} \not{a}_1 \not{k}'}{(k.p_+)}-\frac{\not{a}_1 \not{k} \gamma^\mu \not{k}'}{(k.p_-)}\right)\Bigg]C_n B_n^+=z_n C_n B_n^+,$$

$$\frac{e^2}{4}\;Tr\Bigg[\frac{-\gamma^\mu \not{k} \not{a}_1 \not{a}'_1 \not{k}' \gamma^\nu}{(k.p_+)^2}+\frac{\gamma^\mu \not{k} \not{a}_1 \gamma^\nu \not{k}' \not{a}'_1}{(k.p_+)(k.p_-)}+\frac{\not{a}_1 \not{k} \gamma^\mu \not{a}'_1 \not{k}' \gamma^\nu}{(k.p_+)(k.p_-)}-\frac{\not{a}_1 \not{k} \gamma^\mu \gamma^\nu \not{k}' \not{a}'_1}{(k.p_-)^2}\Bigg]\big|C_n\big|^2=z_n^+\big|C_n\big|^2,$$

$$\frac{e^2}{4}\;Tr\Bigg[\frac{-\gamma^\mu \not{k} \not{a}_1 \not{a}'_2 \not{k}' \gamma^\nu}{(k.p_+)^2}+\frac{\gamma^\mu \not{k} \not{a}_1 \gamma^\nu \not{k}' \not{a}'_2}{(k.p_+)(k.p_-)}+\frac{\not{a}_1 \not{k} \gamma^\mu \not{a}'_2 \not{k}' \gamma^\nu}{(k.p_+)(k.p_-)}-\frac{\not{a}_1 \not{k} \gamma^\mu \gamma^\nu \not{k}' \not{a}'_2}{(k.p_-)^2}\Bigg]C_n D_n^+=k_n C_n D_n^+,$$

$$\frac{e}{2}\;Tr\Bigg[\frac{-\gamma^\mu \not{k} \not{a}_2 \gamma^\nu}{(k.p_+)}+\frac{\not{a}_2 \not{k} \gamma^\mu \gamma^\nu}{(k.p_-)}+\frac{(ea)^2 k^\nu}{2(k.p_+)(k.p_-)}\left(\frac{\gamma^\mu \not{k} \not{a}_2 \not{k}'}{(k.p_+)}-\frac{\not{a}_2 \not{k} \gamma^\mu \not{k}'}{(k.p_-)}\right)\Bigg]D_n B_n^+=k_n^+ D_n B_n^+,$$

$$\frac{e^2}{4}\;\Bigg[\frac{-\gamma^\mu \not{k} \not{a}_2 \not{a}'_1 \not{k}' \gamma^\nu}{(k.p_+)^2}+\frac{\gamma^\mu \not{k} \not{a}_2 \gamma^\nu \not{k}' \not{a}'_1}{(k.p_+)(k.p_-)}+\frac{\not{a}_2 \not{k} \gamma^\mu \not{a}'_1 \not{k}' \gamma^\nu}{(k.p_+)(k.p_-)}-\frac{\not{a}_2 \not{k} \gamma^\mu \gamma^\nu \not{k}' \not{a}'_1}{(k.p_-)^2}\Bigg]D_n C_n^+=w_n D_n C_n^+, and$$

$$\frac{e^2}{4}\;\Bigg[\frac{-\gamma^\mu \not{k} \not{a}_2 \not{a}'_2 \not{k}' \gamma^\nu}{(k.p_+)^2}+\frac{\gamma^\mu \not{k} \not{a}_2 \gamma^\nu \not{k}' \not{a}'_2}{(k.p_+)(k.p_-)}+\frac{\not{a}_2 \not{k} \gamma^\mu \not{a}'_2 \not{k}' \gamma^\nu}{(k.p_+)(k.p_-)}-\frac{\not{a}_2 \not{k} \gamma^\mu \gamma^\nu \not{k}' \not{a}'_2}{(k.p_-)^2}\Bigg]\big|D_n\big|^2=w_n^+\big|D_n\big|^2.$$

Where the abbreviations

$$x_n=4\left[-1+\frac{2(eak)^2}{(k.p_+)(k.p_-)}-\frac{(eak)^4}{[2(k.p_+)(k.p_-)]^2}\right],$$

$$y_n = \frac{e}{2}\left[4(a_1.k)\left(\frac{-1}{(k.p_+)}+\frac{1}{(k.p_-)}\right)+(ea)^2\left(\frac{2\left[k(a_1.k)-a_1k^2\right]}{(k.p_+)^2(k.p_-)}-\frac{2k(k.a_1)-a_1k^2}{2(k.p_+)(k.p_-)^2}\right)\right],$$

$$y_n^+ = \frac{e}{2}\left[4(a_2.k)\left(\frac{-1}{(k.p_+)}+\frac{1}{(k.p_-)}\right)+(ea)^2\left(\frac{2\left[k(a_2.k)-a_2k^2\right]}{(k.p_+)^2(k.p_-)}-\frac{2k(k.a_2)-a_2k^2}{2(k.p_+)(k.p_-)^2}\right)\right],$$

$$z_n = 2e\left[\frac{-(k.a_1)}{(k.p_+)}+(ea)^2k\frac{2k(k.a_1)-a_1k^2}{(k.p_+)^2(k.p_-)}+\frac{(k.a_1)}{(k.p_-)}-(ea)^2k\frac{2k(k.a_1)-a_1k^2}{(k.p_+)(k.p_-)^2}\right],$$

$$z_n^+ = e^2\left[\frac{-(a_1k)^2}{(k.p_+)^2}+\frac{4(a_1.k)^2-8ka_1(a_1.k)+4k^2a_1^2}{(k.p_+)(k.p_-)}-\frac{-(a_1k)^2}{(k.p_-)^2}\right],$$

$$k_n = e^2\Bigg[\frac{-k^2(a_1.a_2)}{(k.p_+)^2}+\frac{4(k.a_2)(k.a1)-2ka_2(a_1.k)-6ka_1(a_2.k)+(a_1.a_2)k^2+2a_1a_2k^2}{(k.p_+)(k.p_-)}$$
$$-\frac{k^2(a_1.a_2)}{(k.p_-)^2}\Bigg],$$

$$k_n^+ = 2e\left[\frac{-(k.a_2)}{(kp_+)}+(ea)^2k\frac{2k(k.a_2)-a_2k^2}{(kp_+)^2(kp_-)}+\frac{(k.a_2)}{(kp_-)}-(ea)^2k\frac{2k(k.a_2)-a_2k^2}{(kp_+)(kp_-)^2}\right],$$

$$w_n = e^2\Bigg[\frac{-k^2(a_1.a_2)}{(k.p_+)^2}+\frac{4(k.a_2)(k.a1)-2ka_1(a_2.k)-6ka_1(a_2.k)+(a_1.a_2)k^2+2a_1a_2k^2}{(k.p_+)(k.p_-)}$$
$$-\frac{k^2(a_1.a_2)}{(k.p_-)^2}\Bigg], and$$

$$w_n^+ = e^2\left[\frac{-(a_2k)^2}{(k.p_+)^2}+\frac{4(a_2.k)^2-8ka_2(a_2.k)+4k^2a_2^2}{(k.p_+)(k.p_-)}-\frac{(-a_2k)^2}{(k.p_-)^2}\right].$$

This implies that

$$Tr[{}_r\Gamma^n_\mu{}_r\bar{\Gamma}^{n'}_\nu] = x_n|B_n|^2+y_nB_nC_n^++y_n^+B_nD_n^++z_nC_nB_n^++z_n^+|C_n|^2$$
$$+k_nC_nD_n^++k_n^+D_nB_n^++w_nD_nC_n^++w_n^+|D_n|^2. \tag{2.2.10}$$

By the same method from the above for the incoming and scattering moun evaluate the trace in Eq. (2.1.39)

$$Tr\left[{}_r\Delta^n_\mu\left(\frac{\not{P}+M}{2M}\right){}_r\ \bar{\Delta}^{n'}{}_\nu\left(\frac{\not{P}'+M}{2M}\right)\right] = \frac{1}{4M^2}Tr\left[{}_r\Delta^n_\mu(\not{P}+M){}_r\bar{\Delta}^{n'}_\nu(\not{P}'+M)\right]$$

$$= \frac{1}{4M^2}Tr\left[{}_r\Delta^n{}_\mu\not{P}{}_r\bar{\Delta}^{n'}_\nu\not{P}' +_r \Delta^n_\mu\not{P}{}_r\bar{\Delta}^{n'}_\nu M +_r \Delta^n_{\mu r}\bar{\Delta}^{n'}_\nu\not{P}'M+M^2_r\Delta^n_{\mu r}\bar{\Delta}^{n'}_\nu\right]$$

$$= \frac{1}{4M^2}\left[Tr\left[{}_r\Delta^n_\mu\not{P}{}_r\bar{\Delta}^{n'}_\nu\not{P}'\right]+MTr\left[{}_r\Delta^n_\mu\not{P}'{}_r\bar{\Delta}^{n'}_\nu\right]+MTr\left[{}_r\Delta^n_{\mu r}\bar{\Delta}^{n'}_\nu\not{P}'\right]-M^2Tr\left[{}_r\Delta^n_{\mu r}\bar{\Delta}^{n'}_\nu\right]\right]. \tag{2.2.11}$$

Due to trace identities involving γ matrices the trace of an odd number of γ matrices is vanishes. From the above Eq. (2.2.11) the terms which vanish are

$$MTr\left[{}_r\Delta^n{}_\mu \not{P}{}_r\bar{\Delta}^{n'}_\nu\right] = MTr\left[{}_r\Delta^n{}_{\mu r}\bar{\Delta n'}_\nu \not{P}'\right] = 0.$$

So that

$$Tr\left[{}_r\Delta^n_\mu\left(\frac{\not{P}+M}{2M}\right){}_r\,\bar{\Delta}^{n'}{}_\nu\left(\frac{\not{P}'+M}{2M}\right)\right] = \frac{1}{4M^2}\left[Tr_r\Delta^n_\mu\not{P}{}_r\bar{\Delta}^{n'}_\nu\not{P}'\right] + M^2Tr\left[{}_r\Delta^n_{\mu r}\bar{\Delta}^{n'}_\nu\right]\right]. \tag{2.2.12}$$

First find out

$$Tr[{}_r\Delta^n_\mu\not{P}{}_r\bar{\Delta}^{n'}_\nu\not{P}']. \tag{2.2.13}$$

substituting the vertex of the projectile pair from Eq. (2.1.34) in to Eq. (2.2.13)

$$\begin{aligned}
&Tr[{}_r\Delta^n_\mu\not{P}{}_r\bar{\Delta}^{n'}_\nu\not{P}'] \\
&= Tr\Bigg[\Bigg\{\left(\gamma^\mu + \frac{e^2a^2k_\mu\not{k}}{2(k.P)(k.P')}\right)F_{r-n} - \frac{e}{2}\Bigg[\left(\frac{\gamma^\mu\not{k}\not{a}_1}{(k.P)} + \frac{\not{a}_1\not{k}\gamma^\mu}{(k.P')}\right)G_{r-n} - \left(\frac{\gamma^\mu\not{k}\not{a}_2}{(k.P)} + \frac{\not{a}_2\not{k}\gamma^\mu}{(k.P')}\right)H_{r-n}\Bigg]\Bigg\}\not{P} \\
&\Bigg\{\left(-\gamma^\nu - \frac{e^2a^2k_\nu\not{k}}{2(k.P)(k.P')}\right)F^+_{r-n} + \frac{e}{2}\Bigg[\left(\frac{\not{a}_1\not{k}\gamma^\nu}{(k.P)} + \frac{\gamma^\nu\not{k}\not{a}_1}{(k.P')}\right)G^+_{r-n} + \left(\frac{\not{a}_2\not{k}\gamma^\nu}{(k.P)} + \frac{\gamma^\nu\not{k}\not{a}_2}{(k.P')}\right)H^+_{r-n}\Bigg]\Bigg\}\not{P}'\Bigg] \\
&= Tr\Bigg[\Bigg\{-\gamma^\mu\not{P}\gamma^\nu\not{P}' - (ea)^2\frac{k^\nu\gamma^\mu\not{P}\not{k}\not{P}' + k^\mu\not{k}\not{P}\gamma^\nu\not{P}'}{2(k.P)(k.P')} - (ea)^4k^\mu k^\nu\frac{\not{k}\not{P}\not{k}\not{P}'}{4((k.P)(k.P'))^2}\Bigg\}\left|F_{r-n}\right|^2 \\
&+\frac{e}{2}\Bigg\{\frac{\gamma^\mu\not{P}\not{a}_1\not{k}\gamma^\nu\not{P}'}{(k.P)} + \frac{\gamma^\mu\not{P}\gamma^\nu\not{k}\not{a}_1\not{P}'}{(k.P')} + \frac{(ea)^2k^\mu}{2(k.P)(k.P')}\left(\frac{\not{k}\not{P}\not{a}_1\not{k}\gamma^\nu\not{P}'}{(k.P)} + \frac{\not{k}\not{P}\gamma^\nu\not{k}\not{a}_1\not{P}'}{(k.P')}\right)\Bigg\}F_{r-n}G^+_{r-n} \\
&+\frac{e}{2}\Bigg\{\frac{\gamma^\mu\not{P}\not{a}_2\not{k}\gamma^\nu\not{P}'}{(k.P)} + \frac{\gamma^\mu\not{P}\gamma^\nu\not{k}\not{a}_2\not{P}'}{(k.P')} + \frac{(ea)^2k^\mu}{2(k.P)(k.P')}\left(\frac{\not{k}\not{P}\not{a}_2\not{k}\gamma^\nu\not{P}'}{(k.P} + \frac{\not{k}\not{P}\gamma^\nu\not{k}\not{a}_2\not{P}'}{k.P'}\right)\Bigg\}F_{r-n}H^+_{r-n} \\
&+\frac{e}{2}\Bigg\{\frac{\gamma^\mu\not{k}\not{a}_1\not{P}\gamma^\nu\not{P}'}{(k.P)} + \frac{\not{a}_1\not{k}\gamma^\mu\not{P}\gamma^\nu\not{P}'}{(k.P')} + \frac{(ea)^2k^\nu}{2(k.P)(k.P)}\left(\frac{\gamma^\mu\not{k}\not{a}_1\not{P}\not{k}\not{P}'}{(k.P)} + \frac{\not{a}_1\not{k}\gamma^\mu\not{P}\not{k}\not{P}'}{(kP')}\right)\Bigg\}G_{r-n}F^+_{r-n} \\
&-\frac{e^2}{4}\Bigg\{\frac{\gamma^\mu\not{k}\not{a}_1\not{P}\not{a}_1\not{k}\gamma^\nu\not{P}'}{(k.P)^2} + \frac{\gamma^\mu\not{k}\not{a}_1\not{P}\gamma^\nu\not{k}\not{a}_1\not{P}'}{(k.P)(k.P')} + \frac{\not{a}_1\not{k}\gamma^\mu\not{P}\not{a}_1\not{k}\gamma^\nu\not{P}'}{(k.P)(k.P')} + \frac{\not{a}_1\not{k}\gamma^\mu\not{P}\gamma^\nu\not{k}\not{a}_1\not{P}'}{(k.P')^2}\Bigg\}\left|G_{r-n}\right|^2 \\
&-\frac{e^2}{4}\Bigg\{\frac{\gamma^\mu\not{k}\not{a}_1\not{P}\not{a}_2\not{k}\gamma^\nu\not{P}'}{(k.P)^2} + \frac{\gamma^\mu\not{k}\not{a}_1\not{P}\gamma^\nu\not{k}\not{a}_2\not{P}'}{(k.P)(k.P')} + \frac{\not{a}_1\not{k}\gamma^\mu\not{P}\not{a}_2\not{k}\gamma^\nu\not{P}'}{(k.P)(k.P')} + \frac{\not{a}_1\not{k}\gamma^\mu\not{P}\gamma^\nu\not{k}\not{a}_2\not{P}'}{(k.P')^2}\Bigg\}G_{r-n}H^+_{r-n} \\
&+\frac{e}{2}\Bigg\{\frac{\gamma^\mu\not{k}\not{a}_2\not{P}\gamma^\nu\not{P}'}{k.P} + \frac{\not{a}_2\not{k}\gamma^\mu\not{P}\gamma^\nu\not{P}'}{k.P'} + \frac{(ea)^2k^\nu}{2(k.P)(k.P')}\left(\frac{\gamma^\mu\not{k}\not{a}_2\not{P}\not{k}\not{P}'}{k.P} + \frac{\not{a}_2\not{k}\gamma^\mu\not{P}\not{k}\not{P}'}{k.P'}\right)\Bigg\}H_{r-n}F^+_{r-n} \\
&-\frac{e^2}{4}\Bigg\{\frac{\gamma^\mu\not{k}\not{a}_2\not{P}\not{a}_1\not{k}\gamma^\nu\not{P}'}{(k.P)^2} + \frac{\gamma^\mu\not{k}\not{a}_2\not{P}\gamma^\nu\not{k}\not{a}_1\not{P}'}{(k.P)(k.P')} + \frac{\not{a}_2\not{k}\gamma^\mu\not{P}\not{a}_1\not{k}\gamma^\nu\not{P}'}{(k.P)(k.P')} + \frac{\not{a}_2\not{k}\gamma^\mu\not{P}\gamma^\nu\not{k}\not{a}_1\not{P}'}{(k.P')^2}\Bigg\}H_{r-n}G^+_{r-n}
\end{aligned}$$

$$-\frac{e^2}{4}\left\{\frac{\gamma^\mu \not{k}\not{a}_2\not{P}\not{a}'_2\not{k}'\gamma^\nu\not{P}'}{(k.P)^2}+\frac{\gamma^\mu \not{k}\not{a}_2\not{P}\gamma^\nu\not{k}'\not{a}'_2\not{P}'}{(k.P)(k.P')}+\frac{\not{a}_2\not{k}\gamma^\mu\not{P}\not{a}'_2\not{k}'\gamma^\nu\not{P}'}{(k.P)(k.P')}+\frac{\not{a}_2\not{k}\gamma^\mu\not{P}\gamma^\nu\not{k}'\not{a}'_2\not{P}'}{(k.P')^2}\right\}\left|H_{r-n}\right|^2\Bigg] \tag{2.2.14}$$

Let as evaluate each trace of Eq. (2.2.14) using trace technology

$$Tr\left[-\gamma^\mu\not{P}\gamma^\nu\not{P}'-(ea)^2\frac{k^\nu\gamma^\mu\not{P}\not{k}'\not{P}'+k^\mu\not{k}\not{P}\gamma^\nu\not{P}'}{2(k.P)(k.P')}-(ea)^4k^\mu k^\nu\frac{\not{k}\not{P}\not{k}'\not{P}'}{4((k.P)(k.P'))^2}\right]\left|F_{r-n}\right|^2=L_n\left|F_{r-n}\right|^2,$$

$$\frac{e}{2}\,Tr\left[\frac{\gamma^\mu\not{P}\not{a}'_1\not{k}'\gamma^\nu\not{P}'}{(k.P)}+\frac{\gamma^\mu\not{P}\gamma^\nu\not{k}'\not{a}'_1\not{P}'}{(k.P)}\right.$$
$$\left.+\frac{(ea)^2k^\mu}{2(k.P)(k.P')}\left(\frac{\not{k}\not{P}\not{a}'_1\not{k}'\gamma^\nu\not{P}'}{(k.P)}+\frac{\not{k}\not{P}\gamma^\nu\not{k}'\not{a}'_1\not{P}'}{(k.P')}\right)\right]F_{r-n}G^+_{r-n}=L^+_nF_{r-n}G^+_{r-n},$$

$$\frac{e}{2}\,Tr\left[\frac{\gamma^\mu\not{P}\not{a}'_2\not{k}'\gamma^\nu\not{P}'}{(k.P)}+\frac{\gamma^\mu\not{p}_+\gamma^\nu\not{k}'\not{a}'_2\not{p}_-}{(k.P')}\right.$$
$$\left.+\frac{(ea)^2k^\mu}{2(k.P)(k.P')}\left(\frac{\not{k}\not{P}\not{a}'_2\not{k}'\gamma^\nu\not{P}'}{(k.P)}+\frac{\not{k}\not{P}\gamma^\nu\not{k}'\not{a}'_2\not{P}'}{(k.P')}\right)\right]F_{r-n}H^+_{r-n}=M_nF_{r-n}H^+_{r-n},$$

$$\frac{e}{2}\,Tr\left[\frac{\gamma^\mu\not{k}\not{a}_1\not{P}\gamma^\nu\not{P}'}{(k.P)}+\frac{\not{a}_1\not{k}\gamma^\mu\not{P}\gamma^\nu\not{P}'}{(k.P')}\right.$$
$$\left.+\frac{(ea)^2k^\nu}{2(k.P)(k.P')}\left(\frac{\gamma^\mu\not{k}\not{a}_1\not{P}\not{k}'\not{P}'}{(k.P)}+\frac{\not{a}_1\not{k}\gamma^\mu\not{P}\not{k}'\not{P}'}{(k.P')}\right)\right]G_{r-n}F^+_{r-n}=M^+_nG_{r-n}F^+_{r-n},$$

$$-\frac{e^2}{4}\,Tr\left[\frac{\gamma^\mu\not{k}\not{a}_1\not{P}\not{a}'_1\not{k}'\gamma^\nu\not{P}'}{(k.P)^2}+\frac{\gamma^\mu\not{k}\not{a}_1\not{P}\gamma^\nu\not{k}'\not{a}'_1\not{P}'}{(k.P)(k.P')}+\frac{\not{a}_1\not{k}\gamma^\mu\not{P}\not{a}'_1\not{k}'\gamma^\nu\not{P}'}{(k.P)(k.P')},\right.$$
$$\left.+\frac{\not{a}_1\not{k}\gamma^\mu\not{P}\gamma^\nu\not{k}'\not{a}'_1\not{P}'}{(k.P')^2}\right]\left|G_{r-n}\right|^2=P_n\left|G_{r-n}\right|^2,$$

$$-\frac{e^2}{4}\,Tr\left[\frac{\gamma^\mu\not{k}\not{a}_1\not{P}\not{a}'_2\not{k}'\gamma^\nu\not{P}'}{(k.P)^2}+\frac{\gamma^\mu\not{k}\not{a}_1\not{P}\gamma^\nu\not{k}'\not{a}'_2\not{P}'}{(k.P)(k.P')}+\frac{\not{a}_1\not{k}\gamma^\mu\not{P}\not{a}'_2\not{k}'\gamma^\nu\not{P}'}{(k.P)(k.P')}\right.$$
$$\left.+\frac{\not{a}_1\not{k}\gamma^\mu\not{P}\gamma^\nu\not{k}'\not{a}'_2\not{P}'}{(k.P')^2}\right]G_{r-n}H^+_{r-n}=P^+_nG_{r-n}H^+_{r-n},$$

$$\frac{e}{2}\,Tr\left[\frac{\gamma^\mu\not{k}\not{a}_2\not{P}\gamma^\nu\not{P}'}{(k.P)}+\frac{\not{a}_2\not{k}\gamma^\mu\not{P}\gamma^\nu\not{P}'}{(k.P')}\right.$$
$$\left.+\frac{(ea)^2k^\nu}{2(k.P)(k.P')}\left(\frac{\gamma^\mu\not{k}\not{a}_2\not{P}\not{k}'\not{P}'}{(k.P)}+\frac{\not{a}_2\not{k}\gamma^\mu\not{P}\not{k}'\not{P}'}{(k.P')}\right)\right]H_{r-n}F^+_{r-n}=S_nH_{r-n}F^+_{r-n},$$

$$-\frac{e^2}{4}\,Tr\left[\frac{\gamma^\mu\not{k}\not{a}_2\not{P}\not{a}'_1\not{k}'\gamma^\nu\not{P}'}{(k.P)^2}+\frac{\gamma^\mu\not{k}\not{a}_2\not{P}\gamma^\nu\not{k}'\not{a}'_1\not{P}'}{(k.P)(k.P')}+\frac{\not{a}_2\not{k}\gamma^\mu P\not{a}'_1\not{k}'\gamma^\nu\not{P}'}{(k.P)(k.P')}\right.$$
$$\left.+\frac{\not{a}_2\not{k}\gamma^\mu\not{P}\gamma^\nu\not{k}'\not{a}'_1\not{P}'}{(k.P')^2}\right]H_{r-n}G^+_{r-n}=S^+_nH_{r-n}G^+_{r-n},and$$

$$-\frac{e^2}{4}\left[\frac{\gamma^\mu\not{k}\not{a}_2\not{P}\not{a}'_2\not{k}'\gamma^\nu\not{P}'}{(k.P)^2}+\frac{\gamma^\mu\not{k}\not{a}_2\not{P}\gamma^\nu\not{k}'\not{a}'_2\not{P}'}{(k.P)(k.P')}+\frac{\not{a}_2\not{k}\gamma^\mu\not{P}\not{a}'_2\not{k}'\gamma^\nu\not{P}}{(k.P)(k.P')}\right.$$

$$+\frac{\not{a}_2\not{k}\gamma^\mu\not{P}\gamma^\nu\not{k}'\not{a}_2\not{P}}{(k.P')^2}\Big]\big|H_{r-n}\big|^2 = T_n\big|H_{r-n}\big|^2.$$

Where the abbreviations

$$L_n = 4\Big[(-P.P') + 2PP' - (ea)^2k\frac{P(k.P') - k(P.P') + P'(k.P)}{(k.P)(kP')} - (ea)^4k^2\frac{2(k.P)(k.P') - k^2(P.P')}{4[(k.P)(k.P')]^2}\Big],$$

$$L_n^+ = 2e\Big[\Big\{(k.a_1)[2PP' - (P'.P)] + (P.a_1)[2kP') - (P'.k)] + (P.k)[-P'a_1] + (a_1.P')\Big\}\frac{1}{(k.P)} + \Big\{(a_1.P')[2kP - (k.P)] + (k.P')[(P.a_1) - 2Pa_1] + (k.a_1)[(P.P') - 2PP']\Big\}\frac{1}{(k.P')} + (ea)^2k\Big\{2(k.P)[k(P'.a_1) - a_-(k.P')] + 2P(k.a_1)(k.P') + k^2[(P.a_1)P' - P(P'.a_1) + a_1(P.P')]\Big\}\frac{1}{(k.P)^2(k.P')} + (ea)^2k\Big\{2(P.k)[P'(k.a_1) - (P'.k)] + k(P.a_1)[2(P'.k) - kP'] + k(P.P')[ka_1 - 2(k.a_1)] + k^2P'(P.a_1)\Big\}\frac{1}{2(k.P)(k.P')^2}\Big],$$

$$M_n = 2e\Big[\Big\{(k.a_2)[2PP' - (P'.P)] + (P.a_2)[2kP') - (P'.k)] + (P.k)[-2P'a_2 + (a_2.P')]\Big\}\frac{1}{(k.p_+)} + \Big\{(a_2.P')[2kP - (k.P)] + (k.P')[(P.a_2) - 2Pa_1] + (k.a_2)[(P.P' - 2PP']\Big\}\frac{1}{(k.P')} + (ea)^2k\Big\{2(k.P)[k(P'.a_2) - a_-(k.P')] + 2P(k.a_2)(k.P') + k^2[(P.a_2)P' - P(P'.a_2) + a_2(P.P')]\Big\}\frac{1}{(k.P)^2(k.P')} + (ea)^2k\Big\{2(P.k)[P'(k.a_2) - (P'.k)] + k(P.a_2)[2(P'.k) - kP'] + k(P.P')[ka_2 - 2(k.a_2)] + k^2P'(P.a_2)\Big\}\frac{1}{2(k.P)(k.P')^2}\Big],$$

$$M_n^+ = 2e\Big[\Big\{(P.a_1)[2kP' - (k.P')] + (k.P')[(P.a_1) - 2Pa_1] + (k.a_1)[2PP' - (P.P')]\Big\}\frac{1}{(k.P)} + \Big\{(a_1.k)[2PP' - (PP')] - 2a_1P(k.P') + kP'(a_1.P) + (a_1.P')[2kP - (k.P)]\Big\}\frac{1}{(k.P')} + (ea)^2k\Big\{2k(a_1.P')(k.P) - 2P(a_1.k)(k.P') - 2(a_1.k)(P.P') - 2a_1(k.P)(k.P')\Big\}\frac{1}{2(k.P)^2(k.P')]} - (ea)^2k\Big\{2k(a_1.k)(P'.P) + 2P'(a_1.k)(k.P) + 2(k.P')[k(a_1.P) - a_1(k.P)] + k^2[a_1(P.P') - P'(a_1.P) + P(a_1.P')]\Big\}\frac{1}{2(k.P)(k.P')^2}\Big],$$

$$
\begin{aligned}
P_n = e^2\Bigg[&-\Bigg\{2k^2(a_1.P)[a_1P' + (a_1.P')] + 2a_1^2(k.P)[(k.P') - kP'] \\
&+ (a_1k)^2PP' - a_1^2kP(k.P') + k^2a_1P(a_1.P') + (a_1.k)(k.P)[6P'k - 4(k.P'k)] \\
&+ ka_1^2(k.Pk) - k^2a_1(a_1.P)\Bigg\}\frac{1}{(k.P)^2} \; -\Bigg\{4(a_1.P')(a_1.k)[2Pk - (k.P)] \\
&+ k^2(a_1.P')[2(a_1.P) - a_1P] + \; 2a_1^2[(k.P')(-2kP + (k.P)) - k^2(PP' + (PP'))]\Bigg\}\frac{1}{(k.P')^2} \\
&-\Bigg\{4k^2(a_1.P)(a_1.P') - 3ka_1(a_1.P')(k.P) + 4ka_1^2P(k.P') + 3ka_1^2P'(k.P) \\
&- 5a_1P'(k.P)(a_1.k) - 7a_1k(k.P')(a_1.P) + 5a_1k(a_1.k)(P.P') - 3a_1^2k^2(P.P') \\
&- 8a_1P(a_1.k)(k.P') + 7PP'(a_1.k)^2 + 3Pa_1k^2(a_1.P') - 4k^2a_1^2PP' \\
&+ 6(k.P)(a_1.k)(k.P') - k^2(a_1.P')(a_1.P) + 6a_1^2(k.P')(k.P)3p_{-}a_1k^2(a_1.P) \\
&- 4P'a_1(k.P')(a_1.k) + 6P'k(a_1.P)(a_1.k) - (a_1.k)^2(P.P') - 2kP'(a_1.P)(a_1.k) \\
&+ 2(k.P')(a_1.P)(a_1.k) + 2k^2P'a_1(a_1.P) - 2a_1^2kP(k.P' - 2kP(a_1.k)(a_1.P')\Bigg\}\frac{1}{(k.P)(k.P')}\Bigg],
\end{aligned}
$$

$$
\begin{aligned}
P_n^+ = e^2\Bigg[&-\Bigg\{2(a_1.P)(k.a_2)\big[2kP' - (k.P')\big] + 2(a_2.P)(k.a_1)\big[2kP' - (k.P')\big] \\
&+ k^2(a_1.a_2)[2PP' - (P.P')] + k^2(a_2.P')[(a_1.P) - 2a_1P] + k^2(a_1.P)\big[-2a_2P' - (a_2.P')\big] \\
&- k^2a_1a_2(P'.P)\Bigg\}\frac{1}{(k.P)^2} - \Bigg\{(a_1.a_2)\big[-2kP(k.P') + k^2\big(2PP' - P.P'\big)\big) - 2kP(k.P') \\
&+ 2(k.P)(k.P')\big] + (a_1.k)\big(2kP(P'.a_2) - 2(k.P)(P'.a_2)\big) - a_1a_2k^2(P.P') \\
&+ k^2(a_1.P)(a_2.P')\Bigg\}\frac{1}{(k.P')^2} - \Bigg\{(k.P)\big[-4a_1P'(k.a_2) + 2(a_1.P')(k.a_2) - 5a_2P'(k.a_1) \\
&+ 2(a_2.P'(k.a_1) + a_1a_2(k.P') - 4a_1k(a_2.P') - 2a_2k(a_1.P')\big] + 2k^2\big[a_2P(a_1.P') \\
&- 2a_2a_1(P'.P) + a_2P'(a_1.P) - (a_1.P)(a_2.P') - (a_2.a_1)(P.P') - 2(a_2.a_1)PP' \\
&+ (a_1.P')(a_2.P)\big] + a_1k\big[kP'(a_2.P) + (k.a_2)(P'.P) + -4(a_2.P)(P'.k)\big] + \\
&2(k.P')\big[-2a_2P(k.a_1) + (a_2.P)(k.a_1) - (a_2.a_1)(k.P) - a_1P(k.a_2) + (a_1.P)(k.a_2) \\
&+ -Pk(a_2.a_1)\big]2(a_2.k)\big[3PP'(a_1.k) - 2(PP')(a_1.k) + (P.a_1)P'k\big] \\
&+ 3k\big[-a_2(a_1.P)(P'.k) + P'(a_2.a_1)(k.P)\big]\Bigg\}\frac{1}{(k.P)(k.P')}\Bigg],
\end{aligned}
$$

$$S_n = 2e\Bigg[\Bigg\{(P.a_2)[2kP' - (k.P')] + (k.P')[(P.a_2) - 2Pa_2] + (k.a_2)[2PP'$$
$$- (P.P')]\Bigg\}\frac{1}{(k.P} + \Bigg\{(a_2.k)[2PP' - (P.P')] - 2a_2P(k.P') + kP'(a_2.P(a_2.P')[2kP$$
$$- (k.P)]\Bigg\}\frac{1}{(kP')} + (ea)^2k\Bigg\{2k(a_2.P')(k.P) - 2P(a_2.k)(k.P') - 2(a_2.k)(P.P')$$
$$- 2a_2(k.P)(k.P')\Bigg\}\frac{1}{2(kP)^2(kP)} + (ea)^2k\Bigg\{2k(a_2.k)(P.P') + 2P'(a_2.k)(k.P)$$
$$+ 2(k.P')[k(a_2.P) - a_2(k.P)] + k^2[a_2(P.P') - P'(a_2.P) + P(a_2.P')]\Bigg\}\frac{1}{2(k.P)(k.P')^2}\Bigg],$$

$$S_n^+ = e^2\Bigg[- \Bigg\{2(a_2.P)(k.a_1)\big[2kp_- - (k.P')\big] + 2(a_1.P)(k.a_2)\big[2kP' - (k.P')\big]$$
$$+ k^2(a_2.a_1)[2PP' - (P.P')] + k^2(a_1.P')[(a_2.P) - 2a_2P] + k^2(a_2.P)\big[- 2a_1P'$$
$$- (a_1.P')\big] - k^2a_2a_2(P.P')\Bigg\}\frac{1}{(k.P)^2} - \Bigg\{(a_1.a_2)\big[- 2kP(k.P') + k^2\big(2PP' - (P.P')\big)$$
$$- 2kP(k.P') + 2(k.P)(k.P')\big] + (a_2.k)\big(2kP(P'.a_1) - 2(k.P)(P'.a_1)\big) - a_2a_2k^2(P.P')$$
$$+ k^2(a_2.P)(a_2.P')\Bigg\}\frac{1}{(k.P')^2} - \Bigg\{(k.P)\big[- 4a_2P'(k.a_1) + 2(a_2.P')(k.a_1) - 5a_2P'(k.a_2)$$
$$+ 2(a_1.P'(k.a_2) + a_2a_2(k.P') - 4a_2k(a_2.P') - 2a_1k(a_2.P')\big] + 2k^2\big[a_1P'(a_2.P')$$
$$- 2a_1a_2(P.P') + a_1P'(a_2.P) - (a_2.P)(a_1.P') - (a_2.a_1)(P.P') - 2(a_2.a_1)PP'$$
$$+ (a_2.P')(a_1.P)\big] + a_1k\big[kP'(a_1.P) + (k.a_1)(P'.P) + -4(a_1.P)(P'.k)\big]$$
$$+ 2(k.P')\big[- 2a_1P(k.a_2) + (a_1.P)(k.a_2) - (a_1.a_2)(k.P) - a_2P(k.a_1) + (a_2.P)(k.a_1)$$
$$- Pk(a_1.a_2)\big] + 2(a_1.k)\big[3PP'(a_2.k) - 2(P.P')(a_2.k) + (P.a_2)P'k\big]$$
$$+ 3k\big[- a_1(a_1.P)(P'.k) + P'(a_2.a_1)(k.P)\big]\Bigg\}\frac{1}{(k.P)(k.P')}\Bigg], and$$

$$T_n = e^2\Big[-\Big\{ 2k^2(a_2.P)[a_2P' + (a_2.P')] + 2a_2^2(k.P)[(k.P') - kP'] + (a_2k)^2PP'$$
$$-a_2^2kP(k.P') + k^2a_2P(a_2.P') + (a_2.k)(k.P)[6P'k - 4(k.P'k)] + ka_2^2(k.Pk)$$
$$-k^2a_2(a_2.P)\Big\}\frac{1}{(k.P)^2} - \Big\{4(a_2.P')(a_2.k)[2Pk - (k.P)] + k^2(a_2.P')[2(a_2.P) - a_2P]$$
$$+2a_2^2[(k.P')(-2kP + (k.P)) - k^2(PP' + (P.P')]\Big\}\frac{1}{(k.P')^2} - \Big\{4k^2(a_2.P)(a_2.P')$$
$$-3ka_2(a_2.P')(k.P) + 4ka_2^2P(k.P') + 3ka_2^2P'(k.P) - 5a_1P'(k.P)(a_1.k)$$
$$-7a_2k(k.P')(a_2.P) + 5a_2k(a_2.k)(P.P') - 3a_2^2k^2(P.P') - 8a_2P(a_2.k)(k.P') + 7PP'(a_2.k)^2$$
$$+3Pa_2k^2(a_2.P') - 4k^2a_2^2PP' + 6(k.P)(a_2.k)(k.P') - k^2(a_2.P')(a_2.P) + 6a_2^2(k.P')(k.P) +$$
$$3P'a_2k^2(a_2.P) - 4P'a_2(k.P)(a_2.k) + 6P'k(a_2.P)(a_2.k) - (a_2.k)^2(P.P') - 2kP'(a_2.P)(a_2.k)$$
$$+2(k.P')(a_2.P)(a_2.k) + 2k^2P'a_2(a_2.P) - 2a_2^2kP(k.P' - 2kP(a_2.k)(a_2.P')\Big\}\frac{1}{(k.P)(k.P')}\Big].$$

Therefor

$$Tr[{}_r\Gamma^n_\mu \not{P}{}_r\bar{\Gamma}^{n'}_\nu \not{P}'] = L_n|F_{r-n}|^2 + L_n^+F_{r-n}G^+_{r-n} + M_nF_{r-n}H^+_{r-n} + M_n^+G_{r-n}F^+_{r-n}$$
$$+ P_n|G_{r-n}|^2 + P_n^+G_{r-n}H^+_{r-n} + S_nH_{r-n}F^+_{r-n} + S_n^+H_{r-n}G^+_{r-n} + T_{r-n}|H_{r-n}|^2. \quad (2.2.15)$$

Secondly we evaluate

$$Tr[{}_r\Delta^n_\mu {}_r\bar{\Delta}^{n'}_\nu]. \quad (2.2.16)$$

Substituting the vertex of the projectile from Eq. (2.1.34) in to Eq. (2.2.16)

$$Tr[{}_r\Delta^n_\mu {}_r\bar{\Delta}^{n'}_\nu]$$
$$= Tr\Bigg[\Bigg\{\left(\gamma^\mu + \frac{e^2a^2k_\mu\not{k}}{2(k.P)(k.P')}\right)F_{r-n} - \frac{e}{2}\Bigg[\left(\frac{\gamma^\mu\not{k}\not{a}_1}{(k.P)} + \frac{\not{a}_1\not{k}\gamma^\mu}{(k.P')}\right)G_{r-n} - \left(\frac{\gamma^\mu\not{k}\not{a}_2}{(k.P)} + \frac{\not{a}_2\not{k}\gamma^\mu}{(k.P')}\right)H_{r-n}\Bigg]\Bigg\}$$
$$\Bigg\{\left(-\gamma^\nu - \frac{e^2a^2k_\nu\not{k}'}{2(k.P)(k.P')}\right)F^+_{r-n} + \frac{e}{2}\Bigg[\left(\frac{\not{a}_1'\not{k}'\gamma^\nu}{(k.P)} + \frac{\gamma^\nu\not{k}'\not{a}_1'}{(k.P')}\right)G^+_{r-n} + \left(\frac{\not{a}_2'\not{k}'\gamma^\nu}{(k.P)} + \frac{\gamma^\nu\not{k}'\not{a}_2'}{(k.P')}\right)H^+_{r-n}\Bigg]\Bigg\}\Bigg]$$
$$= Tr\Bigg[\Bigg\{-\gamma^\mu\gamma^\nu - (ea)^2\frac{k^\nu\gamma^\mu\not{k}' + k^\mu\not{k}\gamma^\nu}{2(k.P)(k.P')} - (ea)^4k^\mu k^\nu\frac{\not{k}\not{k}'}{4((k.P)(k.P)^2}\Bigg\}\big|F_{r-n}\big|^2$$
$$+ \frac{e}{2}\Bigg\{\frac{\gamma^\mu\not{a}_1'\not{k}'\gamma^\nu}{(k.P)} + \frac{\gamma^\mu\gamma^\nu\not{k}'\not{a}_1'\not{p}_-}{(k.P')} + \frac{(ea)^2k^\mu}{2(k.P)(k.P')}\left(\frac{\not{k}\not{a}_1'\not{k}'\gamma^\nu}{(k.P)} + \frac{\not{k}\gamma^\nu\not{k}'\not{a}_1'}{(k.P')}\right)\Bigg\}F_{r-n}G^+_{r-n}$$
$$+\frac{e}{2}\Bigg\{\frac{\gamma^\mu\not{a}_2'\not{k}'\gamma^\nu}{(k.P)} + \frac{\gamma^\mu\gamma^\nu\not{k}'\not{a}_2'}{(k.P')} + \frac{(ea)^2k^\mu}{2(k.P)(k.P')}\left(\frac{\not{k}\not{a}_2'\not{k}'\gamma^\nu}{(k.P)} + \frac{\not{k}\gamma^\nu\not{k}'\not{a}_2'}{(k.P')}\right)\Bigg\}F_{r-n}H^+_{r-n}$$

$$+\frac{e}{2}\left\{\frac{\gamma^\mu \not{k}\not{a}_1\gamma^\nu}{(k.P)}+\frac{\not{a}_1\not{k}\gamma^\mu\gamma^\nu}{(k.P')}+\frac{(ea)^2k^\nu}{2(k.P)(k.P')}\left(\frac{\gamma^\mu\not{k}\not{a}_1\not{k}'}{(k.P)}+\frac{\not{a}_1\not{k}\gamma^\mu\not{k}'}{(k.P')}\right)\right\}G_{r-n}F^+_{r-n}$$

$$-\frac{e^2}{4}\left\{\frac{\gamma^\mu\not{k}\not{a}_1\not{a}_1'\not{k}'\gamma^\nu}{(k.P)^2}+\frac{\gamma^\mu\not{k}\not{a}_1\gamma^\nu\not{k}'\not{a}_1'}{(k.P)(k.P')}+\frac{\not{a}_1\not{k}\gamma^\mu\not{a}_1'\not{k}'\gamma^\nu}{(k.P)(k.P')}+\frac{\not{a}_1\not{k}\gamma^\mu\gamma^\nu\not{k}'\not{a}_1'}{(k.P')^2}\right\}\left|G_{r-n}\right|^2$$

$$-\frac{e^2}{4}\left\{\frac{\gamma^\mu\not{k}\not{a}_1\not{a}_2'\not{k}'\gamma^\nu}{(k.P)^2}+\frac{\gamma^\mu\not{k}\not{a}_1\gamma^\nu\not{k}'\not{a}_2'}{(k.P)(k.P')}+\frac{\not{a}_1\not{k}\gamma^\mu\not{a}_2'\not{k}'\gamma^\nu}{(k.P)(k.P')}+\frac{\not{a}_1\not{k}\gamma^\mu\gamma^\nu\not{k}'\not{a}_2'}{(k.P')^2}\right\}G_{r-n}H^+_{r-n}$$

$$+\frac{e}{2}\left\{\frac{\gamma^\mu \not{k}\not{a}_2\gamma^\nu}{(k.P)}+\frac{\not{a}_2\not{k}\gamma^\mu\gamma^\nu}{(k.P')}+\frac{(ea)^2k^\nu}{2(k.P)(k.P')}\left(\frac{\gamma^\mu\not{k}\not{a}_2\not{k}'}{(k.P)}+\frac{\not{a}_2\not{k}\gamma^\mu\not{k}'}{(k.P')}\right)\right\}H_{r-n}F^+_{r-n}$$

$$-\frac{e^2}{4}\left\{\frac{\gamma^\mu\not{k}\not{a}_2\not{a}_1'\not{k}'\gamma^\nu}{(k.P)^2}+\frac{\gamma^\mu\not{k}\not{a}_2\gamma^\nu\not{k}'\not{a}_1'}{(k.P)(k.P')}+\frac{\not{a}_2\not{k}\gamma^\mu\not{a}_1'\not{k}'\gamma^\nu}{(k.P)(k.P')}+\frac{\not{a}_2\not{k}\gamma^\mu\gamma^\nu\not{k}'\not{a}_1'}{(k.P')^2}\right\}H_{r-n}G^+_{r-n}$$

$$-\frac{e^2}{4}\left\{\frac{\gamma^\mu\not{k}\not{a}_2\not{a}_2'\not{k}'\gamma^\nu}{(k.P)^2}+\frac{\gamma^\mu\not{k}\not{a}_2\gamma^\nu\not{k}'\not{a}_2'}{(k.P)(k.P')}+\frac{\not{a}_2\not{k}\gamma^\mu\not{a}_2'\not{k}'\gamma^\nu}{(k.P)(k.P')}+\frac{\not{a}_2\not{k}\gamma^\mu\gamma^\nu\not{k}'\not{a}_2'}{(k.P')^2}\right\}\left|H_{r-n}\right|^2\Bigg]. \quad (2.2.17)$$

Let as evaluate each trace in Eq. (2.2.17) using trace technology

$$Tr\left[-\gamma^\mu\gamma^\nu-(ea)^2\frac{k^\nu\gamma^\mu\not{k}'+k^\mu\not{k}\gamma^\nu}{2(k.P)(k.P')}-(ea)^4k^\mu k^\nu\frac{\not{k}\not{k}'}{4((k.P)(k.P'))^2}\right]\left|F_{r-n}\right|^2=X_n\left|F_{r-n}\right|^2,$$

$$\frac{e}{2}\,Tr\left[\frac{\gamma^\mu\not{a}_1'\not{k}'\gamma^\nu}{(k.P)}+\frac{\gamma^\mu\gamma^\nu\not{k}'\not{a}_1'}{(k.P')}+\frac{(ea)^2k^\mu}{2(k.P)(k.P')}\left(\frac{\not{k}\not{a}_1'\not{k}'\gamma^\nu}{(k.P)}+\frac{\not{k}\gamma^\nu\not{k}'\not{a}_1'}{(k.P')}\right)\right]F_{r-n}G^+_{r-n}=Y_nF_{r-n}G^+_{r-n},$$

$$\frac{e}{2}\,Tr\left[\frac{\gamma^\mu\not{a}_2'\not{k}'\gamma^\nu}{(k.P)}+\frac{\gamma^\mu\gamma^\nu\not{k}'\not{a}_2'}{(k.P')}+\frac{(ea)^2k^\mu}{2(k.P)(k.P')}\left(\frac{\not{k}\not{a}_2'\not{k}'\gamma^\nu}{(k.P)}+\frac{\not{k}\gamma^\nu\not{k}'\not{a}_2'}{(k.P')}\right)\right]F_{r-n}H^+_{r-n}=Y^+_nF_{r-n}H^+_{r-n},$$

$$\frac{e}{2}\,Tr\left[\frac{\gamma^\mu\not{k}\not{a}_1\gamma^\nu}{(kP)}+\frac{\not{a}_1\not{k}\gamma^\mu\gamma^\nu}{(k.P')}+\frac{(ea)^2k^\nu}{2(k.P)(k.P')}\left(\frac{\gamma^\mu\not{k}\not{a}_1\not{k}'}{(k.P)}+\frac{\not{a}_1\not{k}\gamma^\mu\not{k}'}{(k.P')}\right)\right]G_{r-n}F^+_{r-n}=Z_nG_{r-n}F^+_{r-n},$$

$$-\frac{e^2}{4}\,Tr\left[\frac{\gamma^\mu\not{k}\not{a}_1\not{a}_1'\not{k}'\gamma^\nu}{(k.P)^2}+\frac{\gamma^\mu\not{k}\not{a}_1\gamma^\nu\not{k}'\not{a}_1'}{(k.P)(k.P')}+\frac{\not{a}_1\not{k}\gamma^\mu\not{a}_1'\not{k}'\gamma^\nu}{(k.P)(k.P')}+\frac{\not{a}_1\not{k}\gamma^\mu\gamma^\nu\not{k}'\not{a}_1'}{(k.P')^2}\right]\left|G_{r-n}\right|^2=Z^+_n\left|G_{r-n}\right|^2,$$

$$-\frac{e^2}{4}\,Tr\left[\frac{\gamma^\mu\not{k}\not{a}_1\not{a}_2'\not{k}'\gamma^\nu}{(k.P)^2}+\frac{\gamma^\mu\not{k}\not{a}_1\gamma^\nu\not{k}'\not{a}_2'}{(k.P)(k.P')}+\frac{\not{a}_1\not{k}\gamma^\mu\not{a}_2'\not{k}'\gamma^\nu}{(k.P)(k.P')}+\frac{\not{a}_1\not{k}\gamma^\mu\gamma^\nu\not{k}'\not{a}_2'}{(k.P')^2}\right]G_{r-n}H^+_{r-n}=K_nG_{r-n}H^+_{r-n},$$

$$\frac{e}{2}\,Tr\left[\frac{\gamma^\mu\not{k}\not{a}_2\gamma^\nu}{(k.P)}+\frac{\not{a}_2\not{k}\gamma^\mu\gamma^\nu}{(k.P)}+\frac{(ea)^2k^\nu}{2(k.P)(k.P')}\left(\frac{\gamma^\mu\not{k}\not{a}_2\not{k}'}{(k.P)}+\frac{\not{a}_2\not{k}\gamma^\mu\not{k}'}{(k.P')}\right)\right]H_{r-n}F^+_{r-n}=K^+_nH_{r-n}F^+_{r-n},$$

$$-\frac{e^2}{4}\left[\frac{\gamma^\mu\not{k}\not{a}_2\not{a}_1'\not{k}'\gamma^\nu}{(k.P)^2}+\frac{\gamma^\mu\not{k}\not{a}_2\gamma^\nu\not{k}'\not{a}_1'}{(k.P)(k.P')}+\frac{\not{a}_2\not{k}\gamma^\mu\not{a}_1'\not{k}'\gamma^\nu}{(k.P)(k.P')}+\frac{\not{a}_2\not{k}\gamma^\mu\gamma^\nu\not{k}'\not{a}_1'}{(k.P')^2}\right]H_{r-n}G^+_{r-n}=W_nH_{r-n}G^+_{r-n}\ and$$

$$-\frac{e^2}{4}\left[\frac{\gamma^\mu\not{k}\not{a}_2\not{a}_2'\not{k}'\gamma^\nu}{(k.P)^2}+\frac{\gamma^\mu\not{k}\not{a}_2\gamma^\nu\not{k}'\not{a}_2'}{(k.P)(k.P')}+\frac{\not{a}_2\not{k}\gamma^\mu\not{a}_2'\not{k}'\gamma^\nu}{(k.P)(k.P')}+\frac{\not{a}_2\not{k}\gamma^\mu\gamma^\nu\not{k}'\not{a}_2'}{(k.P')^2}\right]\left|H_{r-n}\right|^2=W^+_n\left|H_{r-n}\right|^2.$$

Where the abbreviations

$$X_n=4\left[-1-\frac{(eak)^2}{(k.P)(k.P')}-\frac{(eak)^4}{[2(k.P)(k.P')]^2}\right],$$

$$Y_n = \frac{e}{2}\left[4(a_1.k)\left(\frac{1}{(k.P)}+\frac{1}{(k.P')}\right)+(ea)^2\left(\frac{2\left[k(a_1.k)-a_1k^2\right]}{(k.P)^2(k.P')}+\frac{2k(k.a_1)-a_1k^2}{2(k.P)(k.P')^2}\right)\right],$$

$$Y_n^+ = \frac{e}{2}\left[4(a_2.k)\left(\frac{1}{(k.P)}+\frac{1}{(k.P')}\right)+(ea)^2\left(\frac{2\left[k(a_2.k)-a_2k^2\right]}{(k.P)^2(k.P')}+\frac{2k(k.a_2)-a_2k^2}{2(k.P)(k.P')^2}\right)\right],$$

$$Z_n = 2e\left[\frac{(k.a_1)}{(k.P)}+(ea)^2k\frac{2k(k.a_1)-a_1k^2}{(k.P)^2(k.P')}+\frac{(k.a_1)}{(k.P')}+(ea)^2k\frac{2k(k.a_1)-a_1k^2}{(k.P)(k.P')^2}\right],$$

$$Z_n^+ = e^2\left[\frac{-(a_1k)^2}{(k.P)^2}-\frac{4(a_1.k)^2-8ka_1(a_1.k)+4k^2a_1^2}{(k.P)(k.P')}-\frac{-(a_1k)^2}{(k.P')^2}\right],$$

$$K_n = e^2\Bigg[\frac{-k^2(a_1.a_2)}{(k.P)^2}-\frac{4(k.a_2)(k.a1)-2ka_2(a_1.k)-6ka_1(a_2.k)+(a_1.a_2)k^2+2a_1a_2k^2}{(k.P)(k.P')}$$
$$-\frac{k^2(a_1.a_2)}{(k.P')^2}\Bigg],$$

$$K_n^+ = 2e\left[\frac{(k.a_2)}{(k.P)}+(ea)^2k\frac{2k(k.a_2)-a_2k^2}{(k.P)^2(k.P')}+\frac{(k.a_2)}{(k.P')}+(ea)^2k\frac{2k(k.a_2)-a_2k^2}{(k.P)(k.P')^2}\right],$$

$$W_n = e^2\Bigg[\frac{-k^2(a_1.a_2)}{(k.P)^2}-\frac{4(k.a_2)(k.a1)-2ka_1(a_2.k)-6ka_1(a_2.k)+(a_1.a_2)k^2+2a_1a_2k^2}{(k.P)(k.P')}$$
$$-\frac{k^2(a_1.a_2)}{(k.P')^2}\Bigg], and$$

$$W_n^+ = e^2\left[\frac{-(a_2k)^2}{(k.P)^2}-\frac{4(a_2.k)^2-8ka_2(a_2.k)+4k^2a_2^2}{(k.P)(k.P')}-\frac{(-a_2k)^2}{(k.P')^2}\right].$$

This implies that

$$Tr[{}_r\Delta^n_{\mu r}\bar{\Delta}^{n'}_{\nu}] = X_n|F_{r-n}|^2+Y_nF_{r-n}G^+_{r-n}+Y_n^+F_{r-n}H^+_{r-n}+Z_nG_{r-n}F^+_{r-n}$$
$$+Z_n^+|G_{r-n}|^2+K_nG_{r-n}H^+_{r-n}+K_n^+H_nF^+_{r-n}+W_nH_{r-n}G^+_{r-n}+W_n^+|H_{r-n}|^2. \qquad (2.2.18)$$

Therefore we can rewrite

$$\sum_{ss'}\sum_{s_+s_-}M_\mu(e^-e^+|n)M_\nu(\mu,\mu'|r-n)M_\mu^+(e^-e^+|n')M_\nu^+(\mu,\mu'|r-n') = T_r^{nn'}$$

$$= \frac{1}{16(mM)^2}\Bigg[l_n|B_n|^2+l_n^+B_nC_n^++m_nB_nD_n^++m_n^+C_nB_n^++p_n|C_n|^2+p_n^+C_nD_n^+$$
$$+s_nD_nB_n^++s_n^+D_nC_n^++t_n|D_n|^2-m^2\Bigg(x_n|B_n|^2+y_nB_nC_n^++y_n^+B_nD_n^++z_nC_nB_n^+$$
$$+z_n^+|C_n|^2+k_nC_nD_n^++k_n^+D_nB_n^++w_nD_nC_n^++w_n^+|D_n|^2\Bigg)\Bigg]\Bigg[L_n|F_{r-n}|^2+L_n^+F_{r-n}G^+_{r-n}$$

$$+M_nF_NH^+_{r-n}+M^+_nG_{r-n}F^+_{r-n}+P_n|G_{r-n}|^2+P^+_nG_{r-n}H^+_{r-n}+S_nH_{r-n}F^+_{r-n}$$

$$+S^+_nH_{r-n}G^+_{r-n}+T_n|H_{r-n}|^2+M^2\bigg(X_n|F_{r-n}|^2+Y_nF_{r-n}G^+_{r-n}+Y^+_nF_{r-n}H^+_{r-n}$$

$$+Z_nG_{r-n}F^+_{r-n}+Z^+_n|G_{r-n}|^2+K_nG_{r-n}H^+_{r-n}+K^+_nH_{r-n}F^+_{r-n}+W_nH_{r-n}G^+_N+W^+_n|H_{r-n}|^2\bigg)\bigg]. \tag{2.2.19}$$

This implies from Eq. (2.2.2) total cross section can be rewrite

$$\sigma=\int\frac{d^3Q'}{(2\pi)^3}\int\frac{d^3q_+}{(2\pi)^3}\int\frac{d^3q_-}{(2\pi)^3}\int d\Omega_f\frac{|N'|^2}{\frac{|v_i|}{V}}\frac{1}{32}\sum_{nn'=-\infty}^{\infty}\frac{1}{(mM)^2}\bigg[l_n|B_n|^2+l^+_nB_nC^+_n+m_nB_nD^+_n$$

$$+m^+_nC_nB^+_n+p_n|C_n|^2+p^+_nC_nD^+_n+s_nD_nB^+_n+s^+_nD_nC^+_n+t_n|D_n|^2-m^2\bigg(x_n|B_n|^2+y_nB_nC^+_n$$

$$+y^+_nB_nD^+_n+z_nC_nB^+_n+z^+_n|C_n|^2+k_nC_nD^+_n+k^+_nD_nB^+_n+w_nD_nC^+_n+w^+_n|D_n|^2\bigg)\bigg]\bigg[L_n|F_{r-n}|^2$$

$$+L^+_nF_{r-n}G^+_{r-n}+M_nF_NH^+_{r-n}+M^+_nG_{r-n}F^+_{r-n}+P_n|G_{r-n}|^2+P^+_nG_{r-n}H^+_{r-n}$$

$$+S_nH_{r-n}F^+_{r-n}+S^+_nH_{r-n}G^+_{r-n}+T_n|H_{r-n}|^2+M^2\bigg(X_n|F_{r-n}|^2+Y_nF_{r-n}G^+_{r-n}+Y^+_nF_{r-n}H^+_{r-n}$$

$$+Z_nG_{r-n}F^+_{r-n}+Z^+_n|G_{r-n}|^2+K_nG_{r-n}H^+_{r-n}+K^+_nH_{r-n}F^+_{r-n}+W_nH_{r-n}G^+_N$$

$$+W^+_n|H_{r-n}|^2\bigg)\bigg]\frac{V\delta^4(q+q_++q_--nk-Q-Nk+Q')}{(2\pi)^4(q_++q_--nk)^2(q_++q_--n'k)^2}. \tag{2.2.20}$$

Chapter 3

The rate and cross section for e^+e^- pair creation by proton impact on intense laser beams

Protons are not fundamental particles. In our case the calculation does not actually apply to them. But since they are $spin^{1/2}$ particles like leptons, we may approximately treat them as effective Dirac particles. We now consider the process $p + r\omega \to p' + e^+ + e^-$ in the presence of a laser field show in the figure below.

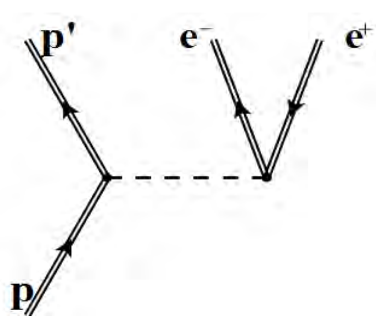

Figure 3.1: Feynman diagram of the considered process in the lowest order in. The dashed line represents the virtual photon propagated between the projectile and the electron positron vertices, and the double lines stand for the exact lepton wave functions in the laser field (Volkov states).

This is similar to the process $\mu + r\omega \to \mu' + e^+ + e^-$ treated in the previous chapter. However p is quite massive compared to e^+ and e^- $(m_p = 1840 m_e)$ as an approximation we can regard p as at rest before and after scattering.

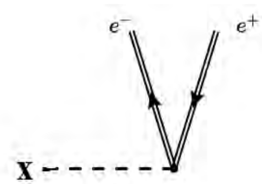

Figure 3.2: Feynman diagram on scattering of e^-e^+ at an external field (x)

We can thus consider the above process as taking place in the Coulomb field of a heavy charged particle (x) at rest shown in Fig. 3.2. We can thus treat the above process in first-order of perturbation theory and work in the rest frame of the proton. From the Feynman diagram in the above using the first-order perturbation theory the scattering amplitude of the transition from negative energy Volkov state to positive Volkov state induced by the Coulomb field A_c of the static proton is

$$S_{fi} = ie \int d^4x \bar{\Psi}_{p_- s_-} \not{A}_c \Psi_{p_+ s_+}, \tag{3.0.1}$$

where a proton in its rest frame can be described by the four potential $A_c^\mu = \left(\frac{e}{|\mathbf{x}|}, 0, 0, 0\right)$, so that $\not{A}_c = -\gamma^i A_c^i + \gamma^o A_c^o$ but $-\gamma^i A_c^i = 0$, this implies $\not{A}_c = \gamma^o A_c^o$.
This implies Eq. (3.0.1) become

$$S_{fi} = ie \int d^4x \bar{\Psi}_{p_- s_-} \gamma^o A_c^o \Psi_{p_+ s_+}. \tag{3.0.2}$$

From cylindrical polarized plane-wave

$$A_\mu = a_1 cos(kx) + a_2 sin(kx). \tag{3.0.3}$$

Substitute the Volkov states expressions of $\bar{\Psi}_{p_- s_-}$ and $\Psi_{p_+ s_+}$ given from the above in section (2.1). This implies the scattering amplitude of Eq. (3.0.2) can be rewrite as

$$S_{fi} = \frac{iem}{V\sqrt{q_+^0 q_-^0}} \int d^4x \bar{u}_{p_- s_-} \left(1 - \frac{e\not{A}\not{k}}{2(k.p_-)}\right) \gamma^o A_c^o \left(1 + \frac{e\not{k}\not{A}}{2(k.p_+)}\right) v_{p_+ s_+} e^{i(S_+ - S_-)}$$

$$= \frac{ie^2 m}{V\sqrt{q_+^0 q_-^0}} \int d^4x \bar{u}_{p_- s_-} \left(1 - \frac{e\not{A}\not{k}}{2(k.p_-)}\right) \frac{\gamma^o}{|\mathbf{x}|} \left(1 + \frac{e\not{k}\not{A}}{2(k.p_+)}\right) v_{p_+ s_+} e^{i(S_+ - S_-)}$$

$$= \frac{ie^2 m}{V\sqrt{q_+^0 q_-^0}} \int \frac{d^4x}{|\mathbf{x}|} \bar{u}_{p_- s_-} \left(\gamma^o + \frac{e\gamma^o \not{k}\not{A}}{2(k.p_+)} - \frac{e\not{A}\not{k}\gamma^o}{2(k.p_-)} - \frac{e^2\not{A}\not{k}\gamma^o}{2(k.p_-)} \frac{\not{k}\not{A}}{2(k.p_+)}\right) v_{p_+ s_+} e^{i(S_+ - S_-)}$$

$$= \frac{ie^2 m}{V\sqrt{q_+^0 q_-^0}} \int \frac{d^4x}{|\mathbf{x}|} \bar{u}_{p_- s_-} \left[\left(\gamma^o - \frac{2e^2 a^2 k^o \not{k}}{2(k.p_+)(k.p_-)}\right) + \frac{e}{2} \left(\frac{\gamma^o \not{k}\not{a}_1}{(k.p_+)} - \frac{\not{a}_1 \not{k}\gamma^o}{(k.p_-)}\right) cos(kx) + \frac{e}{2} \left(\frac{\gamma^o \not{k}\not{a}_2}{(k.p_+)} - \frac{e\not{a}_2 \not{k}\gamma^o}{(k.p_-)}\right) sin(kx) \right] e^{i(S_+ - S_-)} v_{p_+ s_+}. \quad (3.0.4)$$

From section (2.1)

$$e^{i(S_+ - S_-)} = e^{i(q_+ + q_+).x} e^{-i\left(\frac{e(a_1.p_-)}{(k.p_-)} - \frac{e(a_1.p_+)}{(k.p_+)}\right) sin(kx) + i\left(\frac{e(a_2.p_-)}{(k.p_-)} - \frac{e(a_2.p_+)}{(k.p_+)}\right) cos(kx)}.$$

But in this case

$$e^{-i\left(\frac{e(a_1.p_-)}{(k.p_-)} - \frac{e(a_1.p_+)}{(k.p_+)}\right) sin(kx) + i\left(\frac{e(a_2.p_-)}{(k.p_-)} - \frac{e(a_2.p_+)}{(k.p_+)}\right) cos(kx)} = e^{-i\omega sin(kx - \phi_o)}.$$

Therefore

$$e^{i(S_+ - S_-)} = e^{i[(q_+ + q_-).x - \omega sin(kx - \phi_o)]}.$$

From these expressions

$$S_{fi} = \frac{ie^2 m}{V\sqrt{q_+^0 q_-^0}} \int \frac{d^4x}{|\mathbf{x}|} \bar{u}_{p_- s_-} \left[\left(\gamma^o - \frac{e^2 a^2 k^o \not{k}}{2(k.p_+)(k.p_-)}\right) + \frac{e}{2} \left(\frac{\gamma^o \not{k}\not{a}_1}{(k.p_+)} - \frac{\not{a}_1 \not{k}\gamma^\mu}{(k.p_-)}\right) cos(kx) + \frac{e}{2} \left(\frac{\gamma^o \not{k}\not{a}_2}{(k.p_+)} - \frac{e\not{a}_2 \not{k}\gamma^o}{(k.p_-)}\right) sin(kx) \right] e^{i[(q_+ + q_-).x - \omega sin(kx - \phi_o)]} v_{p_+ s_+}. \quad (3.0.5)$$

From cylindrical Bessel function expansion of trigonometric functions in the appendix. Let as consider

$$B_r = e^{ir\phi_o} J_r(\omega),$$

$$C_r = \frac{1}{2}\left(e^{i(r+1)\phi_o} J_{r+1}(\omega) + e^{i(r-1)\phi_o} J_{r-1}(\omega)\right), and$$

$$D_r = \frac{1}{2i}\left(e^{i(r+1)\phi_o} J_{r+1}(\omega) - e^{i(r-1)\phi_o} J_{r-1}(\omega)\right),$$

where

$$\omega_j = \frac{e(a_j.p_-)}{(k.p_-)} - \frac{e(a_j.p_+)}{(k.p_+)}, j = 1, 2.$$

From these equation

$$S_{fi} = \frac{ie^2m}{V\sqrt{q_+^0 q_-^0}} \sum_{r=-\infty}^{\infty} \int \frac{d^4x}{|\mathbf{x}|} \bar{u}_{p_-s_-} \Bigg[\left(\gamma^o - \frac{e^2a^2k^o \not{k}}{2(k.p_+)(k.p_-)} \right) B_r + \frac{e}{2} \left(\frac{\gamma^o \not{k} \not{a}_1}{(k.p_+)} - \frac{\not{a}_1 \not{k} \gamma^o}{(k.p_-)} \right) C_r$$

$$+ \frac{e}{2} \left(\frac{\gamma^o \not{k} \not{a}_2}{(k.p_+)} - \frac{e\not{a}_2 \not{k} \gamma^o}{(k.p_-)} \right) D_r \Bigg] e^{i(q_+ + q_- - rk).x} v_{p_+s_+}$$

$$= \frac{ie^2m}{V\sqrt{q_+^0 q_-^0}} \sum_{r=-\infty}^{\infty} \int \frac{d^4x}{|\mathbf{x}|} M_{p_+p_-}^{(r)} e^{i(q_+ + q_- - rk).x}. \tag{3.0.6}$$

Where the invariant spinor-matrix ($M_{p_+p_-}^{(r)}$)

$$M_{p_+p_-}^{(r)} = \bar{u}_{p_-s_-} \Bigg[\left(\gamma^o - \frac{e^2a^2k^o \not{k}}{2(k.p_+)(k.p_-)} \right) B_r + \frac{e}{2} \left(\frac{\gamma^o \not{k} \not{a}_1}{(k.p_+)} - \frac{\not{a}_1 \not{k} \gamma^o}{(k.p_-)} \right) C_r$$

$$+ \frac{e}{2} \left(\frac{\gamma^o \not{k} \not{a}_2}{(k.p_+)} - \frac{e\not{a}_2 \not{k} \gamma^o}{(k.p_-)} \right) D_r \Bigg] v_{p_+s_+}.$$

To solve the integration of amplitude on Eq. (3.0.6)

$$S_{fi} = \frac{ie^2m}{V\sqrt{q_+^0 q_-^0}} \sum_{r=-\infty}^{\infty} M_{p_+p_-}^{(r)} \iint \frac{d^3\mathbf{x}}{|\mathbf{x}|} e^{i(\mathbf{q}_+ + \mathbf{q}_- - r\mathbf{k}).\mathbf{x}} dx_o^{-i(q_+^o + q_-^0 - rk_o).x_o}. \tag{3.0.7}$$

But

$$\int dx_o e^{-i(q_+^o + q_-^0 - rk_o).x_o} = (2\pi)\delta(q_r^o),$$

$$where \quad q_r^o = (q_+^o + q_-^0 - rk_o).$$

And

$$\int \frac{d^3\mathbf{x}}{|\mathbf{x}|} e^{i(\mathbf{q}_+ + \mathbf{q}_- - r\mathbf{k}).\mathbf{x}},$$

it can be solved easily with the help of the following trick based on partial integration.

$$\int \frac{d^3\mathbf{x}}{|\mathbf{x}|} e^{i(\mathbf{q}_+ + \mathbf{q}_- - r\mathbf{k}).\mathbf{x}} = \frac{1}{(\mathbf{q}_+ + \mathbf{q}_- - r\mathbf{k})^{\mathbf{2}}} \int d^3\mathbf{x} \frac{1}{|\mathbf{x}|} \Delta e^{i(\mathbf{q}_+ + \mathbf{q}_- - r\mathbf{k}).\mathbf{x}}$$

$$= \frac{1}{(\mathbf{q}_r)^2} \int d^3\mathbf{x} (\Delta \frac{1}{|\mathbf{x}|}) e^{i(\mathbf{q_r}).\mathbf{X}}$$

$$= \frac{1}{(\mathbf{q_r})^{\mathbf{2}}} \int d^3\mathbf{x} \left(-4\pi\delta^3(x) \right) e^{i(\mathbf{q_r}).\mathbf{x}}$$

$$= \frac{4\pi}{(\mathbf{q_r})^2},$$

where $\mathbf{q_r} = (\mathbf{q}_+ + \mathbf{q}_- - r\mathbf{k})$.

Therefor

$$S_{fi} = \frac{ie^2m}{V\sqrt{q_+^0 q_-^0}} \sum_{r=-\infty}^{\infty} M_{p_+p_-}^{(r)} \frac{4\pi(2\pi)\delta(q_r^o)}{(\mathbf{q_r})^2}. \tag{3.0.8}$$

3.1 Partial rate corresponding to the process $p+r\omega \rightarrow p' + e^{+} + e^{-}$ taking Proton at rest

The total partial rate corresponding to the photon order r is given by

$$R^{(r)} = \int \frac{Vd^3q_+}{(2\pi)^3} \int \frac{Vd^3q_-}{(2\pi)^3} \frac{1}{T} \sum_{s_+s_-} |s_{fi}|^2. \tag{3.1.1}$$

But the square of the partial transition amplitude

$$|S_{fi}|^2 = \frac{e^4m^2}{V^2q_+^0q_-^0} \sum_{r,r'=-\infty}^{\infty} M^{(r)}_{p_+p_-} M^{(r')+}_{p_+p_-} \frac{(2\pi\delta(q_r^o))^2}{\mathbf{q}_r^4}. \tag{3.1.2}$$

One can show that

$$(2\pi\delta(q_r^o))^2 = 2\pi\delta(q_r^o)T.$$

This implies

$$|S_{fi}|^2 = \frac{e^4m^2}{V^2q_+^0q_-^0} \sum_{r,r'=-\infty}^{\infty} M^{(r)}_{p_+p_-} M^{(r')+}_{p_+p_-} \frac{2\pi\delta(q_r^o)}{\mathbf{q}_r^4}T. \tag{3.1.3}$$

Let as consider

$$\Gamma_r^{\infty} = \left(\gamma^o - \frac{e^2a^2k^o\not{k}}{2(k.p_+)(k.p_-)}\right) B_r + \frac{e}{2}\left(\frac{\gamma^o\not{k}\not{a}_1}{(k.p_+)} - \frac{\not{a}_1\not{k}\gamma^o}{(k.p_-)}\right) C_r + \frac{e}{2}\left(\frac{\gamma^o\not{k}\not{a}_2}{(k.p_+)} - \frac{e\not{a}_2\not{k}\gamma^o}{k.p_-}\right) D_r. \tag{3.1.4}$$

Then the spinor-matricx

$$M^{(r)}_{p_+p_-} = \bar{u}_{p_-s_-}\Gamma_r^{\infty} v_{p_+s_+}.$$

Thus, we can rewrite

$$M^{(r')+}_{p_+p_-} = (\bar{u}_{p_-s_-}\Gamma_{r'}^{\infty} v_{p_+s_+})^+ = v^+_{p_+s_+}\Gamma_{r'}^{\infty+}\bar{u}^+_{p_-s_-}$$

$$= v^+_{p_+s_+}\Gamma_{r'}^{\infty+}(u^+_{p_-s_-}\gamma_4)^+ = v^+_{p_+s_+}\Gamma_{r'}^{\infty+}\gamma_4 u_{p_-s_-}$$

$$= v^+_{p_+s_+}\ \gamma_4\gamma_4\ \Gamma_{r'}^{\infty+}\gamma_4\ u_{p_-s_-} \qquad Because \quad \gamma_4\gamma_4 = I$$

$$= \bar{v}_{p_+s_+}\ \bar{\Gamma}_{r'}^{\infty}\ u_{p_-s_-},$$

where $\quad \bar{v}_{p_+s_+} = v^+_{p_+s_+}\ \gamma_4$ and $\quad \bar{\Gamma}^{\infty}_{r'} = \gamma_4 \Gamma^{\infty+}_{r'} \gamma_4.$

Due to these equation

$$\sum_{s_+s_-} M^{(r)}_{p_+p_-} M^{(r')+}_{p_+p_-} = \sum_{s_+s_-} \bar{u}_{p_-s_-} \Gamma^{\infty}_{r} v_{p_+s_+} \bar{v}_{p_+s_+}\ \bar{\Gamma}^{\infty}_{r'}\ u_{p_-s_-}$$

$$= \sum_{s_+s_-} \Gamma^{\infty}_{r} v_{p_+s_+} \bar{v}_{p_+s_+}\ \bar{\Gamma}^{\infty}_{r'}\ u_{p_-s_-} \bar{u}_{p_-s_-}.$$

From the summation of Dirac spinors of spin in Eq. (2.1.37) and Eq. (2.1.38)

$$\sum_{s_+s_-} M^{(r)}_{p_+p_-} M^{(r')+}_{p_+p_-} = Tr\left[\Gamma^{\infty}_{r} \frac{\not{p}_+ - m}{2m} \bar{\Gamma}^{\infty}_{r'} \frac{\not{p}_- + m}{2m}\right]. \tag{3.1.5}$$

Therefore

$$R^{(r)} = \int \frac{V d^3q_+}{(2\pi)^3} \frac{V d^3q_-}{(2\pi)^3} \int \frac{1}{T} \frac{e^4 m^2}{V^2 q^0_+ q^0_-} \sum_{r,r'=-\infty}^{\infty} \frac{2\pi\delta(q^o_r)}{q^4_r} T \quad Tr\left[\Gamma^{\infty}_{r} \frac{\not{p}_+ - m}{2m} \bar{\Gamma}^{\infty}_{r'} \frac{\not{p}_- + m}{2m}\right]$$

$$= \frac{e^4 m^2}{q^0_+ q^0_-} \int \frac{d^3q_+}{(2\pi)^3} \int \frac{d^3q_-}{(2\pi)^3} \sum_{r,r'=-\infty}^{\infty} \frac{2\pi\delta(q^o_r)}{\mathbf{q}^4_r}\ Tr\left[\Gamma^{\infty}_{r} \frac{\not{p}_+ - m}{2m} \bar{\Gamma}^{\infty}_{r'} \frac{\not{p}_- + m}{2m}\right] \tag{3.1.6}$$

3.2 Total cross section of the process $p + r\omega \rightarrow p' + e^+ + e^-$taking proton at rest

The scattering cross section can be defined as the transition probability per unit time divided by the incoming current of particle given by

$$\frac{d\sigma}{d\Omega_f} = \frac{dR}{\frac{|v_i|}{V}}. \tag{3.2.1}$$

From this equation

$$\sigma = \frac{e^4 m^2}{q^0_+ q^0_-} \int \frac{d^3q_+}{(2\pi)^3} \int \frac{d^3q_-}{(2\pi)^3} \int d\Omega_f \frac{V}{|v_i|} \sum_{r,r'} \frac{2\pi\delta(q^o_r)}{\mathbf{q}^4_r}\ Tr\left[\Gamma^{\infty}_{r} \frac{\not{p}_+ - m}{2m} \bar{\Gamma}^{\infty}_{r'} \frac{\not{p}_- + m}{2m}\right]. \tag{3.2.2}$$

First find out the trace evaluation of

$$Tr\left[\Gamma^{\infty}_{r} \left(\frac{\not{p}_+ - m}{2m}\right) \bar{\Gamma}^{\infty}_{r'} \left(\frac{\not{p}_- + m}{2m}\right)\right] = \frac{1}{4m^2} Tr\left[\Gamma^{\infty}_{r} (\not{p}_+ - m) \bar{\Gamma}^{\infty}_{r'} (\not{p}_- + m)\right]$$

$$= \frac{1}{4m^2} Tr\left[\Gamma_r^{\infty} \not{p}_+ \bar{\Gamma}_{r'}^{\infty} \not{p}_- + \Gamma_r^{\infty} \not{p}_+ \bar{\Gamma}_{r'}^{\infty} m + \Gamma_r^{\infty} \bar{\Gamma}_{r'}^{\infty} \not{p}_- m - m^2 \Gamma_r^{\infty} \bar{\Gamma}_{r'}^{\infty} \right]. \tag{3.2.3}$$

Due to trace identities involving γ matrices the trace of an odd number of γ matrices is vanishes. From the above Eq. (3.2.3) the terms

$$mTr\left[\Gamma_r^{\infty} \not{p}_+ \bar{\Gamma}_{r'}^{\infty}\right] = mTr\left[\Gamma_r^{\infty} \bar{\Gamma}_{r'}^{\infty} \not{p}_-\right] = 0.$$

This implies

$$Tr\left[\Gamma_r^{\infty} \left(\frac{\not{p}_+ - m}{2m} \right) \bar{\Gamma}_{r'}^{\infty} \left(\frac{\not{p}_- + m}{2m} \right) \right] = \frac{1}{4m^2} \left[Tr[\Gamma_r^{\infty} \not{p}_+ \bar{\Gamma}_{r'}^{\infty} \not{p}_-] - m^2 Tr[\Gamma_r^{\infty} \bar{\Gamma}_{r'}^{\infty}] \right]. \tag{3.2.4}$$

First solve the trace

$$Tr[\Gamma_r^{\infty} \not{p}_+ \bar{\Gamma}_{r'}^{\infty} \not{p}_-]. \tag{3.2.5}$$

Substituting the vertex of the produced pair from Eq. (3.1.4) in to Eq. (3.2.5)

$$\begin{aligned}
&Tr[\Gamma_r^{\infty} \not{p}_+ \bar{\Gamma}_{r'}^{\infty} \not{p}_-] \\
&= Tr\Bigg[\Bigg\{ \left(\gamma^o - \frac{e^2 a^2 k^o \not{k}}{2(k.p_+)(k.p_-)} \right) B_r + \frac{e}{2} \Bigg[\left(\frac{\gamma^o \not{k} \not{a}_1}{(k.p_+)} - \frac{\not{a}_1 \not{k} \gamma^o}{(k.p_-)} \right) C_r + \left(\frac{\gamma^o \not{k} \not{a}_2}{(k.p_+)} - \frac{e \not{a}_2 \not{k} \gamma^o}{(k.p_-)} \right) D_r \Bigg] \Bigg\} \not{p}_+ \\
&\Bigg\{ \left(-\gamma^{o'} + \frac{e^2 a^2 k^{o'} \not{k}'}{2(k.p_+)(k.p_-)} \right) B_r^+ + \frac{e}{2} \Bigg[\left(-\frac{\not{a}_1' \not{k}' \gamma^{o'}}{(k.p_+)} + \frac{\gamma^{o'} \not{k}' \not{a}_1'}{(k.p_-)} \right) C_r^+ + \left(-\frac{\not{a}_2' \not{k}' \gamma^{o'}}{(k.p_+)} + \frac{\gamma^{o'} \not{k}' \not{a}_2'}{(k.p_-)} \right) D_n^+ \Bigg] \Bigg\} \not{p}_- \Bigg] \\
&= Tr\Bigg[\Bigg\{ -\gamma^o \not{p}_+ \gamma^{o'} \not{p}_- + (ea)^2 \frac{k^{o'} \gamma^o \not{p}_+ \not{k}' \not{p}_- + k^o \not{k} \not{p}_+ \gamma^{o'} \not{p}_-}{2(k.p_+)(k.p_-)} - (ea)^4 k^o k^{o'} \frac{\not{k} \not{p}_+ \not{k}' \not{p}_-}{4[(k.p_+)(k.p_-)]^2} \Bigg\} \left|B_r\right|^2 \\
&+ \frac{e}{2} \Bigg\{ \frac{-\gamma^o \not{p}_+ \not{a}_1' \not{k}' \gamma^{o'} \not{p}_-}{(k.p_+)} + \frac{\gamma^o \not{p}_+ \gamma^{o'} \not{k}' \not{a}_1' \not{p}_-^o}{(k.p_-)} + \frac{(ea)^2 k^o}{2(k.p_+)(k.p_-)} \left(\frac{\not{k} \not{p}_+ \not{a}_1' \not{k}' \gamma^{o'} \not{p}_-}{(k.p_+)} - \frac{\not{k} \not{p}_+ \gamma^{o'} \not{k}' \not{a}_1' \not{p}_-}{(k.p_-)} \right) \Bigg\} B_r C_r^+ \\
&+ \frac{e}{2} \Bigg\{ \frac{-\gamma^o \not{p}_+ \not{a}_2' \not{k}' \gamma^{o'} \not{p}_-^o}{(k.p_+)} + \frac{\gamma^o \not{p}_+ \gamma^{o'} \not{k}' \not{a}_2' \not{p}_-}{(k.p_-)} + \frac{(ea)^2 k^o}{2(k.p_+)(k.p_-)} \left(\frac{\not{k} \not{p}_+ \not{a}_2' \not{k}' \gamma^{o'} \not{p}_-}{(k.p_+)} - \frac{\not{k} \not{p}_+ \gamma^{o'} \not{k}' \not{a}_2' \not{p}_-}{(k.p_-)} \right) \Bigg\} B_r D_r^+ \\
&+ \frac{e}{2} \Bigg\{ \frac{-\gamma^o \not{k} \not{a}_1 \not{p}_+ \gamma^{o'} \not{p}_-}{(k.p_+)} + \frac{\not{a}_1 \not{k} \gamma^o \not{p}_+ \gamma^{o'} \not{p}_-}{(k.p_-)} + \frac{(ea)^2 k^{o'}}{2(k.p_+)(k.p_-)} \left(\frac{\gamma^o \not{k} \not{a}_1 \not{p}_+ \not{k}' \not{p}_-}{(k.p_+)} - \frac{\not{a}_1 \not{k} \gamma^o \not{p}_+ \not{k}' \not{p}_-}{(k.p_-)} \right) \Bigg\} C_r B_r^+ \\
&+ \frac{e^2}{4} \Bigg\{ \frac{-\gamma^o \not{k} \not{a}_1 \not{p}_+ \not{a}_1' \not{k}' \gamma^{o'} \not{p}_-}{(k.p_+)^2} + \frac{\gamma^o \not{k} \not{a}_1 \not{p}_+ \gamma^{o'} \not{k}' \not{a}_1' \not{p}_-}{(k.p_+)(k.p_-)} + \frac{\not{a}_1 \not{k} \gamma^o p_+ \not{a}_1' \not{k}' \gamma^{o'} \not{p}_-}{(k.p_+)(k.p_-)} - \frac{\not{a}_1 \not{k} \gamma^o p_+ \gamma^{o'} \not{k}' \not{a}_1' \not{p}_-}{((k.p_-))^2} \Bigg\} \left|C_r\right|^2 \\
&+ \frac{e^2}{4} \Bigg\{ \frac{-\gamma^o \not{k} \not{a}_1 \not{p}_+ \not{a}_2' \not{k}' \gamma^{o'} \not{p}_-}{(k.p_+)^2} + \frac{\gamma^o \not{k} \not{a}_1 \not{p}_+ \gamma^{o'} \not{k}' \not{a}_2' \not{p}_-}{(k.p_+)(k.p_-)} + \frac{\not{a}_1 \not{k} \gamma^o p_+ \not{a}_2' \not{k}' \gamma^{o'} \not{p}_-}{(k.p_+)(k.p_-)} - \frac{\not{a}_1 \not{k} \gamma^o p_+ \gamma^{o'} \not{k}' \not{a}_2' \not{p}_-}{(k.p_-)^2} \Bigg\} C_r D_r^+ \\
&+ \frac{e}{2} \Bigg\{ \frac{-\gamma^o \not{k} \not{a}_2 \not{p}_+ \gamma^{o'} \not{p}_-}{(k.p_+)} + \frac{\not{a}_2 \not{k} \gamma^o \not{p}_+ \gamma^{o'} \not{p}_-}{(k.p_-)} + \frac{(ea)^2 k^{o'}}{2(k.p_+)(k.p_-)} \left(\frac{\gamma^o \not{k} \not{a}_2 \not{p}_+ \not{k}' \not{p}_-}{(k.p_+)} - \frac{\not{a}_2 \not{k} \gamma^o \not{p}_+ \not{k}' \not{p}_-}{(k.p_-)} \right) \Bigg\} D_r B_r^+
\end{aligned}$$

$$+\frac{e^2}{4}\left\{\frac{-\gamma^o \not{k}\not{a}_2\not{p}_+\not{a}_1\not{k}\gamma^{o'}\not{p}_-}{(k.p_+)^2}+\frac{\gamma^o \not{k}\not{a}_2\not{p}_+\gamma^{o'}\not{k}\not{a}_1\not{p}_-}{(k.p_+)(k.p_-)}+\frac{\not{a}_2\not{k}\gamma^o\not{p}_+\not{a}_1\not{k}\gamma^{o'}\not{p}_-}{(k.p_+)(k.p_-)}-\frac{\not{a}_2\not{k}\gamma^o\not{p}_+\gamma^{o'}\not{k}\not{a}_1\not{p}_-}{(k.p_-)^2}\right\}D_rC_r^+$$
$$+\frac{e^2}{4}\left\{\frac{-\gamma^o \not{k}\not{a}_2\not{p}_+\not{a}_2\not{k}\gamma^{o'}\not{p}_-}{(k.p_+)^2}+\frac{\gamma^o \not{k}\not{a}_2\not{p}_+\gamma^{o'}\not{k}\not{a}_2\not{p}_-}{(k.p_+)(k.p_-)}+\frac{\not{a}_2\not{k}\gamma^o p_+\not{a}_2\not{k}\gamma^{o'}\not{p}_-}{(k.p_+)(k.p_-)}-\frac{\not{a}_2\not{k}\gamma^o p_+\gamma^{o'}\not{k}\not{a}_2\not{p}_-}{(k.p_-)^2}\right\}\big|D_r\big|^2\Bigg] \tag{3.2.6}$$

Let as evaluate each trace in Eq. (3.2.6) using trace technology

$$Tr\Bigg[-\gamma^o\not{p}_+\gamma^{o'}\not{p}_- + (ea)^2\frac{k^{o'}\gamma^o\not{p}_+\not{k}\not{p}_- + k^o\not{k}\not{p}_+\gamma^{o'}\not{p}_-}{2(k.p_+)(k.p_-)}$$
$$-(ea)^4k^ok^{o'}\frac{\not{k}\not{p}_+\not{k}\not{p}_-}{4[(k.p_+)(k.p_-)]^2}\Bigg]\big|B_r\big|^2 = l_r\big|B_r\big|^2,$$

$$\frac{e}{2}\ Tr\Bigg[\frac{-\gamma^o\not{p}_+\not{a}_1\not{k}\gamma^{o'}\not{p}_-}{(k.p_+)}+\frac{\gamma^o\not{p}_+\gamma^{o'}\not{k}\not{a}_1\not{p}_-}{(k.p_-)}$$
$$+\frac{(ea)^2k^o}{2(k.p_+)(k.p_-)}\left(\frac{\not{k}\not{p}_+\not{a}_1\not{k}\gamma^{o'}\not{p}_-}{(k.p_+)}-\frac{\not{k}\not{p}_+\gamma^{o'}\not{k}\not{a}_1\not{p}_-}{(k.p_-)}\right)\Bigg]B_rC_r^+ = l_r^+B_rC_r^+,$$

$$\frac{e}{2}\ Tr\Bigg[\frac{-\gamma^o\not{p}_+\not{a}_2\not{k}\gamma^{o'}\not{p}_-}{(k.p_+)}+\frac{\gamma^o\not{p}_+\gamma^{o'}\not{k}\not{a}_2\not{p}_-}{(k.p_-)}$$
$$+\frac{(ea)^2k^o}{2(k.p_+)(k.p_-)}\left(\frac{\not{k}\not{p}_+\not{a}_2\not{k}\gamma^{o'}\not{p}_-}{(k.p_+)}-\frac{\not{k}\not{p}_+\gamma^{o'}\not{k}\not{a}_2\not{p}_-}{(k.p_-)}\right)\Bigg]B_rD_r^+ = m_rB_rD_r^+,$$

$$\frac{e}{2}\ Tr\Bigg[\frac{-\gamma^o\not{k}\not{a}_1\not{p}_+\gamma^{o'}\not{p}_-}{(k.p_+)}+\frac{\not{a}_1\not{k}\gamma^o\not{p}_+\gamma^{o'}\not{p}_-}{(k.p_-)}$$
$$+\frac{(ea)^2k^{o'}}{2(k.p_+)(k.p_-)}\left(\frac{\gamma^o\not{k}\not{a}_1\not{p}_+\not{k}\not{p}_-}{(k.p_+)}-\frac{\not{a}_1\not{k}\gamma^o\not{p}_+\not{k}\not{p}_-}{(k.p_-)}\right)\Bigg]C_rB_r^+ = m_r^+C_rB_r^+,$$

$$\frac{e^2}{4}\ Tr\Bigg[\frac{-\gamma^o\not{k}\not{a}_1\not{p}_+\not{a}_1\not{k}\gamma^{o'}\not{p}_-}{(k.p_+)^2}+\frac{\gamma^o\not{k}\not{a}_1\not{p}_+\gamma^{o'}\not{k}\not{a}_1\not{p}_-}{(k.p_+)(k.p_-)}+\frac{\not{a}_1\not{k}\gamma^o p_+\not{a}_1\not{k}\gamma^{o'}\not{p}_-}{(k.p_+)(k.p_-)}$$
$$-\frac{\not{a}_1\not{k}\gamma^o p_+\gamma^{o'}\not{k}\not{a}_1\not{p}_-}{(k.p_-)^2}\Bigg]\big|C_r\big|^2 = p_r\big|C_r\big|^2,$$

$$\frac{e^2}{4}\ Tr\Bigg[\frac{-\gamma^o\not{k}\not{a}_1\not{p}_+\not{a}_2\not{k}\gamma^{o'}\not{p}_-}{(k.p_+)^2}+\frac{\gamma^o\not{k}\not{a}_1\not{p}_+\gamma^{o'}\not{k}\not{a}_2\not{p}_-}{(k.p_+)(k.p_-)}+\frac{\not{a}_1\not{k}\gamma^o p_+\not{a}_2\not{k}\gamma^{o'}\not{p}_-}{(k.p_+)(k.p_-)}$$
$$-\frac{\not{a}_1\not{k}\gamma^o p_+\gamma^{o'}\not{k}\not{a}_2\not{p}_-}{(k.p_-)^2}\Bigg]C_rD_r^+ = p_r^+C_rD_r^+,$$

$$\frac{e}{2}\ Tr\Bigg[\frac{-\gamma^o\not{k}\not{a}_2\not{p}_+\gamma^{o'}\not{p}_-}{(k.p_+)}+\frac{\not{a}_2\not{k}\gamma^o\not{p}_+\gamma^{o'}\not{p}_-}{(k.p_-)}$$
$$+\frac{(ea)^2k^{o'}}{2(k.p_+)(k.p_-)}\left(\frac{\gamma^o\not{k}\not{a}_2\not{p}_+\not{k}\not{p}_-}{(k.p_+)}-\frac{\not{a}_2\not{k}\gamma^o\not{p}_+\not{k}\not{p}_-}{(k.p_-)}\right)\Bigg]D_rB_r^+ = s_rD_rB_r^+,$$

$$\frac{e^2}{4}\Bigg[\frac{-\gamma^o \not{k}\not{a}_2\not{p}_+\not{a}_1\not{k}'\gamma^{o'}\not{p}_-}{(k.p_+)^2}+\frac{\gamma^o\not{k}\not{a}_2\not{p}_+\gamma^{o'}\not{k}'\not{a}_1\not{p}_-}{(k.p_+)(k.p_-)}+\frac{\not{a}_2\not{k}\gamma^o p_+\not{a}_1\not{k}'\gamma^{o'}\not{p}_-}{(k.p_+)(k.p_-)}$$
$$-\frac{\not{a}_2\not{k}\gamma^o p_+\gamma^{o'}\not{k}'\not{a}_1\not{p}_-}{(k.p_-)^2}\Bigg]D_rC_r^+ = s_r^+ D_rC_r^+, and$$
$$\frac{e^2}{4}\Bigg[\frac{-\gamma^o \not{k}\not{a}_2\not{p}_+\not{a}_2\not{k}'\gamma^{o'}\not{p}_-}{(k.p_+)^2}+\frac{\gamma^o\not{k}\not{a}_2\not{p}_+\gamma^{o'}\not{k}'\not{a}_2\not{p}_-}{(k.p_+)(k.p_-)}+\frac{\not{a}_2\not{k}\gamma^o p_+\not{a}_2\not{k}'\gamma^{o'}\not{p}_-}{(k.p_+)(k.p_-)}$$
$$-\frac{\not{a}_2\not{k}\gamma^o p_+\gamma^{o'}\not{k}'\not{a}_2\not{p}_-}{(k.p_-)^2}\Bigg]\big|D_r\big|^2 = t_r\big|D_r\big|^2.$$

Where the abbreviations

$$l_r = 4\Bigg[(p_+^o.p_-^o) - 2p_+^o p_-^o + (ea)^2 k^o \frac{p_+^o(k^o.p_-^o) - k(p_+^o.p_-) + p_-(k.p_+)}{((k.p_+)(k.p_-)}$$
$$-(ea)^4 k^{o2}\frac{2(k^o.p_+^o)(k^o.p_-^o) - k^{o2}(p_+^o.p_-^o)}{4[(k.p_+)(k.p_-)]^2}\Bigg],$$

$$l_r^+ = 2e\Bigg[\Bigg\{-(k^o.a_1^o)[2p_+^o p_-^o + (p_-^o.p_+^o)] + (p_+^o.a_1^o)[2k^o p_-^o)(p_-^o.k^o)] + (p_+^o.k^o)[-2p_-^o a_1^o]$$
$$-(a_1^o.p_-^o)\Bigg\}\frac{1}{(k.p_+)} + \Bigg\{(a_1^o.p_-^o)[2k^o p_+^o - (k^o.p_+^o)] + (k^o.p_-^o)[(p_+^o.a_1^o) - 2p_+^o a_1^o]$$
$$+(k^o.a_1^o)[(p_+^o.p_-^o) - 2p_+^o p_-^o]\Bigg\}\frac{1}{(k.p_-)}$$
$$+(ea^o)^2 k^o\Bigg\{2(k^o.p_+^o)[k^o(p_-^o.a_1)^o - a_-^o(k^o.p_-^o)] + 2p_+^o(k^o.a_1^o)(k^o.p_-^o) + k^{o2}[(p_+^o.a_1^o)p_-^o$$
$$-p_+^o(p_-^o.a_1^o) + a_1^o(p_+^o.p_-^o)]\Bigg\}\frac{1}{(k.p_+)^2(k.p_-)} - (ea^o)^2 k^o\Bigg\{2(p_+^o.k^o)[p_-^o(k^o.a_1^o) - (p_-^o.k^o)]$$
$$+k^o(p_+^o.a_1^o)[2(p_-^o.k^o) - k^o p_-^o] + k^o(p_+^o.p_-^o)[k^o a_1^o - 2(k^o.a_1^o)] + k^{o2}p_-^o(p_+^o.a_1^o)\Bigg\}\frac{1}{2(k.p_+)(k.p_-)}\Bigg],$$

$$m_r = 2e\Bigg[\Bigg\{-(k^o.a_2^o)[2p_+^o p_-^o - (p_-^o.p_+^o)] - (p_+^o.a_2^o)[2k^o p_-^o) - (p_-^o.k^o)] - (p_+^o.k^o)[-2p_-^o a_2^o$$
$$+(a_2^o.p_-^o)]\Bigg\}\frac{1}{(k.p_+)} + \Bigg\{(a_2^o.p_-^o)[2kp_+^o - (k^o.p_+^o)] + (k^o.p_-^o)[(p_+^o.a_2^o) - 2_+^o a_1^o]$$
$$+(k^o.a_2^o)[(p_+^o.p_-^o) - 2p_+^o p_-^o]\Bigg\}\frac{1}{(k.p_-)} + (ea)^2 k^o\Bigg\{2(k^o.p_+^o)[k^o(p_-^o.a_2^o) - a_-^o(k^o.p_-^o)]$$
$$+2p_+^o(k^o.a_2^o)(k^o.p_-^o) + k^{o2}[(p_+^o.a_2^o)p_-^o - p_+^o(p_-^o.a_2^o) + a_2^o(p_+^o.p_-^o)]\Bigg\}\frac{1}{(k.p_+)^2(k.p_-)}$$
$$-(ea^o)^2 k^o\Bigg\{2(p_+^o.k^o)[p_-^o(k^o.a_2^o) - (p_-^o.k^o)] + k^o(p_+^o.a_2^o)[2(p_-^o.k^o) - k^o p_-^o]$$
$$+k^o(p_+^o.p_-^o)[k^o a_2^o - 2(k^o.a_2^o)] + k^{o2}p_-^o(p_+^o.a_2^o)\Bigg\}\frac{1}{2(k.p_+)(k.p_-)^2}\Bigg],$$

$$m_r^+ = 2e\Bigg[\Bigg\{-(p_+^o.a_1^o)[2k^o p_-^o - (k^o.p_-^o)] - (k^o.p_-^o)[(p_+^o.a_1^o) - 2p_+^o a_1^o]$$

$$-(k^o.a_1^o)[2p_+^o p_-^o - (p_+^o.p_-^o)]\Bigg\}\frac{1}{(k.p_+)} + \Bigg\{(a_1^o.k)^o[2p_+^o p_-^o - (p_+^o p_-^o)] - 2a_1^o p_+^o (k^o.p_-^o)$$

$$+k^o p_-^o (a_1^o.p_+^o) + (a_1^o.p_-^o)[2k^o p_+^o - (k^o.p_+^o)]\Bigg\}\frac{1}{(k.p_-)} + (ea^o)^2 k^o \Bigg\{2k^o (a_1^o.p_-^o)(k^o.p_+^o)$$

$$-2p_+^o (a_1^o.k^o)(k^o.p_-^o) - 2(a_1^o.k^o)(p_+^o.p_-^o) - 2a_1^o (k^o.p_+^o)(k^o.p_-^o)\Bigg\}\frac{1}{2(k.p_+)^2(k.p_-)}$$

$$-(ea^o)^2 k^o \Bigg\{2k^o (a_1.k^o)(p_-^o.p_+^o) + 2p_-^o (a_1^o.k^o)(k^o.p_+^o) + 2(k^o.p_-^o)[k^o (a_1^o.p_+^o) - a_1^o (k^o.p_+^o)]$$

$$+k^{o2}[a_1^o (p_-^o.p_+^o) - p_-^o (a_1^o.p_+^o) + p_+^o (a_1^o.p_-^o)]\Bigg\}\frac{1}{2(k.p_+)(k.p_-)^2}\Bigg],$$

$$p_r = e^2\Bigg[-\Bigg\{2k^{o2}(a_1^o.p_+^o)[a_1^o p_-^o + (a_1^o.p_-^o)] + 2a_1^{o2}(k^o.p_+^o)[(k^o.p_-^o) - k^o p_-^o] + (a_1^o k^o)^2 p_+^o p_-^o$$

$$-a_1^{o2} k^o p_+^o (k^o.p_-^o) + k^{o2} a_1^o p_+^o (a_1^o.p_-^o) + (a_1^o.k^o)(k^o.p_+^o)[6p_-^o k - 4(k^o.p_-^o)k^o] + k^o a_1^{o2}(k^o.p_+^o)k^o$$

$$-k^2 a_1^o (a_1^o.p_+^o)\Bigg\}\frac{1}{(k.p_+)^2} - \Bigg\{4(a_1^o.p_-^o)(a_1^o.k^o)[2p_+^o k^o - (k^o.p_+^o)] + k^{o2}(a_1^o.p_-^o)[2(a_1^o.p_+^o)$$

$$-a_1^o p_+^o] + 2a_1^{o2}[(k^o.p_-^o)(-2k^o p_+^o + (k^o.p_+^o)) - k^{o2}(p_+^o p_-^o + (p_+^o p_-^o))]\Bigg\}\frac{1}{(k.p_-)^2}$$

$$+\Bigg\{4k^{o2}(a_1^o.p_+^o)(a_1^o.p_-^o) - 3k^o a_1^o (a_1^o.p_-^o)(k^o.p_+^o) + 4k^o a_1^{o2} p_+^o (k^o.p_-^o) + 3k^o a_1^{o2} p_-^o (k^o.p_+^o)$$

$$-5a_1^o p_-^o (k^o.p_+^o)(a_1^o.k^o) - 7a_1^o k^o (k^o.p_-^o)(a_1^o.p_+^o) + 5a_1^o k^o (a_1^o.k^o)(p_+^o.p_-^o) - 3a_1^{o2} k^{o2}(p_+^o.p_-^o)$$

$$-8a_1^o p_+^o (a_1^o.k^o)(k^o.p_-^o) + 7p_+^o p_-^o (a_1^o.k^o) + 3p_+^o a_1^o k^{o2}(a_1^o.p_-^o) - 4k^{o2} a_1^{o2} p_+^o p_-^o$$

$$+6(k^o.p_+^o)(a_1^o.k^o)(k^o.p_-^o) - k^{o2}(a_1^o.p_-^o)(a_1^o.p_+^o) + 6a_1^{o2}(k^o.p_-^o)(k^o.p_+^o)3p_-^o a_1^o k^{o2}(a_1^o.p_+^o)$$

$$-4p_-^o a_1^o (k^o.p_+^o)(a_1^o.k^o) + 6p_-^o k^o (a_1^o.p_+^o)(a_1^o.k^o) - (a_1^o.k)^{o2}(p_+^o.p_+^o) - 2k^o p_-^o (a_1^o.p_+^o)(a_1^o.k^o)$$

$$+2(k^o.p_-^o)(a_1^o.p_+^o)(a_1^o.k^o) + 2k^{o2} p_-^o a_1^o (a_1^o.p_+^o)$$

$$-2a_1^{o2} k^o p_+^o (k^o.p_-^o 2k^o p_+^o (a_1^o.k^o)(a_1^o.p_-^o)\Bigg\}\frac{1}{(k.p_+)(k.p_-)}\Bigg],$$

$$p_r^+ = e^2\Big[-\Big\{2(a_1^o.p_+^o)(k^o.a_2^o)\big[2k^op_-^o - (k^o.p_-^o)\big] + 2(a_2^o.p_+^o)(k^o.a_1^o)\big[2k^op_-^o - (k^o.p_-^o)\big]$$
$$+k^{o2}(a_1^o.a_2^o)[2p_-^op_+^o - (p_-^o.p_+^o)] + k^{o2}(a_2^o.p_-^o)[(a_1^o.p_+^o) - 2a_1^op_+^o] + k^{o2}(a_1^o.p_+^o)\big[-2a_2^op_-^o$$
$$-(a_2^o.p_-^o)\big] - k^{o2}a_1^oa_2^o(p_-^o.p_+^o)\Big\}\frac{1}{(k.p_+)^2} - \Big\{(a_1^o.a_2^o)\big[-2k^op_+^o(k^o.p_-^o) + k^{o2}\big(2p_+^op_-^o - (p_+^o.p_-^o)\big)$$
$$-2k^op_+^o(k^o.p_-^o) + 2(k^o.p_+^o)(k^o.p_-^o)\big] + (a_1^o.k^o)\big(2k^op_+^o(p_-^o.a_2^o) - 2(k^o.p_+^o)(p_-^o.a_2^o)\big)$$
$$-a_1^oa_2^ok^{o2}(p_+^o.p_-^o) + k^{o2}(a_1^o.p_+^o)(a_2^o.p_-^o)\Big\}\frac{1}{(k.p_-)^2} + \Big\{(k^o.p_+^o)\big[-4a_1^op_-^o(k^o.a_2^o) + 2(a_1^o.p_-^o)(k^o.a_2^o)$$
$$-5a_2^op_-^o(k^o.a_1^o) + 2(a_2^o.p_-^o(k^o.a_1^o) + a_1^oa_2^o(k^o.p_-^o) - 4a_1^ok^o(a_2^o.p_-^o) - 2a_2^ok^o(a_1^o.p_-^o)\big]$$
$$+2k^{o2}\big[a_2^op_+^o(a_1^o.p_-^o) - 2a_2^oa_1^o(p_-^o.p_+^o) + a_2p_-^o(a_1^o.p_+^o) - (a_1^o.p_+^o)(a_2^o.p_-^o) - (a_2^o.a_1^o)(p_-^o.p_+^o)$$
$$-2(a_2^o.a_1^o)p_-^op_+^o + (a_1^o.p_-^o)(a_2^o.p_+^o)\big] + a_1^ok^o\big[k^op_-^o(a_2^o.p_+^o) + (k^o.a_2^o)(p_-^o.p_+^o) - 4(a_2^o.p_+^o)(p_-^o.k^o)\big]$$
$$+2(k^o.p_-^o)\big[-2a_2^op_+^o(k^o.a_1^o) + (a_2^o.p_+^o)(k^o.a_1^o) - (a_2^o.a_1^o)(k^o.p_+^o) - a_1^op_+^o(k^o.a_2^o)$$
$$+(a_1^o.p_+^o)(k^o.a_2^o) - p_+^ok^o(a_2^o.a_1^o)\big] + 2(a_2^o.k^o)\big[3p_+^op_-^o(a_1^o.k^o) - 2(p_+^op_-^o)(a_1^o.k^o) + (p_+^o.a_1^o)p_-^ok^o\big]$$
$$+3k^o\big[-a_2^o(a_1^o.p_+^o)(p_-^o.k^o) + p_-^o(a_2^o.a_1^o)(k^o.p_+^o)\big]\Big\}\frac{1}{(k.p_+)(k.p_-)}\Big],$$

$$s_r = 2e\Big[\Big\{-(p_+^o.a_2^o)[2k^op_-^o - (k^o.p_-^o)] - (k^o.p_-^o)[(p_+^o.a_2^o) - 2p_+^oa_2^o] - (k^o.a_2^o)[2p_+^op_-^o$$
$$-(p_+^o.p_-^o)]\Big\}\frac{1}{(k.p_+)} + \Big\{(a_2^o.k^o)[2p_+^op_-^o - (p_+^op_-^o)] - 2a_2^op_+^o(k^o.p_-^o) + k^op_-^o(a_2^o.p_+^o)$$
$$+(a_2^o.p_-^o)[2k^op_+^o - (k^o.p_+^o)]\Big\}\frac{1}{(k.p_-))} + (ea^o)^2k^o\Big\{2k^o(a_2^o.p_-^o)(k^o.p_+^o) - 2p_+^o(a_2^o.k^o)(k^o.p_-^o)$$
$$-2(a_2^o.k^o)(p_+^o.p_-^o) - 2a_2^o(k^o.p_+^o)(k^o.p_-^o)\Big\}\frac{1}{2(k.p_+)^2(k.p_-)]} - (ea^o)^2k^o\Big\{2k^o(a_2^o.k^o)(p_-^o.p_+^o)$$
$$+2p_-^o(a_2^o.k^o)(k^o.p_+^o) + 2(k^o.p_-^o)[k^o(a_2^o.p_+^o) - a_2^o(k^o.p_+^o)] + k^{o2}[a_2^o(p_-^o.p_+^o) - p_-^o(a_2^o.p_+^o)$$
$$+p_+^o(a_2^o.p_-^o)]\Big\}\frac{1}{2(k.p_+)(k.p_-)^2}\Big],$$

$$s_r^+ = e^2\Big[-\Big\{2(a_2^o.p_+^o)(k^o.a_1^o)\big[2k^op_-^o-(k^o.p_-^o)\big]+2(a_1^o.p_+^o)(k^o.a_2^o)\big[2k^op_-^o-(k^o.p_-^o)\big]$$
$$+k^{o2}(a_2^o.a_1^o)[2p_-^op_+^o-(p_-^o.p_+^o)]+k^{o2}(a_1^o.p_-^o)[(a_2^o.p_+^o)-2a_2^op_+^o]+k^{o2}(a_2^o.p_+^o)\big[-2a_1^op_-^o$$
$$-(a_1^o.p_-^o)\big]-k^{o2}a_2^oa_2^o(p_-^o.p_+^o)\Big\}\frac{1}{(k.p_+)^2}-\Big\{(a_1^o.a_2^o)\big[-2k^op_+^o(k^o.p_-^o)+k^{o2}\big(2p_+^op_-^o-(p_+^o.p_-^o)\big)$$
$$-2k^op_+^o(k^o.p_-^o)+2(k^o.p_+^o)(k^o.p_-^o)\big]+(a_2^o.k^o)\big(2k^op_+^o(p_-^o.a_1^o)-2(k^o.p_+^o)(p_-^o.a_1^o)\big)$$
$$-a_1^oa_2^ok^{o2}(P_+^o.p_-^o)+k^{o2}(a_2^o.p_+^o)(a_2^o.p_-^o)\Big\}\frac{1}{(k.p_-)^2}+\Big\{(k^o.p_+^o)\big[-4a_2^op_-^o(k^o.a_1^o)$$
$$+2(a_2^o.p_-^o)(k^o.a_1^o)-5a_2^op_-^o(k^o.a_2^o)+2(a_1^o.p_-^o(k^o.a_2^o)+a_1^oa_2^o(k^o.p_-^o)-4a_2^ok(a_1^o.p_-^o)$$
$$-2a_1^ok^o(a_2^o.p_-^o)\big]+2k^{o2}\big[a_1^op_+^o(a_2^o.p_-^o)-2a_1^oa_2^o(p_-^o.p_+^o)+a_1^op_-^o(a_2^o.p_+^o)-(a_2^o.p_+^o)(a_1^o.p_-^o)$$
$$-(a_2^o.a_1^o)(p_-^o.p_+^o)-2(a_2^o.a_1^o)p_-^op_+^o+(a_2^o.p_-)^o(a_1^o.p_+^o)\big]+a_1^ok^o\big[k^op_-^o(a_1^o.p_+)^o+(k^o.a_1^o)(p_-^o.p_+^o)$$
$$-4(a_1^o.p_+^o)(p_-^o.k^o)\big]+2(k^o.p_-^o)\big[-2a_1^op_+^ok^o.a_2^o)+(a_1^o.p_+^o(k^o.a_2^o)-(a_1^o.a_2^o)(k.^op_+^o)$$
$$-a_2^op_+^o(^ok.a_1^o)+(a_2^o.p_+^o)(k^o.a_1^o)-p_{+^o}k(a_1.a_2^o)\big]+2(a_1^o.k^o)\big[3p_+^op_-^o(a_2^o.k^o)-2(p_+^op_-^o)(a_2^o.k^o)$$
$$+(p_+^o.a_2^o)p_-k^o\big]+3k^o\big[-a_1^o(a_1^o.p_+^o)(p_-^o.k^o)+p_-^o(a_2^o.a_1^o)(k^o.p_+^o)\big]\Big\}\frac{1}{(k.p_+)(k.p_-)}\Big],\,and$$

$$t_r = e^2\Big[-\Big\{2k^{o2}(a_2^o.p_+^o)[a_2^op_-^o+(a_2^o.p_-^o)]+2a_2^{o2}(k^o.p_+^o)[(k^o.p_-^o)-k^op_-^o]+(a_2^ok)^{o2}p_+^op_-^o$$
$$-a_2^{o2}k^op_+^o(k^o.p_-^o)+k^{o2}a_2^op_+^o(a_2^o.p_-^o)+(a_2^o.k^o)(k^o.p_+^o)[6p_-^ok^o-4(k^o.p_-^ok^o)]+k^oa_2^{o2}(k^o.p_+^ok^o)$$
$$-k^{o2}a_2^o(a_2^o.p_+^o)\Big\}\frac{1}{(k.p_+)^2}-\Big(4(a_2^o.p_-^o)(a_2^o.k^o)[2p_+^ok^o-(k^o.p_+^o)]+k^{o2}(a_2^o.p_-^o)[2(a_2^o.p_+^o)-a_2^op_+^o]$$
$$+2a_2^{o2}[(k^o.p_-^o)(-2k^op_+^o+(k^o.p_+^o))-k^{o2}(p_+^op_-^o+(p_+^op_-^o))]\Big)\frac{1}{(k.p_-)^2}+\Big\{4k^{o2}(a_2^o.p_+^o)(a_2^o.p_-^o)$$
$$-3k^oa_2^o(a_2^o.p_-^o)(k^o.p_+^o)+4k^oa_2^{o2}p_+^o(k^o.p_-^o)+3k^oa_2^{o2}p_-^o(k^o.p_+^o)-5a_1^op_-^o(k^o.p_+^o)(a_1^o.k^o)$$
$$-7a_2^ok^o(k^o.p_-^o)(a_2^op_+^o)+5a_2^ok^o(a_2^o.k^o)(p_+^o.p_-^o)-3a_2^{o2}k^{o2}(p_+^o.p_-^o)-8a_2^op_+^o(a_2^o.k^o)(k^o.p_-^o)$$
$$+7p_+^op_-^o(a_2^o.k)^{o2}+3p_+^oa_2^ok^{o2}(a_2^o.p_-^o)-4k^{o2}a_2^{o2}p_+^op_-^o+6(k^o.p_+^o)(a_2^o.k^o)(k^o.p_-^o)$$
$$-k^{o2}(a_2^o.p_-^o)(a_2^o.p_+^o)+6a_2^{o2}(k^o.p_-^o)(k^o.p_+^o)3p_-^oa_2^ok^{o2}(a_2^o.p_+^o)-4p_-^oa_2^o(k^o.p_+^o)(a_2^o.k^o)$$
$$+6p_-^ok^o(a_2^o.p_+^o)(a_2^o.k^o)-(a_2^o.k)^{o2}(p_+^o.p_+^o)-2k^op_-^o(a_2^o.p_+^o)(a_2^o.k^o)+2(k^o.p_-^o)(a_2^o.p_+^o)(a_2^o.k^o)$$
$$+2k^{o2}p_-^oa_2^o(a_2^o.p_+^o)-2a_2^{o2}k^op_+^o(k^o.p_-^o-2k^op_+^o(a_2^o.k^o)(a_2^o.p_-^o)\Big\}\frac{1}{(k.p_+)(k.p_-)}\Big].$$

Therefor

$$
\begin{aligned}
Tr[\Gamma_r^{\infty}\not{p}_+\bar{\Gamma_r^{\infty}}\not{p}_-] = {} & l_r|B_r|^2 + l_r^+ B_r C_r^+ + m_r B_r D_r^+ + m_r^+ C_r B_r^+ + p_r|C_r|^2 \\
& + p_r^+ C_r D_r^+ + s_r D_r B_r^+ + s_r^+ D_r C_r^+ + t_r|D_r|^2.
\end{aligned}
\tag{3.2.7}
$$

Secondly we can evaluate

$$
Tr[\Gamma_r^{\infty}\bar{\Gamma_r^{\infty}}]. \tag{3.2.8}
$$

Substituting the vertex of the produced pair from Eq. (3.1.4) in to Eq. (3.2.8)

$$
\begin{aligned}
& Tr[\Gamma_r^{\infty}\bar{\Gamma_r^{\infty}}] \\
& = Tr\Bigg[\Bigg\{\left(\gamma^o - \frac{e^2a^2k^o\not{k}}{2(k.p_+)(k.p_-)}\right)B_r + \frac{e}{2}\Bigg[\left(\frac{\gamma^o\not{k}\not{a}_1}{(k.p_+)} - \frac{\not{a}_1\not{k}\gamma^o}{(k.p_-)}\right)C_r + \left(\frac{\gamma^o\not{k}\not{a}_2}{(k.p_+)} - \frac{\not{a}_2\not{k}\gamma^o}{(k.p_-)}\right)D_r\Bigg]\Bigg\} \\
& \Bigg\{\left(-\gamma^{o'} + \frac{e^2a^2k^{o'}\not{k}'}{2(k.p_+)(k.p_-)}\right)B_r^+ + \frac{e}{2}\Bigg[\left(-\frac{\not{a}'_1\not{k}'\gamma^{o'}}{(k.p_+)} + \frac{\gamma^{o'}\not{k}'\not{a}'_1}{(k.p_-)}\right)C_r^+ + \left(-\frac{\not{a}'_2\not{k}'\gamma^{o'}}{(k.p_+)} + \frac{\gamma^{o'}\not{k}'\not{a}'_2}{(k.p_-)}\right)D_r^+\Bigg]\Bigg\}\Bigg] \\
& = Tr\Bigg[\Bigg\{-\gamma^o\gamma^{o'} + (ea)^2\frac{k^{o'}\gamma^o\not{k}' + k^o\not{k}\gamma^{o'}}{2(k.p_+)(k.p_-)} - (ea)^4k^ok^{o'}\frac{\not{k}\not{k}'}{4(k.p_+)(k.p_-)^2}\Bigg\}\big|B_r\big|^2 \\
& + \frac{e}{2}\Bigg\{\frac{-\gamma^o\not{a}'_1\not{k}'\gamma^{o'}}{(k.p_+)} + \frac{\gamma^o\gamma^{o'}\not{k}'\not{a}'_1}{(k.p_-)} + \frac{(ea^o)^2}{2(k.p_+)(k.p_-)}\left(\frac{\not{k}\not{a}'_1\not{k}'\gamma^{o'}}{(k.p_+)} - \frac{\not{k}\gamma^{o'}\not{k}'\not{a}'_1}{(k.p_-)}\right)\Bigg\}B_rC_r^+ \\
& + \frac{e}{2}\Bigg\{\frac{-\gamma^o\not{a}'_2\not{k}'\gamma^{o'}}{(k.p_+)} + \frac{\gamma^o\gamma^{o'}\not{k}'\not{a}'_2}{(k.p_-)} + \frac{(ea^o)^2k^o}{2(k.p_+)(k.p_-)}\left(\frac{\not{k}\not{a}'_2\not{k}'^o\gamma^{o'}}{(k.p_+)} - \frac{\not{k}\gamma^{o'}\not{k}'\not{a}'_2}{(k.p_-)}\right)\Bigg\}B_rD_r^+ \\
& + \frac{e}{2}\Bigg\{\frac{-\gamma^o\not{k}\not{a}_1\gamma^{o'}}{(k.p_+)} + \frac{\not{a}_1\not{k}\gamma^o\gamma^{o'}}{(k.p_-)} + \frac{(ea^o)^2k^{o'}}{2(k.p_+)(k.p_-)}\left(\frac{\gamma^o\not{k}\not{a}_1\not{k}'}{(k.p_+)} - \frac{\not{a}_1\not{k}\gamma^o\not{k}'}{(k.p_-)}\right)\Bigg\}C_rB_r^+ \\
& + \frac{e^2}{4}\Bigg\{\frac{-\gamma^o\not{k}\not{a}_1\not{a}'_1\not{k}'\gamma^{o'}}{(k.p_+)^2} + \frac{\gamma^o\not{k}\not{a}_1\gamma^{o'}\not{k}'\not{a}'_1}{(k.p_+)(k.p_-)} + \frac{\not{a}_1\not{k}\gamma^o\not{a}'_1\not{k}'\gamma^{o'}}{(k.p_+)(k.p_-)} - \frac{\not{a}_1\not{k}\gamma^o\gamma^{o'}\not{k}'\not{a}'_1}{(k.p_-)^2}\Bigg\}\big|C_r\big|^2 \\
& + \frac{e^2}{4}\Bigg\{\frac{-\gamma^o\not{k}\not{a}_1\not{a}'_2\not{k}'\gamma^{o'}}{(k.p_+)^2} + \frac{\gamma^o\not{k}\not{a}_1\gamma^{o'}\not{k}'\not{a}'_2}{(k.p_+)(k.p_-)} + \frac{\not{a}_1\not{k}\gamma^o\not{a}'_2\not{k}'\gamma^{o'}}{(k.p_+)(k.p_-)} - \frac{\not{a}_1\not{k}\gamma^o\gamma^{o'}\not{k}'\not{a}'_2}{(k.p_-)^2}\Bigg\}C_rD_r^+ \\
& + \frac{e}{2}\Bigg\{\frac{-\gamma^o\not{k}\not{a}_2\gamma^{o'}}{(k.p_+)} + \frac{\not{a}_2\not{k}\gamma^o\gamma^{o'}}{(k.p_-)} + \frac{(ea)^2k^{o'}}{2(k.p_+)(k.p_-)}\left(\frac{\gamma^o\not{k}\not{a}_2\not{k}'}{(k.p_+)} - \frac{\not{a}_2\not{k}\gamma^o\not{k}'}{(k.p_-)}\right)\Bigg\}D_rB_r^+ \\
& + \frac{e^2}{4}\Bigg\{\frac{-\gamma^o\not{k}\not{a}_2\not{a}'_1\not{k}'\gamma^{o'}}{(k.p_+)^2} + \frac{\gamma^o\not{k}\not{a}_2\gamma^{o'}\not{k}'\not{a}'_1}{(k.p_+)(k.p_-)} + \frac{\not{a}_2\not{k}\gamma^o\not{a}'_1\not{k}'\gamma^{o'}}{(k.p_+)(k.p_-)} - \frac{\not{a}_2\not{k}\gamma^o\gamma^{o'}\not{k}'\not{a}'_1}{(k.p_-)^2}\Bigg\}D_rC_r^+ \\
& + \frac{e^2}{4}\Bigg\{\frac{-\gamma^o\not{k}\not{a}_2\not{a}'_2\not{k}'\gamma^{o'}}{(k.p_+)^2} + \frac{\gamma^o\not{k}\not{a}_2\gamma^{o'}\not{k}'\not{a}'_2}{(k.p_+)(k.p_-)} + \frac{\not{a}_2\not{k}\gamma^o\not{a}'_2\not{k}'\gamma^{o'}}{(k.p_+)(k.p_-)} - \frac{\not{a}_2\not{k}\gamma^o\gamma^{o'}\not{k}'\not{a}'_2}{(k.p_-)^2}\Bigg\}\big|D_r\big|^2\Bigg].
\end{aligned}
\tag{3.2.9}
$$

Let as evaluate each trace in Eq. (3.2.9) using trace technology

$$Tr\left[-\gamma^o\gamma^{o'}+(ea)^2\frac{k^{o'}\gamma^o\not{k}'+k^o\not{k}\gamma^{o'}}{2(k.p_+)(k.p_-)}-(ea)^4k^ok^{o'}\frac{\not{k}\not{k}'}{4(k.p_+)(k.p_-)^2}\right]\left|B_r\right|^2=x_r\left|B_r\right|^2,$$

$$\frac{e}{2}\,Tr\left[\frac{-\gamma^o\not{a}_1'\not{k}'\gamma^{o'}}{(k.p_+)}+\frac{\gamma^o\gamma^{o'}\not{k}'\not{a}_1'\not{p}_-}{(k.p_-)}+\frac{(ea^o)^2}{2(k.p_+)(k.p_-)}\left(\frac{\not{k}\not{a}_1'\not{k}'\gamma^{o'}}{(k.p_+)}-\frac{\not{k}\gamma^{o'}\not{k}'\not{a}_1'}{(k.p_-)}\right)\right]B_rC_r^+=y_rB_rC_r^+,$$

$$\frac{e}{2}\,Tr\left[\frac{-\gamma^o\not{a}_2'\not{k}'\gamma^{o'}}{(k.p_+)}+\frac{\gamma^o\gamma^{o'}\not{k}'\not{a}_2'}{(k.p_-)}+\frac{(ea^o)^2k^o}{2(k.p_+)(k.p_-)}\left(\frac{\not{k}\not{a}_2'\not{k}'^o\gamma^{o'}}{(k.p_+)}-\frac{\not{k}\gamma^{o'}\not{k}'\not{a}_2'}{(k.p_-)}\right)\right]B_rD_r^+=y_r^+B_rD_r^+,$$

$$\frac{e}{2}\,Tr\left[\frac{-\gamma^o\not{k}\not{a}_1\gamma^{o'}}{(k.p_+)}+\frac{\not{a}_1\not{k}\gamma^o\gamma^{o'}}{(k.p_-)}+\frac{(ea^o)^2k^{o'}}{(k.p_+)(k.p_-)}\left(\frac{\gamma^o\not{k}\not{a}_1\not{k}'}{(k.p_+)}-\frac{\not{a}_1\not{k}\gamma^o\not{k}'}{(k.p_-)}\right)\right]C_rB_r^+=z_rC_rB_r^+,$$

$$\frac{e^2}{4}\,Tr\left[\frac{-\gamma^o\not{k}\not{a}_1\not{a}_1'\not{k}'\gamma^{o'}}{(k.p_+)^2}+\frac{\gamma^o\not{k}\not{a}_1\gamma^{o'}\not{k}'\not{a}_1'}{(k.p_+)(k.p_-)}+\frac{\not{a}_1\not{k}\gamma^o\not{a}_1'\not{k}'\gamma^{o'}}{(k.p_+)(k.p_-)}-\frac{\not{a}_1\not{k}\gamma^o\gamma^{o'}\not{k}'\not{a}_1'}{(k.p_-)^2}\right]\left|C_r\right|^2=z_r^+\left|C_r\right|^2,$$

$$\frac{e^2}{4}\,Tr\left[\frac{-\gamma^o\not{k}\not{a}_1\not{a}_2'\not{k}'\gamma^{o'}}{(k.p_+)^2}+\frac{\gamma^o\not{k}\not{a}_1\gamma^{o'}\not{k}'\not{a}_2'}{(k.p_+)(k.p_-)}+\frac{\not{a}_1\not{k}\gamma^o\not{a}_2'\not{k}'\gamma^{o'}}{(k.p_+)(k.p_-)}-\frac{\not{a}_1\not{k}\gamma^o\gamma^{o'}\not{k}'\not{a}_2'}{(k.p_-)^2}\right]C_rD_r^+=k_rC_rD_r^+,$$

$$\frac{e}{2}\,Tr\left[\frac{-\gamma^o\not{k}\not{a}_2\gamma^{o'}}{(k.p_+)}+\frac{\not{a}_2\not{k}\gamma^o\gamma^{o'}}{(k.p_-)}+\frac{(ea)^2k^{o'}}{2(k.p_+)(k.p_-)}\left(\frac{\gamma^o\not{k}\not{a}_2\not{k}'}{(k.p_+)}-\frac{\not{a}_2\not{k}\gamma^o\not{k}'}{(k.p_-)}\right)\right]D_rB_r^+=k_r^+D_rB_r^+,$$

$$\frac{e^2}{4}\,Tr\left[\frac{-\gamma^o\not{k}\not{a}_2\not{a}_1'\not{k}'\gamma^{o'}}{(k.p_+)^2}+\frac{\gamma^o\not{k}\not{a}_2\gamma^{o'}\not{k}'\not{a}_1'}{(k.p_+)(k.p_-)}+\frac{\not{a}_2\not{k}\gamma^o\not{a}_1'\not{k}'\gamma^{o'}}{(k.p_+)(k.p_-)}-\frac{\not{a}_2\not{k}\gamma^o\gamma^{o'}\not{k}'\not{a}_1'}{(k.p_-)^2}\right]D_rC_r^+=w_rD_rC_r^+,and$$

$$\frac{e^2}{4}\,Tr\left[\frac{-\gamma^o\not{k}\not{a}_2\not{a}_2'\not{k}'\gamma^{o'}}{(k.p_+)^2}+\frac{\gamma^o\not{k}\not{a}_2\gamma^{o'}\not{k}'\not{a}_2'}{(k.p_+)(k.p_-)}+\frac{\not{a}_2\not{k}\gamma^o\not{a}_2'\not{k}'\gamma^{o'}}{((k.p_+)(k.p_-)}-\frac{\not{a}_2\not{k}\gamma^o\gamma^{o'}\not{k}'\not{a}_2'}{(k.p_-)^2}\right]\left|D_r\right|^2\Bigg]=w_r^+\left|D_r\right|^2.$$

Where the abbreviations

$$x_r=4\left[-1+\frac{2(ea^ok^o)^2}{(k.p_+)(k.p_-)}-\frac{(ea^o)^4k^{o2}k^2}{4[(k.p_+)(k.p_-)]^2}\right],$$

$$y_r=\frac{e}{2}\left[4(a_1^o.k^o)\left(\frac{-1}{(k.p_+)}+\frac{1}{(k.p_-)}\right)+(ea^o)^2\left(\frac{2\left[k^o(a_1^o.k^o)-a_1^ok^{o2}\right]}{(k.p_+)^2(k.p_-)}-\frac{2k^o(k^o.a_1^o)-a_1^ok^{o2}}{(k.p_+)(k.p_-)^2}\right)\right],$$

$$y_r^+=\frac{e}{2}\left[4(a_2^o.k)\left(\frac{-1}{(k.p_+)}+\frac{1}{(k.p_-)}\right)+(ea^o)^2\left(\frac{2\left[k^o(a_2^o.k^o)-a_2^ok^{o2}\right]}{(k.p_+)^2(k.p_-)}-\frac{2k^o(k^o.a_2^o)-a_2^ok^{o2}}{(k.p_+)(k.p_-)^2}\right)\right],$$

$$z_r=2e\left[\frac{-(k^o.a_1^o)}{(k.p_+)}+(ea^o)^2k^o\frac{2k^o(k^o.a_1^o)-a_1^ok^{o2}}{(k.p_+)^2(k.p_-)}+\frac{(k^o.a_1^o)}{(k.p_-)}-(ea^o)^2k^o\frac{2k^o(k^o.a_1^o)-a_1^ok^{o2}}{(k.p_+)(k.p_-)^2}\right],$$

$$z_r^+=e^2\left[\frac{-(a_1^ok^o)^2}{(k.p_+)^2}+\frac{4(a_1^o.k)^{o2}-8k^oa_1^o(a_1^o.k^o)+4k^{o2}a_1^{o2}}{(k.p_+)(k.p_-)}-\frac{-(a_1^ok^o)^2}{(k.p_-)^2}\right],$$

$$k_r=e^2\Bigg[\frac{-k^{o2}(a_1^o.a_2^o)}{(k.p_+)}+\frac{4(k^o.a_2^o)(k^o.a1^o)-2k^oa_2^o(a_1^o.k^o)-6k^oa_1^o(a_2^o.k^o)+(a_1^o.a_2^o)k^{o2}+2a_1^oa_2^ok^{o2}}{(k.p_+)(k.p_-)}$$

$$-\frac{k^{o2}(a_1^o.a_2^o)}{(k.p_-)^2}\Bigg],$$

$$k_r^+ = 2e\left[\frac{-(k^o.a_2^o)}{(k.p_+)} + (ea^o)^2k^o\frac{2k^o(k^o.a_2^o) - a_2^ok^{o2}}{(k.p_+)^2(k.p_-)} + \frac{(k^o.a_2^o)}{(k.p_-)} - (ea^o)^2k\frac{2k^o(k^o.a_2^o) - a_2^ok^{o2}}{(k.p_+)(k.p_-)^2}\right],$$

$$w_r = e^2\left[\frac{-k^{o2}(a_1^o.a_2^o)}{(k.p_+)^2} + \frac{4(k^o.a_2^o)(k^o.a1^o) - 2k^oa_1^o(a_2^o.k^o) - 6k^oa_2^o(a_1^o.k^o) + (a_1^o.a_2^o)k^{o2} + 2a_1^oa_2^ok^{o2}}{(k.p_+)(k.p_-)} - \frac{k^{o2}(a_1^o.a_2^o)}{(k.p_-)^2}\right], and$$

$$w_r^+ = e^2\left[\frac{-(a_2^ok^o)^2}{(k.p_+)^2} + \frac{4(a_2^o.k^o)^2 - 8k^oa_2^o(a_2^o.k) + 4k^{o2}a_2^{o2}}{(k.p_+)(k.p_-)} + \frac{(a_2^ok^o)^2}{(k.p_-)^2}\right].$$

This implies that

$$Tr[\Gamma_r^\infty\bar{\Gamma}_{r'}^\infty] = x_r|B_r|^2 + y_rB_rC_r^+ + y_r^+B_rD_r^+ + z_rC_rB_r^+ + z_r^+|C_r|^2 + k_rC_rD_r^+ + k_r^+D_rB_r^+ + w_rD_rC_r^+ + w_r^+|D_r|^2. \quad (3.2.10)$$

Therefor

$$Tr\left[\Gamma_r^\infty\frac{\not{p}_+ - m}{2m}\bar{\Gamma}_{r'}^\infty\frac{\not{p}_- + m}{2m}\right] = \big(l_r - m^2x_n\big)|B_r|^2 + \big(l_r^+ - m^2y_n\big)B_rC_r^+ + \big(m_r - m^2y_n^+\big)B_rD_r^+ + \big(m_r^+ - m^2z_r\big)C_rB_r^+ + \big(p_r - m^2z_r^+\big)|C_r|^2 + \big(p_r^+ - m^2k_r\big)C_rD_r^+ + \big(s_r - m^2k_r^+\big)D_rB_r^+ + \big(s_r^+ - w_rm^2\big)D_rC_r^+ + \big(t_r - m^2w_r^+\big)|D_r|^2. \quad (3.2.11)$$

Then the scattering cross section from Eq(3.2.2) can be rewrite

$$\sigma = \frac{e^4m^2}{q_+^0q_-^0}\int\frac{d^3q_+}{(2\pi)^3}\int\frac{d^3q_-}{(2\pi)^3}\int d\Omega_f\frac{V}{|v_i|}\sum_{r,r'=-\infty}^{\infty}\frac{2\pi\delta(q_r^o)}{\mathbf{q}_r^4}\Big(\big(l_r - m^2x_n\big)|B_r|^2 + \big(l_r^+ - m^2y_n\big)B_rC_r^+ + \big(m_r - m^2y_n^+\big)B_rD_r^+ + \big(m_r^+ - m^2z_r\big)C_rB_r^+ + \big(p_r - m^2z_r^+\big)|C_r|^2 + \big(p_r^+ - m^2k_r\big)C_rD_r^+ + \big(s_r - m^2k_r^+\big)D_rB_r^+ + \big(s_r^+ - w_rm^2\big)D_rC_r^+ + \big(t_r - m^2w_r^+\big)|D_r|^2\Big). \quad (3.2.12)$$

Chapter 4

Conclusion

In this thesis we have first studied in chapter 2 the creation of electron- positron pairs by the projection of a relativistic muon with impact on intense laser beams through the process $\mu+r\omega \to \mu'+e^-e^+$. The Feynman diagrams for this process are given in Fig. 2.1. In this calculation we have considered the contribution of the first four diagrams which are actually third order correction to S_{fi}. However they reduced to second order correction diagrams after that Volkov wave functions for muon, electron as well as positron involved in this process.

We calculate not only the production rate (R), but also the total cross section σ for this process in this thesis. However in [6] only the production rate for this process has been worked out.

Secondly in chapter 3 we studied the creation of electron- positron pairs by the projection of proton in laser field through the process $p + r\omega \to p' + e^-e^+$ and the Feynman diagram is shown in the Fig. 3.1. In this case the diagram is 2^{nd} order process. It is reduced to 1^{st} order process by regarding p as massive in comparison to e^- as well as e^+ and hence does not move during and after interaction. We thus work in p rest frame and the Feynman diagram is shown in Fig. 3.2. In this case also we calculate the production rate and total cross section σ. However the calculation of σ for this process is not done in [6].

Appendix

From cylindrical Bessel function expansion of trigonometric functions relative to muno projectile of electron-positron scattring.

$$e^{-i\alpha sin(kx-\eta_o)} = \sum_{n=-\infty}^{\infty} e^{-nk.x} J_n(\alpha) e^{in\eta_o}$$

$$cos(kx)e^{-i\alpha sin(kx-\eta_o)} = \sum_{n=-\infty}^{\infty} \frac{e^{-nk.x}}{2} \left(e^{(n+1)\eta_o} J_{n+1}(\alpha) + e^{(n-1)\eta_o} J_{n-1}(\alpha)\right)$$

$$sin(kx)e^{-i\alpha sin(kx-\eta_o)} = \sum_{n=-\infty}^{\infty} \frac{e^{-nk.x}}{2i} \left(e^{(n+1)\eta_o} J_{n+1}(\alpha) - e^{(n-1)\eta_o} J_{n-1}(\alpha)\right)$$

From cylindrical Bessel function expansion of trigonometric functions for scattring and incoming muon.

$$e^{-i\beta sin(ky-k_o)} = \sum_{N} J_N(\beta) e^{-iNky} e^{iN\beta k_o}$$

$$cos(kx)e^{-i\beta sin(k-k_o)} = \sum_{N} \frac{e^{-Nk.y}}{2} \left(e^{(N+1)k_o} J_{N+1}(\beta) + e^{(N-1)\beta_o} J_{N-1}(\beta)\right)$$

$$sin(kx)e^{-i\beta sin(ky-k_o)} = \sum_{N} \frac{e^{-Nk.y}}{2i} \left(e^{(n+1)\beta_o} J_{N+1}(\beta) - e^{(N-1)k_o} J_{N-1}(\beta)\right)$$

From cylindrical Bessel function expansion of trigonometric functions relative to proton at rest of electron-positron scattring.

$$e^{-i\omega sin(kx-\phi_o)} = \sum_{r=-\infty}^{\infty} e^{-irkx} e^{ir\phi_o} J_r(\omega)$$

$$cos(kx)e^{-i\omega sin(kx-\phi_o)} = \sum_{r=-\infty}^{\infty} \frac{1}{2} e^{-irkx} \left[e^{i(r+1)\phi_o} J_{r+1}(\omega) + e^{i(r-1)\phi_o} J_{r-1}(\omega)\right]$$

$$sin(kx)e^{-i\omega sin(kx-\phi_o)} = \sum_{r=-\infty}^{\infty} \frac{1}{2i} e^{-irkx} \left[e^{i(r+1)\phi_o} J_{r+1}(\omega) - e^{i(r-1)\phi_o} J_{r-1}(\omega)\right]$$

Bibliography

[1] M. Marklund, P. Shukla, Rev. Mod. Phys. 78, 591 (2006).

[2] Y. I. Salamin, S. X. Hu, K. Z. Hatsagortsyan, and C. H.Keitel, Phys. Rep. 427, 41 (2006).

[3] F. Ehlotzky, K. Krajewska, and J. Z. Kami ski, Rep. Prog. Phys. 72, 046401 (2009).

[4] H. Bethe and W. Heitler, Proc. Roy. Soc. London A 146, 83 (1934).

[5] G. Breit and J. A. Wheeler, Phys. Rev. 46, 1087 (1934).

[6] Sarah J. Muller and Carsten Muller, arXiv: 0907. 5139v1 [hep-ph] (2009).

[7] D. M. Volkov, Z. Phys. 94, 250 (1935).

[8] V. B. Berestetskii, E. M. Lifshitz, and L. P. Pitaevskii, Relativistic Quantum Theory (Pergamon, New York, 1971).

[9] W. Becker and H. Mitter, J. Phys. A 8, 1638 (1975); V. N. Baier et al., Zh. Eksp. Teor. Fiz. 69, 783 (1975) [Sov. Phys. JETP 42, 400 (1976)].

[10] E. Ltstedt, U. D. Jentschura, and C. H. Keitel, Phys. Rev. Lett. 98, 043002 (2007).

[11] M. Abramowitz and I. A. Stegun, Handbook of Mathe- matical Functions (Dover, New York, 1965).

[12] W. Greiner, J. Reinhart, Quantum Electrodynamics, 3rd edn, Springer-Verlag, Berlin, 1994.

Declaration

This thesis is my original work, has not been presented for a degree in any other University and that all the sources of material used for the thesis have been dully acknowledged.

Name: Lijalem Kelemu

Signature:− − − − − − − − − − − −

Place and time of submission: Addis Ababa University, May 2012

This thesis has been submitted for examination with my approval as University advisor.

Name: Dr. S. Bhatnagar

Signature:− − − − − − − − − −−